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Algebraic Aspects of Compatible Poisson Structures

by

Pumei Zhang

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University

June 2012

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Acknowledgement

First of all, I would like to express my deepest gratitude to my supervisor, Professor Alexey Bolsinov for giving me the opportunity to work with him, and who also provided excellent and enthusiastic supervision throughout the pursuit of my research at Loughborough University. Indeed, I could not have imagined having a better advisor and mentor for my PhD. I wish to express my sincerest appreciation for his expert guidance, advise and assistance to improve the work presented in this thesis.

Moreover, I must also thank my parents very much for their continuous unwaivering support and encouragement. My academic achievement would not have been possible without them.

Finally I would be deeply grateful to Paul Jones, who did the proofreading for me, and Wei King Tiong, who helped me with the LaTex coding of the thesis, and Yue Wu, who helped me in many situations, and all the academic and support staff at the Mathematical Sciences Department at Loughborough University for their kind and helpful assistance in many different ways.
Abstract

This thesis consists of three chapters. In Chapter one, we introduce some notions and definitions for basic concepts of the theory of integrable bi-Hamiltonian systems. Brief statements of several open problems related to our main results are also mentioned in this part.

In Chapter two, we applied the so-called Jordan–Kronecker decomposition theorem to study algebraic properties of the pencil $\mathcal{P}$ generated by two constant compatible Poisson structures on a vector space. In particular, we study the linear automorphism group $G_\mathcal{P}$ that preserves $\mathcal{P}$. In classical symplectic geometry, many fundamental results are based on the symplectic group, which preserves the symplectic structure. Therefore in the theory of bi-Hamiltonian structures, we hope $G_\mathcal{P}$ also plays a fundamental role.

In Chapter three, we study one of the famous Poisson pencils which is sometimes called “argument shift pencil”. This pencil is defined on the dual space $\mathfrak{g}^*$ of an arbitrary Lie algebra $\mathfrak{g}$.

This pencil is generated by the Lie-Poisson bracket $\{\ ,\ \}$ and constant bracket $\{\ ,\ \}_a$ for $a \in \mathfrak{g}^*$. Thus we may apply the Jordan–Kronecker decomposition theorem to introduce the so-called Jordan–Kronecker invariants of a finite-dimensional Lie algebra $\mathfrak{g}$. These invariants can be understood as the algebraic type of the canonical Jordan–Kronecker form for the “argument shift pencil” at a generic point.

Jordan–Kronecker invariants are found for all low-dimensional Lie algebras ($\dim \mathfrak{g} \leq 5$) and can be used to construct the families of polynomials in bi-involution. The results are found to be useful in the discussion of the existence of a complete family of polynomials in bi-involution w.r.t. these two brackets $\{\ ,\ \}$ and $\{\ ,\ \}_a$. 
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Chapter 1

Basic notions

1.1 Poisson brackets and Poisson manifolds

Hamiltonian ODE systems are very important subject in Mathematics, Physics and Mechanics. When we study a Hamiltonian ODE system, it is very important to consider Poisson brackets on Poisson manifolds.

1.1.1 Poisson bracket and Poisson manifold

Poisson manifolds play a fundamental role in Hamiltonian mechanics, where they serve as phase space (cotangent bundle). In Poisson geometry, the Poisson manifold we study which is generalized phase space and not necessary even dimensional. When constructing Hamiltonian mechanics, instead of symplectic structure on the manifold, one takes a Poisson bracket as the initial structure, and it is not necessary assumed to be non-degenerate.

Definition 1.1. A Poisson manifold is a smooth manifold $M$ endowed with a Poisson bracket, which is defined to be a bilinear operation $\{\cdot, \cdot\}$ on the space of smooth functions on $M$ [39, 9]:

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying three properties:
• skew-symmetry: \( \{f, g\} = -\{g, f\} \);

• Jacobi identity: \( \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \);

• Leibnitz identity: \( \{f, gh\} = \{f, g\}h + g\{f, h\} \).

for all \( f, g, h \in C^\infty(M) \).

In other words, \( C^\infty(M) \), equipped with the Poisson bracket \( \{ \, , \} \), is a Lie algebra whose Lie bracket satisfies the Leibniz identity.

### 1.1.2 Poisson tensor

Poisson tensor field (see [9], page 5-9) or Poisson structure \( \mathcal{A} \) allows us to define each Poisson bracket in local coordinates. It is easily verified that locally each Poisson bracket can equivalently be defined by the following formula:

\[
\{f(x), g(x)\} = \mathcal{A}(df(x), dg(x)) = \sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.
\]

where \( \mathcal{A} \) is a skew-symmetric \((2, 0)\) tensor field (bivector field) \( A^{ij} \) satisfying the Jacobi identity relation

\[
\sum_{\alpha} \left( A^{i\alpha} \frac{\partial A^{jk}}{\partial x^{\alpha}} + A^{j\alpha} \frac{\partial A^{ki}}{\partial x^{\alpha}} + A^{k\alpha} \frac{\partial A^{ij}}{\partial x^{\alpha}} \right) = 0. \tag{1.1}
\]

Conversely, let us denote by \( TM \wedge TM \) the space of tangent bivectors of the smooth manifold \( M \). Then \( TM \wedge TM \) is a vector bundle over \( M \), whose fiber over each point \( x \in M \) is the space \( (T_xM) \wedge (T_xM) \), which is the exterior (skew-symmetric) product of 2 copies of the tangent space \( T_xM \). In a local coordinate system \( (x_1, \ldots, x_n) \) at \( x \in M \), \( (T_xM) \wedge (T_xM) \) admits a linear basis consisting of the elements \( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \) with \( i < j \).

A Poisson tensor \( \mathcal{A} \), by definition, associates to each point \( x \) of \( M \) a bivector \( \mathcal{A}(x) \in (T_xM) \wedge (T_xM) \) in a smooth way, and satisfies the non-trivial differential relation (1.1) which amounts to the Jacobi identity. In local coordinates, \( \mathcal{A}(x) \) will have a local expression

\[
\mathcal{A}(x) = \sum_{i<j} A^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} = \frac{1}{2} \sum_{i,j} A^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.
\]
where the components \( A^j_i(x) \), called the coefficients of \( A(x) \), are smooth functions and skew-symmetric with respect to the indices. Each Poisson tensor allows us to define a Poisson bracket by setting:

\[
\{ f(x), g(x) \} = A(df(x), dg(x)) = \langle A(x), df(x) \wedge dg(x) \rangle = \left( \sum_{i<j} A^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, 2 \sum_{i<j} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} dx^i \wedge dx^j \right) = 2 \sum_{i<j} A^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.
\]

If the tensor \( A(x) \) is everywhere non-degenerate, i.e. \( \det(A(x)) \neq 0 \), then we can consider the inverse tensor \( (A(x))^{-1} \) which turns out to be a symplectic structure \( \omega(x) = (A(x))^{-1} \). The manifold \( M \) turns out to be a symplectic manifold w.r.t. the symplectic form \( \omega(x) = \sum_{i<j} \omega(x)_{ij} dx^i \wedge dx^j \) obtained from the Poisson tensor by inversion \( (\omega_{ia}A^{aj} = \delta^j_i) \).

### 1.1.3 Rank and singular set

Given a Poisson tensor \( A \) on a smooth manifold \( M \), at each point \( x \in M \), we can define a skew-symmetric matrix \( A(x) \) according to the local coordinate system of \( x \). The rank of each \( A(x) \) does not depend on the choice of local coordinates. When \( \text{rank } A(x) = \text{dim } M \) we say that \( A(x) \) is nondegenerate at \( x \).

**Definition 1.2.** The rank of \( A \) is the maximum of the rank of \( A(x) \) for all \( x \in M \):

\[
\text{rank } A = \max_{x \in M} \text{rank } A(x).
\]

We say that \( x \in M \) is regular if \( \text{rank } A(x) = \text{rank } A \), in other words, the rank of \( A(x) \) at the point \( x \) is maximal. Otherwise, \( y \in M \) is singular, i.e., \( \text{rank } A(y) < \text{rank } A \).

**Definition 1.3.** The *singular set* \( S \) of \( A \) is the set of those points where the rank of the Poisson tensor is not maximal [4]:

\[
S = \{ y \in M \mid \text{rank } A(y) < \text{rank } A \}.
\]
1.1.4 Hamiltonian vector fields

A Hamiltonian vector field on a Poisson manifold $M$ can be naturally defined for any smooth function $f : M \rightarrow \mathbb{R}$ in terms of the corresponding Poisson bracket.

**Definition 1.4.** Hamiltonian vector field for a given function $f$ on a Poisson manifold $M$, is a derivation $X_f$ acting on $C^\infty(M)$ and such that for any $g \in C^\infty(M)$, $X_f$ we have

$$X_f(g) = \{f, g\}.$$ 

To each function $f$ on a Poisson manifold $M$, one can assign a Hamiltonian vector field ($f \mapsto X_f$) in local coordinates:

$$X^j_f = \sum_i A^{ij}(x) \frac{\partial f}{\partial x^i}.$$ 

1.1.5 Splitting theorem, Casimir functions and symplectic leaves

The splitting theorem by A. Weinstein (see [41]) says that locally a Poisson manifold is a direct product of a symplectic manifold with another Poisson manifold whose Poisson tensor vanishes at a point. This splitting theorem together with the Darboux theorem will give us local canonical coordinates for Poisson manifolds.

**Theorem 1.1** (Splitting Weinstein Theorem (see [41] and [9], pages 13-14)). Let $x$ be a singular point of rank $2k$ of a Poisson $m$-dimensional manifold $M$, rank $\mathcal{A}(x) \leq \text{rank } \mathcal{A}$.

Then there is a local system of coordinates $(p_1, \ldots, p_k, q_1, \ldots, q_k, z_1, \ldots, z_{m-2k})$ in a neighborhood of $x$, which satisfy the following conditions:

- $\{q_i, q_j\} = \{p_i, p_j\} = 0$ for all $i, j$; $\{p_i, q_j\} = 0$ if $i \neq j$ and $\{p_i, q_i\} = 1$ for all $i$.
- $\{z_i, p_j\} = \{z_i, q_j\} = 0$ for all $i, j$.
- $\{z_i, z_j\} = h_{ij}(z_1, \ldots, z_{m-2k})$ for all $i, j$.
- $h_{ij}(0) = 0$ for all $i, j$. 

This theorem implies that a Poisson manifold can be split into a collection of symplectic leaves. In the above notation, the symplectic leaf passing through $x$ is defined by the equations $z_1 = z_2 = \cdots = z_{m-2k} = 0$. Each leaf is a non-degenerate submanifold of the Poisson manifold, which itself is a symplectic manifold.

**Definition 1.5.** A smooth function $f : M \to \mathbb{R}$ is called a *Casimir function* of the Poisson bracket $\{ , \}$ if

$$\{ f, g \} \equiv 0 \quad \text{for any smooth function} \quad g \in C^\infty(M).$$

In other words, the Casimir functions are those from the center of $C^\infty(M)$ viewed as the Lie algebra w.r.t. the Poisson bracket. If the Poisson tensor $\mathcal{A}$ is degenerate, i.e., rank $\mathcal{A} < \dim M$, then locally in a neighborhood of a generic point, the Casimir functions always exist. In terms of the Poisson tensor $\mathcal{A}(x)$, a Casimir function $f$ can be characterized by the system of partial differential equations:

$$\sum_i A^{ij}(x) \frac{\partial f}{\partial x^i} = 0 \quad \text{for} \quad j = 1, \ldots, \dim M. \quad (1.2)$$

Moreover, the number of functionally independent Casimir functions is exactly the corank of the Poisson tensor, $\text{corank} \, \mathcal{A} = \dim M - \text{rank} \, \mathcal{A}$, i.e., the differentials of Casimir functions generate the kernel of $\mathcal{A}(x)$ at a regular point $x$. Then a Casimir function $f$ can be characterized by the following condition: $df(x) \in \text{Ker} \, \mathcal{A}(x)$ at each point $x \in M$.

Recall that by a regular point we mean a point $x \in M$, where rank $\mathcal{A}(x) = \text{rank} \, \mathcal{A}$. In a neighborhood of such point we can find a local coordinate system $(p_1, \ldots, p_n, q_1, \ldots, q_n, z_1, \ldots, z_{m-2n})$ as in Theorem 1.1, and it is easy to see that from regularity of $x$ it follows immediately that $\{ z_i, z_j \} \equiv 0$. This implies that locally $z_1, \ldots, z_{m-2n}$ are Casimir functions and, moreover, they generate the space of all Casimir functions. Besides, the symplectic leaves can be characterized as level sets of Casimir functions:

$$\Sigma_c = \{ x \in M \mid z_i(x) = c_i, \quad i = 1, \ldots, m-2n \}.$$  

Strictly speaking, the above conclusion about the number of Casimir functions is true only locally. There are examples of degenerate Poisson tensors for which
global Casimir functions do not exist at all. However, in examples from Mechanics and Mathematical Physics, Casimir functions are usually well defined globally on the whole Poisson manifold $M$.

1.2 Examples of Poisson brackets

1.2.1 Symplectic manifold

Definition 1.6. A symplectic manifold $(M, \omega)$ is a smooth orientable even-dimensional manifold equipped with a non-degenerate closed form $\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$ called a symplectic form.

It is easy to check that the property $d\omega = 0$ is equivalent to the Jacobi identity for the inverse tensor $A = \omega^{-1}$. In other words, the inverse tensor $A = \omega^{-1}$ defines a Poisson structure on $M$.

On a symplectic manifold $(M, \omega)$, the Hamiltonian vector field $X_f$ for a smooth function $f : M \to \mathbb{R}$ can be defined by the identity:

$$\omega(\xi, X_f) = \xi(f),$$

where $\xi$ is an arbitrary vector field on $M$. If $f, g : M \to \mathbb{R}$ are smooth functions, then the Poisson bracket can be defined as $\{f, g\} = \omega(X_f, X_g) = X_f(g)$. Thus, the symplectic manifold can be viewed as a non-degenerate Poisson manifold. Usually local canonical coordinates on a symplectic manifold are denoted by $p_1, \ldots, p_n, q_1, \ldots, q_n$. They can be chosen in such a way that the matrix of the symplectic structure is written as:

$$\left(\omega_{ij}(x)\right) = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix},$$

and the corresponding matrix of the Poisson tensor is the inverse matrix of it:

$$\left(A^{ij}(x)\right) = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}.$$

The simplest example of a symplectic manifold is the symplectic space $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. Another example is a two-dimensional orientable surface. The symplectic structure on it is simply the area form.
1.2.2 Cotangent bundles

Generally speaking the cotangent bundle of a smooth manifold is the vector bundle formed by all cotangent spaces at every point of the manifold. It may be described also as the dual bundle to the tangent bundle.

**Definition 1.7.** The cotangent bundle $T^*N$ of a smooth manifold $N$ is the set of pairs $(x, p)$, where $x \in N$, $p \in T^*_x N$ [6].

The cotangent bundle $T^*N$, since it is a vector bundle, can be regarded as a smooth manifold in its own right. The charts for $T^*N$ are defined as follows. Take a chart $U$ for $N$ with local coordinates $(x_1, \ldots, x_n)$. Consider the subset in $T^*N$ corresponding to $U$:

$$T^*U = \{(x, p) \in T^*N | x \in U\}.$$ 

Since we have fixed a local coordinate system on $U$, each cotangent vector $p \in T^*_x N$, $x \in U$, is uniquely determined by its components $(p_1, \ldots, p_n)$. Thus, there is a natural bijection between $T^*U$ and $\mathbb{R}^{2n}$:

$$(x, p) \leftrightarrow (x_1, \ldots, x_n, p_1, \ldots, p_n).$$

The transition functions between such charts $T^*U_1$ and $T^*U_2$ are smooth and therefore the cotangent bundle $T^*N$ carries a natural structure of a smooth manifold of dimension $2n$.

The symplectic structure on $T^*N$ can be constructed as follows. Denote by $\pi = T^*N \rightarrow N$ the projection which assigns to each covector $p \in T^*_x N$ its base point $x$. Define the so-called **Liouville 1-form** $\alpha$ on $T^*N$. Recall that a 1-form on a manifold is a function that assigns a number to every tangent vector. Let $X$ be a tangent vector to the cotangent bundle at a point $(x, p) \in T^*N$. By definition, we set

$$\alpha(X) = p(\pi_*(X)) \quad \forall \quad X \in T_{(x,p)}(T^*N),$$

where $\pi_* : T(T^*N) \rightarrow TN$ is the natural projection generated by the projection $\pi : T^*N \rightarrow N$. It is easy to see that, in local coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ on $T^*N$, we have $\alpha = \sum_{i=1}^n p_i dx_i$. As the symplectic structure on $M = T^*N$, we take the form $\omega = da$. Obviously, it satisfies all necessary conditions.

In classical mechanics, one often deals with Hamiltonian equations on a cotangent bundle $T^*N$ equipped with the natural symplectic structure, where $N$ is the
configuration space, i.e. the space of all possible configurations or positions; 
\( M = T^*N \) is called the phase space.

### 1.2.3 Constant brackets

Let \( M = \mathbb{R}^n \) with Cartesian coordinates \((x^1, \ldots, x^n)\). Then we can define a constant Poisson bracket of two smooth functions \( f \) and \( g \) on \( M \) such that

\[
\{ f, g \} = \sum_{i,j=1}^{n} A^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},
\]

where each \( A^{ij} \) is constant and does not depend on \( x \). Here, of course, \( A^{ij} = -A^{ji} \), i.e. \( A \) is skew symmetric.

### 1.2.4 Lie-Poisson brackets

Let \( g \) be a finite dimensional Lie algebra and \( e_1, \ldots, e_n \) be a basis of \( g \). Consider the structure constants \( c^k_{ij} \) of \( g \) w.r.t. this basis. Then we define the bivector \( A(x) = (\sum_{k=1}^{n} c^k_{ij} x_k) \) on the dual space \( g^* \), where \( x = (\sum_{i=1}^{n} x_i e^i) \in g^* \) and \( e^1, \ldots, e^n \) is a dual basis of \( g^* \).

**Definition 1.8.** \( A(x) = (\sum_{k=1}^{n} c^k_{ij} x_k) \) is called a Lie-Poisson tensor.

Let \( M = g^* \). Then for any two smooth functions \( f, g : g^* \to \mathbb{R} \) we can define a Lie-Poisson bracket (also called linear Poisson bracket) on \( g^* \) by:

\[
\{ f(x), g(x) \} = x([df(x), dg(x)]) = \sum_{i,j,k} c^k_{ij} x_k \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \quad x \in g^*, df(x), dg(x) \in g.
\]

**Proposition 1.1.** \( A(x) \) is indeed a Poisson tensor, i.e. the Lie-Poisson bracket satisfies the Jacobi identity.

**Proof.** For any three functions \( f, g, h : g^* \to \mathbb{R} \) we have

\[
\{ \{ f, g \}, h \} + \{ \{ g, h \}, f \} + \{ \{ h, f \}, g \} = \sum_{\alpha,\beta} (c^\alpha_{\alpha i} c^\beta_{\beta j} + c^\alpha_{\alpha j} c^\beta_{\beta i}) x^\alpha_{\beta i} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \frac{\partial h}{\partial x^k}
\]

\[
= 0 \quad \text{(since } c^k_{ij} \text{ satisfies the Jacobi identity)}.\]
In other words the Jacobi identity for the Lie-Poisson bracket on $\mathfrak{g}^*$ is equivalent to the Jacobi identity in the Lie algebra $\mathfrak{g}$.

1.3 Compatible Poisson brackets and bi-Hamiltonian systems

In mechanics, mathematical physics and differential geometry, many completely integrable Hamiltonian systems have the remarkable property of being bi-Hamiltonian, i.e., they are Hamiltonian with respect to compatible Poisson brackets.

1.3.1 Compatible Poisson tensors

**Definition 1.9.** Two Poisson tensors $\mathcal{A}$ and $\mathcal{B}$ are said to be *compatible* if their sum (or, equivalently, an arbitrary linear combination of $\mathcal{A}$ and $\mathcal{B}$ with constant coefficients) is again a Poisson tensor. Such a family of Poisson tensors is often called a *pencil* of Poisson tensors $\mathcal{P} = \{\mathcal{A} + \lambda\mathcal{B}\}$. (A similar definition is used for compatible Poisson brackets.) [20, 13, 32, 2]

Let us derive the compatibility equation that guarantees that the sum of two Poisson brackets is again a Poisson bracket. Let $\{\ , \}_\mathcal{A}$ and $\{\ , \}_\mathcal{B}$ be two compatible Poisson brackets w.r.t. the two compatible Poisson tensors $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the sum of the two compatible Poisson brackets $\{\ , \}_\mathcal{A} + \{\ , \}_\mathcal{B} = \{\ , \}_{\mathcal{A} + \mathcal{B}}$ must be again a Poisson bracket with the Poisson tensor $\mathcal{A} + \mathcal{B}$, which is the sum of the two compatible Poisson tensors $\mathcal{A}$ and $\mathcal{B}$. It is clear that the new bracket $\{\ , \}_{\mathcal{A} + \mathcal{B}}$ satisfies the skew-symmetry and Leibnitz identity, but for the Jacobi identity we need to require, in addition, that for any $f, g, h \in C^\infty(M)$ the following identity holds:

$$\{\{f, g\}_\mathcal{A + B}, h\}_\mathcal{A + B} + \{\{g, h\}_\mathcal{A + B}, f\}_\mathcal{A + B} + \{\{h, f\}_\mathcal{A + B}, g\}_\mathcal{A + B} = 0.$$

Taking into account the Jacobi identities for $\mathcal{A}$ and $\mathcal{B}$ separately, we can rewrite this compatibility equation as

$$\{\{f, g\}_\mathcal{B}, h\}_\mathcal{A} + \{\{f, g\}_\mathcal{A}, h\}_\mathcal{B} + \{\{g, h\}_\mathcal{B}, f\}_\mathcal{A} + \{\{g, h\}_\mathcal{A}, f\}_\mathcal{B} + \{\{h, f\}_\mathcal{B}, g\}_\mathcal{A} + \{\{h, f\}_\mathcal{A}, g\}_\mathcal{B} = 0.$$
In local coordinates, we define

$$ \{f, g\}_A = \sum_{i,j} A^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, $$

$$ \{f, g\}_B = \sum_{i,j} B^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, $$

and

$$ \{f, g\}_{A+B} = \{f, g\}_A + \{f, g\}_B = \sum_{i,j} (A^{ij}(x) + B^{ij}(x)) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}. $$

The compatibility equation in the local coordinates is

$$ \sum_\alpha \left( A^{\alpha i} \frac{\partial B^{jk}}{\partial x^\alpha} + B^{\alpha i} \frac{\partial A^{jk}}{\partial x^\alpha} + A^{\alpha j} \frac{\partial B^{ki}}{\partial x^\alpha} + B^{\alpha j} \frac{\partial A^{ki}}{\partial x^\alpha} + A^{\alpha k} \frac{\partial B^{ij}}{\partial x^\alpha} + B^{\alpha k} \frac{\partial A^{ij}}{\partial x^\alpha} \right) = 0. $$

(1.3)

If $A$ and $B$ are both Lie-Poisson tensors such that $A_{ij} = \sum_\beta c^\beta_{ij} x_\beta$ and $B_{ij} = \sum_\beta \tilde{c}^\beta_{ij} x_\beta$, then the above equation 1.3 turns out to be

$$ \sum_{\alpha, \beta} \left( c^\beta_{ai} \frac{\partial (\tilde{c}^\beta_{jk} x_\beta)}{\partial x^\alpha} + c^\beta_{ai} \frac{\partial (\tilde{c}^\beta_{jk} x_\beta)}{\partial x^\alpha} + c^\beta_{aj} \frac{\partial (\tilde{c}^\beta_{ki} x_\beta)}{\partial x^\alpha} + \tilde{c}^\beta_{aj} \frac{\partial (\tilde{c}^\beta_{ki} x_\beta)}{\partial x^\alpha} + c^\beta_{ak} \frac{\partial (\tilde{c}^\beta_{ij} x_\beta)}{\partial x^\alpha} + \tilde{c}^\beta_{ak} \frac{\partial (\tilde{c}^\beta_{ij} x_\beta)}{\partial x^\alpha} \right) x_\beta = 0. $$

(1.4)

Simplifying we obtain

$$ \sum_{\alpha, \beta} \left( c^\beta_{ai} \tilde{c}^\alpha_{jk} + c^\beta_{ai} \tilde{c}^\alpha_{jk} + c^\beta_{aj} \tilde{c}^\alpha_{ki} + \tilde{c}^\beta_{aj} c^\alpha_{ki} + c^\beta_{ak} \tilde{c}^\alpha_{ij} + \tilde{c}^\beta_{ak} c^\alpha_{ij} \right) x_\beta = 0. $$

(1.5)

Let $A$ and $B$ be compatible Poisson tensors and $\mathcal{P} = \{A + \lambda B\}$ be the pencil that is generated by $A$ and $B$. Then the rank of $A$ and $B$ on $M$ is defined by (see Section 1.1.3)

$$ \text{rank } A = \max_{x \in M} \text{rank } A(x), $$

$$ \text{rank } B = \max_{x \in M} \text{rank } B(x), $$

and the rank of $\mathcal{P}$ on $M$ is

$$ \text{rank } \mathcal{P} = \max_{x \in M} \max_{\lambda \in \mathbb{R}} \text{rank } \left( A + \lambda B \right)(x). $$
Let $x \in M$ be a generic point. Then two essentially different situations are possible. It may happen that the rank of $\mathcal{A}(x) + \lambda \mathcal{B}(x)$ is equal to rank $\mathcal{P}$ for all $\lambda \in \bar{C}$ (here we use the following convention: if $\lambda = \infty$ we replace $\mathcal{A} + \lambda \mathcal{B}$ by $\mathcal{B}$). One of the simplest examples of this kind is

$$\mathcal{A} = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}.$$  

We will refer to such pencils as Poisson pencils of (pure) Kronecker type. The meaning of this terminology will be clarified in the next Chapter. The other possibility is that for $x$ (and since $x$ is generic, for all neighboring points) there is a number $\lambda(x)$ (or several numbers $\lambda_1(x), \ldots, \lambda_r(x) \in \mathbb{C}$) such that

$$\text{rank } (\mathcal{A}(x) + \lambda(x) \mathcal{B}(x)) < \text{rank } \mathcal{P}.$$  

**Definition 1.10.** $\lambda(x)$ is called a characteristic number of the pencil $\mathcal{P}$ at the point $x \in M$.

Locally in a neighborhood of a generic point, $\lambda(x)$ can be understood as a smooth function which we will still call a characteristic number of $\mathcal{P}$. A Poisson pencil, of course, may have several characteristic numbers $\lambda_1(x), \ldots, \lambda_r(x)$. If $\mathcal{B}$ is non-degenerate, these numbers are exactly the eigenvalues of the so-called recursion operator $-\mathcal{B}^{-1}\mathcal{A}$.

Notice that for some special (non-generic points) the characteristic numbers may exist even for pencils of Kronecker type. In the general case, the number of distinct characteristic numbers may both drop and increase at non-generic points.

Speaking of characteristic numbers, it is also important to distinguish two cases. It may happen that for certain $\lambda \in \bar{C}$, the rank of $\mathcal{A} + \lambda \mathcal{B}$ drops at each point of $M$, i.e. this characteristic number is constant on the whole manifold $M$ or, equivalently, $\text{rank } (\mathcal{A} + \lambda \mathcal{B}) < \text{rank } \mathcal{P}$. Such brackets in the pencil are sometimes called exceptional or singular. If $\text{rank } (\mathcal{A} + \lambda \mathcal{B}) = \text{rank } \mathcal{P}$, then we call $\mathcal{A} + \lambda \mathcal{B}$ generic. The second possibility is that all brackets in the pencil are generic, but the characteristic numbers $\lambda_i(x)$ still exist. In this case, at each point $x \in M$ some linear combination $\mathcal{A} + \lambda_i(x) \mathcal{B}$ drops rank, but $\lambda_i(x)$ varies from point to point.
1.3.2 Bi-Hamiltonian systems

Let $\mathcal{P} = \{A + \lambda B\}$ be a pencil of compatible Poisson brackets on $M$. There is no common agreement about the definition of a bi-Hamiltonian system in this case. At least three different versions make sense.

**Definition 1.11.** A vector field $X(x)$ on $M$ is called *bi-Hamiltonian* if one of the following conditions hold [4]:

1. $X(x)$ is Hamiltonian w.r.t. $A$ and $B$, i.e. there exist two smooth functions $f$ and $g$ such that $X(x) = A df(x) = B dg(x)$;

2. $X(x)$ is Hamiltonian w.r.t. generic Poisson tensors $A_\lambda \in \mathcal{P}$, i.e. there exists a family of functions $f_\lambda$ such that $X(x) = A_\lambda df_\lambda(x)$, $A_\lambda = A + \lambda B$, $\lambda$ is generic;

3. $X(x)$ is Hamiltonian w.r.t. all Poisson tensors $A_\lambda \in \mathcal{P}$, i.e. there exists a family of functions $f_\lambda$ such that $X(x) = A_\lambda df_\lambda(x)$, $A_\lambda = A + \lambda B$, $\lambda \in \mathbb{R}$.

The corresponding system of ODEs, i.e. $\dot{x} = X(x)$ is called *bi-Hamiltonian system*.

These three cases are not equivalent. The corresponding examples can be easily constructed for a pencil consisting of constant brackets. For example:

1. In $\mathbb{R}^5(x_1, x_2, x_3, x_4, x_5)$, the vector field $X(x) = (0, 0, 0, 1, 0)$ is Hamiltonian w.r.t.

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 0 & 1 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix}.
\]

Indeed, $X(x) = A df(x) = B dg(x)$ with $f = -x_2$ and $g = -x_1$. But there is no function $h$ such that $X(x) = (A + \lambda B)dh(x)$ for $\lambda \neq 0, \infty$.

2. In $\mathbb{R}^2(x_1, x_2)$, the vector field $X(x) = (1, -1)$ is Hamiltonian w.r.t. all generic $A_\lambda = A + \lambda B$, where

\[
A = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
0 & 2 \\
-2 & 0
\end{pmatrix}.
\]
Indeed, $X(x) = A_\lambda df_\lambda(x)$ with $f_\lambda = (\frac{1}{1+2\lambda})(x_1 + x_2)$. But $\lambda = -\frac{1}{2}$ is an exceptional value of the parameter, for which $A - \frac{1}{2}B = 0$, i.e., there exists no non-trivial Hamiltonian vector fields.

3. In $\mathbb{R}^3(x_1, x_2, x_3)$, the vector field $X(x) = (3y-2z, z-3x, 2x-y)$ is Hamiltonian w.r.t. all $A_\lambda \in \mathcal{P}$, where

$$
A = \begin{pmatrix}
0 & x_3 & -x_2 \\
-x_3 & 0 & x_1 \\
x_2 & -x_1 & 0
\end{pmatrix}
$$

and $B = \begin{pmatrix}
0 & 3 & -2 \\
-3 & 0 & 1 \\
2 & -1 & 0
\end{pmatrix}$.

Because $X(x) = A_\lambda df_\lambda(x)$ with $f_\lambda = -x_1 - 2x_2 - 3x_3$ or for $\lambda = \infty$ we have $f_\infty = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$.

Bi-Hamiltonian systems often admit large sets of first integrals, which make them into integrable Hamiltonian systems. Conversely, a vast majority of known integrable systems turn out to be bi-Hamiltonian. The theory of bi-Hamiltonian systems starts with F. Magri (see [20]), I. Gelfand, I. Dorfman (see [13]) and A. Reyman, M. Semenov-Tian-Shansky (see [32]) and there is now a very large number of articles in the subject area.

1.3.3 Integrals of bi-Hamiltonian systems

Bi-Hamiltonian vector field always possesses many first integrals, namely Casimir functions and characteristic numbers.

Let $A$ and $B$ be compatible Poisson tensors, and let $X$ be a bi-Hamiltonian vector field in the sense that $X$ is Hamiltonian w.r.t. any non-trivial linear combination of $A$ and $B$. Then the following 3 well-known propositions hold (see [32], [21], [19], [3]).

**Proposition 1.2.** If $X$ is a Hamiltonian vector field with respect to $C = \lambda A + \mu B$ for all $\lambda, \mu \in \mathbb{R}$, then every Casimir function of $C$ is a first integral of $X$.

**Proof.** Let $X$ be a Hamiltonian vector field w.r.t. $C$ with the local coordinates $(x_1, \ldots, x_n)$, i.e. there exists a Hamiltonian function $H$ such that $X^i = \sum_n C^{ij} \frac{\partial H}{\partial x^j}$. 

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Consider an arbitrary Casimir function $f$ of $\mathcal{C}$, then by the definition, in local coordinates we have

$$X(f) = \sum_j X^j \frac{\partial f}{\partial x^j} = \sum_{ij} C_{ij} \frac{\partial H}{\partial x^i} \frac{\partial f}{\partial x^j} = \{H, f\} = 0,$$

which means $f$ is constant w.r.t. $X$. Thus $f$ is a first integral of $X$.

\[
\square
\]

**Proposition 1.3.** If $X$ is a Hamiltonian vector field w.r.t. $\mathcal{C}$, then the Lie derivative of $\mathcal{C}$ w.r.t. $X$ vanishes, $L_X \mathcal{C} = 0$. In other words, the flow of $X$ preserves $\mathcal{C}$.

**Proof.** The Poisson tensor $\mathcal{C}$ satisfied the Jacobi identity in local coordinates so that (see equation 1.1)

$$\sum_{\alpha} \left( C^{\alpha i} \frac{\partial C^{jk}}{\partial x^\alpha} + C^{\alpha j} \frac{\partial C^{ki}}{\partial x^\alpha} + C^{\alpha k} \frac{\partial C^{ij}}{\partial x^\alpha} \right) = 0.$$

Let $X$ be Hamiltonian vector field w.r.t. $\mathcal{C}$, then there exists a Hamiltonian function $H$ such that $X^\alpha = \sum_i C^{\alpha i} \frac{\partial H}{\partial x^i}$.

Now, we compute the Lie derivative of $\mathcal{C}$ w.r.t. $X$. We have

$$L_X \mathcal{C} = \sum_{\alpha} \left( X^\alpha \frac{\partial C^{jk}}{\partial x^\alpha} - C^{\alpha j} \frac{\partial X^k}{\partial x^\alpha} - C^{\alpha k} \frac{\partial X^j}{\partial x^\alpha} \right)$$

$$= \sum_{\alpha, i} \left( C^{\alpha i} \frac{\partial H}{\partial x^i} \frac{\partial C^{jk}}{\partial x^\alpha} - C^{\alpha j} \frac{\partial C^{ki}}{\partial x^\alpha} \frac{\partial H}{\partial x^i} - C^{\alpha k} \frac{\partial C^{ij}}{\partial x^\alpha} \frac{\partial H}{\partial x^i} \right)$$

$$= \sum_{\alpha, i} \left( C^{\alpha i} \frac{\partial C^{jk}}{\partial x^i} \frac{\partial H}{\partial x^\alpha} - C^{\alpha j} \frac{\partial C^{ki}}{\partial x^\alpha} \frac{\partial H}{\partial x^i} - C^{\alpha k} \frac{\partial C^{ij}}{\partial x^\alpha} \frac{\partial H}{\partial x^i} \right) +$$

$$+ C^{\alpha k} C^{ji} \frac{\partial^2 H}{\partial x^\alpha \partial x^i} - C^{\alpha k} C^{ji} \frac{\partial^2 H}{\partial x^\alpha \partial x^i}$$

$$= \sum_{\alpha, i} \left( C^{\alpha i} \frac{\partial C^{jk}}{\partial x^\alpha} + C^{\alpha j} \frac{\partial C^{ki}}{\partial x^\alpha} + C^{\alpha k} \frac{\partial C^{ij}}{\partial x^\alpha} \right) \frac{\partial H}{\partial x^i}$$

$$= 0,$$

as required. \[
\square
\]

**Proposition 1.4.** If $X$ is a Hamiltonian vector field w.r.t. both $\mathcal{A}$ and $\mathcal{B}$, then all the characteristic numbers $\lambda_i(x)$ of the pair $\mathcal{A}$ and $\mathcal{B}$ are first integrals of $X$.  

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Proof. It follows from Proposition 1.3 that $\mathcal{L}_X A = 0$ and $\mathcal{L}_X B = 0$, i.e., the flow $X$ preserves both $A$ and $B$. Hence this flow preserves all algebraic invariants of the pair $A$ and $B$ and, in particular, the characteristic numbers $\lambda_i(x)$, i.e., $\mathcal{L}_X(\lambda_i(x)) = X(\lambda_i(x)) = 0$.

Instead of characteristic numbers, usually it is more convenient to take symmetric polynomials of them, as characteristic numbers themselves may have singularities or branching points.

Notice that in Proposition 1.4 the vector field $X$ is assumed to be bi-Hamiltonian in the weak sense, i.e., Hamiltonian w.r.t. the two forms $A$ and $B$ only (moreover, the compatibility condition can be omitted). In contrast, Proposition 1.2 requires that $X$ is bi-Hamiltonian in the strong sense (Definition 1.11, case 3). If we assume that $X$ is Hamiltonian w.r.t. $A$ and $B$, then we only can conclude that the family of Casimirs of $A + \lambda B$ remains invariant under the flow of $X$. In other words, this flow $\sigma_X^t$ transforms each Casimir function $f$ again to a Casimir function $f_t$, but in general $f \neq f_t$.

Although Propositions 1.2 and 1.4 guarantee the existence of a large family of first integrals, these integrals are not always sufficient to integrate the system. However, in many concrete examples the bi-Hamiltonian property leads to complete integrability by means of the integrals mentioned above.

1.4 Integrability and complete commutative families of functions

Let $N$ be a smooth symplectic manifold of dimension $2n$, and let $\dot{x} = X_H$ be a Hamiltonian dynamical system with a smooth Hamiltonian function $H$.

**Definition 1.12.** A Hamiltonian system is called Liouville integrable [12] if there exists a set of smooth functions $g_1, \ldots, g_n$ such that

1) $g_1, \ldots, g_n$ are first integrals of $X_H$, i.e., for any $i \in 1, \ldots, n$, $X_H(g_i) = \omega(X_{g_i}, X_H) = \{g_i, H\} = 0$,

2) they are functionally independent on $N$, i.e., their gradients are linearly
independent on $N$ almost everywhere,

3) $\{g_i, g_j\} = 0$ for any $i, j = 1, \ldots, n$,

4) the vector fields $X_{g_i}$ are complete, i.e., the natural parameter on their integral trajectories is defined on the whole real axis.

If we have $n$ functions $g_1, \ldots, g_n$ satisfying the above properties, then as the Hamiltonian $H$ we can take any combination of $g_i$’s or, more generally, any function of the form $H = H(g_1, \ldots, g_n)$. In this thesis, we are interested in the properties of the algebra of integrals rather than in the dynamics of $X_H$. So $H$ does not play any special role in our considerations and can be replaced by any other suitable functions. Also, almost all constructions discussed in the thesis are local, so Condition 4 from the above definition won’t be essential here.

In other words, by an integrable Hamiltonian system we can simply mean (following many other authors) a complete commutative family of functions on a Poisson manifold. This “completness of a commutative family” is not related to the “completeness of vector fields” from Condition 4. In the case of a symplectic manifold $N$, “completness of a commutative family” means that this family contains $n = \frac{1}{2} \dim N$ independent functions. In the case of Poisson manifolds, we will have to slightly modify this condition below.

1.4.1 The Liouville theorem

Although we do not discuss dynamics of integrable systems in the thesis, we would like to mention the classical theorem by Liouville which says that the above integrability condition leads to remarkable dynamical and analytic properties of integrable Hamiltonian systems.

Definition 1.13. The decomposition of the symplectic manifold $N$ into connected components of common level surfaces of the integrals $g_1, \ldots, g_n$ is called the Liouville foliation corresponding to the integrable system $X_H$ [12].

Since $g_1, \ldots, g_n$ are preserved by the flow of $X_H$, every leaf of the Liouville foliation is an invariant surface. The Liouville foliation consists of regular leaves
(filling \(N\) almost in the whole) and singular ones (filling a set of zero measure). The Liouville theorem formulated below describes the structure of the Liouville foliation near regular leaves (see for example, [19], [4]).

Consider a common regular level surface \(T_\xi\) for the functions \(g_1, \ldots, g_n\), that is, \(T_\xi = \{x \in N \mid g_i(x) = \xi_i, i = 1, \ldots, n\}\). The regularity means that all 1-forms \(dg_i\) are linearly independent on \(T_\xi\).

**Theorem 1.2.** Let \(X_H\) be a Liouville integrable Hamiltonian system on \(N\), and let \(T_\xi\) be a regular level surface of the integrals \(g_1, \ldots, g_n\). Then

1) \(T_\xi\) is a smooth Lagrangian submanifold that is invariant with respect to the flows of \(X_H\) and \(X_{g_1}, \ldots, X_{g_n}\);

2) if \(T_\xi\) is connected and compact, then \(T_\xi\) is diffeomorphic to the \(n\)-dimensional torus \(T^n\) (this torus is called the Liouville torus);

3) the Liouville foliation is trivial in some neighborhood of the Liouville torus, that is, a neighborhood \(U\) of the torus \(T_\xi\) is the direct product of the torus \(T^n\) and the disc \(D^n\);

4) in the neighborhood \(U = T^n \times D^n\) there exists a coordinate system \(s_1, \ldots, s_n, \phi_1, \ldots, \phi_n\) (which are called the action-angle variables), where \(s_1, \ldots, s_n\) are coordinates on the disc \(D^n\) and \(\phi_1, \ldots, \phi_n\) are standard angle coordinates on the torus, such that

\[
\omega = \sum d\phi_i \wedge ds_i,
\]

b) the action variables \(s_i\) are functions of the integrals \(g_1, \ldots, g_n\),

c) in the action-angle variables \(s_1, \ldots, s_n, \phi_1, \ldots, \phi_n\), the Hamiltonian flow of \(X_H\) is straightened on each of the Liouville tori in the neighborhood \(U\), that is, \(\dot{s}_i = 0, \ \dot{\phi}_i = q_i(s_1, \ldots, s_n)\) for \(i = 1, 2, \ldots, n\) (this means that the flow of \(X_H\) determines the conditionally periodic motion that generates a rational or irrational rectilinear winding on each of the tori).
1.4.2 Families of commuting functions and completeness condition in the Poisson case (dimension of the maximal isotropic subspace)

Let $M$ be a smooth Poisson manifold with the corresponding Poisson tensor $\mathcal{A}$, and let $X_H$ be a Hamiltonian vector field with a smooth Hamiltonian function $H$.

Consider all functionally independent Casimir functions $f_1, \ldots, f_k$. Then each $f_i$ is a first integral of $X_H$, i.e. $\{f_i, H\} = 0$. The Casimir functions are constant on each generic symplectic leaf $N = \{f_1 = c_1, \ldots, f_k = c_k\}$ of $M$. Thus $N$ is invariant under $X_H$, and each Hamiltonian system of $M$ can be restricted onto $N$.

For the integrability of the Hamiltonian system on $N$ and $\dim N = 2n$, there must exist $n$ functionally independent commuting first integrals $g_1, \ldots, g_n$ of $X_H$ on $N$ such that $\{g_i, H\} = 0$ and $\{g_i, g_j\} = 0$. Thus, the Hamiltonian system on a Poisson manifold $M$ is called integrable, if there are $n + k$ functionally independent commuting functions $f_1, \ldots, f_k, f_{k+1} = g_1, \ldots, f_{k+n} = g_n$, i.e. the set of all Casimir functions on $M$ and commuting first integrals of $X_H$ on $N$. More generally, we will say that $\dot{x} = X_H(x)$ is completely integrable on $M$, if it admits a complete family $\mathcal{F}$ of commuting first integrals. “Completeness” means that $\mathcal{F}$ contains $n + k$ independent functions. This condition can also be reformulated as follows.

Definition 1.14. Let $\mathcal{F}$ be a commutative family of functions on a Poisson manifold $(M, \mathcal{A})$. Consider

$$D_x = \text{span}\{df(x), f \in \mathcal{F}\} \subset T^*_x M,$$

the subspace of $T^*_x M$ generated by the differentials of $f \in \mathcal{F}$. The family $\mathcal{F}$ is called complete on $M$, if $D_x$ is maximal isotropic (Lagrangian) w.r.t. $\mathcal{A}(x)$ for almost all $x \in M$.

Remark 1.1. Notice that functions $f_1, \ldots, f_k$ pairwise commute with respect to a Poisson bracket if and only if their differentials generate a subspace $\text{span}\{df_1(x), \ldots, df_k(x)\} \subset T^*_x M$ which is isotropic with respect to the corresponding Poisson tensor at each point $x \in M$.

Proposition 1.5. A subspace $U \subset V = T^*_x M$ is maximal isotropic w.r.t. $\mathcal{A}$ if and only if $\dim U = \frac{1}{2}(\dim V + \text{corank} \mathcal{A})$.
Proof. First we need to prove that if \( U \) is a maximal isotropic subspace, then \( \dim U = \frac{1}{2}(\dim V + \text{corank } A) \).

Let \( U^\perp = \{ v \in V | A(v, U) = 0 \} \) be orthogonal subspace of \( U \). Since \( U \) is maximal isotropic, we have \( U^\perp = U \) and \( \dim(U^\perp) = \dim U \). On the other hand, because of \( \text{Ker } A \subset U \), we have

\[
\dim(U^\perp) = \dim V - (\dim U - \dim(U \cap \text{Ker } A)) \\
= \dim V - (\dim U - \dim(\text{Ker } A)) \\
= \dim V - \dim U + \text{corank } A,
\]

which implies \( 2 \dim U = \dim V + \text{corank } A \) and

\[
\dim U = \frac{1}{2}(\dim V + \text{corank } A).
\]

Second, we shall prove that if \( U \) is isotropic subspace w.r.t. \( A \) with \( \dim U = \frac{1}{2}(\dim V + \text{corank } A) \), then \( U \) is a maximal isotropic subspace. Since \( U \) is isotropic then \( U \subset U^\perp \), which means \( \dim U \leq \dim(U^\perp) \). On the other hand,

\[
\dim(U^\perp) = \dim V - \dim U + \dim(U \cap \text{Ker } A) \\
\leq \dim V - \dim U + \dim(\text{Ker } A) \\
= \dim V - \dim U + \text{corank } A \\
= (\dim V + \text{corank } A) - \frac{1}{2}(\dim V + \text{corank } A) \\
= \frac{1}{2}(\dim V + \text{corank } A) \\
= \dim U.
\]

This implies \( \dim U = \dim(U^\perp) \). Thus \( U = U^\perp \), which shows \( U \) is a maximal isotropic subspace. \( \square \)

Thus, the completeness condition for a family of commuting functions \( \mathcal{F} \) on a Poisson manifold \((M, A)\) can be stated as follows. \( \mathcal{F} \) is complete if and only if the number of functionally independent functions in \( \mathcal{F} \) is \( \frac{1}{2}(\dim M + \text{corank } A) \).

### 1.5 Examples of compatible Poisson brackets

In this section we discuss some elementary examples of compatible Poisson brackets.
1.5.1 Constant brackets

**Proposition 1.6.** Any two constant Poisson brackets are compatible.

**Proof.** Let $M = \mathbb{R}^m$ with Cartesian coordinates $(x_1, \ldots, x_m)$, and $\{ , \}_A$ and $\{ , \}_B$ be two constant Poisson brackets, then their corresponding Poisson tensors $A(x)$ and $B(x)$ at $x \in M$ turn to be $m \times m$ skew-symmetric matrices with fixed entries, which do not depend on $x$. Then the same is true for $C = A + B$: the matrix of $C$ does not depend on $x$ and, therefore, satisfies the Jacobi identity relation:

$$\sum_{\alpha} \left( C^i_{\alpha} \frac{\partial C^{jk}}{\partial x^\alpha} + C^j_{\alpha} \frac{\partial C^{ki}}{\partial x^\alpha} + C^k_{\alpha} \frac{\partial C^{ij}}{\partial x^\alpha} \right) = 0.$$ 

Hence $C = A + B$ is a Poisson tensor, and $A$ and $B$ are compatible. \qed

Although this example is almost trivial, all algebraic properties of compatible Poisson brackets can be observed in this simplest case.

1.5.2 Linear + constant (argument shift method)

Consider a Lie-Poisson tensor $A(x) = \left( \sum_\beta c^\beta_{ij} x_\beta \right)$ where $c^\beta_{ij}$ are structure constants for a certain Lie algebra $\mathfrak{g}$, and a constant Poisson tensor $B = B_{ij}$. These Poisson tensors are both defined on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. Let us derive the compatibility equation for the linear Poisson tensor $A$ and the constant Poisson bracket $B$.

In local coordinates $x_1, \ldots, x_n$, the compatibility equation (see equation 1.3) takes the following form:

$$\sum_{\alpha, \beta} \left( c^\alpha_{\alpha i} x_\beta \frac{\partial B^{jk}}{\partial x^\alpha} + B^{\alpha i} \frac{\partial (c^\beta_{jk} x_\beta)}{\partial x^\alpha} + c^\beta_{\alpha j} x_\beta \frac{\partial B^{ki}}{\partial x^\alpha} + B^{\alpha j} \frac{\partial (c^\beta_{ki} x_\beta)}{\partial x^\alpha} + c^\beta_{\alpha k} x_\beta \frac{\partial B^{ij}}{\partial x^\alpha} + B^{\alpha k} \frac{\partial (c^\beta_{ij} x_\beta)}{\partial x^\alpha} \right)$$

$$= \sum_{\alpha} \left( B^{\alpha i} c^\alpha_{\alpha j} + B^{\alpha j} c^\alpha_{k i} + B^{\alpha k} c^\alpha_{i j} \right)$$

$$= 0.$$ 

This is a non-trivial algebraic relation for the structure tensor $c^i_{jk}$ and the form $B_{jk}$. The following construction gives a natural way to construct a compatible form $B$ for an arbitrary Lie algebra $\mathfrak{g}$.
Example 1.1. Consider a Lie-Poisson bracket \{ , \} on \mathfrak{g}^*$, i.e.,

\[ \{f, g\}(x) = x([df(x), dg(x)]) \]

and a constant Poisson bracket \{ , \}_a defined by

\[ \{f, g\}_a(x) = a([df(x), dg(x)]), \]

where \( a \in \mathfrak{g}^* \) is a fixed element. Then the corresponding Poisson tensors are

\[ A(x) = \left( \sum_{\beta} c^\beta_{ij} x^\beta \right) \]
\[ B(x) = \left( \sum_{\beta} \tilde{c}^\beta_{ij} a^\beta \right). \]

The compatibility condition reads

\[ \sum_{\alpha, \beta} \left( c^\beta_{ij} c^\alpha_{jk} + c^\alpha_{ij} c^\beta_{ki} + c^\beta_{ij} c^\alpha_{ki} + \tilde{c}^\beta_{ij} c^\alpha_{ij} a^\beta \right) a^\beta = 0. \]

and is obviously fulfilled due to the Jacobi identity for the Lie algebra \( \mathfrak{g} \).

Thus, the brackets \{ , \} and \{ , \}_a are always compatible for every Lie algebra \( \mathfrak{g} \) and every \( a \in \mathfrak{g}^* \). These two brackets are closely related to the shift argument method (see [22], [23]), and will be discussed in detail in Chapter 3.

### 1.5.3 Two linear brackets (Lie pencils)

Now consider the case of two linear Poisson tensors. Let \( A(x) = \sum_{\beta} c^\beta_{ij} x^\beta \) and \( B(x) = \sum_{\beta} \tilde{c}^\beta_{ij} x^\beta \) be two Lie-Poisson tensors related to two different Lie algebras \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \), but with the same dual space \( \mathfrak{g}^* = \tilde{\mathfrak{g}}^* = \mathbb{R}^n \), so that \( c^\beta_{ij} \) are structure constants for \( \mathfrak{g} \) and \( \tilde{c}^\beta_{ij} \) are structure constants for \( \tilde{\mathfrak{g}} \). In other words, we can think of \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \) as the same vector space but with two different commutators \([ , ]_A \) and \([ , ]_B \) respectively.

**Proposition 1.7.** \( A(x) \) and \( B(x) \) are compatible if and only if \([ , ]_A \) and \([ , ]_B \) are compatible in the sense that \([ , ]_{A+B} = [ , ]_A + [ , ]_B \) satisfies the Jacobi identity.

**Proof.** It is easy to see that the compatibility equation for \( A(x) \) and \( B(x) \) (see equation 1.5) can be written in the following way

\[ \sum_{\alpha} \left( c^\beta_{ij} \tilde{c}^\alpha_{jk} + c^\beta_{ij} \tilde{c}^\alpha_{ki} + c^\beta_{ij} \tilde{c}^\alpha_{ki} + \tilde{c}^\beta_{ij} \tilde{c}^\alpha_{ij} + \tilde{c}^\beta_{ij} \tilde{c}^\alpha_{ij} \right) a^\beta = 0. \]

On the other hand, \([ , ]_A \) and \([ , ]_B \) are compatible if and only if
Example 1.3. In the similar way one can construct a Lie pencil on the space 
\[ \{ \xi, \eta, \zeta \}_{A+B} = 0, \]
\[ \{ \xi, [\eta, \zeta]_B \} - \{ [\xi, \eta]_A, \zeta \} = 0, \]
\[ \sum_{\alpha, i,j,k} \left( c^\beta_{\alpha i j k} + \tilde{c}^\beta_{\alpha j i k} + c^\beta_{\alpha j i k} + c^\beta_{\alpha k i j} + \tilde{c}^\beta_{\alpha k i j} + \tilde{c}^\beta_{\alpha i j k} \right) \xi^i \eta^j \epsilon^k = 0, \]
\[ \sum_{\alpha} \left( c^\beta_{\alpha i j k} + \tilde{c}^\beta_{\alpha j i k} + c^\beta_{\alpha j i k} + c^\beta_{\alpha k i j} + \tilde{c}^\beta_{\alpha k i j} + \tilde{c}^\beta_{\alpha i j k} \right) = 0. \]

Here \[ [\xi, \eta]_A^\alpha = \sum_{i,j} c^\alpha_{ij} \xi^i \eta^j \] and \[ [\xi, \eta]_B^\alpha = \sum_{i,j} \tilde{c}^\alpha_{ij} \xi^i \eta^j. \] Thus, the compatibility condition for the Poisson tensors \[ A(x) \] and \[ B(x) \] coincides with the compatibility condition for \[ [\ , \ ]_A \] and \[ [\ , \ ]_B, \] as required.

In other words, \[ A(x) = \sum_{\beta} c^\beta_{ij} x_{\beta} \] and \[ B(x) = \sum_{\beta} \tilde{c}^\beta_{ij} x_{\beta} \] are compatible if and only if for all \( \lambda, \mu \in \mathbb{R}, \lambda c^\beta_{ij} + \mu \tilde{c}^\beta_{ij} \) are structure constants of the linear family of Lie brackets \( \lambda[\ , \ ]_A + \mu[\ , \ ]_B \) (such a family is called a Lie pencil).

Example 1.2. Let \( g \) be the space of all \( n \times n \) matrices with two compatible commutators such that for all \( A, B \in g \)

\[ [A, B]_0 = AB - BA, \]
\[ [A, B]_1 = ACB - BCA, \quad \text{where } C \text{ is a fixed } n \times n \text{ matrix.} \]

Then for any two smooth functions \( f(x), g(x) : g^* \to \mathbb{R}, \) we have two compatible Poisson brackets \( \{\ , \}_0, \{\ , \}_1 \) on \( g^* \):

\[ \{f, g\}_0(x) = x ([df(x), dg(x)]_0) = x (df(x)dg(x) - dg(x)df(x)), \]
\[ \{f, g\}_1(x) = x ([df(x), dg(x)]_1) = x (df(x)Cdg(x) - dg(x)Cdf(x)). \]

If we identify \( g^* \) with \( g \) by means of the inner product \( \langle x, \xi \rangle = \text{Tr} \ x \cdot \xi, \) then the Hamiltonian vector fields corresponding to these two brackets are:

\[ (X_0)_f = xdf(x) - df(x)x = [x, df(x)], \]
\[ (X_1)_f = xdf(x)C - Cdf(x)x \]

Example 1.3. In the similar way one can construct a Lie pencil on the space \( g = so(n) \) of all skew-symmetric matrices by assuming that \( C \) is a fixed \( n \times n \) symmetric matrix.
Example 1.4. A similar construction also works for the space of all \( n \times n \) symmetric matrices. Then the compatible Lie brackets should be defined in the following way:

\[
\begin{align*}
[A, B]_0 &= AC_0B - BC_0A, \\
[A, B]_1 &= AC_1B - BC_1A,
\end{align*}
\]

where \( C_0 \) and \( C_1 \) are two fixed \( n \times n \) skew-symmetric matrices. If \( n \) is even and \( C_0 \) is non-degenerate, then \( \mathfrak{g} \) with the bracket \([ \ , \ ]_0\) is isomorphic to the symplectic Lie algebra \( \mathfrak{sp}(2n) \).

1.6 Open problems and the main results

In the theory of bi-Hamiltonian structures there are several natural questions that still remain open. Let us list some of them. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two compatible Poisson brackets on \( M \), and \( \mathcal{P} = \{ \mathcal{A}_\lambda = \mathcal{A} + \lambda \mathcal{B} \} \) be the corresponding Poisson pencil generated by them.

- Local classification and invariants.

**Definition 1.15.** The pair \((\mathcal{A}_0, \mathcal{B}_0)\) is locally equivalent to \((\mathcal{A}_1, \mathcal{B}_1)\) at \( x \in M \) if there is a diffeomorphism \( \phi : U(x) \to U(x) \), where \( U(x) \subset M \), such that

\[
\begin{align*}
\phi_* (\mathcal{A}_0) &= \mathcal{A}_1, \\
\phi_* (\mathcal{B}_0) &= \mathcal{B}_1.
\end{align*}
\]

The problem is to classify pairs of compatible Poisson brackets up to this equivalence relation, for example by finding an appropriate local normal form for a pair of compatible Poisson structure \( \mathcal{A} \) and \( \mathcal{B} \) in a neighborhood of a generic point \( x \in M \). It is very important to emphasise that the local normal form for the pair \( \mathcal{A} \) and \( \mathcal{B} \) in a neighborhood of \( x \in M \) essentially depend on the algebraic normal form of \( \mathcal{A}(x) \) and \( \mathcal{B}(x) \) considered as a pair of skew symmetric forms on a vector space \( V = T^*_x M \). (A point \( x \) is called generic, if the type of this algebraic normal form does not change in a small neighborhood of \( x \)). The algebraic normal form for \( \mathcal{A}(x) \) and \( \mathcal{B}(x) \) (see Theorem 2.1) is in some sense similar to the Jordan normal form and can be characterised by a
decomposition of $V$ into several blocks of two essentially different types, called *Jordan* and *Kronecker* blocks.

In the case when one of these structures is non-degenerate (equivalently, there are no Kronecker blocks), this problem has been completely solved by J. Turiel (see [38]). The opposite situation (usually called *micro-Kronecker* or just *Kronecker case*) has been studied by I. Zakharevich and I. M. Gelfand (see [15], [16], [42], [14]), in particular, they have obtained a criterion of simultaneous reducibility of $A$ and $B$ to a constant form in this case. However, almost nothing is known about the general case when there both are Jordan and Kronecker blocks at the same time.

By “invariants” in this context we mean those properties of $A$ and $B$ which are preserved by diffeomorphisms. One of such invariants is the algebraic normal form of $A(x)$ and $B(x)$ for a generic point $x \in M$. The problem is to describe a complete set of invariants that allows us to distinguish pairs of compatible Poisson brackets up to (local) diffeomorphisms.

- **Automorphism group of the pencil $P$**.

The description problem for the automorphism group can be asked both in local and global context.

We say that a diffeomorphism $\phi : M \to M$ is an automorphism of the pencil $P$ if $\phi_*(A_\lambda) = A_\lambda$ for each Poisson structure $A_\lambda \in P$. Similarly, we can define a local automorphism $\phi : U(x) \to U(x)$ defined in the neighborhood of a generic point $x \in M$. The collection of all automorphisms form a group $\text{Aut}(P) \subset \text{Diff}(M)$ (or a local group in the local case). What is the structure of this group?

Even the local case is very interesting. Recall that in the case of one single Poisson structure (for simplicity, non-degenerate) the group of automorphisms is generated by Hamiltonian flows. In the case of two compatible brackets, the problem is essentially reduced to the description of (local) bi-Hamiltonian vector fields. Again, this description is known only for very simple particular cases. In general, the problem is still open.

- **Complete family of functions in bi-involution**.

It is well known that the first integrals of a bi-Hamiltonian system, as a rule, commute with respect to the both brackets $A$ and $B$, i.e., are in bi-involution.
Is it true that for each pencil $\mathcal{P} = \{A + \lambda B\}$ of compatible Poisson brackets one can always find a complete family of functions in bi-involution?

There are two natural cases when such a family exists and, moreover, admits a very simple description. The first case is a symplectic pencil $\mathcal{P}$, i.e., one of the brackets, say $B$, is non-degenerate. Then the functions $\lambda_i(x)$ defined from the characteristic equation $\det(A(x) - \lambda(x)B(x)) = 0$ or, equivalently, the eigenvalues of the recursion operator $\mathcal{R} = B^{-1}A$ are in bi-involution. Moreover, if $\mathcal{R}$ admits $n = \frac{1}{2} \dim M$ distinct eigenvalues and none of them is constant, then $\lambda_1(x), \ldots, \lambda_n(x)$ are functionally independent and therefore give a complete family of functions in bi-involution.

The other example is opposite, namely, $\mathcal{P} = \{A + \lambda B\}$ is a Kronecker pencil which can be characterised by the following condition: $\text{rank } (A(x) + \lambda B(x))$ at a generic point $x \in M$ is the same for all $\lambda$. In other words the rank of $A_\lambda$ never drops. In this case, a complete family of functions in bi-involution can be constructed by taking all Casimir functions of all linear combinations $A(x) + \lambda B(x)$.

However, in general case, the construction (and even existence) of such a family $\mathcal{F}$ is not obvious at all. It is clear that the the nature of $\mathcal{F}$ once again essentially depends on the algebraic structure of the corresponding Poisson pencil.

The above discussion leads us to the conclusion that the algebraic properties of the Poisson pencil $\mathcal{P}$ play a very important role in the basic structural theory of compatible Poisson brackets. So they are worth studying in a systematic way. This is exactly the main goal of the present thesis. We mainly focus on purely algebraic aspects of compatible Poisson brackets and we also demonstrate their importance in the context of the theory of integrable Hamiltonian systems on Lie algebras.

Let us briefly comment on the content of this work. We start with discussing the so-called Jordan–Kronecker decomposition theorem (Theorem 2.1) which reduces a pair of skew-symmetric forms $A$ and $B$ to a very simple and convenient canonical form. This result makes it possible to study algebraic properties of the pencil $A + \lambda B$ defined on a vector space $V$. In particular, in Chapter 2

- we describe the linear automorphism group $G_\mathcal{P}$ for the pencil of forms $\mathcal{P} =$
\{A_\lambda = A + \lambda B\}:

\[G_P = \{X \in GL(V) \mid A_\lambda(v, u) = A_\lambda(X(v), X(u)) \text{ for all } v, u \in V, \lambda \in \mathbb{R}\},\]

and its Lie algebra \(g_P\):

- we obtain an explicit formula for the dimension of \(G_P\) in terms of the Jordan–Kronecker canonical form;
- we describe the semisimple Levi subalgebra of \(G_P\) and give necessary and sufficient conditions for solvability of \(G_P\);
- we discuss the properties of common Lagrangian subspaces for \(A_\lambda \in \mathcal{P}\).

Notice that in the classical symplectic geometry, the analogs of all above mentioned objects (symplectic linear transformations, symplectic Lie algebra, Lagrangian subspaces) play a fundamental role. We hope that similar algebraic constructions for pencils will help to understand better geometric properties of compatible Poisson brackets.

In Chapter 3, we apply the results and constructions of Chapter 2 to study one of the famous Poisson pencils which is sometimes called “argument shift pencil”. This pencil is defined on the dual space \(g^*\) of an arbitrary Lie algebra \(g\) and is generated by the standard Lie-Poisson bracket on \(g\):

\[\{f, g\}(x) = \langle x, [df(x), dg(x)] \rangle, \quad f, g \in C^\infty(g^*), x \in g^*,\]

and the constant bracket \(\{\ , \}_a\) defined for every element \(a \in g^*\):

\[\{f, g\}_a(x) = \langle a, [df(x), dg(x)] \rangle.\]

The main problem we aim to discuss is the existence of a complete family of polynomials in bi-involution w.r.t. these two brackets \(\{\ , \}\) and \(\{\ , \}_a\). To deal with it, we introduce the so-called Jordan–Kronecker invariants of a finite-dimensional Lie algebra. These invariants can be understood as the algebraic type of the canonical Jordan–Kronecker form for the Poisson pencil related to the brackets \(\{\ , \}\) and \(\{\ , \}_a\) at a generic point \(x \in g^*\) (more precisely, we assume that the pair \((x, a) \in g^* \times g^*\) is generic).
We study some general properties of these invariants and compute them for several examples. In particular, the Jordan–Kronecker invariants are found for all low-dimensional Lie algebras (\( \dim \mathfrak{g} \leq 5 \)).

For all examples of Lie algebras considered in the thesis, we have explicitly constructed the families of polynomials in bi-involution.
Chapter 2

Algebraic properties of compatible Poisson brackets

In this chapter we study algebraic properties of a pencil generated by two skew-symmetric forms \( A \) and \( B \) defined on a finite-dimensional vector space \( V \):

\[
\left\{ A_\lambda = A + \lambda B, \lambda \in \mathbb{R} \right\}.
\]

In the context of the bi-Hamiltonian dynamics, we can think of \( A \) and \( B \) as constant compatible Poisson tensors on \( M \), which is simply a vector space. Then \( V \) should be treated as \( M^* \). Alternatively, we can consider two non-constant Poisson brackets on some manifold \( M \) and then think of \( A \) and \( B \) as values of the corresponding Poisson tensors at a fixed point \( x \in M \), then \( V = T_x^*M \).

Our aim is to describe the Lie group that preserves the pencil \( \mathcal{P} = \{ A_\lambda \} \):

\[
G_{\mathcal{P}} = \{ X \in GL(V) \mid A_\lambda(v, u) = A_\lambda(X(v), X(u)) \text{ for all } v, u \in V, \lambda \in \mathbb{R} \}.
\]

Since the algebraic structure of this Lie group essentially depends on the properties of a given pencil, we need to consider the classification problem for pencils first.

This chapter is purely algebraic and, for the sake of simplicity, we shall work over \( \mathbb{C} \), the field of complex numbers.
2.1 Jordan–Kronecker decomposition theorem

Given two pairs of skew-symmetric forms $A_0, B_0$ and $A_1, B_1$, we want to know whether or not there is a transformation $\phi : V \rightarrow V$ which sends $A_0$ to $A_1$ and $B_0$ to $B_1$. In other words, we need to find out whether there exists an invertible matrix $P$ such that (see [8])

$$A_1 = P^\top A_0 P$$
$$B_1 = P^\top B_0 P.$$

Equivalently, we want to know if the pencils $P_0 = \{A_0 + \lambda B\}$ and $P_1 = \{A_1 + \lambda B_1\}$ are congruent.

The classification of pairs $A, B$ up to congruence is given by the following theorem which reduces each pair to an elegant canonical block-diagonal form. One usually refers to this result as Jordan–Kronecker decomposition theorem, because their classical papers published in the 19th century contain all the essential ideas and ingredients of the proof. However the complete proof seems to appear first in the paper by R.Thompson (see [34]) in 1991.

**Theorem 2.1.** (Jordan–Kronecker Decomposition Theorem [34, 4]) Let $A$ and $B$ be skew-symmetric bilinear forms defined on a finite-dimensional complex vector space $V$. Then there exists a basis in $V$ in which the pencil $P = \{A + \lambda B\}$ takes a block-diagonal form

$$A + \lambda B = \begin{pmatrix}
C_1(\lambda) & & \\
& C_2(\lambda) & \\
& & \ddots \\
& & & C_k(\lambda)
\end{pmatrix}.$$
with the blocks $C_i(\lambda)$ of three following types:

```
\begin{array}{ccc}
0 & \lambda_i + \lambda & 1 \\
\lambda_i + \lambda & \ddots & \ddots \\
\lambda_i + \lambda & & 1 \\
\end{array}
```

Jordan block for $\lambda_i \in \mathbb{C}$

```
\begin{array}{ccc}
-(\lambda_i + \lambda) & -1 & -(\lambda_i + \lambda) \\
-1 & -\lambda & -1 \\
\ddots & \ddots & \ddots \\
-1 & -(\lambda_i + \lambda) & \ddots & -\lambda & -1 \\
\end{array}
```

Jordan block for $\lambda_i = \infty$

```
\begin{array}{ccc}
0 & 1 & \lambda \\
1 & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots \\
-\lambda & -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{array}
```

Kronecker block

We also allow trivial $(1 \times 1)$-blocks $C_i(\lambda) = (0)$ and refer to them as trivial Kronecker blocks.

If rank $A_{\mu} = \max_{\lambda \in \mathbb{C}} \text{rank} A_{\lambda}$, then the form $A_{\mu}$ is called generic. It is clear, that almost all forms in $\mathcal{P}$ are generic except for a finite number of singular values of $\lambda$ for which the rank of $A_{\lambda}$ drops. It is easy to see that these singular values of $\lambda$ are exactly the numbers $-\lambda_i$ (including $\lambda_i = \infty$) that appear in the above theorem.
Definition 2.1. \( \lambda_i \) are called the characteristic numbers of the pencil \( \mathcal{P} = \{ \mathcal{A} + \lambda \mathcal{B} \} \).

In this theory, characteristic numbers play the same role as eigenvalues in the theory of linear transformations. Moreover, in the “symplectic case” when \( \mathcal{B} \) is non-degenerate, they are just the eigenvalues of the recursion operator \( \mathcal{B}^{-1} \mathcal{A} \). Thus, \( \lambda_i \) are those numbers for which the rank of \( \mathcal{A} + \lambda \mathcal{B} \) for \( \lambda = -\lambda_i \) is not maximal.

The case of Jordan blocks with \( \lambda_i = \infty \) can always be avoided by replacing \( \mathcal{B} \) with \( \mathcal{B}' = \mathcal{B} + \mu \mathcal{A} \) for a suitable \( \mu \). So from now on, we shall assume that \( \infty \) is not a characteristic number, so that no Jordan block with “infinite eigenvalue” appears.

Definition 2.2. A skew-symmetric bilinear form \( \mathcal{A} + \lambda \mathcal{B} \in \mathcal{P} \) is called regular if and only if \( \lambda \neq -\lambda_i \). (Equivalently we may say that the rank of \( \mathcal{A} + \lambda \mathcal{B} \) is maximal in \( \mathcal{P} \), i.e., \( \text{rank} \, \mathcal{A} + \lambda \mathcal{B} = \text{rank} \, \mathcal{P} \).)

Obviously, for congruent pairs \( \mathcal{A}_0, \mathcal{B}_0 \) and \( \mathcal{A}_1, \mathcal{B}_1 \), the groups \( G_{\mathcal{P}_0} \) and \( G_{\mathcal{P}_1} \), are isomorphic, where \( \mathcal{P}_i = \{ \mathcal{A}_i + \lambda \mathcal{B}_i \} \) denote the corresponding pencils. Thus, instead of arbitrary skew-symmetric bilinear forms \( \mathcal{A} \) and \( \mathcal{B} \), we can and shall describe the Lie group \( G_{\mathcal{P}} \) for standard canonical forms only.

2.2 Corollaries: existence of a common Lagrangian subspace, uniqueness, completeness criterion

The Jordan–Kronecker decomposition theorem immediately implies several important facts. These facts are, of course, well known to experts in the field (see, for example, [1, 3, 13, 15, 19, 21, 30, 36, 40]) but, to the best of our knowledge, have not been formulated in purely algebraic terms before. Here we try to summarise them in a more systematic way. First of all, we notice that we can always find a large subspace which is isotropic simultaneously for all forms from the pencil \( \mathcal{P} \). In fact, this result gives an algebraic explanation of the role which compatible Poisson brackets play in the theory of completely integrable systems.

Theorem 2.2. For every pencil \( \mathcal{P} = \{ \mathcal{A} + \lambda \mathcal{B}, \lambda \in \mathbb{C} \} \) on a vector space \( V \), there is a subspace \( U \subset V \) which is isotropic with respect to every form \( \mathcal{A}_\lambda \in \mathcal{P} \) and \( \dim U = \frac{1}{2} \left( \dim V + \text{corank} \, \mathcal{P} \right) \). In other words, \( U \) is maximal isotropic for all regular forms (see Definition 2.2) \( \mathcal{A}_\mu \in \mathcal{P} \), i.e. such that \( \text{rank} \, \mathcal{A}_\mu = \text{rank} \, \mathcal{P} \).
Proof. Let \( \mathcal{P} = \{ \mathcal{A}_\lambda = \mathcal{A} + \lambda \mathcal{B}, \lambda \in \mathbb{C} \} \) be a pencil of skew-symmetric forms on \( V \) and \( \dim V = n, \) \( \text{rank} \ \mathcal{P} = 2r. \) Then we need to find an isotropic subspace \( U \subset V \) and prove that \( \dim U = \frac{1}{2} (\dim V + \text{corank} \ \mathcal{P}) = \frac{1}{2} (n + (n - 2r)) = \frac{1}{2} (2n - 2r) = n - r. \) According to the J-K theorem, \( \mathcal{A} \) and \( \mathcal{B} \) with the size \( n \) and rank \( 2r, \) can be simultaneously reduced to the following form by an appropriate change of basis:

\[
\mathcal{A}_\lambda = \mathcal{A} + \lambda \mathcal{B} = \begin{pmatrix}
\mathcal{J}_1 & & & \\
& \ddots & & \\
& & \mathcal{J}_p & \\
& & & \mathcal{K}_1 \\
& & & & \ddots \\
& & & & & \ddots \\
& & & & & & \mathcal{K}_q
\end{pmatrix},
\]

where

\[
\mathcal{J}_i = \mathcal{J}_i(\lambda) =
\begin{pmatrix}
\lambda_i + \lambda & 1 & & & \\
0 & \ddots & \ddots & \ddots & \\
0 & \ddots & \ddots & 1 & \\
0 & \ddots & \ddots & \ddots & \ddots \\
-1 - \lambda_i - \lambda
\end{pmatrix},
\]

\[
\mathcal{K}_j = \mathcal{K}_j(\lambda) =
\begin{pmatrix}
1 & \lambda & & & \\
0 & \ddots & \ddots & \ddots & \\
0 & \ddots & \ddots & 1 & \\
0 & \ddots & \ddots & \ddots & \ddots \\
-1 & -\lambda & \ddots & \ddots & -1
\end{pmatrix}.
\]

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for $i = 1, \ldots, p$, $j = 1, \ldots, q$. Here some of the $\lambda_i$ may coincide. We have
\[
n = \dim V = \sum_{i=1}^{p} 2m_i + \sum_{j=1}^{q} (2k_j + 1),
\]
\[
2r = \rank \mathcal{P} = \sum_{i=1}^{p} 2m_i + \sum_{j=1}^{q} 2k_j,
\]
\[
n - 2r = \corank \mathcal{P} = q.
\]

Let us denote the basis of the subspace $V_{\mathcal{J}_i} \subset V$ corresponding to the Jordan block $\mathcal{J}_i$ as $\{e_1^i, \ldots, e_{2m_i}^i\}$, and the basis of the subspace $V_{\mathcal{K}_j} \subset V$ related to $\mathcal{K}_j$ as $\{f_1^j, \ldots, f_{2k_j+1}^j\}$. It is clear that $V$ can be split into several subspaces according to $\mathcal{J}_i$ and $\mathcal{K}_j$, namely:
\[
V = V_{\mathcal{J}_1} \oplus \cdots \oplus V_{\mathcal{J}_p} \oplus V_{\mathcal{K}_1} \oplus \cdots \oplus V_{\mathcal{K}_q}.
\]

We can find an isotropic subspace $U_{\mathcal{J}_i} = \span\{e_1^i, \ldots, e_{m_i}^i\} \subset V_{\mathcal{J}_i}$ with respect to $\mathcal{J}_i$. Similarly we can find an isotropic subspace $U_{\mathcal{K}_j} = \span\{f_{k_j+1}^j, \ldots, f_{2k_j+1}^j\} \subset V_{\mathcal{K}_j}$ with respect to $\mathcal{K}_j$. Then we can construct a common isotropic subspace $U$ with respect to $\mathcal{A}_\lambda$ for each $\lambda$:
\[
U = U_{\mathcal{J}_1} \oplus \cdots \oplus U_{\mathcal{J}_p} \oplus U_{\mathcal{K}_1} \oplus \cdots \oplus U_{\mathcal{K}_q}.
\]
and
\[
\dim U = \sum_{i=1}^{p} \dim U_{\mathcal{J}_i} + \sum_{j=1}^{q} \dim U_{\mathcal{K}_j}
\]
\[
= \sum_{i=1}^{p} m_i + \sum_{j=1}^{q} (k_j + 1)
\]
\[
= \left( \sum_{i=1}^{p} 2m_i + \sum_{j=1}^{q} (2k_j + 1) \right) - \left( \sum_{i=1}^{p} m_i + \sum_{j=1}^{q} k_j \right)
\]
\[
= n - r = \frac{1}{2}(\dim V + \corank \mathcal{P}).
\]

Therefore we have shown that such an isotropic subspace $U$ exists. According to Proposition 1.5, the equality $\dim U = \frac{1}{2}(\dim V + \corank \mathcal{P})$ implies that $U$ is a maximal isotropic subspace w.r.t. all regular $\mathcal{A}_\mu \in \mathcal{P}$, $\mu = -\lambda_i$. \hfill \square

Furthermore, under some additional assumptions, the common maximal isotropic subspace from Theorem 2.2 is unique. Namely, we have

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Theorem 2.3. Let $U \subset V$ be a common isotropic subspace for all $A_\lambda \in \mathcal{P}$ such that $\dim U = \frac{1}{2}(\dim V + \text{corank } \mathcal{P})$. Then the following conditions are equivalent:

1. $\mathcal{P}$ is pure Kronecker;
2. $U$ is maximal isotropic for all $A_\lambda \in \mathcal{P}$;
3. $U$ is unique.

Proof. 1 $\Rightarrow$ 2: 

Let $\mathcal{P}$ contain only Kronecker blocks, then each $A_\lambda \in \mathcal{P}$ has no characteristic numbers $\lambda_i$, which means the rank of $A_\lambda$ never drops, i.e., $\text{corank } A_\lambda = \text{corank } \mathcal{P}$ for all $\lambda \in \mathbb{C}$. Thus, $\dim U = \frac{1}{2}(\dim V + \text{corank } A_\lambda)$ for all $A_\lambda \in \mathcal{P}$, and therefore $U$ is maximal isotropic for all $A_\lambda$ by Proposition 1.5.

1 $\Rightarrow$ 3: If $U$ is maximal isotropic subspace for $A_\lambda$, then $\text{Ker } A_\lambda \subset U$. Since $U$ is maximal isotropic w.r.t. all $A_\lambda \in \mathcal{P}$, then $U$ contains all the kernels, i.e.,

$$
\sum_{\lambda} \text{Ker } A_\lambda \subset U.
$$

Since $\mathcal{P}$ is of pure Kronecker type, it can be checked by direct computation that

$$
\dim \left\{ \sum_{\lambda} \text{Ker } A_\lambda \right\} = \frac{1}{2}(\dim V + \text{corank } \mathcal{P}) = \dim U.
$$

Therefore

$$
U = \sum_{\lambda} \text{Ker } A_\lambda,
$$

and this characterization of $U$ shows that $U$ is unique.

3 $\Rightarrow$ 1: Assume by contradiction that the pencil $\mathcal{P} = \{A_\lambda, \lambda \in \mathbb{C}\}$ contains Jordan blocks, i.e.,

$$
A_\lambda = A + \lambda B = \begin{pmatrix}
J_1 & & \\
& \ddots & \\
& & J_p \\
K_1 & & K_q
\end{pmatrix},
$$

where

$$
K_1 = \begin{pmatrix}
& & \\
& \ddots & \\
& & 1
\end{pmatrix},
$$

$$
K_q = \begin{pmatrix}
& & \\
& \ddots & \\
& & 1
\end{pmatrix}.
$$
According to Theorem 2.2, a common isotropic subspace $U$ with $\dim U = \frac{1}{2}(\dim V + \text{corank } \mathcal{P})$ can be defined as
\[
U = \bigoplus_i \text{span} \{e_i^1, \ldots, e_i^{m_i}\} \bigoplus \left\{ \bigoplus_j U_{\mathcal{K}_j} \right\}.
\]
On the other hand, we can similarly define another subspace $\tilde{U}$ with the same properties by
\[
\tilde{U} = \bigoplus_i \text{span} \{e_i^{m_i+1}, \ldots, e_i^{2m_i}\} \bigoplus \left\{ \bigoplus_j U_{\mathcal{K}_j} \right\}.
\]
So both $U$ and $\tilde{U}$ are two different common isotropic subspaces for all $A_\lambda \in \mathcal{P}$ of the same dimension. Thus, such a subspace is not unique, which contradicts to our assumption. Therefore for $U$ to be unique, $\mathcal{P}$ cannot contain any Jordan block, in other words, $\mathcal{P}$ must be of pure Kronecker type.

The proof of Theorem 2.3 shows that a "large" common isotropic subspace for the pencil $\mathcal{P}$ can, in fact, be defined in the following invariant way. Let $A_\mu \in \mathcal{P}$ be regular, in other words, $-\lambda$ is not a characteristic number. Then we define $L \subset V$ as
\[
L = \sum_\mu \text{Ker } A_\mu, \text{ where } A_\mu \in \mathcal{P} \text{ is regular.}
\]
Indeed, since $U$ is maximal isotropic w.r.t. every regular $A_\mu$, then $\text{Ker } A_\mu \subset U$, which implies that $L \subset U$. The next result can be considered as another version of Theorem 2.3.

**Theorem 2.4.** $L$ is isotropic with respect to all $A_\lambda \in \mathcal{P}$, where $\lambda \in \mathbb{C}$ is arbitrary. Moreover, $L$ is maximal isotropic w.r.t. $A_\lambda$ if and only if $\mathcal{P}$ is of pure Kronecker type and in this case $L = U$.

**Proof.** Although this statement easily follows from Theorem 2.3, we give an independent proof by describing the subspace $L$ explicitly. As above, we assume that the pencil $\mathcal{P}$ has been reduced to the Jordan–Kronecker canonical form. Then we have
\[
L = \sum_{\mu \neq \lambda_i} \text{Ker } A_\mu = \sum_{\mu \neq \lambda_i} \text{Ker } J_i(\mu) \bigoplus \sum_{\mu \neq \lambda_i} \text{Ker } K_j(\mu).
\]
If $\mu \neq \lambda_i$ then $J_i(\mu)$ is non-degenerate and $\text{Ker } J_i(\mu) = \{0\}$. The kernel of $K_j$ is one-dimensional and is generated by the vector

$$v_\mu = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{kj} \mu^{kj} \\ \vdots \\ (-1)^2 \mu^2 \\ -\mu \\ 1 \end{pmatrix}.$$ 

If we consider the span of all $v_\mu$, we obtain $\sum_\mu \text{Ker } K_j(\mu) = U_{K_j} = \text{span } \{f_{kj+1}, \ldots, f_{kj+1}^2\}$. Therefore

$$L = \sum_{\mu,j} \text{Ker } K_j(\mu) = \bigoplus_j U_{K_j}.$$ 

It follows that $L$ is a subspace of the common isotropic subspace $U$ constructed in Theorem 2.2. Thus, $L$ is isotropic w.r.t. every $A_\lambda \in \mathcal{P}$. Moreover, the subspace $U$ from Theorem 2.2 has the form

$$U = \left(\bigoplus_i U_{J_i}\right) \bigoplus \left(\bigoplus_j U_{K_j}\right).$$ 

Therefore, $L$ coincides with $U$ if and only if the subspaces $U_{J_i}$ are all trivial, i.e., the Jordan–Kronecker decomposition of $\mathcal{P}$ contains no Jordan blocks. In other words, $L$ is maximal isotropic if and only if $\mathcal{P}$ is of pure Kronecker type.

In bi-Hamiltonian mechanics, one often considers the so-called recursion operator $\mathcal{R} : V \to V$ which, in our algebraic context, can be defined by the following relation:

$$\mathcal{R}v_0 = v_1 \text{ implies } Av_1 = Bv_0.$$ 

If $\mathcal{A}$ is non-degenerate, then $\mathcal{R}$ is simply $A^{-1}B$. In general, this operator $\mathcal{R}$ is not defined, but we can still consider $Av_1 = Bv_0$ as a recursion relation. The following statement says that $L$ is invariant under this recursion relation.
Proposition 2.1. Let $v_0 \in L$ and $v_1$ satisfy $Av_1 = Bv_0$. Then $v_1 \in L$.

Proof. It can be proved by direct computation. Let $\mathcal{P} = \{A_\lambda\}$ be reduced to the Jordan–Kronecker canonical form as in Theorem 2.2. According to Theorem 2.4,

$$L = \bigoplus_j \text{span}\{f_j^{k_j+1}, \ldots, f_j^{2k_j+1}\}.$$

Let $v_0 \in L$ such that

$$v_0 = \sum_j \left( \alpha_j f_j^{k_j+1} + \cdots + \alpha_j f_j^{2k_j+1} \right),$$

we then substitute $v_0$ into $Av_1 = Bv_0$ and obtain

$$v_1 = \sum_j \left( \alpha_j f_j^{k_j+1} + \cdots + \alpha_j f_j^{2k_j} + \gamma_j f_j^{2k_j+1} \right) \in L,$$

where $\gamma_j$ is some arbitrary coefficient.

Together with $L \subset V$, we can define a larger subspace $L' \subset V$ in the following invariant way,

$$L' = \sum_{\lambda} \text{Ker} \ A_\lambda, \quad \text{where } A_\lambda \in \mathcal{P} \text{ is arbitrary.}$$

This subspace also plays an important role in some applications (see [27]), and we want to discuss briefly its properties. Recall that a subspace $U \subset V$ is called co-isotropic w.r.t. $A$ if $A(v, U) = 0$ implies $v \in U$. We start with elementary examples.

Proposition 2.2. Let

$$A_\lambda = \begin{pmatrix} \lambda_i + \lambda \\ -\lambda_i - \lambda \end{pmatrix}$$

be a $2 \times 2$ Jordan block with the characteristic number $\lambda_i$, which is defined on the vector space $V = \text{span}\{e_1, e_2\}$ of dimension 2. Then $L'$ is not an isotropic subspace but a co-isotropic subspace w.r.t. every regular $A_\mu \in \mathcal{P}$.

Proof. First we need to prove that there exist $v_1, v_2 \in L'$ such that $A_\mu(v_1, v_2) \neq 0$ for all $\mu \neq -\lambda_i$. Let $\lambda = -\lambda_i$, then $A_{(-\lambda_i)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\text{Ker} \ A_{(-\lambda_i)} = V$. If $\lambda \neq -\lambda_i$,
then \( \text{Ker } A_\lambda \) is trivial. Thus,

\[
L' = \sum_\lambda \text{Ker } A_\lambda \\
= \sum_{\lambda \neq (-\lambda_i)} \text{Ker } A_\lambda + \text{Ker } A_{(-\lambda_i)} \\
= 0 + \text{span}\{e_1, e_2\} \\
= V.
\]

Since \( e_1, e_2 \in V \), therefore \( e_1, e_2 \in L' \). It is easily to see \( A_\mu(e_1, e_2) = \mu + \lambda_i \neq 0 \) where \( \mu \neq -\lambda_i \).

Second we need to show that \((L')^\perp \subset L'\). According to the definition,

\[
(L')^\perp = V^\top = \{0\}.
\]

Obviously, \( \{0\} \in V \), therefore we have \((L')^\perp \subset L'\), which means that \( L' \) is co-isotropic w.r.t. all regular \( A_\mu \).

**Proposition 2.3.** Let

\[
A_\lambda = \begin{bmatrix}
\lambda_i + \lambda & 1 \\
-\lambda_i - \lambda & \lambda_i + \lambda \\
-1 & -\lambda_i - \lambda
\end{bmatrix}
\]

be a 4 \( \times \) 4 Jordan block with the characteristic number \( \lambda_i \), which is defined on the vector space \( V = \text{span } \{e_1, e_2, e_3, e_4\} \) of dimension 4. Then \( L' \) is both isotropic subspace and co-isotropic subspace w.r.t. every regular \( A_\mu \in \mathcal{P} \). In other words, \( L' \) is a maximal isotropic subspace w.r.t. all \( A_\mu \in \mathcal{P} \) where \( \mu \neq -\lambda_i \).

**Proof.** We need to prove that \( L' \) is a maximal isotropic subspace w.r.t. each \( A_\mu \), i.e., \( L' = (L')^\perp \). Let \( \lambda = -\lambda_i \), then

\[
A_{(-\lambda_i)} = \begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 0 \\
-1 & 0
\end{bmatrix},
\]
and \( \text{Ker } A_{(-\lambda_i)} = \text{span}\{e_2, e_3\} \). If \( \lambda \neq -\lambda_i \), then \( \text{Ker } A_{\lambda} \) is trivial. Thus,

\[
L' = \sum_{\lambda} \text{Ker } A_{\lambda}
= \sum_{\lambda \neq (-\lambda_i)} \text{Ker } A_{\lambda} + \text{Ker } A_{(-\lambda_i)}
= 0 + \text{span}\{e_2, e_3\}
= \text{span}\{e_2, e_3\},
\]

and

\[
(L')^{\perp} = \{ v \in V \mid A_\mu(v, L') = 0 \}
= \{ v \in V \mid A_\mu(v, e_2) = A_\mu(v, e_3) = 0 \}
= \text{span}\{e_2, e_3\}.
\]

Therefore \( L' \) and \((L')^{\perp}\) coincide with each other i.e., \( L' = (L')^{\perp} = \text{span}\{e_2, e_3\} \). It implies \( L' \) is indeed a maximal isotropic subspace w.r.t. all regular \( A_\mu \in \mathcal{P} \).

Proposition 2.4. Let

\[
A_\lambda = \begin{pmatrix}
\lambda_i + \lambda & 1 & & & \\
& \ddots & \ddots & & \\
& & \ddots & 1 & \\
& & & \ddots & \\
-\lambda_i - \lambda & -1 & \cdots & \cdots & -1 -\lambda_i - \lambda
\end{pmatrix}
\]

be a \( 2m \times 2m \) \((m > 2)\) Jordan block with the characteristic number \( \lambda_i \), which is defined on the vector space \( V = \text{span}\{e_1, \cdots, e_m, e_{m+1}, \cdots, e_{2m}\} \) of dimension \( 2m \). Then \( L' \) is an isotropic subspace, but not a co-isotropic subspace w.r.t. every regular \( A_\mu \in \mathcal{P}, \mu \neq -\lambda_i \). In particular, \( L' \) is not a maximal isotropic subspace w.r.t. \( A_\mu \in \mathcal{P}, \mu \neq -\lambda_i \).

Proof. We need to prove that \( L' \) is an isotropic subspace w.r.t. each \( A_\mu \) i.e., \( L' \subset \text{span}\{e_2, e_3\} \). It implies \( L' \) is indeed a maximal isotropic subspace w.r.t. all regular \( A_\mu \in \mathcal{P} \).

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$(L')^\wedge$. But $(L')^\wedge \not\subset L'$. Let $\lambda = -\lambda_i$, then

$$
\mathcal{A}_{(-\lambda_i)} = \begin{bmatrix}
0 & 1 \\
\vdots & \ddots & 1 \\
-1 & \ddots & \ddots & 0 \\
-1 & 0
\end{bmatrix},
$$

and $\text{Ker} \mathcal{A}_{(-\lambda_i)} = \text{span} \{e_m, e_{m+1}\}$. If $\lambda \neq -\lambda_i$, then $\text{Ker} \mathcal{A}_\lambda$ is trivial. Thus,

$$
\begin{align*}
L' &= \sum_\lambda \text{Ker} \mathcal{A}_\lambda \\
&= \sum_{\lambda \neq (-\lambda_i)} \text{Ker} \mathcal{A}_\lambda + \text{Ker} \mathcal{A}_{(-\lambda_i)} \\
&= 0 + \text{span}\{e_m, e_{m+1}\} \\
&= \text{span}\{e_m, e_{m+1}\},
\end{align*}
$$

and

$$(L')^\wedge = \{v \in V \mid \mathcal{A}_\mu(v, L') = 0\}$$

$$
= \{v \in V \mid \mathcal{A}_\mu(v, e_m) = \mathcal{A}_\mu(v, e_{m+1}) = 0\} \\
= \text{span}\{e_2, \ldots, e_m, e_{m+1}, \ldots, e_{2m-1}\}.
$$

It is obvious that $L'$ is isotropic due to $L' \subset (L')^\wedge$, but $(L')^\wedge \not\subset L'$, which means $L'$ is not co-isotropic. Therefore we conclude that $L'$ is not a maximal isotropic subspace $(L' \neq (L')^\wedge)$ w.r.t. all regular $\mathcal{A}_\mu \in \mathcal{P}$. Notice that $(L')^\wedge$ is the same for all regular $\mathcal{A}_\mu$. \qed

Let $\mathcal{A}_\lambda$ and $\mathcal{A}_\mu$ be denoted in the general form as in the Theorem 2.2. The following theorem summarises properties of $L'$ and can be constructed based on the above three propositions.

**Theorem 2.5.** Let $\Lambda = \{-\lambda_1, -\lambda_2, \ldots, -\lambda_k\}$ be the set of negative characteristic numbers for the pencil $\mathcal{P} = \{\mathcal{A} + \lambda \mathcal{B}\}$ and

$$
L' = \sum_\lambda \text{Ker} \mathcal{A}_\lambda \quad (\lambda \text{ is arbitrary}),
$$

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and
\[ L = \sum_{\mu \in \Lambda} \text{Ker } A_{\mu} \quad (\mu \text{ is regular}). \]

be as above. Then

1. \( L \) and \( L' \) are “orthogonal” with respect to all \( A_{\lambda} \in \mathcal{P} \), i.e., \( A_{\lambda}(L, L') = 0 \), for all \( A_{\lambda} \in \mathcal{P} \).

2. \( L' = \sum \text{Ker } A_{\lambda} \) is maximal isotropic with respect to every regular form \( A_{\mu} \in \mathcal{P} \) if and only if each Jordan block of Jordan–Kronecker form of \( \mathcal{P} \) is of size \( 4 \times 4 \).

3. \( L' \) is co-isotropic subspace w.r.t. each regular \( A_{\mu} \in \mathcal{P} \) if and only if all Jordan blocks are of size \( \leq 4 \times 4 \).

4. \( L' \) is isotropic with respect to every regular form \( A_{\mu} \in \mathcal{P} \) if and only if the Jordan–Kronecker form of \( \mathcal{P} \) contains no Jordan blocks of size \( 2 \times 2 \).

**Proof.**

1. According to the definition, we have
\[ L' = \sum_{\lambda} \text{Ker } A_{\lambda} = \bigoplus_{i} \text{span}\{e_{m_{i}}^{m_{i}}, e_{m_{i}+1}^{m_{i}+1}\} \bigoplus L \quad (\lambda \text{ is arbitrary}), \]

and
\[ L = \sum_{\mu} \text{Ker } A_{\mu} = \bigoplus_{j} \text{span}\{f_{j}^{k_{j}+1}, \cdots, f_{j}^{2k_{j}+1}\} \quad (\mu \text{ is regular}). \]

It's follows from Theorem 2.4, that \( A_{\lambda}(L, L) = 0 \). Moreover, we have
\[ A_{\lambda} \left(L, \bigoplus_{i} \text{span}\{e_{i}^{m_{i}}, e_{i}^{m_{i}+1}\}\right) = 0, \]

because Jordan and Kronecker parts ”commute”. Thus, we obtain that \( A_{\lambda}(L, L') = 0 \), for all \( A_{\lambda} \in \mathcal{P} \).

2. First of all we need to prove that if each Jordan block is of size \( 4 \times 4 \), then \( L' \) is maximal isotropic w.r.t. all regular \( A_{\mu} \in \mathcal{P} \) (it is indeed followed from Proposition 2.3). Alternatively, let us denote \( A_{\lambda} \) as in the theorem 2.2, but for all \( i \), the size of each Jordan block \( 2m_{i} = 4 \). For this case we have \( n = 4p + \sum_{j=1}^{q}(2k_{j} + 1) \) and \( r = 2p + \sum_{j=1}^{q} k_{j} \), where \( p \) is the number of Jordan
blocks. In this case we have $A_\mu(L', L') = 0$ (refer to the proof are given in 4). On the other hand,

$$\dim L' = \dim \left\{ \bigoplus_i \text{span}\{e_i^{m_i}, e_i^{m_i+1}\} \bigoplus L \right\}$$

$$= 2p + \sum_{j=1}^q (k_j + 1)$$

$$= n - r$$

$$= \frac{1}{2}(\dim V + \text{corank } P).$$

It means that $L'$ is a maximal isotropic subspace (see Theorem 2.2) w.r.t. $A_\mu$ if each Jordan block is of size $4 \times 4$.

Next, we need to prove that if $L'$ is maximal isotropic w.r.t. each regular $A_\mu$, then the Jordan block is of size $4 \times 4$. It can be proved by contradiction. Assume there exists a $2 \times 2$ Jordan block, then according to Proposition 2.2, $L'$ is not isotropic w.r.t. $A_\mu$, the contradiction appears. Assume there exist a Jordan block of size $2m \times 2m$ where $m \geq 3$. The Proposition 2.4 tells us that $L'$ is not maximal isotropic, therefore a contradiction occurs again.

3. Let each Jordan block be of size either $2 \times 2$ or $4 \times 4$. Then as we defined previously

$$L' = \bigoplus_i \text{span}\{e_i^{m_i}, e_i^{m_i+1}\} \bigoplus L$$

and

$$(L')^\prec = \{ v \in V | A_\mu(v, L') = 0 \}$$

$$= (L')^\prec \bigoplus (L')^{\prec}_K,$$

where

$$(L')^{\prec}_J = \left\{ v \in \bigoplus_i V_{J_i} \big| A_\mu \left( v, \bigoplus_i \text{span} \{e_i^{m_i}, e_i^{m_i+1}\} \right) = 0 \right\}$$

and

$$(L')^{\prec}_K = \left\{ v \in \bigoplus_j V_{K_j} \big| A_\mu(v, L) = 0 \right\}.$$
Theorem 2.4 tells us
\[(L')^\perp K = L.\]
Thus we conclude that \((L')^\perp \subseteq L'\), which means \(L'\) is co-isotropic.

The proof of the inverse statement can also be done by contradiction. Let \(L'\) be a co-isotropic subspace w.r.t. all regular \(A_\mu\). Assume there exists a Jordan block of size \(2m \times 2m\), where \(m \geq 3\). However, the Proposition 2.4 shows us that \(L'\) is not co-isotropic but isotropic, hence a contradiction occurs here.

4. First we shall prove that if for all \(\mu \neq -\lambda_i\) and \(A_\mu(L',L') = 0\), then there is no \(2 \times 2\) Jordan blocks. Assume there exists \(2 \times 2\) sized Jordan blocks, then there exist \(v_1, v_2 \in L'\) such that \(A_\mu(v_1, v_2) \neq 0\) (see Proposition 2.2), which contradicts on \(A_\mu(L',L') = 0\).

Next, we need to show that if no \(2 \times 2\) Jordan blocks exist, in other words, each Jordan block has \(m_i \geq 2\), then \(A_\mu(L',L') = 0\) for all \(\mu \neq \lambda_i\). The result is followed by Proposition 2.3 and Proposition 2.4. Indeed,

\[
L' = \sum_{\lambda} \text{Ker} \ A_\lambda
= \sum_{\lambda = -\lambda_i} \text{Ker} \ A_\lambda + \sum_{\mu \neq -\lambda_i} \text{Ker} \ A_\mu
= \bigoplus_i \text{span}\{e_i^{m_i}, e_i^{m_i+1}\} \bigoplus L.
\]

According to Proposition 2.3 and Proposition 2.4, it follows that

\[
A_\mu \left( \bigoplus_i \text{span}\{e_i^{m_i}, e_i^{m_i+1}\}, \bigoplus_i \text{span}\{e_i^{m_i}, e_i^{m_i+1}\} \right) = 0.
\]

Theorem 2.4 tells us that

\[
A_\mu(L,L) = 0.
\]

Moreover we have

\[
A_\mu \left( L, \bigoplus_i \text{span}\{e_i^{m_i}, e_i^{m_i+1}\} \right) = 0,
\]

because of Kronecker part “commute” with the Jordan part w.r.t. any \(A_\mu\).

One can also suggest the following equivalent explanation. It is easy to see from the Jordan–Kronecker decomposition of \(P\) into blocks, that \(L'\) is isotropic
if and only if this condition holds for each block separately. In other words, the subspaces \( \sum_\lambda \ker (A_\lambda|_{V_{J_i}}) \subset V_{J_i} \) and \( \sum_\lambda \ker (A_\lambda|_{V_{K_j}}) \subset V_{K_j} \) must be isotropic in the corresponding subspaces \( V_{J_i} \) and \( V_{K_j} \). We know already that this condition holds for Kronecker blocks. According to Propositions 2.3 and Propositions 2.4, for a Jordan block of size \( 2m \times 2m \) it is true if and only if \( m \geq 2 \), as stated.

\[ \square \]

2.3 The group \( G_P \): definition and description

Let \( P = \{ A_\lambda = A + \lambda B \} \) be a pencil of skew-symmetric forms on a vector space \( V \) generated by \( A \) and \( B \).

**Definition 2.3.** The linear automorphism group \( G_P \) of the pencil \( P \) is the group of all linear transformations \( Y : V \to V \) that preserve \( P \), i.e.,

\[ G_P = \{ Y \in GL(V) \mid A_\lambda(v,u) = A_\lambda(Y(v),Y(u)) \quad \text{for all} \ v,u \in V, \lambda \in \mathbb{C} \}. \]

By saying that \( G_P \) preserves the pencil \( P \) we mean in fact that \( G_P \) preserves each individual form \( A_\lambda \in P \) separately. This is, of course, equivalent to the fact that \( G_P \) preserves simultaneously two forms \( A \) and \( B \). \( G_P \) is a linear algebraic group. We can interpret \( G_P \) as a stationary subgroup of a certain linear action of \( GL(V) \). Indeed, consider the direct sum of two copies of \( \Lambda^2(V^*) \), i.e. the space whose elements are pairs of skew-symmetric forms \( (A,B) \in \Lambda^2(V^*) \oplus \Lambda^2(V^*) \). Then \( GL(V) \) acts naturally on this space:

\[ (A,B) \to (Y^\top AY, Y^\top BY), \quad Y \in GL(V), \]

and \( G_P \) for \( P = \{ A + \lambda B \} \) is exactly the stationary subgroup of \( (A,B) \) with respect to this action. In other words, the description of the group \( G_P \) for different types of pencils \( P \) is equivalent to the description of orbits of \( GL(V) \) acting on the space \( \Lambda^2(V^*) \oplus \Lambda^2(V^*) \). Since in coordinates \( A_\lambda(v,u) = v^\top A_\lambda u \), where

\[ v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \]

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we have
\[ v^\top A_{\lambda} u = (Yv)^\top A_{\lambda} (Yu) = v^\top Y^\top A_{\lambda} Y u \]
\[ \Rightarrow Y^\top A_{\lambda} Y = A_{\lambda}. \]

Therefore in matrix notation, i.e. as a subgroup of $GL(n, \mathbb{C})$, $G_P$ is defined by:
\[ G_P = \{ Y \in GL(n, \mathbb{C}) \mid Y^\top A_{\lambda} Y = A_{\lambda} \}. \]

Without loss of generality, we consider the case when the pencil is reduced to the Jordan–Kronecker canonical form. There are 3 cases:

- Case(1) The symplectic case with only Jordan blocks (no Kronecker blocks);
- Case(2) Degenerate case with only Kronecker blocks (no Jordan blocks);
- Case(3) Mixed case (with both Jordan and Kronecker blocks).

Instead of studying the Lie group that preserves the pencil $P$, to make it easy in computation, we start with studying the corresponding Lie algebra $g_P$, which is defined as follows:
\[ g_P = \{ X \in \mathfrak{gl}(V) \mid A_{\lambda}(X(v), u) + A_{\lambda}(v, X(u)) = 0 \text{ for all } v, u \in V, \lambda \in \mathbb{C} \}. \]

In the matrix notation:
\[ g_P = \{ X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^\top A_{\lambda} + A_{\lambda} X = 0, \lambda \in \mathbb{C} \}. \]

Notice that $g_P = g_A \cap g_B$.

For the reason of some convenience, we “transpose” this Lie algebra and consider the equivalent equation $X A_{\lambda} + A_{\lambda} X^\top = 0$ instead. The Lie algebra so obtained has, obviously, the same dimension and is isomorphic to $g_P$. We shall still denote it by $g_P$. If we interpret $P$ as a pencil of compatible Poisson brackets, then $V$ is the cotangent space of our Poisson manifold. In this setting, the passage to “transposed” matrices is more natural, because the “transposed” Lie algebra acts then on the tangent space.
2.3.1 Block structure of $G_P$ and $g_P$

Using the Jordan–Kronecker decomposition theorem we may assume without loss of
generality that the pencil $P$ has been already reduced to the canonical form

$$P = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_n \end{pmatrix}.$$ 

Then the Lie algebra $g_P$ is determined by the (family of) matrix equations

$$XP + PX^\top = 0,$$

which splits into several relatively simple matrix equations if we divide $X$ into blocks in the following natural way

$$X = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} \\ \vdots \\ P_{n1} & \cdots & \cdots & P_{nn} \end{pmatrix}.$$ 

These equations are of the following form:

$$P_{ii}C_i + C_i(P_{ii})^\top = 0$$

$$P_{ji}C_i + C_j(P_{ij})^\top = 0$$

or

$$P_{ij}C_j + C_i(P_{ji})^\top = 0,$$

where each $C_i$ is either Kronecker, or Jordan. It is important that the equations for different pairs of $P_{ij}, P_{ji}$ are not linked, so we can solve them independently. As a result, the dimension of the whole Lie algebra can be found as the sum of the dimensions of the spaces of solutions to each system of the above type.

The difficulty is, however, that the structure of the space of solutions essentially depends on $C_i$ and $C_j$ which can be both Kronecker, or both Jordan, or one Jordan and one Kronecker. Moreover, the sizes of blocks and the characteristic numbers (more precisely the fact whether or not they coincide) are very important. These cases will be treated in detail in the following sections. In fact, we need to distinguish 7 essentially different cases. (The result for each case can be found in the Lemmas indicated below.)
\[ i = j \text{ and } C_i \text{ is a Kronecker block (Lemma 2.1)} \]

\[ i = j \text{ and } C_i \text{ is a Jordan block (Lemma 2.3)} \]

\[ C_i \text{ and } C_j \text{ are both Kronecker but of different size (Lemma 2.2)} \]

\[ C_i \text{ and } C_j \text{ are both Kronecker and of the same size (Lemma 2.2)} \]

\[ C_i \text{ and } C_j \text{ are both Jordan with the same characteristic number (Lemma 2.4)} \]

\[ C_i \text{ and } C_j \text{ are both Jordan but with distinct characteristic numbers (Lemma 2.5)} \]

\[ C_i \text{ is Jordan whereas } C_j \text{ is Kronecker (Lemma 2.6)} \]

### 2.4 The dimension formula

#### 2.4.1 Kronecker case

The Bi-Hamiltonian systems related to compatible Poisson structures of pure Kronecker type appear in a very natural way in geometry and mathematical physics, see for example: [1], [2], [4], [3], [15], [16], [17], [14], [22], [23], [27], [28], [32], [42].

**Theorem 2.6.** Consider a pencil \( \{ \mathcal{K}_\lambda = A + \lambda B \} \) of pure Kronecker type as below:

\[
\mathcal{K}_\lambda = \begin{pmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\vdots \\
\mathcal{K}_q
\end{pmatrix},
\]

where \( \mathcal{K}_j \) is an elementary Kronecker block of size \((2k_j + 1)\), i.e.

\[
\mathcal{K}_j = \begin{pmatrix}
\begin{array}{cccc}
1 & \lambda & \cdots & \\
\cdots & \ddots & \ddots & \\
-1 & \cdots & & 1 & \lambda \\
-\lambda & & \ddots & & \\
& \ddots & & \ddots & \\
& & \ddots & & \ddots \\
& & & -1 & \\
& & & -\lambda & \\
\end{array}
\end{pmatrix},
\]

where \( \mathcal{K}_j \) is an elementary Kronecker block of size \((2k_j + 1)\), i.e.

\[
\mathcal{K}_j = \begin{pmatrix}
\begin{array}{cccc}
1 & \lambda & \cdots & \\
\cdots & \ddots & \ddots & \\
-1 & \cdots & & 1 & \lambda \\
-\lambda & & \ddots & & \\
& \ddots & & \ddots & \\
& & \ddots & & \ddots \\
& & & -1 & \\
& & & -\lambda & \\
\end{array}
\end{pmatrix},
\]

where \( \mathcal{K}_j \) is an elementary Kronecker block of size \((2k_j + 1)\), i.e.
Assume that

\[ 0 \leq k_1 \leq k_2 \leq \cdots \leq k_q, \]

1. \( l \) is the number of blocks with distinct sizes,

2. \( m_i \) is the block multiplicity \((i = 1, 2, \cdots, l)\), i.e., the number of times a Kronecker block of a certain fixed size appears in this decomposition:

\[
\underbrace{\mathcal{K}_1 \cdots \mathcal{K}_{m_1}}_{\text{with the same size}} \underbrace{\mathcal{K}_{m_1+1} \cdots \mathcal{K}_{m_1+m_2}}_{\text{with the same size}} \cdots \underbrace{\mathcal{K}_{q-m_l+1} \cdots \mathcal{K}_q}_{\text{with the same size}}
\]

or, equivalently,

\[
k_1 = \cdots = k_{m_1} < k_{m_1+1} = \cdots = k_{m_1+m_2} < \cdots < k_{q-m_l+1} \cdots k_q.
\]

Then the dimension of the Lie algebra \( \mathfrak{g}_{\{\mathcal{K}_\lambda\}} \) is given by the following formula:

\[
\dim \mathfrak{g}_{\{\mathcal{K}_\lambda\}} = \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1).
\]

**Proof.** We consider \( \mathfrak{g}_{\{\mathcal{K}_\lambda\}} \) as a Lie subalgebra of \( gl(n, \mathbb{C}) \). In matrix notation, \( \mathfrak{g}_{\{\mathcal{K}_\lambda\}} \) is defined by one matrix equation:

\[
\mathfrak{g}_{\{\mathcal{K}_\lambda\}} = \{ X \in gl(n, \mathbb{R}) \mid X\mathcal{K}_\lambda + \mathcal{K}_\lambda X^\top = 0 \text{ for all } \lambda \in \mathbb{C} \}
\]

where \( \lambda \) is, however, an arbitrary parameter. We write the elements of \( \mathfrak{g}_{\{\mathcal{K}_\lambda\}} \) in the form

\[
X = \begin{pmatrix}
P_{11} & P_{12} & \cdots & P_{1q} \\
P_{21} & P_{22} & \cdots & P_{2q} \\
\vdots & \ddots & \vdots & \vdots \\
P_{q1} & P_{q2} & \cdots & P_{qq}
\end{pmatrix} \in \mathfrak{g}_{\{\mathcal{K}_\lambda\}},
\]

where \( P_{jj} \) is a square matrix of size \( (2k_j + 1) \times (2k_j + 1) \). We also assume that \( i < j \) when we use \( P_{ij} \) or \( P_{ji} \). By the definition of \( \mathfrak{g}_{\{\mathcal{K}_\lambda\}} \), it is easy to see that the sub-blocks in \( X \) satisfy the matrix equations:

\[ P_{jj} \mathcal{K}_j + \mathcal{K}_j (P_{jj})^\top = 0 \quad (2.2) \]

\[ P_{ij} \mathcal{K}_i + \mathcal{K}_j (P_{ij})^\top = 0. \quad (2.3) \]
We denote by $\{P_{jj}\} \subset g^{\{K_\lambda\}}$ the subspace of matrices of the form
\[
\begin{pmatrix}
0 & 0 & 0 & \cdots \\
0 & P_{jj} & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
similarly $\{P_{ij}\} \subset g^{\{K_\lambda\}}$ is the subspace of matrices
\[
\begin{pmatrix}
0 & P_{ij} & 0 & \cdots \\
P_{ji} & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

The proof given here is based on direct computation. We may break it down into a number of lemmas.

**Lemma 2.1.** (Diagonal block $P_{jj}$ in the case when $C_j = \mathcal{K}_j$ is a Kronecker block). Consider the matrix equation (2.2)
\[
P_{jj}\mathcal{K}_j + \mathcal{K}_j(P_{jj})^\top = 0,
\]
where $\mathcal{K}_j$ is shown in (2.1). Then the space of the solutions ($\{P_{jj}\}$) has dimension $2k_j + 1$, and the matrix $P_{jj}$ has the following explicit form
\[
P_{jj} = \begin{pmatrix}
a & b_1 & b_2 & \cdots & b_{k_j} & b_{k_j+1} \\
 & a & b_2 & \cdots & b_{k_j} & b_{k_j+1} \\
 & & a & b_{k_j} & b_{k_j+1} & \cdots & b_{2k_j} \\
 & & & a & -a & \cdots & \cdots \\
 & & & & a & -a & \cdots \\
 & & & & & a & -a\end{pmatrix}_{k_j \times (k_j+1)},
\]
the parameters $a, b_1, b_2, \cdots, b_{2k_j}$ in the matrix are arbitrary complex numbers.

**Proof.** Let
\[
P_{jj} = \begin{pmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{pmatrix}
\]
be an unknown square matrix, and denote

\[ \mathcal{K}_j = \begin{pmatrix} F \\ -F^\top \end{pmatrix}. \]

By using the matrix equation above, we obtain three independent equations:

\[
\begin{align*}
X_1F &= -FX_1^\top \\
X_2F^\top &= FX_2^\top \\
X_3F &= (X_3F)^\top,
\end{align*}
\]

which can be easily solved separately, and the result is as shown above.

Lemma 2.2. (Blocks \(P_{ij}\) and \(P_{ji}\) in the case when \(C_i = \mathcal{K}_i\) and \(C_j = \mathcal{K}_j\) are both Kronecher).

Consider the matrix equation (2.3)

\[ P_{ji}\mathcal{K}_i + \mathcal{K}_j (P_{ij})^\top = 0, \]

where \(\mathcal{K}_i\) and \(\mathcal{K}_j\) are standard Kronecker blocks defined by (2.1) of size \(2k_i + 1\) and \(2k_j + 1\) respectively. Then the matrices \(P_{ij}\) and \(P_{ji}\) have the following explicit form:

- **When \(k_i < k_j\),** \((C_i = \mathcal{K}_i\) and \(C_j = \mathcal{K}_j\) are both Kronecker but of different size)

\[
\begin{pmatrix}
\begin{array}{cccc}
  c_1 & \cdots & c_{k_j-k_i+1} & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & c_1 & \cdots & c_{k_j-k_i+1}
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
  b_1 \\
  b_2 \\
  \vdots \\
  b_{k_i} \\
  b_{k_i+1}
\end{array}
\end{pmatrix}
\]

\[ k_i \]

\[
\begin{pmatrix}
\begin{array}{cccc}
  c_1 & \cdots & c_{k_j-k_i+1} & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & c_1 & \cdots & c_{k_j-k_i+1}
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
  b_1 \\
  b_2 \\
  \vdots \\
  b_{k_i} \\
  b_{k_i+1}
\end{array}
\end{pmatrix}
\]

\[ k_i + 1 \]
The dimension of the subspace \( \{P_{ij}\} \) is \( 2k_j + 1 \).

When \( k_i = k_j \), \( (C_i = K_i \text{ and } C_j = K_j \text{ are both Kronecker and of the same size}) \)

The dimension of the subspace \( \{P_{ij}\} \) is \( 2k_j + 2 \).
Proof. Let
\[ P_{ij} = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \quad \text{and} \quad P_{ji} = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \]
be unknown matrices. Denote
\[ K_i = \begin{pmatrix} F - F^\top \\ -F^\top \end{pmatrix} \quad \text{and} \quad K_j = \begin{pmatrix} D - D^\top \\ -D^\top \end{pmatrix}. \]
Applying the matrix equation (2.3) for each case, we have 4 following independent equations:

- For \( k_i < k_j \) (\( F \neq D \)):
  \[ Y_1 D = -FZ_4^\top \quad \Rightarrow \quad Y_1 \text{ relates to } Z_4, \]
  \[ Y_2 D^\top = FZ_2^\top \quad \Rightarrow \quad Y_2 \text{ relates to } Z_2, \]
  \[ Y_3 D = F^\top Z_3^\top \quad \Rightarrow \quad Y_3 = Z_3 = \{0\}, \]
  \[ -Y_4 D^\top = F^\top Z_1^\top \quad \Rightarrow \quad Y_4 = Z_1 = \{0\}. \]

- For \( k_i = k_j \) (\( F = D \)):
  \[ Y_1 F = -FZ_4^\top \quad \Rightarrow \quad Y_1 \text{ relates to } Z_4, \]
  \[ Y_2 F^\top = FZ_2^\top \quad \Rightarrow \quad Y_2 \text{ relates to } Z_2, \]
  \[ Y_3 F = F^\top Z_3^\top \quad \Rightarrow \quad Y_3 = Z_3 = \{0\}, \]
  \[ -Y_4 F^\top = F^\top Z_1^\top \quad \Rightarrow \quad Y_4 \text{ relates to } Z_1. \]

Here we say that, for example, \( Y_1 \) relates to \( Z_4 \) in the sense that given any admissible \( Z_4 \) we can uniquely reconstruct \( Y_1 \) and vice versa. In particular, the number of independent parameters in \( Y_1 \) and \( Z_4 \) are the same. After computations we may easily get the structures of \( P_{ij} \) and \( P_{ji} \) in Lemma 2.2. We can see that the space of solutions \( \{P_{ij}\} \) has dimension,
\[
\begin{cases}
2k_j + 1, & \text{if } k_i < k_j; \\
2k_j + 2, & \text{if } k_i = k_j.
\end{cases}
\]

\[ \square \]

Back to the proof of Theorem 2.6, from Lemma 2.1 and Lemma 2.2, we have the following results. First we consider the case when the size of blocks is strictly
increasing, i.e. $k_1 < k_2 < \cdots < k_q$. According to our general scheme, the algebra $\mathfrak{g}_{(K_{\lambda})}$ as a vector space is the direct sum of $\{P_{jj}\}$ and $\{P_{ij}\}$. In particular,

$$\dim \mathfrak{g}_{(K_{\lambda})} = \sum_j \dim \{P_{jj}\} + \sum_{i<j} \dim \{P_{ij}\}.$$ 

Notice that the dimensions of two “block-subspaces” $\{P_{jl}\}$ and $\{P_{jm}\}$ are the same if $l, m < j$. We partition them into natural groups (rows) in the following way:

\[\emptyset, \{P_{21}\}, \{P_{31}, P_{32}\}, \{P_{41}, P_{42}, P_{43}\}, \ldots, \{P_{q1}, P_{q2}, \ldots, P_{q(q-1)}\}.\]

The “block-subspaces” from the $j$th row all have the same dimension $2k_j + 1$ (see Lemma 2.2), and the number of blocks in the $j$th row is $j - 1$. It gives us the dimension of

$$\bigoplus_{i<j} \{P_{ij}\},$$

which is

$$\sum_{j=1}^q (2k_j + 1)(j - 1).$$

The dimension of each diagonal block which corresponds to the $j$th row (i.e. $P_{jj}$) is $2k_j + 1$ (see Lemma 2.1), so the sum of the dimensions of $\{P_{jj}\}$ is $\sum_{j=1}^q (2k_j + 1)$. Thus if $k_1 < k_2 < \cdots < k_q$, the dimension obtained in this case is,

$$\dim \mathfrak{g}_{K_{\lambda}} = \sum_j \dim \{P_{jj}\} + \sum_{i<j} \dim \{P_{ij}\}$$

$$= \sum_{j=1}^q (2k_j + 1) + \sum_{j=1}^q (2k_j + 1)(j - 1)$$

$$= \sum_{j=1}^q (2k_j + 1)j.$$

Next, we consider the case when the sizes of all blocks $(K_i)$ on the diagonal coincide. Suppose there are $m$ blocks (i.e. $K_1, \ldots, K_m$) on the diagonal of $K_{\lambda}$, and $k_1 = k_2 = \cdots = k_m = k$ and $k \geq 0$ in this case. Then each $P_{jj}$ has dimension $2k + 1$ as we mentioned before, thus

$$\dim \left\{ \bigoplus_j^m \{P_{jj}\} \right\} = m(2k + 1).$$
Since \( k_j = k \), each block in \( \{ P_{ij} \} \) has dimension \( 2k + 2 \) (see Lemma 2.2), and there are \( \frac{m(m-1)}{2} \) independent blocks in
\[
\bigoplus_{i<j} \{ P_{ij} \}.
\]

Then if \( k_1 = k_2 = \cdots = k_m = k \), the dimension of this case is
\[
\dim g_{\{K_{\lambda}\}} = \sum \dim \{ P_{jj} \} + \sum \dim \{ P_{ij} \} = m(2k + 1) + \frac{m(m-1)}{2} \cdot (2k + 2) = m(m + mk + k).
\]

Now, for mixture case, we combining the two cases above. Let the condition be \( k_1 \leq k_2 \leq \cdots \leq k_q \) for all \( j \in [1, 2, \cdots, q] \subset \mathbb{N} \) and \( k_1 \geq 0 \). As before, the dimension of diagonal blocks is
\[
\dim \left( \bigoplus_j \{ P_{jj} \} \right) = \sum_{j=1}^q (2k_j + 1),
\]
and the blocks from the \( i \)th row in the partition of \( \{ P_{ij} \} \) have dimension,
\[
\begin{cases} 
2k_j + 1, & \text{for } k_i < k_j \\
2k_j + 2, & \text{for } k_i = k_j.
\end{cases}
\]

Every time when \( m \) multiplicity blocks appear together, for example
\[
k_1 < k_2 < \cdots < k_j < k_{j+1} = \cdots = k_{j+m} < \cdots < k_q
\]
the total dimension of \( g_{\{K_{\lambda}\}} \) increases by \( \frac{m(m-1)}{2} \). Finally we have,
\[
\dim g_{\{K_{\lambda}\}} = \sum_{j=1}^q (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^l m_i(m_i - 1).
\]

This completes the proof. \( \square \)

### 2.4.2 Symplectic case

Non-degenerate compatible Poisson brackets are also related to many important problems in geometry and mathematical physics, see for example: [19], [7], [21], [24], [35], [36], [18], [37].
Theorem 2.7. Let \( \{ J_\lambda = A + \lambda B \} \) be a pencil of symplectic (pure Jordan) type. Denote by \( \lambda_1, \ldots, \lambda_p \) all distinct characteristic numbers and let
\[
\begin{align*}
  k_1(\lambda_1), & \ldots, k_s(\lambda_1), \\
  k_1(\lambda_2), & \ldots, k_s(\lambda_2), \\
  \quad \cdots \\
  k_1(\lambda_p), & \ldots, k_s(\lambda_p),
\end{align*}
\]
where \( k_1(\lambda_t) \geq k_2(\lambda_t) \geq \cdots \) for all \( t = 1, \ldots, p \). Then the dimension of the Lie algebra \( g_{\{ J_\lambda \}} \) is
\[
\sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} (4j - 1) \cdot k_j(\lambda_t) \right).
\]

Proof. In the symplectic case, without loss of generality we may assume that \( J_\lambda \) is reduced to the Jordan–Kronecker canonical form which consists of Jordan blocks only, i.e.,
\[
J_\lambda = \begin{pmatrix}
J_1 & & \\
& \ddots & \\
& & J_n
\end{pmatrix}
\]
and
\[
J_i = \begin{pmatrix}
\begin{array}{cccc}
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
1 \\
\ddots \\
\ddots \\
\ddots \\
\end{pmatrix}
\begin{pmatrix}
\lambda_t + \lambda \\
\ddots \\
\ddots \\
\ddots \\
\end{pmatrix}
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
1 \\
\ddots \\
\ddots \\
\ddots \\
\end{pmatrix}
\begin{pmatrix}
\lambda_t + \lambda \\
\ddots \\
\ddots \\
\ddots \\
\end{pmatrix}
\end{pmatrix}
\]
The \( \lambda_t \) here is some characteristic number (where \( t = 1, \ldots, p \)) related to the block \( J_i \), which means that some of characteristic numbers in different blocks (\( J_i \)) may coincide, and some of them do not. In total, we have \( p \) distinct characteristic numbers: \( \lambda_1, \ldots, \lambda_p \). We arrange the blocks (\( J_i \)) with the same characteristic number next to each other on the diagonal and put them in the order from \( \lambda_1 \) to \( \lambda_p \) so that the blocks are ordered in the following way:
\[
\underbrace{J_1 \ldots J_{s_1}}_{\lambda_1} \underbrace{J_{(s_1+1)} \ldots J_{(s_1+s_2)}}_{\lambda_2} \ldots \underbrace{J_{(n-s_p+1)} \ldots J_n}_{\lambda_p}.
\]
We use the same method as we did in the Kronecker case. First recall the matrix equation that defines $g_{\{J_\lambda\}}$:

$$g_{\{J_\lambda\}} = \{ X \in gl(n, \mathbb{R}) \mid X J_\lambda + J_\lambda X^\top = 0 \text{ for all } \lambda \in \mathbb{C} \},$$

where $\lambda \in \mathbb{C}$ is an arbitrary parameter. Then write the elements of $g_{\{J_\lambda\}}$ in the form

$$X = \begin{pmatrix} P_{11} & P_{12} & \ldots & P_{1n} \\ P_{21} & P_{22} & \ldots & P_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ P_{n1} & P_{n2} & \ldots & P_{nn} \end{pmatrix} \in g_{\{J_\lambda\}}.$$

This time $P_{ii}$ is a square matrix of the size $2k_i \times 2k_i$. We also assume that $i < j$ when we use $P_{ij}$ or $P_{ji}$. From the definition of $g_{\{J_\lambda\}}$, it is easy to see that the sub-blocks in $X$ satisfy the matrix equations:

- For all $i = j$ \( P_{ii} J_i + J_i (P_{ii})^\top = 0 \) \hspace{1cm} (2.5)
- For all $i < j$ \( P_{ji} J_i + J_j (P_{ij})^\top = 0 \) \hspace{1cm} (2.6)

As we did in the previous section, we denote by $\{P_{ii}\} \subset g_{\{J_\lambda\}}$ the subspace of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & \ldots \\ 0 & P_{ii} & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

similarly $\{P_{ij}\} \subset g_{\{J_\lambda\}}$ is the subspace of matrices

$$\begin{pmatrix} 0 & P_{ij} & 0 & \ldots \\ P_{ji} & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now we need the following three lemmas which can be proved just in the same way as Lemma 2.1 and Lemma 2.2 in the previous sections.

**Lemma 2.3.** (Diagonal block $P_{ii}$ in the case of a Jordan block $C_i = J_i$).

Let $P_{ii} J_i + J_i (P_{ii})^\top = 0$, where $J_i$ is a Jordan block of size $2k_i \times 2k_i$ (as we denoted
before). Then \( P_{ii} \) has the following explicit form

\[
P_{ii} = \begin{pmatrix}
\begin{array}{cccc}
  a_1 & a_2 & \cdots & a_{k_i} \\
  & & \cdots & \\
  & & & \ddots \\
  & & & & a_2 \\
  & & & & a_1
\end{array}
& \begin{array}{c}
  b_{k_i} & \cdots & b_2 & b_1
\end{array} \\
\begin{array}{c}
  c_1 \\
  \vdots \\
  \vdots \\
  \vdots \\
  c_1 & c_2 & \cdots & c_{k_i}
\end{array}
& \begin{array}{c}
  -a_1 \\
  \vdots \\
  \vdots \\
  \vdots \\
  -a_{k_i} & \cdots & -a_2 & -a_1
\end{array}
\end{pmatrix},
\]

where \( a_1, \ldots, a_{k_i}, b_1, \ldots, b_{k_i}, c_1, \ldots, c_{k_i} \) are independent arbitrary parameters. In particular, the dimension of \( \{P_{ii}\} \) is \( 3k_i \).

**Lemma 2.4.** (Blocks \( P_{ij} \) and \( P_{ji} \) for \( C_i = J_i \) and \( C_j = J_j \) being both Jordan with the same characteristic number \( \mu \)).

Let \( P_{ji}J_i + J_j(P_{ij})^\top = 0 \), where \( J_i \) and \( J_j \) are Jordan blocks with the same characteristic number \( \mu \) (here \( i < j \) and \( k_i \geq k_j \)):
Then we have

\[
P_{ij} = \begin{pmatrix}
  d_1 & \cdots & d_{k_j} \\
  0 & \ddots & \vdots \\
  \vdots & \ddots & d_1 \\
  \vdots & 0 & 0 \\
  0 & \cdots & 0 \\
  0 & \cdots & 0 & \cdots & 0 \\
  0 & \cdots & 0 & \cdots & 0 \\
  f_1 & \cdots & f_{k_j} & -g_{k_j} & -g_1 \\
\end{pmatrix} \quad \text{and} \quad
P_{ji} = \begin{pmatrix}
  0 & \cdots & 0 & g_1 & \cdots & g_{k_j} \\
  \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & f_1 & \cdots & f_{k_j} \\
  0 & \cdots & 0 & f_1 & \cdots & f_{k_j} \\
\end{pmatrix}
\]

There are \(4k_j\) independent parameters in \(\{P_{ij}\}\), i.e., \(\dim\{P_{ij}\} = 4k_j\).

**Lemma 2.5.** (Block \(P_{ij}\) and \(P_{ji}\) for \(C_i = J_i\) and \(C_j = J_j\) being both Jordan with distinct characteristic numbers \(\lambda_1 \neq \lambda_2\)).

Let \(P_{ji}J_i + J_j(P_{ij})^\top = 0\), where \(J_i\) and \(J_j\) are Jordan blocks with different characteristic numbers \(\lambda_1\) and \(\lambda_2\) respectively, i.e.,

\[
J_i = \begin{pmatrix}
  (\lambda_1 + \lambda) & 1 \\
  1 & (\lambda_1 + \lambda) \\
\end{pmatrix}
\]

\[
J_j = \begin{pmatrix}
  (\lambda_2 + \lambda) & 1 \\
  1 & (\lambda_2 + \lambda) \\
\end{pmatrix}
\]

Then \(P_{ij} = P_{ji} = \{0\}\).
The proof of these three lemmas is straightforward and follows from direct computation. After considering the above three lemmas, we are now in the position to prove Theorem 2.7, by working through these lemmas. Lemma 2.5 tells us that \( \{ P_{ij} \} = \{ 0 \} \) if \( J_i \) and \( J_j \) have distinct characteristic numbers. Thus, if we arrange the blocks \( (J_i) \) in the way indicated in (2.4), then the Lie algebra \( g_{\{J_\lambda\}} \) breaks down into independent diagonal blocks, i.e.,

\[
X = \begin{pmatrix}
\Lambda_1 & 0 \\
\vdots & \ddots \\
0 & \Lambda_p
\end{pmatrix}
\]

each of which splits into smaller sub-blocks

\[
\Lambda_t = \begin{pmatrix}
P_{(r+1)(r+1)} & P_{(r+1)(r+2)} & \cdots & P_{(r+1)(r+s_t)} \\
P_{(r+2)(r+1)} & P_{(r+2)(r+2)} & \cdots & P_{(r+2)(r+s_t)} \\
\vdots & \vdots & \ddots & \vdots \\
P_{(r+s_t)(r+1)} & P_{(r+s_t)(r+2)} & \cdots & P_{(r+s_t)(r+s_t)}
\end{pmatrix},
\]

where \( r = 0, s_1, s_1 + s_2, \cdots, n-s_p \). Here, size \( (P_{(r+1)(r+1)}) \geq \text{size} \ (P_{(r+2)(r+2)}) \geq \cdots \geq \text{size} \ (P_{(r+s_t)(r+s_t)}) \geq 2 \). We relabel the size of each sub-block \( (P_{(r+j)(r+i)}) \) in \( \Lambda_t \) as \( 2k_j(\lambda_t) \), so that \( k_1(\lambda_t) \geq k_2(\lambda_t) \geq \cdots \geq k_{s_t}(\lambda_t) \).

The blocks \( \Lambda_1, \ldots, \Lambda_p \) are “absolutely independent” in the sense that \( g_{\{J_\lambda\}} \), as a vector space, is the direct sum of the block-subspaces \( \{ \Lambda_t \} \), and each \( \Lambda_t \) consists of non-zero \( \{ P_{(r+j)(r+j)} \} \) and \( \{ P_{(r+i)(r+j)} \} \) for \( (r+i) < (r+j) \) and \( i,j \in \{ 0, \ldots, s_t \} \) (a similar notation as we defined \( \{ P_{jj} \} \) and \( \{ P_{ij} \} \)). In other words,

\[
\Lambda_t = \bigoplus_j \{ P_{(r+j)(r+j)} \} + \bigoplus_{i<j} \{ P_{(r+i)(r+j)} \}.
\]

We now follow the same scheme as for the Kronecker case in the previous section. Notice that in the subspace

\[
\bigoplus_{i<j} \{ P_{(r+i)(r+j)} \} \subset \Lambda_t,
\]

the dimensions of two blocks \( P_{(r+j)(r+l)} \) and \( P_{(r+j)(r+m)} \) are the same if \( j,l,m \in \{ 0, \ldots, s_t \} \) and \( (r+l), (r+m) < (r+j) \). We do the same partition as we did in the
Kronecker case:

\[
\emptyset, \\
\{P(r+2)(r+1)\}, \\
\{P(r+3)(r+1), P(r+3)(r+2)\}, \\
\{P(r+4)(r+1), P(r+4)(r+2), P(r+4)(r+3)\}, \\
\ldots, \\
\{P(r+s_t)(r+1)\}, \{P(r+s_t)(r+2)\}, \ldots, \{P(r+s_t)(r+s_t-1)\}.
\]

The “block-spaces” from the same \((r+j)\)th row all have the same dimension \(4k_j\), which is given by Lemma 2.4, and the number of blocks in the \((r+j)\)th row is \(j-1\). Therefore

\[
\sum_{i<j}^{s_t} \dim \{P(r+i)(r+j)\} = \sum_{j=1}^{s_t} 4k_j(j-1).
\]

However in the diagonal block subspace

\[
\bigoplus_j \{P(r+j),(r+j)\} \subset \Lambda_t \quad \text{for} \quad j \in \{0, \ldots, s_t\},
\]

Lemma 2.3 states that the dimension of each \(P(r+j)(r+j)\) is \(3k_j\), i.e.,

\[
\sum_{j=1}^{s_t} \dim \{P(r+j)(r+j)\} = \sum_{j=1}^{s_t} 3k_j.
\]

Thus taking all \(\lambda_t\) into account, we relabel \(k_j\) w.r.t each \(\lambda_t\) with \(k_j(\lambda_t)\) as we mentioned before, we have

\[
\dim g_{\{X_t\}} = \sum_{t=1}^{p} \left( \sum_{i<j}^{s_t} \dim \{P(r+i)(r+j)\} + \sum_{j=1}^{s_t} \dim \{P(r+j)(r+j)\} \right)
\]

\[
= \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} 3k_j(\lambda_t) + \sum_{j=1}^{s_t} 4(j-1)k_j(\lambda_t) \right)
\]

\[
= \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} (4j-1) \cdot k_j(\lambda_t) \right),
\]

as was to be proved. \(\square\)
2.4.3 The mixture of Symplectic and Kronecker blocks

In the general case, the pencil $\mathcal{P} = \{A_\lambda\}$ contains both Jordan blocks and Kronecker blocks. For the sake of simplicity, we partition $A_\lambda$ into two parts on the diagonal, a Jordan part ($J_\lambda$) and a Kronecker part ($K_\lambda$) as follows

$$A_\lambda = \begin{pmatrix} J_\lambda & \ast \\ \ast & K_\lambda \end{pmatrix}.$$

**Theorem 2.8.** Let $\mathcal{P}$ be a canonical pencil of skew-symmetric forms with the mixture of Jordan and Kronecker blocks shown above. If we keep the same notation for $K_\lambda$ as in Theorem 2.6 and for $J_\lambda$ as in Theorem 2.7, then the dimension of the Lie algebra $\mathfrak{g}_\mathcal{P}$ is

$$\sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} (2q + 4j - 1) \cdot k_j(\lambda_t) \right) + \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1),$$

where:

1. $0 \leq k_1 \leq k_2 \leq \cdots \leq k_q$ denote the sizes of Kronecker blocks;
2. $p$ is the number of distinct characteristic numbers;
3. $k_1(\lambda_t) \geq k_2(\lambda_t) \geq \cdots \geq k_{s_t}(\lambda_t) \geq 1$ for any $t = 1, 2, \cdots, p$, which denote the sizes of Jordan blocks;
4. $l$ is the number of Kronecker blocks with distinct sizes;
5. $m_i$ is the Kronecker block multiplicity (the number of Kronecker blocks with the same size).

**Proof.** The definition of $\mathfrak{g}_\mathcal{P}$ states that

$$\mathfrak{g}_\mathcal{P} = \{X \in gl(n, \mathbb{R}) \mid XA_\lambda + A_\lambda X^\top = 0, \lambda \in \mathbb{C}\}.$$

Let us represent $\mathfrak{g}_\mathcal{P}$ in the form

$$\mathfrak{g}_\mathcal{P} = \left\{ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \right\}.$$
The partition is according to the type of different forms (i.e. Jordan and Kronecker forms). By definition, we obtain the following equations:

\[ g_{11}\mathcal{J}_\lambda + \mathcal{J}_\lambda(g_{11})^\top = 0, \]
\[ g_{22}\mathcal{K}_\lambda + \mathcal{K}_\lambda(g_{22})^\top = 0. \]

In other words, the blocks \( g_{11} \) and \( g_{22} \) are exactly the subalgebras \( g_{\{J_\lambda\}} \) and \( g_{\{K_\lambda\}} \) described in two previous sections. Hence,

\[
\dim g_{11} = \dim g_{\{J_\lambda\}} = p \sum_{t=1}^{s_t} \left( \sum_{j=1}^{(4j - 1)} k_j(\lambda_t) \right),
\]
\[
\dim g_{22} = \dim g_{\{K_\lambda\}} = q \sum_{j=1}^{(2k_j + 1)} j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1).
\]

We also have

\[ g_{12}\mathcal{K}_\lambda + \mathcal{J}_\lambda(g_{21})^\top = 0, \]
\[ g_{21}\mathcal{J}_\lambda + \mathcal{K}_\lambda(g_{12})^\top = 0, \]

which implies that \( g_{12} \) and \( g_{21} \) are dependent on each other, so that \( \dim g_{12} = \dim g_{21} \).

The following lemma will help us to compute the dimension of \( g_{12} \).

**Lemma 2.6.** (Blocks \( P_{ij} \) and \( P_{ji} \) in the case when \( C_i = \mathcal{J}_i \) is Jordan whereas \( C_j = \mathcal{K}_j \) is Kronecker).

Let \( C_i \) be a Jordan block of size \( 2k_j(\lambda_t) \times 2k_j(\lambda_t) \) with characteristic number \( \lambda_t \) and \( C_j \) be a Kronecker block of size \( (2k_j + 1) \times (2k_j + 1) \). The solution of the equation \( P_{ji}C_i + C_j(P_{ij})^\top = 0 \) has the following explicit form:

\[
P_{ij} = \begin{pmatrix}
0 & A^k a & \ldots & A^{k+1} a \\
(A^\top)^k b & \ldots & -(A^\top)^2 b & A^\top b & -b
\end{pmatrix}
\]

\[
P_{ji} = \begin{pmatrix}
\ell^\top A^{(k_j - 1)} & \ldots & \ell^\top A^{(k_j - 1)} \\
\vdots & \ddots & \vdots \\
\ell^\top A^2 b & \ell^\top A^2 b
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0
\end{pmatrix}
\]

\[
\text{dim} g_{12} = \dim g_{\{J_\lambda\}} = p \sum_{t=1}^{s_t} \left( \sum_{j=1}^{(4j - 1)} k_j(\lambda_t) \right),
\]
\[
\text{dim} g_{22} = \dim g_{\{K_\lambda\}} = q \sum_{j=1}^{(2k_j + 1)} j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1).
\]
\[
\mathbf{a} = \begin{pmatrix}
\vdots \\
a_{k_i}
\end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix}
\vdots \\
b_{k_i}
\end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix}
\lambda_i & 1 & 0 & 0 \\
0 & \ddots & \ddots & 0 \\
0 & 0 & \ddots & 1 \\
0 & 0 & 0 & \lambda_i
\end{pmatrix}.
\]

The number of independent parameters in \( P_{ij} \), i.e., \( \dim \{ P_{ij} \} \) is \( 2k_j(\lambda_t) \), which is exactly the size of the Jordan block i.e., \( 2k_j(\lambda_t) \times 2k_j(\lambda_t) \).

The above explicit form of \( P_{ij} \) and \( P_{ji} \) indicates that one Jordan block \( J_i \) corresponding to one Kronecker block \( K_j \) give us that \( P_{ij} \) depends on \( 2k_j(\lambda_t) \) parameters. Since in the mixed case, there are total of \( q \) Kronecker blocks \( K_1, \ldots, K_q \) corresponding to each Jordan block \( J_i \), we see that \( P_{i1}, \ldots, P_{iq} \) depend on total of \( 2qk_j(\lambda_t) \) parameters. Therefore taking into account all Jordan blocks corresponding to \( q \) Kronecker blocks gives us the dimension of \( g_{12} \), i.e.,

\[
\dim g_{12} = \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} 2qk_j(\lambda_t) \right).
\]

Since \( g_{21} \) is dependent on \( g_{12} \), moreover \( g_{11}, g_{22} \) and \( g_{12} \) are pairwise independent, we have

\[
\dim g_P = \dim g_{11} + \dim g_{22} + \dim g_{12} \\
= \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} (4j - 1) \cdot k_j(\lambda_t) \right) + \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1) + \\
+ 2q \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} k_j(\lambda_t) \right) \\
= \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} (2q + 4j - 1) \cdot k_j(\lambda_t) \right) + \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1).
\]

Here we give a simple example to clarify the situation.
Example 2.1. Let 

\[ A_\lambda = \begin{pmatrix} J_\lambda & \lambda + 1 \\ K_\lambda & 0 \end{pmatrix} \]

\[ = \begin{pmatrix} \lambda + 1 & 0 & \lambda + 1 \\ -(\lambda + 1) & 0 & -(\lambda + 1) \\ 0 & -1 & 0 \end{pmatrix} \]

Then we obtain

\[ g_P = \begin{bmatrix} a_1 & a_2 & b_1 & b_1 & h & i & u_1 & u_2 & 0 & 0 & A^2x & Ax & x \\ a_1 & b_1 & 0 & 0 & v_1 & v_1 & 0 & 0 & -(A^T)^T y - A^T y - y \\ c_1 & -a_1 & 0 & 0 & j & -k & v_1 & v_2 & 0 & 0 & -\beta \top \beta - B^\top \beta - \beta \top \beta \\ c_1 & c_2 & -a_2 & -a_3 & j & -k & v_2 & 0 & 0 & 0 & -\beta \top \beta - B^\top \beta - \beta \top \beta \\ 0 & k & i & 0 & a_3 & b_3 & p & 0 & 0 & B^2u & B\alpha & \alpha \\ 0 & j & -k & 0 & e_3 & -a_3 & q & 0 & 0 & -(B^T \beta)^T \beta - B^\top \beta - \beta \top \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

where

\[ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \alpha \text{ and } \beta \text{ are 1-dimensional vectors,} \]

\[ A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \quad B = \lambda_1, \text{ which is } 1 \times 1 \text{ matrix.} \]

\[ \dim g_P = \sum_{j=1}^{2} (4 + 4j - 1) \cdot k_j(\lambda_1) + \sum_{j=1}^{1} (2k_j + 1)j \]

\[ = (3 + 4) \cdot 2 + (3 + 8) \cdot 1 + 1 + 10 \]

\[ = 36. \]
2.5 Generic pencils

According to Section 2.4.3, in general we have:

\[ \mathcal{A}_\lambda = \begin{pmatrix} \mathcal{J}_\lambda & \mathcal{K}_\lambda \end{pmatrix} l_1 l_2, \]

\[ g_P = \begin{pmatrix} \mathfrak{g}_{\mathcal{J}_\lambda} & P_{12} \\ P_{21} & \mathfrak{g}_{\mathcal{K}_\lambda} \end{pmatrix} l_1 l_2. \]

Notice, \( \mathcal{A}_\lambda \) is \( n \times n \) matrix and \( l_1 + l_2 = n \geq 1 \). Based on the previous discussions with respect to each kind of blocks, we can easily obtain the following estimates for \( \dim g_P \).

**Lemma 2.7.** Let \( \{ \mathcal{J}_\lambda \} \) be a pencil of Jordan type defined on the space \( V \) of dimension \( l_1 \), then \( \dim \mathfrak{g}_{\{ \mathcal{J}_\lambda \}} \geq \frac{3l_1}{2} \).

**Proof.** According to Theorem 2.7, we have

\[
\dim \mathfrak{g}_{\{ \mathcal{J}_\lambda \}} = \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} (4j - 1) \cdot k_{j}(\lambda_t) \right)
\]

(Since \( j \geq 1 \), then \( 4j - 1 \geq 3 \))

\[
\geq \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} 3k_{j}(\lambda_t) \right)
\]

\[
= 3 \cdot \text{(half of the size of the Jordan part)}
\]

\[
= \frac{3l_1}{2}
\]

\[ \square \]

**Lemma 2.8.** Let \( \{ \mathcal{K}_\lambda \} \) be a pencil of Kronecker type defined on a vector space \( V \) of dimension \( l_2 \), then \( \dim \mathfrak{g}_{\{ \mathcal{K}_\lambda \}} \geq l_2 \).
Proof. According to Theorem 2.6,
\[ \dim g_{\{\kappa, \lambda\}} = \sum_{j=1}^{q} (2k_j + 1)j + \frac{1}{2} \sum_{i=1}^{l} m_i(m_i - 1) \]
\[ \geq \sum_{j=1}^{q} (2k_j + 1) \]
\[ = \text{size of the Kronecker part} \]
\[ = l_2. \]

In particular, for the general case we have

**Lemma 2.9.** \( \dim g_P \geq n. \)

Proof. Applying Lemma 2.7 and Lemma 2.8, the following inequality can be obtained:
\[ \dim g_P = \dim g_{\{\lambda\}} + \dim g_{\{\kappa, \lambda\}} + \dim \{P_{ij}\} \]
\[ \geq \dim g_{\{\lambda\}} + \dim g_{\{\kappa, \lambda\}} \]
\[ \geq \frac{3l_1}{2} + l_2 \]
\[ \geq l_1 + l_2 = n. \]

The following theorem describes generic pencils \( P \), i.e. those for which the dimension of \( g_P \) is minimal. This fact (as well as Theorem 2.10 below) is almost obvious. We consider it as a simple application of our general dimension formula.

**Theorem 2.9.** If \( \dim V = n \) is even, then the minimal possible dimension of \( g_P \) is \( \frac{3n}{2} \). If \( \dim g_P = \frac{3n}{2} \), then \( P \) is of symplectic type and each characteristic number corresponds to exactly one Jordan block (in other words, there are no Jordan blocks with the same characteristic number).

**Remark 2.1.** Notice that this condition of “being generic” can be reformulated as follows. For each characteristic number \( \lambda_i \) we have \( \dim \ker A_{-\lambda_i} = 2. \)
Proof. Here we have 3 possible types of $\mathcal{P}$.

- **Type 1**: $\mathcal{P}$ is of pure Jordan type and for each distinct characteristic number $\lambda_t$ there is exactly one Jordan block (i.e., $s_t = 1$ for all $t$ and $n = l_1 = \sum_{t=1}^{p} 2k(\lambda_t)$), then

  $$\dim g_P = \dim g_{\{J_{\lambda_t}\}} = \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} (4j - 1) \cdot k_j(\lambda_t) \right)$$

  (since $s_t = 1$)

  $$= \frac{3}{2} \sum_{t=1}^{p} 2k(\lambda_t)$$

  $$= \frac{3n}{2}.$$

- **Type 2**: $\mathcal{P}$ is of pure Jordan type and may contain more than one Jordan block corresponding to the same $\lambda$ (i.e., $n = l_1 = \sum_{t=1}^{p} \sum_{j=1}^{s_t} 2k_j(\lambda_t), s_t \geq 2$), then

  $$\dim g_P = \dim g_{\{J_{\lambda}\}}$$

  $$= \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} (4j - 1) \cdot k_j(\lambda_t) \right)$$

  $$> \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} 3k_j(\lambda_t) \right)$$

  $$= \frac{3}{2} \sum_{t=1}^{p} \left( \sum_{j=1}^{s_t} 2k_j(\lambda_t) \right)$$

  $$= \frac{3n}{2}.$$

- **Type 3**: $\mathcal{P}$ contains at least 2 Kronecker blocks (i.e., $n = l_1 + l_2, l_2 = \sum_{j=1}^{q} (2k_j + 1)$ and $q \geq 2, k_2 \geq k_1$), then we have
\[
\dim g_{\{K_{\lambda}\}} = \sum_{j=1}^{q}(2k_j + 1) j + \frac{1}{2} \sum_{i=1}^{l} m_i (m_i - 1) \\
\geq \sum_{j=1}^{2}(2k_j + 1) j \\
= (2k_1 + 1) \cdot 1 + (2k_2 + 1) \cdot 2 \\
= (2k_1 + 1) + \frac{1}{2}(2k_2 + 1) + \frac{3}{2}(2k_2 + 1) \\
> (2k_1 + 1) + \frac{1}{2}(2k_1 + 1) + \frac{3}{2}(2k_2 + 1) \\
= \frac{3l_2}{2},
\]

and therefore
\[
\dim g_P \geq \dim g_{\{J_{\lambda}\}} + \dim g_{\{K_{\lambda}\}} \\
> \frac{3l_1}{2} + \frac{3l_2}{2} \\
= \frac{3}{2}(l_1 + l_2) \\
= \frac{3n}{2}.
\]

As we have shown above the minimum dimension of \( g_P \) is \( \frac{3n}{2} \) for even \( n \). This minimum dimension may only occur in Type 1, where \( P \) contains exactly one Jordan block for each characteristic number \( \lambda_i \).

**Theorem 2.10.** If \( \dim V = n \) is odd, then the minimal possible dimension of \( g_P \) is \( n \), and if \( \dim g_P = n \), then \( P \) is of Kronecker type with one single block only.

**Proof.** There are 3 possible types for \( P \) when \( n \) is odd.

- **Type 1:** \( P \) is one single Kronecker block i.e., \( n = 2k + 1 \). Then by using the formula from the Theorem 2.6, we get \( \dim g_P = 2k + 1 = n \).
- **Type 2:** \( P \) is of pure Kronecker type with at least 3 Kronecker blocks (i.e.,
\[ n = l_2 = \sum_{j=1}^{q} (2k_j + 1) \text{ and } q = 3, k_1 \leq k_2 \leq k_3 \). Then we have

\[
\dim g_P \geq \sum_{j=1}^{3} (2k_j + 1)j \\
= (2k_1 + 1) \cdot 1 + (2k_2 + 1) \cdot 2 + (2k_3 + 1) \cdot 3 \\
= n + (2k_2 + 1) + (2k_3 + 1) \\
> n + (2k_1 + 1) + (2k_2 + 1) + (2k_3 + 1) \\
= 2n > n.
\]

- **Type 3:** \( \mathcal{P} \) contains Jordan blocks and at least one Kronecker block (i.e., \( n = l_1 + l_2 \)). Then we have

\[
\dim g_P > \dim g_{\{J_\lambda\}} + \dim g_{\{K_\lambda\}} \\
\geq \frac{3l_1}{2} + l_2 \\
= \frac{l_1}{2} + (l_1 + l_2) \\
> n.
\]

As we can see, the case when \( \mathcal{P} \) contains only one single Kronecker block is the only possibility to obtain \( \dim g_P = n \).  

### 2.6 Solvability of \( g_\mathcal{P} \)

In this section we discuss the algebraic structure of the Lie algebra \( g_\mathcal{P} \). First of all, we notice that there are cases when this Lie algebra is solvable.

**Theorem 2.11.** \( g_\mathcal{P} \) is solvable if and only if \( \mathcal{P} \) is of pure Kronecker type and the sizes of Kronecker blocks are all different.

**Proof.** The proof of this theorem follows from a number of properties.

**Property 2.1.** If \( \mathcal{P} = \{A_\lambda\} \) contains Jordan blocks, then \( g_\mathcal{P} \) is not solvable.

**Proof.** Let \( A_\lambda \) be a single Jordan block with characteristic number \( \lambda_1 \) i.e.
\[ A_\lambda = \begin{pmatrix}
\lambda + 1 & 1 \\
-\lambda_1 - \lambda & 1 \\
\vdots & \vdots \\
-1 & -\lambda_1 - \lambda \\
\end{pmatrix} \]

so we have

\[ g_P = \begin{pmatrix}
e_1 & \cdots & e_k \\
f_1 & \cdots & f_k \\
a_1 & \cdots & a_k & b_k & \cdots & b_1 \\
\vdots & \vdots & \vdots \\
a_1 & b_1 & c_1 & -a_1 \\
\vdots & \vdots & \vdots \\
c_1 & \cdots & c_k & -a_k & \cdots & -a_1
\end{pmatrix} \]

After the following change of basis:

\[ e_1, e_2, \ldots, e_k, f_1, f_2, \ldots, f_k \rightarrow e_1, f_k, e_2, f_{k-1}, \ldots, e_k, f_1, \]

the Lie algebra \( g_P \) reduces to the following block triangular form which is obviously isomorphic to \( sl(2, \mathbb{C}) \otimes \mathbb{P}_{k-1} \):

\[ g_P \simeq \begin{pmatrix}
a_1 & b_1 & a_2 & b_2 & \cdots & a_k & b_k \\
c_1 & -a_1 & c_2 & -a_2 & \cdots & c_k & -a_k \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
a_2 & b_2 & c_2 & -a_2 & \cdots & a_1 & b_1 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
a_1 & b_1 & c_1 & -a_1 & \cdots \\
\end{pmatrix} = sl(2, \mathbb{C}) \otimes \mathbb{P}_{k-1}. \]

Here \( \mathbb{P}_{k-1} \) is the algebra of truncated polynomials of degree \( \leq k - 1 \):

\[ \mathbb{P}_{k-1} = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1}\}, \]

with the operation induced from the standard multiplication by removing all terms of degree \( > k - 1 \), that is \( x^i x^j = \begin{cases} x^{i+j}, & i + j \leq k - 1 \\ 0, & i + j > k - 1 \end{cases} \). It is not hard to verify that the Lie algebra

\[ sl(2, \mathbb{R}) \otimes \mathbb{P}_{k-1} = \left\{ \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} + x \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix} + \cdots + x^{k-1} \begin{pmatrix} a_k & b_k \\ c_k & -a_k \end{pmatrix} \right\} \]
is isomorphic to our \( \mathfrak{g}_P \) above. It is easy to see that \( \mathfrak{g}_P \) contains a semisimple Lie subalgebra

\[
\begin{pmatrix}
 a_1 & b_1 \\
 c_1 & -a_1 \\
 & \\
 & \\
 & \\
 & \ddots \\
 a_1 & b_1 \\
 c_1 & -a_1
\end{pmatrix}
\]

which means that \( \mathfrak{g}_P \) is not solvable. More generally, for any pencil \((A_\lambda)\) that contains Jordan blocks, the corresponding Lie algebra \(\mathfrak{g}_P\) contains a semisimple Lie subalgebra (as above), which means that \(\mathfrak{g}_P\) is not solvable.

\[\square\]

**Property 2.2.** Let \( \mathcal{P} = \{A_\lambda\} \) be a non-trivial single Kronecker block, then \( \mathfrak{g}_P \) is solvable but not nilpotent.

**Proof.** Consider \( A_\lambda \) to be a single Kronecker block as follows:

\[
A_\lambda =\begin{pmatrix} 
1 & \lambda \\
-1 & \lambda \\
& \ddots & \ddots \\
& & 1 & \lambda \\
& & -1 & \lambda \\
& & & & \ddots & \ddots \\
& & & & & 1 & \lambda \\
& & & & & & -1 & \lambda
\end{pmatrix}
\]

According to Lemma 2.1 the corresponding Lie algebra \( \mathfrak{g}_P \) is

\[
\mathfrak{g}_P = \begin{pmatrix}
 a & b_1 & b_2 & \cdots & b_k & b_{k+1} \\
 & b_1 & b_2 & \cdots & b_k & \cdots \\
 & & b_2 & \cdots & b_k & \cdots \\
 & & & \ddots & \ddots & \cdots \\
 & & & & b_k & \cdots & b_{k+k} \\
 & & & & & b_{k+1} & \cdots & b_{k+k} \\
 & & & & & & & \ddots \\
 & & & & & & & & \ddots \\
 & & & & & & & & & \ddots \\
 & & & & & & & & & & \ddots \\
 & & & & & & & & & & & a \\
 & & & & & & & & & & & -a
\end{pmatrix}
\]

We can see that \( \mathfrak{g}_P \) is a Lie subalgebra of the Lie algebra of upper triangular matrices, and therefore is solvable. Moreover, we have

\[
\mathfrak{g}_P^{(1)} = [\mathfrak{g}_P, \mathfrak{g}_P] = \begin{pmatrix} \ast \end{pmatrix},
\]

\[
\mathfrak{g}_P^{(2)} = [\mathfrak{g}_P^{(1)}, \mathfrak{g}_P^{(1)}] = \{0\},
\]

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where * denotes the entries in the block. Thus, $\mathfrak{g}_P$ is two-step solvable. However, $\mathfrak{g}_P$ is not nilpotent, this can be checked by the direct computation. Indeed, we have

\[
\mathfrak{g}'_P = [\mathfrak{g}_P, \mathfrak{g}_P] = \begin{pmatrix} * \\ -1 \\ -\lambda \\ -1 \\ -\lambda \\ \vdots \\ \vdots \\ 1 \\ \lambda \end{pmatrix},
\]
\[
\mathfrak{g}''_P = [\mathfrak{g}'_P, \mathfrak{g}_P] = \begin{pmatrix} * \\ -1 \\ -\lambda \\ -1 \\ -\lambda \\ \vdots \\ \vdots \\ 1 \\ \lambda \end{pmatrix} = \mathfrak{g}'_P.
\]

Thus, $\mathfrak{g}_P$ is not nilpotent. \hfill \square

**Property 2.3.** If $P$ contains at least 2 equal sized Kronecker blocks, then $\mathfrak{g}_P$ is not solvable.

**Proof.** We first consider $A_{\lambda}$ consisting of 2 Kronecker blocks of the same size and the corresponding Lie algebra $\mathfrak{g}_P$ are

\[
A_{\lambda} = \begin{pmatrix}
\begin{pmatrix}
1 & \lambda \\
-1 & 1 & \lambda \\
-\lambda & -1 & -\lambda \\
\vdots & \vdots & \vdots \\
1 & \lambda \\
-1 & 1 & \lambda \\
-\lambda & -1 & -\lambda \\
\vdots & \vdots & \vdots \\
1 & \lambda \\
-1 & 1 & \lambda \\
-\lambda & -1 & -\lambda \\
\end{pmatrix}
& \begin{pmatrix}
1 & \lambda \\
-1 & 1 & \lambda \\
-\lambda & -1 & -\lambda \\
\vdots & \vdots & \vdots \\
1 & \lambda \\
-1 & 1 & \lambda \\
-\lambda & -1 & -\lambda \\
\vdots & \vdots & \vdots \\
1 & \lambda \\
-1 & 1 & \lambda \\
-\lambda & -1 & -\lambda \\
\end{pmatrix}
\end{pmatrix}
\]

and

\[
\mathfrak{g}_P = \begin{pmatrix}
\begin{pmatrix}
a_1 & b_1 & \ldots & b_{k+1} \\
\vdots & \ddots & \ddots & \vdots \\
a_3 & b_k & \ldots & b_{2k} \\
-a_1 & \ddots & \ddots & \ddots \\
-a_1 & \ddots & \ddots & \ddots \\
-a_3 & \ddots & \ddots & \ddots \\
\end{pmatrix}
& \begin{pmatrix}
a_4 & d_1 & \ldots & d_{k+1} \\
\vdots & \ddots & \ddots & \vdots \\
a_2 & d_k & \ldots & d_{2k} \\
-a_2 & \ddots & \ddots & \ddots \\
-a_2 & \ddots & \ddots & \ddots \\
-a_4 & \ddots & \ddots & \ddots \\
\end{pmatrix}
\end{pmatrix}
\]

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Again, the change of basis: \( e_1 e_2 e_3 e_4 \rightarrow e_1 e_3 e_2 e_4 \), gives us

\[
\mathfrak{g}_P \simeq \begin{pmatrix}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
-\cdot & -\cdot & -\cdot \\
-\cdot & -\cdot & -\cdot \\
-\cdot & -\cdot & -\cdot \\
\end{array}
\end{pmatrix}
\supset gl(2, \mathbb{C}) \simeq sl(2, \mathbb{C}) \oplus \mathbb{C}
\]

which means that \( \mathfrak{g}_P \) contains a semisimple Lie subalgebra isomorphic to \( sl(2, \mathbb{C}) \).

It is easy to see that for any pencil \( \mathcal{P} = \{ \mathcal{A}_\lambda \} \) containing at least 2 equal sized Kronecker blocks, its Lie algebra \( \mathfrak{g}_P \) always contains the above semisimple Lie subalgebra. Therefore \( \mathfrak{g}_P \) is not solvable.

Properties 2.1 and 2.3 tell us that for \( \mathfrak{g}_P \) to be solvable, the pencil \( \mathcal{P} \) should contain neither Jordan blocks nor equal sized Kronecker blocks. Thus the only possible pencil \( \mathcal{P} \) is that with distinct sized Kronecker blocks. Therefore we have:

\( \mathfrak{g}_P \) is solvable \( \Rightarrow \mathcal{A}_\lambda \) consists of Kronecker blocks with distinct sizes.

On the other hand, we need to check the inverse statement. To see this, we shall first consider a pencil \( \mathcal{P} \) that consists of 2 Kronecker blocks of different sizes (i.e. \( k_1 < k_2 \))

\[
\mathcal{A}_\lambda = \begin{pmatrix}
\begin{array}{ccc}
k_1 & 1 & \lambda \\
-1 & -1 & \lambda \\
-\lambda & -\lambda & -1 \\
\end{array}
\end{pmatrix}
\]
Then

\[
\begin{align*}
&\begin{pmatrix}
\begin{array}{cccc}
& e_1 & & \\
& & e_2 & & \\
& & & e_3 & \\
& & & & e_4
\end{array}
\end{pmatrix} \\
&\begin{pmatrix}
\begin{array}{cccc}
& a_1 & b_1 & \ldots & b_{k_1+1} \\
& & & \ldots & \vdots \\
& & a_1 & \ldots & b_{k_2}
\end{array}
\end{pmatrix} \\
&\begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& d_1 & \ldots & d_{k_2+1}
\end{array}
\end{pmatrix} \\
&\begin{pmatrix}
\begin{array}{cccc}
& a_2 \\
& & & \vdots \\
& & a_2 & \ldots & c_{k_2}
\end{array}
\end{pmatrix} \\
&\begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& -a_2
\end{array}
\end{pmatrix}
\end{align*}
\]

\[\mathfrak{g}_P = \begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& d_1 & \ldots & d_{k_2+1}
\end{array}
\end{pmatrix} \rightarrow \begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& d_1 & \ldots & d_{k_2+1}
\end{array}
\end{pmatrix} \rightarrow \begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& d_1 & \ldots & d_{k_2+1}
\end{array}
\end{pmatrix} \rightarrow \begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& -a_2
\end{array}
\end{pmatrix}
\]

After change of basis: \( e_1 e_2 e_3 e_4 \rightarrow e_1 e_3 e_4 e_2 \), the algebra \( \mathfrak{g}_P \) can be reduced to a subalgebra of the Lie algebra of upper triangular matrices

\[
\begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& d_1 & \ldots & d_{k_2+1}
\end{array}
\end{pmatrix} \rightarrow \begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& d_1 & \ldots & d_{k_2+1}
\end{array}
\end{pmatrix} \rightarrow \begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& -a_2
\end{array}
\end{pmatrix} \rightarrow \begin{pmatrix}
\begin{array}{cccc}
& e_1 \ldots e_{k_2-k_1+1} \\
& \vdots & \vdots & \\
& e_1 \ldots e_{k_2-k_1+1} \\
& -a_2
\end{array}
\end{pmatrix}
\]

Similarly, for a pencil \( \mathcal{P} \) that contains \( n \) distinct sized Kronecker blocks, we change the basis in the following way:

\[
e_1 e_2 \cdots e_{2n} \rightarrow e_1 e_3 \cdots e_{2n-1} e_2 e_2 e_2 \cdots e_2,
\]

to reduce \( \mathfrak{g}_P \) to an upper triangular form which means that \( \mathfrak{g}_P \) is solvable.
2.7 Levi subalgebras

For those $\mathfrak{g}_P$ which are not solvable, we may find their Levi subalgebras $\mathfrak{h} \subset \mathfrak{g}_P$ in each case.

2.7.1 Symplectic case

**Theorem 2.12.** Suppose $\{J_\lambda\}$ is a pencil consisting of Jordan blocks only. Let $p$ be the number of distinct characteristic numbers $\lambda_1, \ldots, \lambda_p$, and for each characteristic number $\lambda_i$ we have $m_j = m_j(\lambda_i)$ Jordan blocks of size $2k_j(\lambda_i) \times 2k_j(\lambda_i)$, $j = 1, \ldots, s_t$:

$$k_1(\lambda_1) \ldots k_1(\lambda_t) \ldots k_2(\lambda_1) \ldots k_2(\lambda_t) \ldots k_j(\lambda_1) \ldots k_j(\lambda_t) \ldots k_{s_t}(\lambda_1) \ldots k_{s_t}(\lambda_t).$$

Then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}(J_\lambda)$ is isomorphic to

$$\bigoplus_{t=1}^{p} \bigoplus_{j=1}^{s_t} \text{sp}(2 \times m_j(\lambda_t), \mathbb{C}).$$

**Proof.** The proof is broken down into 4 Propositions as follows.

**Proposition 2.5.** Let $J_\lambda$ be a single Jordan block, then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}(J_\lambda)$ is isomorphic to $\text{sl}(2, \mathbb{C}) = \text{sp}(2, \mathbb{C})$.

**Proof.** See the proof of Property 2.1.

**Proposition 2.6.** Let $J_\lambda$ consist of $n$ Jordan blocks, and all blocks have distinct characteristic numbers. Then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}(J_\lambda)$ is isomorphic to

$$\underbrace{\text{sl}(2, \mathbb{C}) \oplus \cdots \oplus \text{sl}(2, \mathbb{C})}_{n}.$$
characteristic values $\lambda_1$ and $\lambda_2$ respectively,

\[
\mathcal{J}_\lambda = \begin{pmatrix}
\lambda_1 + \lambda & 1 & \cdots & 1 \\
-\lambda_2 - \lambda & \ddots & \ddots & \ddots \\
-1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \lambda_1 + \lambda \\
-\lambda_1 - \lambda & \cdots & -1 & 1 \\
\end{pmatrix}
\]

According to Property 2.1, we know the algebraic structure of $\mathfrak{g}(\mathcal{J}_\lambda)$ w.r.t each characteristic value. Namely, we have

\[
\mathfrak{g}(\mathcal{J}_\lambda) \simeq \bigoplus_{i=1}^n \left( sl(2, \mathbb{C}) \otimes \mathbb{P}_{k_i-1} \right) \bigoplus \left( sl(2, \mathbb{C}) \otimes \mathbb{P}_{k_{n+1}-1} \right)
\]

The semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}(\mathcal{J}_\lambda)$ is isomorphic to $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$.

In general, if $\mathcal{J}_\lambda$ consists of $n$ Jordan blocks of size $2k_1, \ldots, 2k_n$ with distinct characteristic value $\lambda_1, \ldots, \lambda_n$ ($\lambda_i \neq \lambda_j$, $i, j \in \{1, \ldots, n\}$) respectively, and $k_1 \geq \cdots \geq k_n$, we have

\[
\mathfrak{g}(\mathcal{J}_\lambda) \simeq \bigoplus_{i=1}^n \left\{ sl(2, \mathbb{C}) \otimes \mathbb{P}_{k_i-1} \right\}
\]

and

\[
\mathfrak{h} \simeq \left( sl(2, \mathbb{C}) \oplus \cdots \oplus sl(2, \mathbb{C}) \right)_n
\]

\textbf{Proposition 2.7.} Let $\mathcal{J}_\lambda$ consist of $n$ distinct sized Jordan blocks, which all have the same characteristic number, then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}(\mathcal{J}_\lambda)$ is isomorphic to $\left( sl(2, \mathbb{C}) \oplus \cdots \oplus sl(2, \mathbb{C}) \right)_n$. 

\hfill \Box
Proof. First, we consider an example:

Let

$$J_{\lambda} = \begin{pmatrix}
\lambda + \lambda & 1 & 1 \\
-\lambda_1 - \lambda & \lambda_1 + \lambda & 1 \\
-1 & -\lambda_1 - \lambda & \lambda_1 + \lambda
\end{pmatrix}$$

and

$$g(J_{\lambda}) \cong \begin{pmatrix}
a_1 & b_1 \\
c_1 & -a_1
\end{pmatrix}
$$

Again, after an appropriate change of basis:

$$e_1 e_2 e_3 f_1 f_2 \rightarrow e_1 f_1 e_2 f_2 e_3,$$

we get

$$g(J_{\lambda}) \cong \begin{pmatrix}
a_1 & b_1 & m_1 & h_1 \\
c_1 & -a_1 & p_1 & -g_1
\end{pmatrix}$$

It shows
\[
\begin{pmatrix}
  a_1 & b_1 \\
  c_1 & -a_1
\end{pmatrix}
\begin{pmatrix}
  d_1 & e_1 \\
  f_1 & -d_1
\end{pmatrix}
\begin{pmatrix}
  a_1 & b_1 \\
  c_1 & -a_1
\end{pmatrix}
\begin{pmatrix}
  d_1 & e_1 \\
  f_1 & -d_1
\end{pmatrix}
\begin{pmatrix}
  a_1 & b_1 \\
  c_1 & -a_1
\end{pmatrix}
= sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}).
\]

More generally, for \( J_\lambda \) consisting of \( n \) distinct sized Jordan blocks with the same characteristic value, after changing basis in a similar way, the Lie algebra \( g_{\{J_\lambda\}} \) can always be arranged into a Lie subalgebra of block upper triangular matrices, which include a semisimple Lie subalgebra \( h \simeq \sum_{i=1}^{n} sl(2, \mathbb{C}) \) as a Levi subalgebra.

**Proposition 2.8.** Let \( J_\lambda \) consist of \( n \) Jordan blocks of the same size, which all have the same characteristic number, then the semisimple Levi subalgebra \( h \subset g_{\{J_\lambda\}} \) is isomorphic to \( sp(2 \times n, \mathbb{C}) \).

**Proof.** Take \( J_\lambda \) consisting of \( n \) equal sized (i.e., size \( 2k \times 2k \)) Jordan blocks, which have the same characteristic value \( \lambda_1 \), i.e.,

\[
J_\lambda = \begin{pmatrix}
  J_1 \\
  \vdots \\
  J_n
\end{pmatrix}, \quad \text{and each } J_i =
\begin{pmatrix}
  \lambda_1 + \lambda & 1 \\
  \vdots & \ddots \\
  -\lambda_1 - \lambda & \lambda_1 + \lambda
\end{pmatrix}.
\]

Then, after an appropriate change of basis (i.e., by collecting each corresponding entry in each subblock), we have

\[
g_{\{J_\lambda\}} \simeq
\begin{pmatrix}
  A & B & M \\
  C & -A^T & P \\
  Q & -M^T & R
\end{pmatrix}
\]
where

\[
A = \begin{pmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\quad \text{and} \quad
M = \begin{pmatrix}
  m_{11} & \cdots & m_{1n} \\
  \vdots & \ddots & \vdots \\
  m_{n1} & \cdots & m_{nn}
\end{pmatrix}
\]

are arbitrary \(n \times n\) matrices.

\[
B = \begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  b_{21} & & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  b_{n1} & b_{(n-1)n} & \cdots & b_{nn}
\end{pmatrix},
\quad
C = \begin{pmatrix}
  c_{11} & c_{12} & \cdots & c_{1n} \\
  c_{12} & & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  c_{n1} & c_{(n-1)n} & \cdots & c_{nn}
\end{pmatrix},
\quad
P = \begin{pmatrix}
  p_{11} & p_{12} & \cdots & p_{1n} \\
  p_{21} & & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  p_{n1} & p_{(n-1)n} & \cdots & p_{nn}
\end{pmatrix},
\quad
Q = \begin{pmatrix}
  q_{11} & q_{12} & \cdots & q_{1n} \\
  q_{12} & & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  q_{n1} & q_{(n-1)n} & \cdots & q_{nn}
\end{pmatrix}
\]

are arbitrary \(n \times n\) symmetric matrices. Equivalently

\[
\mathfrak{g}_{\{J_{\lambda}\}} \cong \begin{pmatrix}
  A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k \\
  A_1 & A_2 & A_3 & \cdots & A_{k-1} & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k
\end{pmatrix},
\quad \text{where} \quad A_i \in \text{sp}(2 \times n, \mathbb{C}).
\]

It is clear that the Levi subalgebra \(\mathfrak{h} \subset \mathfrak{g}_{\{J_{\lambda}\}}\) is isomorphic to \(\text{sp}(2 \times n, \mathbb{C})\).

To understand better why \(\mathfrak{g}_{\{J_{\lambda}\}}\) takes this form, it is useful to see what happens to the canonical form of \(J_{\lambda}\) after the mentioned change of basis. This “new” canonical form is as follows:

\[
(J_{\lambda})_{\text{new}} = \begin{pmatrix}
  0 & 0 & \cdots & 0 & \Omega & \lambda \Omega \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & \Omega & \lambda \Omega & \cdots & \ddots & \vdots \\
  \Omega & \lambda \Omega & \ddots & \ddots & \ddots & \vdots \\
  0 & \Omega & \lambda \Omega & \ddots & \ddots & \vdots \\
  \lambda \Omega & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix},
\quad \text{where} \quad \Omega = \begin{pmatrix}
  0 & \text{Id}_n \\
  -\text{Id}_n & 0
\end{pmatrix}.
\]

To complete the proof of the theorem, we should notice that the Levi subalgebra is related to the \(m_j(\lambda_t)\) w.r.t. each \(\lambda_t\). Proposition 2.7 and Proposition 2.8 tell us for the Jordan blocks corresponding to each \(\lambda_t\) with the size \(2k_j(\lambda_t) \times 2k_j(\lambda_t)\) and \(j = 1 \ldots s_t\), i.e.,

\[
\underbrace{k_1(\lambda_t) \ldots k_1(\lambda_t)}_{m_1} \underbrace{k_2(\lambda_t) \ldots k_2(\lambda_t)}_{m_2} \ldots \underbrace{k_j(\lambda_t) \ldots k_j(\lambda_t)}_{m_j} \ldots \underbrace{k_{s_t}(\lambda_t) \ldots k_{s_t}(\lambda_t)}_{m_{s_t}},
\]

the corresponding semisimple Levi subalgebra is isomorphic to

\[
\bigoplus_{j=1}^{s_t} \text{sp}(2 \times m_j(\lambda_t), \mathbb{C}).
\]
According to Proposition 2.5 and Proposition 2.6, the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\{J\}}$ for $\{J\}$ mentioned in Theorem 2.12, which is isomorphic to

$$\bigoplus_{t=1}^{p} \bigoplus_{j=1}^{s_t} \text{sp}(2 \times m_j(\lambda_t), \mathbb{C}).$$

Therefore we come to the conclusion that for a general pencil $\{J\}$ of pure Jordan type, the Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\{J\}}$ is given in Theorem 2.12.

\[\square\]

### 2.7.2 Kronecker case

**Theorem 2.13.** Suppose $\{\mathcal{K}_\lambda\}$ is a pencil consisting of Kronecker blocks only, and each elementary Kronecker block $(\mathcal{K}_j)$ with size $(2k_j+1) \times (2k_j+1)$, $j = 1, \ldots, q$. Let $l$ be the number of blocks of distinct sizes and $m_i$ be the block multiplicity, $i = 1, \ldots, l$:

$$\mathcal{K}_1 \cdots \mathcal{K}_{(m_1)} \mathcal{K}_{(m_1+1)} \cdots \mathcal{K}_{m_1+m_2} \cdots \mathcal{K}_{(q-m_l+1)} \cdots \mathcal{K}_q$$

with the same size

or, equivalently,

$$k_1 = \cdots = k_{m_1} < k_{m_1+1} = \cdots = k_{m_1+m_2} < \cdots < k_{q-m_l+1} \cdots k_q.$$

Then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\{\mathcal{K}_\lambda\}}$ is isomorphic to

$$\text{sl}(m_1, \mathbb{C}) \oplus \text{sl}(m_2, \mathbb{C}) \oplus \cdots \oplus \text{sl}(m_l, \mathbb{C}).$$

**Proof.** The proof is broken down into 2 Propositions as follows.

**Proposition 2.9.** For any pencil $\{\mathcal{K}_\lambda\}$ of pure Kronecker type with distinct sized blocks, the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\{\mathcal{K}_\lambda\}}$ is trivial.

**Proof.** This statement immediately follows Theorem 2.15. Since under the above assumptions $\mathfrak{g}_{\{\mathcal{K}_\lambda\}}$ is solvable, it coincides with its radical $\mathfrak{r}$. By the Levi-Malcev decomposition theorem: $\mathfrak{g} = \mathfrak{h} + \mathfrak{r}$. Hence $\mathfrak{h} = \{0\}$.

**Proposition 2.10.** For any pencil $\{\mathcal{K}_\lambda\}$ of pure Kronecker type with $n$ equal sized blocks of size $2k + 1$, the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\{\mathcal{K}_\lambda\}}$ is isomorphic to $\text{sl}(n, \mathbb{C})$.

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Proof. It is not hard to see that there is a change of basis that reduces $\mathcal{K}_\lambda$ to the following “new” canonical form

$$(\mathcal{K}_\lambda)_{\text{new}} = \begin{pmatrix}
\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 1 \\
0 & \ddots & \ddots & 0 \\
-1 & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & -\lambda \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & -\lambda \\
\end{array}
\end{pmatrix} \cdot \begin{pmatrix}
\begin{array}{cccc}
1 & 0 & \cdots & \lambda \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 1 \\
0 & \ddots & \ddots & 0 \\
-1 & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & -\lambda \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & -\lambda \\
\end{array}
\end{pmatrix}.
\end{pmatrix}$$

Then $\mathfrak{g}_{\{\mathcal{K}_\lambda\}}$ takes the following upper triangular block form:

$$\mathfrak{g}_{\{\mathcal{K}_\lambda\}} = \begin{pmatrix}
\begin{pmatrix} B & * & * & \cdots & * \\
B & * & * & \cdots & * \\
-B^T & * & * & \cdots & * \\
-B^T & * & * & \cdots & * \\
-B^T & * & * & \cdots & * \\
\end{pmatrix}
\end{pmatrix}$$

where $B \in \mathfrak{gl}(n, \mathbb{C})$. Thus, the diagonal part of $\mathfrak{g}_{\{\mathcal{K}_\lambda\}}$ is isomorphic to $\mathfrak{gl}(n, \mathbb{C})$. Hence the Levi subalgebra is $\mathfrak{sl}(n, \mathbb{C})$. \hfill \Box

To complete the proof of the theorem, it remains to notice that each group consisting of $n$ Kronecker blocks of equal size will contribute the subalgebra $\mathfrak{sl}(n, \mathbb{C})$ into the whole Levi subalgebra (see Proposition 2.10), whereas Kronecker blocks of distinct sizes do not interact (from the viewpoint of the semisimple part of $\mathfrak{g}_{\{\mathcal{K}_\lambda\}}$) and do not give any contribution to $\mathfrak{h}$ (see Proposition 2.9). \hfill \Box

2.7.3 Mixed case

In the mixed case we can prove the following theorem.

Theorem 2.14. Suppose the pencil $\mathcal{P} = \{A_\lambda\} = \{\mathcal{J}_\lambda \oplus \mathcal{K}_\lambda\}$ consists of both Jordan and Kronecker blocks i.e.,

$$A_\lambda = \begin{pmatrix}
\mathcal{J}_\lambda \\
\mathcal{K}_\lambda
\end{pmatrix}.$$
Then the semisimple Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is isomorphic to

$$\left( \bigoplus_{t=1}^{n} \bigoplus_{j=1}^{s_t} \text{sp} (2 \times m_t(\lambda_t), \mathbb{C}) \right) \bigoplus \left( \bigoplus_{i=1}^{l} \text{sl}(m_i, \mathbb{C}) \right).$$

**Proof.** In fact, the theorem simply says that the Levi subalgebra in the general case is the direct sum of the corresponding subalgebras for the Jordan and Kronecker parts. In other words, the Jordan and Kronecker parts do not interact from the viewpoint of the Levi decomposition. In terms of $P_{ij}$ blocks (see Section 2.5) which are responsible for interaction between Jordan and Kronecker blocks, this fact simply means that all these $P_{ij}$ belong to the radical of $\mathfrak{g}$.

This fact becomes obvious if we change the basis in such a way that $\mathfrak{g}$ takes an upper triangular block form. In particular, all “mixed” $P_{ij}$ after this change will be located above the diagonal. Instead of explaining how to do this change in the general case, we give an example which illustrates all possible situations. Here we have a pencil $\{\mathcal{A}_\lambda\}$ consisting of 4 Jordan blocks (one of them has a characteristic number $\lambda_2$ different from the other three), and 3 Kronecker blocks with two of them (say $K_1$ and $K_2$) having the equal size.

$$\mathcal{A}_\lambda = \begin{pmatrix}
J_1(\lambda_1) & J_2(\lambda_1) \\
J_3(\lambda_1) & J_1(\lambda_2) & K_1 \\
& & J_1(\lambda_2) & K_2 \\
& & & & K_3
\end{pmatrix},$$

where

$$J_1(\lambda_1) = \begin{array}{cccc}
\lambda_1 + \lambda & 1 & 1 & 1 \\
-\lambda_1 - \lambda & \lambda_1 + \lambda & -1 & -1 \\
-1 & -\lambda_1 - \lambda & -1 & -1 \\
-1 & -1 & -\lambda_1 - \lambda & -1
\end{array}, \quad J_1(\lambda_2) = \begin{array}{c}
\lambda_2 + \lambda
\end{array},$$

$$J_2(\lambda_1) = J_3(\lambda_1) = \begin{array}{cc}
\lambda_1 + \lambda & 1 \\
-\lambda_1 - \lambda & \lambda_1 + \lambda \\
-1 & -\lambda_1 - \lambda \\
-1 & -1 & -\lambda_1 - \lambda
\end{array}, \quad K_1 = K_2 = \begin{array}{cc}
1 & \lambda \\
-1 & -1 \\
-\lambda & -1 \\
-\lambda & -\lambda
\end{array}.$$
This $A_\lambda$ can be reduced to a “new” canonical form under a certain change of basis. Namely,

$$(A_\lambda)_{\text{new}} = \begin{pmatrix} (\lambda_2 + \lambda)\Omega_1 & \Omega_2 & 0 & (\lambda_1 + \lambda)\Omega_1 \\ \Omega_1 & 0 & (\lambda_1 + \lambda)\Omega_2 \\ 0 & (\lambda_1 + \lambda)\Omega_2 \\ (\lambda_1 + \lambda)\Omega_1 \end{pmatrix},$$

where

$$\Omega_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 1 & \lambda \\ 1 & \lambda \\ 1 & \lambda \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 1 & 0 & \lambda \\ 1 & 0 & \lambda \\ 1 & 0 & \lambda \end{pmatrix}. $$

Then $g_P$ takes the following upper triangular block form

$$g_P = \begin{pmatrix} B_1 & * \\ B_2 & b_2 \\ b_2 & b_2 \end{pmatrix} \begin{pmatrix} \text{non-zero $P_{ji}$} \\ \text{non-zero $P_{ij}$} \end{pmatrix},$$

where $A_1, A_2 \in sp(2, \mathbb{C}), A_3 \in sp(2 \times 2, \mathbb{C}), B_1 \in gl(2, \mathbb{C})$ and $b_2 \in \mathbb{C}$.

Therefore in this case, for $p = 2$, $l = 2$, $s_1 = 2$, $s_2 = 1$, $m_1(\lambda_1) = 1$, $m_2(\lambda_1) = 2$, $m_1(\lambda_2) = 1$, $m_1(\lambda_1) = 1$, $m_2(\lambda_1) = 2$ and $m_1(\lambda_2) = 1$, it is clear that the semisimple
Levi subalgebra $\mathfrak{h} \subset \mathfrak{g}_P$ is isomorphic to

$$\left( \bigoplus_{t=1}^{p} \bigoplus_{j=1}^{s_t} \text{sp}(2 \times m_j(\lambda_t), \mathbb{C}) \right) \bigoplus \left( \bigoplus_{i=1}^{l} \text{sl}(m_i, \mathbb{C}) \right)$$

$$= \left( \bigoplus_{j=1}^{s_1} \text{sp}(2 \times m_j(\lambda_1), \mathbb{C}) \right) \bigoplus \left( \bigoplus_{j=1}^{s_2} \text{sp}(2 \times m_j(\lambda_2), \mathbb{C}) \right) \bigoplus \left( \bigoplus_{i=1}^{2} \text{sl}(m_i, \mathbb{C}) \right)$$

$$= \text{sp}(2, \mathbb{C}) \oplus \text{sp}(2 \times 2, \mathbb{C}) \oplus \text{sp}(2, \mathbb{C}) \oplus \text{sl}(1, \mathbb{C}) \oplus \text{sl}(2, \mathbb{C}).$$

\[\square\]
This chapter is devoted to some algebraic constructions related to compatible Poisson brackets on finite-dimensional Lie algebras. We start with recalling the necessary definitions and basic facts.

3.1 Lie-Poisson brackets and argument shift method

Definition 3.1. A Lie algebra \( \mathfrak{g} \) is a vector space over the field \( \mathbb{R} \) (or \( \mathbb{C} \)), together with a bilinear map, the Lie bracket \([\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\rightarrow\mathfrak{g}\), which satisfies two properties [11]:

- Skew-symmetry \( [\xi,\eta]=-[\eta,\xi] \) for all \( \xi,\eta\in\mathfrak{g} \);
- Jacobi identity \( [\xi,[\eta,\zeta]]+[\eta,[\zeta,\xi]]+[\zeta,[\xi,\eta]] = 0 \) for all \( \xi,\eta,\zeta\in\mathfrak{g} \).

If \( \mathfrak{g} \) is finite-dimensional, then the above relation can be rewritten in terms of coordinates as follows:
If \((\xi_1, \ldots, \xi_n)\) and \((\eta_1, \ldots, \eta_n)\) are the coordinates of two elements \(\xi, \eta \in g\) w.r.t. the basis \(e_1, \ldots, e_n\) in \(g\), then the coordinates of \([\xi, \eta]\) are

\[ [\xi, \eta]^k = \sum_{i,j=1}^n c^k_{ij} \xi^i \eta^j, \]

where \(c^k_{ij}\) are certain real or complex numbers called the structure coefficients. These coefficients satisfy the two properties:

- Skew-symmetry \(c^k_{ij} = -c^k_{ji}\);
- Jacobi identity \(c^k_{il}c^l_{jm} + c^k_{jl}c^l_{mi} + c^k_{ml}c^l_{ij} = 0\).

### 3.1.1 Lie-Poisson brackets

It is easy to see that there is a natural one-to-one correspondence between linear Poisson tensors and Lie algebras. Indeed, by restriction to linear functions, the Poisson bracket operation defined on \(g^*\) (dual space of \(g\)) gives rise to an operation \([\cdot, \cdot] : g \times g \to g\), which has a Lie algebra structure on \(g\). Conversely, any Lie algebra structure on \(g\) determines a linear Poisson tensor on \(g^*\).

**Definition 3.2.** Let \(f, g : g^* \to \mathbb{R}\) be two smooth functions on the dual space \(g^*\) of a Lie algebra \(g\). Then their differentials \(df(x), dg(x)\) can naturally be treated as elements of \(g\). This allows us to define a Poisson bracket on \(g^*\) (which is called Lie-Poisson bracket) \([3]\) by

\[ \{f, g\}(x) = x([df(x), dg(x)]), \quad x \in g^*, df(x), dg(x) \in g. \]

Equivalently, we can consider the Poisson tensor on \(g^*\) given by

\[ A_{ij}(x) = \sum_k c^k_{ij} x_k. \]

Here \(c^k_{ij}\) are the structure coefficients of \(g\) and \(x_k\) are the coordinates of \(x\) in \(g^*\). Thus,
we have

\[
\{f, g\}(x) = x \left( [df(x), dg(x)] \right) = \sum_k x_k \left( [df(x), dg(x)]^k \right) = \sum_k x_k \left( \sum_{i,j} c_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \right) = \sum_{i,j,k} c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.
\]

The Lie-Poisson bracket is closely related to the coadjoint representation.

\[\text{3.1.2 The adjoint and coadjoint representation of a Lie group and its Lie algebra}\]

Suppose we have a Lie group $G$ and its Lie algebra $\mathfrak{g}$. The adjoint representation of $G$ (resp. $\mathfrak{g}$) on the Lie algebra $\mathfrak{g}$ is defined by

\[\text{Ad} : G \to GL(\mathfrak{g}),\]

where $\text{Ad}_g \eta = \left. \frac{d}{dt} \right|_{t=0} (g \cdot \exp(t\eta) \cdot g^{-1})$, $\eta \in \mathfrak{g}$, $g \in G$;

\[\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}),\]

where $\text{ad}_\xi \eta = \left. \frac{d}{ds} \right|_{s=0} (\text{Ad}_{\exp(s\xi)} \eta) = [\xi, \eta]$, $\xi, \eta \in \mathfrak{g}$. The coadjoint representation $\text{Ad}^*$ is dual to $\text{Ad}$. The space on which $G$ acts is the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ (i.e., $\text{Ad}^* : G \to GL(\mathfrak{g}^*)$). To define the coadjoint representation we need to describe the linear operator $\text{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ for each $g \in G$.

**Definition 3.3.** Let $a \in \mathfrak{g}^*$. Then the result of the action of $\text{Ad}_g^*$ on $a \in \mathfrak{g}^*$ is the linear functional on $\mathfrak{g}$ (i.e. again an element of $\mathfrak{g}^*$) defined by

\[\text{Ad}_g^* a(\eta) = a \left( \text{Ad}_g^{-1} \eta \right).\]

Similarly, for the Lie algebra $\mathfrak{g}$, we define its *coadjoint representation* on $\mathfrak{g}^*$ (ad$^* : \mathfrak{g} \to \text{End}(\mathfrak{g}^*)$) by

\[\text{ad}^*_{\xi} a(\eta) = \left. \left( \frac{d}{ds} \right)_{s=0} (\text{Ad}^{*}_{\exp(s\xi)} a(\eta)) \right| = \left. \frac{d}{ds} \right|_{s=0} a(\text{Ad}_{\exp(s\xi)}^{-1} \eta) = a(-\text{ad}_\xi \eta) = -a([\xi, \eta]).\]
It follows from the definition that
\[ \sum_j (\text{ad}_\xi^* a)_j \eta^j = - \sum_{i,j,k} c^k_{ij} a_k \xi^i \eta^j. \]
Thus, in coordinates we have
\[ (\text{ad}_\xi^* a)_j = - \sum_{i,k} c^k_{ij} a_k \xi^i. \]

### 3.1.3 Basic properties of the Lie-Poisson bracket

Let us recall some basic facts related to the Lie-Poisson bracket see [9] and [12].

**Definition 3.4.** The index of a Lie algebra \( \mathfrak{g} \) is the minimal dimension of the stationary subalgebras for the coadjoint representation of \( \mathfrak{g} \) on its dual space, i.e.,
\[ \text{ind} \mathfrak{g} = \min_{x \in \mathfrak{g}^*} \dim \{ \xi \in \mathfrak{g} | \text{ad}_\xi^* x = 0 \}. \]

The following fact is well-known and can be considered as yet another definition of the index.

**Theorem 3.1.** The index of a Lie algebra is the corank of the corresponding Lie-Poisson tensor.

**Proof.** Let \( \mathfrak{g} \) be a Lie algebra, and \( \{ , \} \) be the Lie-Poisson bracket defined on \( \mathfrak{g}^* \). Then by definition we have
\[ \{ f, g \}(x) = x ([df(x), dg(x)]) = \sum_{i,j,k} c^k_{ij} x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \]
where \( x \in \mathfrak{g}^* \) and \( x \) is regular (see Section 1.1.3), \( df(x), dg(x) \in \mathfrak{g} \). According to the definition, for a regular \( x \), the index of \( \mathfrak{g} \) is the dimension of the space of solutions of \( \text{ad}_\xi^* x = 0 \). In matrix form, the Poisson tensor \( \sum_k c^k_{ij} x_k \) is exactly the matrix of this system of linear equations. Therefore
\[ \text{ind} \mathfrak{g} = \dim \mathfrak{g} - \text{rank} \left\{ \sum_k c^k_{ij} x_k \right\} = \text{corank} \left\{ \sum_k c^k_{ij} x_k \right\}. \]

\( \square \)
Theorem 3.2. The Casimir functions for the Lie-Poisson bracket are exactly the invariants of the coadjoint representation.

Proof. We need to compare the corresponding systems of partial differential equations that define the Casimir functions and the Ad*-invariants. A Casimir function $f$ is characterised by

$$\sum_i A_{ij}(x) \frac{\partial f}{\partial x^i} = 0, \quad j = 1, \ldots, n = \dim g.$$ 

Substituting $A_{ij}(x) = \sum_k c_{ij}^k x_k$ gives:

$$\sum_{i,k} c_{ij}^k x_k \frac{\partial f}{\partial x^i} = 0, \quad j = 1, \ldots, n = \dim g.$$ 

For Ad*-invariants, we know that the invariant function is constant on each coadjoint orbit:

$$f(\text{Ad}_g^*(x)) = f(x) = \text{constant}, \quad g \in G, x \in M \simeq g^*.$$ 

Then

$$(\text{ad}^*_\xi x) f = \left( \frac{d}{dt} \bigg|_{t=0} \text{Ad}^*_{\exp(t\xi)} x \right) f = \left( \frac{d}{dt} \bigg|_{t=0} f(\text{Ad}^*_{\exp(t\xi)} x) \right) = d \left( \frac{d}{dt} \bigg|_{t=0} f(\text{Ad}^*_{\exp(t\xi)} x) \right) = 0,$$

where $\xi \in g, g \in G, x \in g^*$. The converse is also true: if this relation holds identically for all $x \in g^*$ and $\xi \in g$, then $f$ is Ad*-invariant. It follows from Definition 3.1 that

$$(\text{ad}^*_\xi x)_j = -\sum_{i,k} c_{ij}^k x_k \xi^i.$$ 

Thus, we have

$$(\text{ad}^*_\xi x) f = -\sum_{i,j,k} c_{ij}^k x_k \xi^i \frac{\partial f}{\partial x^j} = 0,$$

for any $\xi \in g$. Equivalently,

$$-\sum_{j,k} c_{ij}^k x_k \frac{\partial f}{\partial x^j} = 0, \quad i = 1, \ldots, n = \dim g.$$ 

This coincides with the equation defining the Casimir functions (after interchanging indices $i$ and $j$).

Theorem 3.3. The symplectic leaves of the Lie-Poisson bracket are exactly the orbits of the coadjoint representation.
Proof. For regular orbits the statement holds due to the fact that the Casimir functions coincide with $\text{Ad}^*$-invariants. To prove it for an arbitrary orbit, we need to compare the tangent space for the $\text{Ad}^*$-orbit $O(x)$ and the symplectic leaf $O_x$.

The tangent space of $O(x)$ at $x \in g^*$ consists of all the vectors of the form

$$(\text{ad}^*_\xi x)_j = - \sum_{i,k} c^k_{ij} x_k \xi^i, \quad \xi \in g.$$ 

The tangent space of $O_x$ at $x \in M \simeq g^*$ consists of all Hamiltonian vectors, i.e. $X_h(x)$, where $h$ is an arbitrary function on the dual space $g^*$. In coordinates, we have

$$X_h(x)_j = \sum_i \mathcal{A}_{ij}(x) \frac{\partial h}{\partial x_i} = \sum_{i,k} c^k_{ij} x_k \frac{\partial h}{\partial x_i} = -(\text{ad}^*_h x)_j, \quad dh(x) \in g.$$ 

Thus, the tangent space for $O_x$ and $O(x)$ are the same at any point $x \in g^*$. Hence $O_x = O(x)$, as needed.

3.1.4 Compatible Poisson brackets on $g^*$

As we know from Chapter 1 (Section 1.5.2), on the dual space of $g$ one can always define two natural compatible Poisson brackets. The first one is the Lie-Poisson bracket:

$$\{f, g\}(x) = x([df(x), dg(x)]), \quad x \in g^*, df(x), dg(x) \in g.$$ 

The other is the constant bracket $\{ , \}_a$, which can be defined for every $a \in g^*$ by the following formula:

$$\{f, g\}_a(x) = a([df(x), dg(x)]), \quad x \in g^*, df(x), dg(x) \in g.$$ 

The compatibility condition for $\{ , \}$ and $\{ , \}_a$ easily follows from the Jacobi identity (see Section 1.5.2). For the corresponding Poisson tensors we use below the following notation respectively

$$\mathcal{A}_x = \left\{ \sum_\gamma c^\gamma_{ij} x_\gamma \right\} \quad \text{and} \quad \mathcal{A}_a = \left\{ \sum_\beta c^\beta_{ij} a_\beta \right\}.$$ 

If $x$ is fixed, we can treat $\mathcal{A}_x$ and $\mathcal{A}_a$ as two skew symmetric forms defined on the Lie algebra $g$. The linear combination of brackets $\{ , \} + \lambda \{ , \}_a$ corresponds to the Poisson tensor $\mathcal{A}_x + \lambda \mathcal{A}_a = \mathcal{A}_{x+\lambda a}$. In particular, we see that the properties of the pencil $\mathcal{P} = \{ \mathcal{A}_a + \lambda \mathcal{A}_a \}$ are defined by the properties of the line $x + \lambda a \in g^*$, $\lambda \in \mathbb{C}$. 

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3.1.5 Commutative families of functions related to \{ , \} and \{ , \}_a

The Lie-Poisson bracket \{ , \} is isomorphic to \{ , \} + \lambda\{ , \}_a for a fixed arbitrary \lambda \in \mathbb{C}. One can use this observation to construct a family of commuting functions. The following fact is well-known (see, for example, [23] and [12]).

**Proposition 3.1.** Let \( f(x) : \mathfrak{g}^* \to \mathbb{C} \) be a polynomial Casimir function for \{ , \} of degree \( n \).

1. Then for an arbitrary \( \lambda \in \mathbb{R} \) and a fixed \( a \in \mathfrak{g}^* \), we have \( f(x + \lambda a) : \mathfrak{g}^* \to \mathbb{C} \) is a Casimir function for \{ , \}_{x + \lambda a} = \{ , \} + \lambda\{ , \}_a.

2. Furthermore, if \( f(x + \lambda a) = f_0(x) + \lambda f_1(x, a) + \cdots + \lambda^n f_n(a) \), then the following relations hold:

\[
\begin{align*}
A_x df_0 &= 0, \\
A_x df_1 &= -A_a df_0, \\
&\vdots \\
A_x df_n &= -A_a df_{n-1}, \\
A_a df_n &= 0.
\end{align*}
\]

3. Moreover, \( f_0, f_1, \ldots, f_n \) commute with respect to both \{ , \} and \{ , \}_a.

**Proof.** 1. Since \( f(x) \) is a Casimir function for \{ , \}, by definition we have

\[
\{ f(x), \cdot \} = x([df(x), \cdot]) = \text{ad}_{df(x)}x(\cdot) = 0.
\]

It follows that \( \text{ad}_{df(x)}x = 0 \) for all \( x \in \mathfrak{g}^* \). Then

\[
\begin{align*}
\{ f(x + \lambda a), \cdot \}_{x + \lambda a} &= \{ f(x + \lambda a), \cdot \} + \lambda\{ f(x + \lambda a), \cdot \}_a \\
&= x([df(x + \lambda a), \cdot]) + \lambda([df(x + \lambda a), \cdot]) \\
&= (x + \lambda a)([df(x + \lambda a), \cdot]) \\
&= \text{ad}_{df(x+\lambda a)}(x + \lambda a)(\cdot) \\
&= 0,
\end{align*}
\]

i.e., \( f(x + \lambda a) \) is a Casimir for \{ , \}_{x + \lambda a}.
2. We have proved \( \{ f(x + \lambda a), \cdot \}_{x+\lambda a} = 0 \) in part 1, or equivalently \( \mathcal{A}_{x+\lambda a} df(x + \lambda a) = (\mathcal{A}_x + \lambda \mathcal{A}_a) df(x + \lambda a) = 0 \). Since we can always expand \( df(x + \lambda a) \) in the following way

\[
df(x + \lambda a) = df_0(x) + \lambda df_1(x, a) + \lambda^2 df_2(x, a) + \cdots + \lambda^n df_n(a),
\]

we have

\[
\begin{align*}
(\mathcal{A}_x + \lambda \mathcal{A}_a) df(x + \lambda a) &= (\mathcal{A}_x + \lambda \mathcal{A}_a) (df_0(x) + \lambda df_1(x, a) + \cdots + \lambda^n df_n(a)) \\
&= \mathcal{A}_x df_0(x) + \lambda (\mathcal{A}_x df_1(x, a) + \mathcal{A}_a df_0(x)) + \cdots + \\
&+ \lambda^n (\mathcal{A}_x df_n(a) + \mathcal{A}_a df_{n-1}(x, a)) + \lambda^{n+1} \mathcal{A}_a df_n(a) \\
&= 0.
\end{align*}
\]

As \( \lambda \) is arbitrary, by setting each term to be zero we obtain the relations, which are required.

3. Let \( x \in g^* \) be regular. Then \( \mathcal{A}_x df_0 = 0 \) indicates that \( df_0 \in \ker \mathcal{A}_x \subset L = \sum_\mu \ker \mathcal{A}_{x+\mu a} \) (here we assume that \( x + \mu a \) is regular, see the description of \( L \) in Section 2.2). Then by using Proposition 2.1, we conclude by induction that \( df_0, df_1, \cdots, df_n \in L \). Since \( L \) is isotropic with respect to both \( \mathcal{A}_x \) and \( \mathcal{A}_a \) (see Theorem 2.4), the functions \( f_0, f_1, \cdots, f_n \) commute with each other with respect to both \( \{ , \} \) and \( \{ , \}_a \) (see Remark 1.1).

Strictly speaking, the above argument shows that these functions commute on the set of regular points, but since this set is everywhere dense, by continuity \( \{ f_i, f_j \} = \{ f_i, f_j \}_a = 0 \) holds on \( g^* \) identically.

\[
\square
\]

If the Casimir functions of the Lie-Poisson bracket \( \{ , \} \) on \( g^* \) are not polynomial, then we can prove a similar statement, but to obtain commuting polynomials we need to swap \( x \) and \( a \). See, for example, [1], [5].

**Proposition 3.2.** Let \( a \in g^* \) be a fixed regular element and \( g(x) : g^* \to \mathbb{C} \) be an arbitrary Casimir function for \( \{ , \} \) which is smooth in a neighborhood of \( a \).

1. Then for an arbitrary \( \lambda \in \mathbb{C} \), the function \( g(a + \lambda x) : g^* \to \mathbb{C} \) is Casimir for \( \{ , \}_{a+\lambda x} = \{ , \}_a + \lambda \{ , \} \).
2. Let 
\[ g(a + \lambda x) = g_0(a) + \lambda g_1(a, x) + \cdots + \lambda^k g_k(a, x) + \cdots \]  
be the power series expansion of \( g(a + \lambda x) \) with respect to \( \lambda \) so that \( g_k(a, x) \) is a homogeneous polynomial of degree \( k \) in \( x \) depending on \( a \) as a parameter. Then \( g_k \) satisfy the following relations.
\[
\begin{align*}
A_a dg_1 &= 0, \\
A_a dg_2 &= -A_x dg_1, \\
&\vdots \\
A_a dg_k &= -A_x dg_{k-1}, \\
&\vdots 
\end{align*}
\]

3. The polynomials \( g_0, g_1, \ldots, g_k, \ldots \) commute with respect to both \( \{ \ , \ \} \) and \( \{ \ , \ \}_a \).

Proof. 1. Since \( g(x) \) is a Casimir function for \( \{ \ , \ \} \), we have by definition
\[
\{ g(x), \cdot \} = x ([dg(x), \cdot]) = \text{ad}_{dg(x)} x(\cdot) = 0.
\]
It follows that \( \text{ad}_{dg(x)} x = 0 \) for all \( x \in \mathfrak{g}^* \). Then
\[
\{ g(a + \lambda x), \cdot \}_{a+\lambda x} = \{ g(a + \lambda x), \cdot \}_a + \lambda \{ g(a + \lambda x), \cdot \}
\]
\[= a([\lambda \ dg(a + \lambda x), \cdot]) + \lambda x([\lambda \ dg(a + \lambda x), \cdot])
\]
\[= (a + \lambda x)([\lambda \ dg(a + \lambda x), \cdot])
\]
\[= \text{ad}_{\lambda \ dg(a+\lambda x)}(a + \lambda x)(\cdot)
\]
\[= 0,
\]
i.e., \( g(a + \lambda x) \) is a Casimir function for \( \{ \ , \ \}_{a+\lambda x} \).

2. We have proved \( \{ g(a + \lambda x), \cdot \}_{a+\lambda x} = 0 \) in part 1, or equivalently \( A_{a+\lambda x} dg(a + \lambda x) = (A_a + \lambda A_x) dg(a + \lambda x) = 0 \). Since we can always expand \( dg(a + \lambda x) \) in the following way
\[
dg(a + \lambda x) = dg_0(a) + \lambda dg_1(a, x) + \lambda^2 dg_2(a, x) + \cdots + \lambda^k dg_k(a, x) + \cdots,
\]
then we have
\[
(A_a + \lambda A_x) \, dg(a + \lambda x) = A_a \, dg_0(a) + \lambda (A_a \, dg_1(a, x) + \cdots + A_x \, dg_0(a)) + \cdots + \\
+ \lambda^k (A_a \, dg_k(a, x) + A_x \, dg_{k-1}(a, x)) + \cdots = 0.
\]

By letting each term to be zero we obtain the relations, which are required.

3. Here \( A_a \, dg_1 = 0 \) indicates that \( dg_1 \in \text{Ker} \, A_a \subset L = \sum_\lambda \text{Ker} \, A_{x+\lambda a} \) (again, we assume that \( x + \lambda a \) is regular, see the description of \( L \) in Section 2.2). Then by using Proposition 2.1, we conclude by induction that \( dg_k \in L \) for any \( k \).

Since \( L \) is isotropic with respect to both \( A_x \) and \( A_a \) (see Theorem 2.4), the polynomials \( g_0, g_1, \ldots, g_k, \ldots \) commute with each other with respect to both \{ \, \} and \{ \, \}_a \) (see Remark 1.1).

Strictly speaking, the above argument shows that these functions commute in a neighborhood of \( a \in g^* \), but since they are polynomial, \( \{ g_i, g_j \} = \{ g_i, g_j \}_a = 0 \) holds on \( g^* \) identically.

These two Propositions show that using the Casimir functions of \( g \) and the compatibility of the brackets \{ \, \} and \{ \, \}_a \), we can always construct polynomials in bi-involution. Moreover, the proof shows that the differentials of the polynomials \( g_k \)'s (produced from a given Casimir \( g \)) always belong to the subspace \( L \), which is isotropic w.r.t. the both Poisson tensors \( A_x \) and \( A_a \). This immediately implies that the polynomials \( g_k \) and \( h_m \) produced from two different Casimir functions \( g \) and \( h \) still commute with respect to both brackets. This allows us to construct a large family of commuting functions on \( g^* \). Namely, we consider the family of polynomials

\[
\mathcal{F}_a = \{ g_k(x) \mid g \in \text{Cas}(g), \ k \in \mathbb{N} \}, \tag{3.3}
\]

where \( \text{Cas}(g) \) denotes the set of all Casimir functions and \( g_k \), as before, are homogeneous polynomials obtained from the expansion (3.2) of \( g \) at the point \( a \in g^* \). Notice that in this context we assume that \( a \in g^* \) is regular and the local Casimir functions \( g \) are defined, generally speaking, only in a small neighborhood of \( a \).
This construction of commuting polynomials was introduced by A. Mischenko and A. Fomenko in [23] and is called the argument shift method. The original construction in [23] generalised the idea by S. Manakov [22] and, in fact, was slightly different. They considered the family of functions

$$\mathcal{F}_a = \{ f(x + \lambda a) \mid f \in I(\mathfrak{g}), \lambda \in \mathbb{R} \}, \quad (3.4)$$

where $I(\mathfrak{g})$ denotes the set of Ad $^*$-invariants of $\mathfrak{g}$ (this form of commuting functions explains the terminology: “argument shift”). It is easy to see that in the case of polynomial Casimir functions (Ad $^*$-invariants), these two families (3.3) and (3.4) are equivalent (i.e., they span the same subspace). The modification (3.3) uses local Casimir functions and allows us to obtain commuting polynomials even in the case when the algebra $\mathfrak{g}$ does not have any polynomial Casimir functions, and even in the case when there are no global Casimir functions at all.

The main properties of the family $\mathcal{F}_a$ are given by the following theorem.

**Theorem 3.4** (A.S. Mischenko, A.T. Fomenko [23]).

1) The family $\mathcal{F}_a$ is commutative with respect to both brackets $\{ , \}$ and $\{ , \}_a$.

2) If $\mathfrak{g}$ is semisimple, then $\mathcal{F}_a$ is complete.

However, there are many examples of Lie algebras for which the family $\mathcal{F}_a$ is not complete. Nevertheless, due to the important role of complete commutative families on Lie algebras in the theory of integrable Hamiltonian systems, A. Mischenko and A. Fomenko in 1978 stated the following conjecture.

**Mischenko-Fomenko conjecture.** On the dual space $\mathfrak{g}^*$ of an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$, there exists a complete family of polynomials in involution.

In other words, each Lie algebra admits an integrable Hamiltonian system with polynomial integrals. This conjecture was proved in 2004 by S. Sadetov (see [33]). The proof essentially used the argument shift method, but the main construction was based on different ideas. In the context of bi-Hamiltonian approach, the following conjecture, which can be regarded as a strong version of the Mischenko–Fomenko conjecture, seems to be quite natural.
Generalized argument shift conjecture. On the dual space $g^*$ of an arbitrary finite-dimensional Lie algebra $g$, there exists a complete family $G_a$ of polynomials in bi-involution, i.e., in involution w.r.t. the two brackets $\{,\}$ and $\{,\}_a$.

Notice that the classical algebra of shifts $F_a$, without loss of generality, can be considered as a subalgebra of $G_a$. In other words, the conjecture says that even if $F_a$ is not complete, it can always be extended up to a complete commutative algebra $G_a$.

In this chapter we verify this conjecture for some series of Lie algebras by applying the techniques based on the Jordan–Kronecker decomposition theorem.

### 3.2 Jordan–Kronecker invariants of Lie algebras

Let $g$ be a Lie algebra and $g^*$ be its dual space. Consider the compatible Poisson tensors $A_x$ and $A_a$ (see Section 3.1.4). If we fix $x \in g^*$, then we can think of $A_x$ and $A_a$ as skew-symmetric forms on $T_x^*(g^*) = g$. We can always reduce them to a Jordan–Kronecker normal form, which, in general, depends on both $x$ and $a$.

**Definition 3.5.** We say that $(x,a) \in g^* \times g^*$ is a generic pair if the type of the Jordan–Kronecker decomposition (namely, the number and size of Jordan and Kronecker blocks and multiplicities of characteristic numbers) of $A_x = \sum_k c_{ij}^k x_k$ and $A_a = \sum_k c_{ij}^k a_k$ is the same for all points in the neighborhoods of $x$ and $a$ respectively.

Thus, by definition, the Jordan–Kronecker type of the pencil $P = \{A_{x+\lambda a}\}$ is the same for all generic pairs $(x,a)$. This allows us to give the following definition.

**Definition 3.6.** The type of the Jordan–Kronecker canonical form for the pencil $P = \{A_x + \lambda A_a\}$ for a generic pair $(x,a) \in g^* \times g^*$, is called the Jordan–Kronecker invariant (J-K invariant) of $g$.

In particular, we will say that a Lie algebra $g$ is of:

- Kronecker type,
- Jordan (symplectic) type,
• mixed type,

if the Jordan–Kronecker decomposition for the generic pencil $P = \{A_{x+\lambda a}\}$ consists of

• Kronecker blocks only,
• Jordan blocks only,
• both of Kronecker and Jordan blocks.

The following two theorems can be considered as natural reformulations of some properties of Lie algebras in terms of Jordan–Kronecker invariants (see [1, 2, 10]).

**Theorem 3.5.** The following properties of a Lie algebra $\mathfrak{g}$ are equivalent:

1. $\mathfrak{g}$ is of Kronecker type, i.e. the Jordan–Kronecker decomposition for the generic pencil $P = \{A_{x+\lambda a}\}$ consists of Kronecker blocks only;

2. $\text{codim } S \geq 2$, where

$$S = \{ y \in \mathfrak{g}^* \mid \text{corank } A_y > \text{ind } \mathfrak{g} \} \subset \mathfrak{g}^*$$

is the singular set in $\mathfrak{g}^*$;

3. $\mathcal{F}_a$ is complete.

**Proof.** The equivalence of 2 and 3 was proved by A. Bolsinov (see [1]) (without using the concept of Jordan–Kronecker invariants). From the viewpoint of J-K invariant, the proof becomes very natural and simple.

Indeed, if $\mathfrak{g}$ is of Kronecker type, then the J-K decomposition of a generic pencil $P = \{A_{x+\lambda a}\}$ has no Jordan blocks and characteristic numbers at all. This means that the rank of $A_{x+\lambda a}$ is the same for all $\lambda \in \mathbb{C}$, i.e., the “generic line” $(x + \lambda a)$ does not intersect the singular $S$. This happens if and only if $\text{codim } S > 2$. The converse is also true: if $\text{codim } S > 2$, then a “generic line” $(x + \lambda a)$ does not intersect the singular $S$, i.e., the rank of $A_{x+\lambda a}$ never drops and, therefore, no Jordan block may appear.
Furthermore, from the definition of $F_a$ we can easily see that the subspace in $g$ spanned by the differentials of all $g \in F_a$ at a generic point $x \in g^*$ coincides with the subspace $L = \sum \text{Ker} \ A_{x+\lambda a}$, where the sum is taken over sufficiently small $\lambda \in \mathbb{C}$ or, equivalently, over all $\lambda \in \mathbb{C}$ such that $x + \lambda a \notin S$. The completeness of $F_a$ means that $L$ is maximal isotropic at a generic point. However the properties of $L$ have been studied in detail in Chapter 2, and we know from Theorem 2.4 that this condition is equivalent to the fact that $P = \{A_{x+\lambda a}\}$ is of Kronecker type. This completes the proof.

The next theorem describes Lie algebras of Jordan type.

**Theorem 3.6.** The following properties of a Lie algebra $g$ are equivalent:

1. $g$ is of Jordan type, i.e., the Jordan–Kronecker decomposition for the generic pencil $P = \{A_{x+\lambda a}\}$ consists of Jordan blocks only;

2. a generic form $A_x$ is non-degenerate, i.e., $\text{ind } g = 0$;

3. $F_a$ is trivial, i.e. $F_a = \mathbb{C}$.

**Proof.** This statement is, in fact, a natural reformulation of conditions 2 and 3 in terms of Jordan–Kronecker invariants. Indeed, $\text{ind } g = 0$ means that for a generic $x \in g^*$ the form $A_x$ is non-degenerate. In other words, the generic pencil $P = \{A_{x+\lambda a}\}$ is symplectic, i.e., of Jordan type.

On the other hand, the non-degeneracy of the Poisson tensor $A_x$ at a generic point is equivalent to the absence of non-trivial Casimir functions, i.e., $F_a = \mathbb{C}$.

**Remark 3.1.** It is useful to point out the following generalisation of this theorem. The index of $g$ coincides with the number of Kronecker blocks in the J-K decomposition of a generic pencil $P = \{A_{x+\lambda a}\}$. Indeed, the Jordan–Kronecker decomposition theorem (Theorem 2.1) implies that the number of Kronecker blocks is equal to the corank of the pencil, which is exactly $\text{ind } g$ in our case.

### 3.2.1 Three dimensional Lie algebras

Here we give an explicit description of J-K invariants for all three-dimensional Lie algebras. This can be done by straightforward computation. According to the Bianchi
classification of 3-dimensional Lie algebras $\mathfrak{g}$, there are 9 different types of Lie algebras. Each of them (apart from types VIII and IX) can be constructed as a semidirect product of $\mathbb{R}^2$ and $\mathbb{R}$, with $\mathbb{R}$ acting on $\mathbb{R}^2$ by some $2 \times 2$ matrix $X$, so that the Lie algebras have the following matrix representation: $\mathfrak{g} \rightarrow \mathfrak{gl}(3, \mathbb{R})$ defined by 

$$
e_1 \rightarrow \begin{pmatrix} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where $e_1, e_2, e_3$ are a basis in $\mathfrak{g}$. This representation is faithful unless $X = 0$. Different types of algebras correspond to different types of the matrix $X$ as described below.

- **Type I**: The abelian Lie algebra $\mathbb{R}^3$ with commutation relations: $[e_i, e_j] = 0$ for all $i, j \in 1, 2, 3$, where $e_1, e_2, e_3$ are a basis in $\mathbb{R}^3$, and $X = 0$ in this case.

- **Type II**: The Heisenberg algebra $\mathcal{H}$ with commutation relations: $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$, and $X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- **Type III**: The Lie algebra with the following commutation relations $[e_1, e_2] = e_2$, $[e_1, e_3] = [e_2, e_3] = 0$, and in this case $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. It is a limiting case of type V, where one eigenvalue becomes zero.

- **Type IV**: The Lie algebra with $[e_1, e_2] = e_2$, $[e_1, e_3] = e_2 + e_3$, $[e_2, e_3] = 0$, and $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

- **Type V**: The Lie algebra with $[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$, $[e_2, e_3] = 0$, and $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

- **Type VI**: The Lie algebra with commutation relations $[e_1, e_2] = e_2$, $[e_1, e_3] = \alpha e_3$, $[e_2, e_3] = 0$, and $X = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, where $\alpha \neq 0$.

- **Type VIo**: The Lie algebra with $[e_1, e_2] = e_2$, $[e_1, e_3] = -e_3$, $[e_2, e_3] = 0$, and in this case we have $\text{tr}X = 0$ so that $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. 

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• Type VII: The Lie algebra with 
\[ [e_1, e_2] = e_2 - \alpha e_3, \ [e_1, e_3] = \alpha e_2 + e_3, \ [e_2, e_3] = 0, \]
and 
\[ X = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix}, \]
where \( \alpha \neq 0 \).

• Type VII\(_0\): The Lie algebra with 
\[ [e_1, e_2] = -e_3, \ [e_1, e_3] = e_2, \ [e_2, e_3] = 0, \]
and 
\[ X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

• Type VIII: The Lie algebra \( sl(2, \mathbb{R}) \) of traceless \( 2 \times 2 \) matrices. Let 
\[ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]
be a basis in \( sl(2, \mathbb{R}) \). We have 
\[ [e_1, e_2] = 2e_2, \ [e_1, e_3] = -2e_3, \ [e_2, e_3] = e_1. \]

• Type IX: The Lie algebra \( so(3, \mathbb{R}) \). Let 
\[ e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \]
be a basis in \( so(3, \mathbb{R}) \). We have 
\[ [e_1, e_2] = e_3, \ [e_1, e_3] = -e_2, \ [e_2, e_3] = e_1. \]

In general, the Jordan–Kronecker canonical form for a \( 3 \times 3 \) pencil \( \mathcal{P} \) of skew-symmetric forms can be of three different types, namely:

• three trivial Kronecker blocks,
• one \( 2 \times 2 \) Jordan block and one trivial Kronecker block,
• one \( 3 \times 3 \) Kronecker block.

Each of these cases appears as J-K invariant of some 3-dimensional Lie algebra.

**Theorem 3.7.** Let \( \mathfrak{g} \) be a 3-dimensional Lie algebra and \( \mathcal{P} = \{ A_{x, \lambda \mathfrak{g}} \} \) be a generic pencil, \( a, x \in \mathfrak{g}^* \). Then

1. for the commutative Lie algebra \( \mathfrak{g} \) (type I), the J-K decomposition of \( \mathcal{P} \) consists of three trivial Kronecker blocks;
2. for the Lie algebra \( \mathfrak{g} \) of type II and III (i.e., the Heisenberg algebra and direct sum \( b_2 \oplus \mathbb{R} \), where \( b_2 \) is the non-commutative Lie algebra of dimension 2), the J-K decomposition of \( \mathcal{P} \) consists of one \( 2 \times 2 \) Jordan block and one trivial Kronecker block;

3. for all the other 3-dimensional Lie algebras \( \mathfrak{g} \) (types IV, V, VI, VI\(_0\), VII, VII\(_0\), VIII, IX), the J-K decomposition of \( \mathcal{P} \) consists of one \( 3 \times 3 \) Kronecker block.

**Proof.** Consider a generic pair \((x, a) \in \mathfrak{g}^*\), where \( x = x_1 e_1 + x_2 e_2 + x_3 e_3 \) and \( a = a_1 e_1 + a_2 e_2 + a_3 e_3 \), and \( e_1, e_2, e_3 \) are the basis in \( \mathfrak{g}^* \) dual to \( e_1, e_2, e_3 \) in \( \mathfrak{g} \). Then \( \mathcal{A}_x \) and \( \mathcal{A}_a \) are described below.

**I:** \( \mathcal{A}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

**II:** \( \mathcal{A}_x = \begin{pmatrix} 0 & x_3 & 0 \\ -x_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & a_3 & 0 \\ -a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

**III:** \( \mathcal{A}_x = \begin{pmatrix} 0 & x_2 & 0 \\ -x_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & a_2 & 0 \\ -a_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

**IV:** \( \mathcal{A}_x = \begin{pmatrix} 0 & x_2 & x_2 + x_3 \\ -x_2 & 0 & 0 \\ -x_2 - x_3 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & a_2 & a_2 + a_3 \\ -a_2 & 0 & 0 \\ -a_2 - a_3 & 0 & 0 \end{pmatrix} \)

**V:** \( \mathcal{A}_x = \begin{pmatrix} 0 & x_2 & x_3 \\ -x_2 & 0 & 0 \\ -x_3 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & a_2 & a_3 \\ -a_2 & 0 & 0 \\ -a_3 & 0 & 0 \end{pmatrix} \)

**VI:** \( \mathcal{A}_x = \begin{pmatrix} 0 & x_2 & a x_3 \\ -x_2 & 0 & 0 \\ -a x_3 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & a_2 & a a_3 \\ -a_2 & 0 & 0 \\ -a a_3 & 0 & 0 \end{pmatrix} \)
For the Abelian Lie algebra $g$, the result is trivial. For types II and III, the canonical J-K form for the pencil $P$ is

$$A_x + \lambda a \simeq \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}, \quad A_a = \begin{pmatrix} a_3 & 0 & 0 \\ -a_3 & a_2 & 0 \\ a_2 & -a_1 & 0 \end{pmatrix}.$$

with $\lambda_i = \frac{x_3}{a_3}$ and $\lambda_i = \frac{x_2}{a_2}$ for types II and III respectively. The corresponding change of basis is obvious.

All the other series (for generic $x, a \in g^*$) can be characterised by the property that the corank $A_{x+\lambda a} = 1$ for all $\lambda \in \mathbb{C}$. This means that the index of $g$, which coincides with the number of Kronecker blocks (see Remark 3.1) is 1. Since the rank of all $A_{x+\lambda a}$ is the same there are no Jordan blocks, which gives the required result. \[ \square \]
3.2.2 Semisimple Lie algebras

For semisimple Lie algebras, the description of J-K invariants follows, in fact, from [23], and was explicitly given by A.Panasyuk in [30] (the author used a slightly different terminology).

First of all, it is well-known that the codimension of the singular set $S$ for semisimple Lie algebras is 3. Therefore, they are all of Kronecker type. This fact also follows from the Mischenko–Fomenko Theorem (Theorem 3.4) and Theorem 3.5.

It can be shown (see [30]) that the sizes $k_1, \ldots, k_s$, $s = \text{ind } \mathfrak{g}$, of Kronecker blocks are related to the degrees $m_1, \ldots, m_s$ of the basis polynomial Casimir functions (the so-called exponents of $\mathfrak{g}$) in a very simple way. Namely, $k_i = 2m_i - 1$. The exponents $m_1, \ldots, m_s$ of semisimple Lie algebras are well-known, and in particular, for classical series of simple Lie algebras they are:

- $A_n \simeq sl(n + 1)$: $2, 3, 4, \ldots, n + 1$;
- $B_n \simeq so(2n + 1)$: $2, 4, 6, \ldots, 2n$;
- $C_n \simeq sp(2n)$: $2, 4, 6, \ldots, 2n$;
- $D_n \simeq so(2n)$: $2, 4, 6, \ldots, 2n - 2$ and $n$.

3.2.3 Characteristic numbers, singular set and its codimension, properties and examples

In this section we discuss the properties of the singular set $S \subset \mathfrak{g}^*$ and characteristic numbers of $\mathfrak{g}$, and the relationship between them. First, we recall the definitions.

**Definition 3.7.** The *singular set* $S \subset \mathfrak{g}^*$ is defined to be the set of all those points $x \in \mathfrak{g}^*$ for which $\text{corank } A_x > \text{ind } \mathfrak{g}$, where $A_x$ is the Lie–Poisson tensor at the point $x$.

Equivalently, $S$ can be defined as the union of all coadjoint orbits of non-maximal dimension.
For \( a \in g^* \) and \( x \in g^* \), the characteristic numbers \( \{ \lambda_i \in \mathbb{C} \} \) of the pencil \( \mathcal{P} = \{ A_x + \lambda A_a \} \) can be defined by one of three equivalent ways:

- \( \{ \lambda_i \in \mathbb{C} \} \) such that rank \( (A_x + \lambda_i A_a) \) is not maximal;
- \( \{ \lambda_i \in \mathbb{C} \} \) such that corank \( (A_x + \lambda_i A_a) > \text{ind } g \);
- \( \{ \lambda_i \in \mathbb{C} \} \) such that \( (x + \lambda_i a) \in S \subset g^* \), where \( S \) is the singular set of \( g^* \).

**Definition 3.8.** The characteristic numbers \( \lambda_i \) of the Lie algebra \( g \) are the characteristic numbers of the pencil \( \mathcal{P} = \{ A_x + \lambda A_a \} \) for a generic pair \( (x, a) \in g^* \times g^* \).

In a neighborhood of a generic pair \( (x, a) \), these characteristic numbers are analytic functions of \( x \) and \( a \):

\[
\lambda_i = \lambda_i(x, a).
\]

It follows from the previous discussions that the singular set \( S \subset g^* \) and its codimension are important to define the J-K invariants of \( g \). The following three propositions describe some basic properties of \( S \).

**Proposition 3.3.** The Lie algebra \( g \) is Abelian if and only if its singular set is empty, i.e., \( S = \emptyset \).

**Proof.** Let \( g \) be commutative w.r.t. its Lie bracket \([\cdot, \cdot]\), i.e., its structure constants \( c^k_{ij} \) are all 0. Then the corresponding Lie Poisson tensor of \( g \) is trivial \((c^k_{ij}x_k = 0)\), so that for any \( x \in g^* \) we have \( A_x = 0 \) and \( \text{ind } g = \dim g \). This implies that there is no \( x \in g^* \) such that corank \( A_x > \text{ind } g \), which means the singular set \( S \) is empty for an Abelian Lie algebra \( g \).

On the other hand, suppose we have \( S = \emptyset \), and in particular \( 0 \notin S \) (i.e., 0 is a regular point). It follows that at \( x = 0 \) we have rank \( A_x = 0 \), therefore for each \( x \in g^* \) we have rank \( A_x = 0 \) (rank remains the same at all regular points). In other words, \( A_x = 0 \) for all \( x \in g^* \) (i.e., \( c^k_{ij}x_k = 0 \)). This implies that the structure constants \( c^k_{ij} \) of \( g \) are all 0, and therefore \( g \) is Abelian. □

**Proposition 3.4.** The singular set \( S \) is defined by a system of homogeneous polynomial equations.
Proof. Let a Lie algebra $\mathfrak{g}$ have dimension $n$ and $\text{ind } \mathfrak{g} = k$. According to Definition 3.7,

$$
\mathcal{S} = \{ x \in \mathfrak{g}^* \lvert \text{corank } \mathcal{A}_x > \text{ind } \mathfrak{g} \} = \{ x \in \mathfrak{g}^* \lvert \text{rank } \mathcal{A}_x < (n - k) \}.
$$

This is equivalent to saying that for any $x \in \mathcal{S}$ we have $\text{rank } \mathcal{A}_x < (n - k)$. In other words, for any $x \in \mathcal{S}$, each $(n - k)$-minor of $\mathcal{A}_x$ vanishes. On the other hand, it is not hard to see that each minor of size $(n - k) \times (n - k)$ of $\mathcal{A}_x$, i.e.,

$$
\det\left( \begin{array}{ccc}
\sum_s c^s_{i_1 j_1} x_s & \cdots & \sum_s c^s_{i_1 j_{n-k}} x_s \\
\vdots & \ddots & \vdots \\
\sum_s c^s_{i_{n-k} j_1} x_s & \cdots & \sum_s c^s_{i_{n-k} j_{n-k}} x_s 
\end{array} \right)
$$

is a homogeneous polynomial in $x$ of degree $n - k$. \hfill \Box

**Proposition 3.5.** The singular set $\mathcal{S}$ is a “conical surface” in the sense that for any point $x \in \mathcal{S}$, the whole line through $x$ belongs to $\mathcal{S}$, i.e., $\lambda x \in \mathcal{S}$ for all $\lambda \in \mathbb{C}$.

Proof. Let a Lie algebra $\mathfrak{g}$ have dimension $n$ and $\text{ind } \mathfrak{g} = k$.

We have $\mathcal{S} = \{ x \in \mathfrak{g}^* \lvert \text{rank } \mathcal{A}_x < (n - k) \}$. Take $x \in \mathcal{S}$, then $\text{rank } \mathcal{A}_x < (n - k)$. Now for any $\lambda \in \mathbb{C}$ and $\lambda \neq 0$, we have

$$
\text{rank } \mathcal{A}_{\lambda x} = \text{rank } (c^k_{ij} \lambda x_k) = \text{rank } \lambda (c^k_{ij} x_k) = \text{rank } (\lambda \cdot \mathcal{A}_x) = \text{rank } \mathcal{A}_x < (n - k).
$$

Thus for any $x \in \mathcal{S}$ and $\lambda \in \mathbb{C}$, it is true that $\lambda x \in \mathcal{S}$. \hfill \Box

As we pointed out in Definition 3.8, the characteristic numbers $\lambda_i(x, a)$ of $\mathfrak{g}$ in a neighborhood of a generic pair $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$ are analytic functions of $x$ and $a$. In general, these functions are not globally defined. The following statement shows how to avoid this problem.

**Proposition 3.6.** The symmetric polynomials of characteristic numbers $\lambda_1, \cdots, \lambda_m$ are rational functions of $x$ and $a$. Moreover, if $a \in \mathfrak{g}^*$ is fixed, then they are polynomial in $x$. 

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Proof. According to Theorem 3.5, the existence of characteristic numbers implies that \( \text{codim } S = 1 \). In general, \( S \) may contain several irreducible components of different dimensions. Denote by \( S_0 \) the union of components of codimension 1. It is clear that this algebraic variety is defined by one polynomial equation \( P(x) = 0, \ x \in g^* \). It follows from this that the characteristic numbers \( \lambda_i \) are roots of the equation:

\[
P(x + \lambda a) = 0.
\]

We can rewrite this polynomial as:

\[
P(x + \lambda a) = \sum_{i_1 \ldots i_m} c_{i_1 \ldots i_m} (x_{i_1} + \lambda a_{i_1}) \cdots (x_{i_m} + \lambda a_{i_m})
\]

\[
= P_m(a) \lambda^m + P_{m-1}(x, a) \lambda^{m-1} + \cdots + P_1(x, a) \lambda + P_0(x),
\]

where the coefficients \( P_i(x, a) \) are polynomials in \( x \) and \( a \). Next we are going to use the Viete’s formulas to complete the proof:

Let \( S_j \) be the sum of the products of distinct polynomial roots \( \lambda_i \) of the polynomial equation of degree \( n \):

\[
P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0,
\]

where the roots are taken \( j \) at a time (i.e., \( S_j \) is defined as the symmetric polynomial \( S_j(\lambda_1, \ldots, \lambda_n) \)). Namely \( S_j \) is defined for \( j = 1, \ldots, n \) as

\[
S_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + \lambda_n
\]
\[
S_2 = (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \cdots + \lambda_1 \lambda_n) + (\lambda_2 \lambda_3 + \cdots + \lambda_2 \lambda_n) + \cdots + \lambda_{n-1} \lambda_n
\]
\[
S_3 = (\lambda_1 \lambda_2 \lambda_3 + \cdots + \lambda_1 \lambda_2 \lambda_n) + (\lambda_1 \lambda_3 \lambda_4 + \cdots + \lambda_1 \lambda_3 \lambda_n) + \cdots + \lambda_{n-2} \lambda_{n-1} \lambda_n
\]

\[
\vdots
\]
\[
S_n = \lambda_1 \cdots \lambda_n.
\]

then

\[
S_j = (-1)^j \frac{a_{n-j}}{a_n}.
\]

Therefore according to Viete’s theorem, \( S_j(\lambda_1, \ldots, \lambda_m) = (-1)^j \frac{P_{m-j}(x, a)}{P_m(a)} \). In particular, the symmetric polynomials \( S_j(\lambda_1, \ldots, \lambda_m) \) are polynomial functions in \( x \) (since \( a \) is fixed) and are globally defined on \( g^* \) (as functions depending on \( x \) only). \( \Box \)
Example 3.1. The singular sets $S$ and characteristic numbers for three dimensional Lie algebras $g$.

According to the Bianchi classification, there are 9 types of $g$. For a generic pair $(x, a) \in g^*$, where $x = x_1 e^1 + x_2 e^2 + x_3 e^3$, $a = a_1 e^1 + a_2 e^2 + a_3 e^3$ and $e^1, e^2, e^3$ are basis in $g^*$ (see Theorem 3.7), we have:

- Type I:
  \[ A_x = A_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
  The Jordan–Kronecker decomposition is
  \[ A_x = A_a = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]
  Here, $\text{ind } g = 3$, therefore the singular set $S = \emptyset$, with codim $S = 4$.

- Type II:
  \[ A_x = \begin{pmatrix} 0 & x_3 & 0 \\ -x_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_a = \begin{pmatrix} 0 & a_3 & 0 \\ -a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
  The Jordan–Kronecker decomposition is
  \[ A_x \simeq \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_a \simeq \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
  where $\lambda = -\frac{x_3}{a_3}$.
  Here, $\text{ind } g = 1$ and $S = \{x = (x_1, x_2, x_3) \in g^* | x_3 = 0\}$ with codim $S = 1$.

- Type III:
  \[ A_x = \begin{pmatrix} 0 & x_2 & 0 \\ -x_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_a = \begin{pmatrix} 0 & a_2 & 0 \\ -a_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
  The Jordan–Kronecker decomposition is
  \[ A_x \simeq \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_a \simeq \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
  where $\lambda = -\frac{x_2}{a_2}$.
  Here, $\text{ind } g = 1$ and $S = \{x = (x_1, x_2, x_3) \in g^* | x_2 = 0\}$ with codim $S = 1$. 

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• Type IV:

\[
A_x = \begin{pmatrix}
0 & x_2 & x_2 + x_3 \\
-x_2 & 0 & 0 \\
-x_2 - x_3 & 0 & 0
\end{pmatrix},
\quad A_a = \begin{pmatrix}
0 & a_2 & a_2 + a_3 \\
-a_2 & 0 & 0 \\
-a_2 - a_3 & 0 & 0
\end{pmatrix}.
\]

The Jordan–Kronecker decomposition is

\[
A_x \simeq \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\quad A_a \simeq \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

Here, \( \text{ind } g = 1 \) and \( S = \left\{ x = (x_1, x_2, x_3) \in g^* \left| \begin{array}{l}
x_2 = 0 \\
x_3 = 0
\end{array} \right. \right\} \) with \( \text{codim } S = 2 \).

• Type V:

\[
A_x = \begin{pmatrix}
0 & x_2 & x_3 \\
-x_2 & 0 & 0 \\
-x_3 & 0 & 0
\end{pmatrix},
\quad A_a = \begin{pmatrix}
0 & a_2 & a_3 \\
-a_2 & 0 & 0 \\
-a_3 & 0 & 0
\end{pmatrix}.
\]

The Jordan–Kronecker decomposition is

\[
A_x \simeq \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\quad A_a \simeq \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

Here, \( \text{ind } g = 1 \) and \( S = \left\{ x = (x_1, x_2, x_3) \in g^* \left| \begin{array}{l}
x_2 = 0 \\
x_3 = 0
\end{array} \right. \right\} \) with \( \text{codim } S = 2 \).

• Type VI:

\[
A_x = \begin{pmatrix}
0 & x_2 & \alpha x_3 \\
-x_2 & 0 & 0 \\
-\alpha x_3 & 0 & 0
\end{pmatrix},
\quad A_a = \begin{pmatrix}
0 & a_2 & \alpha a_3 \\
-a_2 & 0 & 0 \\
-\alpha a_3 & 0 & 0
\end{pmatrix}.
\]

The Jordan–Kronecker decomposition is

\[
A_x \simeq \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\quad A_a \simeq \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

Here, \( \text{ind } g = 1 \) and \( S = \left\{ x = (x_1, x_2, x_3) \in g^* \left| \begin{array}{l}
x_2 = 0 \\
x_3 = 0
\end{array} \right. \right\} \) with \( \text{codim } S = 2 \).
• Type $VI_0$:

\[
\mathcal{A}_x = \begin{pmatrix} 0 & x_2 & -x_3 \\ -x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & a_2 & -a_3 \\ -a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{pmatrix}.
\]

The Jordan–Kronecker decomposition is

\[
\mathcal{A}_x \simeq \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a \simeq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

Here, $\text{ind } \mathfrak{g} = 1$ and $S = \left\{ x = (x_1, x_2, x_3) \in \mathfrak{g}^* \mid \begin{array}{l} x_2 = 0 \\ x_3 = 0 \end{array} \right\}$ with codim $S = 2$.

• Type $VII$:

\[
\mathcal{A}_x = \begin{pmatrix} 0 & x_2 - \alpha x_3 & \alpha x_2 + x_3 \\ -x_2 + \alpha x_3 & 0 & 0 \\ -\alpha x_2 - x_3 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & a_2 - \alpha a_3 & \alpha a_2 + a_3 \\ -a_2 + \alpha a_3 & 0 & 0 \\ -\alpha a_2 - a_3 & 0 & 0 \end{pmatrix}.
\]

The Jordan–Kronecker decomposition is

\[
\mathcal{A}_x \simeq \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a \simeq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

Here, $\text{ind } \mathfrak{g} = 1$ and $S = \left\{ x = (x_1, x_2, x_3) \in \mathfrak{g}^* \mid \begin{array}{l} x_2 = 0 \\ x_3 = 0 \end{array} \right\}$ with codim $S = 2$.

• Type $VII_0$:

\[
\mathcal{A}_x = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & 0 \\ -a_2 & 0 & 0 \end{pmatrix}.
\]

The Jordan–Kronecker decomposition is

\[
\mathcal{A}_x \simeq \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_a \simeq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

Here, $\text{ind } \mathfrak{g} = 1$ and $S = \left\{ x = (x_1, x_2, x_3) \in \mathfrak{g}^* \mid \begin{array}{l} x_2 = 0 \\ x_3 = 0 \end{array} \right\}$ with codim $S = 2$. 

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• Type VIII:

\[
A_x = \begin{pmatrix}
0 & 2x_2 & -2x_3 \\
-2x_2 & 0 & x_1 \\
2x_3 & -x_1 & 0
\end{pmatrix}, \quad A_a = \begin{pmatrix}
0 & 2a_2 & -2a_3 \\
-2a_2 & 0 & a_1 \\
2a_3 & -a_1 & 0
\end{pmatrix}.
\]

The Jordan–Kronecker decomposition is

\[
A_x \simeq \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_a \simeq \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

Here, ind \( g = 1 \) and \( S = \{ x = (x_1, x_2, x_3) \in g^* \mid x_1 = 0, x_2 = 0, x_3 = 0 \} \) with codim \( S = 3 \).

• Type IX:

\[
A_x = \begin{pmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{pmatrix}, \quad A_a = \begin{pmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{pmatrix}.
\]

The Jordan–Kronecker decomposition is

\[
A_x \simeq \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_a \simeq \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

Here, ind \( g = 1 \) and \( S = \{ x = (x_1, x_2, x_3) \in g^* \mid x_1 = 0, x_2 = 0, x_3 = 0 \} \) with codim \( S = 3 \).

3.3 Examples

For a given Lie algebra \( g \), we give here some general rules, which will help to obtain the J-K invariants of \( g \). Here are the key steps to follow:

Step 1: Determining whether \( g \) is of Kronecker, or Jordan (symplectic), or mixed type. To do this, we have the following conditions to evaluate:
• \( \text{ind } g = \text{number of Kronecker blocks} \) (see Remark 3.1). In particular, \( g \) is of Jordan type if and only if \( \text{ind } g = 0 \).

• Jordan blocks exist if and only if \( \text{codim } S = 1 \) (see Theorem 3.5). In particular, \( g \) is of Kronecker type if and only if \( \text{codim } S \geq 2 \).

**Step 2 (for Lie algebra of Kronecker type):** We need to know the number and the sizes of the Kronecker blocks. To do this, we have the following conditions to evaluate:

- \( \dim g = \text{sum of the sizes of all Kronecker blocks} \).
- \( \text{ind } g = \text{number of Kronecker blocks} \).
- if \( z \subset g \) is the center of \( g \) and \( \dim z = k \), then there are at least \( k \) trivial Kronecker blocks.

(For low-dimensional \( g \) of Kronecker type, J-K invariants of \( g \) are easily found. However, for multi-dimensional \( g \), some more complicated conditions may also need to be taken into account, see for example [40].)

**Step 2 (for Lie algebra of Jordan (symplectic) type):** We need to know the number and the sizes of the Jordan blocks for each characteristic number \( \lambda_i \), as well as, the number or distinct characteristic numbers. To do this, we have the following conditions to evaluate:

- \( \dim g = \text{sum of the sizes of all Jordan blocks} \).
- Let the singular set \( S \subset g^* \) be given by a polynomial equation \( P(x) = 0 \). Such a polynomial is not unique, but we can always choose the one with minimal degree. Since characteristic numbers are solutions \( \lambda_i \) of the equation \( P(x + \lambda a) = 0 \), we see that the number of distinct characteristic numbers is \( k = \text{degree of } P \).
- \( \dim \{ \text{Ker } A_{(x - \lambda_i a)} \} = 2 \cdot (\text{number of Jordan blocks with } \lambda_i) \).

(Again, for multi-dimensional \( g \), some more conditions may also need to be taken into account, see, for example, Section 3.3.2.)

**Step 2 (for mixed type):** The conditions for both Kronecker type and Jordan type are required to be taken into account to obtain the J-K invariants of \( g \).
Below we give some typical examples to illustrate the above procedure.

### 3.3.1 The Lie algebra $\mathfrak{gl}(n) + \mathbb{C}^n$

In this section we consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n) + \mathbb{C}^n$, the Lie algebra of the affine group $G$, which is a semidirect product of $\mathbb{C}^n$ and $GL(n)$. We may define the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n) + \mathbb{C}^n$ in the following way. Elements of $\mathfrak{gl}(n) + \mathbb{C}^n$ are pairs $(\bar{a}, A)$, where $\bar{a} \in \mathbb{C}^n$, $A \in \mathfrak{gl}(n)$, and $[(\bar{a}_1, A_1), (\bar{a}_2, A_2)] = (A_1\bar{a}_2 - A_2\bar{a}_1, [A_1, A_2])$, for any $A_1, A_2 \in \mathfrak{gl}(n)$ and $\bar{a}_1, \bar{a}_2 \in \mathbb{C}^n$.

The standard matrix representations for $G$, $\mathfrak{g}$ and $\mathfrak{g}^*$ are

\[
G = \begin{pmatrix} GL(n) & \mathbb{C}^n \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{g} = \begin{pmatrix} \mathfrak{gl}(n) & \mathbb{C}^n \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathfrak{g}^* = \begin{pmatrix} \mathfrak{gl}(n) & 0 \\ (\mathbb{C}^n)^\top & 0 \end{pmatrix}
\]

respectively. For any $(\bar{b}, B) \in G$, $(\bar{a}, A) \in \mathfrak{g}$, and $(\bar{c}, C) \in \mathfrak{g}^*$, we have the corresponding coadjoint representation $\text{Ad}^* (\text{or ad}^*)$ of $G$ (or $\mathfrak{g}$) on $\mathfrak{g}^*$:

\[
\text{Ad}^*_{(\bar{b}, B)}(\bar{c}, C) = (\bar{c}\top B^{-1}, \bar{b}c\top B^{-1} + BCB^{-1}),
\]

\[
\text{ad}^*_{(\bar{a}, A)}(\bar{c}, C) = (-\bar{c}\top A, \bar{a}\bar{c}\top + [A, C]).
\]

There following 3 propositions describe important properties of $\mathfrak{g}$.

**Proposition 3.7.** The index of $\mathfrak{g} = \mathfrak{gl}(n) + \mathbb{C}^n$ is zero ($\text{ind} \ \mathfrak{g} = 0$).

**Proof.** The statement can be proved by induction. By definition,

\[
\text{ind} \ \mathfrak{g} = \min_{(\bar{c}, C) \in \mathfrak{g}^*} \dim \{(\bar{a}, A) \in \mathfrak{g} \mid \text{ad}^*_{(\bar{a}, A)}(\bar{c}, C) = (-\bar{c}\top A, \bar{a}\bar{c}\top + [A, C]) = 0\}.
\]

Thus, we need to find an element $(\bar{c}, C) \in \mathfrak{g}^*$, for which the equation $\text{ad}^*_{(\bar{a}, A)}(\bar{c}, C) = 0$ has trivial solution only, i.e., $(\bar{a}, A) = (0, 0)$. For $n = 2$, let $\bar{c}\top = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$, where $c_2 \neq 0$. It is easy to verify that the statement holds for $n = 2$, so that

\[
\text{ind} \ \mathfrak{g} = \min_{(\bar{c}, C) \in \mathfrak{g}^*} \dim \{(\bar{a}, A) \in \mathfrak{g} \mid \begin{cases} \bar{c}\top A = 0 \\ \bar{a}\bar{c}\top + [A, C] = 0 \end{cases} \} = 0.
\]
Assume the proposition also holds for \( n - 1 \), then there exists non-zero \((\tilde{c}_{(n-1)}, C') \in (gl(n-1) + \mathbb{C}^{n-1})^*\) such that for \( A' \in gl(n-1) \), \( \tilde{a} \in \mathbb{C}^{n-1} \), and

\[
\begin{align*}
\tilde{c}^\top_{(n-1)} A' &= 0 \\
\tilde{a} \tilde{c}^\top_{(n-1)} + [A', C'] &= 0,
\end{align*}
\]

implies \( A' = 0, \tilde{a} = 0 \). Now, for the size \( n \), we may take \( \bar{c} \) and \( C \) in the following form:

\[
\bar{c}^\top = \left( \begin{array}{cccc}
1 & 0 & \cdots & 0
\end{array} \right) \quad \text{and} \quad C = \left( \begin{array}{c}
c_{11} \\
c_{21} \\
\vdots \\
c_{n1}
\end{array} \right) \begin{array}{c}
\tilde{c}^\top_{(n-1)} \\
C'
\end{array}.
\]

For these \( \bar{c} \) and \( C \) we now solve the system of linear equations

\[
\begin{align*}
\begin{cases}
\bar{c}^\top A &= 0 \\
\tilde{a} \bar{c}^\top + [A, C] &= 0
\end{cases}.
\end{align*}
\]

This system has the following explicit form:

\[
0 = \bar{c}^\top A = \left( \begin{array}{cccc}
1 & 0 & \cdots & 0
\end{array} \right) \left( \begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1n}
\tilde{a} & A'
\end{array} \right) \\
= (a_{11} \cdots a_{1n})
\]

and

\[
0 = \tilde{a} \bar{c}^\top + [A, C] = \left( \begin{array}{c}
a_1 \\
\vdots \\
a_n
\end{array} \right) \left( \begin{array}{cccc}
1 & 0 & \cdots & 0
\end{array} \right) + \left( \begin{array}{c}
0 \\
\tilde{a} & A'
\end{array} \right) \left( \begin{array}{c}
c_{11} \\
c_{21} \\
\vdots \\
c_{n1}
\end{array} \begin{array}{c}
\tilde{c}^\top_{(n-1)} \\
C'
\end{array} \right) = \left( \begin{array}{c}
a_2 \\
\vdots \\
a_n
\end{array} \right) + (c_{11} I_n - C') \tilde{a} + A' \left( \begin{array}{c}
c_{21} \\
\vdots \\
c_{n1}
\end{array} \begin{array}{c}
\tilde{c}^\top_{(n-1)} \\
A'
\end{array} \right) \tilde{a} \bar{c}^\top + [A', C']
\]

The second “column” of this matrix gives 2 equations

\[
\begin{align*}
\begin{cases}
\tilde{c}^\top_{(n-1)} A' &= 0 \\
\tilde{a} \tilde{c}^\top_{(n-1)} + [A', C'] &= 0
\end{cases}.
\end{align*}
\]
that coincide exactly with those for $n-1$. Hence, $A' = 0$, $\tilde{a} = 0$. Now, substituting
$\tilde{a} = 0$ into
\[
a_1 - \tilde{c}_1^{(n-1)} \tilde{a} = 0,
\]
gives $a_1 = 0$.

Putting $\tilde{a} = 0$ and $A' = 0$ into
\[
\begin{pmatrix}
    a_2 \\
    \vdots \\
    a_n
\end{pmatrix}
+ (c_{11}I_n - C')\tilde{a} + A'
\begin{pmatrix}
    c_{21} \\
    \vdots \\
    c_{n1}
\end{pmatrix}
= 0,
\]
we have
\[
\begin{pmatrix}
    a_2 \\
    \vdots \\
    a_n
\end{pmatrix}
= 0.
\]

Thus we obtain $A = 0$ and $\tilde{a} = 0$ as required, hence $\text{ind } g = 0$. \hfill \Box

**Proposition 3.8.** $(\bar{c}, C) \in g^*$ is regular if and only if $\bar{c}^T, \bar{c}^T C, \bar{c}^T C^2, \cdots, \bar{c}^T C^{n-1}$ are linearly independent. In other words, the singular set $S$ of $g = gl(n, \mathbb{R}) + \mathbb{R}^n$ is
\[
S = \left\{ (\bar{c}, C) \in g^* \left| \begin{pmatrix}
    \bar{c}^T \\
    \bar{c}^T C \\
    \vdots \\
    \bar{c}^T C^{n-1}
\end{pmatrix}
\right. \begin{vmatrix}
    \text{det } \begin{pmatrix}
    \bar{c}^T \\
    \bar{c}^T C \\
    \vdots \\
    \bar{c}^T C^{n-1}
\end{pmatrix}
\right. = 0 \right\}
\]
and $\text{codim } S = 1$.

**Proof.** First we need to prove that “$(\bar{c}, C)$ is regular” implies “$\bar{c}^T, \bar{c}^T C, \bar{c}^T C^2, \cdots, \bar{c}^T C^{n-1}$ are linearly independent”. In other words, we need to show that for any regular $(\bar{c}, C) \in g^*$ the condition $\text{det } \mathcal{C} \neq 0$ holds, where
\[
\mathcal{C} = \begin{pmatrix}
    \bar{c}^T \\
    \bar{c}^T C \\
    \vdots \\
    \bar{c}^T C^{n-1}
\end{pmatrix}.
\]
The set of all regular points in $g^*$ is connected and therefore there exists only one regular orbit. Thus, for each regular point $(\bar{c}, C) \in g^*$, there exists $(\bar{b}, B) \in G$ such
that

\[
\text{Ad}^*_\((b,B)\)\left(\begin{array}{c|c}
C & 0 \\
\hline c^\top & 0
\end{array}\right) = \left(\begin{array}{cccc|c}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0 \\
\hline 1 & 0 & \cdots & 0
\end{array}\right),
\]

the new point (on the orbit) we obtained satisfies the required condition, i.e.,

\[
\det C = \det\left(\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{array}\right) \neq 0.
\]

Therefore we have found one regular point

\[
(\bar{c}, C) = \left(\begin{array}{cccc|c}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0 \\
\hline 1 & 0 & \cdots & 0
\end{array}\right),
\]

which satisfies

\[
\det C \neq 0.
\]

Now, since \(\text{Ad}^*_\((b,B)\)\left(\bar{c}, C\right) = (\bar{c}^\top B^{-1}, \bar{b}c^\top B^{-1} + BCB^{-1})\), for any other point \(\text{Ad}^*_\((b,B)\)\left(\bar{c}, C\right)\) on the regular coadjoint orbit, the condition \(\det C \neq 0\) still holds, i.e.,

\[
\det \begin{pmatrix}
\bar{c}^\top B^{-1} \\
\bar{c}^\top B^{-1}(\bar{b}c^\top B^{-1} + BCB^{-1}) \\
\vdots \\
\bar{c}^\top B^{-1}(\bar{b}c^\top B^{-1} + BCB^{-1})^{n-1}
\end{pmatrix} = \det \begin{pmatrix}
\bar{c}^\top B^{-1} \\
\bar{c}^\top C B^{-1} \\
\vdots \\
\bar{c}^\top C^{n-1}B^{-1}
\end{pmatrix} = \det C \det(B^{-1}) \neq 0
\]

Thus the condition is invariant under the \(\text{Ad}^*\) action. Therefore for any point on the regular orbit \( \{\text{Ad}^*_\((\cdot,\cdot)\)\left(\bar{c}, C\right) | (\cdot, \cdot) \in G\} \subset g^*\), the required condition holds.

Next, we shall prove that “\(\bar{c}^\top, \bar{c}^\top C, \bar{c}^\top C^2, \cdots, \bar{c}^\top C^{n-1}\) are linearly independent” implies “\(\bar{c}, C\) is regular”. It can be done by contradiction. Assume \(\det C \neq 0\), but
\((\bar{c}, C)\) is singular. In other words,
\[
\dim\left\{ (\bar{a}, A) \in g \mid \text{ad}_{(\bar{a}, A)}^*(\bar{c}, C) = 0 \right\} > \text{ind } g = 0.
\]
Then there exists a non-zero \((\bar{a}, A) \in g\) that is a solution of
\[
\text{ad}_{(\bar{a}, A)}^*(\bar{c}, C) = (\bar{c}^\top A, \bar{a} \bar{c}^\top + [A, C]) = 0.
\]
Suppose we have \(\bar{a} \neq 0\). Take
\[
\mathcal{C} = \begin{pmatrix}
\bar{c}^\top \\
\bar{c}^\top C \\
\bar{c}^\top C^2 \\
\vdots \\
\bar{c}^\top C^{n-1}
\end{pmatrix}
\]
and multiply it by \(\bar{a}\) (i.e., \(\mathcal{C} \cdot \bar{a}\)). Then the \(k\)'th component of the column-vector so obtained is \(\bar{c}^\top C^{k-1} \bar{a}\), since the product is a scalar,
\[
\bar{c}^\top C^{k-1} \bar{a} = \text{tr}\left(\bar{c}^\top C^{k-1} \bar{a}\right) = \text{tr}\left(\bar{a} \bar{c}^\top C^{k-1}\right) = \text{tr}\left(-[A, C] C^{k-1}\right) = \text{tr}\left(C^k A\right) - \text{tr}\left(AC^k\right) = 0,
\]
thus \(\mathcal{C} \bar{a} = 0\). This implies \(\det(\mathcal{C}) = 0\), which contradicts to the assumption. Suppose \(\bar{a} = 0\) and \(A \neq 0\), then we have
\[
\left\{\begin{array}{l}
\bar{c}^\top A = 0 \\
AC = CA
\end{array}\right.
\]
Consider the product of \(\mathcal{C}\) and \(A\)
\[
\mathcal{C} A = \begin{pmatrix}
\bar{c}^\top A \\
\bar{c}^\top C A \\
\bar{c}^\top C^2 A \\
\vdots \\
\bar{c}^\top C^{n-1} A
\end{pmatrix} = \begin{pmatrix}
\bar{c}^\top A \\
\bar{c}^\top AC \\
\bar{c}^\top AC^2 \\
\vdots \\
\bar{c}^\top AC^{n-1}
\end{pmatrix} = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}.
\]
We have \(\mathcal{C} A = 0\), but \(A \neq 0\). Therefore \(\det(\mathcal{C}) = 0\) again contradicts to the assumption. Thus for \(\bar{c}^\top, \bar{c}^\top C, \bar{c}^\top C^2, \ldots, \bar{c}^\top C^{n-1}\) to be linearly independent, \((\bar{c}, C)\) is required to be regular.
\(\square\)
Since \( \text{ind} \, g = 0 \), the form \( A_x \) is non-degenerate for a generic \( x \in g^* \), and therefore the singular set \( S \) can be defined by the following polynomial equation:

\[
\det A_x = \det \left( \sum_k c^k_{ij} x_k \right) = 0.
\]

In particular, the characteristic numbers \( \lambda_i(x, a) \) are roots of the polynomial \( P(\lambda) = \det A_{x+\lambda a} \).

**Proposition 3.9.** For a generic pair \( (x, a) \in g^* \times g^* \), the polynomial \( P(\lambda) = \det A_{x+\lambda a} = \det \left( \sum_k c^k_{ij} (x_k + \lambda a_k) \right) = 0 \) has \( \frac{n^2 + n}{2} \) distinct roots. In other words, \( g \) possesses \( \frac{n^2 + n}{2} \) distinct characteristic numbers.

**Proof.** Let \( x + \lambda a = (\bar{c}(\lambda), C(\lambda)) \), according to Proposition 3.8, this element is singular, i.e., belongs to \( S \), if and only if \( \bar{c}^T(-\lambda_i), \bar{c}^T(-\lambda_i)C(-\lambda_i), \bar{c}^T(-\lambda_i)C^2(-\lambda_i), \ldots, \bar{c}^T(-\lambda_i)C^{n-1}(-\lambda_i) \) are linearly dependent. Therefore solving the following polynomial equation

\[
P(\lambda) = 0
\]

is equivalent to solving

\[
\det \mathcal{C} = \det \begin{pmatrix}
\bar{c}^T(\lambda) \\
\bar{c}^T(\lambda)C(\lambda) \\
\bar{c}^T(\lambda)C^2(\lambda) \\
\vdots \\
\bar{c}^T(\lambda)C^{n-1}(\lambda)
\end{pmatrix} = 0.
\]

Note that \( \det \mathcal{C} \) is a polynomial of \( \lambda \) of degree \( \frac{n}{2}(n + 1) \). Thus, there are at most \( \frac{n^2 + n}{2} \) distinct roots of \( \lambda \). Let us take \( x \) and \( a \) of the following form (which is easy for
computation):

\[
x = \left( \frac{X}{\bar{x}^\top} \right) = \begin{pmatrix} x_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_{nn} \end{pmatrix},
\]

\[
a = \left( \frac{A}{\bar{a}^\top} \right) = \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix},
\]

\[
(x + \lambda a) = \left( \frac{C(\lambda)}{\bar{c}^\top(\lambda)} \right) = \begin{pmatrix} x_{11} + \lambda a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_{nn} + \lambda a_{nn} \end{pmatrix},
\]

then

\[
P(\lambda) = \det \begin{pmatrix} \bar{c}^\top(\lambda) \\ \bar{c}^\top(\lambda)C(\lambda) \\ \bar{c}^\top(\lambda)C^2(\lambda) \\ \vdots \\ \bar{c}^\top(\lambda)C^{n-1}(\lambda) \end{pmatrix} = (x_1 + \lambda a_1) \cdots (x_n + \lambda a_n)
\]

\[
= \prod_{i=1}^{n} (x_i + \lambda_i a_i) \prod_{i>j}^{n} [(x_{ii} - x_{jj}) + \lambda (a_{ii} - a_{jj})]
\]

\[
= 0.
\]

We have \( n \) distinct roots \( \lambda = -\frac{x_i}{a_i} \) from \( \prod_{i=1}^{n} (x_i + \lambda_i) = 0 \), and other \( \frac{(n^2 - n)}{2} \) distinct roots \( \lambda = -\frac{x_{ii} - x_{jj}}{a_{ii} - a_{jj}} \) from \( \prod_{i>j}^{n} [(x_{ii} - x_{jj}) + \lambda (a_{ii} - a_{jj})] = 0 \). Since the above particular forms of \( x \) and \( a \) are certainly regular, the multiplicities of characteristic numbers are preserved on all other regular pairs \((x, a)\). Therefore for a generic pair \((x, a)\) there exist \( \frac{n^2 + n}{2} \) distinct roots, as required. \( \square \)
The following theorem describes the J-K invariant for $g$.

**Theorem 3.8.** The Lie algebra $g = gl(n) + \mathbb{C}^n$ is of pure Jordan type. Its J-K invariant “consists” of $\frac{n^2+n}{2}$ Jordan blocks of size 2, with distinct characteristic numbers in each block.

**Proof.** Proposition 3.7 (ind $g = 0$) tells us that for a generic pair $(x, a)$, $A_x + \lambda a$ has no Kronecker blocks, i.e., $g$ is of Jordan type (Definition 3.6). For some $\lambda = -\lambda_i$, $(x - \lambda_i a)$ becomes singular. The number of characteristic numbers $\lambda_i$ that exist in the pencil $A_{x+\lambda a}$ determines the number and the sizes of Jordan blocks. Proposition 3.9 tells us that there are a totally $\frac{n^2+n}{2}$ distinct characteristic numbers of the pencil $A_{x+\lambda a}$, and $\dim g = n^2 + n$, which indicates that the J-K invariant of $g$ consists of $\frac{n^2+n}{2}$ Jordan blocks (with distinct characteristic numbers) of size 2. \qed

Recall that by Definition 1.14 and Proposition 1.5, a complete commutative family $F$ is a family of $N$ functions $f_1, \cdots, f_N$ on $g^*$ (where $N = \frac{1}{2} (\dim g + \ind g)$) such that the subspace span$\{df_1(x), \cdots, df_N(x)\} \subset g$ is maximal isotropic w.r.t. $A_x$ for a generic $x \in g^*$. Equivalently, all functions in $F$ commute with each other, i.e., $\{F, F\} = 0$, and are independent. Now, in addition, we want to require a very strong condition for $F$ (see Theorem 3.4). Namely, we want to find a family $G_a$ being commutative and complete w.r.t. all brackets from the pencil $A_{x+\lambda a}$ for $g = gl(n) + \mathbb{C}^n$. In other words, our goal is to verify the simultaneously generalised argument shift conjecture for $g = gl(n) + \mathbb{C}^n$.

**Theorem 3.9.** Let $g = gl(n) + \mathbb{C}^n$ and $\lambda_1, \cdots, \lambda_N$, $N = \frac{n^2+n}{2}$, be distinct characteristic numbers of $g$. Then the symmetric polynomials $S_j(\lambda_1, \cdots, \lambda_N)$ for $j = 1, \cdots, N$ form a complete family $G_a$ of polynomials in bi-involution.

**Proof.** Obviously, $N = \frac{n^2+n}{2} = \frac{1}{2} (\dim g + \ind g)$. So we need to show that these functions are independent and commute w.r.t. all brackets from the pencil $A_{x+\lambda a}$. Also it is clear that $\lambda_i$’s are not constant (see Proposition 3.6). In such a situation the conclusion follows from the following fact well known in the theory of bi-Hamiltonian systems (see, for example, [2], [29]).

Let $A$ and $B$ be non-degenerate compatible Poisson tensors on $M^{2N}$. Assume that their characteristic numbers $\lambda_1, \cdots, \lambda_N$ are all distinct and not constant.
Then they are functionally independent and \(\{\lambda_i, \lambda_j\}_A = \{\lambda_i, \lambda_j\}_B = 0\) for all \(i, j = 1, \ldots, N\).

This fact is an easy corollary from the “canonical form” theorem for non-degenerate compatible brackets proved by J. Turiel (see [38]) which says, in particular, that under the above assumptions there is a local coordinate system \(x_1, y_1, x_2, y_2, \ldots, x_N, y_N\) such that

\[
A = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \ldots + x_N \frac{\partial}{\partial x_N} \wedge \frac{\partial}{\partial y_N},
\]

and

\[
B = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \ldots + \frac{\partial}{\partial x_N} \wedge \frac{\partial}{\partial y_N}.
\]

This immediately implies that \(\lambda_i = x_i\) and the above conclusion becomes obvious.

### 3.3.2 The Lie algebra \(gl(n) + \mathbb{C}^{(n^2)}\)

In this section we discuss the Lie algebra \(g = gl(n) + \mathbb{C}^{(n^2)}\), i.e., the Lie algebra of the group \(G\), which is a semidirect product of \(\mathbb{C}^{(n^2)}\) and \(GL(n)\). We may define the Lie algebra \(g = gl(n) + \mathbb{C}^{(n^2)}\) in the following way. Elements of \(gl(n) + \mathbb{C}^{(n^2)}\) are pairs \((\bar{A}, A)\), where \(\bar{A} \in \mathbb{C}^{(n^2)}\) (here \(\mathbb{C}^{(n^2)}\) is identified with the space of complex \(n \times n\) matrices), \(A \in gl(n)\), and

\[
[(\bar{A}_1, A_1), (\bar{A}_2, A_2)] = (A_1 \bar{A}_2 - A_2 \bar{A}_1, [A_1, A_2]),
\]

for any \(A_1, A_2 \in gl(n), \bar{A}_1, \bar{A}_2 \in \mathbb{C}^{(n^2)}\).

The standard matrix representations of \(G\), \(g\) and \(g^\ast\) are

\[
G = \begin{pmatrix} GL(n) & \mathbb{C}^{(n^2)} \\ 0 & \mathbb{I}_n \end{pmatrix}, \quad g = \begin{pmatrix} gl(n) & \mathbb{C}^{(n^2)} \\ 0 & 0 \end{pmatrix}, \quad \text{and } g^\ast = \begin{pmatrix} gl(n) \\ (\mathbb{C}^{(n^2)})^\top \end{pmatrix}
\]

respectively. For any

\[
(\bar{B}, B) = \begin{pmatrix} B & \bar{B} \\ 0 & \mathbb{I}_n \end{pmatrix} \in G, \quad (\bar{A}, A) = \begin{pmatrix} A & \bar{A} \\ 0 & 0 \end{pmatrix} \in g, \quad (\bar{C}, C) = \begin{pmatrix} \bar{C} & 0 \\ C & 0 \end{pmatrix} \in g^\ast,
\]

the corresponding coadjoint representation \(Ad^\ast\) (or \(ad^\ast\)) of \(G\) (or \(g\)) on \(g^\ast\) are

\[
Ad^\ast_{(\bar{B}, B)}(\bar{C}, C) = (\bar{C}B^{-1}, \bar{B}\bar{C}B^{-1} + BC B^{-1}),
\]

\[
ad^\ast_{(\bar{A}, A)}(\bar{C}, C) = (-\bar{C}A, \bar{A}\bar{C} + [A, C]).
\]

The following 5 propositions describe some important properties of the Lie algebra \(g = gl(n) + \mathbb{C}^{(n^2)}\).
Proposition 3.10. \( \text{ind } \mathfrak{g} = 0. \)

Proof. By definition,

\[
\text{ind } \mathfrak{g} = \min_{(C, C) \in \mathfrak{g}^*} \dim \left\{ (\tilde{A}, A) \bigg| \text{ad}_{(\tilde{A}, A)}^*(\tilde{C}, C) = (-\tilde{C}A, \tilde{A}\tilde{C} + [A, C]) = 0 \right\}.
\]

We need to show that for regular \((\tilde{C}, C) \in \mathfrak{g}^*\), the system

\[
\begin{align*}
\tilde{C}A &= 0 \\
\tilde{A}\tilde{C} + [A, C] &= 0
\end{align*}
\]

has only the trivial solution. Assume \(\tilde{C}\) is non-degenerate (\(\det \tilde{C} \neq 0\)), then there exists \(\tilde{C}^{-1}\), which give us \(A = 0\) from the first equation. Substituting \(A = 0\) into the second equation, we have \(\tilde{A}\tilde{C} = 0\) and consequently, \(\tilde{A} = 0\) as \(\tilde{C}\) is invertible. Thus \((\tilde{A}, A) = 0\) is the only solution of the above system, which means that \(\text{ind } \mathfrak{g} = 0. \)

Proposition 3.11. \((\tilde{C}, C) \in \mathfrak{g}^*\) is regular if and only if \(\det \tilde{C} \neq 0. \) In other words, the singular set \(S\) of \(\mathfrak{g} = \text{gl}(n, \mathbb{R}) + \mathbb{R}^{(n^2)}\) is

\[
S = \left\{ (\tilde{C}, C) \in \mathfrak{g}^* \bigg| \det \tilde{C} = 0 \right\},
\]

and \(\text{codim } S = 1. \)

Proof. It follows from the Proposition 3.10 that \((\tilde{C}, C)\) is regular if \(\det \tilde{C} \neq 0. \)

The “only if” condition is proved by contradiction. Assume \((\tilde{C}, C)\) is regular, but \(\tilde{C}\) is degenerate (i.e., \(\det \tilde{C} = 0\)). To solve

\[
\begin{align*}
\tilde{C}A &= 0 \\
\tilde{A}\tilde{C} + [A, C] &= 0
\end{align*}
\]

suppose \(A = 0, \) then \(\tilde{A}\tilde{C} = 0. \) This matrix equation has a non trivial solution \(\tilde{A} \neq 0\) since \(\det \tilde{C} = 0. \) Thus, in this case

\[
\dim \left\{ (A, A) \bigg| \text{ad}_{(A, A)}^*(C, C) = 0 \right\} > \text{ind } \mathfrak{g} = 0,
\]

and therefore \((\tilde{C}, C)\) is singular, which contradicts the assumption. We conclude that for regular \((\tilde{C}, C) \in \mathfrak{g}\) we always have \(\det \tilde{C} \neq 0. \)

\(\square\)
As in the previous section, the condition \( \text{ind } g = 0 \) implies that the singular set \( S \subset g^* \) can be defined by the following polynomial equation

\[
\det A_x = \det \left( \sum_k c_{ij}^k x_k \right) = 0
\]

Moreover, the characteristic numbers \( \lambda_i(x, a) \) are exactly the roots of the polynomial \( P(\lambda) = \det A_{x+\lambda a} \) with multiplicity.

**Proposition 3.12.** \( P(\lambda) = \det A_{x+\lambda a} = \det \left( \sum_k c_{ij}^k (x_k + \lambda a_k) \right) = 0 \) has \( n \) distinct roots for all generic pairs \( (x, a) \in g^* \times g^* \).

**Proof.** Let \( x + \lambda a = (\bar{C}(\lambda), C(\lambda)) \), according to Proposition 3.11, this element is singular, i.e., belongs to \( S \), if and only if \( \det \bar{C}(\lambda) = 0 \). Therefore solving the polynomial equation

\[
P(\lambda) = 0
\]

is equivalent to solving

\[
\det \bar{C}(\lambda) = 0.
\]

Note that \( \det \bar{C}(\lambda) \) is a polynomial in \( \lambda \) of degree \( n \), and therefore has at most \( n \) distinct roots. Let us show that for generic \( (x, a) \), the number of roots is exactly \( n \).

Take \( x \) and \( a \) of the following form (which is easy for computation)

\[
x = \left( \begin{array}{c|c} X & 0 \\ \hline x & 0 \end{array} \right) \quad \text{and} \quad \bar{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \neq x_j \text{ for } i \neq j,
\]

\[
a = \left( \begin{array}{c|c} A & 0 \\ \hline A & 0 \end{array} \right) \quad \text{and} \quad \bar{A} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.
\]

We have:

\[
(x + \lambda a) = \left( \begin{array}{c|c} C & 0 \\ \hline \bar{C} & 0 \end{array} \right) \quad \text{and} \quad \bar{C} = \begin{pmatrix} x_1 + \lambda \\ \vdots \\ x_n + \lambda \end{pmatrix},
\]

then

\[
\det \bar{C} = \prod_{i=1}^n (x_i + \lambda) = 0.
\]
and we have \( n \) distinct roots for \( \lambda_i = -x_i \). Since \((x, a)\) is generic, the characteristic numbers are preserved on all other generic pairs in \( g^* \times g^* \). Therefore there exist a total of \( n \) distinct roots for the generic pairs \((x, a) \in g^* \times g^*\).

**Proposition 3.13.** For the generic pairs \((x, a) \in g^* \times g^*\), the multiplicity of each root of \( P(\lambda) = 0 \) is \( 2n \).

**Proof.** Let \( x = (\bar{C}, C) \in g^* \), the equations

\[
\det \mathcal{A}_x = 0,
\]

and

\[
\det \bar{C} = 0
\]

both define the singular set and therefore has the same set of solutions. We know that \( \det \mathcal{A}_x \) is a polynomial of \( \lambda \) of degree \( 2n^2 \) and \( \det \bar{C} \) is a polynomial of degree \( n \), moreover \( \det \bar{C} \) is irreducible. Then \( \det \mathcal{A}_x = \text{const} \cdot (\det \bar{C}(\lambda))^{2n} \), which implies each root of \( P(\lambda) = \det \mathcal{A}_{x+\lambda a} \) appears with multiplicity \( 2n \). \qed

**Proposition 3.14.** Let \((\bar{C}, C) \in g^* \) be singular, then

\[
\dim \left\{ (\bar{A}, A) \mid \text{ad}^*_{(A, A)}(\bar{C}, C) = 0 \right\} \geq 2(n - 1).
\]

Moreover, for almost all singular elements ("generic singular") \((\bar{C}, C)\) we have the equality:

\[
\dim \left\{ (\bar{A}, A) \mid \text{ad}^*_{(A, A)}(\bar{C}, C) = 0 \right\} = 2(n - 1).
\]

**Proof.** To find \( \dim \left\{ (\bar{A}, A) \mid \text{ad}^*_{(A, A)}(\bar{C}, C) = (\mathcal{C} \mathcal{A}, \bar{A} \mathcal{C} + [A, C]) = 0 \right\} \), we need to solve

\[
\begin{align*}
\bar{C}A &= 0 \\
\bar{A}C + [A, C] &= 0
\end{align*}
\]

Since \((\bar{C}, C)\) is singular and \( \det \bar{C} = 0 \), then rank \( \bar{C} = n - 1 \) for "generic" \( C \) and without loss of generality, we may assume that

\[
C = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
C = \begin{pmatrix}
\hat{c}_1 & \hat{c}_2 \\
\hat{c}_3 & c_4
\end{pmatrix}
\quad \text{(suppose} \hat{c}_2 \neq 0 \text{ for "generic singular" elements)}.
\]

Let

\[
\bar{A} = \begin{pmatrix}
\bar{a}_1 \\
\bar{a}_3
\end{pmatrix}
\quad \text{and} \quad
A = \begin{pmatrix}
A_1 \\
\hat{a}_2 \\
\hat{a}_3 \\
\hat{a}_4 \\
a_1 \cdots a_{n-1} \quad a_n
\end{pmatrix},
\]

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solving the above system of matrix equations, we see that $\bar{C}A = 0$ gives us $A_1 = 0$, $\hat{a}_2 = 0$, and furthermore,

$$\bar{A}\bar{C} + [A, C] = \begin{pmatrix} \bar{A}_1 & 0 \\ \bar{a}_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a_1 \cdots a_{n-1} & a_n \end{pmatrix} \begin{pmatrix} C_1 & \hat{c}_2 \\ \hat{c}_3 & c_4 \end{pmatrix} = 0.$$

Note, for non-zero $\hat{c}_2$, $-a_n\hat{c}_2 = 0$ means $a_n = 0$. So we have $a_n\hat{c}_3 = 0$. Next $(a_1 \cdots a_{n-1})\hat{c}_2 = 0$ means there are $n - 2$ independent parameters in $(a_1 \cdots a_{n-1})$. On the other hand $\bar{A}_1$ and $\bar{a}_3$ are both dependent on $(a_1 \cdots a_{n-1})$.

Summarizing, we have $A_1 = 0$, $\hat{a}_2 = 0$, and $a_n = 0$ and there are totally $n - 2$ independent parameters in $\bar{A}_1$, $\bar{a}_3$ and $(a_1 \cdots a_{n-1})$. Since there is no restriction on $\bar{a}_2$ and $a_4$, we obtain $n$ more independent parameters in the solution. Thus we conclude that

$$\dim \left\{ (\bar{A}, A) \mid \text{ad}^*_{(\bar{A}, A)}(\bar{C}, C) = 0 \right\} = (n - 2) + n = 2(n - 1).$$

For any other singular $(\bar{C}, C) \in \mathfrak{g}^*$ (i.e., rank $\bar{C} \leq (n - 1)$ or $\hat{c}_2 = 0$) we shall obtain

$$\dim \left\{ (\bar{A}, A) \mid \text{ad}^*_{(\bar{A}, A)}(\bar{C}, C) = 0 \right\} \geq 2(n - 1).$$

Now we are ready to describe the J-K invariants of $\mathfrak{g}$.

**Theorem 3.10.** The Lie algebra $\mathfrak{g} = \mathfrak{gl}(n) + \mathbb{C}^{(a^2)}$ is of pure Jordan type with $n$ distinct characteristic numbers $\lambda_1, \ldots, \lambda_n$. The Jordan–Kronecker decomposition of a generic pencil $A_{x+\lambda_0}$ contains $n - 1$ Jordan blocks related to each $\lambda_i$, $i \in 1, \ldots, n$. The corresponding sizes of $n - 1$ Jordan blocks associated with each $\lambda_i$ are $2, \ldots, 2, 4$.

**Proof.** Proposition 3.10 (ind $\mathfrak{g} = 0$) tells us that the J-K invariant of $\mathfrak{g}^*$ is of pure Jordan type. It has $n$ totally distinct characteristic numbers $(\lambda_i, i = 1, \ldots, n)$ for the Jordan blocks (from Proposition 3.12). Since $\dim \mathfrak{g} = 2n^2 > 2n$, there may exist more than one block with respect to each $\lambda_i$. Therefore we need to know the multiplicity.
for each $\lambda_i$ in the J-K invariant. Proposition 3.13 shows that the multiplicity of each characteristic number $\lambda_i$, is $2n$.

Next, we shall find the number of Jordan blocks (and their sizes) for each of $\lambda_i$’s. Let $(\bar{C}(\lambda), C(\lambda)) = (x + \lambda a)$ and for any regular pair $(x, a) \in g^* \times g^*$,

$$
\dim \{ \text{Ker } A_{(x-\lambda_i a)} \} = \dim \left\{ (\bar{A}, A) \left| \text{ad}_{\bar{A}, A}^*(\bar{C}(\lambda_i), C(\lambda_i)) = 0 \right. \right\}
= 2 \cdot (\text{number of Jordan blocks with } \lambda_i).
$$

Notice, here $(\bar{C}(-\lambda_i), C(-\lambda_i)) = (x - \lambda_i a)$ is always “generic singular”, since $(x, a) \in g^* \times g^*$ is required to be regular for J-K invariant in first place. Now, according to Proposition 3.14, we have

$$
\dim \left\{ (\bar{A}, A) \left| \text{ad}_{\bar{A}, A}^*(\bar{C}(\lambda_i), C(\lambda_i)) = 0 \right. \right\} = 2(n - 1)
= 2 \cdot (\text{number of Jordan blocks with } \lambda_i).
$$

It means that the number of Jordan blocks corresponding to each $\lambda_i$ is $n - 1$, which obviously indicate there is one block of size $4 \times 4$ and $n - 2$ blocks of size $2 \times 2$. □

Now, we want to show that the generalized argument shift conjecture holds for this Lie algebra $g$, i.e., we can find a complete family $G_a$ of polynomials in bi-involution. In this case, this family $G_a$ is very simple. The construction is based on the following general fact.

**Proposition 3.15.** Let $g$ be a Lie algebra and $h \subset g$ be a commutative subalgebra i.e., $[h, h] = 0$. Then a basis $e_1, \ldots, e_N$ of $h$, which can be considered as a family of linear functions on $g^*$, is a commutative family w.r.t. all $\{ \cdot , \} _{x + \lambda a}$. Moreover, if $\dim h = \frac{1}{2}(\dim g + \text{ind } g)$, then this family is complete.

**Proof.** Since $e_1, \ldots, e_N$ is a basis of $h \subset g$, and $[h, h] = 0$, we have

$$
[e_i, e_j] = \sum_k c_{ij}^k e_k = 0,
$$

for all $i, j \in 1, \ldots, N$. On the other hand, $e_1, \ldots, e_N$ can be considered as linear functions on $g^*$. Therefore we may define Lie-Poisson brackets (from the pencil $A_{(x + \lambda a)}$) on $h$:

$$
\{ e_i, e_j \}_{x + \lambda a} = \sum_k c_{ij}^k(x_k + \lambda a_k) = 0 \quad \text{for all } \lambda \in \mathbb{C}.
$$

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If \( N = \frac{1}{2}(\dim g + \text{ind } g) \) then \( \{e_1, \cdots, e_N\} \) is a complete family (see Definition 1.14 and Proposition 1.5).

Let
\[
g = gl(n) + \mathbb{C}^{(n^2)} = \begin{pmatrix} gl(n) & \mathbb{C}^{(n^2)} \\ 0 & 0 \end{pmatrix}
\]
and
\[
h = \begin{pmatrix} 0 & \mathbb{C}^{(n^2)} \\ 0 & 0 \end{pmatrix} \subset g,
\]
which is a commutative subalgebra, i.e., \([h, h] = 0\), moreover, \(\dim h = n^2 = \frac{1}{2}(\dim g + \text{ind } g)\). Then applying Proposition 3.15, we immediately obtain the following theorem.

**Theorem 3.11.** A basis of \( h \) i.e.,
\[
E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \cdots,
\]
\[
E_{nn} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},
\]
forms a complete commutative family \( G_a \) w.r.t. all \( \{ , \} \) and \( \{ , \}_{x+\lambda a} \).

In other words, for \( g = gl(n) + \mathbb{C}^{(n^2)} \), the generalized argument shift conjecture holds true.

### 3.3.3 Low-dimensional Lie algebras

In this section we check that the generalized argument shift conjecture holds true for all low-dimensional Lie algebras, and describe their J-K invariants.

**Theorem 3.12.** Let \( g \) be a low-dimensional Lie algebra, i.e., \( \dim g \leq 5 \), and \( a \in g^\ast \) be a fixed regular point. Then there exists a complete family \( G_a \) in bi-involution w.r.t. \( \{ , \} \) and \( \{ , \}_{a} \).
In the following Table 3.1 (on page 133), we give the J-K invariants, singular set \( S \) and complete commutative family \( G_a \) for all real Lie algebras of dimension 3, 4 and 5. A complete list of these algebras is given by Mubarakzyanov (see [25], [26], and [31]). We use the notation from [31].

Here, we give a short explanation (by using a few typical examples) of computing the table results.

- For example, we take \( A_{3,1} \) defined by the following commutation relation,

\[ [e_2, e_3] = e_1. \]

We can easily notice that

\[
A_x = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & x_1 \\
0 & -x_1 & 0
\end{pmatrix}
\]

and

\[
A_a = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & a_1 \\
0 & -a_1 & 0
\end{pmatrix}
\]

are already in Jordan–Kronecker decomposition, which consists of one trivial Kronecker block and one \( 2 \times 2 \) Jordan block with characteristic number \(-\frac{x_1}{a_1}\).

Obviously, the maximal isotropic subspace is \( \text{span}\{e_1, e_2\} \). Therefore \( G_a \) is generated by \( x_1 \) and \( x_2 \).

- For the Lie algebra \( A_{3,8} \), the non-zero commutation relations are

\[ [e_1, e_3] = -2e_2 \], \([e_1, e_2] = e_1\), and \([e_2, e_3] = e_3\).

In this case

\[
A_x = \begin{pmatrix}
0 & x_1 & -2x_2 \\
-x_1 & 0 & x_3 \\
2x_2 & -x_3 & 0
\end{pmatrix}
\]

has rank 2, which means corank = index = 1 (there exists one Kronecker block). It is easily seen that \( \text{rank} A_x < 2 \) happens only if \( x_1 = x_2 = x_3 = 0 \), thus

\[ S = \{x_1 = 0, x_2 = 0, x_3 = 0\} \]
and \( \text{codim } S = 3 > 1 \) (there is no Jordan block, see Theorem 3.5). Therefore we conclude that the J-K invariant is one single Kronecker block of size \( 3 \times 3 \).

According to Theorem 2.2, the number of independent functions in \( G_a \) is

\[
\frac{1}{2} (\dim A_{3,8} + \text{corank } A_x) = \frac{1}{2} (3 + 1) = 2,
\]

and in this case we cannot easily see the maximal isotropic subspace from the structure of \( A_x \). However, we can still obtain the complete commutative family \( G_a \) by the argument shift method (see Proposition 3.1). In this case, since the coadjoint invariant function (or Casimir function to \( A_x \)) is the polynomial (refer to [31])

\[
f(x) = 2x_2^2 + x_1 x_3 + x_3 x_1,
\]

we have

\[
f(x + \lambda a) = 2(x_2 + \lambda a_2)^2 + (x_1 + \lambda a_1)(x_3 + \lambda a_3) + (x_3 + \lambda a_3)(x_1 + \lambda a_1);
\]

\[
f_0(x) = 2(x_2^2 + x_1 x_3);
\]

\[
f_1(x, a) = 2(a_3 x_1 + 2 a_2 x_2 + a_1 x_3);
\]

\[
f_2(a) = 2(a_2^2 + a_1 a_3).
\]

As generators of \( G_a \) we can therefore take \( f_0 \) and \( f_1 \).

- For the Lie algebra \( A_{4,12} \), the non-zero commutation relations are

\[
[e_1, e_3] = e_1 \ , \ [e_2, e_3] = e_2 \ , \ [e_1, e_4] = -e_2 \ , \text{ and } [e_2, e_4] = e_1.
\]

In this case

\[
A_x = \begin{array}{cccc}
0 & 0 & x_1 & -x_2 \\
0 & 0 & x_2 & x_1 \\
-x_1 & -x_2 & 0 & 0 \\
x_2 & -x_1 & 0 & 0
\end{array}
\]

has full rank, therefore \( \text{ind } A_{4,12} = 0 \) (no Kronecker blocks). The rank drops at least by 2 if \( x_1^2 + x_2^2 = 0 \), so the singular set is

\[
S = \{ x_1^2 + x_2^2 = 0 \}.
\]

If we replace \( x \) by \( (x + \lambda a) \), for any \( \lambda \in \mathbb{C} \), then the rank of

\[
A_{x+\lambda a} = \begin{array}{cccc}
0 & 0 & x_1 + a_1 & -x_2 - a_2 \\
0 & 0 & x_2 + a_2 & x_1 + a_1 \\
-x_1 - a_1 & -x_2 - a_2 & 0 & 0 \\
x_2 + a_2 & -x_1 - a_1 & 0 & 0
\end{array}
\]
drops if

\[(x_1 + \lambda a_1)^2 + (x_2 + \lambda a_2)^2 = 0,\]

which gives us two complex characteristic numbers \(\lambda_1 = -\frac{x_1 + ix_2}{a_1 + ia_2}\) and \(\lambda_2 = -\frac{x_1 - ix_2}{a_1 - ia_2}\). Therefore the J-K invariant consists of two \(2 \times 2\) Jordan blocks with distinct characteristic numbers.

Since the maximal isotropic subspace is \(\text{span}\{e_1, e_2\}\), which is easily to note, then for any \(df_1 \in \text{span}\{e_1\}\) and \(df_2 \in \text{span}\{e_2\}\) we have \(A_x(df_1, df_2) = 0\). Therefore \(f_1 = x_1\) and \(f_2 = x_2\) generate the complete family \(G_a\).

- For the Lie algebra \(A_{5,4}\), the non-zero commutation relations are

\([e_2, e_4] = e_1\), and \([e_3, e_5] = e_1\).

Its index is 1 and dimension of the center is 1, therefore there is only one Kronecker block, and it is trivial (also can be observed by inspection).

Again by replacing \(x\) with \((x + \lambda a)\) we have

\[
A_{x+\lambda a} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 + \lambda a_1 & 0 \\
0 & 0 & 0 & 0 & x_1 + \lambda a_1 \\
0 & -x_1 - \lambda a_1 & 0 & 0 & 0 \\
0 & 0 & -x_1 - \lambda a_1 & 0 & 0
\end{pmatrix}.
\]

\(\text{rank } A_{x+\lambda a}\) can only be dropped to 0 once \(x_1 + \lambda a_1 = 0\), so \(\lambda_1 = -\frac{x_1}{a_1}\) and the singular set is

\[S = \{x_1 = 0\}\) and \(\lambda = -\frac{x_1}{a_1}\).

Therefore the only possible type of J-K invariant consists of two \(2 \times 2\) Jordan blocks with the same characteristic number and one trivial Kronecker block.

By observation, \(G_a\) is generated by \(x_1, x_2\) and \(x_3\).

- For the Lie algebra \(A_{5,5}\), the non-zero commutation relations are

\([e_3, e_4] = e_1\), \([e_2, e_5] = e_1\), and \([e_3, e_5] = e_2\),
and

\[
A_{(x+\lambda a)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_1 + \lambda a_1 \\
0 & 0 & 0 & x_1 + \lambda a_1 & x_2 + \lambda a_2 \\
0 & 0 & -(x_1 + \lambda a_1) & 0 & 0 \\
0 & -(x_1 + \lambda a_1) & -(x_2 + \lambda a_2) & 0 & 0
\end{pmatrix}.
\]

By inspection we see that there exists one Kronecker block (index=1) but it is not trivial. \(\text{rank } A_{(x+\lambda a)}\) can be dropped by 2 once we have \(x_1 + \lambda a_1 = 0\), which means that the only characteristic number is \(\lambda = -\frac{x_1}{a_1}\). Therefore the only possible type of J-K invariant consists of one \(2 \times 2\) Jordan block and one \(3 \times 3\) Kronecker block.

By observation, \(\mathcal{G}_a\) is generated by \(x_1, x_2\) and \(x_3\).

• For the Lie algebra \(A_{5,21}\), the non-zero commutation relations are

\[
[e_2, e_3] = e_1, \ [e_1, e_5] = 2e_1, \ [e_2, e_5] = e_2 + e_3, \ [e_3, e_5] = e_3 + e_4, \ \text{and} \ [e_4, e_5] = e_4,
\]

and

\[
A_{(x+\lambda a)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 2(x_1 + \lambda a_1) \\
0 & 0 & x_1 + \lambda a_1 & 0 & (x_2 + x_3) + \lambda(a_2 + a_3) \\
0 & -(x_1 + \lambda a_1) & 0 & 0 & (x_3 + x_4) + \lambda(a_3 + a_4) \\
0 & 0 & 0 & 0 & (x_4 + \lambda a_4) \\
-2(x_1 + \lambda a_1) & -(x_2 + x_3) - \lambda(a_2 + a_3) & -(x_3 + x_4) - \lambda(a_3 + a_4) & -(x_4 + \lambda a_4) & 0
\end{pmatrix}.
\]

By inspection we see there exists one trivial Kronecker block and \(\text{rank } A_{(x+\lambda a)}\) can drop from 4 to 2 once we have \(x_1 + \lambda a_1 = 0\), which means that the only characteristic number is \(\lambda = -\frac{x_1}{a_1}\). It implies that there exists only one Jordan block, which has the size \(4 \times 4\). Therefore the J-K invariant of \(A_{5,5}\) consists of one \(4 \times 4\) Jordan block and one trivial Kronecker block.

By observation, \(\mathcal{G}_a\) is generated by \(x_1, x_2\) and \(x_3\).

• For the Lie algebra \(A_{5,36}\), the non-zero commutation relations are

\[
[e_2, e_3] = e_1, \ [e_1, e_4] = e_1, \ [e_2, e_4] = e_2, \ [e_2, e_5] = -e_2, \ \text{and} \ [e_3, e_5] = e_3,
\]
and

\[
A_x = \begin{bmatrix}
0 & 0 & 0 & e_1 & 0 \\
0 & 0 & e_1 & e_2 & -e_2 \\
0 & -e_1 & 0 & 0 & e_3 \\
-e_1 & -e_2 & 0 & 0 & 0 \\
0 & e_2 & -e_3 & 0 & 0
\end{bmatrix}.
\]

We know the index is 1, which means that there is one Kronecker block. The rank is dropped by 2, when \(x^2 = 0\) and \(x_2x_3 = 0\). Therefore the singular set is

\[
S = \left\{ \begin{array}{l}
x_1 = 0 \\
x_2x_3 = 0
\end{array} \right\}
\]

and \(\text{codim } S = 2 > 1\),

which indicates that there are no Jordan blocks (see Theorem 3.5). Therefore the J-K invariant of \(A_{5,36}\) is of one single \(5 \times 5\) Kronecker block. The number of independent functions in \(G_a\) is 3. Again we compute them by using the argument shift method (See Proposition 3.2). This means that we should consider the following expansion

\[
g(a + \lambda x) = g_0(a) + \lambda g_1(a, x) + \lambda^2 f_2(a, x) + \lambda^3 g_3(x) + \cdots,
\]

where \(g\) is a Casimir function. Then we have

\[
\begin{align*}
A_a dg_1 &= 0 \\
A_a dg_2 &= A_x dg_1 \\
A_a dg_3 &= A_x dg_2 \\
&\vdots
\end{align*}
\]

where \(g_1, g_2, g_3 \in G_a\) and are independent. In this case, the coadjoint invariant function (or Casimir function of \(A_x\)) is not a polynomial, but a rational function (refer to [31])

\[
g(x) = \frac{1}{x_1}(x_2x_3 + x_3x_2 + 2x_1x_5),
\]
By using Taylor expansion we have

\[ g(a + \lambda x) = (a_5 + \lambda x_5) + \frac{1}{a_1}(a_2 + \lambda x_2)(a_3 + \lambda x_3)(1 - \lambda x_1^1 a_1^-1 + \lambda^2 x_1^2 a_1^-2 - \lambda^3 x_1^3 a_1^-3 + \cdots); \]

\[ g_0(a) = a_5 + \frac{1}{a_1}a_2a_3; \]

\[ g_1(a, x) = \frac{1}{a_1^2}(-a_2a_3x_1 + a_1a_3x_2 + a_1a_2x_3 + a_1^2x_5); \]

\[ g_2(a, x) = \frac{1}{a_1^3}(a_1^2x_2x_3 - a_1a_3x_1x_2 - a_1a_2x_1x_3 + a_2a_3x_1^2); \]

\[ g_3(a, x) = \frac{1}{a_1^4}(-a_1^2x_1x_2x_3 + a_1a_3x_1^2x_2 + a_1a_2x_1^2x_3 - a_2a_3x_1^3). \]

Similarly, all other examples can be worked out as shown above. In the table below we use “\( \lambda \)-block of size \( k \times k \)” to denote “the \( k \times k \) Jordan block with characteristic number \( \lambda \)”. Similarly, “\( K \)-block of size \( k \times k \)” means “the Kronecker block with the size \( k \times k \)”. 
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<td>$[e_2, e_3] = e_1$</td>
<td>$\lambda$-block of size $2 \times 2$, and $K$-block of size $1 \times 1$</td>
<td>$\lambda = -\frac{x_1}{a_1}$</td>
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<td>$A_{3,2}$ (ind = 1)</td>
<td>$[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$</td>
<td>$K$-block of size $3 \times 3$</td>
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<td>$x_1 = 0$, $x_2 = 0$, codim $S = 2$</td>
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<tr>
<td>$A_{3,3}$ (ind = 1)</td>
<td>$[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$</td>
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<tr>
<td>$A_{3,4}$ (ind = 1)</td>
<td>$[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$</td>
<td>$K$-block of size $3 \times 3$</td>
<td></td>
<td>$x_1 = 0$, $x_2 = 0$, codim $S = 2$</td>
<td>$x_1, x_2$</td>
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<td>$A_{3,5}$ (ind = 1)</td>
<td>$[e_1, e_3] = e_1$, $[e_2, e_3] = ae_2$ $(0 &lt;</td>
<td>a</td>
<td>&lt; 1)$</td>
<td>$K$-block of size $3 \times 3$</td>
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<td>$A_{3,6}$ (ind = 1)</td>
<td>$[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$</td>
<td>$K$-block of size $3 \times 3$</td>
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<td>$x_1 = 0$, $x_2 = 0$, codim $S = 2$</td>
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<td>$A_{3,7}$ (ind = 1)</td>
<td>$[e_1, e_3] = ae_1 - e_2$, $[e_2, e_3] = e_1 + ae_2$ $(a &gt; 0)$</td>
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<td>$x_1 = 0$, $x_2 = 0$, codim $S = 2$</td>
<td>$x_1, x_2$</td>
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<tr>
<td>$A_{3,8}$ (ind = 1)</td>
<td>$[e_1, e_3] = -2e_2$, $[e_1, e_2] = e_1$, $[e_2, e_3] = e_3$</td>
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<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 3$</td>
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<td>$A_{3,9}$ (ind = 1)</td>
<td>$[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 3$</td>
<td>$x_1^2 + x_2^2 + x_3^2$, $2(a_1 x_1 + a_2 x_2 + a_3 x_3)$</td>
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<tr>
<td>$A_{4,1}$ (ind = 2)</td>
<td>$[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 2$</td>
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<td>$A_{4,2}^a$ (ind = 2)</td>
<td>$[e_1, e_4] = a e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$, $(a \neq 0)$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 3$</td>
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<tr>
<td>$A_{4,3}$ (ind = 2)</td>
<td>$[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 2$</td>
<td>$x_1, x_2, x_3$</td>
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<tr>
<td>$A_{4,4}$ (ind = 2)</td>
<td>$[e_1, e_4] = e_1$, $[e_2, e_4] = e_1 + e_2$, $[e_3, e_4] = e_2 + e_3$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 3$</td>
<td>$x_1, x_2, x_3$</td>
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<tr>
<td>$A_{4,5}^a$ (ind = 2)</td>
<td>$[e_1, e_4] = e_1$, $[e_2, e_4] = a e_2$, $[e_3, e_4] = b e_3$, $(a b \neq 0)$, $(-1 \leq a \leq b \leq 1)$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 3$</td>
<td>$x_1, x_2, x_3$</td>
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<tr>
<td>$A_{4,6}^b$ (ind = 2)</td>
<td>$[e_1, e_4] = a e_1$, $[e_2, e_4] = b e_2 - e_3$, $[e_3, e_4] = e_2 + b e_3$, $(a \neq 0)$, $(b \geq 0)$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$</td>
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<td>$A_{4.7}$ (ind = 0)</td>
<td>$[e_2, e_3] = e_1$, $[e_1, e_4] = 2e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$</td>
<td>$\lambda$-block of size $4 \times 4$</td>
<td>$\lambda = -\frac{a_1}{a_1}$</td>
<td>$x_1 = 0$, codim $S = 1$</td>
<td>$x_1, x_2$</td>
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<tr>
<td>$A_{4.8}$ (ind = 2)</td>
<td>$[e_2, e_3] = e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = -e_3$</td>
<td>$K$-block of size $3 \times 3$, $K$-block of size $1 \times 1$</td>
<td></td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 3$</td>
<td>$2(-a_4 x_1 + a_3 x_2 + a_2 x_3 - a_1 x_4)$</td>
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<tr>
<td>$A_{4.9}$ (ind = 0)</td>
<td>$[e_2, e_3] = e_1$, $[e_1, e_4] = (1 + b)e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = be_3$, $(-1 &lt; b \leq 1)$</td>
<td>$\lambda$-block of size $4 \times 4$</td>
<td>$\lambda = -\frac{a_1}{a_1}$</td>
<td>$x_1 = 0$, codim $S = 1$</td>
<td>$x_1, x_2$</td>
</tr>
<tr>
<td>$A_{4.10}$ (ind = 2)</td>
<td>$[e_2, e_3] = e_1$, $[e_2, e_4] = -e_3$, $[e_3, e_4] = e_2$</td>
<td>$K$-block of size $3 \times 3$, $K$-block of size $1 \times 1$</td>
<td></td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, codim $S = 3$</td>
<td>$2(a_4 x_1 + a_3 x_2 + a_2 x_3 + a_1 x_4)$</td>
</tr>
<tr>
<td>$A_{4.11}$ (ind = 0)</td>
<td>$[e_2, e_3] = e_1$, $[e_1, e_4] = 2ae_1$, $[e_2, e_4] = a e_2 - e_3$, $[e_3, e_4] = e_2 + a e_3$, $(a &gt; 0)$</td>
<td>$\lambda$-block of size $4 \times 4$</td>
<td>$\lambda = -\frac{a_1}{a_1}$</td>
<td>$x_1 = 0$, codim $S = 1$</td>
<td>$x_1, x_2$</td>
</tr>
<tr>
<td>$A_{4.12}$ (ind = 0)</td>
<td>$[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = -e_2$, $[e_2, e_4] = e_1$</td>
<td>$\lambda_1$-block of size $2 \times 2$, $\lambda_2$-block of size $2 \times 2$, $\lambda_1 \neq \lambda_2$</td>
<td>$\lambda_1 = -\frac{a_1 + 122}{a_1 + 102}$, $\lambda_2 = -\frac{a_1 - 122}{a_1 - 102}$</td>
<td>$x_1^2 + x_2^2 = 0$, codim $S = 1$</td>
<td>$x_1, x_2$</td>
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<td>$A_{5,1}$ (ind = 3)</td>
<td>$[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $\text{codim } S = 2$</td>
<td>$x_1, x_2, x_3, x_4$</td>
<td></td>
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<tr>
<td>$A_{5,2}$ (ind = 3)</td>
<td>$[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $\text{codim } S = 3$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,3}$ (ind = 3)</td>
<td>$[e_3, e_4] = e_2$, $[e_3, e_5] = e_1$, $[e_4, e_5] = e_3$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $\text{codim } S = 3$</td>
<td>2($-a_4 x_1 + a_5 x_2 + a_3 x_3 - a_1 x_4 + a_2 x_5$)</td>
<td></td>
</tr>
<tr>
<td>$A_{5,4}$ (ind = 1)</td>
<td>$[e_2, e_4] = e_1$, $[e_3, e_5] = e_1$</td>
<td>$\lambda$-block of size $2 \times 2$, $\lambda$-block of size $2 \times 2$, and $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$\lambda = -\frac{x_4}{a_1}$, $x_1 = 0$, $\text{codim } S = 1$</td>
<td>$x_1, x_2, x_3$</td>
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<tr>
<td>$A_{5,5}$ (ind = 1)</td>
<td>$[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$</td>
<td>$\lambda$-block of size $4 \times 4$, and $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$\lambda = -\frac{x_4}{a_1}$, $x_1 = 0$, $\text{codim } S = 1$</td>
<td>$x_1, x_2, x_3$</td>
<td></td>
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<tr>
<td>$A_{5,6}$ (ind = 1)</td>
<td>$[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$</td>
<td>$\lambda$-block of size $4 \times 4$, and $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$\lambda = -\frac{x_4}{a_1}$, $x_1 = 0$, $\text{codim } S = 1$</td>
<td>$x_1, x_2, x_3$</td>
<td></td>
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<tr>
<td>$A_{5,7}^{\text{abc}}$ (ind = 3)</td>
<td>$[e_1, e_5] = e_1$, $[e_2, e_5] = ae_2$, $[e_3, e_5] = be_3$, $[e_4, e_5] = ce_4$, $(abc \neq 0)$, $(−1 ≤ c ≤ b ≤ a ≤ 1)$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $\text{codim } S = 4$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<td>$A_{5,8}^c$ (ind = 3)</td>
<td>$[e_2, e_5] = e_1$, $[e_3, e_5] = e_3$, $[e_4, e_5] = ce_4$, $(-1 &lt;</td>
<td>c</td>
<td>\leq 1)$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_3 = 0$, $x_4 = 0$, codim $S = 3$</td>
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<tr>
<td>$A_{5,9}^{bc}$ (ind = 3)</td>
<td>$[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = be_3$, $[e_4, e_5] = ce_4$, $(0 \neq c \leq b)$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, codim $S = 4$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,10}$ (ind = 3)</td>
<td>$[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_4$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, codim $S = 3$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,11}^c$ (ind = 3)</td>
<td>$[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = e_2 + e_3$, $[e_4, e_5] = ce_4$, $(c \neq 0)$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, codim $S = 4$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,12}$ (ind = 3)</td>
<td>$[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = e_2 + e_3$, $[e_4, e_5] = e_3 + e_4$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, codim $S = 4$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,13}^{apq}$ (ind = 3)</td>
<td>$[e_1, e_5] = e_1$, $[e_2, e_5] = ae_2$, $[e_3, e_5] = pe_3 - qe_4$, $[e_4, e_5] = qe_3 + pe_4$, $(aq \neq 0)$, $(</td>
<td>a</td>
<td>\leq 1)$</td>
<td>$\mathcal{K}$-block of size $3 \times 3$, $\mathcal{K}$-block of size $1 \times 1$, $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, codim $S = 4$</td>
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<td>$A_{5,14}^p$ (ind = 3)</td>
<td>$[e_2, e_5] = e_1,$ $[e_3, e_5] = pe_3 - e_4,$ $[e_4, e_5] = e_3 + pe_4$</td>
<td>$K$-block of size $3 \times 3$, $K$-block of size $1 \times 1$, $K$-block of size $1 \times 1$</td>
<td>$x_1 = 0,$ $x_3 = 0,$ $x_4 = 0,$ codim $S = 3$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,15}^q$ (ind = 3)</td>
<td>$[e_1, e_5] = e_1,$ $[e_2, e_5] = e_1 + e_2,$ $[e_3, e_5] = ae_3,$ $[e_4, e_5] = e_3 + ae_4,$ $(</td>
<td>a</td>
<td>\leq 1)$</td>
<td>$K$-block of size $3 \times 3$, $K$-block of size $1 \times 1$, $K$-block of size $1 \times 1$</td>
<td>$x_1 = 0,$ $x_2 = 0,$ $x_3 = 0,$ $x_4 = 0,$ codim $S = 4$</td>
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<tr>
<td>$A_{5,16}^{pq}$ (ind = 3)</td>
<td>$[e_1, e_5] = e_1,$ $[e_2, e_5] = e_1 + e_2,$ $[e_3, e_5] = pe_3 - qe_4,$ $[e_4, e_5] = qe_3 + pe_4,$ $(q \neq 0)$</td>
<td>$K$-block of size $3 \times 3$, $K$-block of size $1 \times 1$, $K$-block of size $1 \times 1$</td>
<td>$x_1 = 0,$ $x_2 = 0,$ $x_3 = 0,$ $x_4 = 0,$ codim $S = 4$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,17}^{pq}$ (ind = 3)</td>
<td>$[e_1, e_5] = pe_1 - e_2,$ $[e_2, e_5] = e_1 + pe_2,$ $[e_3, e_5] = qe_3 - se_4,$ $[e_4, e_5] = se_3 + qe_4,$ $(s \neq 0)$</td>
<td>$K$-block of size $3 \times 3$, $K$-block of size $1 \times 1$, $K$-block of size $1 \times 1$</td>
<td>$x_1 = 0,$ $x_2 = 0,$ $x_3 = 0,$ $x_4 = 0,$ codim $S = 4$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,18}^p$ (ind = 3)</td>
<td>$[e_1, e_5] = pe_1 - e_2,$ $[e_2, e_5] = e_1 + pe_2,$ $[e_3, e_5] = e_1 + pe_3 - e_4,$ $[e_4, e_5] = e_2 + e_3 + pe_4,$ $(p \leq 0)$</td>
<td>$K$-block of size $3 \times 3$, $K$-block of size $1 \times 1$, $K$-block of size $1 \times 1$</td>
<td>$x_1 = 0,$ $x_2 = 0,$ $x_3 = 0,$ $x_4 = 0,$ codim $S = 4$</td>
<td>$x_1, x_2, x_3, x_4$</td>
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<tr>
<td>$A_{5,19}^q$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = ae_1,$ $[e_2, e_5] = e_2,$ $[e_3, e_5] = (a - 1)e_3,$ $[e_4, e_5] = be_4,$ $(b \neq 0)$</td>
<td>$\lambda$-block of size $2 \times 2$, and $K$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{x_4}{x_1}$, codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<td>$A_5^{a,20}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = ae_1,$ $[e_2, e_5] = e_2,$ $[e_3, e_5] = (a - 1)e_3,$ $[e_4, e_5] = e_1 + ae_4$</td>
<td>$\lambda$-block of size $2 \times 2,$ and $\mathcal{K}$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{a_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<tr>
<td>$A_5^{a,21}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = 2e_1,$ $[e_2, e_5] = e_2 + e_3,$ $[e_3, e_5] = e_3 + e_4,$ $[e_4, e_5] = e_4$</td>
<td>$\lambda$-block of size $2 \times 2,$ and $\mathcal{K}$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{a_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<td>$A_5^{a,22}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_2, e_5] = e_3,$ $[e_4, e_5] = e_4$</td>
<td>$\lambda_1$-block of size $2 \times 2,$ $\lambda_2$-block of size $2 \times 2,$ $\lambda_1 \neq \lambda_2$ and $\mathcal{K}$-block of size $1 \times 1$</td>
<td>$\lambda_1 = -\frac{a_1}{a_1}$ $\lambda_2 = -\frac{a_1}{a_4}$</td>
<td>$x_1 x_4 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<tr>
<td>$A_5^{a,23}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = 2e_1,$ $[e_2, e_5] = e_2 + e_3,$ $[e_3, e_5] = e_3,$ $[e_4, e_5] = be_4,$ $(b \neq 0)$</td>
<td>$\lambda$-block of size $2 \times 2,$ and $\mathcal{K}$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{a_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<tr>
<td>$A_5^{a,24}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = 2e_1,$ $[e_2, e_5] = e_2 + e_3,$ $[e_3, e_5] = e_3,$ $[e_4, e_5] = e_4 + 2e_4,$ $(\epsilon = \pm 1)$</td>
<td>$\lambda$-block of size $2 \times 2,$ and $\mathcal{K}$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{a_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<td>$A_{5,25}^{by}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = 2pe_1,$ $[e_2, e_5] = pe_2 + e_3,$ $[e_3, e_5] = pe_3 - e_2,$ $[e_4, e_5] = be_4,$ ($b \neq 0$)</td>
<td>$\lambda$-block of size $2 \times 2$, and $K$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{x_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<tr>
<td>$A_{5,26}^{pe}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = 2pe_1,$ $[e_2, e_5] = pe_2 + e_3,$ $[e_3, e_5] = pe_3 - e_2,$ $[e_4, e_5] = e_1 + 2e_4,$ ($\epsilon = \pm 1$)</td>
<td>$\lambda$-block of size $2 \times 2$, and $K$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{x_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<tr>
<td>$A_{5,27}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = e_1,$ $[e_3, e_5] = e_3 + e_4,$ $[e_4, e_5] = e_1 + e_4$</td>
<td>$\lambda$-block of size $2 \times 2$, and $K$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{x_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<td>$A_{5,28}^e$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1,$ $[e_1, e_5] = ae_1,$ $[e_2, e_5] = (a - 1)e_2,$ $[e_3, e_5] = e_3 + e_4,$ $[e_4, e_5] = e_4$</td>
<td>$\lambda$-block of size $2 \times 2$, and $K$-block of size $3 \times 3$</td>
<td>$\lambda = -\frac{x_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_4$</td>
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<td>$A_{5,29}$ (ind = 1)</td>
<td>$[e_2, e_4] = e_1,$ $[e_1, e_5] = e_1,$ $[e_2, e_5] = e_2,$ $[e_4, e_5] = e_3$</td>
<td>$\lambda$-block of size $4 \times 4$, and $K$-block of size $1 \times 1$</td>
<td>$\lambda = -\frac{x_1}{a_1}$</td>
<td>$x_1 = 0,$ codim $S = 1$</td>
<td>$x_1, x_2, x_3$</td>
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<tr>
<td>$A_{5,30}^e$ (ind = 1)</td>
<td>$[e_2, e_4] = e_1,$ $[e_3, e_4] = e_2,$ $[e_1, e_5] = (a + 1)e_1,$ $[e_2, e_5] = ae_2,$ $[e_3, e_5] = (a - 1)e_3,$ $[e_4, e_5] = e_4$</td>
<td>$K$-block of size $5 \times 5$</td>
<td>$x_1 = 0,$ $x_2 = 0,$ codim $S = 2$</td>
<td>$x_1, x_2, x_3$</td>
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| $A_{5,31}$    | $[e_2, e_4] = e_1,$  
               | $[e_3, e_4] = e_2,$  
               | $[e_1, e_5] = 3e_1,$  
               | $[e_2, e_5] = 2e_2,$  
               | $[e_3, e_5] = e_3,$  
               | $[e_4, e_5] = e_3 + e_4$ | $\mathcal{K}$-block of size $5 \times 5$ | $x_1 = 0,$  
               | $x_2 = 0,$  
               | codim $S = 2$ | $x_1, x_2, x_3$ |
| $A_{5,32}$    | $[e_2, e_4] = e_1,$  
               | $[e_3, e_4] = e_2,$  
               | $[e_1, e_5] = e_1,$  
               | $[e_2, e_5] = e_2,$  
               | $[e_3, e_5] = ae_1 + e_3$ | $\mathcal{K}$-block of size $5 \times 5$ | $x_1 = 0,$  
               | $x_2 = 0,$  
               | codim $S = 2$ | $x_1, x_2, x_3$ |
| $A_{5,33}$    | $[e_1, e_4] = e_1,$  
               | $[e_3, e_4] = be_3,$  
               | $[e_2, e_5] = e_2,$  
               | $[e_3, e_5] = ae_3,$  
               | $(a^2 + b^2 \neq 0)$ | $\mathcal{K}$-block of size $5 \times 5$ | $x_1 = 0,$  
               | $x_2 = 0,$  
               | $x_1x_2 = 0,$  
               | $x_1x_3 = 0,$  
               | $x_2x_3 = 0,$  
               | codim $S = 3$ | $x_1, x_2, x_3$ |
| $A_{5,34}$    | $[e_1, e_4] = ae_1,$  
               | $[e_2, e_4] = e_2,$  
               | $[e_3, e_4] = e_3,$  
               | $[e_1, e_5] = e_1,$  
               | $[e_3, e_5] = e_2$ | $\mathcal{K}$-block of size $5 \times 5$ | $x_2 = 0,$  
               | $x_1x_2 = 0,$  
               | $x_1x_3 = 0,$  
               | codim $S = 2$ | $x_1, x_2, x_3$ |
| $A_{5,35}$    | $[e_1, e_4] = be_1,$  
               | $[e_2, e_4] = e_2,$  
               | $[e_3, e_4] = e_3,$  
               | $[e_1, e_5] = ae_1,$  
               | $[e_2, e_5] = -e_3,$  
               | $[e_3, e_5] = e_2,$  
               | $(a^2 + b^2 \neq 0)$ | $\mathcal{K}$-block of size $5 \times 5$ | $x_1x_2 = 0,$  
               | $x_2^2 + x_3^2 = 0,$  
<pre><code>           | codim $S = 2$ | $x_1, x_2, x_3$ |
</code></pre>
<table>
<thead>
<tr>
<th>Name and Index</th>
<th>Relations</th>
<th>Jordan–Kronecker invariant</th>
<th>Char. number</th>
<th>Singular set</th>
<th>Family $G_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{5,36}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1$, $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_2, e_5] = -e_2$, $[e_3, e_5] = e_3$</td>
<td>$\mathcal{K}$-block of size 5×5</td>
<td>$x_1 = 0$, $x_2x_3 = 0$, codim $S = 2$</td>
<td>$\frac{1}{a_1^2}(a_1^2x_5 + a_1a_3x_2 + a_1a_2x_3 - 2a_3x_1)$, $\frac{1}{a_1}(-a_1^2x_1x_3 + a_1a_3^2x_2 + a_1a_2x_1x_3 - 2a_3x_1^3)$</td>
<td></td>
</tr>
<tr>
<td>$A_{5,37}$ (ind = 1)</td>
<td>$[e_2, e_3] = e_1$, $[e_1, e_4] = 2e_1$, $[e_2, e_4] = e_3$, $[e_2, e_5] = -e_3$, $[e_3, e_5] = e_2$</td>
<td>$\mathcal{K}$-block of size 5×5</td>
<td>$x_1 = 0$, $x_2 + x_3 = 0$, codim $S = 2$</td>
<td>$\frac{1}{a_1^2}(-a_2^2 + a_3^2)x_1 + 2a_1a_2x_2 + 2a_1a_3x_3 + 2a_2^2x_3$</td>
<td></td>
</tr>
<tr>
<td>$A_{5,38}$ (ind = 1)</td>
<td>$[e_1, e_4] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_3$</td>
<td>$\lambda_1$-block of size 2×2, $\lambda_2$-block of size 2×2, $\lambda_1 \neq \lambda_2$ and $\mathcal{K}$-block of size 1×1</td>
<td>$x_1x_2 = 0$, codim $S = 1$</td>
<td>$x_1, x_2, x_3$</td>
<td></td>
</tr>
<tr>
<td>$A_{5,39}$ (ind = 1)</td>
<td>$[e_1, e_4] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_3$, $[e_1, e_5] = -e_2$, $[e_2, e_5] = e_1$, $[e_2, e_6] = e_1$, $[e_4, e_5] = e_3$</td>
<td>$\lambda_1$-block of size 2×2, $\lambda_2$-block of size 2×2, $\lambda_1 \neq \lambda_2$ and $\mathcal{K}$-block of size 1×1</td>
<td>$x_1^2 + x_2^2 = 0$, codim $S = 1$</td>
<td>$x_1, x_2, x_3$</td>
<td></td>
</tr>
<tr>
<td>$A_{5,40}$ (ind = 1)</td>
<td>$[e_1, e_2] = 2e_1$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = 2e_3$, $[e_1, e_4] = e_5$, $[e_2, e_4] = e_4$, $[e_2, e_5] = -e_5$, $[e_3, e_5] = e_4$</td>
<td>$\mathcal{K}$-block of size 5×5</td>
<td>$x_4 = 0$, $x_5 = 0$, codim $S = 2$</td>
<td>$2a_4x_1x_4 + a_1^2x_2^2 - a_5x_2x_4 - 2a_5x_5 - a_4x_2x_5$, $2a_4x_1x_4 + a_1^2x_2^2 - a_5x_2x_4 - 2a_5x_5 - a_4x_2x_5$, $a_2^2x_1 - a_4a_5x_2 - a_5^2x_3 + (2a_1a_4 - a_2a_5)x_4 - (a_2a_4 + 2a_3a_5)x_5$</td>
<td></td>
</tr>
</tbody>
</table>
Conclusion

In the present thesis we focus on the algebraic properties of compatible Poisson brackets.

A pair of compatible Poisson structures $\mathcal{A}$ and $\mathcal{B}$ can locally be considered as two skew-symmetric forms on the contangent bundle that smoothly depend on local coordinates and are related by a certain differential relation. Many properties of bi-Hamiltonian systems related to $\mathcal{A}$ and $\mathcal{B}$ are determined just by the pencil of skew-symmetric forms $\mathcal{P}(x) = \{\mathcal{A}(x) + \lambda \mathcal{B}(x)\}$ at a generic point $x \in M$.

By algebraic properties we understand the properties of this pencil and other objects related to $\mathcal{P}(x)$ such as, for example, the automorphism group of $\mathcal{P}(x)$, common isotropic subspaces and so on. These algebraic constructions (i.e., those related to a pair of arbitrary skew-symmetric forms) are presented in Chapter 2.

The contents of Chapter 3 can be considered as an application of some ideas and results developed in the preceding chapter to one very important pencil of compatible Poisson brackets. This pencil is generated by the brackets $\{\ , \}$ and $\{\ , \}_a$ (see Example 1.1) defined on the dual space of a finite-dimensional Lie algebra $\mathfrak{g}$. The first of them is the standard Lie-Poisson bracket on $\mathfrak{g}^*$, the other is constant and can be associated with an arbitrary fixed element $a \in \mathfrak{g}^*$.

The main results of this thesis are as follows:

- By using the Jordan–Kronecker decomposition for a pencil $\mathcal{P}$ of skew-symmetric forms, we describe the automorphism group $G_\mathcal{P}$ and study some of its properties. In particular,
  - we obtain an explicit formula for the dimension of the corresponding Lie
algebra $\mathfrak{g}_P$ in terms of the Jordan-Kronecker decomposition of $\mathcal{P}$ (Theorems 2.6, 2.7 and 2.8);

- we found necessary and sufficient conditions for solvability of $\mathfrak{g}_P$ (Theorem 2.11);

- for non-solvable Lie algebras $\mathfrak{g}_P$, we described the Levi-subalgebras $\mathfrak{h} \subset \mathfrak{g}_P$ (Theorem 2.12, 2.13 and 2.14).

- We introduce *Jordan-Kronecker invariants* of a finite-dimensional Lie algebra and discuss their basic properties (Theorems 3.5, 3.6).

- We explicitly describe Jordan-Kronecker invariants for low-dimensional Lie algebras (Theorem 3.7 and Table 3.1 on page 133) and the Lie algebras $gl(n) + \mathbb{R}^n$ and $gl(n) + \mathbb{R}^{n^2}$ (Theorems 3.8, 3.10).

- We state the so-called *Generalised Argument Shift Conjecture*:

  On the dual space of every finite-dimensional Lie algebra, there exists a complete commutative family $\mathcal{G}_a$ of polynomials in bi-involution w.r.t. $\{ \cdot, \cdot \}$ and $\{ \cdot, \cdot \}_a$.

  This conjecture has been verified for all Lie algebras of dimension $\leq 5$ and some other examples of Lie algebras (Theorems 3.12, 3.9 and 3.11).

In our opinion, the results of the thesis can be considered as an algebraic background of the theory of compatible Poisson brackets in the same sense as Linear Symplectic Geometry serves as a basis of the theory of symplectic manifolds and Hamiltonian mechanics. In particular, the automorphism group $G_P$ plays the same role in bi-Hamiltonian theory as the classical symplectic group $SP(2n, \mathbb{R})$ plays in Hamiltonian mechanics. This analogy allows us to hope that the results of the thesis could be useful in the local theory of compatible Poisson structures.

We also hope that Jordan–Kronecker invariants of finite-dimensional Lie algebras, introduced in the thesis as a formalisation of a number of ideas and results obtained by many authors, can be an additional convenient instrument in the theory of Lie algebras, especially for the questions related to the coadjoint representation.
Bibliography


