Hamiltonian systems of hydrodynamic type in $2 + 1$ dimensions and their dispersive deformations

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Additional Information:

- A Doctoral Thesis. Submitted in partial fulfillment of the requirements for the award of Doctor of Philosophy of Loughborough University.

Metadata Record: [https://dspace.lboro.ac.uk/2134/10183](https://dspace.lboro.ac.uk/2134/10183)

Publisher: © Nikola M. Stoilov

Please cite the published version.
This item was submitted to Loughborough’s Institutional Repository (https://dspace.lboro.ac.uk/) by the author and is made available under the following Creative Commons Licence conditions.

For the full text of this licence, please go to:
http://creativecommons.org/licenses/by-nc-nd/2.5/
HAMILTONIAN SYSTEMS OF HYDRODYNAMIC TYPE IN 2 + 1 DIMENSIONS AND THEIR DISPERSIVE DEFORMATIONS

by

NIKOLA M. STOILOV

Doctoral thesis
submitted in partial fulfilment of the requirements
for the award of

DOCTOR OF PHILOSOPHY

of Loughborough University

©by Nikola M. Stoilov, 2011
Abstract

Hamiltonian systems of hydrodynamic type occur in a wide range of applications including fluid dynamics, the Whitham averaging procedure and the theory of Frobenius manifolds. In $1+1$ dimensions, the requirement of the integrability of such systems by the generalised hodograph transform implies that integrable Hamiltonians depend on a certain number of arbitrary functions of two variables. On the contrary, in $2+1$ dimensions the requirement of the integrability by the method of hydrodynamic reductions, which is a natural analogue of the generalised hodograph transform in higher dimensions, leads to finite-dimensional moduli spaces of integrable Hamiltonians. We classify integrable two-component Hamiltonian systems of hydrodynamic type for all existing classes of differential-geometric Poisson brackets in 2D, establishing a parametrisation of integrable Hamiltonians via elliptic/hypergeometric functions. Our approach is based on the Godunov-type representation of Hamiltonian systems, and utilises a novel construction of Godunov’s systems in terms of generalised hypergeometric functions. Furthermore, we develop a theory of integrable dispersive deformations of these Hamiltonian systems following a scheme similar to that proposed by Dubrovin and his collaborators in $1+1$ dimensions. Our results show that the multi-dimensional situation is far more rigid, and generic Hamiltonians are not deformable. As an illustration we discuss a particular class of two-component Hamiltonian systems, establishing triviality of first order deformations and classifying Hamiltonians possessing nontrivial deformations of the second order.
Acknowledgements

First of all, I would like to express my deepest gratitude to my supervisor Evgeny Ferapontov, who guided me through my three years of research with openness, warmth and great patience.

I would like to thank all the staff at the Department of Mathematical Sciences for all the support, help, advice and fruitful discussions.

It is a great pleasure to thank my friends and colleagues Dragomir Tsoniev, Vladimir Novikov, Benoit Huard, Antonio Moro, Georgi Grahovski, Karima Khusnutdinova and Gennady El for their support, encouragement and advice.

I would like to thank the Department of Mathematical Sciences of Loughborough University for the financial support during my PhD studies.

Finally, I want to thank my girlfriend, Desislava, as well as my parents Olia and Mihail and my brother Stefan for all the love, support and encouragement they gave me.
## Contents

1 Introduction 5

2 Hydrodynamic type systems in 1+1 dimensions. The method of hydrodynamic reductions 12
   2.1 Quasilinear systems in 1+1 dimensions. Examples. Hyperbolicity 13
   2.2 Riemann invariants 15
   2.3 Hamiltonian structures of hydrodynamic type 18
   2.4 Semi-Hamiltonian systems 20
   2.5 Conservation Laws and Commuting Flows 24
   2.6 Generalized Hodograph Method 27
   2.7 The Method of Hydrodynamic Reductions. Example of dKP 29
       2.7.1 Using the method of hydrodynamic reductions as an integrability test. Example of dKP 32

3 Classification of integrable two-component Hamiltonian systems of hydrodynamic type in 2+1 dimensions 34
   3.1 Introduction 35
   3.2 Hamiltonian systems of type I 42
       3.2.1 Classification of integrable potentials 45
   3.3 Hamiltonian systems of type II 46
       3.3.1 Classification of integrable potentials 51
3.3.2 Dispersionless Lax pairs .......................... 54
3.4 Hamiltonian systems of type III ...................... 60
  3.4.1 Classification of integrable potentials ............ 65
  3.4.2 Dispersionless Lax pairs ......................... 73
3.5 Godunov systems and generalized hypergeometric functions .. 75
  3.5.1 Godunov systems .................................... 75
  3.5.2 Godunov form of integrable quasilinear systems and
geneneralized hypergeometric functions .................. 77
  3.5.3 Application to integrable potentials \( H(V, W) \) ....... 81
3.6 Appendix. Point and contact symmetry groups for differential
equations. .................................................. 83
  3.6.1 Local one-parameter point transformation groups .... 83
  3.6.2 Prolongation formulas ............................. 85
  3.6.3 Groups admitted by differential equations ........ 86
  3.6.4 Contact transformations .......................... 90

4 Dispersive deformations of Hamiltonian systems in 2+1 dimensions

  4.1 Introduction ......................................... 92
  4.2 Dispersive deformations in 2 + 1 dimensions .......... 97
  4.3 Triviality of \( \epsilon \)-deformations .................. 98
  4.4 Reconstruction of \( \epsilon^2 \)-deformations ............ 101
  4.5 Example 1: dispersive deformation of the Boyer-Finley equation 104
  4.6 Example 2: deformation of the dKP equation .......... 106
  4.7 Deformable Hamiltonians of type I ................. 107
  4.8 Deformable Hamiltonians of type III ............... 109

5 Concluding remarks and outline .......................... 110
Chapter 1

Introduction

The classification of integrable systems has been an area of intensive research since the very beginning of soliton theory. Many differential equations coming from physics and mathematics are in fact Hamiltonian systems of hydrodynamic type on an infinite dimensional phase space of vector functions. The theory of such systems was first developed in 1 + 1 dimensions for a vector variable $u(x,t)$, $u = (u^1, u^2, \ldots, u^n)$. The understanding of integrability in this context is based on symmetry approach, i.e. on the existence of infinitely many Hamiltonian flows, also called symmetries, commuting with the original one. S.P. Novikov made a conjecture that a Hamiltonian system in 1 + 1 dimensions, $u_t = \{H,u\}$, is integrable if it can be brought to a diagonal form. S.P. Tsarev [61] later proved this conjecture and also found out that the Hamiltonian condition could be relaxed. He introduced the semi-Hamiltonian property, and showed that it is necessary and sufficient condition for integrability. Coordinate-invariant criterion to establish diagonalisability using the Haantjes tensor was first applied in the paper [32]. Finally, a tensor criterion was found to test the semi-Hamiltonian property [55]. Detailed discussion of the theory is presented in Chapter 2.

Since the 1+ 1 dimensional situation was completely understood, the focus was moved to higher dimensional problems. Although the symmetry
approach generalises to 2+1 dimensions, difficulties due to non-local variables [43] make this route non-appealing. The method of hydrodynamic reductions was first proposed as a test for integrability in the paper [23]. In a nutshell, given a 2+1 dimensional system of hydrodynamic type,

\[ u_t = Ax + Bu_y \]

one looks for a solution \( u = u(R^1, R^2, \ldots, R^N) \), where \( R^i \) is a function of \( x, y \) and \( t \) and \( R_t^i = \lambda^i R_x^i, R_y^i = \mu^i R_x^i \) (no summation!), with the condition that these flows commute, \( R_{ty}^i = R_{yt}^i \). The existence of infinitely many such reductions for any \( N \) imposes very strong conditions on the matrices \( A \) and \( B \) and can be used to establish integrability for a given system. Details as well as an example of the application of this method as an integrability test are given at the end of Chapter 2.

Using the method of hydrodynamic reductions as an integrability test, we have successfully classified all possible two component, integrable Hamiltonian systems in 2+1 dimensions. The results are published in the article [31] and are presented in Chapter 3 of the present work. We note here that in the 1+1 dimensional case, if we fix signature, up to a coordinate change there exists only one non-degenerate Poisson bracket. In contrast, in the two component 2+1 dimensional case things are different - there are three essentially different types of Poisson brackets and up to a coordinate change any non-degenerate Poisson bracket could be brought to one of these three. The general theory and classification of these Poisson brackets is briefly described in Section 3.2. The first two could be brought to a constant coefficient form

\[ P = \begin{pmatrix} d/dx & 0 \\ 0 & d/dy \end{pmatrix}, \quad P = \begin{pmatrix} 0 & d/dx \\ d/dx & d/dy \end{pmatrix}, \]
whereas the third one is essentially non constant:

\[
P = \begin{pmatrix} 2v & w \\ w & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & v \\ v & 2w \end{pmatrix} \frac{d}{dy} + \begin{pmatrix} v_x & v_y \\ w_x & w_y \end{pmatrix}.
\]

We call them type I, type II and type III respectively. Here \(v, w\) are the dependent variables. Hamiltonian of hydrodynamic type is a functional \(F = \int h(v, w) dx dy\), for which the density \(h(v, w)\) depends on \(v\) and \(w\) only. Then the equations of motion for Poisson brackets of type I in explicit form are

\[
v_t = (h_v)_x, \quad w_t = (h_w)_y,
\]

for type II we have

\[
v_t = (h_w)_x, \quad w_t = (h_v)_x + (h_w)_y,
\]

and for type III

\[
v_t = (2vh_v + wh_w - h)_x + (vh_w)_y, \quad w_t = (wh_v)_x + (2wh_w + vh_v - h)_y.
\]

It turns out that the analysis of these systems is considerably simplified if we use the Legendre transform of the equations. We call the Legendre transformed variables \(V\) and \(W\) and the Legendre transformed Hamiltonian \(H\). The transformations itself is \(v \rightarrow H_V, \quad w \rightarrow H_W\) and \(H \rightarrow vh_v + wh_w - h\). In the rest of this work will call both \(h\) and \(H\) potentials, but \(h\) will always refer to a Hamiltonian density, whereas \(H\) - to its Legendre transform.

We were able to derive integrability conditions and classify all possible integrable Hamiltonians for each type, using the method of hydrodynamic reductions. In all cases the integrability conditions constitute an overdetermined system of fourth order differential equations for the Hamiltonian density \(h(v, w)\). For Poisson bracket of type I the integrability conditions were derived in [23]. The classification of this system’s solutions, hence the
classification of integrable potentials, was accomplished in [26]. The results of these papers are briefly presented in Sect. 3.3. Type II and type III brackets were studied in author’s joint work with E.V. Ferapontov and A.V. Odeskii [31]. Both cases are fully understood and the results are presented in Chapter 3. For type II Poisson bracket the following result is proved.

**Theorem 10** The generic Legendre transformed integrable potential of type II is given by the formula

\[ H = V \ln \frac{V}{\sigma(W)} \]

where \( \sigma \) is the Weierstrass sigma-function: \( \sigma'/\sigma = \zeta, \quad \zeta' = -\wp, \quad \wp^2 = 4\wp^3 - g_3 \). Its degenerations correspond to

\[ H = V \ln \frac{V}{W}, \quad H = V \ln V, \quad H = \frac{V^2}{2W} + \alpha W^7, \]

as well as the following polynomial potentials:

\[ H = \frac{V^2}{2} + \frac{VW^2}{2} + \frac{W^4}{4}, \quad H = \frac{V^2}{2} + \frac{W^3}{6}. \]

Details are given in Section 3.4. We were also able to find Lax pairs for all these potentials to make a connection to integrability in classical sense.

Type III systems proved to be a considerably more difficult nut to crack. In order to understand them we studied the associated point and contact symmetry groups. Furthermore we used Godunov systems and generalised hypergeometric functions in order to establish the following result.

**Theorem 11** The generic integrable potential of type III is given by the series

\[ H = \frac{1}{Wg(V)} \left( 1 + \frac{1}{g_1(V)W^6} + \frac{1}{g_2(V)W^{12}} + \ldots \right) . \]
Here \(g(V)\) satisfies the fourth order ODE,

\[
g^{iv}(2gg'^2 - g^2g'') + 2g^2g'''^2 - 20gg'g''' + 16g^2g''' + 18gg'^3 - 18g'^2g'' = 0,
\]

whose general solution can be represented in parametric form as

\[
g = w_2(t), \quad V = \frac{w_1(t)}{w_2(t)},
\]

where \(w_1(t)\) and \(w_2(t)\) are two linearly independent solutions of the hypergeometric equation \(t(1-t)d^2w/dt^2 - \frac{2}{5}w = 0\). The coefficients \(g_i(V)\) are certain explicit expressions in terms of \(g(V)\), e.g., \(g_1(V) = gg'' - 2g'^2\) and so on. Degenerations of this solution correspond to

\[
H = \frac{1}{Wg(V)}, \quad H = \frac{1}{WV}, \quad H = V - W \log W, \quad H = V - W^2/2.
\]

We were also able to find a representation of the generic solution based solely on generalised hypergeometric functions. Details are presented in Section 3.5. This part of our joint work is essentially due to A. V. Odeskii.

After the classification of hydrodynamic integrable potentials was completed, it seemed natural to look for integrable dispersive deformations of these systems. This is inspired by a similar program in 1+1 dimensions, developed by B.A. Dubrovin and his collaborators. The basic idea in 1+1 dimensions is to look for deformation of the symmetries of a given integrable system, that preserve the commutativity of the respective flows [22]. This approach is presented in the introduction to Chapter 4. Since the symmetries in the case of 2+1 systems are non-local, and very difficult to find, the understanding of integrability in 2+1 dimensions is based on hydrodynamic reductions and we are looking for dispersive deformations of the original Hamiltonian system, that inherit all of its hydrodynamic reductions. This method is outlined in Section 4.2. We studied integrable dispersive defor-
mations for all three types of Poisson brackets. The results of this work are published in a joint paper of the author with E.V. Ferapontov and V.S. Novikov [29]. In this paper we present solid evidence to the conjecture that the generic Hamiltonian has no integrable dispersive deformations for any of the three types of brackets. There are, however, deformable examples.

First of all we show that deformations at level $\epsilon$ are trivial. It turns out that the case of Type II Poisson bracket is the simplest one, and at the same time involves all essential phenomena, so we concentrate our discussion on it.

**Theorem 14** A Hamiltonian $H_0 = \int h(v, w) \, dx \, dy$ of type II possesses a nontrivial integrable deformation to the order $\epsilon^2$ if and only if, along with the integrability conditions, it satisfies the additional differential constraints

\[
h_{vvv} h_{vwv} - h_{vvw}^2 = 0, \quad h_{vvv} h_{wwv} - h_{vvw} h_{vww} = 0, \quad h_{wwv} h_{vww} - h_{vwv}^2 = 0,
\]

that is,

\[
\text{rank} \begin{pmatrix} h_{vvv} & h_{vwv} & h_{vww} \\ h_{vvv} & h_{vwv} & h_{vww} \end{pmatrix} = 1. \tag{1.1}
\]

Modulo equivalence transformations, this gives two types of deformable densities:

\[
h(v, w) = \frac{w^2}{2} + e^v, \quad h(v, w) = \frac{v^2}{2} + \beta vw + f(w),
\]

where $f(w)$ satisfies the integrability condition $f''' f'' (\alpha f'' - \beta^2) = f'''^2 (3\alpha f'' - 2\beta^2)$.

In the case of Type I Poisson bracket the following conjecture is proposed

**Conjecture 2** A Hamiltonian $H_0 = \int h(v, w) \, dx \, dy$ of type I possesses a nontrivial integrable deformation to the order $\epsilon^2$ if and only if, along with the integrability conditions, presented in Chapter 3, it satisfies the additional
differential constraints

\[ h_{vvv}h_{vw} - h_{vww}^2 = 0, \quad h_{vvv}h_{wvw} - h_{vwv}h_{vw} = 0, \quad h_{wvw}h_{vww} - h_{vww}^2 = 0, \]

or, equivalently,

\[ \text{rank} \left( \begin{array}{ccc} h_{vvv} & h_{vw} & h_{vw} \\ h_{vvv} & h_{vw} & h_{vw} \\ h_{vvv} & h_{vw} & h_{vw} \end{array} \right) = 1. \]

Modulo equivalence transformations, this gives three types of deformable densities:

\[ h(v, w) = vw + \alpha v^3, \quad h(v, w) = \frac{1}{2}(v + w)^2 + \alpha^v, \]
\[ h(v, w) = \frac{\alpha}{2}v^2 + \beta vw + \frac{\gamma}{2}w^2 + f(v + w), \]

where \( f \) satisfies the integrability condition

\[ (\beta + f'')Qf''' = f''^2[3Q + (\beta - \alpha)(\beta - \gamma)], \]

here \( Q = (\beta + f'')^2 - (\alpha + f'')(\gamma + f''). \)

The respective deformations are shown in Section 4.7.

Direct computations support the conjecture that in the case of Type III Poisson bracket there are no deformable potentials.
Chapter 2

Hydrodynamic type systems in $1+1$ dimensions. The method of hydrodynamic reductions

In this Chapter we include the basic necessary information on quasilinear systems in $1+1$ dimensions. Our main goal is to develop understanding and fully illustrate the following conjecture, first suggested by S. P. Novikov and later proved by S. P. Tsarev [61]:

Conjecture 1 A quasilinear system in $1+1$ dimensions is integrable if it is both diagonalisable and Hamiltonian.

Furthermore, we will present algorithmic and invariant criteria in order to establish if a given system is indeed diagonalisable and semi-Hamiltonian, and we will study the generalised hodograph method [61] which allows us to find solutions to such systems. Finally we will look at the extension of these ideas to higher dimensions, and at the method of hydrodynamic reductions.
2.1 Quasilinear systems in 1+1 dimensions. Examples. Hyperbolicity

Systems of the following type:

\[ u_t^i = \sum_{j=1}^{n} v^j_i(u)u_x^j, \]  

(2.1)

are called quasi-linear systems in 1 + 1 dimensions, or systems of hydrodynamic type. Here \( u = (u^1(t,x), u^2(t,x), \ldots, u^n(t,x)) \) is a \( n \)-component vector of dependent variables. The functions \( v^j_i(u) \), which could also be considered as matrix elements of \( n \times n \) matrix \( V \), are assumed sufficiently smooth and, in general, non-constant. Systems of this type arise in many backgrounds including differential geometry, general relativity, magneto-fluid dynamics, etc.

For example, the equations of motion for ideal barotropic gas,

\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
u_t + uu_x + p_x/\rho &= 0,
\end{align*}

(2.2)

where \( p = p(\rho) \) is the equation of state, can be re-written in the form (2.1) in the following way

\[ \left( \begin{array}{c} \rho \\ u \end{array} \right)_t + \left( \begin{array}{cc} u & \rho \\ p/\rho & u \end{array} \right) \left( \begin{array}{c} \rho \\ u \end{array} \right)_x = 0, \]

so we have

\[ u = \left( \begin{array}{c} \rho \\ u \end{array} \right), \quad V = -\left( \begin{array}{cc} u & \rho \\ p/\rho & u \end{array} \right). \]

Other well-known examples are the Benney’s equations [62], describing a
multi-layered system of fluids, with \( \eta_i \) being the height and \( u_i \) the velocity of each layer,

\[
\begin{align*}
\eta_i' + (u_i \eta_i)_x &= 0, \\
u_i' + u_i u_i^i + f \left( \sum_{i=1}^{n} \eta_i^i \right)_x &= 0,
\end{align*}
\]

and the equations of ideal chromatography,

\[
\begin{align*}
cu_i^i + (a^i(u) + u^i)_t &= 0, \quad i = 1, \ldots, n.
\end{align*}
\]

The last system describes the passage of an \( n \)-component mixture through an absorbing medium [58]. Here \( c \) is a constant, and \( u^i \) and \( a^i \) are the concentrations of the non-absorbed and absorbed \( i \)th component, respectively.

In order to bring this system to the familiar form (2.1) we introduce the variable \( \tau = ct - x \) and obtain

\[
u_i^i + a^i(u)_\tau = 0,
\]

which may be rewritten in the hydrodynamic type form \( u_i^i_x - \partial_j a^i(u) u_i^j = u_i^i_x - v_j^i(u) u_i^j = 0 \). To define this system completely, one needs to specify an isotherm, an explicit form of dependence \( a^i = a^i(u) \). For instance, in the simplest case of the classical Langmuir isotherm one has:

\[
a^i = k_i u^i / W, \quad W := 1 + \sum_{s=1}^{n} k_s u^s.
\]

Here we need to give the following definition, which will play an important role in our discussion.

**Definition 1** The system (2.1) is called strictly hyperbolic if and only if all eigenvalues of \( V \) are real and distinct.

All systems that we will consider in this work are assumed to be strictly
hyperbolic. We note that all examples mentioned above are hyperbolic in an appropriate region of the hodograph space.

2.2 Riemann invariants

Let us perform a locally invertible change of coordinates for the system (2.1) such that $u^i = u^i(p^1, p^2, \ldots, p^n)$. Using the chain rule we find

$$\frac{\partial u^i}{\partial p^j} p^j_t = v^i_k(u(p)) \frac{\partial u^k}{\partial p^l} p^l_x,$$

and equivalently

$$p^j_t = \frac{\partial p^j}{\partial u^i} v^i_k(u(p)) \frac{\partial u^k}{\partial p^l} p^l_x,$$

which means that $V$ transforms as a tensor of type $(1, 1)$.

If there exists a change of variables $u^i = u^i(R^1, R^2, \ldots, R^n)$ such that the matrix $V$ becomes diagonal in the new coordinates $R^i$ we say that this system is diagonalisable. The $R^i$ coordinates are called Riemann invariants, and in this new set of variables the system has the form

$$R^i_t = v_i(R) R^i_x,$$

(2.3)

(no summation!). Provided that Riemann invariants exist, there is a general algorithm to find them. Let $v_1, \ldots, v_n$ be the eigenvalues of the matrix $V$, which, due to the assumed hyperbolicity, are pairwise distinct and satisfy the characteristic equation $\det(v^j_i - v_i \delta^j_i) = 0$. Let us denote the corresponding left eigenvectors by $\xi^p_j$

$$\xi^p_i v^i_j = v^p \xi^p_j, \quad p = 1, \ldots, n.$$

Suppose that for each eigenvector $\xi^p_j$ there exists an integrating factor $e^p$ a
function $R^p$ such that
\[ \xi^p_j = \partial R^p / \partial u^j. \] (2.4)

Then $\xi^p_j$ appear to be the components of a gradient, and the functions $R^1, \ldots, R^n$ are the desired Riemann invariants, because
\[ R^p_t = \frac{\partial R^p}{\partial u^j} \frac{\partial u^j}{\partial t} = \xi^p_j \xi^p_k \frac{\partial u^k}{\partial x} = v^p \xi^p_k \frac{\partial u^k}{\partial x} = v^p \frac{\partial R^p}{\partial u^k} \frac{\partial u^k}{\partial x} = v^p R^p. \]

Let us look how this procedure works for the system of equations of gas dynamics (2.2) in the case of polytropic equation of state $p(\rho) = \gamma \rho^{\gamma - 1}$:
\[ \rho_t + u_x \rho + u \rho_x = 0, \quad u_t + uu_x + \gamma \rho^{\gamma - 2} \rho_x = 0. \] (2.5)

First we have to solve the characteristic equation, $\gamma \rho^{\gamma - 1} - (u - v)^2 = 0$, which has solutions,
\[ v_1, v_2 = u \pm (\gamma \rho^{\gamma - 1})^{1/2}. \]

Next, in order to find the Riemann invariants we require,
\[ \left( \begin{array}{cc} R^i_\rho & R^i_u \end{array} \right) \left( \begin{array}{cc} u - v_i & \rho \\ \gamma \rho^{\gamma - 2} & u - v_i \end{array} \right) = 0, \]
where $i = 1, 2$. This allows us to find $R^1$ and $R^2$ in terms of $u$ and $\rho$,
\[ R^1 = u + \frac{2\gamma^{1/2} \rho^{\gamma - 1}}{\gamma - 1}, \quad R^2 = u - \frac{2\gamma^{1/2} \rho^{\gamma - 1}}{\gamma - 1}. \]

Finally we re-express the eigenvalues $v_1, v_2$ as functions of $R_1$ and $R_2$,
\[ v_1 = \frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^1 - R^2)}{4}, \quad v_2 = \frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^2 - R^1)}{4}. \]
Thus we can re-write the system in diagonal form

\[ R_t^1 + \left( \frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^1 - R^2)}{4} \right) R_x^1 = 0, \]

\[ R_t^2 + \left( \frac{R^1 + R^2}{2} + \frac{(\gamma - 1)(R^2 - R^1)}{4} \right) R_x^2 = 0. \]

(2.6)

A direct check ensures that the change of variables

\[ \frac{R^1 + R^2}{2} = u, \quad R^1 - R^2 = \frac{4\gamma^2 \rho^{\frac{\gamma - 1}{2}}}{\gamma - 1}, \]

brings (2.5) to (2.6). We note here that we explicitly needed the hydrodynamic system (2.5) to be strictly hyperbolic.

The method we have just discussed has two obvious weak points. First of all, for \( n \geq 5 \) it is not always possible to solve the characteristic equation in explicit form. Second, for \( n \geq 3 \) the integrating factor \( c^p \) in (2.4) may not exist. However, Novikov’s **Conjecture 1** does not require the explicit diagonal form of the system in question, it just requires the existence of coordinates in which the system (2.1) is diagonal. There exists an invariant differential-geometric criterion for diagonalisability due to Haantjes [38]. One needs to construct the Nijenhuis tensor from a strictly hyperbolic matrix \( V = v^i_j \),

\[ N^{ij}_{jk} := \sum_{p=1}^{n} \sum_{q=1}^{n} \left( v^j_s \partial_s v^i_k - v^k_s \partial_s v^i_j - v^i_s (\partial_s v^k_j - \partial_k v^i_j) \right), \]

and then use it to find the Haantjes tensor,

\[ H^{ij}_{jk} := \sum_{p=1}^{n} \sum_{q=1}^{n} \left( N^{i}_{qp} v^q_k - N^{q}_{kp} v^q_i \right) v^p_j - v^i_p (N^{i}_{qp} v^q_k - N^{q}_{kp} v^q_i). \]

The following theorem then holds

**Theorem 1** [38] A strictly hyperbolic matrix \( v^i_j \) is diagonalisable if and only
if the corresponding Haantjes tensor \( H_{jk}^i \) is identically zero.

This test was first applied in the field of integrable systems to classify isotherms of absorption for which the equations of chromatography possess Riemann invariants [32].

### 2.3 Hamiltonian structures of hydrodynamic type

Let us now look at the Hamiltonian properties of a hydrodynamic type system. The Hamiltonian structure is defined by a Poisson bracket which is said to be of hydrodynamic type when it has the following form:

\[
\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} A^{ij} \frac{\delta J}{\delta u^j(x)} \, dx. \tag{2.7}
\]

Here \( A^{ij} \) is a Hamiltonian differential operator

\[
A^{ij} = g^{ij}(u(x, t)) \frac{d}{dx} + b_k^{ij}(u(x, t)) u^k. \tag{2.8}
\]

The following important theorem, due to Dubrovin and Novikov holds:

**Theorem 2** [21]

1. Under local changes of coordinates, \( g^{ij} \) transforms like a \((2,0)\) - tensor. Furthermore if \( \det g^{ij} \neq 0 \) then \( b_k^{ij} \) transforms like the expression \( -g^{is} \Gamma^i_{sk} \), where \( \Gamma^i_{sk} \) is an affine connection.

2. The skew-symmetry of (2.7) imposes that \( g^{ij} \) must be symmetric and thus it can be considered as a pseudo-Riemannian metric on the space of field variables \( u \). Moreover, the connection \( \Gamma^i_{sk} \) must be compatible with this metric, \( \nabla_k g_{ij} = 0 \).

3. In order that the bracket (2.7) satisfies the Jacoby identity,

\[
\{\{I, J\}, K\} + \{\{J, K\}, I\} + \{\{K, I\}, J\} = 0,
\]
It is necessary and sufficient that the connection $\Gamma^i_{sk}$ is torsionless and flat i.e. $\Gamma^i_{sk} = \Gamma^i_{ks}$ and the Riemann curvature tensor $R^i_{jkl} = 0$.

Hamiltonian of hydrodynamic type is a functional $H = \int h(u)dx$ such that its density $h(u)$ depends only on the field variables $u(x,t)$, but is independent of their spatial derivatives $u_x, u_{xx}, \ldots$. Hamiltonian system of hydrodynamic type $u^i_t = \{u^i, H\}$ is, explicitly,

$$u^i_t = \left( g^{jk} \frac{\partial^2 h}{\partial u^k \partial u^j} + b^j_k \frac{\partial h}{\partial u^k} \right) u^j_x,$$

and, as a direct result of Theorem 2,

$$v^i_j = g^{ik} \frac{\partial^2 h}{\partial u^k \partial u^j} + b^j_k \frac{\partial h}{\partial u^k} = g^{is} \nabla_s \nabla_j h = \nabla_i \nabla_j h.$$ 

We will call such matrices \(V\) Hamiltonian.

**Lemma 3** [61] The matrix \(V\) with elements \(v^i_j(u)\) is a matrix of a Hamiltonian system of hydrodynamic type if and only if there exists a non-degenerate flat metric $g_{ij}$ such that $g_{ik}v^k_j = g_{jk}v^k_i$ and also $\nabla_i v^k_j = \nabla_j v^k_i$.

**Proof** Let $v^i_j = \nabla^i \nabla_j h$, for a connection with zero curvature. Since the metric is flat it follows that $g_{ik}v^k_j = \nabla_i \nabla_j h = \nabla_j \nabla_i h = g_{jk}v^k_i$. The same argument can be applied for $\nabla_i v^k_j = \nabla_j v^k_i = \nabla_i \nabla^k \nabla_j h = \nabla_j \nabla^k \nabla_i h$. \(\blacksquare\)

If the conditions of the Lemma are fulfilled for a zero curvature metric $g^{ij}$, we can take a set of coordinates in which $g^{ij}$ is constant-coefficient. In these coordinates the expression $M_{ij} = g_{ik}v^k_j$ is symmetric with respect to $i$ and $j$ and also $\partial_i M_{jk} = \partial_j M_{ik}$, where $\partial_i = \frac{\partial}{\partial u^i}$. Then there exists a vector $N_k$ such that $M_{ik} = \partial_i N_k$, and also, since $M_{ij} = M_{jk}$, $N_k = \partial_k h$ for some functional $h$. It follows that, in flat coordinates,

$$v^i_j = g^{ik} \frac{\partial^2 h}{\partial u^k \partial u^j}.$$
It turns out that the Hamiltonian property is not necessary for integrability. We will now discuss a way to weaken the condition in an appropriate way as found by S.P. Tsarev in [61].

2.4 Semi-Hamiltonian systems

Let us now look at diagonal Hamiltonian systems, which, as mentioned in the beginning of this Chapter, play central role in the theory:

\[ u_t^i = v_i(u)u_x^i. \]

Here we have introduced the notation \( v_j^i = v_i \delta^i_j \). We note that in the rest of this Section we do not sum over repeated indices, unless explicitly noted. We remind that \( v_i(u) \) are pairwise distinct due to hyperbolicity. Let us apply Lemma 3 to diagonal matrices \( v_j^i = v_i \delta^i_j \),

\[ 0 = \nabla_i v^k_i - \nabla_j v^k_j = \partial_i v_j \delta^k_i - \partial_j v_i \delta^k_j + \Gamma^k_{ij}(v_j - v_i). \]

This means that \( \Gamma^k_{ij} = 0 \) for \( i \neq j \neq k \) and

\[ \partial_i v_k = \Gamma^k_{ki}(v_i - v_k), \quad i \neq k. \quad (2.9) \]

On the other hand, for diagonal matrices \( v_j^i = v_i \delta^i_j \) the condition \( \sum_k g_{ik} v_j^k = \sum_k g_{jk} v_i^k \) directly implies that the associated metric \( g_{ik} \) is diagonal since,

\[ \sum_{k=1}^n g_{ik} v_j^k - g_{jk} v_i^k = g_{ij}(v_j - v_i) = 0, \]

and \( v_i \neq v_j \) for \( i \neq j \) due to hyperbolicity. From \( \Gamma^k_{ij} = \sum_s \frac{1}{2} g^{ks}(\partial_j g_{si} + \partial_i g_{sj} - \partial_s g_{ij}) \) we find

\[ \Gamma^k_{ki} = \frac{1}{2} \partial_i \ln g_{kk}. \quad (2.10) \]
From the last equation it directly follows that the consistency condition
\[ \partial_j \partial_i \ln g_{kk} - \partial_i \partial_j \ln g_{kk} = \partial_j \Gamma^k_{ki} - \partial_i \Gamma^k_{kj} = 0 \]
is equivalent to
\[ \partial_j \left( \frac{\partial_i v_k}{v_i - v_k} \right) = \partial_i \left( \frac{\partial_j v_k}{v_j - v_k} \right), \quad i \neq j \neq k. \quad (2.11) \]

In differential geometry the existence of a flat diagonal metric is equivalent
to the existence of an orthogonal curvilinear coordinate system in (pseudo-)
Euclidean space. If we are given an orthogonal curvilinear system of coordinates it is natural to ask how to find the relevant Hamiltonian matrices \( v^i \).
Evidently, from Lemma 3 and the arguments in this section, the sole requirement for a diagonal matrix to be Hamiltonian and respectively possess an associated flat metric is (2.9).

On the other hand, we can think of (2.9) as a linear system of equations
for \( v_i(u) \), which is compatible if and only if
\[ \partial_i \Gamma^k_{kj} - \Gamma^k_{kj} \Gamma^j_{ij} - \Gamma^k_{ki} \Gamma^i_{ij} + \Gamma^k_{ki} \Gamma^k_{kj} = R^k_{jik} = 0, \quad (2.12) \]
from which follows
\[ \partial_j \Gamma^k_{ki} - \partial_i \Gamma^k_{kj} = R^k_{kij} = 0, \]
where \( R^k_{kij} \) are components of the Riemann curvature tensor. These conditions are trivially satisfied for a flat metric \( g_{ij} \).

The solutions of systems of type (2.9) are discussed in classical differential
gometry [11]. One can prove that the solution of the system (2.9) depends
on \( n \) functions of a single variable. This implies that for each orthogonal
curvilinear coordinate system there exist a family of Hamiltonian matrices,
locally parametrised by \( n \) functions of a single variable.

The following theorem [61] summarises the results discussed in this sec-
tion.

**Theorem 4** The metric, associated with the Hamiltonian matrix \( V, v^i_j(u) = v_j \delta^i_j \), is diagonal and the variables \( u \) constitute curvilinear orthogonal system
of coordinates. On the other hand for each curvilinear orthogonal system of coordinates there exists a family of Hamiltonian matrices, which are diagonal in this system of coordinates. This family is locally parametrized by \( n \) functions of a single variable. For all matrices \( V \) with different diagonal coefficients \( v_i \) belonging to this family the following relations hold:

\[
\partial_i v_k = \Gamma_{ki}^k(v_i - v_k), \quad \partial_j \left( \frac{\partial_i v_k}{v_i - v_k} \right) = \partial_i \left( \frac{\partial_j v_k}{v_j - v_k} \right).
\]

Here we make an important observation. The conditions (2.12) follow directly from (2.11). Precisely speaking, given a diagonal system \( u_i^t = v_i(u)u_x^i \) with pairwise distinct coefficients, usually called characteristic speeds, that satisfy (2.11) then the system of equations for \( w_i \),

\[
\partial_i w_k = \Gamma_{ki}^k(w_i - w_k), \quad \Gamma_{ki}^k = \frac{\partial_k v_i}{v_k - v_j}, \tag{2.13}
\]

is compatible.

**Definition 2** [61] A diagonal hydrodynamic system \( u_i^t = v_i^i u_x^i \) is called semi-Hamiltonian if it is hyperbolic and its coefficients satisfy the condition

\[
\partial_j \left( \frac{\partial_i v_k}{v_i - v_k} \right) = \partial_i \left( \frac{\partial_j v_k}{v_j - v_k} \right).
\]

As it turns out, the semi-Hamiltonian condition is necessary and sufficient for integrability. We will however hold this discussion until the next section.

Following **Conjecture 1** the procedure to check if a hydrodynamic type system is integrable has three steps - one needs to check if the system is diagonalisable by Theorem 1, then actually bring it to diagonal form and then check the semi-Hamiltonian property by virtue of (2.11). As we noted in the discussion of Riemann invariants, the second step is a particularly weak point of the procedure, which for some time was making the entire routine inapplicable. A way to overcome the problem was found in [55], by
the construction of a tensor criterion to establish the semi-Hamiltonian property.

**Theorem 5** [55] A generic diagonalisable operator \( V \) with pairwise distinct eigenvalues is semi-Hamiltonian if and only if the associated tensor \( P \) is identically 0,

\[
P^s_{kij} = \sum_{p=1}^{n} \sum_{q=1}^{n} (v^s_p Q^p_{kqj} v^q_i + v^s_p Q^p_{kij} v^q_j - v^s_q v^q_p Q^p_{kij} - Q^s_{kpq} v^p_i v^q_j) \equiv 0.
\]

Here the components of the tensor \( Q \) are

\[
Q^s_{kij} = \sum_{p=1}^{n} \sum_{q=1}^{n} (v^p_k K^s_{pqj} v^q_i + v^p_k K^s_{piq} v^q_j - v^p_q v^q_k K^s_{pij} - K^s_{kpq} A^p_i A^q_j) + \sum_{p=0}^{n} (4 v^p_k M^s_{pij} - 2 M^s_{kpq} A^p_i - 2 M^s_{kpq} A^q_j),
\]

where \( M \), in component form, is

\[
M^s_{kij} = \sum_{p=1}^{n} \sum_{q=1}^{n} (N^s_{kp} v^p_i N^q_j + N^s_{pq} v^p_k N^q_i - N^s_{pq} N^p_i v^q_j - N^s_{kp} N^p_k v^q_j - N^s_{kp} N^p_q v^q_i).
\]

Here \( N \) is the Nijenhuis tensor and to compute \( K \) we take \( B = V^2 \), so that

\[
K^s_{kij} = \sum_{p=1}^{n} (B^p_k \partial_k N^p_i N^s_j + B^p_k \partial_p N^s_i N^p_j + N^p_i \partial_p B^p_j + N^p_k \partial_k B^p_i) + \ldots.
\]

with the dots in the last equation representing ten summands which are obtained from the five written out by the cyclic permutations of \( i,j,k \).

Although being too complicated for direct computations, above formulas provide an option for the use of software packages, and provided sufficient computational power, it could be a useful tool. It could also be proven that the existence of \( n \) conservation laws together with diagonalisability is
sufficient condition for semi-Hamiltonian property.

2.5 Conservation Laws and Commuting Flows

When we study a finite dimensional Hamiltonian system we say it is integrable when the number of involutive first integrals of the system is equal to the dimension of this system. Let us look how a similar construction works for the considered $1 + 1$ hydrodynamic type systems. We will distinguish a narrow class of first integrals, namely hydrodynamic type first integrals $I$, functionals of the type

$$I = \int P(u)dx,$$

with density $P(u)$ independent of the spatial derivatives of the field variables $u_x, u_{xx}, \ldots$, which Poisson commute with the Hamiltonian. The functional $I$, together with the action of the Poisson bracket generates the flow

$$u^i_t = \{u^i, I\} = w^i_j(u)u^j_x = Wu_x.$$

Since the first integral and the Hamiltonian Poisson commute, $\{H, I\} = 0$, then from the Jacobi identity it directly follows that $u_{tt} = u_{rt}$ (as a functional, $u^i(x) = \int u^i(y)\delta(x - y)dy$).

The following lemma holds:

**Lemma 6** [61] The functional $I$ from (2.14) is a first integral for the Hamiltonian system $u_t = \{u, H\} = Vu_x$ if and only if $VW = WV$.

**Proof** Let us consider the trivial identity $I_t = \int \partial_t Pdx = \int \partial_t P v^i_j w^j_x dx = 0$. Its variational derivative

$$\frac{\delta}{\delta w} \int \partial_t P v^i_j w^j_x dx \equiv 0,$$

must be trivially zero, but if we apply Lemma 3 and use $\partial_i \partial_j P = \nabla_i \partial_j P -$
The results lead to the non-trivial result:

\[
\left[ \partial_k (\partial_i P v^i_j) - \partial_j (\partial_i P v^i_k) \right] u^i_x = \left[ (\nabla_k \nabla_i P) v^i_j - (\nabla_j \nabla_i P) v^i_k \right] u^i_x = 0,
\]

and due to \( g^{lk} v^i_l = g^{li} v^i_k \),

\[
g^{kl} \left[ (\nabla_i \nabla_j P) v^i_j - (\nabla_j \nabla_i P) v^i_k \right] = (\nabla^k \nabla_i P) v^i_j - v^k_i (\nabla^k \nabla_j P) = u^i_k v^j_i - v^k_i u^j_i = 0.
\]

Conversely, if \( v^i_j \) and \( w^i_j \) commute, the previous argument shows that \( \partial_i (\partial_k P v^k_j) = \partial_j (\partial_k P v^k_i) \), which implies the existence of the function \( Q(u) \) such that \( \partial_i Q = \partial_i P v^k_j \), i.e.

\[
I_t = \int \partial_k P v^k_j u^j_x dx = \int \frac{d}{dx} Q(u) dx = 0. \tag{2.16}
\]

We can now introduce the notion of conservation law, namely if a Hamiltonian system of the type \( u_t = \{u, H\} \) possesses a hydrodynamic integral \( I \) (2.14) there exists a corresponding function \( Q(u) \) such that we have a conservation law

\[
P(u)_t = Q(u)_x.
\]

In order for \( I \) (2.14) to be first integral of the semi-Hamiltonian system (2.11) it is necessary and sufficient that

\[
\partial_i \partial_j P - \Gamma^{ij}_{ki} \partial_i P - \Gamma^{ij}_{ji} \partial_j P = 0,
\]

where \( \Gamma^{ij}_{ji} \) is defined by (2.9). Let us introduce new field variables \( z \) such that \( \partial_i P = z^i \). Then the system above could be rewritten as

\[
\partial_i z^j = \Gamma^{ij}_{ki} z^i + \Gamma^{ij}_{ji} z^j.
\]

We can compute the consistency conditions for this system,

\[
\partial_k \partial_i z^j - \partial_i \partial_k z^j = z^i R^k_{kji} + z^k R^i_{kji} - z^j (\partial_k \Gamma^j_{ji} - \partial_i \Gamma^j_{jk}) = 0.
\]
This condition is satisfied because of (2.12), which is a direct consequence of the semi-Hamiltonian property.

**Theorem 7** [61] A semi-Hamiltonian system has infinitely many commuting flows, parametrised locally by $n$ functions of a single variable. These flows commute with each other, their matrices are all diagonal and all hydrodynamic type integrals of the initial semi-Hamiltonian system are also their integrals.

**Proof** Denoting $v^i_j = v_j \delta^i_j$, let us compute the consistency condition

$$\partial_t \partial_x u^i - \partial_t \partial_x u^i = \left( \partial^k w^j_p v^k_p - \partial^k v^j_p w^k_p \right) u^i_p u^j_x + \left( w^i_j v^j_q - v^i_j w^j_q \right) u^q_p u^i_p$$

$$+ (w^i_j v^j_p - v^j_i w^i_p) u^p_x,$$

where we sum over repeating indices. We want this expression to be trivially zero, so if we consider it as a polynomial in the $x$-derivatives of the field variables $u^i_x, u^{ix}_{xx}, \text{etc.}$, each of the coefficients of this polynomial should be trivially zero. This means that $(w^i_j v^j_p - v^i_j w^j_p) = 0$, and since $v^i_j = v_j \delta^i_j$ then $w^i_j$ is also diagonal, and we can denote it as $w^i_j = w_j \delta^i_j$. Setting all coefficients of the remaining part of the consistency condition,

$$\partial_t \partial_x u^i - \partial_t \partial_x u^i = \sum_{k=1}^n \left( \partial^i_k v_i (w_k - w_i) - \partial^i_k w_i (v_k - v_i) \right) u^i_x u^k_x = 0, \quad i \neq k, \quad (2.17)$$

is equivalent to the conditions (2.13). Any two diagonal flows automatically commute due to (2.17) and (2.13). Finally, if we consider a hydrodynamic integral of the original semi-Hamiltonian system $I = \int P dt$ it follows that it is a first integral for each of the flows commuting with the original system, because

$$(w_j - w_i) \left( \partial_i \partial_j P - \Gamma^i_{ij} \partial_i P - \Gamma^i_{j} \partial_j P \right) = (w_j - w_i) \partial_i \partial_j P - \partial_i P \partial_j w_i - \partial_j P \partial_i w_j = 0.$$
We will now discuss the generalized hodograph method which allows us to find solutions of semi-Hamiltonian systems.

### 2.6 Generalized Hodograph Method

Let us consider \( \mathbf{u} \) to be a two-dimensional vector \( \mathbf{u} = (u^1, u^2) \). The hyperbolic system \( u_i^t = v_j^i u_j^i \), which is automatically semi-Hamiltonian, could be solved using the classical hodograph method. To do this we need \( u^1 = u^1(t, x) \) and \( u^2 = u^2(t, x) \) to be locally invertible. We can then invert this system by considering \( t \) and \( x \) as functions of \( u^1 \) and \( u^2 \). Next we expand the relations \( \partial_t t = \partial_x x = 1 \) and \( \partial_t x = \partial_x t = 0 \) by the chain rule and reduce them modulo the initial system. We end up with a linear system for the functions \( x_{u^1}, x_{u^2}, t_{u^1}, t_{u^2} \).

For example, in the case of shallow water equations,

\[
\begin{align*}
  u_t + uu_x + h_x &= 0, \\
  h_t + (hu)_x &= 0,
\end{align*}
\]

following the described procedure we obtain a linear system of equations for the functions \( x(u, h) \) and \( t(u, h) \),

\[
\begin{align*}
  x_{h} &= ut_{h} - t_{u}, \\
  -x_{u} &= ht_{h} - ut_{u},
\end{align*}
\]

which can be written in the linear form

\[
\begin{align*}
  (x - ut)_h &= -t_{u}, \\
  (x - ut)_u &= -ht_{h} - t.
\end{align*}
\]

In a similar manner, if we consider a \( n \)-dimensional semi-Hamiltonian diagonal system \( u_i^t = v_i u_j^j \), it can be solved by virtue of the generalised hodograph method.
graph method [61]. As we have already discussed, a system of the considered type has infinitely many commuting flows \( u^i_r = w^i_{rx} \), which satisfy the equations
\[
\frac{\partial_i w_k}{w_i - w_k} = \frac{\partial_i v_k}{v_i - v_k}, \quad i \neq k. \tag{2.18}
\]
Let us now construct a system of \( n \) equations for the field variables \( u \),
\[
w_i(u) = v_i(u)t + x. \tag{2.19}
\]
Here \( x \) and \( t \) are parameters, \( v_i \) are the coefficients of the original semi-Hamiltonian matrix, and \( w_i \) are the coefficients of any flow commuting with the original system. The following theorem then holds:

**Theorem 8** [61] *Any smooth solution of (2.19) is a solution of the semi-Hamiltonian system \( u_i^t = v_i u_i^x \). Furthermore, any solution of the given system \( u_i^t = v_i u_i^x \) may be represented as a solution of (2.19) in a neighborhood of a point \((t_0, x_0)\) such that \( u_i^x(t_0, x_0) \neq 0 \).*

**Proof** Let us differentiate (2.19) by \( t \) and then denote \( M_{ik} = \partial_k w_i(u) - \partial_i v_k(u)t \), then
\[
\sum_{k=1}^n M_{ik} u_i^k = v_i, \quad \sum_{k=1}^n M_{ik} u_x^k = 1.
\]
Taking into account (2.19) and the semi-Hamiltonian property (2.18) we have that \( M_{ik} = 0 \), \( M_{ii} u_i^t = v_i \) and \( M_{ii} u_x^k = 1 \). It directly follows that \( u_i^t = v_i u_x^i \) (we note the condition \( u_x^i \neq 0 \)).

To prove this in the reverse direction, suppose that in a neighborhood of \( u(t_0, x_0) \) we have \( u_i^t(t, x) \neq 0 \) and \( u_i^t = v_i u_x^i \). Then the initial conditions \( u_0^i = u_i^t(t_0, x) \) give rise to initial Cauchy data
\[
w_i(u_0(x)) = v_i(u_0(x))t + x, \tag{2.20}
\]
on the curve \( u_0(x) \) for the system (2.18). Since we assume that \( u_x^i(x_0) \neq 0 \) it follows that in a neighbourhood of \( u_0(x) \) there exists a unique solution.
of (2.18) with the initial conditions above. With the initial data (2.20) the system (2.19) has a unique solution in a neighbourhood of \((x_0, t_0, u^i_0)\) since the Jacoby matrix \(M_{ik}\) of (2.19) is diagonal at \((x_0, t_0, u^i_0)\) and \(M_{ii} = (u^i)^{-1} = 0\). Since the solution of the system \(u^i_t = v^i u^i\) and the considered curve coincide, \(\bar{u}(t_0, x) = u^i_0(x)\), then, in the considered neighbourhood of \((t_0, x_0)\), \(\bar{u}(t, x) = u^i(x)\).  

2.7 The Method of Hydrodynamic Reductions. Example of dKP

The theory of integrable 1 + 1 dimensional systems provides a base for the exploration of higher dimensional hydrodynamic type systems. The method of hydrodynamic reductions [23] extensively relies on the results discussed in the previous sections and provides a way to investigate the integrability of higher dimensional systems of hydrodynamic type. We will be interested in the 2 + 1 dimensional case, and we will present the method in this context.

Let us consider a 2 + 1 dimensional system of hydrodynamic type in the following form:

\[
\begin{align*}
    u_t &= A(u)u_x + B(u)u_y. \\
    (2.21)
\end{align*}
\]

Here \(u\) is a \(n\)-component vector, \(A(u)\) and \(B(u)\) are \(n \times n\) matrices, with \(n\) being the number of equations. We consider the following ansatz

\[
    u(x, y, t) = u(R^1, R^2, \ldots, R^N),
\]

with \(R^i(x, y, t)\) required to satisfy the equations,

\[
    R^i_t = \lambda^i(R)R^i_x, \quad R^i_x = \mu^i(R)R^i_t. \quad (2.22)
\]

In the literature these reductions are also known as "non-linear interactions of \(N\) planar simple waves" or "multi-phase solutions", and are discussed
extensively in gas dynamics and in the context of the dispersionless KP hierarchy. We can think of this as a decoupling of the original $2 + 1$ dimensional system (2.21) into two separate commuting $1 + 1$ dimensional systems of hydrodynamic type from where the name hydrodynamic reductions comes. Substituting the ansatz into (2.21) we arrive at the following equation

$$(\lambda^i I - A - \mu^i B)\partial_i u = 0,$$ (2.23)

which implies that $\lambda^i$ and $\mu^i$ must satisfy the dispersion relation $\det(\lambda I - A - \mu B) = 0$.

We can now compute the commutativity condition $R_{ty} = R_{yt}$ to find that

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j.$$ (2.24)

Provided (2.24) holds, the solution of the system (2.21) is given by the generalised hodograph formula

$$v^i(R) = x + \lambda^i(R)t + \mu^i(R)y,$$ (2.25)

where $v^i$ is the general solution of the linear system

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j.$$ (2.26)

**Definition 3 [23]** We will call the system (2.21) integrable if it possesses sufficiently many $N$-component reductions, parametrized by $N$ functions of a single variable.

We note here that this definition agrees with any other definitions of integrability for this kind of systems. It was extensively used for classification purposes, see for instance [23, 25, 26].

A number of remarks are in place. First of all, this procedure works only if $A$ and $B$ do not commute $[A, B] \neq 0$. This related to the irreducibility of
the dispersion relation $\det(\lambda I - A - \mu B) = 0$.

Second, let us consider $N = 1$, then we have $u = u(R)$, with $R$ being a solution of
\[
R_t = \lambda(R)R_x, \quad R_y = \mu(R)R_x.
\]
We automatically have $R_{ty} = R_{yt}$. The hodograph formula in the scalar case gives $f(R) = x + \lambda(R)t + \mu(R)y$ with $f(R)$ being an arbitrary function. This means $u(R)$ is constant along one parameter family of planes. This kind of solutions exist for all multi-dimensional quasilinear systems and cannot be used to check integrability. When we consider a two component reduction, i.e. $u = u(R^1, R^2)$ with $R^1$ and $R^2$ satisfying (2.22) we have the general solution given by the formula
\[
v^1(R) = x + \lambda^1(R)t + \mu^1(R)y, \\
v^2(R) = x + \lambda^2(R)t + \mu^2(R)y.
\]
Setting $R = \text{const.}$ results in a two-parameter family of lines in the space $(x,y,t)$ which means $u$ is constant along the lines of a two parameter family. This is also not too restrictive condition - for instance if $n = 2$ any system (2.21) possesses infinitely many 2-component reductions.

On the other hand, the existence of three component reductions is a very strong condition. This is evident if we consider (2.23) and (2.24) as a Gibbons-Tsarev-type system [35]. The compatibility conditions of this system involve only triplets of indices $i \neq j \neq k$. This means that the existence of infinitely many three-phase reductions guaranties the existence of higher phase reductions, and hence integrability.
2.7.1 Using the method of hydrodynamic reductions as an integrability test. Example of dKP

To illustrate the way that the method can be used as a test to detect integrability we will try to find the class of functions $f(u)$ for which the system

$$
u_t = f(u)\nu_x + w_y, \quad w_x = u_y,$$  \hspace{1cm} (2.27)

is integrable. We make a hydrodynamic reductions of the system $u = u(R^1, R^2, \ldots, R^N)$, $w = w(R^1, R^2, \ldots, R^N)$, where the phases $R^i$ satisfy

$$R^i_t = \lambda^i R^i_x, \quad R^i_y = \mu^i R^i_x.$$

Substituting in (2.27) produces the system

$$\lambda^i \nu^i = f(u)\nu_i + \mu^i w_i, \quad w_i = \mu^i u_i,$$

(no summation!) where $u_i$ denotes $\partial_{R^i} u$. We can eliminate $w_i$, reducing the first equation of the last system to

$$\lambda^i = f(u) + \mu^i \nu^2.$$

We now enforce the condition (2.24), which provides

$$\partial_j \nu^i = \frac{f'(u) u_j}{\mu^i - \mu^j}. \hspace{1cm} (2.28)$$

On the other hand the equation $w_i = \mu^i u_i$ itself must be consistent i.e. \( w_{ij} = w_{ji} \), which leads to the relation

$$u_{ij} = \frac{\partial_j \nu^i}{\mu^j - \mu^i} u_i + \frac{\partial_i \nu^j}{\mu^i - \mu^j} u_j.$$
Finally we require that (2.28) is in involution, \( \partial_k \partial_j \mu^i - \partial_j \partial_k \mu^i = 0 \). In general we obtain \( P \partial_j \partial_k u = 0 \) with \( P \) being a rational expression in \( \mu^i, i = 1, \ldots, N \). The numerator of \( P \) is a polynomial in \( \mu^i, i = 1, \ldots, n \) and require that all its coefficients are zero. The equations generated this way constitute the integrability conditions.

In the considered problem (2.27) , the situation is simple since the only non-vanishing term appears to be

\[
 f''(u) \left( \frac{u_k}{\mu^j - \mu^i} + \frac{u_j}{\mu^k - \mu^i} \right) \partial_k u,
\]

from which we can conclude that \( f(u) \) is a linear function, which, up to a suitable change of variables, means that the only integrable equation in this class is the dKP.
Chapter 3

Classification of integrable
two-component Hamiltonian
systems of hydrodynamic type in
2+1 dimensions

In this Chapter, based on the author’s joint work with E.V. Ferapontov and
A.V. Odesskii [31], we provide a complete classification of integrable two com-
ponent 2 +1 dimensional Hamiltonian systems of hydrodynamic type. Our
approach relies on the method of hydrodynamic reductions, as discussed in
Chapter 1. We begin with a classification of two component Hamiltonian op-
erators in 2+1 dimensions. Next, we calculate the integrability conditions for
each class, and obtain a list of integrable potentials. Our results demonstrate
that the moduli spaces of integrable Hamiltonians in 2 +1 dimensions are
finite-dimensional: the integrability conditions for the corresponding Hamil-
tonian densities constitute over-determined involutive systems of finite type.
Modulo natural equivalence groups, this leads to finite lists of integrable
Hamiltonians parametrised by elliptic or hypergeometric functions. We also
present the relevant dispersionless Lax pairs, whose existence is equivalent
to integrability.

3.1 Introduction

We consider 2 + 1 dimensional quasilinear systems

\[ u_t = P(u)u_x + Q(u)u_y. \]  \hspace{1cm} (3.1)

The Hamiltonian structure in 2+1 dimensions is introduced in a way analogous to the approach discussed in Chapter 1 for the 1+1 dimensional case, with Poisson bracket of two functionals \( I \) and \( J \) defined as

\[ \{ I, J \} = \int \frac{\delta I}{\delta u^i(x,y)} A^{ij} \frac{\delta J}{\delta u^j(x,y)} dxdy. \]  \hspace{1cm} (3.2)

Here \( A = A^{ij} \) is a two-dimensional Hamiltonian operator of differential-geometric type,

\[ A^{ij} = g^{ij}(u) \frac{d}{dx} + b^{ij}_k(u)u^k_x + \tilde{g}^{ij}(u) \frac{d}{dy} + \tilde{b}^{ij}_k(u)u^k_y. \]

Operators of this form are generated by a pair of metrics \( g^{ij}, \tilde{g}^{ij} \). Theorem 2 holds for \( g^{ij}, b^{ij}_k \) and \( \tilde{g}^{ij}(u), \tilde{b}^{ij}_k \) separately. Despite the similar structure there is a substantial difference to the 1 + 1 dimensional case. Although both metrics \( g^{ij} \) and \( \tilde{g}^{ij} \) must necessarily be flat, it may not be possible to bring them to constant coefficient form simultaneously: there exist obstruction tensors.

The two-component case completely understood and classified in [13], [44] and [45]. Any two component operator \( A^{ij} \) (3.1) can be brought to one of the following three forms by an appropriate change of coordinates,

\[ A = \begin{pmatrix} d/dx & 0 \\ 0 & d/dy \end{pmatrix}, \]  \hspace{1cm} (3.3)
\[ A = \begin{pmatrix} 0 & d/dx \\ d/dx & d/dy \end{pmatrix}, \] (3.4)

and a third one, which is essentially non-constant,

\[ A = \begin{pmatrix} 2v & w \\ w & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & v \\ v & 2w \end{pmatrix} \frac{d}{dy} + \begin{pmatrix} v_x & v_y \\ w_x & w_y \end{pmatrix}. \] (3.5)

Here \( v, w \) are components of the vector \( u \). We will call them Hamiltonian operators of type I, type II, and type III, respectively, and refer to the Poisson brackets generated as brackets of type I, type II and type III.

Hamiltonian of hydrodynamic type in \( 2 + 1 \) dimensions is a functional \( H = \int h(u) dxdy \). Thus Hamiltonian systems generated by the operators (3.3) - (3.5) take the form

\[ v_t = (h_v)_x, \quad w_t = (h_w)_y, \] (3.6)

\[ v_t = (h_w)_x, \quad w_t = (h_v)_x + (h_w)_y, \] (3.7)

and

\[ v_t = (2vh_v + wh_w - h)_x + (vh_w)_y, \quad w_t = (wh_v)_x + (2wh_w + vh_v - h)_y, \] (3.8)

respectively.

Our main result is a description of Hamiltonian densities \( h(v, w) \) for which the corresponding systems (3.6) - (3.8) are integrable by the method of hydrodynamic reductions. Let us point out that the integrability conditions for the general class of two-component systems (3.1) were derived in [24] in the coordinates where the first matrix \( P \) is diagonal. It was demonstrated in [46, 48] that, again in special coordinates, the matrices \( P \) and \( Q \) can be parametrised by hypergeometric functions. However, it is not clear how to isolate Hamiltonian cases within these general descriptions.

We adopt a straightforward approach and derive the integrability con-
ditions by directly applying the method of hydrodynamic reductions to the systems (3.6) - (3.8) in the same way it was applied to the example of dKP. For Hamiltonian systems of type (3.6) this was done in [23]. Applied to equations (3.6) - (3.8), this method results in involutive systems of fourth order PDEs for the Hamiltonian densities $h(v, w)$ which are, in general, quite complicated.

It was observed that these systems simplify considerably under the Legendre transformation $h(v, w) \to H(V, W)$ defined as

$$V = h_v, \quad W = h_w, \quad H = vh_v + wh_w - h, \quad H_V = v, \quad H_W = w.$$  

In the new variables, equations (3.6) - (3.8) take the so-called Godunov form,

$$\begin{align*}
(H_V)_t &= V_x, & (H_W)_t &= W_y, \quad (3.9) \\
(H_V)_t &= W_x, & (H_W)_t &= V_x + W_y, \quad (3.10)
\end{align*}$$

and

$$\begin{align*}
(H_V)_t &= (VH_V + H)_x + (WH_V)_y, & (H_W)_t &= (VH_W)_x + (WH_W + H)_y, \quad (3.11)
\end{align*}$$

respectively. All our classification results will be formulated in terms of the Legendre-transformed Hamiltonian densities $H(V, W)$. We recall that a system (3.1) is said to be in Godunov form [37] if, for appropriate potentials $F_\alpha(u)$, $\alpha = 0, 1, 2$, it possesses a conservative representation

$$(F_{0,i})_t + (F_{1,i})_x + (F_{2,i})_y = 0,$$

$F_{\alpha,i} = \partial F_\alpha/\partial u^i$. In particular, in the case (3.11) one has $F_0 = -H, F_1 = VH, F_2 = WH$. This representation will be utilised in Section 5 to provide a parametrisation of the generic integrable potential of type (3.11) by generalised hypergeometric functions.
Let us first review the known results for systems of type I. Here the integrability conditions take the form

\[ H_{VW}H_{VVV} = 2H_{VYV}H_{VVW}, \]

\[ H_{VW}H_{VVW} = 2H_{VYV}H_{VW}, \]

\[ H_{VW}H_{VWV} = H_{VYV}H_{VW} + H_{VVV}H_{WWW}, \]

\[ H_{VW}H_{WWW} = 2H_{VYV}H_{WVW}, \]

\[ H_{VW}H_{WWV} = 2H_{VYV}H_{WWW}, \]

see [23]. These equations are in involution and can be solved in closed form. Up to the action of a natural equivalence group (see Appendix), this provides a complete list of integrable potentials.

**Theorem 9** [26] The generic integrable potential of type I is given by the formula

\[ H = Z(V + W) + \epsilon Z(V + \epsilon W) + \epsilon^2 Z(V + \epsilon^2 W) \]

where \( \epsilon = e^{2\pi i/3} \) and \( Z''(s) = \zeta(s) \). Here \( \zeta \) is the Weierstrass zeta-function, \( \zeta' = -\wp, \ (\wp')^2 = 4\wp^3 - g_3 \). Degenerations of this solution correspond to

\[ H = (V + W) \log(V + W) + \epsilon(V + \epsilon W) \log(V + \epsilon W) + \epsilon^2(V + \epsilon^2 W) \log(V + \epsilon^2 W), \]

\[ H = \frac{1}{2} V^2 \zeta(W), \quad H = \frac{V^2}{2W}, \quad H = (V + W) \ln(V + W), \]

as well as the following polynomial potentials:

\[ H = V^2 W^2, \quad H = VW^2 + \frac{\alpha}{5} W^5, \quad H = VW + \frac{1}{6} W^3. \]

We refer to Section 2 of the present Chapter for further details.
In Section 3 we demonstrate that for systems of type II the integrability conditions take the form

\[
\begin{align*}
H_{VVVV} &= \frac{2H_{VVV}^2}{H_{VV}}, \\
H_{VWVV} &= \frac{2H_{VWW}H_{VVV}}{H_{VV}}, \\
H_{VWWV} &= \frac{2H_{VWW}^2}{H_{VV}}, \\
H_{VWWW} &= \frac{3H_{VWW}H_{VVV} - H_{WWW}H_{VVV}}{H_{VV}}, \\
H_{WWWV} &= \frac{6H_{VWW}^2 - 4H_{WWW}H_{VWW}}{H_{VV}}.
\end{align*}
\]

These equations are also in involution, and can be solved in a closed form. Up to the action of a natural equivalence group, we obtain a complete list of integrable potentials.

**Theorem 10** [31] The generic integrable potential of type II is given by the formula

\[H = V \ln \frac{V}{\sigma(W)}\]

where \(\sigma\) is the Weierstrass sigma-function: \(\sigma'/\sigma = \zeta, \ \zeta' = -\wp, \ \wp'^2 = 4\wp^3 - g_3\). Its degenerations correspond to

\[H = V \ln \frac{V}{W}, \quad H = V \ln V, \quad H = \frac{V^2}{2W} + \alpha W^7,\]

as well as the following polynomial potentials:

\[H = \frac{V^2}{2} + \frac{VW^2}{2} + \frac{W^4}{4}, \quad H = \frac{V^2}{2} + \frac{W^3}{6}.\]

Further details, as well as dispersionless Lax pairs corresponding to integrable potentials from Theorem 10, are provided in Section 3 of the present Chapter.

**Remark.** For \(H = V \ln V\) the equations (3.10) take the form \(V_t = VW_x, \ V_x = \)
−W_y, implying the Boyer-Finley equations for V: V_{xx} + (\ln V)_{yy} = 0. Similarly, the choice \( H = \frac{V^2}{2} + \frac{W^3}{6} \) results in the equations \( V_t = W_x, \ V_x = WW_t - W_y, \) implying the dKP equation for W: \( W_{xx} = (WW_t - W_y)_t. \) These Hamiltonian representations have appeared previously in the literature, see e.g. \([4]\).

The case III turns out to be considerably more complicated. The integrability conditions constitute an involutive system of fourth order PDEs for the potential \( H \) which is not presented here due to its complexity (see Sect. 4). However, it possesses a remarkable \( SL(3,R) \)-invariance which reflects the invariance of the Hamiltonian formalism (3.8) under linear transformations of the independent variables \( x,y,t. \) Based on this invariance, we were able to classify integrable potentials.

**Theorem 11** \([31]\) The generic integrable potential of type III is given by the series
\[
H = \frac{1}{Wg(V)} \left( 1 + \frac{1}{g_1(V)W^6} + \frac{1}{g_2(V)W^{12}} + \ldots \right).
\]

Here \( g(V) \) satisfies the fourth order ODE,
\[
g'''(2gg'^2 - g^2g'') + 2g^2g'''^2 - 20gg''g''' + 16g''^3g'' + 18gg''^3 - 18g'^2g'' = 0,
\]
whose general solution can be represented in parametric form as
\[
g = w_2(t), \quad V = \frac{w_1(t)}{w_2(t)},
\]
where \( w_1(t) \) and \( w_2(t) \) are two linearly independent solutions of the hypergeometric equation
\[
t(1-t)d^2w/dt^2 - \frac{2}{9}w = 0.
\]

The coefficients \( g_i(V) \) are certain explicit expressions in terms of \( g(V), \) e.g.,
\( g_1(V) = g g'' - 2g'^2 \) and so on. Degenerations of this solution correspond to

\[
H = \frac{1}{W g(V)}, \quad H = \frac{1}{W V}, \quad H = V - W \log W, \quad H = V - W^2/2.
\]

We refer to Section 4 for further details. In particular, we obtained a parametrisation of the generic integrable potential \( H(V, W) \) by generalised hypergeometric functions:

\[
H = G \left( \frac{u_1 - 1}{u_1} \right), \quad V = \frac{h_1}{h_0}, \quad W = \frac{h_2}{h_0}.
\]  

(3.13)

Here \( G'(t) = \frac{1}{t^{2/3}(t-1)^{2/3}} \) and \( h_0, h_1, h_2 \) are linearly independent solution of the hypergeometric system

\[
\begin{align*}
& u_1(1 - u_1)h_{u_1, u_1} - \frac{4}{3} u_1 h_{u_1} - \frac{2}{9} h = \frac{2}{3} u_1 (u_1 - 1) h_{u_1} + \frac{2}{3} u_2 (u_2 - 1) h_{u_2}, \\
& u_2(1 - u_2)h_{u_2, u_2} - \frac{4}{3} u_2 h_{u_2} - \frac{2}{9} h = \frac{2}{3} u_1 (u_1 - 1) h_{u_1} + \frac{2}{3} u_2 (u_2 - 1) h_{u_2}, \\
& h_{u_1, u_2} = \frac{2}{3} \frac{h_{u_2} - h_{u_1}}{u_2 - u_1},
\end{align*}
\]

which can be viewed as a natural two-dimensional generalisation of the hypergeometric equation \( t(1 - t)d^2w/dt^2 - \frac{2}{9} w = 0 \). This parametrisation is based on a novel construction of integrable quasilinear systems in Godunov’s form in terms of generalised hypergeometric functions which builds on [46, 48]. We believe that this construction is of independent interest, and can be applied to a whole variety of similar classification problems. The details are provided in Theorem 12, to be found in Section 5 of the present Chapter.

Our results lead to the following general observations:

The moduli spaces of integrable Hamiltonians in \( 2 + 1 \) dimensions are finite dimensional.
In the two-component case, the actions of the natural equivalence groups on the moduli spaces of integrable Hamiltonians possess open orbits. This leads to a remarkable conclusion that for any two-component Poisson bracket in $2 + 1$ dimensions there exists a unique ‘generic’ integrable Hamiltonian, all other cases can be obtained as its appropriate degenerations.

We anticipate that the results presented in this Chapter will find applications in the theory of infinite-dimensional Frobenius manifolds, see [9] for the first steps in this direction.

3.2 Hamiltonian systems of type I

In this section we review the classification of integrable systems of the form (3.6),

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = \begin{pmatrix} d/dx & 0 \\ 0 & d/dy \end{pmatrix} \begin{pmatrix} h_v \\ h_w \end{pmatrix},$$

or, explicitly,

$$v_t = (h_v)_x, \quad w_t = (h_w)_y,$$

following [23, 26]. The integrability conditions were first derived in [23] based on the method of hydrodynamic reductions. These conditions constitute a
system of fourth order PDEs for the Hamiltonian density \( h(v, w) \),

\[
\begin{align*}
    h_{vw}(h^2_{vw} - h_{vv}h_{ww})h_{vvvv} &= 4h_{vw}h_{vv}(h_{vw}h_{vvw} - h_{vw}h_{vww}) \\
    &+ 3h_{vv}h_{vw}h^2_{vvw} - 2h_{vv}h_{wvw}h_{vvw} - h_{vw}h_{ww}h^2_{vvv}, \\
    h_{vw}(h^2_{vw} - h_{vv}h_{ww})h_{vvww} &= -h_{vw}h_{vv}(h_{vw}h_{wvw} + h_{wv}h_{vww}) \\
    &+ 3h^2_{vw}h^2_{vww} - 2h_{vv}h_{wvw}h_{wvw} + h^2_{vw}h_{vww}, \\
    h_{vw}(h^2_{vw} - h_{vv}h_{ww})h_{vwvv} &= 4h^2_{vw}h_{vw}h_{vw} - h_{vw}h_{ww}h_{wvw} + h_{vw}h_{ww}h_{wvw}, \\
    h_{vw}(h^2_{vw} - h_{vv}h_{ww})h_{wwww} &= 4h_{vw}h_{ww}(h_{vw}h_{wvw} - h_{wv}h_{vww}) \\
    &+ 3h_{vw}h_{vv}h^2_{vww} - 2h_{vw}h_{wvw}h_{wvw}h_{vwv} - h_{vw}h_{ww}h^2_{vww}.
\end{align*}
\]  

(3.14)

This system is in involution, and its solution space is 10-dimensional. Eqs. (3.14) can be represented in compact form as

\[
sd^4 h = d^3 h ds + 3h_{vw}(2dh_vdh_w - h_{vw}d^2 h) \det(dn)
\]

where \( s = h^2_{vw}(h_{vw}h_{wv} - h^2_{vw}) \), \( d^k h \) denotes \( k \)-th symmetric differential of \( h \), and \( n \) is the Hessian matrix of \( h \). The contact symmetry group of the system (3.14) (see Sect 3.6) is also 10-dimensional, consisting of the 8-parameter group of Lie-point symmetries,

\[
\begin{align*}
    v &\rightarrow av + b, \\
w &\rightarrow cw + d, \\
h &\rightarrow \alpha h + \beta v + \gamma w + \delta,
\end{align*}
\]
along with the two purely contact infinitesimal generators,

\[-\Omega_{hv} \frac{\partial}{\partial v} - \Omega_{hw} \frac{\partial}{\partial w} + (\Omega - h_v \Omega_{hv} - h_w \Omega_{hw}) \frac{\partial}{\partial h} + (\Omega_v + h_v \Omega_v) \frac{\partial}{\partial h_v} + (\Omega_w + h_w \Omega_h) \frac{\partial}{\partial h_w},\]

with the generating functions \( \Omega = h_v^2 \) and \( \Omega = h_w^2 \), respectively. It was observed in [23] that the integrability conditions simplify under Legendre transformation,

\[V = h_v, \ W = h_w, \ H = vh_v + wh_w - h, \ H_V = v, \ H_W = w,\]

which brings Eqs. (3.6) into the Godunov form (3.9),

\[(H_V)_t = V_x, \ (H_W)_t = W_y.\]

The integrability conditions (3.14) simplify to

\[H_{VW} H_{VVV} = 2H_{VVV} H_{VWW},\]

\[H_{VW} H_{VVW} = 2H_{VVV} H_{VWW},\]

\[H_{VW} H_{VWW} = H_{VVW} H_{VWW} + H_{VVV} H_{WWW},\]

(3.15)

\[H_{VW} H_{VWWW} = 2H_{VVW} H_{WWW},\]

\[H_{VW} H_{WWWW} = 2H_{VVW} H_{WWW}.\]

This system is also in involution, and can be represented in compact form as

\[Sd^3H = d^3HdS + 6H_{VW} dVdW \det(dN)\]
where $S = H_{VW}^2$, and $N$ is the Hessian matrix of $H$. The system (3.15) is invariant under a 10-parameter group of Lie-point symmetries (see Appendix),

$$
V \rightarrow aV + b,
W \rightarrow cW + d,
H \rightarrow \alpha H + \beta V^2 + \gamma W^2 + \mu V + \nu W + \delta.
$$

Thus, both contact symmetry generators of the system (3.14) are mapped by the Legendre transformation to point symmetries of the system (3.15). One can show that the action of the symmetry group on the moduli space of solutions of the system (3.15) possesses an open orbit. The classification of integrable potentials $H(V,W)$ is performed modulo this equivalence.

### 3.2.1 Classification of integrable potentials

The paper [26] provides a complete list of integrable potentials:

**Theorem 9** The generic integrable potential of type I is given by the formula

$$
H = Z(V + W) + \epsilon Z(V + \epsilon W) + \epsilon^2 Z(V + \epsilon^2 W)
$$

where $\epsilon = e^{2\pi i / 3}$ and $Z''(s) = \zeta(s)$. Here $\zeta$ is the Weierstrass zeta-function, $\zeta' = -\wp, (\wp')^2 = 4\wp^3 - g_3$. Degenerations of this solution correspond to

$$
H = (V+W) \log(V+W) + \epsilon (V+\epsilon W) \log(V+\epsilon W) + \epsilon^2 (V+\epsilon^2 W) \log(V+\epsilon^2 W),
$$

$$
H = \frac{1}{2} V^2 \zeta(W), \quad H = \frac{V^2}{2W}, \quad H = (V + W) \ln(V + W),
$$

as well as the following polynomial potentials:

$$
H = V^2 W^2, \quad H = V W^2 + \frac{\alpha}{5} W^5, \quad H = V W + \frac{1}{6} W^3.
$$
The corresponding systems (3.9) possess dispersionless Lax pairs, see [26] for further details.

3.3 Hamiltonian systems of type II

In this section we consider Hamiltonian systems of the form (3.7),

\[
\begin{pmatrix}
  v \\
  w
\end{pmatrix}_t = \begin{pmatrix}
  0 & d/dx \\
  d/dx & d/dy
\end{pmatrix} \begin{pmatrix}
  h_v \\
  h_w
\end{pmatrix},
\]

or, explicitly,

\[v_t = (h_w)_x, \quad w_t = (h_v)_x + (h_w)_y.\]

Let us first mention two well-known examples which fall into this class.

**Example 1.** The Hamiltonian

\[h = \frac{w^2}{2} + e^v\]

generates the system

\[v_t = w_x, \quad w_t = e^v v_x + w_y.\]

Under the substitution \(v = u_x, \ w = u_t\) these equations reduce to

\[u_{tt} - u_{ty} = e^{u_x} u_{xx},\]

which is an equivalent form of the Boyer-Finley equation [7].

**Example 2.** The Hamiltonian

\[h = \frac{v^2}{2} + \frac{2\sqrt{2}}{3} w \sqrt{w}\]
generates the system
\[ v_t = \frac{1}{\sqrt{2w}} w_x, \quad w_t = v_x + \frac{1}{\sqrt{2w}} w_y. \]

Introducing the variables \( V = v, \ W = \sqrt{2w} \), we obtain
\[ V_t = W_x, \quad WW_t = V_x + W_y. \]

Setting \( W = u_t, \ V = u_x \) we arrive at the dKP equation,
\[ u_{ty} - u_t u_{tt} + u_{xx} = 0. \]

Let us now demonstrate how to find all Hamiltonians \( h(v, w) \) for which the system (3.7) is integrable by the method of hydrodynamic reductions. Rewriting Eqs. (3.7) as
\[ v_t = h_{vw} v_x + h_{ww} w_x, \quad w_t = h_{vv} v_x + h_{vw} w_x + h_{ww} v_y + h_{ww} w_y, \quad (3.16) \]
we seek \( n \)-phase solutions in the form
\[ v = v(R^1, R^2, \ldots, R^n), \quad w = w(R^1, R^2, \ldots, R^n), \]
where the phases (Riemann invariants) \( R^i \) satisfy the equations
\[ R^i_t = \lambda^i(R) R^i_x, \quad R^i_y = \mu^i(R) R^i_x. \]

Here the characteristic speeds \( \lambda^i \) and \( \mu^i \) satisfy the commutativity conditions
\[ \frac{\partial_j \lambda^i}{\lambda^i - \lambda^j} = \frac{\partial_j \mu^i}{\mu^i - \mu^j}. \]
The substitution of this ansatz into (3.16) implies the relations
\[ \begin{align*}
    v_i \lambda^i &= h_{vw} v_i + h_{ww} w_i, \\
    w_i \lambda^i &= h_{vw} v_i + h_{ww} w_i + h_{vw} v_i \mu^i + h_{ww} w_i \mu^i,
\end{align*} \tag{3.17} \]
here \( v_i = \partial_i v \), \( w_i = \partial_i w \), \( \partial_i = \partial/\partial R^i \). The condition of their non-trivial solvability implies the dispersion relation for \( \lambda^i \) and \( \mu^i \),

\[ -\lambda^i \lambda^i + h_{ww} \lambda^i \mu^i + 2 h_{vw} \lambda^i + h_{vv} h_{ww} - h_{vw}^2 = 0. \]

In what follows we assume that the dispersion relation defines an irreducible conic, which is equivalent to the requirement \( h_{ww} \neq 0, \ h_{vv} h_{ww} - h_{vw}^2 \neq 0 \). Setting \( v_i = \phi^i w_i \), we can rewrite (3.17) in the from

\[ \begin{align*}
    \phi^i \lambda^i &= h_{vw} \phi^i + h_{ww}, \\
    \lambda^i &= h_{vv} \phi^i + h_{vw} \mu^i \phi^i + h_{ww} \mu^i + h_{vw},
\end{align*} \tag{3.18} \]

We also require the compatibility of the relations \( v_i = \phi^i w_i \), which gives

\[ \partial_i \partial_j w = \frac{\partial_j \phi^i}{\phi^i - \phi^j} \partial_i w + \frac{\partial_i \phi^j}{\phi^i - \phi^j} \partial_j w. \tag{3.19} \]

Expressing \( \lambda^i, \mu^i \) in terms of \( \phi^i \) from (3.18),

\[ \begin{align*}
    \lambda^i &= \frac{h_{ww} \phi^i + h_{ww}}{\phi^i}, \quad \mu^i = \frac{h_{ww} - h_{vv} \phi^i}{\phi^i(h_{vv} \phi^i + h_{ww})},
\end{align*} \]
and substituting these expressions into the commutativity conditions \( \partial_i \lambda^j / (\lambda^i - \lambda^j) = \partial_i \mu^j / (\mu^i - \mu^j) \) we obtain relations of the form \( \partial_j \phi^i = (\ldots) \), \( \partial_i \partial_j w = (\ldots) \partial_i w \partial_j w \) where dots denote certain rational expressions in \( \phi^i, \phi^j \) whose coefficients depend on the Hamiltonian density \( h(v, w) \) and its derivatives up to the order three. The compatibility conditions \( \partial_j \partial_k \phi^i = \partial_k \partial_j \phi^i \) take the form \( P \partial_j w \partial_k w = 0 \) where \( P \) is a polynomial in \( \phi_i, \phi_j, \phi_k \). Setting all
coefficients of this polynomial equal to zero one obtains the integrability conditions, which constitute a system of fourth order PDEs for the Hamiltonian density \( h(v, w) \),

\[
\begin{align*}
  h_{ww} (h_{vv} h_{ww} - h_{vw}^2) h_{vvvw} &= -6h_{vw}^2 h_{vvw}^2 + 3h_{ww} h_{vv} h_{vw}^2 \\
  &+ 4h_{vw}^2 h_{vww} h_{vww} - 2h_{ww} h_{vww} h_{vvw} + h_{ww}^2 h_{vww}^2, \\
  h_{ww} (h_{vv} h_{ww} - h_{vw}^2) h_{vvvv} &= -3h_{vw}^2 h_{vww} h_{vvw} + 3h_{ww} h_{vww} h_{vw} h_{vv} \\
  &- 3h_{ww} h_{vw} h_{wvw} h_{vww} + h_{ww} h_{vww} h_{wvw} + h_{ww} h_{vww} h_{vww} + h_{ww}^2 h_{vww} h_{vww}, \\
  h_{ww} (h_{vv} h_{ww} - h_{vw}^2) h_{vwww} &= -2h_{vw}^2 h_{vww}^2 + 2h_{vw} h_{vww} h_{ww} \\
  &- 3h_{ww} h_{vww} h_{vww} h_{vww} + h_{ww} h_{vww} h_{vww} + h_{ww} h_{vww} h_{vww} + h_{ww}^2 h_{vww} h_{vww}, \\
  h_{ww} (h_{vv} h_{ww} - h_{vw}^2) h_{vwwv} &= -2h_{vw}^2 h_{vww}^2 - 3h_{ww} h_{vww} h_{vww} h_{ww} \\
  &+ 3h_{ww} h_{vww} h_{vww} h_{ww} + h_{ww} h_{vww} h_{vww} + h_{ww}^2 h_{vww} h_{vww}, \\
  h_{ww} (h_{vv} h_{ww} - h_{vw}^2) h_{vwwv} &= -2h_{vw}^2 h_{vww}^2 - 2h_{ww} h_{vww} h_{vvw} h_{ww} \\
  &- 3h_{ww} h_{vww}^2 h_{vww} + 3h_{ww} h_{vww} h_{vww} h_{vww} + 4h_{ww}^2 h_{vww} h_{vww}.
\end{align*}
\]

We have verified that system (3.20) is in involution, and its solution space is 10-dimensional. The integrability conditions can be represented in compact form as

\[
\text{sd}^4 h = d^3 h ds + 3h_{ww} (2(dh_w)^2 - h_{ww} d^2 h) \det(dn)
\]

where \( s = h_{ww}^2 (h_{vv} h_{ww} - h_{vw}^2) \), \( d^k h \) denotes \( k \)-th symmetric differential of \( h \), and \( n \) is the Hessian matrix of \( h \). This system is invariant under an 11-dimensional group of contact symmetries which consists of the 9-dimensional
group of Lie-point symmetries,

\[ v \rightarrow av + b, \]
\[ w \rightarrow pv + cw + d, \]
\[ h \rightarrow ah + \beta v + \gamma w + \delta, \]

along with the two purely contact infinitesimal generators,

\[-\Omega_h \frac{\partial}{\partial v} - \Omega_w \frac{\partial}{\partial w} + (\Omega - h_v \Omega_h - h_w \Omega_h) \frac{\partial}{\partial h} + (\Omega_v + h_v \Omega_h) \frac{\partial}{\partial h_v} + (\Omega_w + h_w \Omega_h) \frac{\partial}{\partial h_w}, \]

with the generating functions \( \Omega = h_v h_w \) and \( \Omega = h_w^2 \), respectively. To solve this system we apply the Legendre transformation,

\[ V = h_v, \ W = h_w, \ H = vh_v + wh_w - h, \ H_V = v, \ H_W = w. \]

The transformed equations (3.7) take the Godunov form (3.10),

\[(H_V)_t = W_x, \ (H_W)_t = V_x + W_y, \]

and the integrability conditions (3.20) simplify to

\[ H_{VVVV} = \frac{2H_{V^2}}{H_{VV}}, \]
\[ H_{VVVW} = \frac{2H_{VVW}H_{VVV}}{H_{VV}} , \]
\[ H_{VVWW} = \frac{2H_{V^2}W}{H_{VV}}, \]
\[ H_{VVWV} = \frac{3H_{VVW}H_{VVV} - H_{VV}H_{WV}}{H_{VV}}, \]
\[ H_{WVWW} = \frac{6H_{VVW}^2 - 4H_{WW}H_{VVW}}{H_{VV}}. \]
This system is also in involution, and can be represented in compact form as

\[ Sd^4H = d^3HdS - 6H_{VV}(dW)^2 \det(dN), \]

where \( S = H_{VV}^2 \), and \( N \) is the Hessian matrix of \( H \). The system (3.21) is invariant under an 11-parameter group of Lie-point symmetries,

\[
\begin{align*}
V &\rightarrow aV + bW + p, \\
W &\rightarrow cW + d, \\
H &\rightarrow \alpha H + \beta W^2 + \gamma VW + \epsilon V + \kappa W + \eta.
\end{align*}
\] (3.22)

Thus, both contact symmetry generators of the system (3.20) are mapped to point symmetries of the system (3.21). All our classification results will be performed modulo this equivalence.

### 3.3.1 Classification of integrable potentials

The main result of this subsection is a complete classification of solutions of the system (3.21):

**Theorem 10** Modulo the equivalence group (3.22), the generic integrable potential \( H(V,W) \) is given by the formula

\[
H = V \ln \frac{V}{\sigma(W)},
\] (3.23)

where \( \sigma \) is the Weierstrass sigma-function: \( \sigma'/\sigma = \zeta, \quad \zeta' = -\varphi, \quad \varphi'^2 = 4\varphi^3 - g_3 \). Its degenerations correspond to

\[
\begin{align*}
H &= V \ln \frac{V}{W}, \quad \text{(3.24)} \\
H &= V \ln V, \quad \text{(3.25)} \\
H &= \frac{V^2}{2W} + \alpha W^7, \quad \text{(3.26)}
\end{align*}
\]
as well as the following polynomial potentials:

\[
H = \frac{V^2}{2} + \frac{VW^2}{2} + \frac{W^4}{4},
\]

(3.27)

\[
H = \frac{V^2}{2} + \frac{W^3}{6}.
\]

(3.28)

**Proof:**

In the analysis of Eqs. (3.21) it is convenient to consider two cases, \( H_{VVV} \neq 0 \) and \( H_{VVV} = 0 \).

**Case 1:** \( H_{VVV} \neq 0 \). Then Eqs. (3.21)\(_1\) and (3.21)\(_2\) imply \( H_{VVV} = cH^2_{VV} \), \( c = \text{const} \), so that \( H_{VV} = -\frac{1}{cV + f(W)} \). The substitution into (3.21)\(_3\) implies that \( f(W) \) must be linear. Modulo the equivalence group we can thus assume that \( H_{VV} = \frac{1}{V} \), which gives

\[
H = V \ln(V) + g(W)V + h(W).
\]

The substitution of this ansatz into (3.21)\(_4\) gives \( h''' = 0 \), so that we can set \( h \) equal to zero modulo the equivalence group. Ultimately, the substitution into the last equation (3.21)\(_5\) gives

\[
g''' = 6g''^2.
\]

The general solution of this equation is \( g = -\ln \sigma(W) \), where \( \sigma \) is the Weierstrass sigma-function defined as \((\ln \sigma)' = \zeta, \, \zeta' = -\wp\) with \( \wp \) satisfying \( \wp^2 = 4\wp^3 - g_3 \) (notice that \( g_2 = 0 \)). Degenerations of this solution give \( g = -\ln W \) and \( g = 0 \). This leads to the three cases

\[
H = V \ln \frac{V}{\sigma(W)}, \quad H = V \ln \frac{V}{W}, \quad H = V \ln V.
\]
Case 2: $H_{VVV} = 0$. Then $H_{V} = a(W)$, and (3.21)$_3$ implies

$$a'' = 2\frac{a'^2}{a},$$

so that $a(W) = \frac{1}{pW + q}$. There are again two subcases to consider depending on whether $H_{VVW}$ is zero or non-zero. If $H_{VVW} \neq 0$ then $p \neq 0$ and, modulo the equivalence group, one can set $H_{V} = \frac{1}{W}$. Integrating twice with respect to $V$ we obtain

$$H = \frac{V^2}{2W} + b(W)V + c(W).$$

The substitution into (3.21)$_4$ gives $b'' = -3b''/W$. Modulo the equivalence group, one can set $b(W) = 1/W$. Thus we get

$$H = \frac{V^2}{2W} + \frac{V}{W} + c(W).$$

By a translation of $V$ one can remove the intermediate term $\frac{V}{W}$, leaving $H = \frac{V^2}{2W} + c(W)$. Then the substitution into (3.21)$_5$ gives

$$c''' - \frac{4c''}{w} = 0.$$

Modulo the equivalence group, this gives

$$H = \frac{V^2}{2W} + \alpha W^7.$$

In the second subcase, $H_{VVV} = H_{VWW} = 0$, one can normalize $H$ as $H = V^2/2 + a(W)V + b(W)$. With this ansatz Eqs. (3.21)$_1$ - (3.21)$_3$ are trivial, while the last two equations imply, modulo the equivalence group, that

$$H = \frac{V^2}{2} + \alpha VW^2 + \alpha^2 W^4 + \beta W^3.$$
The case $\alpha = 0$ leads to
\[ H = \frac{V^2}{2} + \frac{W^3}{6}. \]
The case $\alpha \neq 0$ leads, on appropriate rescalings, to
\[ H = \frac{V^2}{2} + \frac{VW^2}{2} + \frac{W^4}{4}. \]
This finishes the proof of Theorem 10.

3.3.2 Dispersionless Lax pairs

We recall that a quasilinear system (3.1) is said to possess a dispersionless Lax pair,
\[ S_x = F(u, S_t), \quad S_y = G(u, S_t), \tag{3.29} \]
if it can be recovered from the consistency condition $S_{xy} = S_{yx}$ (we point out that the dependence of $F$ and $G$ on $S_t$ is generally non-linear). Dispersionless Lax pairs first appeared in the construction of the universal Whitham hierarchy, see [41] and references therein. It was observed in [63] that Lax pairs of this kind naturally arise from the usual ‘solitonic’ Lax pairs in the dispersionless limit. It was demonstrated in [24, 25] that, for a number of particularly interesting classes of systems, the existence of a dispersionless Lax pair is equivalent to the existence of hydrodynamic reductions and, thus, to the integrability. Lax pairs form a basis of the dispersionless $\bar{\partial}$-dressing method [6] and a novel version of the inverse scattering transform associated with parameter-dependent vector fields [42]. In this section we calculate dispersionless Lax pairs,
\[ S_x = F(V, W, S_t), \quad S_y = G(V, W, S_t), \]
for all integrable potentials appearing in Theorem 10. Thus, we require that the consistency condition $S_{xy} = S_{yx}$ results in the corresponding equations
We will demonstrate the standard technique on the case \( H = V^2/2 + W^3/6 \). For all other cases we will only present the final results. Let us point out that in all examples discussed below the first equation of the Lax pair does not explicitly depend on \( V \). The general class of Lax pairs of this kind was discussed in [47].

**Case (3.28):** \( H = \frac{V^2}{2} + \frac{W^3}{6} \). The equations (3.10) take the form

\[
\begin{align*}
V_t &= W_x, \\
WW_t &= V_x + W_y.
\end{align*}
\]

(3.30)

This system has dispersionless Lax pairs of the form

\[
\begin{align*}
S_x &= F(V, W, S_t), \\
S_y &= G(V, W, S_t)
\end{align*}
\]

(3.31)

and the consistency condition \( S_{xy} = S_{yx} \) is

\[
\begin{align*}
F_V V_y + F_W W_y + F_{\xi} G_V V_t + F_{\xi} G_W W_t &= \\
G_V V_x + G_W W_x + G_{\xi} F_V V_t + G_{\xi} F_W W_t,
\end{align*}
\]

(3.32)

where \( S_t = \xi \). We now substitute \( V_t \) and \( V_x \) from (3.30), and comparing coefficients in front of \( V_y, W_x, W_y, W_t \) we obtain the system

\[
\begin{align*}
F_V &= 0, \\
F_W &= -G_V, \\
F_{\xi} G_V &= G_W, \\
F_{\xi} G_W &= G_{\xi} F_W + W G_V.
\end{align*}
\]

(3.33)

From (3.33)_1 and (3.33)_2 we find \( G = A(\xi)V + B(W, \xi), F_W = -A(\xi) \). Sub-
stituting these into (3.33)₃ provides

\[ F_{\xi}A = (A_W V + B_W). \]

From this it directly follows that \( A_W = 0 \), so we can put \( A = a_{\xi} \), as well as \( F = -a(\xi)W - c(\xi) \). Also we find \( B(W, \xi) = a(-a'W + c) \). Next we substitute these in (3.33)₄ to find

\[ (-a'W + c')² = W - (a'V + B_{\xi}). \]

This means that \( a' = 0 \), and without loss of generality we set \( a = 1 \). Also we have

\[ B_{\xi} = W - c'^{2}. \]

Then the consistency condition \( B_{\xi}W = B_{W\xi} \) results in \( c'' = 1 \). Finally

\[ F = -W + \frac{\xi^2}{2}, \]
\[ G = V + \xi W - \frac{\xi^3}{3}, \]

Hence the corresponding dispersionless Lax pair is

\[ S_x = \frac{S_1^2}{2} - W, \]
\[ S_y = -\frac{S_3^3}{3} + S_1 W + V. \]

**Case (3.27):** \( H = \frac{V^2}{2} + \frac{W^2}{2} + \frac{W^4}{4} \). The equations (3.10) take the form

\[ V_t + W W_t = W_x, \]
\[ V W_t + W V_t + 3W^2 W_t = V_x + W_y. \]
The corresponding dispersionless Lax pair is
\[
S_x = \frac{S_t^4}{4} - S_t W,
S_y = -\frac{S_t^7}{7} + W S_t^4 + S_t (V - W^2).
\]

**Case (3.26):** \(H = \frac{V^2}{2W}, \alpha = 0\). The equations (3.10) take the form
\[
\frac{V}{W} - \frac{V W_t}{W^2} = W_x,
-\frac{V V_t}{W^2} + \frac{V^2 W_t}{W^3} = V_x + W_y.
\]

The corresponding dispersionless Lax pair is
\[
S_x = -\frac{W^2}{2S_t^2},
S_y = \frac{V W}{S_t^2} + \frac{W^5}{5S_t^5}.
\]

**Case (3.26):** \(H = \frac{V^2}{2W} + \alpha W^7\). Without any loss of generality we will set \(\alpha = 1/168\). The equations (3.10) take the form
\[
\frac{V}{W} - \frac{V W_t}{W^2} = W_x,
-\frac{V V_t}{W^2} + \frac{V^2 W_t}{W^3} + \frac{1}{4} W^5 W_t = V_x + W_y.
\]

The corresponding dispersionless Lax pair is
\[
S_x = -\frac{a W^2}{2},
S_y = a V W - \frac{aa' W^5}{10},
\]
here \(a = a(S_t)\) satisfies the ODE \(3a'^2 - 5 = 2aa''\). When we integrate once we
obtain $a'^2 = \lambda a^3 + 5/3$. We can then take $\lambda = 4$ to get $a(S_t) = \varphi(S_t, 0, -5/3)$

**Case (3.25):** $H = V \log V$. The equations (3.10) take the form

\[
\begin{align*}
V_t &= VW_x, \\
0 &= V_x + W_y.
\end{align*}
\]

The corresponding dispersionless Lax pair is

\[
\begin{align*}
S_x &= -\log(W + S_t), \\
S_y &= \frac{V}{W + S_t}.
\end{align*}
\]

**Case (3.24):** $H = V \log \frac{V}{W}$. The equations (3.10) take the form

\[
\begin{align*}
\frac{V_t}{V} - \frac{W_t}{W} &= W_x, \\
\frac{VW_t}{W^2} - \frac{V_t}{W} &= V_x + W_y.
\end{align*}
\]

The corresponding dispersionless Lax pair is

\[
\begin{align*}
S_x &= -\ln(S_t + W) - \epsilon \ln(S_t + \epsilon W) - \epsilon \ln(S_t + \epsilon^2 W), \quad \epsilon = e^{\frac{2\pi i}{3}}, \\
S_y &= -3 \frac{VWS_t}{W^3 + S_t^3}.
\end{align*}
\]

**Case (3.23):** $H = V \log \frac{V}{\sigma'(W)}$. We recall that $(\log \sigma)' = \zeta$ and $\zeta' = -\varphi$

where $\varphi = \varphi(w, 0, g_3)$ is the Weierstrass $\varphi$-function, $\varphi'^2 = 4\varphi^3 - g_3$. The equations (3.10) take the form

\[
\begin{align*}
\frac{1}{V} V_t - \zeta(W) W_t &= W_x, \\
-\zeta(W) V_t + V \varphi(W) W_t &= V_x + W_y.
\end{align*}
\]
The corresponding dispersionless Lax pair is of the form

\[ S_x = F(W, S_t), \]
\[ S_y = VA(W, S_t), \]

where the functions \( F \) and \( A \) satisfy the equations

\[ F_W = -A, \quad F_\xi = A_W \frac{A}{A} - \zeta(W) \]

and

\[ A_\xi = \wp(W) - A_W^2 \frac{A^2}{A^2}, \quad A_{WW} = 2A_W^2 \frac{A^2}{A} - 2A\wp(W), \]

respectively (here \( \xi = S_t \)). Setting \( A = 1/u \) one can rewrite the last two equations for \( A \) in the equivalent form,

\[ u_\xi = u_W W_2 - \wp(W) u^2, \quad u_{WW} = 2\wp(W) u. \quad (3.34) \]

Here the second equation is a particular case of the Lame equation. It has two linearly independent solutions \([3]\),

\[ e^{-W_\zeta(\alpha)} \frac{\sigma(W + \alpha)}{\sigma(W)\sigma(\alpha)}, \quad e^{W_\zeta(\alpha)} \frac{\sigma(W - \alpha)}{\sigma(W)\sigma(\alpha)}; \]

where \( \alpha \) is the zero of the \( \wp \)-function: \( \wp(\pm \alpha) = 0 \). Thus, one can set

\[ u = -e^{-W_\zeta(\alpha)} \frac{\sigma(W + \alpha)}{\sigma(W)\sigma(\alpha)} a(\xi) + e^{W_\zeta(\alpha)} \frac{\sigma(W - \alpha)}{\sigma(W)\sigma(\alpha)} b(\xi). \]

The substitution into \((3.34)_1\) implies a pair of ODEs for \( a(\xi) \) and \( b(\xi) \),

\[ a' = \wp'(\alpha)b^2, \quad b' = -\wp'(\alpha)a^2. \]
This system can be solved in the form
\[ a(\xi) = \tau \varphi'(\xi, 0, g_3) - 1, \quad b(\xi) = c \varphi(\xi, 0, g_3) \]
where the constants \( c, \tau \) and \( \gamma \) are defined as
\[ \tau = \frac{1}{\sqrt{3}} g_3^{-1/2}, \quad \gamma = 2 g_3^{-1/3}, \quad c = \frac{\sqrt{3}}{\varphi'(\alpha) g_3^{1/6}}. \]
Setting \( g_3 = 1, \varphi'(\alpha) = i \) one obtains
\[ a(\xi) = -i \sqrt{3} \frac{\varphi'(\xi, 0, 1) - \sqrt{3}}{\varphi'(\xi, 0, 1) + \sqrt{3}}, \quad b(\xi) = -i \frac{6 \varphi(\xi, 0, 1)}{\varphi'(\xi, 0, 1) + \sqrt{3}}. \]
Ultimately,
\[ u = i \sqrt{3} e^{-W \zeta(\alpha)} \frac{\sigma(W + \alpha)}{\sigma(W) \sigma(\alpha)} \frac{\varphi'(\xi) - \sqrt{3}}{\varphi'(\xi) + \sqrt{3}} - 6 i e^{W \zeta(\alpha)} \frac{\sigma(W - \alpha)}{\sigma(W) \sigma(\alpha)} \frac{\varphi(\xi)}{\varphi'(\xi) + \sqrt{3}}. \]
The functions \( A \) and \( F \) can be reconstructed via \( A = 1/u \),
\[ F_W = -1/u, \quad F_\xi = -\frac{u_W}{u} - \zeta(W). \]

### 3.4 Hamiltonian systems of type III

In this section we consider Hamiltonian systems of the form III,
\[ \left( \begin{array}{c} v \\ w \end{array} \right)_t = \left[ \left( \begin{array}{cc} 2v & w \\ w & 0 \end{array} \right) \frac{d}{dx} + \left( \begin{array}{cc} 0 & v \\ v & 2w \end{array} \right) \frac{d}{dy} + \left( \begin{array}{cc} v_x & v_y \\ w_x & w_y \end{array} \right) \right] \left( \begin{array}{c} h_v \\ h_w \end{array} \right), \]
or, explicitly,
\[ v_t = (2vh_v + wh_w - h)_x + (vh_w)_y, \quad w_t = (wh_v)_x + (2wh_w + vh_v - h)_y. \]
The integrability conditions constitute a system of fourth order PDEs for the Hamiltonian density \( h(v, w) \) which is not presented here due to its complexity. We have verified that this system is in involution, and its solution space is 10-dimensional. It is invariant under an 8-dimensional group of Lie-point symmetries,

\[
v \to av + bw, \\
w \to cv + dw, \\
h \to ah + \beta v + \gamma w + \delta,
\]

along with the two purely contact infinitesimal generators,

\[
-\Omega_{hv} \frac{\partial}{\partial v} - \Omega_{hw} \frac{\partial}{\partial w} + (\Omega - h_v \Omega_{hv} - h_w \Omega_{hw}) \frac{\partial}{\partial h} + (\Omega_v + h_v \Omega_h) \frac{\partial}{\partial h_v} + (\Omega_w + h_w \Omega_h) \frac{\partial}{\partial h_w},
\]

with the generating functions \( \Omega = h_v (vh_v + wh_w) \) and \( \Omega = h_w (vh_v + wh_w) \), respectively. As in the previous two cases, the integrability conditions for the Hamiltonian density \( h(v, w) \) simplify under the Legendre transformation,

\[
V = h_v, \quad W = h_w, \quad H = vh_v + wh_w - h, \quad H_V = v, \quad H_W = w,
\]

which brings the corresponding equations (3.8) into the Godunov form (3.11),

\[
(H_V)_t = (VH_V + H)_x + (WH_V)_y, \quad (H_W)_t = (VH_W)_x + (WH_W + H)_y.
\]

**Remark.** Equations (3.11) coincide with the so-called EPDiff equations [39],

\[
m_t - u \times \text{curl } m + \nabla(u, m) + m \text{ div } u = 0,
\]

where \( u = -(V, W, 0) \) and \( m = (H_V, H_W, 0) \).
The integrability conditions for the system (3.11) take the form

\[
SH_{VVVV} = 2(H_W H_{VVV} - H_V H_{VWV})^2 \\
+ 4H_V^2(H_{VW}^2 - H_{VVV} H_{VWW}) \\
+ 12H_W H_V^2 H_{VVW} - 12H_W H_V H_{VW} H_{VVV} \\
- 24H_V H_{VV} H_{VW} H_{VWW} + 8H_V H_{VV} H_{VWV} + 6H_V H_V^2 H_{VWW} \\
+ 10H_V H_{VV} H_{WWW} H_{VVV} + 12H_V^2(H_{VV}^2 - H_{VW} H_{WWW}),
\]

(3.35)

\[
SH_{VW} = H_{VVV}(2H_W^2 H_{VWV} - 4H_V H_W H_{VWV} - H_V^2 H_{WWW}) \\
+ 3H_V^2 H_{VVW} H_{VWW} - 3H_W H_{VV} H_{VW} H_{VVV} \\
- 4H_W H_V^2 H_{VVV} - 6H_V H_W^2 H_{VWW} + \frac{15}{2}H_W H_V^2 H_{VWV} \\
- 6H_V H_{VV} H_{VW} H_{VWW} + \frac{3}{2}H_V H_V^2 H_{WWW} \\
- \frac{1}{2}H_W H_{VV} H_{WWW} H_{VVV} + \frac{3}{2}H_V H_{VV} H_{WWW} H_{VWV} \\
+ 9H_V H_{VW} H_{WWW} H_{VVV} + 12H_V H_{VW}(H_{VW}^2 - H_{VV} H_{WWW}),
\]

\[
SH_{VVVV} = 2(H_V H_{VWV} - H_W H_{VVV})^2 \\
+ 2H_V H_W(H_{VVW} H_{VWV} - H_{VVV} H_{WWW}) - 8H_V^2(H_W H_{VVW} + H_V H_{VWW}) \\
+ 6H_W H_{VV} H_{VW} H_{VWW} + 6H_V H_{VW} H_{WWW} H_{VVV} \\
- H_{VV} H_{WWW}(H_W H_{VWV} + H_V H_{WWW}) + 3H_W H_V^2 H_{WWW} \\
+ 3H_V H_W^2 H_{VWW} + 4(2H_V^2 + H_{VV} H_{WWW})(H_{VV}^2 - H_{VW} H_{WWW}),
\]

\[
SH_{WWW} = H_{WWW}(2H_W^2 H_{VWW} - 4H_V H_W H_{VWW} - H_V^2 H_{VWW}) \\
+ 3H_W^2 H_{VWW} H_{VWW} - 3H_V H_{WWW} H_{VW} H_{WWW} \\
- 4H_V H_W^2 H_{WWW} - 6H_W H_V^2 H_{WVV} + \frac{15}{2}H_V H_W^2 H_{WWW} \\
- 6H_W H_{WWW} H_{VVW} H_{VWW} + \frac{3}{2}H_W H_V^2 H_{WWW} H_{VWW} - \frac{1}{2}H_V H_{WWW} H_{WWW} H_{WWW} \\
+ \frac{3}{2}H_W H_{VV} H_{WWW} H_{WWW} + 9H_W H_{VWW} H_{WWW} H_{WWW} \\
+ 12H_W H_{WWW}(H_{VW}^2 - H_{VW} H_{WWW}),
\]
\[ \begin{align*} 
SH_{WWW} &= 2(H_V H_{WWW} - H_W H_{VWW})^2 \\
&+ 4H_W^2(H^2_{WWW} - H_{WWW} H_{WWW}) + 12H_V H_W H_{WWW} \\
&- 12H_V H_{WWW} H_{VWW} - 24H_W H_{WWW} H_{VWW} \\
&+ 8H_W H_V^2 H_{WWW} + 6H_W H^2_{WWW} H_{VWW} \\
&+ 10H_W H_{WWW} H_{VWW} + 12H^2_{WWW}(H^2_{VW} - H_{VWW} H_{WWW}), \\
\end{align*} \]

where
\[ S = H_W^2 H_{VV} - 2H_V H_W H_{VW} + H_V^2 H_{WW}. \]

This system is manifestly symmetric under the interchange of \( V \) and \( W \), and can be represented in compact form as
\[ S d^4 H = 2d^3 H dS - 6(dH)^2 \det(dN) + 6dH d^2 H d(\det N) + \\
12(H_V dH_W - H_W dH_V)(d^2 H_V dH_W - d^2 H_W dH_V) - \\
12 \det N(d^2 H)^2, \quad (3.36) \]

where \( N \) is the Hessian matrix of \( H \). We have verified that the system (3.36) is in involution, and is invariant under a 10-dimensional group of Lie-point symmetries generated by projective transformations of \( V \) and \( W \) along with affine transformations of \( H \),
\[ \begin{align*} 
V &\rightarrow \frac{aV + bW + c}{pV + qW + r}, \quad W \rightarrow \frac{\alpha V + \beta W + \gamma}{pV + qW + r}, \quad H \rightarrow \mu H + \nu. 
\end{align*} \]
The corresponding infinitesimal generators are:

2 translations: \( \frac{\partial}{\partial V}, \frac{\partial}{\partial W} \);

4 linear transformations: \( V \frac{\partial}{\partial V}, W \frac{\partial}{\partial V}, V \frac{\partial}{\partial W}, W \frac{\partial}{\partial W} \);

2 projective transformations: \( V^2 \frac{\partial}{\partial V} + VW \frac{\partial}{\partial W}, VW \frac{\partial}{\partial V} + W^2 \frac{\partial}{\partial W} \);

2 affine transformations of \( H \): \( \frac{\partial}{\partial H}, H \frac{\partial}{\partial H} \).

(3.37)

Projective transformations (3.37) constitute the group of ‘canonical’ transformations of equations (3.11); combined with appropriate linear changes of the independent variables \( x, y, t \), they leave equations (3.11) form-invariant. Let us consider, for instance, the projective transformation \( \tilde{V} = 1/V, \tilde{W} = W/V \).

A direct calculation shows that the transformed equations take the form

\[
(\tilde{V} H_{\tilde{V}} + H)_t = (H_{\tilde{V}})_x + (\tilde{W} H_{\tilde{V}})_y, \quad (\tilde{V} H_{\tilde{W}})_t = (H_{\tilde{W}})_x + (\tilde{W} H_{\tilde{W}} + H)_y.
\]

They assume the original form (3.11) on the identification \( t \rightarrow x, x \rightarrow t, y \rightarrow -y \). In other words, the projective invariance of the integrability conditions is a manifestation of the invariance of the Hamiltonian formalism (3.8) under arbitrary linear transformations of the independent variables. This is analogous to the well-known invariance of Hamiltonian structures of hydrodynamic type in 1 + 1 dimensions under linear transformations of \( x \) and \( t \) [53]. We emphasise that this invariance is not present in the case of constant coefficient Poisson brackets.
### 3.4.1 Classification of integrable potentials

The analysis of this section is somewhat similar to the classification of integrable Lagrangians of the form $\int u_t g(u_x, u_y) \, dx \, dy \, dt$ proposed in [30]. This suggests that there may be a closer link between these two classes of equations. The main result of this section is the following

**Theorem 11** The generic integrable potential of type (3.11) is given by the series

$$H = \frac{1}{W g(V)} \left(1 + \frac{1}{g_1(V) W^6} + \frac{1}{g_2(V) W^{12}} + \ldots\right).$$

Here $g(V)$ satisfies the fourth order ODE,

$$g'''(2g g'^2 - g^3 g'') + 2g^2 g''' - 20g' g'' g''' + 16g'^3 g'' + 18g' g'^3 - 18g'^2 g''^2 = 0,$$

whose general solution can be represented in parametric form as

$$g = w_2(t), \quad V = \frac{w_1(t)}{w_2(t)},$$

where $w_1(t)$ and $w_2(t)$ are two linearly independent solutions of the hypergeometric equation $t(1-t) d^2 w/dt^2 - \frac{2}{9} w = 0$. The coefficients $g_i(V)$ are certain explicit expressions in terms of $g(V)$, e.g., $g_1(V) = gg'' - 2g'^2$ and so on. Degenerations of this solution correspond to

$$H = \frac{1}{W g(V)}, \quad H = \frac{1}{W V}, \quad H = V - W \log W, \quad H = V - W^2/2.$$

In the rest of this section we provide details of the classification, and discuss various properties and representations of the generic solution. The system (3.36) for $H(V, W)$ is not straightforward to solve explicitly. We will start with the investigation of special solutions which are invariant under various one-parameter subgroups of the equivalence group. In this case the system of integrability conditions for $H(V, W)$ reduces to ODEs which are easier
to solve. Up to conjugation and normalisation, there exist four essentially
different one-parameter subgroups of the projective group $SL(3)$, with the
infinitesimal generators
\[ \alpha V \frac{\partial}{\partial V} + W \frac{\partial}{\partial W}, \quad V \frac{\partial}{\partial V} + \frac{\partial}{\partial W}, \quad W \frac{\partial}{\partial V} + \frac{\partial}{\partial W}, \quad \frac{\partial}{\partial V}. \]

Combined with the operators $\partial/\partial H$, $H\partial/\partial H$ this leads to the following
list of eleven essentially different ‘ansatzes’ governing invariant solutions (in
what follows we do not consider degenerate solutions for which the expression
$S = H_V^2 H_{VV} - 2H_V H_W H_{VW} + H_W^2 H_{WW}$ equals zero):

**Case 1.** Solutions invariant under the operator $\partial/\partial V + \partial/\partial H$ are described
by the ansatz $H = V + F(W)$. The integrability conditions imply $F''F'''' - 2F'''^2 = 0$. Modulo the equivalence transformations this leads to integrable
potentials $H = V - W^2/2$ and $H = V - W \log W$, which constitute the last
two cases of Theorem 11. Applying to the first potential transformations
from the equivalence group we obtain integrable potentials of the form
\[
H(V,W) = \frac{Q(V,W)}{l^2(V,W)}
\]
where $Q$ and $l$ are quadratic and linear forms, respectively (not necessarily
homogeneous). The integrability implies that the line $l = 0$ is tangential to
the conic $Q = 0$ on the $V,W$ plane. Any such potential can be reduced to
the form $H = V - W^2/2$ by a projective transformation which sends $l$ to the
line at infinity.

**Case 2.** Solutions invariant under the operator $\partial/\partial W + H\partial/\partial H$ are
described by the ansatz $H = e^W F(V)$. This case gives no non-trivial solutions.

**Case 3.** Solutions invariant under the operator $W\partial/\partial V + \partial/\partial W$ are
described by the ansatz $H = F(V - W^2/2)$. A simple analysis leads to the
only polynomial potential $H = V - W^2/2$, the same as in Case 1.

**Case 4.** Solutions invariant under the operator $W\partial/\partial V + \partial/\partial W + \partial/\partial H$
are described by the ansatz $H = W + F(V - W^2/2)$. In this case $F$ turns
out to be linear so that, modulo translations in $W$, we again arrive at the same potential as in Case 1.

**Case 5.** Solutions invariant under the operator $W \partial/\partial V + \partial/\partial W + H \partial/\partial H$ are described by the ansatz $H = e^W F(V - W^2/2)$. A detailed analysis shows that this case gives no non-trivial solutions.

**Case 6.** Solutions invariant under the operator $V \partial/\partial V + \partial/\partial W$ are described by the ansatz $H = F(W - \ln V)$. This case gives no non-trivial solutions.

**Case 7.** Solutions invariant under the operator $V \partial/\partial V + \partial/\partial W + \partial/\partial H$ are described by the ansatz $H = W + F(W - \ln V)$. This case gives no non-trivial solutions.

**Case 8.** Solutions invariant under the operator $V \partial/\partial V + \partial/\partial W + \mu H \partial/\partial H$ are described by the ansatz $H = e^{\mu W} F(W - \ln V)$, also no non-trivial solutions.

**Case 9.** Solutions invariant under the operator $\alpha V \partial/\partial V + W \partial/\partial W$ are described by the ansatz $H = F(W^\alpha/V)$. Here one can assume $\alpha \neq 0, 1$. A straightforward substitution implies that, without any loss of generality, one can assume $F$ to be linear, while the parameter $a$ can only take two values: $a = 2$ or $a = -1$. This results in the two rational potentials $H = W^2/V$ and $H = 1/VW$. Notice that they are related by the projective transformation $W \to 1/W$, $V \to V/W$.

**Case 10.** Solutions invariant under the operator $\alpha V \partial/\partial V + W \partial/\partial W + \partial/\partial H$ are described by the ansatz $H = \ln W + F(W^\alpha/V)$. This case gives no non-trivial solutions.

**Case 11.** Solutions invariant under the operator $\alpha V \partial/\partial V + W \partial/\partial W + \mu H \partial/\partial H$ are described by the ansatz $H = W^\mu F(W^\alpha/V)$. A detailed analysis shows that, modulo the equivalence group and the solutions already discussed above, the only essentially new possibility corresponds to the ansatz $H = \ldots$
It leads to the fourth order ODE for $g = g(z)$,

$$
g''''(2gg'' - g^2g'') + 2g^2g''^2 - 20gg'g''' + 16g^3g''' + 18gg''^2 - 18g'^2g''^2 = 0, \tag{3.38}
$$

which possesses a remarkable $SL(2, R)$-invariance inherited from (3.37):

$$
\tilde{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \tilde{g} = (\gamma z + \delta)g; \tag{3.39}
$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants such that $\alpha\delta - \beta\gamma = 1$. Moreover, there is an obvious scaling symmetry $g \rightarrow \lambda g$. The equation (3.38) can be linearised as follows. Introducing $h = g'/g$, which means factoring out the scaling symmetry, we first rewrite it in the form

$$
h''''(h' - h^2) - 2hh'' - 4h'^2h'' - 15h'^3 + 9h'h'^2 - 3h^4h' + h^6 = 0; \tag{3.40}
$$

the corresponding symmetry group modifies to

$$
\tilde{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \tilde{h} = (\gamma z + \delta)^2 h + \gamma(\gamma z + \delta). \tag{3.41}
$$

We point out that the same symmetry occurs in the case of the Chazy equation, see [1], p. 342. The presence of the $SL(2, R)$-symmetry of this type implies the linearisability of the equation under study. One can formulate the following general statement which is, in fact, contained in [10].

**Proposition 1.** Any third order ODE of the form $F(z, h, h', h'', h''') = 0$, which is invariant under the action of $SL(2, R)$ as specified by (3.41), can be linearised by a substitution

$$
h = \frac{d}{dz} \ln w_2, \quad z = \frac{w_1}{w_2} \tag{3.42}
$$

where $w_1(t)$ and $w_2(t)$ are two linearly independent solutions of a linear equa-
tion $d^2w/dt^2 = V(t)w$ with the Wronskian $W$ normalised as $W = w_2dw_1/dt - w_1dw_2/dt = 1$. The potential $V(t)$ depends on the given third order ODE, and can be efficiently reconstructed.

In particular, the general solution of the equation (3.40) is given by parametric formulae (3.42) where $w_1(t)$ and $w_2(t)$ are two linearly independent solutions of the hypergeometric equation $d^2w/dt^2 = \frac{2}{9t(1-t)}w$ with $W = 1$.

Proof:

Our presentation follows [30]. Let us consider a linear ODE $d^2w/dt^2 = V(t)w$, take two linearly independent solutions $w_1(t), w_2(t)$ with the Wronskian $W = 1$, and introduce new dependent and independent variables $h, z$ by parametric relations

$$h = \frac{d}{dz} \ln w_2, \quad z = \frac{w_1}{w_2}.$$

Using the formulae $dt/dz = w_2^2$ and $h = w_2dw_2/dt$, one obtains the identities

$$h' - h^2 = w_2^4V,$$

$$h'' - 6hh' + 4h^3 = w_2^6 \frac{dV}{dt},$$

$$h''' - 12hh'' - 6(h')^2 + 48h^2h' - 24h^4 = w_2^8 \frac{d^2V}{dt^2},$$

where prime denotes differentiation with respect to $z$. Thus, one arrives at the relations

$$I_1 = \frac{(h'' - 6hh' + 4h^3)^2}{(h' - h^2)^3} = \frac{(dV/dt)^2}{V^3},$$

$$I_2 = \frac{h''' - 12hh'' - 6(h')^2 + 48h^2h' - 24h^4}{(h' - h^2)^2} = \frac{d^2V/dt^2}{V^2}.$$

We point out that $I_1$ and $I_2$ are the simplest second and third order differential invariants of the action (3.41) whose infinitesimal generators, prolonged
to the third order jets \( z, h, h', h'', h''' \), are of the form

\[
X_1 = \partial_z, \quad X_2 = z\partial_z - h\partial_h - 2h'\partial_{h'} - 3h''\partial_{h''} - 4h'''\partial_{h'''};
\]

\[
X_3 = z^2\partial_z - (2zh + 1)\partial_h - (2h + 4zh')\partial_{h'}
- (6h' + 6zh'')\partial_{h''} - (12h'' + 8zh''')\partial_{h'''};
\]

notice the standard commutation relations

\[
[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.
\]

One can verify that the Lie derivatives of \( I_1, I_2 \) with respect to \( X_1, X_2, X_3 \) are indeed zero. Thus, any third order ODE which is invariant under the \( SL(2, R) \)-action (3.41), can be represented in the form \( I_2 = F(I_1) \) where \( F \) is an arbitrary function of one variable. The corresponding potential \( V(t) \) has to satisfy the equation

\[
\frac{d^2V}{dt^2} = F \left( \frac{(dV/dt)^2}{V^3} \right).
\]

This simple scheme produces many the well-known equations, for instance, the relation \( I_2 = -24 \) implies the Chazy equation for \( h \), that is, \( h''' - 12hh'' + 18(h')^2 = 0 \). The corresponding potential satisfies the equation \( d^2V/dt^2 = -24V^2 \).

Similarly, the choice \( I_2 = I_1 - 8 \) results in the ODE \( h''' = 4hh'' - 2(h')^2 + \frac{(h'' - 2hh')^2}{h' - h^2} \) which, under the substitution \( h = y/2 \), coincides with the equation (4.7) from [2]. The potential \( V \) satisfies the equation \( Vd^2V/dt^2 = (dV/dt)^2 - 8V^3 \).

The relation \( I_2 = I_1 - 9 \) gives \( h'''(h' - h^2) = h^6 - 3h^4h' + 9h^2(h')^2 - 3(h')^3 - 4h^3h'' + (h'')^2 \). This equation appeared in [30] in the context of first order integrable Lagrangians. The corresponding potential \( V \) satisfies the equation \( Vd^2V/dt^2 = (dV/dt)^2 - 9V^3 \).

Finally, the relation \( I_2 = 2I_1 + 9 \) coincides with the equation (3.40). The
corresponding potential $V$ satisfies the equation $V d^2V/dt^2 = 2(dV/dt)^2 + 9V^3$. It remains to point out that the general solution to the last equation for $V$ is given by $V = -\frac{2}{9} t^2 + \frac{1}{t^2} + a t + b$. Without any loss of generality one can set $V = \frac{2}{9} \frac{1}{t(1-t)}$. It this case the linear equation $d^2w/dt^2 = V(t)w$ takes the hypergeometric form corresponding to the parameter values $a = -\frac{1}{3}$, $b = -\frac{2}{3}$, $c = 0$: $(1 - t)d^2w/dt^2 - \frac{2}{9} w = 0$. This finishes the proof of proposition 1.

As $h = g'/g$, this immediately implies the following formula for the general solution of (3.38):

**Proposition 2.** The general solution of the equation (3.38) is given by parametric formulae

$$g = w_2, \quad z = \frac{w_1}{w_2},$$

where $w_1$ and $w_2$ are two linearly independent solutions to the hypergeometric equation $t(1-t)d^2w/dt^2 - \frac{2}{9} w = 0$.

**Remark.** Expansions at zero give

$$w_1 = w_2 \ln t + \frac{9}{2} + \frac{t^2}{3} + \frac{211t^3}{1458} + ..., \quad w_2 = t + \frac{t^2}{9} + \frac{10t^3}{243} + ..., $$

so that

$$g = w_2 = t + \frac{t^2}{9} + \frac{10t^3}{243} + ..., \quad z = \frac{w_1}{w_2} = \ln t + \frac{9}{2t} - \frac{1}{2} + \frac{11t}{54} + \frac{34t^2}{729} + \frac{715t^3}{39366} + ....$$

Solving the first relation for $t$ in terms of $g$ and substituting into the second, one gets an implicit relation connecting $g$ and $z$,

$$z = \ln g - \frac{9}{g} + \frac{1}{2} + \frac{112g}{27} + \frac{289g^2}{486} + \frac{4381g^3}{39366} + ....$$

Similarly, expansions at infinity give
\[ w_1 = t^{2/3} \left( 1 - \frac{1}{3t} - \frac{2}{45t^2} - \ldots \right), \quad w_2 = t^{1/3} \left( 1 - \frac{1}{6t} - \frac{5}{126t^2} - \ldots \right), \]

\[ z = \frac{w_1}{w_2} = t^{1/3} \left( 1 - \frac{1}{6t} - \frac{41}{1260t^2} - \ldots \right), \]

so that \( g = w_2 \) can be represented explicitly as

\[ g(z) = z \left( 1 - \frac{a}{6} - \frac{1165}{143} \frac{a^2}{z^6} - \frac{5280035}{46189} \frac{a^3}{z^{12}} - \ldots \right), \]

here \( a = 1/140 \). For other values of \( a \) this formula represents the general solution to (3.38) in the form \( g = z(1 - a/z^6 - b/z^{12} - c/z^{18} - \ldots) \). The corresponding potential \( H(V, W) \) takes the form

\[ H(V, W) = \frac{1}{Wg(V)} = \frac{1}{VW} \left( 1 + \frac{a}{V^6} + \frac{1308}{143} \frac{a^2}{V^{12}} + \ldots \right). \]

It can be viewed as a perturbation of the potential \( H = \frac{1}{VW} \) constructed before. Computer experiments show that the ‘generic’ integrable potential \( H(V, W) \) can be represented as

\[ H = \sum_{j,k \geq 0} \frac{a_{jk}}{V^{6j+1}W^{6k+1}} = \frac{1}{VW} \left( 1 + \frac{1}{V^6} + \frac{1}{W^6} - \frac{24}{V^6W^6} + \frac{1308}{143V^{12}} + \frac{1308}{143W^{12}} + \ldots \right). \]

An alternative representation of this solution,

\[ H = \frac{1}{Wg(V)} \left( 1 + \frac{1}{g_1(V)V^6} + \frac{1}{g_2(V)W^{12}} + \ldots \right), \]

can be obtained by rearranging terms in the above sum. The substitution of this expression into the integrability conditions implies that \( g(V) \) has to
satisfy the fourth order ODE (3.38), \( g_1(V) \) is expressed in terms of \( g(V) \) via the Rankin-Cohen-type operation, \( g_1(V) = gg'' - 2g^2 \) (recall that, due to (3.39), \( g(V) \) transforms as a modular form of weight 1), and so on. In Sect. 3.5 we provide a parametrisation of the generic solution by generalised hypergeometric functions. This finishes the proof of Theorem 11.

3.4.2 Dispersionless Lax pairs

Here we present Lax pairs for some of the simplest potentials found in the previous section.

**Case \( H = V - W^2/2 \).** The corresponding system (3.11) takes the form

\[
W_y = WW_x - 2V_x, \quad W_t = VW_x - 5WV_x + 3W^2W_x - V_y,
\]

and possesses the Lax pair

\[
S_y = -WS_x + \frac{1}{S_x}, \quad S_t = (V - W^2)S_x + \frac{W}{S_x^2} - \frac{2}{5S_x^5}.
\]

**Case \( H = V - W \log W \).** The corresponding system (3.11) takes the form

\[
W_y = (W \ln W - 2V)_x, \quad W_t/W = (V \ln W + V)_x + (2W \ln W + W - V)_y,
\]

and possesses the Lax pair

\[
S_y = -S_x(\ln W + 1) + p(S_x), \quad S_t = S_x(V - W \ln W - W) - \frac{1}{2}S_xp'(S_x)W
\]

where \( p(z) \) satisfies the ODE \( zp'' + p'^2 + 3p' = 0 \). This gives \( p' = \frac{3}{z^3 - 1} \) so that

\[
p(S_x) = \ln(S_x - 1) + \epsilon \ln(S_x - \epsilon) + \epsilon^2 \ln(S_x - \epsilon^2), \quad \epsilon = e^{\frac{2\pi}{3}}. \quad (3.43)
\]
Case $H = \frac{1}{WV}$. The corresponding system (3.11) takes the form

\[
\left( \frac{1}{V^2 W} \right)_t = \left( \frac{1}{V^2} \right)_y, \quad \left( \frac{1}{V W^2} \right)_t = \left( \frac{1}{W^2} \right)_x,
\]

and possesses the Lax pair

\[
S_x = a(S_t)/V, \quad S_y = b(S_t)/W
\]

where the functions $a(z)$ and $b(z)$, $z = S_t$, satisfy a pair of ODEs

\[
a' = 1 - \frac{a}{2b}, \quad b' = 1 - \frac{b}{2a}.
\]

These equations can be solved in parametric form as

\[
a(z) = \frac{\varphi'(p) + \lambda}{2\varphi(p)}, \quad b(z) = \frac{\varphi'(p) - \lambda}{2\varphi(p)}, \quad z = -2\zeta(p),
\]

where $\varphi(p)$ and $\zeta(p)$ are the Weierstrass functions, $\varphi'^2 = 4\varphi^3 + \lambda^2$, $\zeta' = -\varphi$.

The resulting Lax pair can be written in parametric form as

\[
S_x = \frac{1}{V} \frac{\varphi'(p) + \lambda}{2\varphi(p)}, \quad S_y = \frac{1}{W} \frac{\varphi'(p) - \lambda}{2\varphi(p)}, \quad S_t = -2\zeta(p),
\]

or, equivalently,

\[
S_x S_y (VS_x - WS_y) = \frac{\lambda}{VW}, \quad S_x S_y = \frac{\varphi(p)}{VW}, \quad S_t = -2\zeta(p).
\]

Notice that equations (3.44) coincide with the Euler-Lagrange equations corresponding to the Lagrangian density $\int \frac{u_x u_y}{u_t} dx dy dt$ upon setting $V = \frac{u_t}{2u_x}$, $W = \frac{u_t}{2u_y}$ as pointed out by Maxim Pavlov. In this context, the above Lax pair appeared previously in [52].
3.5 Godunov systems and generalized hypergeometric functions

In this section we include the general theory of Godunov’s systems [37], and describe the Godunov form of $n$-component quasilinear systems constructed in [46, 48] in terms of generalised hypergeometric functions. In particular, this provides a Godunov representation for any generic 2-component integrable system [24, 46]. Applied to the system (3.11), this construction gives a parametrisation of the generic integrable potential $H(V, W)$ in the form (3.13). Most of these results are due to A. V. Odeskii.

In Sect. 3.5.1 we recall the main aspects of the Godunov representation, and clarify its symmetry properties. In Sect. 3.5.2 we construct a Godunov form for quasilinear systems found in [46, 48] in terms of generalised hypergeometric functions. In Sect. 3.5.3 we specialise this construction to integrable potentials $H(V, W)$ of the type (3.11).

3.5.1 Godunov systems

A $2 + 1$ dimensional quasilinear system in $n$ unknowns $\mathbf{v} = (v^1, ..., v^n)$ is said to possess a Godunov representation [37] if it can be written in the conservative form,

$$
(F_0, i)_t + (F_1, i)_x + (F_2, i)_y = 0, \quad i = 1, \ldots, n, \quad (3.45)
$$

where potentials $F_0, F_1, F_2$ are functions of $\mathbf{v}$, and $F_{\alpha,j} = \partial F_{\alpha}/\partial v^j$, $\alpha = 0, 1, 2$. Any such system automatically possesses an extra conservation law,

$$
\mathcal{L}(F_0)_t + \mathcal{L}(F_1)_x + \mathcal{L}(F_2)_y = 0, \quad (3.46)
$$

where $\mathcal{L}$ denotes the Legendre transform: $\mathcal{L}(F_{\alpha}) = F_{\alpha,k}v^k - F_{\alpha}$. Many systems of physical origin are known to be representable in the Godunov
form. This representation is widely used for analytical/numerical treatment of quasilinear systems.

Note that the systems (3.9)-(3.11) are written in the Godunov form. For example, in the case (3.11) we have \( n = 2 \), \( v^1 = V \), \( v^2 = W \), \( F_0 = -H \), \( F_1 = VH \), \( F_2 = WH \). Recall that a 2 + 1 dimensional quasilinear system of \( n \) equations for \( n \) unknowns possesses a Godunov form iff it possesses \( n + 1 \) conservation laws of hydrodynamic type. The following fact will be useful:

**Proposition 1.** The Godunov representation (3.45) is form-invariant under the projective action of \( GL_{n+1} \) defined as

\[
\tilde{v}^i = \frac{l^i(v)}{l(v)}, \quad \tilde{F}_\alpha = \frac{F_\alpha}{l(v)},
\]

here \( l^i \), \( l \) are linear (inhomogeneous) forms in \( v \).

**Proof:**

With any Godunov system we associate the following geometric objects. Let us consider an auxiliary \((n + 1)\)-dimensional affine space \( A^{n+1} \) with coordinates \( x_1, \ldots, x_n, x_{n+1} \). With any potential \( F_\alpha \) we associate an \( n \)-parameter family of hyperplanes,

\[
x_{n+1} - x_k v^k + F_\alpha(v) = 0.
\] (3.47)

The envelope of this family is a hypersurface \( M_{F_\alpha}^n \subset A^{n+1} \) defined parametrically as

\[(x_1, \ldots, x_n, x_{n+1}) = (F_{\alpha,1}, \ldots, F_{\alpha,n}, L(F_\alpha)).\]

Notice that components of its position vector are the conserved densities appearing in Eqs. (3.45), (3.46). By construction, hypersurfaces \( M_{F_0}^n \), \( M_{F_1}^n \) and \( M_{F_2}^n \) have parallel tangent hyperplanes at the points corresponding to the same values of the parameters \( v^i \). Let us now apply an arbitrary affine transformation in \( A^{n+1} \), \( \tilde{x} = Ax \), where \( A \) is a constant \((n + 1) \times (n + 1)\)
matrix. This will transform Eq. (3.47) to

$$\tilde{x}_{n+1} - \tilde{x}_k \tilde{v}^k + \tilde{F}_\alpha(\tilde{v}) = 0$$  \hspace{1cm} (3.48)

where the transformation \( v \rightarrow \tilde{v} \) will automatically be projective:

$$\tilde{v}^i = \frac{l^i(v)}{l(v)}, \quad \tilde{F}_\alpha = \frac{F_\alpha}{l(v)};$$

here \( l^i, l \) are linear (inhomogeneous) forms in \( v \). Since affine transformations preserve the properties of being parallel/tangential, they naturally act on the class of Godunov’s systems.

### 3.5.2 Godunov form of integrable quasilinear systems and generalized hypergeometric functions

Let \( H_{s_1, \ldots, s_{n+2}} \) be the space of solutions of the system

$$\frac{\partial^2 h}{\partial u_i \partial u_j} = \frac{s_i}{u_i - u_j} \cdot \frac{\partial h}{\partial u_j} + \frac{s_j}{u_j - u_i} \cdot \frac{\partial h}{\partial u_i}, \quad i, j = 1, \ldots, n, \quad i \neq j, \quad (3.49)$$

and

$$\frac{\partial^2 h}{\partial u_i \partial u_i} = -\left(1 + \sum_{j=1}^{n+2} s_j \right) \frac{s_i}{u_i(u_i - 1)} h + \frac{s_i}{u_i(u_i - 1)} \sum_{j \neq i}^n \frac{u_j(u_j - 1)}{u_j - u_i} \cdot \frac{\partial h}{\partial u_j} + \ldots \hspace{1cm} (3.50)$$

for one unknown function \( h(u_1, \ldots, u_n) \). Here \( s_1, \ldots, s_{n+2} \) are arbitrary constants. Elements of \( H_{s_1, \ldots, s_{n+2}} \) are examples of the so-called generalised hypergeometric functions, see [33], [48]. It is known that \( \dim H_{s_1, \ldots, s_{n+2}} = n + 1 \). Let \( g_0, g_1, \ldots, g_n \) be a basis of \( H_{s_1, \ldots, s_{n+2}} \). Choose a time \( t_q \) for each element

77
This basis. Here \( q \) runs from 0 to \( n \). It is known [46], [48] that for each pairwise distinct \( q, r, s \) running from 0 to \( n \) the system

\[
\sum_{1 \leq i \leq n, i \neq j} (g_{q,u}g_{r,u} - g_{r,u}g_{q,u}) \frac{u_i(u_j-1)u_i-t_s - u_i(u_i-1)u_i-t_s}{u_j-u_i} +
\]

\[
\sigma \cdot (g_{q}g_{r,u} - g_{r}g_{q,u}) u_{j,t_s} +
\]

\[
\sum_{1 \leq i \leq n, i \neq j} (g_{r,u}g_{s,u} - g_{s,u}g_{r,u}) \frac{u_j(u_i-1)u_i-t_q - u_i(u_i-1)u_i-t_q}{u_j-u_i} +
\]

\[
\sigma \cdot (g_{r}g_{s,u} - g_{s}g_{r,u}) u_{j,t_q} +
\]

\[
\sum_{1 \leq i \leq n, i \neq j} (g_{s,u}g_{q,u} - g_{q,u}g_{s,u}) \frac{u_j(u_i-1)u_i-t_r - u_i(u_i-1)u_i-t_r}{u_j-u_i} +
\]

\[
\sigma \cdot (g_{s}g_{q,u} - g_{q}g_{s,u}) u_{j,t_r} = 0,
\]

where \( j = 1, ..., n \), \( \sigma = 1 + s_1 + ... + s_{n+2} \), possesses a dispersionless Lax representation and an infinity of hydrodynamic reductions. Moreover, any generic integrable 3-dimensional hydrodynamic type system with two unknowns is isomorphic to a system of the form (3.51), see [46]. It is also known that the system (3.51) possesses \( n + 1 \) conservation laws of hydrodynamic type, see [24] for \( n = 2 \) and [48] for general \( n \). Therefore, it possesses a Godunov representation which can be constructed explicitly in the following way.

**Theorem 12** Let \( h_0, h_1, ..., h_n \) be a basis of \( H_{2s_1, ..., 2s_{n+2}} \). For each \( \alpha \neq \beta = 0, 1, ..., n \) let \( f_{\alpha, \beta} \) be a solution of the system

\[
\frac{\partial^2 h}{\partial u_i \partial u_j} = \frac{2s_i}{u_i - u_j} \cdot \frac{\partial h}{\partial u_j} + \frac{2s_j}{u_j - u_i} \cdot \frac{\partial h}{\partial u_i} + \frac{\partial g_{\alpha,u} \partial g_{\beta,u}}{\partial u_i \partial u_j} - \frac{\partial g_{\alpha,u} \partial g_{\beta,u}}{\partial u_i \partial u_i},
\]

(3.51)

\[
i, j = 1, ..., n, \quad i \neq j.
\]
\[
\frac{\partial^2 h}{\partial u_i \partial u_i} = -(1 + 2 \sum_{j=1}^{n+2} s_j) \frac{2s_i}{u_i(u_i - 1)} h + \frac{2s_i}{u_i(u_i - 1)} \sum_{j \neq i}^{n} \frac{u_j(u_j - 1)}{u_j - u_i} \cdot \frac{\partial h}{\partial u_j} + \left( \sum_{j \neq i}^{n} \frac{2s_j}{u_i - u_j} + \frac{2s_i + 2s_{n+1}}{u_i} + \frac{2s_i + 2s_{n+2}}{u_i} \right) \frac{\partial h}{\partial u_i} - \\
\sum_{j \neq i}^{n} \frac{\partial g_{\alpha}}{\partial u_i} \frac{\partial g_{\beta}}{\partial u_j} \cdot \frac{u_j(u_j - 1)}{u_i(u_i - 1)} - (1 + s_1 + \ldots + s_{n+2}) \frac{\partial g_{\alpha}}{\partial u_i} g_{\beta} - \frac{\partial g_{\beta}}{\partial u_i} g_{\alpha} \frac{u_i(u_i - 1)}{u_i(u_i - 1)}.
\]

(3.52)

Define new coordinates \( v_1, \ldots, v_n \) and functions \( F_{\alpha,\beta}(v_1, \ldots, v_n) \) by
\[
v_i = \frac{h_i(u_1, \ldots, u_n)}{h_0(u_1, \ldots, u_n)}, \quad F_{\alpha,\beta} = \frac{f_{\alpha,\beta}}{h_0(u_1, \ldots, u_n)}. \quad (3.53)
\]

Then, in coordinates \( v_1, \ldots, v_n \), the system (3.51) takes the Godunov form
\[
\left( \frac{\partial F_{r,s}}{\partial v_i} \right)_{t_q} + \left( \frac{\partial F_{s,q}}{\partial v_i} \right)_{t_r} + \left( \frac{\partial F_{q,r}}{\partial v_i} \right)_{t_s} = 0, \quad i = 1, \ldots, n. \quad (3.54)
\]

**Proof:**

Substituting (3.53) into (3.54) and calculating derivatives of \( f_{\alpha,\beta} \) by virtue of (3.51), (3.52) we obtain conservation laws of the system (3.51) as found in [48]. More precisely, the left hand side of (3.54) is equal to
\[
\sum_{j=1}^{n} (-1)^{i+j} \det W_{i,j} R_j
\]

where \( R_j \) is the left hand side of (3.51), the matrix \( W \) is defined as \( W = (w_{\alpha,\beta}) \) where \( w_{0,\beta} = h_{\beta}(u_1, \ldots, u_n), \ w_{\alpha,\beta} = \frac{\partial h_{\beta}}{\partial u_{\alpha}} \) if \( \alpha \neq 0 \) and \( W_{i,j} \) is the \( n \times n \)-minor of \( W \) obtained by eliminating a column with \( h_i \) and a row with \( \frac{\partial}{\partial u_j} \). In other
words, the left hand side of (3.54) is equal to

$$\sum_{j=1}^{n} (W^{-1})_{i,j} R_j;$$

so that (3.54) holds identically modulo (3.51).

**Remark 1.** The system (3.54) possesses a natural action of the group $GL_{n+1} \times GL_{n+1}$. Namely, the first copy of $GL_{n+1}$ acts on the times $t_0, \ldots, t_n$, and in the same way on the basis $g_0, \ldots, g_n$. This action corresponds to a change of basis $g_0, \ldots, g_n$ in $H_{s_1, \ldots, s_{n+2}}$. The second copy of $GL_{n+1}$ acts according to the Proposition 1. This action corresponds to a change of basis $h_0, \ldots, h_n$ in $H_{2s_1, \ldots, 2s_{n+2}}$.

**Remark 2.** Note that $F_{\alpha,\beta}$ is defined up to an arbitrary linear combination of $1, v_1, \ldots, v_n$ and, therefore, $f_{\alpha,\beta}$ is defined up to an arbitrary linear combination of $h_0, h_1, \ldots, h_n$. This explains why $f_{\alpha,\beta}$ satisfies a linear non-homogeneous system with the same homogeneous part as for elements from $H_{2s_1, \ldots, 2s_{n+2}}$. Moreover, the structure of the non-homogeneous part of the system (3.51), (3.52) containing linear combinations of $\frac{\partial g_{\alpha}}{\partial u_i} \frac{\partial g_{\beta}}{\partial u_j} - \frac{\partial g_{\alpha}}{\partial u_j} \frac{\partial g_{\beta}}{\partial u_i}$ and $\frac{\partial g_{\alpha}}{\partial u_i} g_{\beta} - \frac{\partial g_{\beta}}{\partial u_i} g_{\alpha}$ is dictated by the action of $GL_{n+1}$ on the times $t_0, \ldots, t_n$, and in the same way on $g_0, \ldots, g_n$, compare with (3.51).

**Remark 3.** It was proven in [46] that any 2-component integrable system is isomorphic to a member of the family (3.51) with $n = 2$, or its appropriate limit. Therefore, Theorem 12 gives, in particular, a description of the Godunov form for any generic 2-components integrable system. If $n > 2$ there exist $n$-component integrable systems which do not belong to the family (3.51) or its degenerations (see, for example, [49]). However, $n$-component systems constructed in [49]) have $n$ conservation laws only and, therefore, do not possess a Godunov form. Probably, the only integrable systems possessing Godunov’s form should belong to the family (3.51) or its appropriate degenerations.
3.5.3 Application to integrable potentials $H(V, W)$

Let us now apply these results to the system (3.11). Set $n = 2$, choose a triple of indices $s = 0$, $q = 1$, $r = 2$ and set $t_0 = -t$, $t_1 = x$, $t_2 = y$ where $t$, $x$, $y$ are the independent variables in (3.11). Note that the bases in $H_{s_1,\ldots,s_{n+2}}$ and $H_{2s_1,\ldots,2s_{n+2}}$ are no longer independent since we have constraints on the functions $F_{1,2}$, $F_{2,0}$, $F_{0,1}$, namely, $F_{2,0} = v_1 F_{1,2}$, $F_{0,1} = v_2 F_{1,2}$. To obtain (3.11) it remains to set $F_{1,2} = H$, $v_1 = V$, $v_2 = W$. Therefore, $f_{1,2} = Hh_0$, $f_{2,0} = Hh_1$, $f_{0,1} = Hh_2$. Moreover, we must have

$$h_i = a(u_1, u_2)(g_j g_{k,u_1} - g_k g_{j,u_1}) + b(u_1, u_2)(g_j g_{k,u_2} - g_k g_{j,u_2}) + c(u_1, u_2)(g_{j,u_2} g_{k,u_1} - g_{j,u_1} g_{k,u_2})$$

where $i, j, k$ is a cyclic permutation of 0, 1, 2. Indeed, $h_i$ must have the same structure in $g_0$, $g_1$, $g_2$ as coefficients at $u_{j,i}$ in (3.51). Substituting the expressions for $h_0, h_1, h_2$ and $f_{1,2}$, $f_{2,0}$, $f_{0,1}$ into the equations for $H_{2s_1,\ldots,2s_{n+2}}$ and (3.51), (3.52), respectively, and using the equations (3.49), (3.50) for $g_0, g_1, g_2$, we obtain that $s_1 = s_2 = -s_3 = s_4 = -\frac{1}{3}$, along with the following expressions for the functions $h_0, h_1, h_2$:

$$h_0(u_1, u_2) = C\left(\frac{u_2}{u_2 - 1}(g_1 g_{2,u_1} - g_2 g_{1,u_1}) + \frac{u_1}{u_1 - 1}(g_1 g_{2,u_2} - g_2 g_{1,u_2}) + 3(u_1 - u_2)(g_{1,u_2} g_{2,u_1} - g_{1,u_1} g_{2,u_2})\right),$$

$$h_1(u_1, u_2) = C\left(\frac{u_2}{u_2 - 1}(g_2 g_{0,u_1} - g_0 g_{2,u_1}) + \frac{u_1}{u_1 - 1}(g_2 g_{0,u_2} - g_0 g_{2,u_2}) + 3(u_1 - u_2)(g_{2,u_2} g_{0,u_1} - g_{2,u_1} g_{0,u_2})\right),$$

$$h_2(u_1, u_2) = C\left(\frac{u_2}{u_2 - 1}(g_0 g_{1,u_1} - g_1 g_{0,u_1}) + \frac{u_1}{u_1 - 1}(g_0 g_{1,u_2} - g_1 g_{0,u_2}) + 3(u_1 - u_2)(g_{0,u_2} g_{1,u_1} - g_{0,u_1} g_{1,u_2})\right),$$

81
where
\[ C = (u_1 - 1)^{2/3}(u_2 - 1)^{2/3}(u_1 - u_2)^{-1/3}. \]

We also obtain the following system for \( H \),
\[ H_{u_1} = \frac{u_2 - 1}{C(u_1 - u_2)}, \quad H_{u_2} = \frac{u_1 - 1}{C(u_2 - u_1)}. \]

The solution of this system reads
\[ H(u_1, u_2) = G\left(\frac{u_1 - 1}{u_2 - 1}\right) \]
where \( G'(t) = \frac{1}{t^{2/3}(t-1)^{2/3}} \). Summarizing, we obtain the following

**Proposition 3.** Let the potential \( H(V, W) \) be defined parametrically in the form
\[ H = G\left(\frac{u_1 - 1}{u_2 - 1}\right), \quad V = \frac{h_1}{h_0}, \quad W = \frac{h_2}{h_0}, \]
where \( G'(t) = \frac{1}{t^{2/3}(t-1)^{2/3}} \) and \( h_0, h_1, h_2 \) are linearly independent solution of the hypergeometric system
\[ u_1(1 - u_1)h_{u_1,u_1} - \frac{4}{3} u_1 h_{u_1} - \frac{2}{9} h = \frac{2}{3} u_1(u_1 - 1)h_{u_1} + \frac{2}{3} u_2(u_2 - 1)h_{u_2}, \]
\[ u_2(1 - u_2)h_{u_2,u_2} - \frac{4}{3} u_2 h_{u_2} - \frac{2}{9} h = \frac{2}{3} u_1(u_1 - 1)h_{u_1} + \frac{2}{3} u_2(u_2 - 1)h_{u_2}, \]
\[ h_{u_1,u_2} = \frac{2}{3} \frac{h_{u_2} - h_{u_1}}{u_2 - u_1}; \]
(this system coincides with (3.49), (3.50) for the values of constants \( s_1 = s_2 = -s_3 = s_4 = -\frac{2}{3} \)). Then \( H(V, W) \) provides the generic solution to the system (3.36). In particular, it does not possess any continuous symmetry from the equivalence group.
3.6 Appendix. Point and contact symmetry groups for differential equations.

In this Appendix we present the group methods used to obtain the point and contact symmetry groups of the systems (3.6) - (3.8). Our discussion is based on the book [40].

3.6.1 Local one-parameter point transformation groups

Let us consider a function \( y(x) \) depending on a variable \( x \). We look at locally invertible transformation \( \bar{x} = \phi(x, y, a) \), \( \bar{y} = \psi(x, y, a) \), depending upon a real parameter \( a \). We require that,

\[
\phi|_{a=0} = x, \quad \psi|_{a=0} = y.
\]

These transformations form a group iff two successive transformations are equivalent to another transformation. We can always choose the parameter in such a way that

\[
\phi(\bar{x}, \bar{y}, b) = \phi(x, y, a + b),
\]

\[
\psi(\bar{x}, \bar{y}, b) = \psi(x, y, a + b).
\]

We note that for our purposes it is sufficient if the group property is satisfied only for sufficiently small \( a \) and \( b \), i.e. we consider only local groups.

Let us denote the transformation \( \bar{x} = \phi(x, y, a) \), \( \bar{y} = \psi(x, y, a) \) as \( T_a \). The set \( G \) of transformations \( T_a \) is a one-parameter group iff for \( a \) and \( b \) sufficiently small,

\[
T_0 = I \in G,
\]

\[
T_a T_b = T_{a+b} \in G,
\]

\[
T_a^{-1} = T_{-a} \in G.
\]
Since $a$ is considered small, we can study the infinitesimal transformation by expanding $\bar{x} = \phi(x, y, a)$ and $\bar{y} = \psi(x, y, a)$ around the point $a = 0$,

$$\bar{x} = x + \xi(x, y)a, \quad \bar{y} = y + \eta(x, y)a.$$ 

Here we have denoted 

$$\xi(x, y) = \left. \frac{\partial \phi(x, y, a)}{\partial a} \right|_{a=0}, \quad \eta(x, y) = \left. \frac{\partial \psi(x, y, a)}{\partial a} \right|_{a=0}.$$

Given an infinitesimal transformation the respective one-parameter group can be found by solving the Lie equations with the appropriate initial conditions:

$$\frac{d\phi}{dx} = \xi(x, y), \quad \phi|_{a=0} = x,$$
$$\frac{d\psi}{dx} = \eta(x, y), \quad \psi|_{a=0} = y.$$ 

A tangent vector field can be then written in terms of first-order differential operator

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}.$$ 

$X$ is called the group operator.

The function $F(x, y)$ is called the invariant of a group of if for each point $(x, y)$ it is constant along the trajectory determined by the action of the group, $F(x, y) = F(\bar{x}, \bar{y})$.

**Theorem 13** The function $F(x, y)$ is an invariant of the group $G$ with infinitesimal operator $X$ if and only if $XF = 0$.

From the last theorem it follows that every one-parameter group on the plane has only one functionally independent invariant, which is taken as the first integral of the characteristic equation

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}.$$
3.6.2 Prolongation formulas

We now look at the way derivatives $y', y'', \ldots$ transform under the action of a one-parameter group. We introduce the operator of total differentiation

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \ldots.$$  

The derivatives transform as

$$\bar{y}' \equiv \frac{dy}{dx} = \frac{D\psi}{D\phi} = \frac{\psi_x + y'\psi_y}{\phi_x + y'\phi_y} \equiv P(x, y, y', a),$$

$$\bar{y}'' \equiv \frac{dy'}{dx} = \frac{Dy'}{D\phi} = \frac{P_x + y'Py + y''P_y}{\phi_x + y'\phi_y}. \quad (3.55)$$

If we start from a group $G$ and add the action of $(3.55)_1$ we obtain the first prolongation $G^1$, acting on the space of three variables $(x, y, y')$. If we further add $(3.55)_2$ we obtain the second prolongation $G^2$, acting on the space $(x, y, y', y'')$.

Let us now look at the infinitesimal form of transformations $(3.55)$. By keeping only linear terms in $a$, we find

$$\bar{y}' = \frac{y' + aD(\eta)}{1 + aD(\xi)} = (y' + aD(\eta))(1 - aD(\xi)) =$$

$$y' + a(D(\eta) - y'D(\xi)) \equiv y' + a\xi_1,$$

$$\bar{y}'' = \frac{y'' + aD(\xi_1)}{1 + D(\xi)} = (y'' + aD(\xi_1))(1 - aD(\xi)) =$$

$$y'' + a(D(\xi_1) - y''D(\xi)) \equiv y'' + a\xi_2$$

Form these follows that the infinitesimal operators of the groups $G^1$ and $G^2$
are respectively

\[ X_1 = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial x} + \xi_1 \frac{\partial}{\partial y'}, \]

\[ X_2 = X_1 + \xi_2 \frac{\partial}{\partial y'}. \]

This construction could be easily generalised to a multidimensional situation, with \( x^i, i = 1, \ldots, n \), being independent variables and \( u = u(x^1, \ldots, x^n) \).

Then the operator of a one-parameter group is

\[ X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}. \]

In a similar manner the first and second prolongations are

\[ X_1 = X + \zeta \frac{\partial}{\partial u_i}, \quad X_2 = X_1 + \zeta_{ij} \frac{\partial}{\partial u_{ij}}. \]

Here \( u_i = \partial_{x^i} u \),

\[ \zeta^i = D_i(\eta) - u_j D_i(\xi^j), \]

\[ \zeta^i_{ij} = D_j(\zeta) - u_j D_i(\xi^j), \]

and

\[ D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \ldots. \] (3.56)

### 3.6.3 Groups admitted by differential equations

Let us consider a group of point transformations \( G \) with first and second prolongation \( G_1 \) and \( G_2 \) respectively. We say that a first order differential equation

\[ F(x, y, y') = 0 \]
admits the group $G$ if the two-dimensional surface $F(x, y, p) = 0$ in the three
dimensional space of variables $x$, $y$, $p$ is invariant under the first prolongation
$G_1$ modulo $p = y'$. Analogously a second-order differential equation

$$F(x, y, y', y'') = 0$$

admits a group $G$ if the surface $F(x, y, p, q)$ is invariant under the second
prolongation $G_2$ under the condition $p = y'$, $q = y''$. This construction
is easily generalised to higher order equations, as well as partial differential
equations. We note that under the action of a group admitted by a differential
equation a solution of the equation is transformed into a solution of the same
equation. Let us now construct the maximal group admitted by a second
order differential equation. The infinitesimal criterion for invariance is

$$X_2 F|_{F=0} \equiv (\xi F_x + \eta F_y + \zeta_1 F_{y'} + \zeta_2 F_{y''})|_{F=0} = 0. \quad (3.57)$$

The equation (3.57) is called determining equation for the group $G$. For
simplicity we will consider differential equations written in the form $y'' = f(x, y, y')$. The determining equation then becomes

$$\eta_x x + (2\eta_{xy} - \xi_x x)y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^2\xi_{yy} +
(\eta_y - 2\xi_x - 3y'\xi_y) f - (\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y) f_{y'} - \xi f_x - \eta f_y = 0. \quad (3.58)$$

Here $f(x, y, y')$ is a known function, whereas $\xi$ and $\eta$ are unknown functions
of $x$ and $y$. Because in this equation $y'$ is considered independent variable,
we can use it to split the determining equation into a overdetermined system
of PDE’s for $\xi$ and $\eta$. To do this we simply need to consider the left-hand
side as a polynomial in $y'$ and require that all its coefficients are trivially
zero. By solving this system we can find all operators admitted by the given
differential equation.
Example Let us find all operators $X = \xi \partial_x + \eta \partial_y$ admitted by

$$y'' + \frac{y'}{x} - e^y = 0.$$

We substitute $f = e^y - \frac{y'}{x}$ in (3.58) and obtain

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3 \xi_{yy} +$$

$$(\eta_y - 2\xi_x - 3y'\xi_y)(e^y - \frac{y'}{x}) - (\eta_x + (\eta_y - \xi_x)y' - y'^2 \xi_y) \frac{1}{x} -$$

$$\xi \frac{y'}{x^2} - \eta e^y = 0.$$

The left-hand side of this equation is a third order polynomial in $y'$. We require that all its coefficients are identically zero and obtain the following system of equations:

$$\xi_{yy} = 0,$$

$$\eta_y - 2\xi_{xy} + \frac{2}{x} \xi_y = 0,$$

$$2\eta_{xy} - \xi_{xx} + \frac{\xi_x}{x} - \frac{\xi}{x^2} - 3\xi_y e^y = 0,$$

$$\eta + \frac{\eta_x}{x} + (\eta_y - 2\xi_x - \eta)e^y = 0.$$

The general solution for this system is

$$\xi = C_1 \ln x + C_2 x, \quad \eta = -2 (C_1 (1 + \ln x) + C_2).$$

Here $C_1$ and $C_2$ are constants of integration. Due to the linearity of the determining equations the general solution can be represented as a linear combination of the following independent solutions

$$\xi_1 = x \ln x, \quad \eta_1 = -2(1 + \ln x),$$

$$\xi_2 = x, \quad \eta_2 = -2.$$
This means the equation \( y'' + \frac{y'}{x} - e^y = 0 \) admits two independent operators

\[
X^1 = x \ln x \partial_x - 2(1 + \ln x) \partial_y, \quad X^2 = x \partial_x - 2 \partial_y.
\]

Hence the set of all operators admitted by \( y'' + \frac{y'}{x} - e^y = 0 \) is a two dimensional vector space with basis given by the above operators.

In the case of partial differential equations the procedure follows the procedure for ODEs. To find the point group of symmetries of (3.20), one needs to compute all prolongations up to forth order, then compute the determining equation. Due the size of the formulas we only present the final result. The general solution for the determining equations is

\[
\begin{align*}
\xi_v &= -c_6 - c_5 v, \\
\xi_w &= -c_4 - c_2 v - c_3 w, \\
\eta &= c_{11} + c_{10} h + c_8 v + c_9 w.
\end{align*}
\]

Hence the vector space of operators admitted by the system (3.20) in nine dimensional. It is spanned by two translations \( \partial_v \) and \( \partial_w \), three linear transformations \( v \partial_v, v \partial_w \) and \( w \partial_w \), linear transformations of \( h \) generated by \( v \partial_h \), \( w \partial_h \) and affine transformations of \( h \) generated by \( h \partial_h \) and \( \partial_h \). The natural equivalence group that is obtained from this infinitesimal group is

\[
\begin{align*}
v &\rightarrow av + b, \\
w &\rightarrow pv + cw + d, \\
h &\rightarrow \alpha h + \beta v + \gamma w + \delta.
\end{align*}
\]

We will now move the discussion to a wider class of transformations - the so called contact or tangent transformations.
3.6.4 Contact transformations

Transformations that depend on the first derivatives as well as on the dependent and independent variable are called contact transformations. Here we will consider the case of a variable $u$ depending on $n$ variables $x^i$. Let us denote the set of first derivatives $\partial_i u = u_i$ by $u'$. Now consider the transformation

$$\bar{x} = \phi^i(x, u, u', a), \quad \bar{u} = \psi(x, u, u', a), \quad \bar{u}_i = \omega(x, u, u', a)$$

which acts in the $2n+1$ space $(x, u, u')$. The transformation is called contact under the condition $\bar{u}_i = \partial \bar{u} / \partial \bar{x}^i$. We can write the transformation in the infinitesimal form

$$\bar{x}^i = x^i + \xi^i(x, u, u')a, \quad \bar{u} = u + \eta^i(x, u, u')a, \quad \bar{u}_i = u_i + \zeta_i(x, u, u')a.$$ 

In these terms the transformation is called contact if

$$\zeta_i = D_i(\eta) - u_j D_i(\xi^j).$$

The differential operator defining a contact transformation is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i},$$

(3.59)

if and only if

$$\xi^i = \frac{\partial W}{\partial u_i}, \quad \eta = W - u_i \frac{\partial W}{\partial u_i}, \quad \zeta = \frac{\partial W}{\partial x^i} - u_i \frac{\partial W}{\partial u},$$

(3.60)

for some function $W(x, u, u')$. This function is called the characteristic function of the contact transformation group. To find it we need to find the determining equation $XF|_{F=0} = 0$ with the symbol $(3.59)$. Here we need to substitute $\xi$, $\eta$ and $\zeta$ from (3.60)
In the case (3.7) considered above the characteristic function is

\[ W = c_1 h_w^2 + (c_2 v + c_3 w + c_4) h_w + (c_5 v + c_6) h_v + \\
+ c_7 h_w h_v + c_8 v + w c_9 + c_{10} h + c_{11}, \]

and, as noted in the main text it gives rise to two purely contact generators

\[ h_w \partial_v + h_v \partial_w + h_v h_w \partial_h, \quad 2h_w \partial_w + h_w^2 \partial_h. \]

We note here that for the Legendre transformed system the point symmetry group coincides with the contact symmetry group.
Chapter 4

Dispersive deformations of Hamiltonian systems in 2+1 dimensions

In this chapter, based on the work [29], we develop a theory of integrable dispersive deformations of 2 + 1 dimensional Hamiltonian systems of hydrodynamic type. We use a scheme analogous to the scheme proposed by B. A. Dubrovin and his collaborators in 1 + 1 dimensions. Our results show that the multi-dimensional situation is far more rigid, and generic Hamiltonians are not deformable. As an illustration we discuss in detail Type II Hamiltonian systems, establishing triviality of first order deformations and classifying Hamiltonians possessing nontrivial deformations of the second order.

4.1 Introduction

The deformation theory of 1 + 1 dimensional Hamiltonian systems is based on the theory of integrability of hydrodynamic type Hamiltonian systems in 1 + 1 dimensions, presented in Chapter 2. We recall that a Hamiltonian
system of hydrodynamic type

\[ u_i^j = \{u^i, H_0\} = P^{ij} \frac{\delta H_0}{\delta u^j}, \quad (4.1) \]

\( i, j = 1, \ldots, n \), where \( P^{ij} = \epsilon^{ij} \delta^i_j \frac{d}{dx} \) is the Hamiltonian operator and \( H_0 = \int h(u) \, dx \) is the Hamiltonian with the density \( h(u) \), is integrable if and only if it is diagonalisable. Integrability in this case is understood as the existence of infinitely many functionals \( F = \int f(u) \, dx \), commuting with the Hamiltonian, \( \{H, F\} = 0 \). The functionals \( F \) are parametrized by \( n \) arbitrary functions of one variable.

There are several approaches in studying deformations of integrable dispersionless equations. These include deformations of Lie algebra homomorphisms [22] and dressing operator method applied to Moyal algebra valued loop group [59]. The theory of dispersive deformations of 1+1 dimensional systems has been developed and thoroughly investigated by Dubrovin and his collaborators in a series of papers [16, 17, 18, 19, 20]. Given an integrable Hamiltonian system of hydrodynamic type one considers a deformation of the original Hamiltonian in the form

\[ H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots \quad (4.2) \]

where the density of \( H_i \) is assumed to be a homogeneous differential polynomial of degree \( i \) in the \( x \)-derivatives of \( u \) (\( u^{i}_{xxx}, u^{i}_x u^{j}_x \) and \( u^{i}_x^3 \) all have degree three) with coefficients being functions of \( u^i \) themselves. Here the Hamiltonian operator \( P^{ij} \) can be assumed undeformed due to the general results of [34, 12]. Deformation (4.2) is called integrable (to the order \( \epsilon^m \)) if any hydrodynamic Hamiltonian \( F_0 = \int f(u) \, dx \) commuting with \( H_0 \) can be deformed in such a way that \( \{H, F\} = 0 \) (mod \( \epsilon^{m+1} \)). The classification of integrable deformations is performed modulo canonical transformations of the form

\[ H \rightarrow H + \epsilon \{K, H\} + \frac{\epsilon^2}{2} \{K, \{K, H\}\} + \ldots \quad (4.3) \]
where $K$ is any functional of the form (4.2). The richness of this deformation scheme is due to the following facts:

The variety of integrable ‘seed’ Hamiltonians $H_0$ is parametrised by $n(n - 1)/2$ arbitrary functions of two variables;

For a fixed integrable Hamiltonian $H_0$, the deformation procedure introduces extra arbitrary functions of one variable known, in bi-Hamiltonian context, as ‘central invariants’. One should point out that it is still an open problem to extend a deformation, for arbitrary values of these functions, to all orders in the deformation parameter $\epsilon$.

The main goal of this Chapter is to present the analogous deformation scheme in $2 + 1$ dimensions, developed in the context of hydrodynamic reductions. Since the definition of integrability according to the method of hydrodynamic reductions is in a sense intrinsic, based on a reduction of the given system, the route to deform this system also is intrinsic. We will say that a deformed system is integrable (integrable to order $\epsilon^m$) if and only if all its hydrodynamic reductions inherited from the original system could be deformed without breach of commutativity conditions. The details for this procedure based on [27, 28] are given in Section 4.3. We note here that the Hamiltonian operator will be assumed undeformed (although we are not aware of any results establishing the triviality of Poisson cohomology in higher dimensions). In the previous Chapter 3 we have developed the theory of integrability of two-component systems of type I, type II and type III Poisson brackets, respectively,

\[ P = \begin{pmatrix} d/dx & 0 \\ 0 & d/dy \end{pmatrix}, \quad P = \begin{pmatrix} 0 & d/dx \\ d/dx & d/dy \end{pmatrix}, \quad P = \left( \begin{array}{cc} 2v & w \\ w & 0 \end{array} \right) \frac{d}{dx} + \left( \begin{array}{cc} 0 & v \\ v & 2w \end{array} \right) \frac{d}{dy} + \left( \begin{array}{cc} v_x & v_y \\ w_x & w_y \end{array} \right), \]
with \( v, w \) being the dependent variables.

We will concentrate on the deformation of type II Hamiltonian systems, as it turns out to be the most interesting of the three. As we already noticed, the corresponding Hamiltonian systems take the form

\[
\begin{pmatrix}
v \\
w
\end{pmatrix}_t = \begin{pmatrix}
0 & d/dx & d/dy \\
d/dx & \delta H_0/\delta v & \delta H_0/\delta w
\end{pmatrix} \begin{pmatrix}
\delta H_0/\delta v \\
\delta H_0/\delta w
\end{pmatrix},
\]

(4.4)

\( H_0 = \int h(v, w) \, dx \, dy \), or, explicitly,

\[
v_t = (h_w)_x, \quad w_t = (h_v)_x + (h_w)_y.
\]

We will be looking at deformations of the form (4.2) where the density of \( H_i \) is a homogeneous polynomial of degree \( i \) in the \( x \)- and \( y \)-derivatives of \( v \) and \( w \) with coefficients explicitly depending on \( v \) and \( w \).

The main features of the \( 2 + 1 \) dimensional deformation scheme can be summarised as follows:

The variety of integrable ‘seed’ Hamiltonians \( H_0 \) is finite dimensional.

Generic integrable Hamiltonians \( H_0 \) possess no nontrivial deformations.

Nevertheless, there exist deformable (non-generic) Hamiltonians.

**Example 1.** Let \( H_0 = \int \frac{w^2}{2} + f(v) \, dx \, dy \). In this case the integrability conditions reduce to a single fourth order ODE, \( f''' f'' = f''^2 \), so that without any loss of generality one can set \( f(v) = e^v \). Modulo canonical transformations, this Hamiltonian possesses a unique integrable dispersive deformation of the form

\[
H = \int \frac{w^2}{2} + f(v) - \frac{\epsilon^2}{3!} f''' v_x^2 + O(\epsilon^4) \, dx \, dy.
\]
For $f(v) = e^v$ it can be rewritten in the equivalent form

$$H = \int \frac{w^2}{2} + e^v + \frac{\epsilon^2}{3!} e^v v_{xx} + O(\epsilon^4) \, dx \, dy.$$ 

It is quite remarkable that this deformation can be extended to all orders in the deformation parameter $\epsilon$, providing a Hamiltonian formulation of the 2D Toda system,

$$H = \int \frac{w^2}{2} + \exp \left( v + \frac{\epsilon^2}{3!} v_{xx} + \frac{\epsilon^4}{5!} v_{xxxx} + \ldots \right) \, dx \, dy,$$

see Sect. 4.6 for further details.

**Example 2.** Let $H_0 = \int \alpha v^2 + \beta vw + f(w) \, dx \, dy$. Here the integrability conditions reduce to a single fourth order ODE, $f'''(\alpha f'' - \beta^2) = f''''(3\alpha f'' - 2\beta^2)$. Modulo canonical transformations, this Hamiltonian possesses a unique integrable dispersive deformation of the form

$$H = \int \frac{\alpha v^2}{2} + \beta vw + f(w) +$$

$$\epsilon^2 f'''\left( -\frac{\alpha^2}{f''} v_x^2 + \frac{\beta^2}{f''} v_y^2 + 2\alpha w_x^2 + 2\beta v_y w_y + f'' w_y^2 \right) + O(\epsilon^4) \, dx \, dy.$$

Although for $\beta = 0$ the Hamiltonian $H_0$ gives rise to the dispersionless KP (dKP) equation, the deformation presented here is not equivalent to the full KP equation: see Sect. 4.7 for further discussion.

It will be demonstrated (Theorem 14 of Section 4.4) that, modulo certain equivalence transformations, these two examples exhaust the list of Hamiltonians of type II which possess nontrivial integrable deformations to the order $\epsilon^2$. In Sect. 4.3 we prove the triviality of $\epsilon$-deformations. The structure of $\epsilon^2$-deformations is analysed in Sect. 4.4. The classification of deformable Hamiltonians of type I is summarised in Sect. 4.7.
4.2 Dispersive deformations in 2+1 dimensions

We remind that the classification of hydrodynamic systems developed in Chapter 2 is based on the method of hydrodynamic reductions, which requires the existence of infinitely many $N$-phase solutions of the form

$$v = v(R^1, R^2, \ldots, R^N), \quad w = w(R^1, R^2, \ldots, R^N),$$

(4.5)

where the phases $R^i(x, y, t)$ satisfy the commuting equations

$$R^i_t = \lambda^i(R)R^i_x, \quad R^i_y = \mu^i(R)R^i_x;$$

(4.6)

recall that the assumption of commutativity $R^i_{xy} = R^i_{yx}$ imposes the following restrictions on the characteristic speeds $\lambda^i$ and $\mu^i$:

$$\frac{\partial_j \lambda^i}{\lambda^i - \lambda^j} = \frac{\partial_j \mu^i}{\mu^i - \mu^j},$$

(4.7)

$\partial_j = \partial/\partial R^j, \ i \neq j$, see [61]. Equations (4.6) are said to define an $N$-component hydrodynamic reduction of the original system (4.4). Given a Hamiltonian system of the form (4.4), its deformation $H = H_0 + \epsilon H_1 + \cdots + \epsilon^m H_m + O(\epsilon^{m+1})$ will be called integrable (to the order $\epsilon^m$) if both equations (4.5) and (4.6) defining $N$-phase solutions can be deformed to the same order in $\epsilon$. In other words, the deformed dispersive system is required to ‘inherit’ all hydrodynamic reductions of its dispersionless limit [27, 28]. More precisely, we require the existence of expansions

$$v = v(R^1, R^2, \ldots, R^N) + \epsilon v_1 + \cdots + \epsilon^m v_m + O(\epsilon^{m+1}),$$

$$w = w(R^1, R^2, \ldots, R^N) + \epsilon w_1 + \cdots + \epsilon^m w_m + O(\epsilon^{m+1}),$$

(4.8)

where $v_i$ and $w_i$ are assumed to be homogeneous polynomials of degree $i$ in the $x$-derivatives of $R$’s (thus, both $R^i_{xx}$ and $R^i_xR^i_x$ have degree two, etc).
Similarly, hydrodynamic reductions (4.6) are deformed as

\[ R^i_t = \lambda^i(R)R^i_x + \epsilon a_1 + \cdots + \epsilon^m a_m + O(\epsilon^{m+1}), \]

\[ R^i_y = \mu^i(R)R^i_x + \epsilon b_1 + \cdots + \epsilon^m b_m + O(\epsilon^{m+1}), \]

where \( a_i \) and \( b_i \) are assumed to be homogeneous polynomials of degree \( i+1 \) in the \( x \)-derivatives of \( R \)'s. We require that the substitution of (4.8), (4.9) into the deformed system (4.4), as well as the commutativity conditions \( R^i_t = R^i_y \) are satisfied up to the order \( O(\epsilon^{m+1}) \). This requirement proves to be very restrictive indeed, and imposes strong constraints on the structure of the deformed Hamiltonian \( H \).

**Remark.** Expansions (4.8)-(4.9) are invariant under Miura-type transformations of the form

\[ R^i \to R^i + \epsilon r^i_1 + \epsilon^2 r^i_2 + \ldots, \]

where \( r^i \) denote terms which are polynomial of degree \( i \) in the \( x \)-derivatives of \( R \)'s. These transformations can be used to simplify calculations. For instance, working with one-phase solutions one can assume that \( v \) remains undeformed. Similarly, working with two-phase solutions one can assume that both \( v \) and \( w \) remain undeformed. For three-phase solutions this normalisation still leaves some extra Miura-freedom which can be used to simplify expressions for \( a_i \) and \( b_i \) (to the best of our knowledge there exist no general theory of normal forms under Miura-type transformations).

### 4.3 Triviality of \( \epsilon \)-deformations

In this section we prove that all \( \epsilon \)-deformations are trivial and can be eliminated by an appropriate canonical transformation. Thus, we consider defor-
mations of the form
\[
\begin{pmatrix}
  v \\
  w
\end{pmatrix}_t = \begin{pmatrix}
  0 & d/dx \\
  d/dx & d/dy
\end{pmatrix} \begin{pmatrix}
  \delta H/\delta v \\
  \delta H/\delta w
\end{pmatrix}
\] (4.10)
where
\[
H = \int h(v, w) + \epsilon (av_x + bv_y + pw_x + qw_y) + O(\epsilon^2) \, dx \, dy.
\]

Here \(a, b, p, q\) are functions of \(v\) and \(w\). We require that all \(N\)-phase solutions (4.5) can be extended to the order \(\epsilon\),
\[
v = v(R^1, R^2, \ldots, R^N) + \epsilon v_1 + O(\epsilon^2),
\]
\[
w = w(R^1, R^2, \ldots, R^N) + \epsilon w_1 + O(\epsilon^2),
\] (4.11)
where \(v_1\) and \(w_1\) are polynomials of order one in the \(x\)-derivatives of \(R\)'s.

Similarly, hydrodynamic reductions (4.6) are deformed as
\[
R^i_t = \lambda^i(R) R^i_x + \epsilon a_1 + O(\epsilon^2), \quad R^i_y = \mu^i(R) R^i_x + \epsilon b_1 + O(\epsilon^2),
\] (4.12)
where \(a_1\) and \(b_1\) are polynomials of order two in the \(x\)-derivatives of \(R\)'s. We thus require that relations (4.11), (4.12) satisfy the original system (4.10) up to the order \(O(\epsilon^2)\).

It was verified by a direct calculation that all one- and two-component reductions can be deformed in this way, for any \(a, b, p, q\) and any density \(h(v, w)\), not necessarily integrable. On the contrary, the requirement of the inheritance of three-component reductions (recall that the existence of three-component reductions forces \(h(v, w)\) to satisfy the integrability conditions (3.20)), is nontrivial, and leads to the following single relation:
\[
\begin{pmatrix}
  h_{vw}N_w - h_{vw}(M_w - N_v) \\
  h_{vw}h_{ww} - h^2_{vw}
\end{pmatrix}_w = \begin{pmatrix}
  h_{vw}N_w - h_{ww}(M_w - N_v) \\
  h_{vw}h_{ww} - h^2_{vw}
\end{pmatrix}_v,
\] (4.13)
here $M = (a_w - p_v)/h_{wv}$, $N = (b_w - q_v)/h_{ww}$. It remains to show that the relation (4.13) is necessary and sufficient for the existence of a canonical transformation of the form

$$H \to H + \epsilon\{K, H\} + O(\epsilon^2),$$

with $K = \int k(v, w) \, dx dy$, which eliminates all $\epsilon$-terms. Since the density of the functional $H + \epsilon\{K, H\}$ is given by the formula

$$h(v, w) + \epsilon (av_x + bv_y + pw_x + qw_y) + \epsilon (k_v, k_w) \left( \begin{array}{cc} 0 & d/dx \\ d/dx & d/dy \end{array} \right) \left( \begin{array}{c} h_v \\ h_w \end{array} \right) + O(\epsilon^2) =$$

$$h(v, w) + \epsilon (Av_x + Bv_y + Pw_x + Qw_y) + O(\epsilon^2),$$

where

$$A = a + k_v h_{vw} + k_w h_{v}, \quad B = b + k_w h_{ww},$$
$$P = p + k_v h_{ww} + k_w h_{vw}, \quad Q = q + k_w h_{ww},$$

the conditions that $\epsilon$-terms are trivial (form a total derivative), take the form $A_w = P_v, B_w = Q_v$. This leads to the following linear system for $k(v, w)$:

$$k_{v} v_{w} h_{w} - k_{w} w_{v} h_{v} = a_{w} - p_{v}, \quad k_{v} w_{v} h_{w} - k_{w} v_{w} h_{v} = b_{w} - q_{v}.$$

The compatibility conditions of these equations for $k$ can be obtained by introducing the auxiliary variable $p$ via the relation $k_{w} w_{v} = ph_{wv}$, and solving for the remaining second order derivatives of $k$,

$$k_{v} v = M + ph_{wv}, \quad k_{v} w = N + ph_{vw}, \quad k_{w} w = ph_{ww}.$$

Cross-differentiating and solving for $p_v$ and $p_w$ we obtain

$$p_v = \frac{h_{v} v N_w - h_{w} w (M_w - N_v)}{h_{v} v h_{w} w - h_{w} w^2}, \quad p_w = \frac{h_{v} v N_w - h_{w} w (M_w - N_v)}{h_{v} v h_{w} w - h_{w} w^2}.$$
Ultimately, the compatibility condition \( p_{vw} = p_{wv} \) gives the required relation (4.13), thus finishing the proof.

4.4 Reconstruction of \( \epsilon^2 \)-deformations

In this section we analyse the structure of \( \epsilon^2 \)-deformations. The result of the previous section allows us to set all \( \epsilon \)-terms equal to zero. Thus, we consider deformations of the form

\[
\begin{pmatrix}
  v \\
  w
\end{pmatrix}
_t =
\begin{pmatrix}
  0 & d/dx \\
  d/dx & d/dy
\end{pmatrix}
\begin{pmatrix}
  \delta H/\delta v \\
  \delta H/\delta w
\end{pmatrix}
\tag{4.14}
\]

where

\[
H = \int h(v, w) + \epsilon^2 h_2(v, w, v_x, w_x, v_y, w_y) + O(\epsilon^3) \, dxdy.
\]

Here \( h_2 \) is assumed to be of second order in the \( x \)- and \( y \)-derivatives of \( v \) and \( w \),

\[
h_2 = f_1 v_x^2 + f_2 v_y^2 + f_3 w_x^2 + f_4 w_y^2 + f_5 v_x w_x + \\
f_6(v_x w_y + v_y w_x) + f_7 v_y w_y + f_8 v_x v_y + f_9 w_x w_y,
\]

where \( f_1, \ldots, f_9 \) are functions of \( v \) and \( w \). Note that all terms which are linear in the second order derivatives of \( v \) and \( w \) can be removed via integration by parts. Furthermore, any expression of the form \( f(v, w)(v_x w_y - v_y w_x) \) can be omitted, since its variational derivative is identically zero. We require that all \( N \)-phase solutions (4.5) can be extended to the order \( \epsilon^2 \),

\[
\begin{align*}
  v &= v(R^1, R^2, \ldots, R^N) + \epsilon^2 v_2 + O(\epsilon^3), \\
  w &= w(R^1, R^2, \ldots, R^N) + \epsilon^2 w_2 + O(\epsilon^3),
\end{align*}
\tag{4.15}
\]
where $v_2$ and $w_2$ are polynomials of order two in the $x$-derivatives of $R$'s. Similarly, hydrodynamic reductions (4.6) are deformed as

$$R_i^x = \lambda^i(R)R_i^x + \epsilon^2 a_2 + O(\epsilon^3), \quad R_i^y = \mu^i(R)R_i^y + \epsilon^2 b_2 + O(\epsilon^3), \quad (4.16)$$

where $a_2$ and $b_2$ are polynomials of order three in the $x$-derivatives of $R$'s. We thus require that relations (4.15), (4.16) satisfy the deformed system (4.14) up to the order $O(\epsilon^3)$. The classification is performed modulo canonical transformations of the form

$$H \rightarrow H + \epsilon\{K, H\} + O(\epsilon^3),$$

here $K = \epsilon \int (av_x + bw_y + pv_x + qw_y) \, dx dy$. Note that the density of the functional $\epsilon\{K, H\}$ is given by the following formula (set $m = aw - pv, \, n = bw - qw$):

$$\epsilon^2 \left( \frac{\delta K}{\delta v}, \frac{\delta K}{\delta w} \right) \left( \begin{array}{c} 0 \\ \frac{d}{dx} \\ \frac{d}{dy} \end{array} \right) \left( \begin{array}{c} h_v \\ h_w \end{array} \right) + O(\epsilon^3) =$$

$$\epsilon^2 m(h_{vv}v_x^2 + h_{vw}v_x v_y + h_{ww}v_x w_y - h_{ww}w_x^2) +$$

$$\epsilon^2 n(h_{vv}v_y^2 + h_{vw}v_y v_x + h_{ww}v_y w_x - h_{ww}w_y^2) + O(\epsilon^3).$$

Our calculations demonstrate that generic integrable Hamiltonians $H_0$ do not possess nontrivial dispersive deformations. To be precise, these deformations are parametrised by two arbitrary functions, analogous to $m$ and $n$ above, which can be eliminated by a canonical transformation. There are cases, however, where dispersive deformations are parametrised by two arbitrary functions and a constant. It is exactly this extra constant which gives rise to a non-trivial deformation. We emphasize that canonical transformations can be used from the very beginning to bring the deformation to a ‘normal form’: since $h_{ww} \neq 0$ one can set, say, $f_6 = f_9 = 0$. This normalisation simplifies all subsequent calculations. Our results can be summarised as follows.
Theorem 14 A Hamiltonian \( H_0 = \int h(v, w) \, dxdy \) of type II possesses a nontrivial integrable deformation to the order \( \epsilon^2 \) if and only if, along with the integrability conditions (3.20), it satisfies the additional differential constraints

\[
\begin{align*}
    h_{vvv}h_{ww} - h_{vww}^2 &= 0, \\
    h_{vvv}h_{ww} - h_{vww}h_{vww} &= 0, \\
    h_{ww}h_{vww} - h_{vww}^2 &= 0,
\end{align*}
\]

that is,

\[
\text{rank} \begin{pmatrix} h_{vvv} & h_{vww} & h_{vww} \\ h_{vww} & h_{vvv} & h_{vww} \\ h_{ww} & h_{vww} & h_{ww} \end{pmatrix} = 1.
\]

(4.17)

Modulo equivalence transformations, this gives two types of deformable densities:

\[
\begin{align*}
    h(v, w) &= \frac{w^2}{2} + e^v, \\
    h(v, w) &= \frac{\alpha v^2}{2} + \beta vw + f(w),
\end{align*}
\]

where \( f(w) \) satisfies the integrability condition \( f^{m''}f''(\alpha f'' - \beta^2) = f''^2(3\alpha f'' - 2\beta^2) \).

Proof:

In contrast to the case of \( \epsilon \)-corrections where all constraints were coming from deformations of three-component reductions, at the order \( \epsilon^2 \) the main constraints appear at the level of one-component reductions already. Furthermore, it was verified by a direct calculation that multi-component reductions impose no extra conditions. Since the third order derivative \( h_{ww} \) appears as a factor in all deformation formulae, there are two cases to consider.

Case 1: \( h_{ww} = 0 \). Then the integrability conditions imply \( h_{vvv} = 0 \). The further analysis shows that one has to impose an extra condition, namely \( h_{vww} = 0 \), otherwise all deformations are trivial. Notice that conditions \( h_{ww} = h_{vww} = h_{vww} = 0 \) clearly imply (4.17). Modulo equivalence transformations, this is the case of the Hamiltonian density \( h(v, w) = \frac{w^2}{2} + e^v \). Its dispersive deformation is given in Example 1 of the Sect. 4.1.

Case 2: \( h_{ww} \neq 0 \). In this case one gets a system of equations for the coefficients \( f_1, \ldots, f_9 \) which contains \( f_4 \) as a factor. If \( f_4 \) equals zero, all de-
formations are trivial. In the case $f_4 \neq 0$ one can express $f_1, f_2, f_3, f_5, f_7, f_8$ in terms of $f_4, f_6, f_9$. What is left will be a system of two compatible first order PDEs for $f_4$, and a system of additional differential constraints for $h(v, w)$ which coincides with (4.17). Solving equations for $f_4$ we obtain a constant of integration which is responsible for non-trivial dispersive deformations. To find integrable Hamiltonian densities satisfying (4.17) we set $h_{ww} = q, h_{ww} = pq$. Then the remaining third order derivatives of $h$ can be parametrised as

$$h_{ww} = q, \quad h_{ww} = pq, \quad h_{ww} = p^2 q, \quad h_{ww} = p^3 q.$$ 

Calculating the compatibility conditions we obtain $p_v = pp_w, q_v = (pq)_w$. With this ansatz the integrability conditions (3.20) imply $p=\text{const}$ so that $q = F(w + pv + c)$ where $f$ is a function of one variable. Thus, $h$ can be represented in the form $h(v, w) = f(w + pv + c) + Q(v, w)$, where $Q(v, w)$ is an arbitrary quadratic form. Modulo the equivalence group any such density can be written in the form $h(v, w) = \alpha w^2 + \beta vw + f(w)$, and the substitution into (3.20) gives a fourth order ODE for $f$. The dispersive deformation of this Hamiltonian is presented in Example 2. We believe that both Hamiltonians from Theorem 14 can be deformed to all orders in $\epsilon$.

4.5 Example 1: dispersive deformation of the Boyer-Finley equation

In this section we discuss the key example where dispersive deformations can be reconstructed explicitly at all orders of the deformation parameter $\epsilon$. Let us consider the system (4.4) with the Hamiltonian density $h = \frac{w^2}{2} + e^v$,

$$v_t = w_x, \quad w_t = e^v v_x + w_y.$$
On the elimination of \( w \), it reduces to the Boyer-Finley equation \([7]\),

\[
v_{tt} - v_{ty} = (e^v)_{xx},
\]

(the left hand side can be put into the standard form \( v_{ty} \) by a linear transformation of \( t \) and \( y \)). An integrable dispersive deformation of this example is closely related to the 2D Toda equation, see \([5]\) for an equivalent construction based on the central extension procedure. Let us introduce the auxiliary Hamiltonian system

\[
\begin{pmatrix}
  u \\
  w
\end{pmatrix}_t =
\begin{pmatrix}
  0 & \frac{1}{\epsilon} \sinh(\epsilon d/dx) \\
  \frac{1}{\epsilon} \sinh(\epsilon d/dx) & d/dy
\end{pmatrix}
\begin{pmatrix}
  h_u \\
  h_w
\end{pmatrix},
\]

\( (4.18) \)

where the Hamiltonian density \( h \) is the same as above, \( h(u,w) = \frac{w^2}{2} + e^u \) (the exact relation between \( u \) and \( v \) is specified below). Explicitly, this gives

\[
u_t = \frac{1}{\epsilon} \sinh(\epsilon d/dx)w, \quad w_t = \frac{1}{\epsilon} \sinh(\epsilon d/dx)e^u + w_y,
\]

which, on elimination of \( w \), leads to the integrable 2D Toda equation,

\[
u_{tt} - v_{ty} = \frac{1}{\epsilon^2} (\sinh(\epsilon d/dx))^2 e^u = \frac{1}{4\epsilon^2} \left( e^{u(x+2\epsilon)} + e^{u(x-2\epsilon)} - 2e^{u(x)} \right).
\]

Introducing the change of variables \( u \leftrightarrow v \) by the formula

\[
u = \frac{1}{\epsilon} (d/dx)^{-1} \sinh(\epsilon d/dx) v
\]

\[
= (d/dx)^{-1} \left( \frac{v(x+\epsilon) - v(x-\epsilon)}{2\epsilon} \right)
= v + \frac{\epsilon^2}{3!} v_{xx} + \frac{\epsilon^4}{5!} v_{xxxx} + \ldots,
\]

one can verify that the Hamiltonian operator in \((4.18)\) transforms into the Hamiltonian operator in \((4.4)\), while the Hamiltonian density \( h(u,w) = \frac{w^2}{2} + \)

e^v takes the form

\[ h(v, w) = \frac{w^2}{2} + \exp \left( \frac{1}{\epsilon} (d/dx)^{-1} \sinh(\epsilon d/dx)v \right) = \]
\[ \frac{w^2}{2} + \exp \left( v + \frac{\epsilon^2}{3!} v_{xx} + \frac{\epsilon^4}{5!} v_{xxxx} + \ldots \right) = \]
\[ \frac{w^2}{2} + e^v \left( 1 + \frac{\epsilon^2}{3!} v_{xx} + \frac{\epsilon^4}{5!} (v_{xxxx} + \frac{5}{3} v_{xx}^2) + \ldots \right). \]

This provides the required integrable deformation for the Hamiltonian density \( h = \frac{w^2}{2} + e^v \).

### 4.6 Example 2: deformation of the dKP equation

For \( \beta = 0, \alpha = 1 \) the Hamiltonian density from Example 2 takes the form

\[ h(v, w) = \frac{v^2}{2} + f(w) \] where \( f^{'''}f'' = 3f^{'''}^2 \). The corresponding deformation assumes the form

\[ H = \int \frac{v^2}{2} + f(w) + \epsilon^2 f^{'''} \left( -\frac{1}{f''} v_x^2 + 2w_x^2 + f'' w_y^2 \right) + O(\epsilon^4) \, dx dy. \]

Without any loss of generality one can set \( f(w) = \frac{2\sqrt{2}}{3} w^{3/2} \). In this case the dispersionless system takes the form

\[ v_t = (\sqrt{2w})_x, \quad w_t = v_x + (\sqrt{2w})_y. \]

Introducing the new variable \( u = \sqrt{2w} \) one obtains

\[ v_t = u_x, \quad uu_t = v_x + u_y, \]

which, on elimination of \( v \), leads to the dKP equation \((u_y - uu_t)_t + u_{xx} = 0\), with ‘non-standard’ notation for the independent variables. The correspond-
ing KP equation, \((u_y - uu_t)_t + u_{xx} + \epsilon^2 u_{tttt} = 0\), gives rise to the following integrable deformation of the original system:

\[
v_t = (\sqrt{2w})_x, \quad w_t = v_x + (\sqrt{2w})_y + \epsilon^2(\sqrt{2w})_{ttt}.
\]

This, however, is clearly outside the class of Hamiltonian deformations.

### 4.7 Deformable Hamiltonians of type I

In this section we summarise our results on deformations of Hamiltonian systems of the form

\[
\left(\begin{array}{c}
v \\
w
\end{array}\right)_t = \left(\begin{array}{cc}
d/dx & 0 \\
0 & d/dy
\end{array}\right)\left(\begin{array}{c}
h_v \\
h_w
\end{array}\right),
\]

or, explicitly,

\[
v_t = (h_v)_x, \quad w_t = (h_w)_y.
\]

As discussed in Chapter 2, the integrability conditions constitute a system of fourth order PDEs (3.14) for the Hamiltonian density \(h(v, w)\).

**Conjecture 2** A Hamiltonian \(H_0 = \int h(v, w) \, dx \, dy\) of type I possesses a nontrivial integrable deformation to the order \(\epsilon^2\) if and only if, along with the integrability conditions (3.14), it satisfies the additional differential constraints

\[
\begin{align*}
h_{vvv}h_{vw} - h_{vww}^2 &= 0, \\
h_{vww}h_{ww} - h_{vvw}^2 &= 0, \\
h_{wvw}h_{vww} - h_{vw}^2 &= 0,
\end{align*}
\]

or, equivalently,

\[
\text{rank} \left(\begin{array}{ccc}
h_{vvv} & h_{vvw} & h_{vww} \\
h_{vww} & h_{vww} & h_{vw} \\
h_{wvw} & h_{vw} & h_{ww}
\end{array}\right) = 1.
\]

Modulo equivalence transformations, this gives three types of deformable den-
where $f$ satisfies the integrability condition

$$(\beta + f'')Qf''' = f''''[3Q + (\beta - \alpha)(\beta - \gamma)],$$

here $Q = (\beta + f'')^2 - (\alpha + f'')\gamma + f'''.

Dispersive deformations of these Hamiltonians are given by the following formulae (we use the normalisation $f_8 = f_9 = 0$ which can always be achieved by a canonical transformation; furthermore, all $\epsilon$-deformations are trivial, and have been set equal to zero):

$$H = \int vw + \alpha v^3 + e^2(6\alpha v_x^2 + v_xw_x) + O(\epsilon^4) \, dxdy,$$

$$H = \int \frac{1}{2}(v^2 + w^2 + e^2 + e^2e^v(2(1 + e^v)v_x^2 - 2w_x^2 - 2\epsilon^v v_x w_x + v_x w_y + v_y w_x) + O(\epsilon^4) \, dxdy.$$  

The third case is somewhat more complicated:

$$H = \int \frac{\alpha}{2} v^2 + \beta vw + \frac{\gamma}{2} w^2 + f(v + w) + \epsilon^2 h_2 + O(\epsilon^4) \, dxdy,$$

where $h_2 = f_1 v_x^2 + f_2 v_y^2 + f_3 w_x^2 + f_4 w_y^2 + f_5 v_x w_x + f_6 (v_x w_y + v_y w_x) + f_7 v_y w_y,$

recall that we use the normalisation $f_8 = f_9 = 0$. The coefficients $f_1 - f_7$ are defined as follows:

$$f_1 = (\alpha + f'')(\beta + f'')^2 Q, \quad f_4 = (\gamma + f'')(\beta + f'')^2 Q,$$

$$f_2 = (4\beta - \alpha + 3f'')(\beta + f'')^2 Q, \quad f_3 = (4\beta - \gamma + 3f'')(\beta + f'')^2 Q,$$
\[ f_5 = (\beta + f'')Q[Q + 4(\alpha + f'')(\beta + f'')], \quad f_7 = (\beta + f'')Q[Q + 4(\gamma + f'')(\beta + f'')], \]
\[ f_6 = \frac{1}{2}(\beta + f'')Q[2Q + (2\beta - \alpha - \gamma)^2]. \]

We conjecture that these Hamiltonians from can be deformed to all orders in \( \epsilon \).

### 4.8 Deformable Hamiltonians of type III

We consider Hamiltonian systems of type III,

Our calculations suggest that none of these examples are deformable. We propose the following:

**Conjecture 3** For Hamiltonians of type III, all deformations of the order \( \epsilon^2 \) are trivial.
Chapter 5

Concluding remarks and outline

In the discussion of hydrodynamic systems we restricted ourselves to the case of local Poisson brackets of hydrodynamic type, and local Hamiltonians. The main reason for this is the existence of a well-developed theory of differential-geometric Poisson brackets, allowing their efficient classification. On the other hand, the approach of [4] based on the Dirac reduction suggests that there exists a whole variety of integrable models corresponding to non-local brackets/Hamiltonians, involving variables of the form $d_y^{-1}d_x v$, $d_y^{-1}d_x w$, etc. It is a challenging problem to extend our classification program to non-local Hamiltonian systems: the construction of a covariant theory of non-local Poisson brackets would be the first step in this direction. The question for the description of the corresponding Hamiltonian hierarchies meets the same difficulty - higher symmetries /conservation laws are non-local. Furthermore, the study of the associated Hamiltonian hydrodynamic chains requires introduction of a canonical set of non-local variables reducing all higher flows of the hierarchy to infinite component systems of hydrodynamic type.

For dispersive systems our results demonstrate that, already at the order $\epsilon^2$, the requirement of the existence of nontrivial deformations is very restrictive so that ‘generic’ integrable Hamiltonians are not deformable. The main reason for this is apparently the assumption that all higher order dispersive
corrections are *local* expressions in the dependent variables $v, w$ and $x, y$-derivatives thereof. It would be of interest to extend this scheme to the case of nonlocal brackets/Hamiltonians, see [4] for particular examples obtained via Dirac reduction.

Furthermore, to the best of our knowledge the theory of deformations of multi-dimensional Poisson brackets of hydrodynamic type has not been constructed: is it true that all such deformations are trivial, as in the $1 + 1$ dimensional case?

Moreover, calculations leading to Example 1 of Sect. 2 show that any Hamiltonian of the form $H_0 = \int \frac{w^2}{2} + f(v) \, dx dy$, where the function $f$ is arbitrary, possesses a unique dispersive deformation of the form

$$H = \int \frac{w^2}{2} + f(v) + \frac{\epsilon^2}{3!} f''' v_x^2 + O(\epsilon^4) \, dx dy,$$

which inherits all one-phase solutions to the order $\epsilon^2$. Thus, one can speak of ‘partial integrability’ of a certain kind. However, already the requirement of the inheritance of two-phase solutions forces $f$ to satisfy the integrability condition $f''' f'' = f'''^2$.

Finally, the general theory of the studied systems suggests that the dispersionless case breaks down (one of the spatial derivatives becomes infinite) in finite time, whereas the deformed system should not have such behaviour, and a numerical study might be carried out in this direction.
Bibliography


