The influence of the basic electronic calculator on the teaching and learning of mathematics in the 11-16 age range

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**Due Date:**

- 6 Oct 1995
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- 6 Feb 1995
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THE INFLUENCE OF THE BASIC ELECTRONIC CALCULATOR ON THE
TEACHING AND LEARNING OF MATHEMATICS IN THE 11 - 16 AGE RANGE

by

PAUL GEORGE EDMONDS, B.Sc.

A Master's Dissertation submitted in partial fulfilment of
the requirements for the award of the degree of M.Sc. in
Mathematical Education of the Loughborough University of

Supervisor : Dr. D.R. Green, M.Sc., M.Ed., A.R.I.M.A.

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ABSTRACT

The electronic calculator is now invariably the device used by people in employment and everyday life to deal with complicated and tedious calculations. The aim of this dissertation is to examine the effect it may have on the secondary school mathematics curriculum and, especially, to examine its potential as a powerful teaching aid which can be used to help pupils to acquire understanding of mathematical concepts.

Chapter 1 investigates the contribution the basic calculator makes as a calculating aid which should cause the teacher to reassess the place of the standard pencil and paper algorithms in the curriculum. Some of the fears associated with this innovation are also discussed. The final section emphasises the importance of knowing the idiosyncrasies of different calculators.

Chapter 2 suggests, in some detail, ways in which the teacher may use the calculator to enhance the understanding of certain topics such as fractions and place value. Applications of the calculator to everyday life problems, such as compound interest, are also included as well as the possibility of more interesting and enjoyable topics being introduced into the syllabus. New methods, such as iterative procedures, are discussed and the potential of the calculator as an aid to investigations is ascertained.

Chapter 3 looks at the beneficial influence of the calculator on the mathematics curriculum generally and the possible effect on the mathematical content in particular with further suggestions following on from Chapter 2. Some contentious issues are considered and it is emphasised that more must be done to encourage the effective use of the calculator and not allow it to be overshadowed by its more 'glamorous' counterpart - the microcomputer.
ACKNOWLEDGMENTS

I would like to express my gratitude to all of the people who have helped me with this dissertation. In particular, I would like to thank my supervisor, Dr. D.R. Green of C.A.M.E.T., Loughborough University, for his guidance, Alan Graham of the Open University, for his encouragement, and the mathematics staff and pupils of Collenswood School, Stevenage, for their help in testing the calculator games.

My special thanks to my wife for typing the manuscript and for her general support and forbearance.
I declare that this dissertation is entirely my own work.
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Chapter 1: THE CALCULATOR AS A CALCULATING AID

1) Historical Background
2) Attitudes Towards the Calculator
3) Numeracy and the Standard Written Algorithms
4) Getting to Know Your Calculator

It is generally accepted that the man who designed and produced the first calculating machine was Blaise Pascal (1623-62), a Frenchman whose father was a government official. The father always seemed to be weighed down with tedious calculations, so Blaise at the age of 19 years produced his first machine in 1642.

"by means of which you alone may, without any effort, perform all the operations of arithmetic and may be relieved of the work which has often fatigued your spirit ........"
HISTORICAL BACKGROUND

Most people are aware that there is a great debate going on about mathematical education in primary and secondary schools, but most people are not aware that it has been going for over 100 years.

In recent years there has been considerable discussion of the nature of mathematical understanding. There is general agreement that understanding in mathematics implies an ability to recognise and to make use of a mathematical concept in a variety of settings, including some which are not immediately familiar. The idea that "the ability to solve problems is at the heart of mathematics" (Cockcroft, 1982, 231) is certainly not a new one. Polya (1961) put this well many years ago: "Mathematical know-how is the ability to solve problems - not merely routine problems but problems requiring some degree of independence, judgement, originality and creativity".

The debate has centred mainly on the emphasis that should be put on the computational techniques involved in arithmetic and on the ability to understand and solve the problem and how this understanding may be achieved.

The following comment was made by an H.M.I. in 1895: "The accuracy of the work is all that can be desired, and in many cases marvellous: at the same time the oral test shows that the children are working in the dark .... (This) shows itself in the inability .... to solve very simple problems".

The situation seems to have changed little over the years. The first report from the Assessment of Performance Unit based on a 1978 survey, concludes: "Most 11 year olds can do maths involving fundamental concepts and skills and do simple applications. There is a fairly sharp decline as understanding is
probed more deeply into more complex settings and unfamiliar contexts."

One factor which has contributed to the difficulty in problem-solving is that the standards by which the results of mathematical education are judged too often only concern the childrens' facility with tools - multiplication, adding fractions and so on. The pressure on the teacher is predominantly in these terms. Far less external concern is expressed about childrens' problem-solving. The Cockcroft Committee received very many submissions which showed concern for the 'basics' usually defined in terms of purely arithmetic skills, with stress on the operations of addition, subtraction, multiplication and division treated in isolation from application to real situations. The skills which are basic are presumably needed as a basis either for the mathematics required in employment or in adult life or for further study. Although many of the requirements may be considered to be 'elementary' in terms of their position within the hierarchy of learning mathematics and the stage of schooling at which they are first introduced, it does not follow that they are necessarily either simple or straightforward for most pupils to learn, and, more importantly, to apply.

The most significant event in recent times in mathematical education has been the introduction of so-called 'modern' mathematics. A direct outcome of a conference held in Southampton in 1961 was the setting up of the School Mathematics Project. A Director of S.M.P. wrote in his report for 1962-63 that "a major aim of the syllabus is to make school mathematics more exciting and more enjoyable, and to impart a knowledge of the nature of mathematics and its uses in the modern world".

The course was designed for those pupils whose mathematical attainment was in the top quarter of their age group and the
classroom materials designed for the course took for granted that the teachers using them would possess sufficient mathematical insight and experience to enable them to work in the ways intended. However, as the movement spread the materials were used by lower attaining pupils and by teachers who did not have the necessary mathematical background and training to appreciate the intentions underlying the new course. The materials were often presented as a collection of disconnected topics whose relevance to the mathematics course as a whole did not become apparent to the pupils. Many teachers who had previously concentrated all their efforts on the basic skills now went to the other extreme and neglected them. This led to claims and counter-claims by opponents and proponents of the movement and at one stage a teacher claimed to be either 'modern' or 'traditional'. Public concern for standards grew to such an extent, that pressure was created in some quarters for a 'back to the basics' movement which culminated in the renowned Ruskin College speech by James Callaghan in 1976.

So pity the poor teacher who has been tugged this way and that way in a relatively short period of time. It is no wonder that the 1979 H.M.I. report on secondary education referred to the problem in these terms: "Many schools are not finding it easy to strike an appropriate balance between the varied demands which are being made on them at the moment ..... The dilemma .... is accentuated by demands for 'greater numeracy' because the notion of numeracy is so often ill-defined. In a great many cases the schools are responding by concentrating narrowly on computational skills, devoid of context or application, in a way which easily becomes counter productive. The notion of numeracy should certainly include more than accurate computation .... it should include .... the ability to apply knowledge in fresh circumstances".

The importance of developing the skills of computation for all members of society has never been doubted. The problem has been
that so much time must be devoted to mastering these skills for many pupils, that the reason for developing them is too often neglected - that is, their use which enables an individual to cope with the practical demands of his everyday life. There have been times when the educational system has been in danger of producing a nation of clerks. Too many children and adults who have been quite incapable, for a variety of reasons, of mastering these underestimated skills have become disillusioned at best and at worst left with "feelings of fear, helplessness, anxiety and even guilt" (Cockcroft, 20)

What joy if somehow these skills could be mastered more quickly or their importance nullified to some extent! More avenues would then be opened to the really rewarding, enjoyable and even exciting aspects of mathematics.

About 30 years ago the electronic calculator was invented. In the mid 1970's their price started to fall so rapidly that educationalists recognised the very great implications for the teaching of mathematics in schools.
ATTITUDES TOWARDS THE USE OF CALCULATORS IN SCHOOL

These implications are far reaching and include the ways in which calculators can be used to assist and improve the teaching of mathematics in the classroom and the extent to which they should change the content of what is taught and the relative stress which is placed on different topics within the mathematics syllabus.

However, the real controversy surrounding the introduction of calculators is about the use of them by children who have not yet mastered the traditional pencil-and-paper methods of computation. It is feared that children who use them too early will not acquire fluency in computation nor confident recall of basic number facts. These fears are understandable and may be illustrated by views expressed in submissions to the Cockcroft Report (375) For instance:

"Exercise of the basic skills should not depend upon use of calculators: these should be limited to higher education".

"Mathematics must be a compulsory subject, taught to a reasonable standard using one's loaf and not a calculator".

These views are not just shared by some teachers and parents but also by employers. The Bath and Nottingham studies, reporting to the Cockcroft Committee, believes that there is an ambivalent attitude to the use of calculators in industry and commerce at the present time. In many types of employment which require a considerable amount of calculation and analysis of data, they are regarded as desirable aids to speed and accuracy. However, their use is still viewed with suspicion by some shop floor managers and supervisors (especially those who supervise engineering and other technical apprentices and craftsmen of various kinds) who themselves were trained to use slide rules or logarithm tables. However, the majority of young employees who were seen to be using calculators at work had not been trained in their use either at school or on the job. In consequence, calculators
were frequently not being used in the most effective way. On one fact, however, there is general agreement - calculators are here to stay. They are becoming increasingly available to young and old, in all walks of life, and the proper use of them must be taught at some stage. Most would also agree that this is best accomplished at school. However, questions still surround the stage at which they should be introduced in the school life of the student; their use in examinations; which ability groups would benefit most from them; and so on.

Some would argue that low-attaining pupils should not use them at all but should concentrate on mastering the basics, while others support the view that it is these pupils who would benefit most from the use of a calculator.

"Calculators have revolutionised computation and barely numerate students can overcome their weaknesses with these". (Cockcroft 375)

Proponents stress that the availability of a calculator in no way reduces the need for mathematical understanding on the part of the person who is using it. For example, knowing how to multiply is a different skill from knowing when to multiply. A calculator is useless until the student knows how to solve the problem.

There remains too, some uncertainty about introducing calculators in the primary classrooms. Views vary from complete freedom to replace pencil and paper calculations to a complete ban on them. There is little evidence about the eventual balance to be obtained at the primary stage between calculations carried out mentally, on paper, or with a calculator. However, one submission to the Cockcroft Committee from a parent may well prove significant: "Following professional advice of mathematical colleagues we kept calculators away from our children until their late teens. But the youngest at age 6 got hold of a calculator to help out with his 'tables' and found it such fun that he has been much
more mathematically inclined since. So perhaps it would be wise to introduce simple calculators at an early age".

In the secondary school, support for the use of the calculator is much more prevalent.

In their 1981 questionnaire to Heads of Department, the Mathematics Advisers of Hertfordshire asked for details of policy concerning the use of the calculator. Most replies from schools indicated that a policy did exist with only a few making reference to a 'flexible policy' or 'at the teacher's discretion'. Most schools gave a stage at which its use was first allowed which indicated the popular attitude was for mastering basic computational skills with pencil and paper first and that other uses of the calculator did not rate highly. The information is:

<table>
<thead>
<tr>
<th>Stage</th>
<th>Percentage</th>
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<tbody>
<tr>
<td>1</td>
<td>6%</td>
</tr>
<tr>
<td>2</td>
<td>8%</td>
</tr>
<tr>
<td>3</td>
<td>12%</td>
</tr>
<tr>
<td>4</td>
<td>45%</td>
</tr>
<tr>
<td>5</td>
<td>8%</td>
</tr>
</tbody>
</table>

However, the Advisers have reason to believe that this information is already out of date. Since the publishing of the Cockcroft Report there is evidence that its strong advocacy of the use of calculators has already encourage their greater use in the classroom.

Two significant events in 1983 provide evidence that the move towards accepting the calculator in the classroom is gaining momentum. In January 1983, the G.C.E. and C.S.E. Boards Joint Council for 16+ National Criteria published its recommendations for mathematics. The written submissions from all walks of life overwhelmingly demanded that electronic calculators should be used
in at least part of the assessment. The list of assessment objectives includes 'the use of the electronic calculator' and the content list includes 'the efficient use of an electronic calculator' for all children. Thus if the criteria are approved by the Education Secretary, it will be assumed that all children from 11 to 16 will use them throughout secondary school and will take them into public examinations in their final year.

An equally significant event will be the introduction of the calculator-based course. In September 1983 the new S.M.P. 11 - 16 course will be available to all schools. At its inception 5 years ago, the team of teachers who planned the course assumed that calculators would be available to all pupils, and devised the materials accordingly. The course has been highly acclaimed by schools involved in the pilot scheme. Much of the material necessitates the use of a calculator and takes advantage of the power of the calculator to promote understanding as well as allow realistic problems to be introduced. Other material requires work to be done without its use. To ensure mental skills are not neglected the team is devising oral tests for all pupils when they leave school.
NUMERACY AND THE STANDARD WRITTEN ALGORITHMS

Numeracy

The concept of 'numeracy' and the word itself were introduced in the Crowther Report published in 1959. The definition is intended to imply a quite sophisticated level of mathematical understanding, but over the last few years the meaning seems to have changed and it is now associated with the ability to perform basic arithmetic operations. However, Cockcroft emphasises the wider aspect and would wish the word 'numerate' to imply the possession of two attributes. The first of these is an 'at-homeness' with numbers and an ability to cope with the practical mathematical demands of everyday life. The second is an ability to have some appreciation and understanding of information which is presented in mathematical terms, for instance graphs, charts or tables or by reference to percentage increase or decrease.

"Our concern is that those who set out to make their pupils 'numerate' should pay attention to the wider aspects of numeracy and not be content merely to develop the skills of computation". (39)

Perhaps the most significant definition of numeracy is the one given by Girling (1977)

"Basic numeracy is the ability to use a 4-function electronic calculator sensibly".

The implications of this definition are more far-reaching than is at first apparent. In brief, Girling believes this implies:

1) the need to be able to check that the user and/or calculator have not made a mistake
2) the need to understand the relative size of numbers
3) the need to be able to perform mental calculations for speed, convenience and for use in the commercial and industrial world.
The extract below comes from 'Practical Arithmetic' a book published in 1884.

DIVISION OF DECIMALS.

Example 4—Divide 690.23 by 41, so as to have six decimal places in the quotient.

Divide and dividend are extended by repeating the figures of the respective recurring decimal numbers. The operation is then carried on in the same way as in the contracted process in Example 3. It is necessary to observe that when the divisor does not contain so many figures as those required in the quotient, the division must be carried on entire, till the figures required in the quotient be one less than the figures of the divisor, before we begin to cut off figures from the divisor. In this example, for instance, there is one place of whole numbers (this is obvious from dividing 308, the integral part of the dividend, by 41, the integral part of the divisor), and six places of decimals, so that the divisor must contain at least seven figures to secure greater accuracy, a figure more is taken in the annexed example.

Reason or the Rule.—The removal of the decimal points in the divisor and dividend to the same number of places to the right, is just multiplying the divisor and dividend by the same number; and this operation does not affect the value of the quotient. The divisor being now a whole number, the number of decimal places in the quotient must be equal to the number of decimal places in the dividend, since the dividend is the product of the divisor and quotient.

Exercise.

1. Divide 4.29213 by 0.73, and 73. Ans. 60.24 and 6624.
2. 0.54644881676 by 6.852, and 0.005651.
   Ans. 73.93325, and 79693.25.
3. 795 by 0.0188, and 18.3125, Ans. 4019.2, and 40192.
4. 10.7255 by 99.7255, Ans. 1.053297.
5. 11.237 by 0.009184.
   Ans. 1.259-9837.
6. 2.0122 by 271.884, Ans. 78.157.
7. 17.331 by 765, so as to have six decimal places in the quotient.
   Ans. 0.0054965.
8. 2.4587 by 90, so as to have five decimal places in the quotient.
   Ans. 0.00006.

To divide by 10, 100, or 1 with any other number of noughts annexed, it is only necessary to remove the decimal point as many places to the left, as there are noughts in the divisor; thus—124.5 divided by 100, becomes 1.245. Also,

9785 + 10 = 9795, for 9375 + 10 = 9385 = 0.003765,

Similarly, 9785 + 1000 = 0.003765, &c.

Exercise.—Divide 7.2754 by 10, 100, 10000, 1000000.
The Standard Written Algorithms

If we accept that the calculator will change but by no means eliminate the need for reasonable mental and written skills of calculation, an important question arises. What is to be regarded as appropriate levels of skills in the calculator age?

Go into any classroom in this country and it is a fair bet that the children have been taught the same standard written algorithms for the four rules of number. With a few minor variations they will probably look like this:

\[
\begin{array}{cccc}
68 & \text{567} & 64 & \text{13} \\
+20 & -49 & \times 8 & 5765 \\
97 & 18 & 512 & \\
\end{array}
\]

And yet is this the way people really calculate given a free choice of method? It may well be an illuminating exercise for the experienced, traditional teacher to ask his pupils how they would tackle similar calculations mentally given a free choice. It may well be an interesting exercise for the same teacher to examine how he approached the calculations himself. A fascinating variety of methods would probably emerge.

There are, of course, important reasons why the written algorithms have been developed and persevered with over many years. They are written, so the calculation is permanent and correctible. They are standardised, that is, it is possible to arrange that everyone does the same thing. They are general, in the sense that they will work for any numbers, large or small, whole or decimal. This is possible because the methods require breaking a number up into hundreds, tens and units and dealing with these as digits separately. However, this process does not correspond to the ways in which people tend to think about numbers, and the
methods are usually carried out automatically without a real need to understand what is going on. Rote learning is, thus, usually associated with the teaching of these methods. However, they have remained attractive because they are so general and efficient and because they are much easier to manage and mark than mental techniques. Add to this the fact that they are traditional and we see why they remain so sacrosanct in the eyes of the general public.

Do real alternatives to these methods exist?

In 1973, D.A. Jones, investigated the methods used by each of 80 11 year olds to calculate

\[
67 + 38, 83 - 26, 17 \times 6, \text{ and } 116 \div 4^2
\]

The questions were written in this form, and the children were free to use written or mental methods. Over half of the 320 calculations were successfully completed by non-standard methods despite the heavy teaching of standard algorithms. This suggests, at the least, that the standard methods are not suitable for mental work. At the same time, quite often, they are not understood by children. Mathematics teachers will, no doubt, recognise these errors:

\[
\begin{array}{cccc}
44 & -28 & 45 & 43 \\
+32 & -37 & \times 2 & 2 \times 218 \\
61 & 12 & 2012 & 19
\end{array}
\]

Further, the standard algorithms are misused. How often do we see calculations such as these?

\[
\begin{array}{cccc}
1003 & -924 & 36 & 101 \div 7320 \\
& \times 100 & 3600 & 732 \\
& 009 & 000 & 00 \\
& 00 & 3600 &
\end{array}
\]
At least the mental algorithms require understanding, and although they are not designed for recording they can be if desired. They are also flexible and can be adapted to suit the numbers involved. They work with complete numbers rather than separated tens and units digits e.g.

\[
\begin{align*}
4 \times 45 &= 2 \times 90 = 180; \\
6 \times 28 &= 6 \times 30 - 12 = 168
\end{align*}
\]

Unfortunately, they are limited in the sense that they cannot be applied to the most difficult calculations, which is why they have not been developed further. However, with a greater emphasis on understanding predominating these days, it would be sensible to develop these mental techniques at the expense of the traditional written algorithms for the easier calculations, providing an alternative method can be found for the more difficult calculations.

With the advent of the electronic calculator, Stuart Plunkett (1979), believes a real alternative now exists, which may remove the necessity of spending an inordinate amount of time teaching the standard algorithms. He illustrates his proposals by examining the wide range of calculations we can do with numbers.

Plunkett, for convenience, divides the range into five rough bands

<table>
<thead>
<tr>
<th>Red</th>
<th>Orange</th>
<th>Yellow</th>
<th>Green</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>6+9</td>
<td>145+100</td>
<td>148+39</td>
<td>693+387</td>
<td>4974+6872+4567+7928</td>
</tr>
<tr>
<td>13-8</td>
<td>85-20</td>
<td>85-27</td>
<td>682-376</td>
<td></td>
</tr>
<tr>
<td>5x7</td>
<td>5x30</td>
<td>18x3</td>
<td>841x8</td>
<td>891x678</td>
</tr>
<tr>
<td>45-5</td>
<td>60-3</td>
<td>72-4</td>
<td>693-7</td>
<td>8391-57</td>
</tr>
</tbody>
</table>

They are in order of increasing difficulty and yet probably in order of decreasing frequency of use. The Red band contains number bonds up to 10+10 and 10x10 and their inverses, and these facts should be available for instant
recall.
The Orange band can be done by one-step mental methods given thorough knowledge of the Red band.
The Yellow band calculations are also appropriate for mental methods and the average person in the street should be able to do these in his head.
The Green band could be done mentally but it would probably be more appropriate to reach for pencil and paper.
In a practical situation it would be absurd to use a mental process for the Blue band, but if a calculator were available, it would be equally absurd to use a written method.

So Plunkett proposes that there is now a place for mental algorithms, the use of the calculator and non-standard written methods. He believes that the standard written algorithms are out of date because the large amount of time that is at present wasted on attempts to teach and learn them so often leads to frustration, unhappiness and a deteriorating attitude towards mathematics. The Cockcroft Committee will vouch for this.

Three stages are proposed that children may go through, progressing according to ability.
Stage 1. The acquisition of mental techniques for calculations in the red, orange, and yellow bands.
Stage 2. The use of calculators for green and blue calculations.
Stage 3. The development of some casual written methods.

Following on from this stage, an understanding of larger numbers should be made easier, and so children will be more likely to make sensible use of the calculator for more difficult problems. It will be less likely that they will press the wrong button, but their understanding of orange band calculations will enable them to check the machine’s answer for reasonableness. In other words, the calculator will make sense.
Stage 3 would be necessary when calculators are not available in an everyday life situation, especially money problems. Non-standard written methods for dealing with these can be adopted by children and teachers from mental algorithms used for more manageable numbers.

The advent of the calculator has, therefore, provided us with a great opportunity. We do not need to teach methods of dealing with calculations of great complexity for the average citizen at least. We can now foster methods more suitable to the minds and purposes of the users. People who need complex calculations for their jobs will find a calculator a much more valuable aid than pencil and paper.

However, we are expected to be able to calculate, and the standard algorithms are the traditional ones. It would be unthinkable for many people that other methods and a calculator should be allowed to replace them. But the argument that most everyday calculations are mental ones and much time is wasted teaching the standard written ones is strong. As Plunkett concludes:

"The calculator should be regarded as a sensible tool for difficult calculations, the ideal complement to mental arithmetic".
GETTING TO KNOW YOUR CALCULATOR

It is inevitable that pupils will possess different makes and models of calculators, with the consequent difficulties of co-ordinating their usage in the classroom. Of immediate concern are the problems which will arise when younger secondary pupils, particularly, bring to their mathematics lessons calculators which operate in the reverse Polish system or those which automatically display numbers in exponential form. Clearly the range of characteristics of individual makes of calculator is greater than the two referred to above, but these are particularly critical to the conduct of lessons in which the use of a calculator plays a fundamental part. Fortunately, more and more schools, often with encouragement from Mathematics Advisers, are able to make available a class set of a single model (or at least a half class set). Free calculators have been received by all Suffolk schools with pupils in the age range 7-16, "sufficient to launch them into a new era", while the advisers of Northampton provide a similar service after the teachers have attended an in-service course run by them on calculators. Further, schools are also advising parents as to the type of calculator used by the school and encouraging them to buy the recommended model. The problems of differences between calculators will gradually recede and this chapter will be more concerned with fundamental differences between basic calculators. (Green and Lewis, 1978, provide a very full account of all the differences the teacher may encounter and also look at the inside of the calculator).

Alan Graham, Centre for Mathematics Education, Open University, identifies three important aspects in which basic calculators differ:

1. logic - arithmetic or algebraic
2. methods of correcting keying-in errors
3. operation of the constant function - K or automatic
As an introduction a class, who have been requested to bring in their own calculators where available, may be given a set of calculator sequences to predict the answer and then to find the actual calculator answer:

<table>
<thead>
<tr>
<th>GUESS</th>
<th>ANSWER</th>
</tr>
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<tbody>
<tr>
<td>(a)</td>
<td>2 x 3 + 4 =</td>
</tr>
<tr>
<td>(b)</td>
<td>2 + 3 x 4 =</td>
</tr>
<tr>
<td>(c)</td>
<td>4 + 3 =</td>
</tr>
<tr>
<td>(d)</td>
<td>3 - 2 =</td>
</tr>
<tr>
<td>(e)</td>
<td>2 + 3 ON/C 4 =</td>
</tr>
<tr>
<td>(f)</td>
<td>3 + 2 = =</td>
</tr>
</tbody>
</table>

On the Texas TI 30 the answers were as follows; (a) 10 (b) 14 (c) 1 (d) 1 (e) 6

It is important that the inconsistencies that are likely to crop up be dealt with before any further work is done. A few simple tips will be of help to the pupils.

1. Type of Logic - Algebraic or Arithmetic

This is perhaps the most basic difference between calculators. At the cheap end of the calculator market, most machines are programmed to carry out the operations arithmetically - i.e. in the order in which they are fed into the machine. Thus, for sequence (b)

$$2 + 3 \times 4 =$$
a calculator with arithmetic logic will perform the addition of 2 and 3 before multiplying the result by 4. A machine with algebraic logic, however, obeys the conventions of precedence in algebra, where multiplication and division must be carried out before addition and subtraction. The pupils may discover ways of coaxing an 'algebraic' calculator to calculate 'arithmetically'

1. Key stroke errors — and how to correct them

A later chapter emphasises the importance of estimating an answer to a problem so that a ridiculous answer cannot be accepted. However, quite often we make mistakes in pressing the wrong button part of the way through a problem and realise the error at the time. Do we start again?

Sequences (c) and (d) show what course to take if we press the wrong operation by mistake. On many machines, including the TI 30, all but the last of a sequence of operations will be ignored. So if you accidently press $-$ instead of $+$, this will be corrected by pressing $+$ next time. This does not work for some calculators. Pressing $6 + -$ may confuse some machines into displaying 12, while $6 - +$ may give 0.

Sequence (e) should confirm how the calculator copes with pressing the wrong number. What happens with this sequence:

2. The constant key —

A constant facility enables the user to set the calculator up to perform the same 'sum' repeatedly at the touch of the $=$ key. If the machine has a key marked $K$, then this can be done directly.
For example, the TI 30 requires $5 + \overline{5}$ to add on 5 to any number chosen followed by $=$. Thus:

\[
\begin{align*}
4 \overline{5} & \text{ gives 9} \\
10 \overline{5} & \text{ gives 15}
\end{align*}
\]

and so on.

If the machine does not have a $K$ key, it is likely that the constant facility comes on automatically after a suitable calculation is keyed in.

For example, if the following sum is keyed in:

```
3 \overline{4} \overline{5}
```

Now press

```
1 \overline{1} \overline{1} \overline{0} \overline{8} \overline{1} \overline{1} \overline{0} \overline{1} \overline{0}
```

With many calculators the addition sum ($\overline{4}$) is carried over as a constant addition and so 4 will be added to 1, 10, 100, giving 5, 14, 104. Other models take the $3 \overline{4}$ part of the original sum as its constant function. A few models bring in their constant function when an operation is pressed twice in succession and kills it when any key other than a number or $=$ is pressed.

After practice with an exercise of 'Guess and Press', for example, 2 $\overline{x} \overline{1} \overline{1} \overline{0}$, further practice may be given in the form of a game:

**Guess the Number**

Player A chooses any number between 1 and 100 (say 40) and, unseen by B presses 40 $\overline{x} \overline{K} \overline{0}$ (zero wipes 40 from display)

Player B has then to guess which number (denominator) A has chosen by trying different numbers (numerators) and pressing $=$

B's aim is to guess A's number as quickly as possible, B then chooses a number for A to guess.
Sample Play: B's attempts:

<table>
<thead>
<tr>
<th>B Presses</th>
<th>Display</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.125</td>
<td>5 is too small</td>
</tr>
<tr>
<td>24</td>
<td>0.6</td>
<td>24 is too small</td>
</tr>
<tr>
<td>50</td>
<td>1.25</td>
<td>50 is too big</td>
</tr>
<tr>
<td>40</td>
<td>1</td>
<td>40 is the denominator</td>
</tr>
</tbody>
</table>

B's score is 4

The game can be adapted if a calculator's constant is set up in a different way.

These three important features, i.e. logic, keying-in errors, constant facility should be checked systematically by the pupils as described previously, before the calculator can be used effectively.

Accuracy and Overflow

Another problem which arises in the use of different calculators is why, for example, \( \frac{1}{3} \times 3 \) appears on some as 0.999 999 9, on others as 1.000 000 2 and on others simply as 1; or why \( \sqrt{2} \) may turn out to be 1.999 999 8. The answer is connected entirely with the conventions which have been built into the machine.

Some 'round off', i.e. give the number corrected to the number of figures on display, some truncate, i.e. cut the end off the number, some round up, some hold more figures than they display, some work in scientific (exponent) form, even when they display in index-free form. There are those machines which convert to exponent form as soon as numbers become too large or too small for display and others merely display 'E' when this happens.
successfully eradicating all trace of your calculations so far!

The subject can become confusing but can also be an interesting source of investigation. One useful quick test enabling the teacher to diagnose the accuracy to which any particular machine works, when dividing, is to perform:

\[(7 \div 9) \approx 0.7777777\]

Some calculators are designed particularly for use in scientific problems and can work in exponential notation. Such a calculator will usually display an answer, of say, 38 000 000 as 3.8 07, meaning \(3.8 \times 10^7\), or will display 0.000 000 76 as 7.6 -07, meaning \(7.6 \times 10^{-7}\). However, calculators which do not work in this way, do not have enough room to display a very large or very small number, and usually have special symbols to indicate overflow or underflow. If this is the case, matters may be improved by using standard form. For example:

\[
\begin{align*}
5 180 \times 69 5000 &= 5.18 \times 10^3 \times 6.95 \times 10^4 \\
&= (5.18 \times 6.95) \times 10^7 \\
&= 36.001 \times 10^7 \text{ or } 360 010 000
\end{align*}
\]

More able pupils may be asked to investigate the problem

\[
\frac{0.00086}{1162} \times 438
\]

by comparing and discussing answers to each of the following methods:

1. \((0.00086 \div 1162) \times 438;
2. \((0.00086 \times 438) \div 1162;
3. \((438 \div 1162) \times 0.00086;
4. \frac{8.6 \times 10^{-4} \times 4.38 \times 10^2}{1.162 \times 10^3}
\]

Recommended Calculators

A sensible question would seem to be "why can't calculators be
It is likely that all L.E.A.'s have set up working parties to discuss and recommend essential and desirable features of a calculator and then try to find one on the market that best fits these features. The Hertfordshire Calculator Working Party meets regularly to revise its recommendations according to availability of new models. Liquid Crystal Display is high on its list, together with an automatic 'power off' device, a positive 'feel' or 'click' when a key is depressed, as well as strength and a reasonable size.

Algebraic logic, is regarded as essential and for lower secondary classes the following factors, together with those above, are thought to be particularly important:

**Keyboard:** Facility to include the following:

(a) Arithmetic Keys:

- Sign change key
- Reciprocal key
- Square Root key
- Power key

(b) Clear and entry (may be combined as a single key)

(c) Memory

(d) Keys for percentage, squares and 'pi' were felt to be useful but less important.

**Display:** No more than eight digits should appear in the display.

For sixth form (or equivalent) use, a scientific calculator should incorporate the following additional facilities:
Keyboard

Arithmetic keys \( n^{th} \) root \( \sqrt[\prime]{\prime} \)
factorial \( ! \)

Trigonometric functions \( \sin \) \( \cos \) \( \tan \)
with both degrees and radians. Inverse and trigonometric functions should be available by means of the same keys in conjunction with the inverse INV key.

Logarithmic functions \( e^x \) \( 10^x \) \( \log^x \) \( \ln^x \)

Brackets; facility for at least three levels of brackets

Statistics \( \Sigma x \) \( \Sigma x^2 \)
mean \( \bar{x} \)
standard deviation SD

Display

(a) A high degree of accuracy is required. The calculator should work with at least three digits more than are displayed.

(b) When the answer to a calculation exceeds normal display capacity, the machine should automatically switch to scientific notation.
Chapter 2: THE CALCULATOR AS A TEACHING AID

1) Estimation, Errors and Accuracy
2) Fractions
3) A Classroom Investigation into Recurring Decimals
4) Place Value and Decimals
5) The Ordering of Operations
6) Growth and Decay
7) Sequence, Series and Limits
8) The Fibonacci Sequence
9) Iterative Methods
ESTIMATION, ERRORS AND ACCURACY

Whenever a computation is performed, whether mentally, pencil and paper or with a calculator, due consideration must be given to whether the answer is a 'reasonable' one. A 'reasonable' answer may be considered from two important aspects. One, we should first estimate the answer and consider whether the estimate is sensible in terms of size, and two, after being satisfied that it is a sensible estimate, is the answer given to reasonable degree of accuracy in the context of the question?

Estimation of answers

Estimation is already part of today's curriculum but with the availability of calculators, the skill of estimation is even more important. However, we no longer need the precision in our estimates which was often demanded for checking hand calculation, and, in fact, to check a calculator result, one is primarily concerned with order of magnitude and hence the only real need is the ability to do single digit arithmetic and work with powers of 10. For example, to estimate $358 \times 7294$ we may think of this as $300 \times 7000$, that is we need only consider the leading digit rather than round. Thus, our estimate is $3 \times 100 \times 7 \times 1000 = 2100000$. More ambitious students may prefer a better estimate and so rounding off may be introduced together with some number relationship notions and some common-sense.

There are no fixed rules for estimating answers and different procedures can result in somewhat different answers. The main point to keep in mind is that we want to get an estimate that will tell us whether or not we have made any major error in the use of the calculator. Some more difficult examples may help to illustrate the points made.
Example 1)

\[
0.0243 \times 11.92 + 14.07 \times 0.0039
\]
\[
\approx 0.02 \times 12 + 14 \times 0.004
\]
\[
= 0.24 + 0.056 \approx 0.2 + 0.1 = 0.3
\]

Notice that we have alternated a round off down with a round off up and our estimate compares quite closely to the calculator answer of 3.44529.

Example 2)

\[
14.87 \times 9.25 \times 0.0027 - 0.1443
\]
\[
\approx 14 \times 10 \times 0.003 - 0.14
\]
\[
= 140 \times 0.003 - 0.14 = 0.42 - 0.14 = 0.28
\]

Here we have not obeyed the accepted rule for rounding off when giving an answer to the required degree of accuracy. In fact, we round up 9.25 because multiplying by 10 is easy and rounded down 14.87 and 0.0027 to compensate. The estimated answer is close enough to the calculator answer of 2.2707825 to indicate that no major error has been made by the user.

In estimating the answer to a multiplication or an addition we try to alternate up and down roundings since questions may arise where obeying the usual rules would result in an estimate which is really too big or too small. In estimating the answer to a division or subtraction problem, however, both of the numbers involved should be rounded up or both rounded down in order to balance the approximation.

Example 3)

\[
(655.45 \div 3.22) - (27.655 \times 5.87)
\]
\[
- (600 \div 3) - (27 \times 6)
\]
\[
= 200 - 162
\]
\[
= 38
\]

Calculator answer = 41.22105
Other forms of practice would include those such as given by S.M.P. Book 4 (1968):

'Say which answers you consider are reasonable estimates, and in the case of the others, put down what you think would be better'

1) \[ 12.3 \times 2.9 = 36 \text{ approx.} \]
2) \[ 0.105 \times 0.1 = 0.1 \text{ approx.} \]
3) \[ 9.6 \times 26.2 = 250 \text{ approx.} \]
4) \[ 1023 \times 19 = 2000 \text{ approx.} \]
5) \[ (20.2)^2 = 400 \text{ approx.} \]
6) \[ \sqrt{170} = 13 \text{ approx.} \]
7) \[ \sqrt[3]{0.08} = 0.2 \text{ approx.} \]
8) \[ \frac{16 \times 1.1}{3.4} = 5 \text{ approx.} \]
9) \[ \sqrt{(3.1^2 + 6.9^2)} = 10 \text{ approx.} \]
10) \[ 62.9 \times 0.9 \times 0.49 = 3 \text{ approx.} \]

Cockcroft (257) makes it clear that "the ability to estimate is important not only in many kinds of employment but in the ordinary activities of adult life". Industry and commerce rely extensively on the ability to estimate but although these skills develop on the job, employers often complain that young entrants to industry and commerce lack a 'feel' for both number and measurement. Cockcroft has observed that although estimation is included in most schemes of work, it is not practiced in many classrooms. It is felt that teachers do not appreciate how much is implied by the words 'ability to estimate' and how long it takes to develop this ability. The conceptual difficulty of realising whether an answer is reasonable in the context of a question, however, must have been experienced by most mathematics teachers. For instance, we can take those children who have been quite happy to accept their answer for the area of a desk top even though that answer is close the the area of a football pitch or a postage stamp.

There are, therefore, other aspects of estimation besides that required for computational work on a calculator. Children need
to estimate lengths, areas, capacities, and weights. They should be able to estimate the amount of wallpaper needed for a room, the amount of petrol needed for a journey and so on.

**Limits of Accuracy and Errors**

When we have made our estimation of a given computation we then perform the computation on the calculator. We now have to satisfy ourselves that the answer shown on the calculator, say 7.382 074 8, is close enough to our estimation to feel confident that we have not made any major error in pushing the calculator buttons. Do we then write down 7.382 074 8 as our answer and feel pleased with ourselves? Many children are perfectly happy to do this or just forget a few figures on the end and write down, say 7.38. Just as bad, perhaps, is that many textbooks tell them to round off the answer to 3 significant figures or 2 decimal places without explanation or without challenging the pupils to round off to a sensible answer in the context of the question.

Much of our calculation depends on measurement and all measurements are approximate. We must, therefore, think very carefully about the data being fed into the calculator and relate our answer to these data. For example, with an eight-digit display calculator and an ordinary ruler, is there much point in saying that the answer to a particular problem based on measurement with the ruler is 7.382 074 8?

The topic of limits of accuracy and errors takes even greater importance now that we are getting our answers from the calculator. The concept of the reasonableness of answers in terms of physical reality and a feel for errors may be introduced before the child reaches secondary school and then the mathematics of errors may be taught at various levels.
In any measurement, we speak of the GREATEST POSSIBLE ERROR which is always one-half of the unit of measure used in making the measurement, and the child should appreciate that when two measurements are compared for their PRECISION, the more precise of the two measurements is the one for which the greatest possible error is smaller. It is important that measurements be stated in a way which shows clearly how precise they are. One way of doing this is to make certain agreements about what is meant when we write the number for a measure in decimal form. For example, when we write 4.52 cm. for a certain measurement, we understand that the measurement has been made correct to the nearest 0.01 cm. and that the greatest possible error is \( \pm 0.005 \) cm. This is equivalent to writing \( (4.52 \pm 0.005) \) cm. for the measurement. By this agreement each of the digits in 4.52 is SIGNIFICANT.

Non-zero digits in a measure are always significant, but zeros may or may not be. The child should appreciate and understand the rules for zeros in a measure. He should appreciate, for example, that 4.80 cm. is a more precise measurement than 4.8 cm. and that the zero here is significant, while the zeros in 0.0034 are not. Further, that if the distance between two objects is given as 470 cm, we usually assume that the zero is not significant and that the measurement is precise to the nearest 10 cm. \( (470 \pm 10) \) cm. If we indicate that the zero is significant (for example, as 470 cm.) then the greatest possible error is 0.5 cm. Similarly, if the measurement of 93,000,000 miles is precise to the nearest 10,000 miles we may indicate this as 93,000,000 but then tie this in with a more familiar way of expressing very large or very small numbers, that is, by scientific notation. For example, \( 9.3 \times 10^7 \) miles has two significant digits, while \( 9.30 \times 10^7 \) miles has three.

What about computations with numbers involving measurements?
If we are given two measurements of, say, 2.63 cm. and 6.8 cm., is it significant to give their sum as 9.43 cm.? The child should appreciate that, by considering the sum of the smallest possible measure and the sum of the largest possible measure, the sum of the two measures is somewhere between 9.375 and 9.485 and that it is only reasonable to give an answer with the same precision as the least precise measurement. Thus, we round each measure to the same precision as the least precise measure involved and then perform the addition (or subtraction).

Similarly, a rule frequently used when multiplying numbers obtained from measurements or from approximations of any kind ('keep as many significant digits in the product as there are in the factor with the fewest significant digits') may be illustrated by considering the difference between the smallest possible area of a rectangle and the largest possible area. For example, the approximate area of a rectangle whose dimensions are 72 cm. by 42 cm. may be written as $3024 \pm 57$ cm.$^2$, and we see that the computed area cannot be more accurate than two significant digits and may not even be that accurate.

The mathematical notion of RELATIVE AND ABSOLUTE errors should probably be introduced after pupils have more facility with fractions. Pupils should have an idea of the importance of the concepts and how to actually calculate them numerically. That is, the pupil should be able to tell you that dropping the 8 in $0.700\ 8$ is a small relative error, while dropping the 8 in $0.007\ 8$ is a large relative error. The absolute error for both, however, is the same.

The discussion about data will, no doubt, lead on to the implications of interpreting calculator answers when the initial data is known to be EXACT. An example is money. An amount such as £3.27 is exact and may be written with as many zeros after it as you wish i.e. £3.270 000 .... Suppose we found the
simple interest on this amount for one year at 12\% per annum.
The interest would be 0.057 225 0, that is 5.722 5 pence.
This is an exact answer and in the circumstances is quite correct.
How you pay it is another matter! However, since this interest
is payable yearly, the decimal parts of a penny can make a
difference over a number of years.

However, the situation is potentially confusing to the child.
Science teachers tend to use theoretical models in their
everyday problems. A physics teacher asking a child to find
the density of a 3 kg. mass, having volume 4 m.\(^3\), probably
anticipates the answer 0.75 kg.m.\(^{-3}\), not 0.8 kg.m.\(^{-3}\). Here he
is more interested in checking understanding and/or knowledge
of the definition of density, than awareness of accuracy.
Mathematics teachers (and examiners) are often thoroughly
careless about making it clear whether a problem, apparently
involving measurements, is intended to be regarded as a
description of a real model situation or a model. In 'A circle
of 1.5 cm. ........', is 1.5 a number x such that 1.45 < x < 1.55
or is it an exact pure number?

Nigel Webb (1980) suggests simplified versions of some rules
of accuracy already discussed. They are intended as rules of
thumb for everyday use, but he emphasises the need for the
teacher to discuss problems of accuracy and the anomalies
arising. In brief, he suggests that:

1) Where specific units of measurement are involved,
the number of significant figures given in the measurement
will be assumed to indicate the accuracy of the measurement.
In this case the result of a calculation should be given to the
same number of significant figures as the least accurate item
of data. However, care must be taken in formulating questions
otherwise the following may result:
The two sides of a rectangle are of length 3 cm. and 4 cm. respectively.

The Area = 3 x 4 cm²
= 12 cm²
= 10 cm² (to 1 sig. fig.)

2) Where specific units of measurement are not involved, numbers given as data for calculation will be assumed to be exact, pure numbers unless otherwise described. Then the result of a calculation should be given to the maximum accuracy of the calculator.

However, John Hersee (1981) argues that measurements are implicit wherever any real situation is being modelled. "We want to develop and encourage a critical sense in young people, an attitude which assesses the reasonableness of any result achieved in the light of the original situation from which it arose. In the classroom, this critical attitude will be developed only through constant attention; it must be part and parcel of any 'modelling' of any attempt to apply mathematics to a problem".

Hersee adds that it is for this reason that the S.M.P. take the view that arbitrary, or apparently arbitrary rules should be avoided. Each situation should be assessed as it arises, unless a rule does exist. However, simplifying rules may be used for the time being, if a new topic is being introduced and the pupil needs to concentrate on the new idea and not be distracted by worries about accuracy. It is also probable that some simplifying implication of exactness may be required in the artificial environment of examining.

All this is not new, calculators have helped to draw it to our attention afresh.
FRACTIONS

The mathematics curriculum has always placed great emphasis on the skills of computations involving fractions, and the experienced mathematics teacher will tell you that it is one of the most difficult and frustrating topics to teach. Most adults will remember with trepidation the seemingly endless exercises involving computations such as \( \left( \frac{3}{4} + \frac{5}{7} \right) \div \left( \frac{1}{2} \times \frac{2}{3} \right) \), but how many could remember how to do them now? And, indeed, why should they? How often are they used in everyday life or even industry?

The Bath and Nottingham study (Cockcroft, 75-6) reports that although fractions are still widely used within engineering and some other craft work these are almost always fractions whose denominators are included in the sequence 2, 4, 8, .... 64.

The need to perform operations such as \( \frac{2}{5} + \frac{3}{7} \) does not normally arise, and the manipulation of fractions of the kind which is commonly practiced in the classroom is hardly ever carried out. In the rare instances in which it is necessary to multiply or divide, it is usual to convert each to a decimal before performing the operation, if necessary with the help of a calculator". Further, "the notation of fractions appears in some clerical and retail jobs, for instance \( 4\frac{3}{7} \) to represent 4 weeks and 3 days or \( 2\frac{5}{12} \) to represent 2 dozens and 5 singles. However, school-type manipulation is rarely found and then only in very simple cases; for instance, the calculation required to find the charge for 3 days based on a weekly rate, is division by 7, followed by multiplication by 3".

It would seem therefore, that the calculator has enabled us to shift the emphasis from fraction notation and manipulation to decimal notation and manipulation in order to describe the world around us when whole numbers are not adequate.
In any case, the advisability of introducing fraction manipulation before secondary school age has been reassessed, and even new research has concluded that for many the abstraction of fraction manipulation is entirely inappropriate for many children who leave school still at Piaget's concrete stage.

The Concept in Secondary Mathematics and Science project (1979) concludes that many children do not feel confident with the use of fractions and try whenever possible to apply the rules of whole numbers to operations on fractions. Most children avoided the use of the addition and subtraction algorithms when another method was available. When the addition rule was used the younger children were more successful than the older ones because the rules were fresh in their memories. Other common difficulties found, which any mathematics teacher must be only too familiar with, include:

- finding equivalent fractions such as \( \frac{2}{7} = \frac{10}{35} \);
- comparing the sizes of fractions which do not have the same denominator;
- adding fractions using 'add tops add bottoms';
- multiplying the whole numbers and fractions separately, e.g. \( \frac{3}{4} \times \frac{2}{3} = \frac{6}{12} \);
- their own concept that 'multiplying only makes it bigger';
- application of fractions to problems.

What skills should be taught then to secondary school children? Cockcroft's list (455-8) suggests that all pupils should be able to:

- "Perform calculations involving the word 'of' such as \( \frac{1}{3} \) of £4.50.
- Be able to add and subtract fractions with denominators 2, 4, 8, or 16 in the context of measurement."
Know the decimal equivalents of $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{10}$, $\frac{1}{4}$ and also that $\frac{1}{3}$ is about 0.33.

Be able to convert fractions to decimals with the help of a calculator.

However, the higher attaining pupils should not be limited to these concepts but should be introduced to the whole range of fractions especially those who would be studying higher mathematics. Some of the ideas that follow therefore, are mainly aimed at this ability group. While the emphasis for the average and below average ability group is on understanding the concepts behind easier fractions. Hopefully, some of the examples are interesting and worthwhile in their own right.

After the idea of EQUVALENCE has been introduced, it may be verified by using the calculator. Once pupils are convinced that a principle is true, and a calculator is a powerful tool to this end, they are more inclined to try to understand and learn the principle. Thus,

$$\frac{15}{8} = 1.875 \quad \text{and}$$

$$\frac{15 \times 7}{8 \times 7} = \frac{105}{56} = 1.875$$

and the pupils are convinced that the two fractions are the same size. Conversely, they may be asked to check $\frac{2}{5}$, $\frac{10}{25}$, $\frac{14}{35}$ etc. and to find a rule for themselves.

Equivalence is fundamental to comparing sizes of fractions with unequal denominators and addition of fractions. The following questions may be set and verified with the calculator:
Using the four numbers 2, 5, 7, 9

(a) Write as many fractions as you can (you may use a number more than once)

(b) Which fraction is the smallest? The largest?

(c) Which fractions are less than 1? Greater than 1? Equal to 1?

(d) Which fractions are equivalent?

(e) Write down the fractions in order of size.

After ADDITION of fractions has been introduced with the aid of equivalence, practice with more interesting problems may be achieved by posing problems of the following type:

(1) Arrange the numbers 5, 6, 7, and 8 into two fractions that will give

(a) the largest sum

(b) the smallest sum (Do not use any number more than once)

(2) Consider the following sequence of numbers

1, \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \) etc.

(a) Add the first two numbers in the sequence

(b) Add the first three numbers in the sequence

(c) Add the first four numbers in the sequence

Do you begin to see a pattern? 

(d) Predict the sum of the first ten numbers in the sequence.

Dubisch/Hood (1979) provide many similar exercises.

By considering the area of a rectangle, the rule for MULTIPLICATION of fractions can be developed, that is,

\[
\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}
\]

again, we can use a calculator to check this rule. For example,
to check that

\[ \frac{7}{6} \times \frac{5}{3} = \frac{7}{6} \times \frac{5}{3}, \text{ check on the calculator} \]

that \( \frac{7}{6} = 1.1666667 \) and \( \frac{5}{3} = 1.6666667 \)

so \( \frac{7}{6} \times \frac{5}{3} = 1.6666667 \times 1.6666667 = 1.9444445 \)

on the other hand,

\[ \frac{7}{6} \times \frac{5}{3} = \frac{7 \times 5}{6 \times 3} = \frac{35}{18} = 1.9444444 \]

The small difference in the answer may provoke some discussion, another desirable feature of calculator usage.

After the procedure for \textsc{MULTIPLICATION OF MIXED NUMBERS} has been explained and verified on the calculator, the more-able pupil may be able to discover a method of multiplying without recourse to changing to improper fractions first. For example,

\[ 2\frac{2}{3} \times 1\frac{4}{5} = \frac{8}{3} \times \frac{9}{5} = \frac{44}{15} \text{ or } 4.8 \text{ (on the calculator)} \]

alternatively,

\[ 2\frac{2}{3} \times 1\frac{4}{5} = (2 + \frac{2}{3}) \times (1 + \frac{4}{5}) \]

\[ = (2 \times 1) + (2 \times \frac{4}{5}) + (\frac{2}{3} + 1) + (\frac{2}{3} \times \frac{4}{5}) \]

\[ = 2 + \frac{8}{5} + \frac{2}{3} + \frac{8}{15} \]

\[ = 4.7999999 \text{ (on the calculator)} \]

Again, the small difference in answers should provoke the question: Is 4.8 the same as 4.7999999?
The usual rule for division of fractions, \( \frac{a}{b} ÷ \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} \)
may also be easily checked on the calculator. For example,

\[
\frac{28}{93} ÷ \frac{17}{27} = \frac{28}{93} \times \frac{27}{17}
\]

\[
= 0.47817836
\]

To check,

\[
\frac{28}{93} ÷ \frac{17}{27} = 0.30107526 ÷ 0.62962962 = 0.47817836
\]

It may be shown also that

\[
\frac{a}{b} ÷ \frac{c}{d} = \frac{a ÷ c}{b ÷ d}
\]

with the above question, although it does not usually tell us what the answer is as a fraction.

Similarly, it may be shown, for example, that

\[3\frac{1}{2} ÷ 2\frac{1}{5} \text{ is equal to } (3 ÷ 2\frac{1}{5}) + (\frac{1}{2} ÷ 2\frac{1}{5})\]

rather than

(a) \((3 ÷ 2) + (\frac{1}{2} ÷ \frac{1}{5})\) or

(b) \((3\frac{1}{2} ÷ 2) + (3\frac{1}{2} ÷ \frac{1}{5})\) or

(c) \((3\frac{1}{2} ÷ 2) + \frac{1}{5}\)

In the C.S.M.S. tests 35% of pupils opted for (a) and only 5.1% of 15 year olds chose the correct answer. Although it
was recognised that if they had been asked to do the computation, the children may have used an algorithm which gave a correct answer, but the 35% certainly did not check the correspondence between this version and the rule.

To test the ingenuity of the pupils, they may be asked to find the value of $\frac{5}{7} + \frac{8}{9}$ on the calculator without using the memory or writing down an intermediate value. Of course, a simple calculator cannot cope with the task as it stands but if it is rearranged as

$$\left(\frac{5}{7} \times 9 + 8\right) \div 9$$

it can be dealt with, usually with the key sequence

$$5 \div 7 \times 9 + 8 \div 9 \text{ = }$$

Can the method be extended to a sequence of fractions, involving either addition or subtraction i.e.

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \ldots \text{ etc?}$$

Careful thought will show that it can. For example,

$$\frac{5}{7} + \frac{8}{9} + \frac{2}{3} \text{ may be rearranged as }$$

$$\left(\frac{5}{7} \times 9 + 8\right) + \frac{2}{3}$$

$$= \left[\frac{\left(\frac{5}{7} \times 9 + 8\right)}{9}\right] \times 3 + 2 \div 3$$

Possible with key sequence

$$5 \div 7 \times 9 + 8 \text{ - } 9 \times 3 + 2 \div 3 \text{ = }$$
A more interesting exercise involves manipulation with continued fractions. The child gains practice with several skills of fractions and is also required to predict a pattern from the results. The calculator becomes useful for converting the fractions to decimals so that the child may concentrate on the manipulation and predictions themselves.

An example of a continued fraction is:

\[
\frac{1}{1 + \frac{2}{1 + \frac{2}{1 + \ldots}}}
\]

The exercise, illustrated by Green (1981), is to calculate each stage as follows:

\[
(1) \quad 1 = 1.0000 \\
(2) \quad 1 + \frac{2}{1} = \frac{3}{1} = 3.0000 \\
(3) \quad 1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1}}} = 1 + \frac{2}{1 + \frac{2}{1}} = 1 + \frac{2}{3} \approx 1.667 \\
(4) \quad \frac{1 + 2}{1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1}}}} = \frac{1 + 2}{1 + \frac{2}{3}} = \frac{1 + 2}{3} = \frac{5}{3} = 2.2000 \\
(5) \quad 1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1}}}}} = 1 + \frac{2}{1 + \frac{2}{11}} = 1 + \frac{2}{12} \approx 1.909
\]
The children may then be asked to predict the next few results by spotting a pattern in the fractional part:

\[
\begin{align*}
\text{Double previous denominator} + 2 & \quad \text{THEN} & \quad \text{Double} - 2 \\
\text{Double previous numerator} - 1 & \quad & \text{Double} + 1
\end{align*}
\]

A check may be desirable.

Some may also spot that the decimal answers are getting closer and closer to 2 and a reasonable conclusion is that:

\[
\frac{1 + 2}{1 + 2} = 2 \text{ eventually if enough stages are taken.}
\]

Further investigations for rules or patterns may be carried out such as:

\[
\begin{align*}
(1) \quad & \frac{2 + 3}{2 + 3} \\
& \frac{2 + 3}{2 + 3} \\
& \frac{2 + 3}{2 + 3} \\
\text{and } & \frac{3 + 4}{3 + 4} \\
& \frac{3 + 4}{3 + 4} \\
& \frac{3 + 4}{3 + 4}
\end{align*}
\]
It is surprising how much interest is generated by this type of investigation, and the next section will illustrate how the pupils' enthusiasm and energies may be nurtured by an experienced and sympathetic teacher.
A CLASSROOM INVESTIGATION INTO RECURRING DECIMALS

Most mathematics teachers would agree that some of the most successful lessons spring up from unexpected questions (and sometimes unexpected interest) during routine lessons. It can be most rewarding if the teacher allows a particular line of enquiry to continue and even branch out as long as interest is maintained. This was the particular experience of David Wiseman, Head of Mathematics at a Hertfordshire secondary school, during a routine fractions lesson with an average ability third year form. He gives a full account of how the topic developed and the questions and reactions of his pupils in a series of articles for Hertsmaths (1981-2). The initial work can be done without a calculator, but a calculator allows the lesson to proceed more fluently and quickly. This is particularly important with a low ability class who are more likely to make computational mistakes and so miss the main point of the exercise. Boredom and frustration are likely to follow unless a calculator is used at an early stage.

In short, it all began when he was discussing the decimal equivalence of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, and subsequently $\frac{1}{6}$, $\frac{3}{6}$, $\frac{5}{6}$, and $\frac{7}{6}$ with 3rd year set 4. They worked out $\frac{1}{6}$ and $\frac{3}{6}$ by "dividing the bottom into the top", but one boy, of his own accord, decided to try $\frac{1}{3}$ and with a mixture of worry and excitement exclaimed "it keeps going on". This all too rare opportunity in the class for child-initiated discussion led to other children trying other fractions and resulted in:

\[
\begin{align*}
\frac{1}{3} &= 0.3333 \\
\frac{2}{3} &= 0.6666 \\
\frac{1}{6} &= 0.1666 \\
\frac{5}{6} &= 0.8333
\end{align*}
\]
The word 'RECURRANCE' was then introduced, and although they surprisingly refused the aid that, for example,

\[
\frac{2}{3} = 2 \times \frac{1}{3} = 2 \times 0.3333 = 0.6666
\]

all members of the class found decimal equivalents of each member of the 'ninth family':

\[
\begin{align*}
\frac{1}{9} &= 0.\overline{1} \\
\frac{2}{9} &= 0.\overline{2} \\
\frac{3}{9} &= 0.\overline{3}
\end{align*}
\]

A reinforcement of fractional equivalence was experienced when a few realised that \(\frac{2}{3} = \frac{1}{\frac{3}{2}}\).

Some children noticed the pattern in the remainders, and as this particular class had already done some work on "Clock" or "Modular" Arithmetic, the explanation for the increasing size of the remainder was given by reference to the module NINE clock:

\[
\begin{align*}
10 &\equiv 1 \pmod{9} \\
20 &\equiv 2 \pmod{9} \\
30 &\equiv 3 \pmod{9}
\end{align*}
\]

A homework assignment on the elevenths family resulted in excited discussion during the following lesson:
Again patterns were observed in the remainders and relationships between digits in the quotients and remainders. These observations may be explained through knowledge of Modular Arithmetic.

It was during the following term that David Wiseman was asked the question "When can we do some more of those divides that go on, Sir?"

After a rapid revision of the previous lessons, a discussion arose as to a suitable notation to save both time and space. Several ideas were put forward by the pupils, but finally they agreed to adopt the familiar "dot" above the recurring number, or numbers, symbol:

\[ \begin{align*}
1 \div 11 & = 1 \div 11 = 11 \left( \begin{array}{c}
0.0 \ 0 \ 0 \ 0 \\
1.0 \ 0 \ 0 \ 0 \ 0
\end{array} \right) \\
2 \div 11 & = 2 \div 11 = 11 \left( \begin{array}{c}
0.1 \ 8 \ 1 \ 8 \ 1 \\
2.0 \ 9 \ 2 \ 9 \ 2 \ 0
\end{array} \right) \\
3 \div 11 & = 3 \div 11 = 11 \left( \begin{array}{c}
0.2 \ 7 \ 2 \ 7 \ 2 \\
3.0 \ 8 \ 3 \ 8 \ 3 \ 0
\end{array} \right) \\
4 \div 11 & = 4 \div 11 = 11 \left( \begin{array}{c}
0.3 \ 6 \ 3 \ 6 \ 3 \\
4.0 \ 7 \ 0 \ 7 \ 0 \ 0 \ 0
\end{array} \right)
\end{align*} \text{etc.}
\]

This invitation to the children to devise their own shorthand helps them to recognize the importance of a clear unambiguous but concise notation. One pupil coined the phrase "dotty number" and another pupil asked whether it was possible to get a "dotty number", in which more than two digits recurred. It was decided to investigate the family of sevenths. Although it takes longer, it is again more fruitful to find each recurring decimal by division (and leave the calculator for checking) as
there is an additional pattern in the remainders which is also worthy of analysis.

\[
\begin{align*}
\frac{1}{7} &= 7 \left(0.\overline{142857}\right) = 0.142857 \\
\frac{2}{7} &= 7 \left(0.\overline{285714}\right) = 0.285714 \\
\frac{3}{7} &= 7 \left(0.\overline{428571}\right) = 0.428571 \\
\frac{4}{7} &= 7 \left(0.\overline{571428}\right) = 0.571428 \\
\frac{5}{7} &= 7 \left(0.\overline{714285}\right) = 0.714285 \\
\frac{6}{7} &= 7 \left(0.\overline{857142}\right) = 0.857142
\end{align*}
\]

Nearly all pupils discovered that each fraction recurred in a cycle of six, and that each decimal equivalent used the same digit; 1, 4, 2, 8, 5, 7 and in that order. Only a few discovered a similar recurring decimal in the remainders; 4, 5, 1, 3, 2, 6. An explanation may be given to the pupils in terms of "modular" arithmetic.

For example, the cycle of six can be explained in terms of the set of possible remainders 1, 2, 3, 4, 5, 6 when we divide unity by seven. At each stage of the division one of these elements is the remainder. Once all the members of this set have appeared as the remainder, then one of them appears again.
MODULO SEVEN

10 ≡ 3 (mod 7)
30 ≡ 2 (mod 7)
20 ≡ 6 (mod 7)
60 ≡ 4 (mod 7)
40 ≡ 5 (mod 7)
50 ≡ 1 (mod 7)
10 ≡ 3 (mod 7) etc.

COUNT ROUND 10

Further, this suggests that if the remainders are re-arranged they will form an arithmetic sequence in modulo seven:

3, 6, 2, 5, 1, 4
+3 +3 +3 +3 +3

What number completes the cyclic sequence? Why?

Does the recurring cycle 1, 4, 2, 8, 5, 7 .... form an arithmetic sequence? If so, in which modulo?

7, 4, 1, 8, 5, 2
+7 +7 +7 +7 +7

OR

1 x 7 ≡ 7 (mod 10)
2 x 7 ≡ 4 (mod 10)
3 x 7 ≡ 1 (mod 10)
4 x 7 ≡ 8 (mod 10)
5 x 7 ≡ 5 (mod 10)
6 x 7 ≡ 2 (mod 10)
How does this sequence continue? Why do only the first six members of the sequence occur in the recurring cycle?

Finally, one observant pupil noticed that with $\frac{1}{7}$ there is a connection between the first four digits:

\[
i.e. \ 0.1428 \ \ldots \ \ \\
28 = 2 \times 14 \\
\]

Is there any interest in this?

\[
\begin{array}{c}
.14 \\
28 \\
56 \\
112 \\
224 \\
\hline \\
ADD \quad 448 \ etc. \\
\hline \\
.1428571428 \ \ldots \\
\end{array}
\]

And so the decimal equivalent of $\frac{1}{7}$ is obtained another way.
"What about the twelths?" asked Nigel.

Wiseman was able to show the class, through a question and answer exchange, that each member of the twelths was in fact related to families already considered.

\[
\frac{1}{12} = \frac{1}{2 \times 6} = \frac{1}{2} \text{ of } \frac{1}{6} = 2 \times 0.1666 \ldots = 0.083 \ldots
\]

\[
\frac{2}{12} = \frac{1}{6} = 0.1666 \ldots = 0.16
\]

\[
\frac{3}{12} = \frac{1}{4} = 0.25
\]

\[
\frac{4}{12} = \frac{1}{3} = 0.3333 \ldots = 0.3
\]

\[
\frac{5}{12} = 5 \times \frac{1}{12} = 5 \times 0.0833 \ldots = 0.416, \text{ etc.}
\]

It was established that the thirteenths was the next family unrelated to any previous as were the thirds, fifths, sevenths, elevenths. So a valuable reinforcement of the concept of PRIME NUMBERS developed.

The class then began to find the decimal equivalent of the thirteenths family, and it is at this stage that a calculator would prove invaluable. However, the divisions still had to be done by long hand because the remainders were required, but with the assistance of the calculator.

\[
\begin{array}{c}
0.0 \overline{7} 6 9 2 3 0 \\
13 \overline{1} 0 0^{9} 12^{3} 0^{4} 1^{0} 0^{1}
\end{array}
\]

Notice the cycle of six in the decimal equivalent:

\[\{0, 7, 6, 9, 2, 3}\]
and, as you would expect, a matching cycle of six in the remainders:

\[
\{9, 12, 3, 4, 1, 10\}
\]

\[
\frac{2}{13} = 13 \overline{0.153846}
\]

with cycle \(\{1, 5, 3, 8, 4, 6\}\)

Discussion followed about other possible cycles for

\[
\frac{3}{13} = 13 \overline{0.230769}
\]

Which has the same cycle and remainder cycle as \(\frac{1}{13}\)

What about \(\frac{4}{13}\)? Would this have the same cycle as \(\frac{2}{13}\)?

After only 3 steps:

\[
\frac{0.307}{14} = 0.\overline{010100}
\]

an excited voice rang out "It's part of the first cycle. It's like \(\frac{1}{13}\) ...

The class were then able to write in the remaining figures with further division from:

\[
\{0, 7, 6, 9, 2, 3\}
\]

so \(\frac{4}{3} = 0.\overline{307692}\)

"I wonder whether any other members of the thirteenth family have a decimal equivalence belonging to the second cycle, 1, 5, 3, 8, 4, 6?"
The investigation continued and pupils' answers pooled to discover:

\[
\begin{array}{cccccc}
1 & 2 & 4 & 9 & 10 & 12 \\
\hline
13 & 13 & 13 & 13 & 13 & 13
\end{array}
\]

and \( \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\hline
13 & 13 & 13 & 13
\end{array} \)

had cycle pattern

with remainder cycle

\[
\begin{array}{cccc}
3 & 0 & 7 & 2 \\
\hline
9 & 6 & 7 & 9
\end{array}
\]

while

\[
\begin{array}{cccc}
2 & 6 & 7 & 8 \\
\hline
13 & 13 & 13 & 13
\end{array}
\]

and \( \begin{array}{cccc}
11 & 13 \\
\hline
13 & 13
\end{array} \)

had cycle pattern

with remainder cycle

\[
\begin{array}{cccc}
6 & 1 & 4 & 5 \\
\hline
3 & 4 & 8 & 6
\end{array}
\]

Other questions which may be investigated:

Why are there two and only two cycles?
Can you detect any symmetries in the 'bicycle' pattern?
Explanations can be given again in terms of modular arithmetic.
Further, it is interesting to note that the decimal equivalence of \( \frac{1}{13} \) can be obtained in a similar kind of way to \( \frac{1}{7} \):

\[
\begin{align*}
7 & = 0.07 \\
7 \times 9 & = 63 \\
7 \times 9^2 & = 567 \\
7 \times 9^3 & = 5103 \\
7 \times 9^4 & = 45927 \\
7 \times 9^5 & = 413343 \\
\hline
.0769230 & = ...
\end{align*}
\]

It was at this stage that the classroom investigation was curtailed. However, there is usually one enthusiastic pupil who has become gripped with the work and he may be encouraged to continue with the 'seventeenths' and 'nineteenths' families on his own.

\[
\frac{1}{17} = 0.0588235 2941176470 ...
\]

but as there is no special merit in continuing a study of remainders by long division, the work may proceed much more quickly with the calculator doing all the computation.

The result raises interesting questions such as:

Why do you get a sixteen digit cycle?
Would \( \frac{2}{17} \) reproduce the same cycle, or would a second cycle appear on the scene as is the case of the thirteenths family?

For \( \frac{2}{17} \) the Texas T.I. 30 calculator displays

\[
0.1176471
\]
and prediction and checking for $\frac{2}{17}$ and $\frac{3}{17}$ would continue until the sixteen members of the cycle are confirmed.

It is then easy to write down the decimal equivalents of the other members of the family of seventeenths without doing any more working:

\[
\frac{4}{17} = .2352941276470588\]

\[
\frac{5}{17} = .2941176470588235\]

......

......

\[
\frac{16}{17} = .9411764705882352\]

These results could be summarised in a table and this is illustrated with the family of nineteenths working on the assumption that there is only one cycle and this has 18 digits:
\[
\begin{array}{|c|c|c|}
\hline
\text{CALCULATOR DISPLAY} & \text{OBSERVATIONS} & \text{CUMULATIVE DEDUCTIONS} \\
\hline
\frac{1}{19} &= \boxed{0.0526315} & & \\
\hline
\frac{2}{19} &= \boxed{0.1052631} & \text{052631 belongs to the cycle. The initial "1" may be final digit of cycle.} & \boxed{0.052631} \\
\hline
\frac{3}{19} &= \boxed{0.1578947} & \ldots \text{15 ... could belong to the cycle} & \boxed{0.052631578947} \\
\hline
\frac{4}{19} &= \boxed{0.2105263} & \text{Since the cycle begins 05263, it must end with 21} & \boxed{0.052631578947} \\
\hline
\end{array}
\]

To find these four digits \( \frac{18}{19} \) which is approximately equal to 0.9 was tried next.

\[
\frac{18}{19} = \boxed{0.9473684} \\
\text{947 must be followed by 3684.} \\
\text{Since there are 18 digits in the cycle it suggests ..}
\]

To check \( \frac{16}{19} = \boxed{0.8421052} \) and the result was confirmed.

Other members of the family could now be written out without difficulty.

Finally, in a similar way the calculator, rather than long division, can be profitably used to find the two cycles of the family of thirteenths.
The topic of recurring decimals has long fascinated teachers and pupils alike and many interesting accounts of their experiences may be found in articles such as Beldon (1975, pp 38), Hewitt (1982, pp 48), Pallister (1978, pp 53), Ounsted (1978, pp 54).

The calculator, therefore, has enabled this type of investigation to become accessible to many more pupils than otherwise would have been possible. In return, it gives useful calculator practice and provides one of the many different opportunities for children to appreciate the power and limitations of this computational device; reinforcing, for example, the fact that many of its answers are approximate rather than exact.

To conclude with a quote from F.R. Watson (1979)

"..... most pupils can be lead to an interest in the properties of numbers, provided the associated computational effort involved is not excessive. An important aspect of mathematics, and one the pupils find attractive, is the search for pattern and regularity".
PLACE VALUE AND DECIMALS

The aims of this area of the C.S.M.S work was to find whether children could meaningfully use the base-ten place-value notation for both whole numbers and decimals, in the sense of both understanding how it worked and applying it to appropriate situations. The study concentrated on the area of decimals since most children in the 11 - 15 year age group were expected to have a reasonably sound basic knowledge of whole numbers. There was a major problem in trying to differentiate between the 'relational' understanding of properties of number, and a mere 'instrumental' ability to carry out a technique.

An important conclusion arrived at by the team was that the learning of whole numbers and decimals is not just a matter of recalling some place-names and a few rules of computation. Indeed the children who did rely blindly on the rules more often misapplied them than not.

Instead it involves internalising a whole chain of relationships and connections, some with place-structure itself (e.g. 0.9 is equivalent to 0.90); some linking to other concepts like those of fractions (e.g. the notion of one hundredth and its relationship to one tenth); some visual correspondences and some connecting to applications in the 'real' world.

In the interviews, the weakest group of children displayed only a tenuous grasp of place value when asked questions involving adding one to 6399 (with answers such as 06310 and 06439) and to adding ten to 3597. Writing and verbalising large numbers also caused problems with some children. and although the calculator's visual display can help with these concepts, these children still need other visual models to bring out the relationships in a more concrete way.
More problems were found with the concept of decimals and some children had difficulty in understanding that the figures after the point indicated the part of the number which was less than one unit. This came out with answers to the questions:

Ring the BIGGER of the two numbers:
0.75 or 0.8.
Why is it bigger?

Ring the BIGGER number:
4.06 or 4.5

Ring the number NEAREST IN SIZE to
0.18 or 0.1 / 0.2 / 0.2 / 0.1

The interviews revealed some weaknesses in comparing the sizes of 4.9 and 4.90, adding one-tenth to 2.9 (typical answers 2.19, 2.10), and multiplying 5.13 by 10. In the latter case, the rule for whole numbers was usually known (add a nought) but wrongly applied to decimals resulting in answers such as 5.130, 50.130.

There seemed to be a marked reluctance to admit that the answer to the division of one whole number by another might be expressible as a number containing decimals, or even one containing fractions. This difficulty surfaced in the responses to questions such as:

"Divide by twenty the number 24", and dividing a smaller number by a larger one caused even more problems:

"Divide by twenty the number 16"

Over 50% of 12 year olds responded "there is no answer" to the latter question, although this reduced to 23% for 15 year olds. This may arise because you cannot share fairly 16 sweets among 20 people.
It was also clear that the idea that "multiplication always makes it bigger, division always makes it smaller" was very much entrenched. The question below illustrated this:

Ring the one which gives the BIGGER answer
(a) $8 \times 4$ or $8 \div 4$
(b) $8 \times 0.4$ or $8 \div 0.4$
(c) $0.8 \times 0.4$ or $0.8 \div 0.4$

A reasonable response was given to the question:
"Write down any number between 0.41 and 0.42"
(37% correct from 12 year olds to 71% for 15 year olds), but the poorest response of all items resulted from the question:
"How many different numbers could you write down which lie between 0.41 and 0.42?"

It is important that all mathematics teachers should be familiar with the misconceptions and mistakes that children make which have been highlighted here. The advent of the calculator will obviously make children more familiar with decimal representation, but without careful structuring of the work it may just be used to produce meaningless answers which can be copied down faithfully to eight decimal places. The calculator may be used effectively by allowing the pupil to use it to generate output, with the purpose that the output will demonstrate a concept or relationship, or with the actual generation of the output serving to help reinforce a concept which has been taught previously.

Several of the problems encountered in the C.S.M.S. tests might be overcome by using the calculator for concept-reinforcement. At a very basic level after the number system has been introduced by using a concrete method such as Dienes base 10 blocks or even just squared paper divided, the relationships between adjacent columns could be reinforced by successive
division by 10. For example,

\[
\begin{align*}
1000 \div 10 &= 100 \\
100 \div 10 &= 10 \\
10 \div 10 &= 1 \\
1 \div 10 &= 0.1 \\
0.1 \div 10 &= 0.01 \text{ etc.}
\end{align*}
\]

or by multiplying by 10

The number 273.4894 may be written in expanded notation as

\[
200 + 70 + 3 + 0.4 + 0.08 + 0.009 + 0.0004
\]

and then actually performed on the calculator.

Other difficulties which arose may be approached in a similar way or by carefully guiding the pupil to a desired result using the technique of concept-demonstration:

What happens if you multiply 8 by 3, then 8 by 1, 8 by 0.2, 8 by 0.06 etc?

Then concept-reinforcement may be used to check other numbers multiplied by a number less than 1. This concept may be illustrated by WHIRLPOOLS.

Go round the whirlpool on your calculator. Multiply the numbers together:

\[
1.7 \times 0.8 \times 1.1 \times 0.3 \times 1.7 \times 0.8 \times \text{ and so on.}
\]

What happens to the answers?

Sometimes the answer increases and sometimes decreases, but the answer shrinks eventually.

Follow the whirlpools:

\[\begin{array}{c}
\text{a)}
\end{array}\]

For each, say whether the answers grow or shrink eventually.
First estimate whether the answers will grow or shrink and then check using your calculator.

Similarly for dividing by a number less than one; dividing a number by a larger number; and developing rules for multiplying and dividing by powers of 10.

The calculator may also be of some help when comparing the size of numbers involving decimals, for example, 4.75 and 4.8, by subtracting the chosen smaller from the chosen larger. A plus sign shows a correct answer and a negative sign an incorrect answer. If the answer is incorrect, interest may be stimulated in the pupil to discover why.

At a higher level, the traditional approach to decimal multiplication is to show how the multiplication is done by a number of examples in which the decimal fraction is first converted to the standard fraction representation with denominators of ten, hundred etc. The examples are used to justify the rule regarding the placement of the decimal point. Thus, the usual pedagogical sequence is first to justify the new rule and then actually to implement it in practice with decimals. Some pupils overlook the justification stage and tend to concentrate on memorizing the rule often with disastrous effects. An alternative approach could be to first ask the pupils to look for a pattern when the calculator is used to do some calculations (discover a rule), then justify the result with a few, well-chosen examples, and then practice. In other words, using the calculator for concept-demonstration - Johnson (1978), particularly illustrates this approach.
The emphasis is on finding the pattern and then asking why it works.

Use your calculator to find the products:

- $84 \times 0.2$
- $0.9 \times 0.7$
- $4.5 \times 0.8$
- $4.3 \times 2.7$
- $0.04 \times 0.35$
- $1.33 \times 4.75$ etc.

None of the items should have a zero in the trailing digit in one of the factors, or a 5 as a trailing digit when the other factor has an even number as the trailing digit.

The class may then arrive at, or be guided to, a generalisation about the placement of the decimal point, further examples given as a check and then the justification given. For example:

Now try the following with your calculator:

- $3.54 \times 0.45$
- $146 \times 0.35$
- $4.70 \times 0.60$

Does your rule still work? What's wrong?

This is used to help reinforce the idea that the rule still holds as the calculator suppresses the trailing zeros in the result. Also this reinforces the idea that two-tenths is twenty-hundredths (further reinforced by converting $\frac{2}{10}$ to $\frac{20}{100}$, $\frac{200}{1000}$ etc. on the calculator).

Some calculator games have the potential for a very real contribution to the learning of mathematics, especially for concept-reinforcement in the field of decimals. One such game is Wipeout.

The game involves entering a given number, with all digits different, and then asking the pupils to use subtraction to remove the specified digit (i.e. replace it with a zero)
without changing any other digit in the number. For example, if the game is used to reinforce concepts of place-value, one might start with something like 749.65128 and ask pupils to remove the 2 and then the 5 and so on. This reinforces place-values, since to remove the 2 requires the subtraction of .0002. It is important that the game is complemented by other forms of place-value practices as it could become a rote activity.

Other calculator games will be demonstrated and discussed in a later chapter.

The C.S.M.S. Project team conclude that pencil and paper computation techniques with complicated whole numbers or decimals are no longer essential, but anyone who understands enough about decimals to be able to use a calculator sensibly should be able, given time, to work out most of the 'rules' from first principles anyway, as a number of children managed to do during the interviews.

"It is certainly to be hoped that the presence of calculators will shift the emphasis from routine techniques, which did not seem to be performed very reliably, to the understanding of the principles, especially since this latter aspect seems to have been neglected at secondary school level in this particular area" (Hart, 1981 pp. 65)
THE ORDERING OF OPERATIONS

The suggestion has been made by Booth (1981) and by Hart (1981) that part of the difficulty which some people experience in mathematics is due to their use of intuitive "child-methods" which, while being adequate for the "easy" questions, do not generalize to the "hard" ones. One of the consequences of the intuitive approach is that you do not have to be so rigorous in what you write down, since this is always to be interpreted in terms of 'common-sense', or in terms of the context of the question. Consequently, as Brown (1981) points out, in answer to the question:

<table>
<thead>
<tr>
<th>A bar of chocolate can be broken into 12 squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>There are 3 squares in a row</td>
</tr>
<tr>
<td>How do you work out how many rows there are?</td>
</tr>
<tr>
<td>12 + 3</td>
</tr>
<tr>
<td>3 x 4</td>
</tr>
<tr>
<td>12 x 3</td>
</tr>
<tr>
<td>3 - 12</td>
</tr>
<tr>
<td>6 + 6</td>
</tr>
<tr>
<td>12 - 3</td>
</tr>
<tr>
<td>12 - 3</td>
</tr>
<tr>
<td>3 - 12</td>
</tr>
</tbody>
</table>

for children operating in this way, there is no ambiguity between $12 ÷ 3$ and $3 ÷ 12$ since the context has determined the meaning of the expression. Either way means "the number of 3's in 12". (However, entering both expressions in the written order on a calculator will, of course, produce different answers.)

Further, Kiernan (1979) remarks that such an approach also eliminates the need for such rules as "order of operations" and "use of brackets", since the obvious thing to do with $3 \times 2 + 4$, for example, is to work it out the way it comes, unless, of course, the context requires that the addition be done first, in which case the expression means $3 \times (2 + 4)$ and not $(3 \times 2) + 4$. However, where the child is concerned, there is no need to actually use brackets to indicate which
interpretation is intended for the meaning is defined by the question. If the child can handle many problems without worrying about conventions for ordering operations or brackets, one might ask how important a knowledge of these conventions is. Since algebraic operations such as \( x + y \) cannot be 'performed' in the sense of being replaced by a single value, any need to record a sequence of operations immediately requires consideration of these conventions. If the child has not seen the need for them in arithmetic problems, it is perhaps unlikely that he will concern himself with them in generalised arithmetic.

The S.E.S.M. Project (1982) uncovered this possibility when investigating the mistakes that children make in beginning algebra or 'generalised arithmetic'. Forty-eight second, third and fourth form children from middle mathematics streams were interviewed and were posed questions of the type:

\begin{tabular}{|l|}
\hline
(1) What can you write for the area of this rectangle? \\
\begin{tikzpicture}
\draw (0,0) rectangle (4,3);
\end{tikzpicture} \\
(2) Multiply \( k + 2 \) by 3 \\
(3) Which of the following can you write for \( e + 2 \) multiplied by 3? Tick everyone you think is correct. \\
e + 6 \\
3 \times e + 2 \\
3e + 6 \\
none correct \\
\hline
\end{tabular}

Over one third of the children interviewed could explain how questions 1 and 2 were solved, but only one realised that it was necessary to use brackets in recording the answer. The remaining children recorded their answers as:

\[ m + 4 \times 3 \text{ or } 3 \times m + 4 \text{ and } \]
\[ 3 \times k + 2 \text{ or } k + 2 \times 3, \]
even though they stated that each problem required the addition to be performed first. In question 3, only one child selected both bracket expressions and nothing else. Some children answered "you can put the brackets in if you want, its just the same", while others excluded them altogether because they did not know what they meant.

Consequently, if we do in fact require children to have an understanding of these conventions (and for those children who are going to be taught algebra such conventions may be essential), then it would seem that we must first address ourselves to the problem of creating an awareness of the need for such conventions. The calculator can prove a useful aid to understanding this need and provides a useful source for practicing the conventions.

The class may be asked to work out $5 + 4 \times 3$ mentally and then on the calculator. This is one occasion when different types of logic, algebraic and arithmetic, on different calculators would prove an asset. It soon becomes obvious to the class that some rule must be formulated to avoid ambiguity.

The other rules may be built up in a similar way culminating in the overall M.D.A.S. convention (sometimes recalled as 'My Dear Aunt Sally') Brackets may then be introduced to emphasise the M.D.A.S. rule, but the problem of using the calculator for expressions such as $(173 \times 42) - (1638 \div 234) + (427 \times 14)$ must be discussed.

Some basic calculators now offer brackets (or parenthesis) either singly or multiple nested, which act as a form of memory, storing operands and pending operations. This means that expressions are evaluated properly before being combined with other expressions. However, if these keys are not
available then the memory may be used. For example,

\[(3 \times 4) + (5 \times 6)\] would require:

\[
\begin{array}{c}
3 \times 4 = \text{STO} \\
5 \times 6 + \text{RCL} =
\end{array}
\]

If the memory was not available then the result of each bracket would have to be written down first before the final answer could be achieved. It may even be desirable to use this latter method first even if the calculator does have a memory for the emphasis is on use of brackets and ordering of operations. Children at this stage may not be familiar with the use of the memory.

When the M.D.A.S. rule needs to be broken, then the use of brackets becomes essential (the BONDAS convention may now be introduced). Further calculations should be tried, firstly by writing down an intermediate answer from the brackets, and then the final answer calculated, and secondly by writing down the final answer only. For example:

\[
\begin{align*}
(279 - 142) & \times 15 \\
28 & \times (192 + 45) \\
(372 + 639) & \div 55 \\
43 + 17 & \times 8 \\
51 & \times 16 - 13
\end{align*}
\]

The check without memory may seem tedious, but at an early stage, it may be necessary for the confidence of the child.

A third kind of check involves ESTIMATION, and this process is important in its own right, since we are very often not so much concerned with the exact answer as we are with an approximate answer: the distance is about 50 miles, the tank will hold about 50 gallons, etc. That is, learning to estimate is useful in general as well as an aid in detecting any major errors in computation as already discussed.
All students should be asked to check the validity of the two basic properties of addition and multiplication: the commutative and associative properties, by comparing results using the calculator. At some later stage, they should compare answers to questions such as:

Compute $352 \times (1894 + 693)$
and Compute $(352 \times 1894) + (352 \times 693)$

Clearly, the first is easier to compute than the second and this leads to establishing the usefulness of another basic property: the distributive property of multiplication over addition. In general, for any numbers $a$, $b$, $c$,

$$a \times (b + c) = (a \times b) + (a \times c)$$

Also by the commutative property of multiplication, we have

$$(b + c) \times a = (b \times a) + (c \times a)$$

By devising suitable numeric examples, other properties which are used to simplify calculations, can be validated by checking a given result or even encouraging the student to discover the property himself. For example,

$$a - (b - c) = a - b + c$$
$$a - (b + c) = a - b - c$$
$$(a + b) \div c = (a \div c) + (b \div c)$$

Dubisch / Hood provide a rich source of suitable material to demonstrate these basic principles.

The calculator, then, provides a useful teaching aid to the topic 'ordering of operations', on the other hand, it becomes apparent that the topic itself has become even more important because an understanding of it is essential to the proper use of the calculator.
How money grows.

A study of annual interest rates and compound interest can be a useful introduction to develop the idea of a growth factor and to exhibit the power of the calculator in evaluating exponential growth. Interest rates are a common feature in the high-street windows of banks and building societies and are prominent in their press advertisements.

The work can be aimed at third year and above and assumes some experience of percentages and their decimal equivalents, simple sequences, indices and formation of simple algebraic formulae.

After practice with simple interest rates for one year, followed by calculating compound interest rates one year at a time using the calculator, the results may be displayed as a sequence of growing amounts of money:

\[
\begin{align*}
£200 & \rightarrow £224 \rightarrow £250.88 \rightarrow £280.99 \rightarrow \\
\end{align*}
\]

To search for a pattern of change from one number to the next two different plans may be tried:

Plan 1 uses subtraction:

\[
\begin{align*}
224 - 200 &= 24 \\
250.88 - 224 &= 26.88 \\
280.99 - 250.88 &= 30.11 \\
\end{align*}
\]

These results give the interest for successive years but do not suggest how you might continue the sequence after £280.99
Plan 2 uses division (a calculator is a great help here)

\[
\begin{align*}
224 \div 200 &= 1.12 \\
250.88 \div 224 &= 1.12 \\
280.99 \div 250.88 &= 1.12
\end{align*}
\]

The questioning may continue as follows:

What number should you multiply £224 by to get £250.88?
Continue the sequence beyond £280.99 and calculate the number of years it takes to almost double £200 with an interest rate of 12% per annum.

What interest rate would produce this sequence?

\[
\begin{align*}
£125 & \rightarrow £135 \rightarrow £145.80 \rightarrow ? \\
£75 & \rightarrow £84 \rightarrow £96.60 \rightarrow £115.92 \rightarrow ?
\end{align*}
\]
(Change of interest rate)

The amount at the end of any year can be calculated by repeated multiplication by a constant GROWTH FACTOR.

For an interest rate of 15% the growth factor is 1.15, for 13\(\frac{1}{2}\)% it is 1.135, for 8\(\frac{3}{4}\)% it is 1.0875 and so on.

For a principal of £200 and a 12% p.a. rate of compound interest, the pattern of growth can be seen from:

\[
\begin{align*}
£200 &= \text{multiply by } (1.12)^1 \rightarrow £224 \text{ after 1 year} \\
£200 &= \text{multiply by } (1.12)^2 \rightarrow £250.88 \text{ after 2 years} \\
£200 &= \text{multiply by } (1.12)^3 \rightarrow £280.99 \text{ after 3 years} \\
& \vdots \\
£200 &= \text{multiply by } (1.12)^7 \rightarrow £442.14 \text{ after 7 years} \\
& \text{and so on.}
\end{align*}
\]

Further questions to consider:

Make a formula for calculating the amount after n years
Make a formula which applies to any principal £P and any growth factor R.
Growth Factors and Decay Factors.

The compound interest formula

\[
\text{Amount after n years} = \text{Principal} \times (\text{Growth Factor})
\]

introduces the formation and use of growth factors and now the idea of a decay factor may be explored. If scientific calculators are being used it is worth paying some attention to the use of a \( y^x \) key (or its equivalent), and where a calculator result is to be followed by a realistic approximation, it is a good idea to ask pupils to mark the calculator result in some way e.g. \( \circ \)

A growth factor is always greater than 1.

\[ 1.12 = (1 + 0.12) \text{ for a } 12\% \text{ rate of growth or appreciation.} \]

A decay factor is always less than 1.

\[ 0.88 = (1 - 0.12) \text{ for a } 12\% \text{ rate of decay or depreciation.} \]

Further exercises highlight the differences between growth and decay factors and introduce everyday terminology associated with them:

(1) If inflation runs at \( s\% \) p.a. for the next five years, how much would I expect to pay at the end of that time for a leather coat which costs £50 now?

(2) What value would a £1 note have, under constant devaluation of \( 12\% \) per year after 10 years?

(3) A car depreciates in value by \( 30\% \) in its first year and then by \( 20\% \) in each year after that. What \% of its value at new will the car have after 5 years?
Regular Savings

Many children will be saving regularly with a building society using the idea of the sum of a geometric progression. If there are \( n \) payments of \( £P \) each, and interest is added at a rate of \( R% \), then the final amount is:

\[
£P \left( \frac{\left(1 + \frac{R}{100}\right)^n - 1}{\frac{R}{100}} \right)
\]

The power of the calculator may be used to combat formidable but impressive formulae such as the modification of the above formula when interest is compounded more frequently than annually:

\[
\text{Amount} = P \left( k + \frac{(k + 1)}{2} \right) x \frac{R}{100} \left[ \frac{\left(1 + \frac{R}{100}\right)^{2n/k} - 1}{\frac{R}{100}} \right]
\]

where \( k \) is the number of months before compounding occurs. (Green and Lewis, 1978 pp. 77)

Regular Payments

Many children would have heard of a repayment mortgage, for a house purchase, and, although it would not be of any immediate practical significance, the calculator provides another opportunity for the more-able child to take advantage of the challenge of following its workings.

For example, payment per year for a loan of £5000 at 8\% per annum for 25 years may be built up as follows:

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount owed</th>
<th>Amount plus interest</th>
<th>Repayment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5000</td>
<td>5000 x 1.085</td>
<td>A</td>
</tr>
</tbody>
</table>
Year 2 : Amount owed = 5000 x 1.085 - A  
Amount plus interest = (5000 x 1.085 - A) x 1.085  
= 5000 x 1.085² - A x 1.085  
Repayment = A

Year 3 : Amount owed = 5000 x 1.085² - A x 1.085 - A  
Amount plus interest = (5000 x 1.085² - A x 1.085  
- A) x 1.085  
= 5000 x 1.085³ - A x 1.085²  
- A x 1.085  
Repayment = A

........................

Year 10 : 5000 x 1.085⁹ - A x 1.085⁸ - ...... - A x 1.085 - A  
Year 20 : 5000 x 1.085¹⁹ - A x 1.085¹⁸ - ... - A x 1.085 - A

The end of the 25th year must see the loan completely repaid.  
The amount at the start of the 26th year must therefore be  
zero:

\[ 5000 x 1.085^{25} - A x 1.085^{24} \ldots - A x 1.085 - A = 0 \]

\[ 5000 x 1.085^{25} = A(1.085^{24} + 1.085^{23} + \ldots + 1.085 + 1) \]

\[ 5000 x 1.085^{25} = \frac{A(1.085^{25} - 1)}{1.085 - 1} \]

(the sum of a G.P. has already been developed)

\[ = 38433.8 = A \times 78.6678 \]

\[ A = 488.56 \]

The annual payment is £488.56

The following table shows how the loan decreases during the  
25 years and the proportions of interest and repayment, the  
yearly payment change.
<table>
<thead>
<tr>
<th>Year</th>
<th>Outstanding Loan</th>
<th>Interest Payment (8% of loan)</th>
<th>Loan repayment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>£5000</td>
<td>£425</td>
<td>£63.56</td>
</tr>
<tr>
<td>2</td>
<td>£4936.44</td>
<td>£419.60</td>
<td>£68.96</td>
</tr>
<tr>
<td>3</td>
<td>£4867.48</td>
<td>£413.74</td>
<td>£74.82</td>
</tr>
<tr>
<td>23</td>
<td>£1247.80</td>
<td>£106.06</td>
<td>£362.50</td>
</tr>
<tr>
<td>24</td>
<td>£865.30</td>
<td>£73.55</td>
<td>£415.01</td>
</tr>
<tr>
<td>25</td>
<td>£450.29</td>
<td>£38.27</td>
<td>£450.29</td>
</tr>
</tbody>
</table>

Total £488.56
The study of simple sequences and series can give pupils valuable insight into interesting mathematical ideas and generalisations. This can lead into the use of algebraic formulae for the terms of a sequence and the sum of a series. The role of the calculator in the topic is simply to carry out additions, subtractions, multiplication and division and so enable the pupil to concentrate on investigating patterns and relationships more easily and quickly.

The series of odd numbers is considered first, including investigating

a) the number of terms in the series and its sum and
b) the last term and the sum of the series.

Further pupil material develops the topic by working with the sequences of whole numbers and triangle numbers.

Sequence of Odd Numbers

Although the sequence of odd numbers may not be of any practical value to the vast majority of pupils, interest can be heightened by presenting it in a physical context which aids the imagination. This may be achieved by considering stacking tins of beans in a supermarket, an idea included in "calculators Count", Schools Council (1983)
<table>
<thead>
<tr>
<th>Number of layers in the stack</th>
<th>Number of tins in the bottom layer</th>
<th>Total number of Tins in the stack</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>$1 + 3 = 4$</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>$1 + 3 + 5 = 9$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The build up to the concept may proceed with a similar sequence of questions and instructions to:

Use a calculator to complete the table.

How can you quickly calculate the numbers of tins needed to make the stack 15 layers high? The more able pupils should be encouraged to express the relationship algebraically i.e. $S_n = n^2$ when $n$ is the number of terms.

Check the relationship using the calculator

All children should find that:

Number of layers $\rightarrow$ squared $\rightarrow$ number of tins needed

How many layers could be built with a) 169 tins

b) 247 tins?

The square root button may be used or a trial-and-error method.

From the table, what has to be done to each number in the second column to make the corresponding number in the first column?

How quickly can you calculate how many tins are needed if there are to be 27 tins in the bottom layer?

This exercise is designed to develop the general relationship between the last term in the series and its sum. With many pupils it is enough to express the relationship in words.

Number of tins in the bottom layer $\rightarrow$ Number of tins needed altogether
Mor able children should be encouraged to go on to express it as an algebraic formula

\[ S = \left[ \frac{1 + l}{2} \right]^2 \]

where \( l \) is the last term in the series.

How many tins would there be altogether if the bottom layer contained

a) 33 tins
b) 397 tins?

Challenge: One of the stacks has some layers missing at the top. The bottom layer had 89 tins. The top layer has 11 tins. Can you calculate how many tins were in the stack?

Whole Numbers and their Sum

To investigate the relationship between the number of terms in the series of whole numbers and its sum, the tins may be stacked in a different way
Following a similar build up to odd numbers, most children should be able to work out a relationship by considering the table for 12 layers:

\[
\begin{align*}
12 & \\
12 + 1 &= 13 & \text{WORK OUT} & \quad 12 \times 13 & \text{DIVIDE BY 2} & \quad 78 \text{ tins}
\end{align*}
\]

More able children should be able to express this algebraically:

\[
e.g. \quad S = \frac{n(n + 1)}{2} \quad \text{when } n \text{ is the number of terms.}
\]

It should be noted that each sum is called a TRIANGLE number.

**Triangle Numbers and their sum**

To introduce the series formed by the sequence of triangle numbers, and the relationship between the number of terms and its sum, stacking in the supermarket may be used again. This time, however, we consider stacking oranges in a pyramid that has a triangular base with:

- 4 oranges in the top layer
- 3 oranges in the second layer
- 6 oranges in the third layer, and so on

It will be difficult for many pupils to visualize the successive layers and so table-tennis balls glued together can be used for each separate layer.

Is there an easy way to calculate the number of oranges needed?
A table of results leads to: (for 12 layers)

\[
\begin{align*}
12 & \\
12 + 1 = 13 & \text{WORK OUT} \\
12 + 2 = 14 & \text{DIVIDE} \\
12 \times 13 \times 14 & \text{BY 6} \\
\end{align*}
\]

364 oranges

Again the more-able pupils may express the relationship as

\[S = \frac{n(n + 1)(n + 2)}{6}\]

After using the calculator to practice the generalisation, the applications of the relationship in reverse may be tested and this involves seeking three consecutive whole numbers whose product is known.

\[\text{e.g. How many layers could be built with 1540 oranges?}\]

All pupils should be encouraged to experiment with their calculators initially, using trial and error to find the solution. For many pupils this may be as far as they can go. However, it should be possible for more able pupils, with a suitable background of knowledge, to develop a better strategy using the integer immediately below the cube root of the known product as a starting point:

\[\text{e.g. } n(n + 1)(n + 2) = 9240 \text{ (For 1540 oranges)}\]

\[3 \sqrt{9240} = 20.98 \text{ using the } [x^y] \text{ key}\]

so try \(n = 20\).

check \(20 \times 21 \times 22 = ?\)

"When you multiply any three consecutive numbers together and then divide the answer by 6, you always finish up with a whole number".
Can you explain why this MUST happen?

Many pupils will again benefit from experimenting with their calculators before trying to suggest a formal answer. The following style of argument should be expected from pupils:

At least one of the consecutive numbers must be even and have 2 as a factor.

Exactly one of the consecutive numbers must be in the '3 times' table and have 3 as a factor.

Therefore the product must have $6(=2 \times 3)$ as a factor.

More searching questions:

A big stack had some layers missing from the top.
The bottom layer had 120 oranges and the top layer had 55 oranges.

How many layers were missing?

How many oranges are in the stack?

The Geometric Series and a Limit

To study some properties of a simple convergent geometric series and to introduce the idea of a mathematical limit, the topic, at its simplest level, can be approached by using decimals alone, in conjunction with a calculator. At a more advanced level fractions should be used alongside decimals.

A jumping frog is used to introduce sequences in which successive terms are generated by a process of halving. For example:

A clockwork frog can leap 32 cm when fully wound up.
Its next leap is half as long (16 cm)
Its next leap is half again (8 cm) and so on.

Copy the table and use it to answer the questions:

<table>
<thead>
<tr>
<th>Number of jumps</th>
<th>Length of jump (cm)</th>
<th>Total length jumped (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32</td>
<td>32</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>48</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>56</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
If the frog had 12 jumps, how far would it jump altogether?

At its basic level, this exercise provides an opportunity for discussing place value and calculator technique e.g. use of a memory, if present. If fractions are also used, it is easier to tackle the further question:

If a frog had an unlimited number of jumps, could it ever jump 64 cm altogether?

In considering the seventh jump onwards, this could lead to the sequence \(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \ldots\) and \(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}\) as the corresponding partial sums.

The idea of a mathematical limit can then become most apparent.

Another clockwork frog can leap only 20 cm when fully wound up.

Its next leap is half as long - and so on.

In only 8 jumps, can this frog cross a rug that is 40 cm wide?

A 'super-de-luxe' frog has an adjustable first jump.

This can be set in centimetres and tenths of a centimetre; for example 21.3

1) What is the smallest first jump that must be set so that this frog can cross a 40 cm rug in 5 leaps?

2) Find the smallest jump that must be set so that this frog can cross the rug in 7 leaps.

At a basic level, these questions give an opportunity for systematic decimal search (restricted to one decimal place) using a calculator. At a more advanced level, pupils might be asked to predict the answer on theoretical grounds, before checking with a calculator.

The argument might run as follows .....
First jump is 1 unit long
Second jump is \( \frac{1}{2} \) unit long
Third jump is \( \frac{1}{4} \) unit long
Fourth jump is \( \frac{1}{8} \) unit long
Fifth jump is \( \frac{1}{16} \) unit long

In 5 jumps a total of \( \frac{15}{16} \) units

So

\[
40 \text{ cm} = \frac{31}{16} \text{ units}
\]

\[
40 \text{ cm} - 31 = \frac{1}{16} \text{ unit}
\]

\[
(40 \text{ cm} - 31) \times 16 = 1 \text{ unit}
\]

Thus, length of first jump should be 20.645 cm (theoretically)

Hence, the 'super-de-luxe' frog should be set at 20.7 cm in order to cross the rug in 5 leaps

In general, if the frog is allowed \( n \) jumps, then

\[
40 \text{ cm} = \frac{2^n - 1}{2^n -1} \text{ units}
\]
THE FIBONACCI SEQUENCE

In the year 1202 the Italian mathematician Leonardo Fibonacci completed his book 'Liber abaci'. In the book was the following problem, which has inspired mathematicians up to the present day.

"A pair of rabbits can breed when they are two months old. They will produce another pair of rabbits each month after that. The new pair of rabbits will also be able to breed after two months and will also produce a new pair of rabbits each month. Starting from a single pair, how many rabbits will there be in 12 months?"

The illustration shows part of the solution.

Tree graph for Fibonacci's rabbits
What has interested mathematicians is not the solution itself but the sequence of numbers which is generated in the finding of it. Each number in the series can be made by adding the two previous numbers together: 1, 1, 2, 3, 5, 8, ........
This sequence is known as the Fibonacci Numbers.

Many other examples can be used to demonstrate their generation. One way is to list all the ways of using 1p coins and 2p coins to make up 1p, 2p, 3p, etc.

| 1p | 0 | 1 |
| 2p | 0, 0 | 2 |
| 3p | 0, 0, 0 | 3 |
| 4p | 0, 0, 0, 0 | 5 |
| 5p | 0, 0, 0, 0, 0 | 8 |

Other ways include the dichotomous branching of a tree year by year and a progression of increasing sets of notes which is musically very tidy and in line with the conventional development of western music.

The Fibonacci Numbers provide a wealth of patterns which may be investigated using the calculator. This material can be made suitable for a wide ability range.

The most interesting and significant sequence can be produced by working out the ratios of neighbouring pairs of numbers in the Fibonacci sequence as follows:

<table>
<thead>
<tr>
<th>Fibonacci numbers</th>
<th>Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{1} = 1.0000$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{1} = 1.0000$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{1} = 2.0000$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{2} = 1.5000$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{5}{3} = 1.6667$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{8}{5} = 1.6000$</td>
</tr>
</tbody>
</table>
Continue this pattern writing down the ratios to 4 decimal places. The calculator will enable pupils to conclude that the ratios get closer and closer to 1.6180... a number known as the 'GOLDEN RATIO'.

The pupils may then be asked to investigate the sequence produced by using the same rule but with different starting numbers. They should be encouraged to use different starting numbers amongst themselves too. They should all arrive at the same limit as the numbers get bigger - the Golden Ratio - 1.6180.

A further investigation would change the rule by adding three consecutive terms to produce the next. Will this sequence also have neighbouring numbers whose ratios get closer to 1.6180? A discussion should precede the calculations. The results of investigating adding 4, 5 or more numbers may be practiced in a table and graphically.

Since the time of the ancient Greeks the Golden Rectangle has played an important part in art and in architecture. A Golden Rectangle can be drawn using the ratio 1:1.618. It is said to be one of the most pleasing of all geometric shapes and the dotted lines in the drawing show how the Golden Rectangle fits the famous Parthenon at Athens.
The pupils may produce an approximate one, without measuring, by adding squares on squared paper several times as follows:

After many stages you will have an almost perfect Golden Rectangle.

It follows that if you draw a Golden Rectangle and take away a square you are left with another Golden Rectangle. If you take away a square from this Golden Rectangle you get ... another Golden Rectangle. The Golden Rectangle is unique in this way.

The more able pupil may be shown this in another way:

Given a rectangle of width 1 unit and length \( x \), find \( x \) such that if a square of side 1 unit is cut away, the remaining rectangle is similar to the original.

\[
\frac{x}{x-1} = x^2 - x - 1 = 0
\]

\[
x = \frac{1 \pm \sqrt{5}}{2}
\]

\[
\frac{1 + \sqrt{5}}{2} \text{ is the Golden Ratio}
\]

The continued fraction \( 1 + \frac{1}{1 + \frac{1}{1 + \ldots}} \) may also be investigated and be shown to approach the Golden Ratio.
Successive stages give 1, $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, ... 

The calculator proves an invaluable aid to checking Binet's formula for the $n$th term of a Fibonacci Sequence, namely:

$$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)$$

Furthermore as $n \to \infty \left( \frac{1 - \sqrt{5}}{2} \right)^n \to 0$

and so $F_n \to \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \approx 0.447214 \times (1.618034)^n$

Other interesting properties of the Fibonacci Sequence may be investigated by studying patterns and extending them.

e.g.  
a) $1 + 1 = 3 - 1$  
$1 + 1 + 2 = 5 - 1$  
$\ldots$  
$1^2 + 1^2 = 1 \times 2$  
$1^2 + 1^2 + 2^2 = 2 \times 3$  
\ldots  
\ldots  

b) $1 + 2 = 3$  
$1 + 2 + 3 = 8$  
\ldots  
\ldots  

c) $1^2 + 1^2 = 1 \times 2$  
$1^2 + 1^2 + 2^2 = 2 \times 3$  
\ldots  
\ldots  

d) $1 \times 2 = 1^2 + 1$  
$1 \times 3 = 2^2 - 1$  
$2 \times 5 = 3^2 + 1$  
\ldots  
\ldots  

Pupils should be encouraged to explain them.

The Fibonacci Sequence is only one of many sequences and number patterns that may be investigated more thoroughly and enjoyably now a calculator is available to deal quickly with calculations.
ITERATIVE METHODS

One method of finding a reasonably close answer to a problem which does not have an exact solution is to find an approximate one, and then use this to obtain a second approximation which is more accurate than the first. The process may be repeated so that we obtain a third approximation which is more accurate than the second. We may go on like this until we obtain the degree of accuracy that we desire. Unfortunately, some of the calculations involved are rather difficult, although many of the methods themselves are fairly simple. Now that calculators (and computers) have become available to tackle the tedious series of operations, the methods are a more feasible proposition for the 11 - 16 year range. The methods are called ITERATIVE METHODS and the process is ITERATION.

Perhaps the simplest place to start for this range is determining the square root of a number correct to a given number of decimal places by the method of decimal search. It is an example of the way in which the systematic use of trial and error can help solve a problem and illustrates a particular advantage which calculators can bring to problem solving.

After some initial work on the concept of a square and a square root the pupils may be asked to find $\sqrt{13}$ to a given number of decimal places.

The number line is a useful visual aid to many pupils.

Step 1

$\sqrt{13}$

$\sqrt{13}$ is between 3 and 4, but is it nearer to 3 or 4?
Step 2

\[
\begin{array}{cccccccccc}
& & & & & & S & & & L \\
3.0 & 3.1 & 3.2 & 3.3 & 3.4 & 3.5 & 3.6 & 3.7 & 3.8 & 3.9 & 4.0 \\
\end{array}
\]

3.0 is too small (S) and 4.0 is too large (L)

Try half way between 3 and 4, that is 3.5

The calculator is used to test if this is too small or too large.

3.5 is too small, so try 3.6, 3.7 until one is too large.

3.7 is too large.

Mark on the diagram therefore, \( \sqrt{13} \) is between 3.6 and 3.7.

Step 3

\[
\begin{array}{cccccccccc}
& & & & & & S & & & L \\
3.60 & 3.61 & 3.62 & 3.63 & 3.64 & 3.65 & 3.66 & 3.67 & 3.68 & 3.69 & 3.70 \\
\end{array}
\]

Continue by trying 3.65. Mark all the numbers on the diagram with S or L.

Continue until \( \sqrt{13} \) is found correct to two decimal places.

The method also increases an awareness and understanding of place value in decimals, especially in relation to the number line.
A similar pattern could be used to find the cube root of a number.

A more interesting method than 'interval halving' for finding the square root of a number is the iterative method known as Hero's Method. However, pupils should not gain the impression that square roots should normally be obtained in this manner!

The square root is used here as a vehicle for the introduction of iterative methods and when a square root is needed in a calculation the usual way is to use a calculator with a square root key.
In order to find the square root of 70, the calculator is used to fill in the table

<table>
<thead>
<tr>
<th>Estimate (E)</th>
<th>70 ÷ E (F)</th>
<th>New Estimate (F + E) ÷ 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2nd</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3rd</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The plausibility of 8 as a first estimate should be discussed and why \( F > \sqrt{70} \) if \( E < \sqrt{70} \) (and vice-versa).

The iterative formula for producing the sequence of estimates of the square root of some number \( x \), that is \( E' = \left(\frac{x}{E} + 2\right) ÷ 2 \) when \( E' \) is the new estimate calculated from the previous estimate \( E \), should then be investigated.

When \( E' \) agrees with \( E \) to calculator accuracy, we could say that \( E' \) and \( E \) are both approximately equal to some number \( N \). We could then write the formula as

\[
N \approx \left(\frac{x}{E} + N\right) ÷ 2
\]

Rearrange this formula so that the connection between \( N \) and \( x \) becomes very clear.

It will be seen that the formula approaches the answer very quickly, but what happens with other starting points? How many steps?

Using a similar build up the pupils may be introduced to

\[
N \approx \left(\frac{x}{N} + 2N\right) ÷ 3,
\]

and then be asked to rearrange so that the connection between \( N \) and \( x \) becomes very clear.
Does this require as many steps as
\[ N \approx \left( \frac{x}{N^2} + N \right) \div 2? \]

What could this new iterative method be used for?

Try to write down a similar iterative formula which might be used to find the fifth root of \( x \).

(Terms such as 'converge' and 'recursive' may be introduced at the discretion of the teacher and the more able pupils may be shown the use of the more traditional notation used in many textbooks such as
\[ x_{n+1} = \frac{1}{2} \left( x_n + \frac{70}{x_n} \right) \]

Advantages of this method include the fact that even if the first guess is a poor one, the correct answer will still emerge. Further, even if a mistake is made the correct answer will still emerge although, of course, after more steps than would have been necessary otherwise.

It may be desirable at some stage to encourage the use of the constant key on the calculator and then the use of the memory, but these should not be rushed or assumed to be understood by the pupils. Some pupils may well prefer to continue setting out each intermediate step as a check.

Pupils should appreciate that most iterative methods would be too lengthy and too complicated to continually carry out on a basic or scientific calculator - a programable calculator or computer would be more appropriate. However, by following a set of simple instructions, pupils may be able to work out what a particular iterative procedure does. Take, for example, the following set of instructions. Choose a number from 8, 64, 125 and call it \( x \).
1) Start with 1
2) Multiply by your number x
3) Take the square root
4) And again!
5) Record your result but do not clear your calculator
6) Repeat steps 2, 3, 4, 5 over and over again until successive recorded results agree when corrected to three decimal places.
7) Write out your last result correct to 3 decimal places
8) Stop

What relationship has your final result with your number x?

The pupils have, in effect, been using the following iterative formula to produce a sequence of estimates to the cube root of x. It has been achieved using a calculator with a square root button.

$$E' = \sqrt[3]{Ex}$$

Rearrange $N = \sqrt[3]{Ex}$ to show the connection between x and N.

What might you expect these iterative formulae to lead to?

a) $E' = \sqrt[3]{Ex}$ ?

b) $E = \sqrt[3]{\frac{1}{E} \times x}$ ? Test your idea

An alternative form of the instructions given may be a flow chart.
Start

Set E = 1

Calculate $F = \sqrt{Ex}$

Print E

Does $F = E$ Correct to 3 decimal places?

No

Set $E = F$

Yes

Print F Correct to 3 decimal places

Stop
SOLUTION OF EQUATIONS

The work on iterative methods, sequences and limits may be extended to the more able fifth year pupils and more traditional notation used.

If we require the solution to a problem such as

\[ x^2 - 2x - 2 = 0 \]

we might rearrange as follows:

\[ x^2 = 2x + 2 \]
\[ x = \pm \sqrt{2x + 2} \]

If the corresponding iterative process:

\[ t_{n+1} = \sqrt{(2t_n + 2)} \]

or alternatively \( x_{\text{new}} = \sqrt{2x + 2} \)

results in a limit then it will be a solution of the original equation. Putting \( t_1 = 1 \) will soon confirm that this leads to 2.732 as a solution of \( x^2 - 2x - 2 = 0 \)

The process may be summarised as follows:

1) Rewrite the equation to be solved in the form \( x = f(x) \) for some \( f(x) \)
2) Use \( x_{\text{new}} = f(x) \) to generate a sequence of \( x \)-values
3) If a limit results, then this value satisfies \( x = f(x) \) and so is a solution of the original equation.

Some important questions arise including:

Will the processes always yield a solution no matter how the equation is rearranged?
Is the starting point for the iterative process important? Well chosen examples will show that convergence may arise for some starting points but not others and that not all arrangements will, in fact, lead to a solution.

The situation may be confusing for the pupils and so a graphical approach can be employed to illustrate why some sequences diverge and others converge onto one particular solution of the related equation. Further, it should assist in deciding which particular rearrangement might be profitable.

We may adopt an investigatory approach to the solution of an equation in order to illustrate these main ideas.

Consider \( x^2 - 7x - 3 = 0 \)

First rearrange in the form \( x = f(x) \)

There are many possibilities, for example

\[
x^2 = 7x + 3 \quad \text{leads to} \quad x = \sqrt{(7x + 3)} \quad \text{and} \quad x = \frac{x^2 - 3}{7}
\]

\[
x(x - 7) - 3 = 0 \quad \text{leads to} \quad x = \frac{3}{x - 7}
\]

At this stage we could choose one of the above, together with a starting value, and see what emerged from the related iterative process, but this is rather hit-or-miss in approach to a complete solution. For example

\[
x_{n+1} = \sqrt{(7x_n + 3)}
\]

with \( x_1 = 10 \) gives 7.405 as a limit but

\[
x_{n+1} = \frac{x_n^2 - 3}{7}
\]

with \( x_1 = 10 \) does not give any limit.
If, however, we first examine the graphs $y = x^2$ and $y = 7x + 3$ then we can see that there are two solutions to be found.

$x = \sqrt{(7x + 3)}$

gives rise to a path going 'vertical' to the line and 'horizontal' to the curve and thus any positive starting value leads to a larger solution as illustrated. However,

$$x = \frac{x^2 - 3}{7}$$

is effectively an inverse process but with the arrows reversed. Thus, a too large starting value (7.405) produces a divergent sequence, whereas for $x < 7.405$ (and $> -7.405$) the resulting sequence converges to the other solution.
If we choose, say, \( x_1 = 3 \)

\[
x = \sqrt{(7x + 3)} \text{ yields } 7.405
\]

and \( x = \frac{x^2 - 3}{7} \) yields \(-0.405\)

which furnishes a complete solution of the equation.

Pupils appreciate visually the idea of spiralling towards a point of intersection but it is not so easy to understand how to work the corresponding numerical process. A step-by-step illustration is desirable and even then considerable teacher assistance will be necessary. Unless a graph-drawing exercise using the calculator is to be incorporated as part of the lesson, it would be useful to have the graphs already prepared by the teacher.
CHAPTER 3 - THE CALCULATOR AS AN INFLUENCE ON CURRICULUM DESIGN

1 The Effect of the Calculator on the Mathematics Curriculum

2 Some outstanding issues on the effective use of the Calculator

3 The Effect of the Calculator on Mathematical Content

"Look - if you have five pocket calculators and I take two away, how many have you got left?"

WITH ACKNOWLEDGEMENTS TO PUNCH
THE EFFECT OF THE CALCULATOR ON THE MATHEMATICS CURRICULUM

Over the past few years the introduction of the calculator into schools has become less of a contentious issue. However, it is doubtful whether the impact of this technological tool has been as dramatic as has been predicted and hoped for by many educators. This chapter will discuss some of the advantages the calculator could bring to the mathematics curriculum in general, the effects it might have on mathematics content in particular, and some outstanding issues in its effective use.

In brief, the calculator can be used:

1) To assist in UNDERSTANDING some mathematical CONCEPTS.
2) As an aid to PROBLEM-SOLVING.
3) To encourage discovery, exploration and creativity by introducing appropriate INVESTIGATIONS and mathematical ideas.
4) To introduce 'REAL WORLD' DATA into problems.
5) To MOTIVATE pupils by introducing more interesting and challenging topics.
6) As a SERVICE FUNCTION across the curriculum.
7) As an aid to COMPUTATION.
8) As an aid to more FLEXIBLE TEACHING programmes. For example, mixed ability teaching and individualised learning by 'freeing' the teacher and closing the gap between ability groups.

Add to the list a statement from Cockcroft(389):

"We believe that there is one over-riding reason why all secondary pupils should, as part of their mathematics course, be taught and allowed to use a calculator. This arises from the increasing use which is being made of calculators both in employment and adult life".

and we see how strong the arguments for its use are.
THE CALCULATOR AS AN AID TO UNDERSTANDING SOME MATHEMATICAL CONCEPTS

The calculator is now invariably the device used by those people involved in regular day to day calculations and it has largely replace such aids as logarithm tables, slide rules and ready reckoners in shops, offices and factories. In school, however, the position is somewhat different for research has shown that the calculator has a specially important contribution to make in the teaching and learning of mathematics. The calculator is potentially a powerful teaching aid which can be used to considerable effect in helping pupils to acquire understanding of mathematical concepts.

In his excellent article 'Calculator: Abuses and Uses' David Johnson (1978) suggests that the pupil uses the calculator to generate output with the purpose that the output will demonstrate a concept or relationship, or that the actual generation of the output will serve to help reinforce a concept which has been taught previously.

In CONCEPT - DEMONSTRATION, the pupils are first asked to look for a pattern when the calculator is used to do some calculations the pupils have not yet learned how to do (discover a rule), then justify the result with a few, well-chosen examples, and then practise.

When a particular piece of mathematical content has already been introduced and the calculator activity is planned to provide an opportunity to practise or apply what has been learned, this can be thought of a Concept - reinforcement.

Chapter 2 attempts to demonstrate how certain basic topics, which have been identified by the C.S.M.S and S.E.S.M. research teams as presenting particular difficulties to many pupils, may be illuminated by using the calculator. These topics:
fractions, decimals and place value, and order of operations are not the only ones identified which may be helped. Others include percentage, proportion, negative number, and graphs, where greater use of the unitary method may be made for the understanding of proportion; greater emphasis on the link between decimals and percentage rather than fractions and percentage; easier plotting of points of a graph and testing if points lie on a line; and so on.

The standard pencil and paper ALGORITHMS may, in time, be removed from the curriculum but that does not mean that pupils are introduced to fewer algorithms. There are many, more powerful and more general, waiting to be explored. In the words of Polya (1965) "we must teach 'guessing'". From unstructured trial-and-error we can move to guided trial and error, search methods and more general iterative procedures. Blakeley (1980) suggests that pupils should be able to:

a) follow an algorithm
b) modify an algorithm to produce an alternative required result; and
c) design algorithms, involving them in analysis of problems.

Iteration can be a powerful and widely applicable technique and can be used to aid the understanding of such concepts as square root and cube root as described in Chapter 2. Further iteration can be used to attack all equations met in an o-level course should the teacher so desire. Although traditional methods of factorisation, completing the square and the formula for quadratic equations may often be more easily applied at this level, iterative methods may be more readily understood by the pupils.

As an example of a possible style of algorithm we may ask a
pupil to DESCRIBE the method used with the calculator when
testing if certain numbers (say 901, 1009) are prime. We might
expect something like the following:

1) Enter the number
2) Divide by - smallest prime
   - next odd number until this exceeds x
   (x may be n, n/2 or even n depending
    on the ability of the child)
3) Check for remainder (i.e. a decimal fraction in output)
4) If no remainder - Stop (not prime)
5) If remainder go back to 1

This style may be preferable to a flow chart for the majority
of pupils.

The emphasis will be on flexibility and clarity with real
discussion of pupils' suggestions, rather than the learning of
'standard' algorithms.

A better understanding of the concepts of square root and square
will naturally lead to a better understanding of Pythagoras;
an understanding of ratio and proportion leads to enlargement,
scale factors, lengths and areas of similar figures. Before,
circumference and area of circles can be understood, pupils
should have an appreciation of 'pi'. The Schools Council
(Calculators Count, 1983) provides a gradual approach by
considering the perimeter/diagonal relationship for a square,
then for a hexagon and finally the circumference/diameter
relationship for a circle. More ambitious pupils may like
to study the history of 'pi' and appreciate (but not necessarily
understand) the significance of the attempts made by such men
as Archimedes, Wallis, Vieta and Leibnitz to obtain approximations
for the elusive ratio. Turinose (Maths Teaching, No.89)
gives a brief outline of these. Rade and Kaufman (1980) too
give a fascinating account of the history of 'pi' as well as
other interesting and sometimes puzzling areas of real mathematics
via a pocket calculator.
Trigonometrical concepts, too, may be enhanced by the use of the calculator, and this section of work is now more accessible for pupils of moderate ability.

The ability to estimate an answer and to give an answer correct to a sensible number of significant figures is also highlighted in Chapter 2. Great importance is attached to these skills and the difficulties that pupils may have in understanding the concepts is not 'underestimated'.

However, although the use of the calculator in the classroom is becoming increasingly accepted, it is under this heading - 'Understanding of Concepts' - where least work has been done. There has been a steady trickle of articles and publications which have been designed to be easily usable in the classroom and related to the current mathematical curriculum. It is the need for these materials which is most pressing. To quote from Cockcroft:

"There is an urgent need for an increase in the limited amount of work which is at present being undertaken to develop classroom materials designed to develop understanding of fundamental principles". (392)
Mathematics is only 'useful' to the extent to which it can be applied to a particular situation and it is the ability to apply mathematics to a variety of situations to which we give the name 'problem-solving'.

In 1980, the first of N.C.T.M.'s Recommendations for School Mathematics of the 1980's states:

"Problem solving must be the focus of school mathematics in the 1980's".

This recommendation not only indicates the importance of problem-solving, it also implies that a concentrated effort is needed in order to establish problem-solving as an integral part of the mathematics curriculum. Further, today's research, is concentrating on processes or the set of steps students use to find a solution rather than focusing on the actual solution to the problem or the answer to the exercises. It has been found that correct solutions to problems involve setting up a plan, however brief, for the solution (Kantowski, 1977, 1980) and that different students approach the same problem in a variety of ways, indicating the existence of a style or preference.

To many classroom teachers and other educators, a problem is simply a word problem or an exercise stated in verbal form. An example might be:

"Clare bought a sandwich for 70p and a drink of squash for 30p. If the service charge is 10%, how much change would she receive from £2.00?"

Such problems are easily solved by most students by application of algorithms that are part of standard instruction.

To other educators a problem exists if a situation is non-routine, that is, if the student has no algorithm at hand that
will guarantee a solution. The student must put together the available knowledge in a new way to find a solution. The possibilities must be tabulated and some trial and error attempted. Moreover, more than one solution may be possible. An example might be:

"Clare has exactly £3.00 and would like to spend it all on her lunch. The menu includes hot dogs at 80p, hamburgers at 90p, onion rings at 60p, french fries at 50p and drinks at 30p, 40p, and 50p. The service charge is 10%. What would Clare have for lunch?"

A third type of problem can be called application or 'real problems'. A real problem involves a complex real-life situation that must somehow be resolved. Often there is not an exact solution, but one that is determined to be optimal to fit the conditions. The Spode Group have produced a compilation of such problems (Solving Real Problems, 1982) and include 'Easing the Traffic Flow', 'Car-Park Layout', 'Where to place a Telephone Box' as typical problems. Most of these problems include substantial computation.

What part then can the calculator play in this new emphasis on problem-solving in the curriculum? We must certainly be cautious about claims that it is a panacea. Even with the verbal question above, the pupil must still be able to select the correct buttons to push. As Cockcroft states:

"We wish to stress that the availability of a calculator in no way reduces the need for mathematical understanding on the part of the person who is using it" (378)

Suydan (1981) points to evidence from research in America to support the belief that problem-solving requires more than computational skills. Several reports conclude that the use of calculators does not affect problem-solving scores significantly.
However, there is also evidence to support the use of calculators in problem-solving in the same Suydan review. Other findings show that calculators are useful if the problems are within the range of students' paper-and-pencil computational ability; that students are less afraid to tackle difficult problems when using calculators; and that students use more varied problem-solving strategies when using calculators.

The calculator then has a useful, if somewhat limited, role to play. Polya (1973) considers four phases in the solution of a problem, one of which is the computational skill. If the calculator can considerably simplify this phase then the student may focus his attention on the other phases. Further, the student will have more confidence and less fear of the problem.

The calculator, too, may be useful in some of the thought processes in both non-routine and real problems. If we consider three stages of specialising (trying something out), conjecturing (guessing a pattern) and verifying (explaining the pattern) of a non-routine problem, various roles of the calculator may be linked to these stages. These roles may be considered as

1) providing the correct answer
2) enabling attention to remain on the problem
3) giving confidence to try specific cases.

Thus the calculator supports non-routine problems by allowing the swift numerical exploration of particular cases as a basis for 'guessing' and 'explaining'.

The calculator overcomes the main objection to using realistic problems in the syllabus. Knowing a calculator is at hand to cope with the arithmetic is enough of a confidence booster to get the pupils started. Often, if you can just get a feel for what happens when a couple of specific cases are tried, this can provide the impetus for more involvement in the problem. Interest
will be more likely to be maintained if the pupils know they will not get stuck on the arithmetic, and more time may be spent on interpreting the result. It is as a direct result of the use of calculators that statistics questions now tend to concentrate on interpretation of answers and less on calculations.

However, one note of caution should be heeded. Realistic numbers should not necessarily be introduced when a new concept is to be learned. It is still good teaching practice to use easy, manufactured and convenient numbers to illustrate a new concept.

"Problems involving 'messy' numbers rather than 'neat' numbers are not necessarily appropriate merely because of calculator availability. 'Messy' numbers may be as much a distraction with calculators as they are without them" (Kirst, 1980)

There is a further advantage that may be accrued from having the calculator available in the mathematics curriculum, although more indirect to problem solving. As Kantowski (1981) reflects:

"Too much emphasis on computational drill may be counter productive to the development of the flexibility needed for problem-solving. How can this demand for emphasis on basic skills be reconciled with the need for development of problem-solving ability in the limited time available for mathematics instruction".

Perhaps this time may now be more wisely spent. Kantowski sees the present decade as an exciting one with the curriculum promising to be more child-centred if the new technology (and he includes computers here) can be used to advantage in dealing with the diversity of problem-solving styles.

Problem-solving is a basic skill and we must teach problem-solving:

"Anyone who can execute standard routines but is incapable of solving a new problem has no skill" (Van Dormolen, 1976).
To encourage discovery, exploration and creativity by introducing appropriate investigations and mathematical ideas

"The idea of investigation is fundamental both to the study of mathematics itself and also to an understanding of the ways in which mathematics can be used to extend knowledge and to solve problems in very many fields". - Cockcroft (250)

A mathematical investigation is often thought of as an extensive piece of work or 'project' which will take quite a long time to complete and will probably be undertaken individually or as a member of a small group. However, investigations need be neither lengthy nor difficult. An investigation may be planned by the teacher or it may arise spontaneously from a pupil's question such as "what happens if ...?" The teacher must be prepared to pursue the particular question which may have arisen from a routine lesson, and the result may be a brief, interesting discussion or may lead to a full investigation. Chapter 2 includes such an enquiry from a pupil which leads to an interesting and even exciting series of investigations on recurring decimals.

The teacher should not be afraid to veer from the syllabus nor to curtail the investigation because of lack of time, but must be aware of when the investigation has served its purpose and the pupils have lost interest.

The calculator may be thought of as having two significant roles in investigations. On the one hand it may prove an invaluable aid to computation in investigations which would otherwise have been too time-consuming and too difficult to pursue to any worthwhile length. Just as the ideas involved become interesting the arithmetic becomes boring and difficult. Watson's 'Exploring Numbers: Some Investigations with a
Calculator (1979) contains activities concerning the usual number patterns, arrow games, factors, Pascal's Triangle, factors and questions are asked about rules, conjectures, tests and proofs. The activities are interesting and worthwhile, but the use of the calculator is only incidental and, indeed, the word does not appear in the text. Chapter 2 contains investigations on the Fibonacci Series and sequences and limits, where the calculator plays the same sort of useful but 'passive' role.

The other role is when the investigation centres on the calculator itself and its distinct features. The answer to the question:

"In how may different ways can you carry out this calculation on your calculator: which way requires the least number of steps?"

depends on the particular model of calculator which is used, and pupils who undertake an investigation of this kind will produce a variety of answers, all of which are equally valid.

Keane and Fenby (Things to do with a Calculator) ask the reader to investigate:

"What is the biggest / smallest number you can show on your calculator? What is the largest number you can square and still get a result? Show 0.0123456 on the display by pressing no more than 5 keys. Calculate powers of three on your calculator and powers of 9 without one"

There are some purely artificial activities which rely on the idiosyncrasies of the calculator. At a higher level, an interesting exploration by John Clarke (Sept 1978) involves reversing a random 4-digit number not ending in zero by combining it using any of the four operations - though not the same twice in succession - with any 2-digit numbers (the first digit being non-zero) made up from the digits of the number being reversed. Digits after a decimal point produced
by division are omitted before the next operation.

Example:  
1119 - 19 = 1100  
1100 x 91 = 100100  
100100 - 11 = 9100  
9100 + 11 = 9111

Dr. Clarke says the maximum number of steps necessary so far is 10, and there are some obvious variations.

David Fielker (M.T. number 89) begins some sessions with teachers by saying:

"I divided one whole number by another - each under 1000 - on my calculator and obtained 0.6786 389: I can't remember what the two numbers were; can you find out?"

After anything from 15 to 40 minutes there is usually a digressional discussion about what makes a good mathematical problem! But sooner or later it is realised that the real problem is to find an efficient algorithm rather than to find the two numbers.

Other calculator investigations of varying degrees of difficulty abound and Haylock (M.T. 101) is even able to investigate the mathematics of a dud calculator which had a defective zero key.

One other very significant value of the calculator is worth mentioning at this stage. The less-able pupil is now able to join in investigational work, to proceed at his own pace without the distraction of difficult arithmetic, and discover the pleasure of real mathematics. There still exists a minority of teachers who believe that the calculator should be excluded from this type of pupil, but this is surely a misguided view, for it is the less-able pupil who must have more to gain from the availability of the calculator.
Mathematical puzzles and games of various sorts also offer valuable opportunities for investigational work for these and other pupils, and their value will be discussed later.

It is perhaps appropriate to conclude this section with the thoughts of Brian Davies, I.L.E.A. (M.T. number 93) after two months of calculator activities with juniors:

"How exciting and interesting it has all been. How it has converted the teacher to an investigational approach. What a liberating force the calculators have been, enabling the children to tackle calculations that would have been tedious, time-consuming and error-prone without them. How important is the teacher's attitude, giving the children time, taking their ideas seriously, taking time for discussion, and being interested in the process as well as the answer".
To Introduce 'real world' Data in Problems

"How fast do you think James Hunt was going when he completed a lap at Silverstone in the British Grand Prix in 1 minute and 18.81 seconds?", the teacher asked the class of twelve year old children.

The overhead projector displayed the following information:

<table>
<thead>
<tr>
<th>One Lap</th>
<th>Record Time</th>
<th>Driver</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silverstone</td>
<td>2.932 miles</td>
<td>1 min 18.81 secs</td>
<td>James Hunt</td>
</tr>
<tr>
<td>Brands Harch</td>
<td>2.614 miles</td>
<td>1 min 18.60 secs</td>
<td>Niki Lauda</td>
</tr>
</tbody>
</table>

First pupil: "Are those real numbers?"
Teacher : "Yes, they are the current records"
First pupil: "I'll bet it isn't as fast as Niki Lauda"
Second pupil: "You're wrong! It's faster"
Teacher : "Let's use our calculator and figure the speed for both records"

Hutton (1980) quotes this lively discussion as typical of many during a course involving the use of calculators.

Bell et al (1978) argue that calculators, by allowing numbers of realistic size and actual measurements and observations to be handled, fulfil an important role in emphasising the usefulness of mathematics. Too often we restrict children's experience of the mathematical environment through the choice of convenient examples and problems. The children show more interest and commitment to problems which they know contain real data.

Johnson (1978) uses applications of everyday living to illustrate
that the calculator can help us to become 'wise consumers' e.g. comparative shopping; personal consumption of water, gas, electricity. The calculator, too, helps us to make decisions which have implications for benefiting society as a whole e.g. population growth; conservation of resources; health and health care; inflation; and so on.

Chapter 2 illustrates the use of realistic mortgages and compound interest calculations, while statistics must be one of the areas of mathematics which benefits most from the availability of the calculator. Actual data from surveys may now be used; the teacher will no longer have to use a convenient 30 when taking readings from his class of 29; realistic pie-charts may now be drawn, too.

Green (1981) invites his readers to examine and investigate the incredible distances between the bodies in the solar system and the sizes of these bodies, while the Spode Group (1982) deal with actual attendances at soccer matches and transfer fees.

However, as usual, some care must be taken. Reys (1980) warns that merely displaying very large numbers on the calculator may be falsely interpreted as understanding. Research has shown, he says, that understanding the formulation of numbers is a delicate cognitive process developed over a long period of time, and the calculator may not bypass the development of this concept.

It is with the use of very large, very small and realistic numbers, generally, that the understanding of estimation and accuracy really become of major importance.
To Motivate Pupils

During every mathematics lesson a child is not only learning, or failing to learn mathematics as a result of the work he is doing, but is also developing his attitude towards the subject. Once attitudes are formed they can be very persistent and difficult to change. Positive attitudes assist the learning of mathematics; negative attitudes not only inhibit learning but very often persist into adult life and affect choice of job. The challenge for the teacher is to present mathematics in a way which continues to be interesting and enjoyable and so allows understanding to develop. The teacher must nurture the young child’s natural uninhibited enthusiasm and curiosity in the primary school and, hopefully, this can be maintained in the secondary school. Any aid to this difficult task is very welcome. There can be little doubt of the motivating effect which calculators have for very many children. As Shumway (1981, pp. 169) emphasises:

"One of the most powerful and consistently reported effects of student use of calculators is the high enthusiasm and valuing students have for calculator aided mathematics activities. Any device which causes so much pleasure to be associated with mathematics ...... deserves special note".

Shumway also believes that parental openness to calculator use can be influenced significantly by the enthusiastic response children exhibit to calculator aided mathematics.

Hutton (1980) too, reports that teachers in her study group believed the calculators not only improved attitudes towards the mathematics in general, but specifically improved interest in problem-solving.

The calculator engenders confidence in the child by enabling him to experience success. This allays the fear and anxiety
that Cockcroft emphasises, and especially in those less-able children who usually experience failure.

The calculator is an alternative source of the correct answer and is 'neutral'. As one young pupil said:

"I like the calculator because it is easy to work with. It is easy and it does not get angry with you when you get it wrong." (Open University, 1982)

The child's inquisitive nature is stimulated as soon as he encounters a calculator. However, Johnson (1978) warns that although there is no harm in pressing buttons to see what happens just for fun, there is concern if the child is asked for responses or procedures for which he lacks the necessary background for understanding.

There seems to be some disagreement in the educational world about using the calculator for checking answers. On the one hand some, and Johnson is one, believe that the calculator is the best device for doing the calculation in the first place, while others such as Cockcroft believe that the child can gain confidence by checking 'mental' answers or written answers.

The calculator, then, can motivate by not only increasing the child's chances of success but it can also allow the child to experience a much more exciting mathematical world. Amongst the most rewarding sources of calculator based activities are calculator games and puzzles.

Games need no justification as an aid to teaching as long as they have a mathematical objective. Children play games with great concentration and involvement and most enjoy the competitive nature of them. Games should be fun to play and should
have a logical reasoning component, but they should also illustrate some mathematical idea or practice some useful skill. The teacher should plan the type of game he uses, where he introduces it into the curriculum and the reason for doing so. It is too often used as an end of term treat or as a casual addition to a lesson to relieve boredom without much forethought. For maximum benefit the child should be encouraged to keep records. The teacher should be able to exploit the flexibility of the activity, for example by adjusting the difficulty of the game. Further, the teacher must plan the teaching sequence and decide how the game can be located within the usual teaching of mathematical concepts. A motivating and effective strategy is to start with a game and then teach a relevant topic and finally let the children return to the game.

The appendices of this project include games that have been tried by the second year pupils of Collenswood School, Stevenage. The games were from various sources such as F.R. Watson (1982) and those drawn from the S.M.P. 11 - 16 syllabus. Those included were considered by the pupils and staff to be the most interesting, useful and appropriate of all the ones attempted. Only a sample has been included.
The Calculator as a Service Function

There is one important function that the calculator offers that is often neglected when the use of calculators is discussed. As Kaner (1980) remarks:

"It is in the service aspects of mathematics that the use of the calculator is such an enormous blessing. The barrier of poor arithmetic skills, that has kept so many children from effective study in the sciences, geography, home economics etc., has now been swept away. All that is needed is a small supply of calculators in the laboratory to be used whenever a calculation is to be carried out".

The science teacher should no longer be able to blame the mathematics department for children's lack of success in his subject. Hart (1981) highlights many of the problems that exist between the two departments but remarks that:

"Calculators may serve a useful purpose in that children may be able to concentrate on the essence of the problem rather than the calculations involved".

However, it is the duty of the mathematics teacher to do two things. First he must make sure that his science colleagues are fully aware of the help that a calculator can be to ordinary children in learning science. Second, he must make sure that all his pupils are fully competent with the electronic calculator.

It is important that, not only mathematics teachers, but other teachers too, take advantage of in-service courses. Research must be initiated into how the calculators can be used effectively in other subjects. One such piece of recent research is:

'An Experiment in Using Calculators in Teaching Home Economics' (Parry and Rothery 1983)
The Calculator as an Aid to Computation

It is well to note, obvious as it is, that calculators are the quickest, most accurate computational algorithm available to children today. In fact, the primary function of a calculator is to compute, and in the hands of children, the calculator serves the computational function better than any other technique or device in existence. They make logarithmic tables and slide-rules obsolete and have caused us to think hard about what written computational skills the children now need and the effect the calculator has had on these and their mental skills. (discussed at some length in Chapter 1)

Calculators are practical, convenient, and efficient. They remove the drudgery and save time on tedious calculations. They are less frustrating, especially for low achievers. They encourage speed and accuracy.

The point has been made in this chapter and others that the ability to compute can be very beneficial to various aspects of the mathematics curriculum such as problem-solving, inclusion of real data, motivation and so on. The calculator, therefore, narrows the gap between children's abilities and widens the scope of mathematics teaching. It widens the educational horizons of many pupils who have, until now, been held back by poor arithmetic skills. Further, Blakely (1980) remarks that the calculator puts numbers back into mathematics and also provides a portable number laboratory, which can be taken home and used there by the children.

However, there still exists some contentious issues surrounding the use of the calculator, and these are mainly concerned with not if they should be used but how and when.
OUTSTANDING ISSUES IN THE EFFECTIVE USE OF THE CALCULATOR

So far this chapter has been concerned with advantages attributed to the use of the calculator in the secondary school curriculum. To restore the balance of the argument it would seem fair to consider any disadvantages that may arise. There are, perhaps, two objections that have constantly been put forward to limit their use, and even these seem to have been discredited over the years.

Firstly, it has been argued that calculators encourage mental laziness and that they are detrimental to the mastery of basic facts and algorithms. It was this fear that prompted many calculator investigations into the possible effects on computational skills. However, reviews and summaries of these investigations, especially by Suydam (1977), Suydam (1978) and again Suydam (1979) conclude that the calculator has no adverse effect on these skills. A review by Roberts (1980) and year long study by Moser (1979) agree with this conclusion. Further, Shumway (1981) observed that:

"... children ... did not develop any of the feared debilitations when tested without calculators because of calculator use for instruction".

Seldom is research literature so clear that there is no cause for concern or alarm about this particular calculator effect.

Secondly, it is argued that they are not available to all students and so leave some at a disadvantage. It is evident that most students now have access to a calculator either at home or school and that more and more schools are able to offer their pupils the use of one. For example, a survey (Saunders 1980) of 100 'random' occupations reports that fully 98% of those persons involved used a calculator.

Hopefully, therefore, we are able to avoid creating another
false dichotomy, as in new mathematics versus old mathematics, discovery versus expository lessons. It is not appropriate to consider calculator programmes versus non-calculator programmes. The real question is not should calculators be used but when, where and how they can be used most effectively.

Fielker (1979) believes that the cartoon at the beginning of this chapter is only one step removed from reality because, he says, judging by some of the literature that is being produced, it is sometimes a struggle to find the right sort of mathematical questions to ask when calculators are readily available to children. He believes that there are too many books and articles which feature the word 'calculator' in the title and yet use of the calculator is only incidental. More research should be put into activities that use the power of the calculator more directly.

The recent publication 'Calculators Count' (1983) demonstrates various topics that can be taught more effectively with the aid of the calculator. Perhaps, as important as the ideas themselves, is the authors' suggestions where these topics may be usefully placed into existing mathematics courses. The appendices include their suggestions for both the S.M.P. lettered books and the S.M.G. modern mathematics for schools courses.

S.M.P. have produced their own calculator series of booklets to complement their existing 'O' level course and the new S.M.P. 11 - 16 course has been devised with the assumption of calculator availability in mind.

Fielker, too, is one of the many educationalists outside the school classroom who believe that the calculator should be available to all children at all times (assuming correct use).
To those teachers who would advocate the learning of some arithmetic before the calculator is used he would argue that this is rather like not allowing anyone to pass their driving test until they can run at 40 m.p.h.

"Children were born with brains, and they should use them",
one teacher had said to him at a course. He replied that she was born with legs, and had she walked the twenty miles to the course? However, it might well have been pointed out that walking is a beneficial exercise, and that many people find it enjoyable and a challenge.

Perhaps a more appropriate analogy would concern the advent of sewing machines. It was feared that many of the traditional hand-sewing skills would disappear, and it is true they have, but who could deny that the use of a sewing machine requires new skills and, some would agree, better and more versatile results.

The classroom teacher may well consider Fielker's view as rather extreme. The ability to carry out simple calculations by all children and more difficult ones for the more able is still valued by parents, employers and the pupils themselves. To them it is a matter of pride, and, to some, enjoyment.

It must be useful, at times, therefore to prohibit the use of the calculator in order to test these skills. Mental arithmetic is regarded as even more important in the calculator age. If asked

"What is 8.16 x 25?",
most children, if they have access to a calculator, will use it and quickly get the right answer, but it is very easy to answer this question without one.

The ability to find short cuts and tricks (an interesting
and worthwhile activity in its own right) may well disappear if some limitation, however slight, is not put on calculator use. Also, well-worn arguments such as the calculator breaking down or forgetting your calculator, do have a ring of truth.

It is unlikely that any external examination in mathematics now forbids the use of calculators on all of its papers. Cockcroft recommends that:

"Examination Boards should design their syllabus and examinations on the assumption that all candidates will have access to a calculator by 1985". (395)

However, despite strong views that calculators should be allowed at all times, most examinations like to include a section which forbids its use. This usually involves questions of a simple, computational nature.

Using calculators to check paper and pencil calculations is often cited as an important use for pupils. In 1979 P.R.I.S.M. reported that 93% of professional people sampled by them gave 'checking answers' as the best use of calculators. It is not surprising that this use is cited, but it is surprising that it was quoted as the most important use by so many. Reys (1979), like many other educators, believes there is little point in checking computations done without a calculator when the calculator could have been used in the first place.

It soon becomes obvious that the mere possession of a calculator by a pupil is not enough. There is still a good deal of convincing to do that the calculator offers more than the ability to ease the burden of computation. Great claims have been made as to the effect the calculator will have on the actual content of the mathematics curriculum and some of these claims will be examined briefly in the next section.
EFFECT OF THE CALCULATOR ON MATHEMATICS CONTENT

In considering the impact of the calculator on the mathematics syllabus at secondary level, we may ask three questions:

"What comes in?"
"What goes out?"
"What do we do about what is left?"

Pollak (1977) writes of two partial orderings of the curriculum, one supplied by the discipline of mathematics (the partial ordering of prerequisites) and the other by society (the partial ordering of importance). He argues that both will be affected in a major way by the calculator. Certainly both will be borne in mind when considering the needs of our pupils when they are 10 or 20 years into their adult life.

The following suggestions do not purport to be a comprehensive or definitive list, and they will certainly cause some disagreement. However, it is clear that due consideration must be given to the syllabus now. Many factors, of course, have to be considered when devising a syllabus for the future; the calculator is only one of them.

What goes out?

There seems to be general agreement that the calculator will make logarithm tables and the slide rule redundant as calculating aids. Trigonometrical tables, too, will soon be superseded by the scientific calculators. The exclusion of the long division algorithm for many pupils and more difficult fractions may cause a little more controversy but, hopefully, not too much.

Other minor areas, such as the use of coding for calculation of the mean and standard deviation, and other less obvious areas
will need careful consideration. In the latter category, Blakeley (1980) looked at the S.M.P. 'O' level course which, as he comments, was devised before calculators were taken into account, and he was able to eliminate 22 chapters out of 124 by adding to the list:

linear programming,
latitude and longitude,
and the matrices/transformations link.

He does not reason strongly for these and suggests other topics could be argued against by other people. His 'hatchet job' is really to create room in an already overcrowded syllabus.

For a start, a conservative list might be:

LOGARITHMIC TABLES
TRIGONOMETRICAL TABLES
SLIDE RULES
FRACTIONS IN DEPTH
DIVISION ALGORITHMS FOR MOST PUPILS

What comes in?

A long list of additions to the syllabus might appear rather daunting to teachers, but few of these are entirely new. Some of them, such as patterns, estimation and accuracy have been on the fringe for some time while others, such as iterative methods, may replace more traditional techniques. Further, it must be assumed that more time will be available with less emphasis on the standard pencil-and-paper algorithms.

MATHEMATICAL MODELLING involves translating a real situation into a mathematical equation which can be used to explain
known phenomena and even predict. Excessive calculations have, in the past, hampered the inclusion of this topic in the syllabus.

ESTIMATION OF ANSWERS AND ACCURACY take on a new importance in the calculator/computer age, and more emphasis can be placed on PROBLEM-SOLVING. The concept of ALGORITHM and the use of ITERATIVE PROCEDURES, too, are basic in today's computer age. Children need to be given the opportunity to design and modify new algorithms, and iterative methods may be given as an alternative or even replace such traditional techniques as solving quadratic equations by factorising or by the formula. Certainly, more interesting and higher order equations could be investigated, as well as more interesting graphs.

Studying more practical applications of geometry and trigonometry, such as TECHNICAL DRAWING AND SURVEYING, has been reported by Wilson (1977) to be much more successful for pupils of moderate ability, provided calculators were used.

GROWTH AND DECAY RATES should gain greater prominence on the new syllabus. Compound Interest is a vital topic to us all ranging from personal overdrafts and hire purchase and mortgages to major economic problems of the National Debt.

The exploration of PATTERN has already become a popular topic in new books and courses related to the calculator. However, if it is to be worthwhile, it must be used to promote mathematical thinking. While the main purpose of the calculator is to generate the pattern, the main purpose of the activity should be the search for the pattern, with some possible explanation of the pattern, a generalisation and then the calculator may be used again to verify the result. Johnson (1978) examines patterns in more detail in his article 'Calculators: Use and Abuses'.

Most of the books on calculators have given prominence to
GAMES AND PUZZLES, APPLICATIONS of all types and to INVESTIGATIONS, and the calculator allows much greater scope for these. Arguments for the inclusion of SEQUENCES, SERIES AND LIMITS, would be strong.

We must not forget what might be thought of as the two most important additions to the syllabus: the PROPER USE OF THE CALCULATOR and MENTAL TECHNIQUES as suggested by Plunkett to replace the standard written algorithms.

The list might look like this:
- MATHEMATICAL MODELLING
- ESTIMATION OF ANSWERS
- ACCURACY AND ERRORS
- ALGORITHMS
- ITERATIVE METHODS
- POLYNOMIALS
- PROBLEM-SOLVING TECHNIQUES
- GROWTH AND DECAY APPLICATIONS
- PRACTICAL GEOMETRIC AND TRIGONOMETRIC APPLICATIONS (Surveying and technical drawing)
- PATTERNS
- APPLICATION TO THE 'REAL WORLD' PROBLEMS WITH REAL DATA
- INVESTIGATIONS
- GAMES AND PUZZLES
- SEQUENCES, SERIES AND LIMITS
- PROPER USE OF THE CALCULATOR
- NEW MENTAL TECHNIQUES TO REPLACE STANDARD WRITTEN ALGORITHMS

No doubt, a strong case could be made for several other topics and it is inevitable that the 'inclusion' list will far exceed the 'exclusion' list.
What do we do about what is left?

It is up to all educators to examine each topic in the syllabus and decide if and how the calculator can make a real contribution to the UNDERSTANDING and LEARNING of that topic. It is up to every teacher to exploit its very special properties. However, the application of calculators should not be forced if better aids, and methods exist.

Suggestions for ways in which the calculator can aid the understanding of some topics, have already been described. Many other topics can also benefit, including the promotion of all pupils 'numeracy' skills.

The ORDER in which topics are introduced into the syllabus will also need rethinking although other factors, such as the work of PIAGET and C.S.M.S., will have to be taken into consideration. For example, decimals may be introduced before fractions, negative numbers may be taught earlier.
CONCLUSION

If we have any doubts about the tremendous interest that the advent of the calculator has had in educational research, then we should heed the words of Suydam (1979, pp. 3) in the second of her state-of-the-art reviews she asserted that:

"Almost 100 studies on the effect of calculator use have been conducted during the past four or five years. This is more investigation than almost any other topic or tool or technique in mathematics instruction during this century".

Reports of calculator use in school settings have continued to be released, more or less unabated, ever since.

However, research is not an end in itself. We should do something as a result of research. The teacher should incorporate the conclusion of research (after taking into account his/her own unique situation and circumstances) into his/her daily teaching. We may not be inclined to take the dramatic action some educators strongly recommend by allowing the use of the calculator at all times. When Shumway (1981) thinks of his own children, for example, he wants to say:

"Don't waste their time doing trivia! Give them a calculator and get on to teaching them the mathematics they cannot now do".

The classroom teacher may have to take a more conservative line but recent support for the use of the calculator by Cockcroft has given the impetus to take action. The Head of the Mathematics Department has an important role to play here:

1) He has to decide where effective calculator based activities can be fitted into his existing syllabus, and alter
aspects of the curriculum with the calculator in mind.

2) He should rethink his philosophy about the standard pencil-and-paper algorithms. Whatever his decision, he should liaise closely with the primary school and employers. He should convince parents and colleagues, too, that the calculator has no detrimental effects on the pupils own skills.

3) He should initiate closer co-operation between mathematics and other departments by emphasising the service aspect of calculators. As Green (1980) reports:

"..... there is a need for closer co-operation. Maybe the calculator can be the catalyst".

The L.E.A's must provide the finance and in-service training, and the teachers and other educators must carry on with the research, especially for material where the calculator can be used as a teaching and learning aid.

There is an increasing danger, however, that the calculator will be overshadowed by its technological 'big-brother' - the microcomputer. In his article 'Remember the Calculator', David E. Williams (1983) reports on a conversation between two mathematics supervisors:

"I'm putting all my money into microcomputers this year ......."

"Have you ordered any calculators lately?"

"Calculators are dead! Nobody cares about calculators any more. Everyone wants to order microcomputers ......."

"But even with micros being purchased, calculators are still valuable tools and aids for students and teachers to use".

"Don't waste your time and money on calculators. Microcomputers are going to revolutionise mathematics education. Get on with it - you're really behind the times".
Mathematics educators are rushing to purchase computer hardware, and yet it is curious that many who questioned the educational value of the role of the calculator in the mathematics education of primary school children have no reservations about putting microcomputers in the primary classroom. (The reader might enjoy 'The Discovery of the Mathematics Textbook', an amusing, tongue-in-cheek article by Michael Cornelius, 1982). We must, of course, continue to develop ways to use both calculators and computers in our classrooms.

Others may see the danger coming from another source. On a more frivolous note Watson (1980) draws a picture of what may be in store for us. Some have concluded that, just as with the invention of the motor car, mankind is losing the ability to walk, so we shall lose all computational facility in the calculator age.

You will notice that the pupil has no legs (car travel) and that his television aerials are 'built in'. The pointed index finger is for pressing the tiny buttons of his wrist-watch calculator, and the bifocals are for reading the tiny displays.

However, Watson does not share this pessimistic view.

Weaver, (1980), too, is confident that we can embark on school mathematics programmes with "freedom from fear" - freedom from fear that calculator use will have harmful or debilitating effects on students' mathematical achievement. This can no longer be
used as a reason, nor excuse, for not welcoming and including calculators among the instructional aids and materials that have potential contributions to make in connection with school mathematics programmes.

He goes on to conclude:

"Now, as we enter the 1980's, we are in a position to reformulate school mathematics programmes in a manner that will free them from the shackles of the attainment of computational skills with pencil-and-paper algorithms as the basis upon which instruction is initiated, organised and sequenced at the pre-secondary level; and will have analogous reorganisational implications for programmes at the secondary level. THE CALCULATOR IS THE KEY. Now is the time to turn that key in all earnestness".
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Possible fit of some of the project materials to this series of textbooks.

<table>
<thead>
<tr>
<th>BOOK 1</th>
<th>PROJECT MATERIAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8 Understanding the (x^2) key</td>
</tr>
<tr>
<td></td>
<td>15 Odd numbers and their sum</td>
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</tbody>
</table>

| BOOK 2 ARITHMETIC - Chapter 1 | 2 Multiplication of decimals |
|                               | 3 Division by a decimal |
|                               | 6 Fractions and decimals |
|                               | 20 Counting and estimation |

| BOOK 2 ALGEBRA - Chapter 4 | 4 Introduction to the distributive law |

| BOOK 2 GEOMETRY - Chapter 2 | 5 Areas of rectangles and decimal multiplication |

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| BOOK 5 TRIGONOMETRY - Chapter 1 | 33 Introduction to trigonometric ratios |

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|                            | 38 Box making |

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| BOOK 7 ARITHMETIC - Chapter 1 | 27 Errors and accuracy: areas |
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| BOOK 8 CALCULUS - Chapter 1 | |
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## APPENDIX 2

### S.M.P. Lettered Books

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<th>BOOK/CHAPTER</th>
<th>PROJECT MATERIAL</th>
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<tbody>
<tr>
<td>After A1</td>
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</tr>
<tr>
<td>After A8</td>
<td>16 Whole numbers and their sum</td>
</tr>
<tr>
<td>B5</td>
<td>17 Triangle numbers and their sum</td>
</tr>
<tr>
<td>After B3 (or as late as X5)</td>
<td>6 Fractions and decimals</td>
</tr>
<tr>
<td>B9 or C11</td>
<td>23 Errors and accuracy: length</td>
</tr>
<tr>
<td>C4</td>
<td>24 Errors and accuracy: weight</td>
</tr>
<tr>
<td>With C4</td>
<td>25 Errors and accuracy: time</td>
</tr>
<tr>
<td>After C4</td>
<td>26 Errors and accuracy: angles</td>
</tr>
<tr>
<td>After C4 (or X5)</td>
<td>22 Letter frequency</td>
</tr>
<tr>
<td>C11</td>
<td>2 Multiplication of decimals</td>
</tr>
<tr>
<td>D2</td>
<td>3 Division by a decimal</td>
</tr>
<tr>
<td>With D2 (or X5)</td>
<td>5 Areas of rectangles and decimal multiplication</td>
</tr>
<tr>
<td>D5 or earlier</td>
<td>39 Electricity meters and electricity bills</td>
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<tr>
<td>D5</td>
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<tr>
<td>D8</td>
<td>40 Conversion between metric and imperial units</td>
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<td>D10</td>
<td>27 Errors and accuracy: area</td>
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<tr>
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<td>21 Word length</td>
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<tr>
<td>E9</td>
<td>35 Lengths of similar figures</td>
</tr>
<tr>
<td>After E9</td>
<td>28 Errors and accuracy: comparing lengths</td>
</tr>
<tr>
<td>E10</td>
<td>1 Number messages</td>
</tr>
<tr>
<td>E11</td>
<td>4 Introduction to the distributive law</td>
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<td>8 Understanding the ((x^2)) key</td>
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<td>30 Introduction to Pythagoras' rule</td>
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<tr>
<td></td>
<td>9 Introduction to square roots</td>
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<td></td>
<td>10 Square roots by decimal search</td>
</tr>
<tr>
<td></td>
<td>37 Gutter design</td>
</tr>
<tr>
<td></td>
<td>38 Box making</td>
</tr>
<tr>
<td></td>
<td>11 Introduction to cube roots</td>
</tr>
<tr>
<td></td>
<td>12 Cube roots by decimal search</td>
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<td>33 Introduction to trigonometric ratios</td>
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<table>
<thead>
<tr>
<th>BOOK/CHAPTER</th>
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<tbody>
<tr>
<td>F8</td>
<td>36 Areas of similar figures</td>
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<tr>
<td>F12 or earlier</td>
<td>13 An interactive method for finding square roots</td>
</tr>
<tr>
<td>F12</td>
<td>14 An iterative method using a square root key</td>
</tr>
<tr>
<td>G9</td>
<td>41 How money grows</td>
</tr>
<tr>
<td>After G9</td>
<td>42 Growth factors and decay factors</td>
</tr>
<tr>
<td>X5</td>
<td>43 Mathematics and the guitar</td>
</tr>
<tr>
<td>Y1</td>
<td>29 Errors and accuracy : speed</td>
</tr>
<tr>
<td>After Y</td>
<td>19 Introduction to rates of change for non-linear relationships</td>
</tr>
<tr>
<td></td>
<td>34 The sine rule</td>
</tr>
</tbody>
</table>
APPENDIX 3

Calculator Games

(1) For two players and one calculator.

Player A enters any number.

Player B multiplies by any number he or she chooses to try to make the display show
1.0******
where the asterisks may be any digits. If B succeeds he or she wins.

Player A multiplies the number now in the display by any number he or she chooses to try to make the display show
1.0******
They continue until either A or B succeeds.

For a harder game try 1.00*****
or 1.000****

PRACTICE: Estimation and showing that multiplication does not necessarily increase a number.

(2) For any number of players, 1 calculator per player, pencil and paper.

Choose a two-digit number as the target.

Now choose four digits as the ammunition.

Try to hit the target using only the specified ammunition and the operations +, -, x, -.

Score 10 for a direct hit
7 for an "inner" (within 2 of target)
3 for an "outer" (within 5 of target)
Example - Target 58 Ammunition 4, 9, 3, 7.

Player A: (4x9) + (3x7) = 57
B: (9+3) x 4 + 7 = 55
C: 7x(4+3) + 9 = 58

PRACTICE: Ordering of Operations

(3) Twenty Questions

A thinks of a whole number < 10000
B tries to guess it in 20 questions
Both players have calculators; answers to questions can only be YES/NO.

Questions which involve the words "less than", "more than" count double!
(Easier versions for younger pupils:
Number must be less than 1000 (100)
Remove the restriction on "less than"/"more than")

PRACTICE: Properties of Numbers

(4) Calculator Snooker

Player A enters any two-digit number, B takes a 'shot' by performing a multiplication sum. To 'pot' a ball, the first digit of the answer must be correct according to the table shown. (The degree of accuracy can be varied according to experience.)

<table>
<thead>
<tr>
<th>BALL</th>
<th>Red</th>
<th>Yellow</th>
<th>Green</th>
<th>Brown</th>
<th>Blue</th>
<th>Pink</th>
<th>Black</th>
</tr>
</thead>
<tbody>
<tr>
<td>RESULT NEEDED</td>
<td>1..</td>
<td>2..</td>
<td>3..</td>
<td>4..</td>
<td>5..</td>
<td>6..</td>
<td>7..</td>
</tr>
<tr>
<td>SCORE</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>
Otherwise the rules are similar to 'real' snooker. There are 10 (or 15) reds and one of each of the six 'colours'. A player must score in the order red, colour, red, colour, and so on, until all the reds have gone. (Note that the colours are replaced but the reds are not.) When the last red has gone, the colours are potted 'in order' and are not replaced.

For example, one sequence of plays was

<table>
<thead>
<tr>
<th>Player</th>
<th>Enters</th>
<th>Display</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jimmy</td>
<td>69</td>
<td>69</td>
<td></td>
</tr>
<tr>
<td>Peter</td>
<td>x 2 = 138</td>
<td>690</td>
<td>Peter pots the first red, he elects to go for blue ... and misses.</td>
</tr>
<tr>
<td>Karen</td>
<td>x 2 = 138</td>
<td>7590</td>
<td>Karen pots the second red, she elects to go for black ... and pots it.</td>
</tr>
</tbody>
</table>

PRACTICE: Estimation skills

(5) Target Practice (2 players)

The players agree to aim for a 'target area' which is a range of decimal numbers (say 0.2 to 0.5 inclusive). Each player in turn must then choose two whole numbers which are then divided on the calculator. If the result falls into the target area, then they score a hit.

Sample Play

<table>
<thead>
<tr>
<th>Player</th>
<th>Numbers chosen</th>
<th>Key Sequence</th>
<th>Answer</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8, 4</td>
<td>8 ÷ 4 =</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>4, 8</td>
<td>4 ÷ 8 =</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>A</td>
<td>1, 4</td>
<td>1 ÷ 4 =</td>
<td>0.25</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>3, 5</td>
<td>3 ÷ 5 =</td>
<td>0.6</td>
<td>0</td>
</tr>
</tbody>
</table>

etc
Note
Keeping a record in a table is slow and rather unexciting. Much better is to record the play on graph paper, thus:

![Graph Paper Diagram]

PRACTICE: Decimals and Estimation

(6) Percentage Maze

Work through this maze, starting with 100 on your calculator. Find which route gets to 46.1916 (the finish) Choose from:
ABE, ADE, ADG, CDE, CDG, CFG.

PRACTICE: Percentage
You need a calculator.

Write down the digits 1, 2, 3, ..., 9.

This is the check list.

Enter 1 into the calculator.

Players take it in turns to subtract a number from the display.

The number you subtract can have 2 digits at most.

You can use 0 as many times as you like in the numbers you subtract.

But each number in the check list can only be used once in the game.

Cross off the digits in the check list as you use them.

The winner is the first person to get 0.01 in the display.

The game is a draw if no-one can get 0.01.

If you get a negative number in the display, you lose straight away.

PRACTICE: Place Value and Decimals

In this game you only use the numbers from 1 to 9.

Start with 31 on the display.

The first player picks any number from 1 to 9.

They subtract this off the display.

The second player picks a number.

Their number must be next to the key the first player used.

The second player subtracts their number.

Then the first player picks a number next to the second player's, and so on.

The first player to get a negative answer loses.

PRACTICE: Mainly a game of strategy
(9) **Target 100**

Similar to Game Number 1 but the target is 100.****
by multiplication.

For harder games the target becomes 100.0***
or 100.00** etc.

Now try playing "target" when you are only allowed to press
the \[ \div \] button!

PRACTICE : Estimation

(10) **4-in-a-row**

This is a calculator game for two players.
You need one calculator, some graph paper and a pencil.
Copy the number line below onto your graph paper. (Let
2 cm represent 1 unit)

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

Player 1 chooses two numbers from the table below, and an
operation (either \( x \) or \(+\))

<table>
<thead>
<tr>
<th></th>
<th>0.4</th>
<th>0.07</th>
</tr>
</thead>
<tbody>
<tr>
<td>103</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>1.5</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>0.5</td>
<td>8.3</td>
</tr>
</tbody>
</table>

Player 1 writes down this sum and works out the answer
using a calculator (for example: \( 25 \times 0.3 = 7.5 \))

He marks his answer on the number line with a cross (\( x \))

\[
\begin{array}{cccc}
6 & 7 & 8 & 9 \\
\end{array}
\]

Player 2 then chooses two more numbers and an operation
\[ \times \] or \[ \div \]
He marks his answer with a blob (•)

If the point cannot be marked, (off the edge or already taken), then the player misses a go.

The first player to get 4 of his points in a consecutive row wins!

i.e. •••••••• is a win for player 2.

PRACTICE: Use of scale and estimation.

(11) Home

2 Players (more-able children)

One calculator with $\sqrt{x}$, $x^2$ keys

1. A sets any number in a calculator display - 8 or 9 is a reasonable choice.

2. B adds any number Z, presses $\sqrt{x}$ and then subtracts Z, presses $\sqrt{x}$, and then $\sqrt{x}$

3. A does the same as B, (choosing a different no. Z), and so on.

(The display is NOT cleared at each stage, so each player carries on where the other leaves off).

First to get 10.0**** in the display is the winner.

NOTE
i) The number Z can be NEGATIVE (i.e. subtract 1st and then add)

ii) **** can be any digits, but the display must start with 10.0 ...

Harder version "Home" is 10.00**

"You are the commander of a space craft, which is malfunctioning. In order to ensure safe return to Earth (before the motor goes critical) you have to set the quasor-warp-force display at 10.0 ...

PRACTICE: Behaviour of functions $\sqrt{x}$ and $x^2$