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AN ATTEMPT TO REPRESENT GEOMETRICALLY
THE IMAGINARY OF ALGEBRA

by

Ruth K Tobias

A doctoral thesis submitted in partial fulfilment of
the requirements for the award of Doctor of Philosophy
of the Loughborough University of Technology, 1987

Supervisor:  Professor A C Bajpai
Director of CAMET and Head of
Department of Engineering Mathematics
Loughborough University of Technology

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ABSTRACT

An Attempt to Represent Geometrically
the Imaginary of Algebra

by Ruth K Tobias
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In 1981* the author submitted that "many of the (then) more recent school syllabuses remain disjointed and give expression still to a school mathematics course as step-by-step progression through a list of disparate topics".

The position has not changed. It is not yet generally accepted that there can no longer be an accepted body of mathematical knowledge that needs to be taught. The rapid development of new technology and the introduction of the microcomputer should enable the 'modern' mathematics of the early 1960's to enhance the mathematical experiences of pupils in a practical and comprehensible way and prompt a new style of teaching and learning mathematics.

There is, however, a fundamental core of mathematics which must inevitably find a place in the school mathematics curriculum. In Part I of the thesis the emphasis is on a method of presentation of certain key topics which illustrate the basic pattern of a group structure.

Former complications at school level of putting plane geometry on a logical footing have to be avoided. The use of complex numbers highlights significant and sometimes rather difficult geometrical ideas. In Part II the author attempts to show how some of these ideas may be presented to extend the basic pattern to that of linear algebra.

The work culminates in Part III with the use of linear complex algebra to present more vividly the symmetries of the Platonic solids. The author anticipates the realistic presentation of the aesthetic side of 3-dimensional geometry and takes a look at its possible presentation through the medium of the microcomputer.

At this early stage of the development of the ideas to be discussed, there can be no formal testing of the results by quantitative analysis. Evaluation of the viability of the proposals will be qualitative and the comments of 'critical academic friends' will be included.

The originality demanded of a piece of research goes beyond the exposition. Here it will consist of new insights into ideas appropriate to senior pupils in schools and a rewriting of existing material often thought to be beyond their scope. The work is supported by suggested lesson sequences, transcripts of recorded presentations, and examples of students' work. Subsequent development must face the question of assessment and evaluation at sixth-form level of the proposed new style of teaching mathematics. The author makes some suggestions in the concluding chapter.

Key words
ACKNOWLEDGEMENTS

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To Tim for his enthusiastic appreciation of the use of the language of linear algebra as 'sums of multiples', for his permission to use examples from Ampleforth Mathematics which arose directly out of this thesis, and together with Bruce for reading the typescript.
To the typist for somehow succeeding in translating the modern mathematical symbolism of the manuscript into golf-ball characters to produce this very 'fair copy'.

To all my friends in the Mathematical Association who have been responsible for my own training and experience: in particular to the late Mr J T Combridge - Theo as he was known to me has been my revered friend and confidante throughout my professional career; to Dr H Martyn Cundy for the loan of his unpublished 'masterpiece' as it was described by the late Dr Frank J Budden who was sympathetic to this work.

To the patient and sympathetic support of all spouses and families without whose moral support this work could not have been achieved.

And last but by no means least to all the teachers and students whose comments as 'critical friends' appear in the text and too numerous to mention by name.
CONTENTS

PART I

THE PATTERN OF A GROUP STRUCTURE

CHAPTER 1 INTRODUCTION

1.1 Author's background
1.2 Historical background
1.3 Elementary foundations
1.4 The scope of the thesis
   (1) Part I
   (2) Part II
   (3) Part III
1.5 Evaluation

CHAPTER 2 EXTENDING THE NUMBER SYSTEM – A POSITIVE APPROACH TO 'NEGATIVENESS'

2.1 Physical take-away
2.2 The need to extend the natural-number system
2.3 Links with the natural-number system
2.4 Negativeness
2.5 The construction of the directed whole-numbers as equivalence classes of ordered differences of natural numbers
2.6 A VTR lesson sequence: The game of 'cross-tots'
2.7 Graphical representation: The extended natural-number line
2.8 Addition of the 'cross-totting' number-families: The pattern of a group structure
2.9 Graphical representation of the addition rule
2.10 Subtraction as inverse addition
2.11 The 'rule of signs'
2.12 Multiplication of the 'cross-totting' number-families
2.13 Alternative methods of multiplication of the directed whole-numbers
2.14 A student's comments
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.15</td>
<td>Group structure under multiplication denied</td>
<td>45</td>
</tr>
<tr>
<td>2.16</td>
<td>Graphical representation of multiplication</td>
<td>45</td>
</tr>
<tr>
<td>2.17</td>
<td>Conclusion: A look ahead to the rational-number system</td>
<td>48</td>
</tr>
<tr>
<td>3.1</td>
<td>Elementary fractions - the pitfalls</td>
<td>50</td>
</tr>
<tr>
<td>3.2</td>
<td>The need to extend the integer-number system</td>
<td>52</td>
</tr>
<tr>
<td>3.3</td>
<td>The construction of the rationals as equivalence classes of ordered ratios of directed whole-numbers</td>
<td>54</td>
</tr>
<tr>
<td>3.4</td>
<td>The game of 'cross-products'</td>
<td>58</td>
</tr>
<tr>
<td>3.5</td>
<td>Graphical representation</td>
<td>61</td>
</tr>
<tr>
<td>3.6</td>
<td>The multiplication of the 'cross-product' number-families</td>
<td>65</td>
</tr>
<tr>
<td>3.7</td>
<td>Pupils' comments</td>
<td>68</td>
</tr>
<tr>
<td>3.8</td>
<td>The ordering of the 'cross-product' number-families</td>
<td>70</td>
</tr>
<tr>
<td>3.9</td>
<td>The extension of the integer-number line</td>
<td>76</td>
</tr>
<tr>
<td>3.10</td>
<td>A lesson sequence - equivalences</td>
<td>79</td>
</tr>
<tr>
<td>3.11</td>
<td>Addition of fractions using equivalences</td>
<td>84</td>
</tr>
<tr>
<td>3.12</td>
<td>An audio-tape recorded lesson - equivalent fractions</td>
<td>88</td>
</tr>
<tr>
<td>3.13</td>
<td>A note on the diagonal scale</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 4</th>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>SYMMETRY</td>
<td>4.1</td>
<td>What is symmetry?</td>
<td>102</td>
</tr>
<tr>
<td></td>
<td>4.2</td>
<td>Rigidness? or Rigidity?</td>
<td>105</td>
</tr>
<tr>
<td></td>
<td>4.3</td>
<td>Euclid transformed: the rigid motions of the plane</td>
<td>107</td>
</tr>
<tr>
<td></td>
<td>4.4</td>
<td>Stage A</td>
<td>110</td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td>Movement</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td>Invariance</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>Stage B - Fundamental Theorem I</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td>The set of isometries</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td>The missing link</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>Key theorem I</td>
<td></td>
</tr>
</tbody>
</table>
PART II
THE LINEAR ALGEBRA OF A VECTOR SPACE

CHAPTER 5
EXTENDING THE PATTERN – FROM GROUPS TO LINEAR ALGEBRA

5.1 The need to extend the pattern
5.2 The solution of a set of simultaneous linear equations
5.3 The elimination method of solution
5.4 Elimination? – or augmented matrix?
5.5 The inverse matrix
5.6 The inverse of a matrix by row-reduction
5.7 The solution of 3 simultaneous linear equations in 3 unknowns
5.8 Pupil reaction
5.9 Further developments
5.10 Sums of multiples
5.11 The pattern of a vector space
5.12 Linear dependence: basis; dimension
5.13 Arithmetic progressions as a vector space
5.14 Arithmetic progressions: linear dependence
   (1) Dimension
   (2) Choice of basis
5.15 Summation of the arithmetic progression
5.16 The arithmetic progression: graphical representation
5.17 Further developments

CHAPTER 6
ISOMETRIC MATRICES

6.1 Algebra-geometry coordinated
6.2 Linear equations as transformations
6.3 Invariance
   (1) Invariant points
   (2) Discriminant? or determinant?
   (3) Invariant lines
6.4 Distance preserved 186
6.5 The algebraic representation of the isometries 187
6.6 The transformation matrix 190
6.7 Translation 195
6.8 An algebraic law of composition 196
6.9 Investigations on the symmetries of the regular polygons
   (1)
   (2) Links with stage A
   (3) The equilateral triangle
   (4) Algebraic representation
6.10 Tidying-up the loose ends 204

CHAPTER 7 ENLARGEMENT

7.1 From symmetry to similarity 206
7.2 Similarity? or enlargement? 208
7.3 Dilatation - central similarity 209
7.4 The transformation \( \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \lambda \begin{bmatrix} x \\ y \end{bmatrix} : \lambda > 1 \) 212
7.5 Invariance 214
7.6 The properties of a central similarity 216
7.7 The law of composition 'followed-by' 220
7.8 Properties of the triangle 224
7.9 The story so far 228

PART III THE GEOMETRICAL REPRESENTATION OF THE COMPLEX IN ALGEBRA 234

CHAPTER 8 THE SIXTH FORM COURSE

8.1 'Modern' foundations 235
8.2 New teaching styles 237
8.3 The new technology 240
8.4 The sixth form course 243
<table>
<thead>
<tr>
<th>CHAPTER 9</th>
<th>COMPLEX NUMBERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.1</td>
<td>The algebra of $2 \times 2$ matrices</td>
</tr>
<tr>
<td></td>
<td>(1) Equivalence</td>
</tr>
<tr>
<td></td>
<td>(2) Addition</td>
</tr>
<tr>
<td></td>
<td>(3) Multiplication</td>
</tr>
<tr>
<td>9.2</td>
<td>Extending the number system - a realistic approach to complex numbers</td>
</tr>
<tr>
<td>9.3</td>
<td>Quadruples - A special subset of $2 \times 2$ matrices</td>
</tr>
<tr>
<td>9.4</td>
<td>Isomorphisms</td>
</tr>
<tr>
<td>9.5</td>
<td>Geometrical representation of the quadruples</td>
</tr>
<tr>
<td>9.6</td>
<td>The linear algebra of a 2-dimensional vector space</td>
</tr>
<tr>
<td>9.7</td>
<td>An algebra of the special subset of quadruples</td>
</tr>
<tr>
<td>9.8</td>
<td>Real-imaginary links</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 10</th>
<th>HARMONIC RANGE AND CROSS-RATIO</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1</td>
<td>The cross-ratio of a harmonic range</td>
</tr>
<tr>
<td>10.2</td>
<td>The group of cross-ratios</td>
</tr>
<tr>
<td>10.3</td>
<td>The isometries in 3-dimensional space</td>
</tr>
<tr>
<td>10.4</td>
<td>The regular tetrahedron</td>
</tr>
<tr>
<td>10.5</td>
<td>The platonic solids</td>
</tr>
<tr>
<td></td>
<td>(1) The symmetries of the cube</td>
</tr>
<tr>
<td></td>
<td>(2) The symmetries of the octahedron</td>
</tr>
<tr>
<td></td>
<td>(3) The symmetries of the icosahedron</td>
</tr>
<tr>
<td></td>
<td>(4) The symmetries of the dodecahedron</td>
</tr>
<tr>
<td></td>
<td>(5) An investigation</td>
</tr>
<tr>
<td>10.6</td>
<td>Isomorphisms</td>
</tr>
<tr>
<td>10.7</td>
<td>Algebraic representation in 3-dimensional space</td>
</tr>
<tr>
<td>10.8</td>
<td>Transformations of the complex plane</td>
</tr>
<tr>
<td>10.9</td>
<td>The complex cross-ratio</td>
</tr>
<tr>
<td>10.10</td>
<td>The game continues ...</td>
</tr>
</tbody>
</table>

245 249 250 253 256 257 259 265 267 269 271 272 280 297 300 305 307 308
CHAPTER 11

SUMMARY, RECOMMENDATIONS AND SUGGESTIONS FOR FURTHER RESEARCH

11.1 Linear algebra - its place in the school curriculum

11.2 Ampleforth Mathematics

11.3 The syllabus
   (1) 9 to 16
   (2) 16 to 18

11.4 The microcomputer

11.5 Further research

11.6 Conclusion

REFERENCES

APPENDIX A

SOLUTIONS TO EXERCISE
(see page 226)

A(i) - A(viii)

APPENDIX B

NEW TEACHING STYLES
Specimen Questions (see page 239)

B(i) - B(v)

APPENDIX C

ALGEBRAIC REPRESENTATION IN 3-DIMENSIONAL SPACE
(see pages 301, 304)

C(i) - C(ii)

APPENDIX D

AMPLEFORTH MATHEMATICS
An edited transcript of an audio-recorded lesson on Coaxal Circles and the Radical Axis with micro-computer illustration

D(i) - D(xxiii)
PART I

THE PATTERN OF A GROUP STRUCTURE
CHAPTER 1

INTRODUCTION

1.1 Author's Background

This thesis is the embodiment of the author's experience of learning and teaching mathematics through four revolutionary periods of mathematical education. Firstly, as a pupil, her formal introduction to Euclid in which she was required, for example, to reproduce proofs of the tests of congruence was, at the age of 12, reversed. She reluctantly had to make a completely fresh start, in an apparently more enlightened situation, with the not-so-recent new stage A-B-C approach.

Secondly, as a probationary teacher, she saw the introduction of pre-School Certificate calculus as a result of the Fleming Report: 1947.

Thirdly, in the 1960's, she joined the 'band-wagon' of the 'modernists' and proudly accepted an invitation to join the team of the Nuffield Mathematics Teaching Project. Later when pioneering the in-service training of teachers, supported by the Shell Petroleum Company at the University of Nottingham, she had the opportunity of consolidating 'Nuffield Mathematics' of the primary/lower-secondary years and of investigating developments with teachers and their pupils towards the O- and A-level project syllabuses of the General Certificate Examination and the examinations of the then new Certificate of Secondary Education Mode 3. Progress was slow and inhibitions on the part of teachers were very real. The work now to be described is the result of experiences based on the experimental
project syllabuses. It instances a style of teaching in the classroom which has evolved from methods dictated by the traditionally disparate subjects of arithmetic, algebra and geometry, through the O-level Syllabus A/Alternative Syllabus B (with Calculus), and on to a 'modern' O-level Syllabus C of which the author was one of the two first Chief Examiners for the corresponding school examinations board.

Fourthly, the current discussion of the more recent experiences with students on initial and in-service courses of teacher training are now being overtaken by the dictates of the General Certificate of Secondary Education at 16+ and the rapid onslaught of new technology and the potential use of the microcomputer in the mathematics classroom. It has taken approximately 20 years for 'Nuffield' discovery methods to be still only partially accepted at primary level. Only now are they being adapted to secondary mathematics for the GCSE initiative. The author exhorts the reader to encourage a more enlightened approach to the teaching of mathematics at the sixth-form and tertiary levels as a matter of urgency.

1.2 Historical Background [1]

Mathematics was introduced as an integral part of the school curriculum in the mid-nineteenth century. The teaching of geometry was based almost entirely on the Elements of Euclid as a classical masterpiece and as a means of developing one's powers of logical deduction. The first real challenge to a Euclidean approach was made by J M Wilson, later Canon Wilson, Headmaster of Clifton College, Bristol in 1868 with the publication of a text on the teaching of geometry. The aim was to lay less emphasis on the formal proofs of Euclid and to suggest a
more practical approach. It was felt that too much importance had been attached to the ability to reproduce a logical sequence of geometrical propositions based on a formal set of assumed axioms - self-evident truths - which were often far from self-evident to the majority of pupils in the schools. For the future it was felt that pupils should be encouraged to rely less on the memory and the parrot-fashion rote-learning of formal proofs and to tackle intelligently problems devised to illustrate the fundamental relationships involved.

Ultimately [2] three successive stages A, B, C were identified in the teaching and learning processes of geometry. Stage A was to be essentially practical and geometrical relationships were to be experienced through accurate ruler and compass constructions; Stage B was intended to introduce pupils gradually to simple ideas of mathematical proof: such proofs usually took the form of the syllogisms of Lewis Carroll consisting of a three-statement argument in which a conclusion could be deduced logically from a pair of given premises; Stage C set out to organise these results into a sequential development. The modification of the mathematics curriculum on these lines was slow and gradual.

The Association for the Improvement of Geometrical Teaching (later re-named the Mathematical Association) was inaugurated in 1871; but it was not until 1923 that a first "Report on the Teaching of Geometry" was published.

In any attempts to facilitate the learning of mathematics by the pupils, however, teachers still have to ensure that such learning is based on sure foundations. The author is often astounded at the inability of the young trainee teacher of today to appreciate the circular argument which results from an unacceptable construction for the proof by congruence of the equality of the base angles of an isosceles triangle in which two sides are equal [3]. The logical
foundations which require the isosceles triangle property to establish the right-angle/hypotenuse/one-other-side (RHS) test of congruence have all too often been insecurely laid. These foundations dictate the bisection of the vertical angle for the application of the side/angle/side (SAS) test of congruence: no other alternative description of that construction line can establish an acceptable logical deduction of the equality of the base angles.

In the opinion of the author, such logical fallacies can be avoided if congruence, for example, is discussed with pupils in a practical investigation of the symmetry transformations (see Chapter 4). This is just one example from the current thesis in which the author attempts to put the flesh and bones of a mathematical structure on to the skeletal framework discovered from the delights and joys of an enlightened mathematical classroom situation in the primary years of schooling.

1.3 Elementary Foundations

The author assumes the foundations of a well-informed primary experience. She requires a working knowledge of the operations on the counting numbers of ordinary arithmetic: the sums and products of classical algebra. It is anticipated that pupils will have been introduced to another kind of algebra in which the elements need not be numbers: operations on a universal set of non-numerical elements resulting in sums and products which are the intersections and unions of the Boolean algebra of sets. It will be helpful, if not essential, for pupils to have had experience of systems in which the usual properties of addition and multiplication are satisfied: commutativity, associativity and the distributivity of one over the other. Counter examples of systems in which these properties are denied serve as useful reinforcements of the main issues. The ideas of identity and inverse pairs of elements within a mathe-
mational system required of a group structure are useful pre-requisites for the development of ideas in this thesis.

It is important, too, that pupils will have had practical experiences of 2- and 3-dimensional space and of the elementary notions which arise from an acceptance of Euclid's parallel axiom. They may perhaps have been prepared for later acceptance of different geometries in which 2-dimensional space is not necessarily 'flat'. Practical investigations of the puzzles and problems of a topological nature serve a useful purpose if - but only if - they lead to a meaningful mathematical discussion at the next stage of mathematical development. Let it never be said that primary teachers have taken the gilt off the gingerbread.

It is expected that pupils will have been encouraged to make an extensive use of language in an attempt to explain in their own words the results of their activities and to describe some of the properties resulting from their investigations. Early experiences of pattern with numbers, of geometrical pattern, of puzzles and problems and mathematical games with strategies for their solution and success, give plenty of scope for the use of language in their own words, followed by expression in the written language of the mother tongue, only then to be followed by the symbolic expression in the appropriate algebraic system.

An extensive use is made in this thesis of graphs to link number relationships with pattern in geometry. The intelligent representation in, and use of, symbols will facilitate, later, algebraic manipulation. Throughout the work, the author has identified at each and every stage an abstract mathematical notion as an imaginary entity in the mind of the pupil. Its geometrical, or graphical, representation enhances the understanding of that idea.
1.4 The Scope of the Thesis

(1) In Part I, the practical experiences of an enlightened primary education are used and developed into a more structured representation of the number system on the one hand and a sequential development of geometrical ideas on the other hand. Well-defined and meaningful mathematical activity and investigative work in the form of mathematical games are described for the pupils. The risk of an overloaded curriculum by these time-consuming exercises may be overcome, the author suggests, by the non-formal teaching of algebra. For her, algebra is the symbolic language which is used, rather than taught, to translate the spoken and written verbal description and results of the investigative programmes into manipulative symbolism.

The system of counting numbers is extended, in Chapter 2 to the integer number system to enable subtraction to be defined as inverse addition. For the pupil, the imaginary 'negativeness' is represented graphically in terms of vectors. Then in Chapter 3 a further extension of the number system is made to the rational number system to enable division to be defined as inverse multiplication. A sequential development of geometrical symmetry in Chapter 4 is intended as a concurrent section of the mathematical curriculum of the lower-secondary years.

Pupils are encouraged throughout to be able to recognise and identify the pattern of a group in mathematics. The algebraic representation of these early secondary experiences will eventually require the use of an algebra of 'sums of multiples'.

(2) In Part II of the thesis, a distinction is made between the restrictive practices of classical algebra - the generalised arithmetic of the number systems; the Boolean algebra of sets which requires a binary operation combining a pair of
elements in the universal set to give a product element which also belongs to that same universe; and a modern linear algebra which requires in addition a unary operation resulting in the product of a single element of the universe with a real number (scalar) multiple - the universe must be closed under the operation of a 'sum of scalar multiples'. This last, linear algebra, is used (not taught) in Chapter 5 in the treatment of a set of linear equations and again in Chapter 6 in the matrix representation of the linear transformations of motion geometry.

Linear algebra is seen to be a unifying factor in school mathematics when, in Chapter 7, the discussion of geometrical similarity completes the analysis of cases in which a solution to a set of linear equations exists. As a language, it is the means of forging horizontal links between arithmetic and geometry at different age/ability levels. It is also the means of forging vertical links between the different stages of an upward spiral in the mathematical development of the pupils: primary/secondary, lower-secondary/upper-secondary, upper-secondary/sixth-form and tertiary levels. From early experiences of size and shape with environmental materials and structural apparatus and the numerical tabulation and graphical representation of the results obtained, the pupil is led upwards towards the need for an extension of the number system and an investigation of its pattern or structure together with an investigation of the geometry of the 2-dimensional Euclidean plane.

(3) In case the reader should be alarmed at the sophistication of the development, in Part III, at sixth-form level attention is drawn in Chapter 8 to the rapid onslaught of the technical era and the invasion of schools by the microcomputer. It is very obvious to the author that a change in teaching methods must be made. The use of the microcomputer in the mathematics classroom will inevitably encourage and make possible the
re-introduction of sixth-form geometry and the extensions to 3-dimensional space.

The reader is invited, in Chapter 9, to continue the upward movement in a vertical spiral. At sixth-form level the number system is again extended to the system of complex numbers to make possible a solution of the equation $x^2 = r$ for all positive and negative real numbers, $r$. The system is represented in terms of linear algebra by a special subset of $2 \times 2$ matrices which, in turn, are regarded as operations and illustrated by the geometrical transformations of the complex plane.

In Chapter 10 the author has described some of her own experiences with sixth-formers in informal recreational mathematics sessions by courtesy and invitation of schools in many parts of the country. The latest results of current student activity are submitted in Chapter 10 in which an investigation was made of the symmetries of the complete set of the Platonic solids. (See also Exhibits)

1.5 Evaluation

It is to be noted that, at this level, it is difficult to engage in formal experiment with topics which have all but disappeared from the syllabus of the public examinations. All the work described has been extensively tested in schools and the results of classroom experiences with teachers and students have been built in. Many of the ideas, particularly at the more advanced level, are at an early stage of development and the results and achievements have not been formally tested by quantitative analysis. Qualitative evaluation in the form of suggestions and comments of "critical friends" has been included at each and every stage.
2.1 Physical Take-Away

In the real world we can always find the difference in size between two measurable quantities; the elementary arithmetic of the counting numbers provides a mathematical model in which the numerical difference between two counting numbers can always be determined. Again, in the real world, we can always physically take-away a smaller quantity from a larger quantity; in the elementary arithmetic of the counting numbers a smaller number can always be taken-away from a larger number. Sometimes, in the real world, we need to physically take-away a larger quantity from a smaller quantity; the problem cannot be solved in the elementary arithmetic of the counting numbers.

Example:

It is required to make a junket and a cup of coffee. Milk for the junket is put into pan A and heated to a temperature of 37°C (blood heat). Milk for the coffee is put into pan B and heated to a temperature of 90°C (just under boiling point).

The answer to the question: "How much \textbf{hotter} is the milk in pan B than the milk in pan A?" is given by the answer to the physical take-away sum: Temperature B - Temperature A. i.e. \[90^\circ - 37^\circ\]
Using the mathematical model: $90 - 37$ of take-away in the arithmetic of the counting numbers and interpreting back to the real world situation, the answer to the problem is:

Milk in pan B is $53^\circ$ hotter than milk in pan A.

The answer to the question: "How much hotter is the milk in pan A than the milk in pan B?" is given by the answer to the physical take-away sum: Temperature A - Temperature B i.e. $37^\circ - 90^\circ$

The model in the arithmetic of the counting numbers is: $37 - 90$ and an answer cannot be obtained within the system of counting numbers.

Common sense dictates the question: "How much cooler is the milk in pan A than the milk in pan B?". We can manipulate the opposites of heat and cold to fit a mathematical model in which a smaller counting number is taken-away from a larger counting number to enable us to give an answer to the newly-phrased question. But there is no way in which the take-away operation in the arithmetic of the counting numbers can be used to interpret the original question: "How much hotter is the milk in pan A than the milk in pan B?" The good mathematics teacher would surely prefer to provide the pupils with the mathematical means of providing an answer to the problem rather than leaving them to rely on verbal manipulation of the question.

Mathematically, for the teacher, if $a, b$ belong to the universal set of counting numbers, or natural numbers $N$ as they are normally known, we write: $a,b \in N$. The numerical difference, $a - b$, between $a$ and $b$ can always be determined as another natural number $x$:

$$x = \begin{cases} a - b & \text{according as: } \begin{cases} a > b \\ b - a & \text{a} < b \end{cases} \end{cases}$$

The take-away sum: $a - b$ determines a counting number $x$ if and only if $b$ is less than $a$. There is no counting number $x$ such that: $a - b = x$ in the case when $b$ is greater than $a$. 
The numerical difference, \( a - b \), between \( a \) and \( b \) may be illustrated as \( x \) in the diagrams:

\[
\begin{align*}
\text{a > b: } x &= a - b \\
\text{OR } a < b: x &= b - a
\end{align*}
\]

If we look again at the case when \( a \) is greater than \( b \), \( x \) is obtained as: \( a - b \) and may be regarded as the natural number which added to the smaller number \( b \) gives the larger number \( a \). Physical take-away in the arithmetic of natural numbers may be illustrated by \( x \) in the form:

\[
\begin{align*}
a &= b + x
\end{align*}
\]

Yet again, in the case when \( a \) is greater than \( b \), \( x \) may be regarded as the natural number which taken-away from the larger number \( a \) leaves the smaller number \( b \). The situation may be illustrated by \( x \) in the form:

\[
\begin{align*}
b &= a - x
\end{align*}
\]

These three representations: \( a - b = x \)  
\( a = b + x \)  
\( b = a - x \) 
of the natural number \( x \) in terms of \( a \) and \( b \), when \( a \) is greater than \( b \), are three different aspects of the more sophisticated operation of \textit{subtraction}. In the opinion of
the author, it is not until the pupils have had substantial experience of all three situations - not of course all at once - and has been introduced to numerical difference, complementary addition, and decomposition, that they are ready for the respectable approach to the operation of subtraction. There need be no controversy as to which method of subtraction is the more appropriate in elementary arithmetic. Greater understanding of subtraction is achieved if a pupil is introduced to all three ways of getting the same answer.

Subsequently, if we are to avoid the hit-and-miss manipulation of opposites as in the example discussed on page 10 we need to search for a system of numbers in which an answer to subtraction can always be found. We need to be able to find a number \( x \) such that: \( a - b = x \) even when \( a \) is less than \( b \).

We need to extend the natural-number system of arithmetic to the system of directed whole-numbers in which subtraction can always be performed.

2.2 The Need to Extend the Natural-Number System

In school mathematics the extension of the natural number system to the system of directed numbers is usually demonstrated by means of pairs of opposites which occur in everyday life. These may include:

- temperatures of heat and cold (see example, page 10);
- credits and debits of a bank account;
- distances measured upwards and downwards on a hill-slope;
- distances measured forwards and backwards on a line;

and may be found in many self-respecting school textbooks in algebra [for example 4.1]. These, and many others, afford excellent examples of the way in which the natural number line will already have been extended backwards. Equally spaced points on the extension will have been labelled with often meaningless symbols: \(-1, -2, -3, \ldots\), or more modernly \(-1, -2, -3, \ldots\), which represent numbers with properties very different from the natural numbers and which will not
have been used in any meaningful way. Sometimes, but not always, the natural-number names: 1, 2, 3, ..., on the original natural-number section of the line will have been replaced by number symbols: +1, +2, +3, ..., or more modernly: +1, +2, +3, .... More often than not, in the experience of the author in her work in schools with students on initial and in-service teacher training courses, these latter represent numbers which are regarded as synonymous with the natural numbers they replace. A number nought or zero, represented by the symbol '0', is often introduced as the origin from which measurements are taken. The extended natural-number line takes the form:

```
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
```

The extended natural-number line is then used as an aid for performing subtraction as backward addition often in the form of a slide-rule consisting of two such lines. Thus:

```
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
```

illustrates that: +5 - +2 = +3 from: +3 + +2 = +5. Such an aid will inevitably go a long way towards convincing the average pupil that an answer can be found to the problem of finding a solution in directed whole-numbers to the equation: \(a - b = x\), even when \(a\) is less than \(b\). For example:

```
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
```

illustrates that: +2 - +5 = +3 from -3 + +5 = +2.
The good teacher must however be aware of problems that will lie immediately ahead in the mathematical development of the pupils. Pupils will have had the experience of working in the system of natural numbers not only under the operation of addition but also under the operation of multiplication. The pupil will be expected to know that it is always possible to find a natural number which is the product of two natural numbers a and b. He must be encouraged to expect to be able to multiply together two numbers in the new system of directed whole-numbers which he has used to name the positions on the extended natural-number line but will not yet know very much, if anything, of their significance other than as mere labels to points on a line.

Furthermore, just as subtraction cannot always be performed in the system of natural numbers, so too must he be encouraged to 'discover' that division may not always be possible in the new system of directed numbers. A further extension of the directed whole-number line to the rational-number line will be required (Chapter 3).

In attempting an extension of the natural-number system under addition and multiplication the author would want to give pupils some idea of the nature of the new numbers to be constructed before they are used as meaningless names and represented by meaningless symbols on the extended natural-number line.

2.3 **Links with the Natural-Number System**

The good teacher will also think back to what his pupils have so far achieved and will build on their experience and understanding. Much of their early arithmetic will have been illustrated by the natural-number track. This may have been constructed as points reached from an origin, a selected starting point called *zero* and marked with the symbol '0.'
(nought), in a succession of equal paces in a straight line. The points reached will have been given whole-number names according to the number of paces required to reach them. The natural-number line:

```
0 1 2 3 4 5 6 7 8
```

will have been used in conjunction with considerable number experience at infant and junior levels. It will have been used also as a measuring instrument in physical situations - a ruler, a thermometer scale, a scale on a spring balance, a yard-stick for measuring heights; and as a slide-rule to check the accuracy of number bonds in addition and multiplication as repeated addition.

It is then customary to represent pairs of opposite physical quantities such as those listed on page 13 by steps forwards and backwards from the origin. Those points which are reached as a result of steps backwards are said to represent the minus or negative numbers and are labelled with a minus (-) sign. The natural numbers which had already been marked on the natural-number line are re-named the plus or positive numbers and are regarded as the positive counterparts of their negative opposites. Only occasionally [4.1] do we see the plus (+) sign written in to distinguish the positive from the natural numbers. Only occasionally [4.1] are we given an explanation of the use of the plus and minus signs as labels as distinct from their former use to denote the operations of addition and take-away. All too often in the teaching of directed numbers, no clear distinction is made for the pupil between the familiar natural numbers and the positive-number subset of a newly defined set of directed whole-numbers, the positives-and-negatives.

```
The Negatives
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
```

The natural number line has been extended backwards and points reached by equal steps have been given names in terms of
numbers which have, as yet, no meaning for the pupil and represented by symbols which can be nothing more than confusing.

Before extending the natural-number line we need to 'construct' in some way the new set of directed whole-numbers which are to be used to name the points on the extended line.

Negativeness is perhaps one of the most difficult concepts in school mathematics for pupils to grasp. There is for the average pupil no ready-made concrete example of a negative number. It is purely abstract and, indeed, purely imaginary at his level of understanding in an exactly analogous way in which a complex number can be purely imaginary to the sixth-form pupil. The author urges the teacher to give the pupils an opportunity to come to grips with negativeness; actually to handle the abstract notion of a negative number. A five or six year-old pupil experiences, say, the 'fiveness-of-5' by playing with a multitude of sets of 5 physical objects and by making up a large number of sets each containing 5 things. The pupil is thus enabled to abstract the property that is common to all the sets of 5 objects, namely their 'fiveness'.

In an exactly similar way the author, based on her work with the Nuffield Mathematics Teaching Project in the 1960's, has attempted to show [S1.2, 5.2] how the 'negative-fiveness-of-negative-5', say, can and should be experienced and handled by the average upper-junior/lower-secondary pupil. It is essential also for, say, the 'positive-twoness-of-positive-2' to be so handled and distinguished from the natural-number 2 which it will replace. The natural-number line and its extension to the directed whole-number line are separate and distinct representations of two different number systems. Admittedly, under certain conditions, the positive directed whole-numbers behave like their natural-number counterparts, as do the negatives under other certain conditions. But this is not to say that the positives are the same as, nor can the negatives be the same as, the naturals.
The author regards it as desirable, if not essential, for even the slower learners to have some idea of what actually are the negative- and the positive- whole numbers that are now needed for practical mathematics in a real world. What is their connection, if any, with the natural numbers that have so far served our purpose? What, too, is the part played by the zero of the natural-number system and represented by the origin on the natural-number line? These are important questions which must be in the mind of every good teacher of mathematics who will surely wish to attempt to put an answer within the grasp of at least some of the pupils.

It will now be shown how pupils may be encouraged to construct the directed whole-numbers, the 'imaginary' negatives and positives in algebra, and to represent them geometrically in a graph before they are used to name the points on the extended natural-number line. It will further be shown how the 'imaginary' arithmetic, the addition and multiplication of the directed whole-numbers, may be represented geometrically as vectors, thus preparing the way for a meaningful approach to the further development of mathematics both for the user and for the specialist mathematician.

2.4 Negativeness

'Negativeness' is abstract: there can be no concrete representation of it at the upper-junior/lower-secondary level of mathematical experience. Such abstract ideas in school mathematics often lend themselves, however, to number games. A game requires motivation and interest for the players, rules of play, strategies for winning. Pupils can often be motivated, even excited, by a challenge to explore the possibility of 'discovering' a piece of mathematics for themselves. The teacher may present that challenge in the form of a game.
He must first set the scene: describe the arena - the court, the pitch, the board - in which 'play' will take place. He will describe the 'pieces' to be moved or manipulated by the 'players' and he will explain the 'object' of the game in terms of mathematical objectives. The players who are the pupils are then invited to suggest the 'rules' of the game which determine the way in which 'moves' can be made to achieve the final objectives. 'Winning strategies' involve an adjustment of the rules to enable more ambitious objectives to be achieved. The rules, or rather the mathematical operations which they represent, must provide a pastime which is not only enjoyable and satisfying to the pupils but which also gives results which are seen to be the 'right answers' when translated into terms of real-life situations in the physical world. The teacher must be prepared to go along with (almost) any suggestion made by the pupils however trivial or unfruitful that suggestion might appear. Guidance there must be if a chosen rule is to be adopted as 'sensible' in the mathematical sense of a well-defined, consistent or compatible operation, and 'respectable' in the sense of conformity with mathematical experience so far developed.

A lesson sequence presented by the author to a good-average 11 to 12 year-old class of pupils has been video-tape recorded [S1]. The sound-track has been transcribed and published [5.2]. It is not the intention here to reproduce this earlier work. The object now is to explain in more detail the nature of the mathematics given in the form of a game called 'cross-tots' and the potential mathematical value for the pupils to whom it was presented.

The main purpose of the lesson sequence is to encourage the pupils to suggest a piece of mathematics to find a solution to a specific problem: that of subtracting a larger number from a smaller number. The solution to the problem, albeit an important one from the point of view of practical needs,
is not an end in itself but rather a means to the end of getting pupils to think and talk mathematically and then to translate that mathematics into symbolic and pictorial language.

At each and every stage the development of the number work may be illustrated geometrically in the form of a graph or diagram. The algebraic symbolic language of mathematics is used to describe the numerical and geometrical relationships obtained. A knowledge is assumed of the natural numbers under the operations of addition and multiplication. The number zero is included. The pupils will have been accustomed to working with number-pairs in elementary work with coordinates, graphical representation, collection of statistical data, etc. The scene is set.

The arena is the universal set $E$ of ordered pairs of natural numbers denoted by $(x,y)$. A relation $R$ is defined on $E$:

$$(a,b) \sim (x,y) \text{ if and only if } a + y = b + x \text{ for all } (a,b), (x,y) \in E$$

The relation $R$ is an equivalence relation: the commutativity, associativity and the cancellation properties of the natural numbers under addition ensure that the relation $R$ is:

**(i) reflexive:** $(a,b) \sim (a,b)$ since $a + b = b + a$

**(ii) symmetric:** $(a,b) \sim (x,y) \Rightarrow a + y = b + x \Rightarrow b + x = a + y \Rightarrow x + b = y + a \Rightarrow (x,y) \sim (a,b)$

**(iii) transitive:** For $(c,d)$ also in $E$ we have:

$$(x,y) \sim (a,b) \text{ and } (a,b) \sim (c,d) \Rightarrow x + b = y + a \text{ and } a + d = b + c$$

$$\Rightarrow x + b + a + d = y + a + b + c$$

$$\Rightarrow x + d = y + c$$

$$\Rightarrow (x,y) \sim (c,d)$$

The equivalence relation $R$ partitions the universal set $E$ into equivalence classes, disjoint sets or families, of ordered
pairs of natural numbers. These families of natural numbers are the pieces in the game of 'cross-tots' to be manipulated according to a set of rules chosen by the pupils for performing arithmetical operations in the new-number system.

2.5 The Construction of the Directed Whole-Numbers as Equivalence Classes of Ordered Differences of Natural Numbers

The method of presentation and teaching style need explanation. In the article [5.2] the arithmetic, the addition and multiplication, of the natural numbers is taken as the starting point. It is assumed that the pupils can add and multiply together two natural numbers, that they can illustrate addition using the natural-number line as a slide-rule with the origin marked with the number 'zero'. For this reason the author has included zero in the set of natural numbers used in the game of 'cross-tots' described. It is helpful when naming the 'families' of new numbers which the pupils construct, and convenient in the graphical representation of these families, to have the \( 0,0 \) family rather than the \( 1,0 \) family arising as the new origin on the extended number line. Family names are put in boxes, '□□s', to avoid confusion with their representative member pairs which are put in brackets, '(□□)s'.
The actual game, and/or the apparatus used, for the motivation of the work is immaterial. The number-pair approach of teaching the directed whole-numbers to young pupils is not original. It was investigated extensively by the author in her work with the Nuffield Mathematics Teaching Project as a result of the practical teaching experiences of J.W. Boucher, a member of the Nuffield Team. Many and varied pieces of equipment are described [6.3] and some attempts have been made by other courageous pioneers to make the abstract notion of 'negativeness' more positively accessible to even the less able pupil. The author is attracted to the two piles of coloured bricks described in an article by Tim Rowland [7] to represent the two elements of an ordered pair of natural numbers and written, incidentally, after the production of the video-tape recording of the 'cross-tot' lesson sequence [S1.2].

More recently, in 1986, one of the author's students on the Postgraduate Certificate in Education course of initial teacher training at the University of Hull, describes her experiences on teaching practice undertaken in one of the toughest local 13 to 18 age-range secondary schools at a time when not only teacher disruptions were at their height but also the local bus crew withdrew the school-bus service. She noted that her pupils had difficulty with the usual ready-made examples of the extended natural-number line in the form of a thermometer, time-chart, ladder, ..., and its use as a slide-rule for the addition and subtraction of the directed whole-numbers. She felt that advantage would be gained if more time could be spent on the number line itself and some alternative method found for its extension. Interestingly she indicates that:

"a suitable method could be YIN and YANG [8]...

... Chinese symbols for all pairs of opposites appearing in nature.

YIN is dark, calm and negative
YANG is bright, lively and positive

Together they combine to form a perfect whole:"

* Vicky Harrison, now teaching at Copleston High School, Ipswich
She suggests that cards could be constructed in the form:

![Fig 2.1](image1)

of BLACK and RED halves of a circle to represent \( +1 \) and \( -1 \) respectively. When placed together they make a 'zero': i.e. they cancel each other out. Any positive or negative number may be represented by the corresponding number of BLACK for positive or RED for negative cards. When two BLACK cards - representing (say) 2 - are placed to fit against three RED cards - representing (say) 3 - to form circles, or zeros, we see that: \( +2 + -3 = -1 \):

![Fig 2.2](image2)

The cards may also be used for subtraction in which RED cards can be physically 'taken-away' to represent subtraction of a negative number.

The student suggests that a still more concrete representation of the whole circle such as a football, or some similar object of interest to pupils, could be used instead of cards, but still retaining the condition that the whole is made up of a positive half and a negative half so that when placed together they constitute the 'whole' which represents 'zero'.

The arena may change, the pawns in the game may change, but the rules of play and the objectives do not. We need to emphasize for the pupils:
(i) the need for the extension of the natural-number line

(ii) the possibility of laying down any definitions we please for our various operations

(iii) the criterion which, out of several, we employ so that what we do shall correspond with what goes on in the natural-number system.

The directed whole-numbers are constructed as a completely new set of numbers. Once we know what sort of numbers we are playing with, we can then illustrate them more vividly as geometrical points on a line or lines in a plane. The author returns now to a discussion of the game of 'cross-tots' as recorded on video-tape.

2.6 A VTR Lesson Sequence: The Game of 'Cross-Tots'

Irrespective of the apparatus selected for the game, the pupils are given a rule which enables them to construct families of ordered pairs of natural numbers under the equivalence relation $R$ defined on page 20. For the pupils, $R$ is replaced by 'cross-tots with'. This form of words for the relation gives a rule which enables them to:

(i) find as many natural-number pairs $(x,y)$ as they like which 'belong to' the same family as the given natural-number pair $(a,b)$:

A given 'cross-tots' with $x \times y$ if and only if $a + y = b + x$

(ii) ascertain that:

(a) any given $(x,y)$ can be put into a definite and unique family

(b) any given $(x,y)$ cannot belong to more than one family.
The pupils are in fact 'handling' the idea of a partitioning of a universal set under an equivalence relation without the sophistication of the formal mathematics indicated on page 20.

Of course the teacher must know the conditions under which such a partitioning is assured. In this case it is easy to see that:

we can find infinitely many \((x,y)\) such that given \((a,b)\) \(a + y = b + x\). Thus:

\[
given: \quad a \quad b \\
for any \quad x \quad y \\
\]

we choose any \(y\) to determine \(x\) and choose any \(x\) to determine \(y\) \{from \(a + y = b + x\)

and we can find the unique family to which a given \((x,y)\) belongs. The pupils are enabled to give a name to a family by selection of the representative natural-number pair which has zero as one of its elements. The problem becomes:

either: is there an \((a,0)\) to which \((x,y)\) belongs?
This requires that \(a + y = 0 + x\) which has a unique solution \(a \in \mathbb{N}\) if \(x > y\)
or: is there a \((0,b)\) to which \((x,y)\) belongs?
This requires that \(0 + y = b + x\) which has a unique solution \(b \in \mathbb{N}\) if \(x < y\)

Note that if \(x = y\) then \(a = 0 = b\) and \((x,y) \in 0,0\) family.

Some excellent preliminary foundation work is done by T J Fletcher [9] to illustrate the breakdown of a partitioning in the case of a non-equivalence relation. The relation 'is the brother of' in a set of males, for example, leaves an only son of a family isolated and not a member of any subset.
of the partition. The situation of a partitioning is restored if and only if the relation is made reflexive by regarding the only son as 'a brother of' himself!

In the lesson sequence "Un-natural Numbers for Ordinary Children" of the video-tape recording, the directed whole-numbers are introduced by the game called 'cross-tots'. The author wishes now to draw attention to some of the joys that arose from the second lesson given in the film sequence.

When working with experienced teachers, she has often come up against the proverbial brick wall: if the underlying mathematics is discussed at teacher level, the audience grows impatient and is convinced that the work is beyond the capabilities of its pupils; 'demonstration' on the other hand with a group of pupils all too often elicits the comment that "the pupils have been specially selected". Transporting a live classroom situation by means of video-tape recordings has done much for the author to overcome such prejudices. The lesson sequence was repeated with a fourth year class of low mixed-ability pupils at the request of the class teacher, a non-specialist mathematician who "had learned a lot of mathematics from the experience"! The real justification for what must appear to be a somewhat abstract approach surely came with the expressions on the faces of two of the pupils, Neil Kehoe and Stephen Atkins, as seen in the photographs overleaf and whose contribution to the work is now described.
2.7 Graphical Representation: The Extended Natural-Number Line

Pupils were asked to work in pairs to produce wall charts of the graphs of the natural-number pair families. This particular class produced a large selection of graph-work. The author finds it difficult to learn the names of children in the random selection of classes offered to her in the interests of initial and in-service trainee teachers and school-based in-service training. One illustration in particular seemed to convey better than the rest the object of the exercise: the extension of the natural-number line. The attention of the reader is drawn to the way in which the natural number-pair families were labelled on the graph reproduced below and prepared by Neil Kehoe and his mate Stephen Atkins:

![Graph](image)

The author was told that these two boys were in fact remedial pupils who had re-entered their class for mathematics just two weeks before this repeat of the lesson sequence began. Notice how Neil and Stephen have labelled the natural-number pair families in the graph. What more could any-one ask of pupils who have used their newly-constructed number families, illustrated them graphically as a set of parallel straight lines and labelled those lines with the simplest names of the natural number-pair families which they represent: the something, 0 s and the 0, something s. Furthermore, the
new-number names have been inserted neatly on the horizontal line parallel to the first natural-number axis. If now we transfer those names to the first natural-number axis we have a very clear indication of the one-to-one correspondence between the positions 0, 1, 2, 3, ... of the natural numbers and the new \textit{something, 0} subset of the new-number families. It will be 'obvious' to some of the pupils that the new \textit{something, 0}s are not the same as the natural-numbers that they now replace but there is some connection between them. This connection will be explored in terms of, for the teacher, mathematical isomorphisms of the new-number system with the natural-numbers under addition and under multiplication.

In the following diagram, produced by two other boys Barrie Holland and Carl Hulme in the class, the lines representing the \textit{0, something} families have been produced backwards to meet the first natural-number line also extended backwards, in positions which can be labelled by their corresponding new-number family names.

Fig 2.4
The first natural-number line has been extended backwards and re-labelled with the newly constructed number-family names. What more, indeed, can we expect of low-ability children who have played and, it is suggested, enjoyed a game of 'cross-tots' to construct new-number families, who have illustrated them graphically, have named them at points where they actually cut and evenly graduate the extended natural-number line? The mathematics is sophisticated; the game is apparently enjoyable. It was at this point with this second lower-ability class of pupils that it was decided here to translate the new-number family names into their familiar 'little-tiny-positive-at-the-top' and 'little-tiny-negative-at-the-top' positive and negative directed whole-number symbols, as shown in Fig 2.4.

![Fig 2.5](image)

The author was further delighted to see one pupil actually to correct her mistaken graph (Fig 2.5). Such a correction of her own geometrical representation surely enhances her understanding of her 'imaginary' negativeness! The same pupil, too, noticed that a line was missing from her set of parallel lines representing the new-number families. She in fact 'discovered' the existence of the important additive identity new-number family \(0,0\) of the new-number system (Fig 2.6).
Exercise

The reader is invited to plot the 'cross-totting' families of natural-number pairs in a system of polar coordinates in which the first number of a natural-number pair is plotted against a set of concentric circles of radii 0, 1, 2, 3, ..., with centre at the origin; and the second number of a natural-number pair is plotted against a set of half-lines from the origin at equiangular distances numbered 0, 1, 2, 3, ... from a starting position OA and numbered 0. The number families will appear as a set of Archimedean spirals.

Strategies must now be developed for 'winning the game'. The pupils are next encouraged to construct for themselves an arithmetic of the new numbers - the 'cross-totting' new-number families. With the first class of pupils, the film sequence continued with the definition of rules for addition and subtraction and the simplest number-pair family names were retained for this purpose.
2.8 Addition of the 'Cross-Totting' Number-Families: The Pattern of a Group Structure

In Part III of the lesson sequence [5.2]: "Plus pos-and-neg", the isomorphism of the subset, the something, 0 s, of 'cross-totting' families with the natural numbers under addition prompts the conventional and expected rule for addition:

\[ \{a, b\} \oplus \{c, d\} = \{a + c, b + d\} \]

and suggested by the pupils. The selected rule for addition must be seen by the pupils to be 'sensible' - the word used by the author to give experience of the mathematical language of compatibility of an operation on the partition of a universal set of classes of elements as distinct from the pupils' own experience so far of combining the natural numbers - which must, to them, appear to be single entities - by adding them together.

The strategy of the game dictates that if the operation performed by the chosen rule for addition is to be called addition then it must also be seen by the pupils to be 'respectable' - in the sense that it behaves like, and gives results analogous to, those of the ordinary addition of the natural numbers. Pupils are encouraged to test the new system for closure, associativity and commutativity. Only when these properties are established are we really entitled to call the operation 'addition' and to write it with the plus sign without the circle: '+'.

The 'discovery' of a unique additive identity 'cross-totting' family and of corresponding unique pairs of additive inverses prompts the exploration of subtraction defined as inverse addition [see page 34]. Incidentally the rule for addition gives also isomorphism of the 0, something s with the natural numbers under addition. Of course if this, or any other, property is suggested by the pupils as a starting point, an
alternative approach can often be developed which may (or may not!) lead to the same conventional mathematical system with the same conventional properties. It requires courage on the part of the teacher but, if we are selective and wary of the proverbial 'red herring', some interesting results are often produced.

The author wants her pupils to learn to use the language of mathematics and talks with the pupils in that language when presenting a mathematical topic - in this case the directed whole-numbers - which appears in every self-respecting scheme of school mathematics. In the video-tape recorded lesson sequence the language of sets that is used has been simplified so that it may be understood by all - well, almost all - the pupils in the class and to enable them to 'play' mathematics in a meaningful way.

The language is still further developed in the investigation of the operation of addition in the new system to give experience of the structure of a group in mathematics. This is the language that the potential mathematician will need for his further development of mathematics for its own sake. The game of 'cross-tots' enables the pupils to handle and to work with the useful technical skills involved in the directed whole-numbers.

It is important to mention here that we are not teaching or 'doing' sets and groups. We are using the language of these highly sophisticated mathematical concepts to explain and work mathematically with the technical skills of useful mathematics.

2.9 Graphical Representation of the Addition Rule

The graphical representation of the addition rule is interesting. Historically it gives an excellent elementary application of the parallelogram law of addition of the 'cross-totting' number-
families as vectors. Modernly it enables pupils to use the idea of a vector in a step towards the language of modern (linear) algebra. We require a graphical representation of the sum of a pair of 'cross-totting' number-families, using any chosen representative members for performing the addition.

Example:

The sum of the 'cross-totting' families \( \mathbf{2,0} \) and \( \mathbf{0,3} \) is illustrated (Fig 2.7) using representative members \( (5,3) \) and \( (1,4) \) respectively.

![Diagram showing the addition of vectors]

The addition rule will have been tested for compatibility: graphically the result of combining any pair of representative members of the 'cross-totting' number-families to be added will give a point, by the parallelogram law of addition of the position vectors for the selected representative members, which lies on the unique answer 'cross-totting' number-family line: in this case \( \mathbf{0,1} \).

The properties of commutativity, associativity, identity for addition, additive inverses, can all be similarly tested graphically. The addition of, for the pupil, the 'imaginary'
positive and negative whole-numbers has been meaningfully represented geometrically.

2.10 Subtraction as Inverse Addition

Subtraction is defined as addition of the additive inverse. The lesson sequence [5.2] is designed to give pupils the experience of attempting a solution to the problem posed in §2.1 now phrased in the form '5 from 3 you cannot do'. By using the isomorphism under addition of the 'cross-totting' number-families, the [something, 0]s, with the natural numbers, the problem is interpreted as: [3,0] - [5,0] in the new system.

Just as in a physical take-away situation in the arithmetic of natural numbers we use 'shop-keeper's addition': ie to take 3 from 5 we look for a natural number which added to 3 gives 5; so now in order to perform:

[3,0] - [5,0]

we look for a 'cross-totting' number-family which added to [5,0] gives [3,0]. The discussion involved in the search for such a family in the new system provides a lot of mathematical thought on the part of the pupils and with carefully selected questions, the author has almost never failed to get a positive response from at least one member of even low-ability classes. This, it is suggested, makes the work very worth-while: it stretches the more able pupil and it does not do the weaker members of a class any harm to make some attempt for a wee while! Having done a certain amount of manipulating and having used a rather random hit-and-miss method of finding a solution to the problem, pupils are invited to perform the corresponding addition sums in which the subtrahends are replaced by their additive inverses.
The teacher must of course be able to justify for himself his definition of subtraction if he is to avoid leading his pupils astray and keep them along a sensible track by a sensible sequence of questions. He will require for his background knowledge a justification on the following lines:

Let \( a, b \) denote elements in any universal set \( E \) in which a binary law of addition is defined. It is assumed that \((E,+)\) forms a group. We require to find \( x \in E \) such that:

\[
x = a - b
\]

We know (shopkeeper's addition) that:

\[
x + b = a
\]

\[
\Rightarrow (x + b) + (-b) = a + (-b) \quad (-b \text{ is inverse of } b)
\]

\[
\Rightarrow x + (b + (-b)) = a + (-b) \quad \text{(associativity)}
\]

\[
\Rightarrow x + 0 = a + (-b) \quad \text{(inverse property)}
\]

\[
\Rightarrow x = a + (-b) \quad \text{(identity property)}
\]

ie \( x \) is obtained as inverse of \( b \) added to \( a \).

The definition of subtraction as inverse addition is valid in any mathematical system whose elements need not be numbers and which forms a group under addition. Attention here is drawn to the value of not having to take a different view of subtraction when dealing with a universal set of (say) vector elements.

It is evident that, as for addition so with subtraction, the results can be illustrated graphically as the vector sum of 'cross-totting' number-families.

If the introduction of the familiar traditional positive and negative labels for the 'cross-totting' number-families has been postponed, this could be an appropriate stage at which to dispense with the number-pair family notation. The author urges the reader to bear with her a little longer, however, for an interesting development of the number-pair method to a definition of multiplication for the directed whole numbers. This section of the work was unfortunately not video-tape recorded.
2.11 The 'Rule of Signs'

The 'rule of signs' for the multiplication of the directed whole numbers has always been difficult to motivate. At the early secondary stage there is no readily available concrete illustration of 'negativeness multiplied'. Elementary kinematics, even time and motion studies in a straight line, are not normally dealt with before the fourth year of a secondary mathematics course. More general rates of change in which a deceleration is regarded as a negative acceleration and the application of the formula:

\[
distance = velocity \times time
\]

and, more abstractly, the positive and negative 1st and 2nd order rates of change of a function \( y = f(x) \) in the differential calculus are more appropriate to the fifth and first year sixth-form levels. Area under a curve \( y = f(x) \) is treated as negative for negative \( y \) but integration is normally not taken until the fifth and sixth years of study.

Perhaps the earliest application of the product of two natural numbers is in finding the area of a rectangle or other rectilinear shape. But what pupil at second or third year secondary level is able to visualise a shape with negative area? Negative area can have no meaning for the average/low-ability pupil, at lower-/middle-secondary level, who all too often identifies area of whatever shape as 'length x breadth'. Our best hope of motivating multiplication involving 'negativeness' is to invent a strategy in our number game. We need a rule for multiplication that is 'sensible' - mathematically compatible with the equivalence relation on which the 'cross-totting' number-families are constructed.

The teacher will know that the 'cross-totting' number-families are, in fact, classes of ordered differences. The rule that we are looking for may be obtained from a consideration of:

\[ a, b \odot c, d \] as:
(a - b) \times (c - d) = ac + bd - ad - bc
= ac + bd - \frac{ad + bc}{ad + bc}

which is the ordered difference: \([ad + bc, ac + bd]\).

2.12 Multiplication of the 'Cross-Totting' Number-Families

The following approach was attempted by the author with a small group of above-average 12 year-old girls at a school in Nottingham. After reaching subtraction in the game of 'cross-tots' with the whole class, twelve of the girls were enthusiastic and keen to proceed with the development to multiplication in time out-of-school!

The strategy used for defining a rule for the addition of two 'cross-totting' number-families is repeated. It is suggested as a starting point that we try to arrange for the 'cross-totting' number-families: the \([\text{something, 0}]\)'s to be isomorphic with the natural numbers under multiplication. After all, they do occupy the same positions, even if they are not exactly the same, as the natural numbers on the extended natural-number line. This condition dictates that just as:

natural number \times natural number = natural number

so now \([\text{something, 0}] \times [\text{something, 0}] = [\text{something, 0}]\).

Some pupils will try to be awkward, even obstinate. If they insist on trying to get an isomorphism with the \([0, \text{something}]\)'s it is advisable to go along with them - having ascertained in advance that a perfectly viable mathematical multiplication rule can be defined but that results may not give the conventional rule of signs. If no such viable alternative exists, pupils can always be invited to see how far they can get.
For isomorphism with the $3, 0 \otimes 2, 0$ which, it is agreed, behaves like: $3 \times 2$. We have: $3 \times 2 = 6$ which corresponds in the isomorphism to $6, 0$ and so:

$$3, 0 \otimes 2, 0 = 6, 0$$

We next try to define a rule for multiplying together two 'cross-totting' number-families using any two representative members. Obviously, the simplest members $(3, 0)$ and $(2, 0)$ suggest:

$$(3, 0) \otimes (2, 0) = (3 \times 2, 0 \times 0) = (6, 0)$$

Invariably this must elicit the suggestion:

$$(a, b) \otimes (c, d) = (a \times c, b \times d)$$

It is easy to produce 'un-sensible' counter-examples: incidentally a very important and valuable teaching technique which almost never fails to convince pupils of the viability of a mathematical fact.

eg $(5, 2) \otimes (3, 1) = (5 \times 3, 2 \times 1) = (15, 2) \notin 6, 0$

Reverting to a consideration of multiplication of the natural numbers, a closer look is made of multiplication defined as repeated addition. Care is needed in getting the pupils to interpret their understanding of:

$3 \times 2$ as '3 multiplied by 2': ie '2 lots of 3'
or $3 \times 2$ as '3 times 2': ie 'thrice 2'

It may be necessary to consider separately the two interpretations using repeated addition:

$3 \times 2$ as 2 lots of $3 = 3 + 3 = 6$

$3 \times 2$ as thrice 2 $= 2 + 2 + 2 = 6$.

Using isomorphism in the new system we see that:
\[3,0 \otimes 2,0\] is interpreted as \(3,0 + 3,0\)

or as \(2,0 + 2,0 + 2,0\)

Using representative member-pairs we can calculate \(3,0 \otimes 2,0\) as:

repeated addition using member-pairs \((9,6), (7,5)\) giving:

\[(9,6) \otimes (2,0) = (9,6) + (9,6) = (18,12) \in 6,0\]

[Notice that both parts of \((9,6)\) are multiplied by the 2 of \((2,0)\)]

or \((3,0) \otimes (7,5) = (7,5) + (7,5) + (7,5) = (21,15) \in 6,0\]

[Notice that both parts of \((7,5)\) are multiplied by the 3 of \((3,0)\)]

Multiplication using the correspondence to repeated addition in the isomorphism of the natural numbers with the

something, 0's appears to be compatible. Pupils can verify that this is so by selecting different member-pairs with which to perform the multiplication, with the challenge to try to produce a counter-example if they doubt it!

We are however not much nearer to a rule for multiplication using representative member-pairs. The teacher may wish to generalise the position reached so far: we have \([a,b] \otimes [c,d]\) as:

(i) \((a,b) \otimes (x,0) = (ax,bx)\)

\((y,0) \otimes (c,d) = (yc,yd)\)

and the results are 'sensible': ie mathematically compatible.

It is suggested that next we try: \(3,0 \otimes 0,2\) as:

\((3,0) \otimes (0,2) = (0,2) + (0,2) + (0,2) = (0,6) \in 0,6\)
Using representative members \((3,0), (5,7)\) (say), we have:

\[
(3,0) \otimes (5,7) = (5,7) + (5,7) + (5,7) = (15,21) \in \mathbb{Z}
\]

[Notice that both parts of \((5,7)\) are again multiplied by the 3 of \((3,0)\)]

Is it then too much to encourage thought towards \(3,0 \otimes 0,2\) as:

\[
(9,6) \otimes (0,2) = (6,9) + (6,9) = (12,18) \in \mathbb{Z}
\]

for compatibility 'by reversing the order' of the parts 9,6 before multiplying by the 2 of \([0,2]\)?

We now have:

\[
(ii) \quad (a,b) \otimes (0,y) \text{ as } (by,ay).
\]

For (i): \((a,b) \otimes (x,0)\) and (ii): \((a,b) \times (0,y)\) all possible products of the elements \(ax, bx, ay, by\) appear to contribute to the answer. The author did get one, but only one, girl in that first experimental group to arrive at:

\[
\begin{align*}
(a,b) &\otimes (x,0) \text{ as: } (a,b) \otimes (x,0) = (ax+by, ay+bx). \\
\end{align*}
\]

The obvious next step in systematising the thinking is to combine (i) and (ii) and agree to try to ensure *distributivity of multiplication over addition*. After all, this is merely a step towards 'respectability' of the operation. We then have:

\[
\begin{align*}
\sum (a,b \otimes (x,0) + (0,y)) &= \{(a,b) \otimes (x,0)\} + \{(a,b) \times (0,y)\} \\
&= (ax, bx) + (by, ay) \\
&= (ax+by, bx+ay)
\end{align*}
\]

Notice, here, that we are also using commutativity of multiplication to achieve the desired result. 'Respectability' is being assured in the development of a 'sensible' rule.
The method does stretch the more able pupils, but this surely is not a bad thing. In the opinion of the author, if it can be done at all, it is worth spending a little while to provide the opportunity for mathematical thought. It need not discourage the slower learner. The time involved is not excessive.

A student in a written examination for a higher degree remarked that mathematics here is "being presented from the top down and the more sophisticated ideas do not impinge on the less able pupil, whereas the elementary simplistic approach lacks the spark of mathematical creativity that one likes to see". Tony Wilson specialises in the teaching of mathematics to the less able and remedial pupils. If the approach proves unsuccessful we have, working downwards as he suggests, the well-tried traditional methods outlined for example in the Nuffield Mathematics Teaching Projects [6.3]: after translating the 'cross-totting' number-family names to the familiar symbols with positive (*) and negative (−) labels, the familiar 'rule of signs' can be obtained from:

- acceptance of isomorphism of the positives with the naturals under multiplication
- acceptance of distributivity of multiplication over addition
- use of the identity for addition, the zero, of the new system which corresponds to the natural zero in the isomorphism
- use of the corresponding pairs of additive inverses which are readily seen to add up to the zero of the new system.

In short: we can obtain, for example, \(^*3 \times (-2 + ^*2)\) by evaluating \(^*3 \times (-2 + ^*2)\) in two ways, in the form of a commutative rectangle:

\[
\begin{align*}
^*3 \times (-2 + ^*2) & \iff (^*3 \times -2) + (^*3 \times ^*2) \\
& \downarrow \text{by distributivity} \\
& \downarrow \text{by inverses} \\
& ^*3 \times 0 \iff (^*3 \times -2) + ^*6 \\
& \downarrow \text{by isomorphism} \\
& 0 \iff (?) + ^*6
\end{align*}
\]
6 is the only possible interpretation of \((3 \times -2)\) to replace the question mark (?) if the answer is to agree with zero obtained on the left-hand side.

This method, too, may be too difficult for all but the average to good-average pupil, but again it is stressed that it can do no harm to the rest of a class to witness the process.

2.13 Alternative Methods of Multiplication of the Directed Whole-Numbers

It should be remarked perhaps that the author does advocate working in 'setted' classes for mathematics! Ultimately we still have the familiar pattern of the multiplication tables plotted for the positives on a rectangular frame of reference consisting of two perpendicularly extended natural-number lines and completed backwards, across and down, in sequential pattern. But let the reader be warned! The author has never ever yet succeeded in getting all the members of a class to arrive at the same conventionally accepted pattern!!

It is perhaps worth commenting on yet another alternative approach which is to be found in traditional courses with adequate and reasonably acceptable explanation and also in the modern School Mathematics Project texts [4.2]. In the latter there is a serious possible source of confusion for the pupils when invited to treat the multiplier as a scale factor operator. The SMP team have, in all their early secondary work directed the teacher and the pupil towards an experience of set language, the use of the algebra of sets and the properties of a group structure. Universal sets of elements, which need not be numbers - sets of geometrical transformations, for example, have been considered - have been
associated with a *binary operation* defined so as to give closure within the system. The author, not unnaturally perhaps, *expected* the product of two directed whole-numbers to be defined as a binary operation resulting in a directed whole-number answer. But the reader of the text, who could be the pupil, is asked to use a *positive or negative scale factor* as an operator acting on and enlarging a single element in the universal set of directed whole-numbers. It will be seen that fractions will have already been dealt with and the scale factor is allowed to take positive and negative fractional values, thus giving an enlargement or dilatation of the *multiplicand*, which is a directed *whole-number*.

The approach is, of course, perfectly valid and mathematically viable. But it needs to be made clear to the reader of the text that he is no longer working within a system which is a group under a binary operation of multiplication. Instead, the SMP team has, inadvertently (?), taken the reader into a mathematical system forming a vector space in which the directed whole-numbers are the vector elements of the universal set and in which a permitted operation is the *unary operation* of multiplication by a *scalar* element selected from the *real number field* outside of the universe of vectors. This aspect of the work is developed in Part II where the experience of a *Boolean algebra of sets* in the situation of a group structure is extended to the experience and use of the language of a *vector space* and a modern *linear algebra*.

2.14  *A Student's Comments*

The following use of a linear algebra situation was suggested by Tony Wilson (page 32) when working with pupils in preparation for the examination for the degree of Master of Arts. He illustrates a modelling of multiplication of the directed whole-numbers as shifts along a number line. Imagine, if you will, the situation of a skating competition with
international competitors and judges. The judges are to award points within the range -5 to +5. The marks awarded to one of the skaters are as follows:

<table>
<thead>
<tr>
<th>Judge</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marks awarded</td>
<td>+3</td>
<td>+2</td>
<td>+2</td>
<td>-1</td>
<td>+6</td>
</tr>
</tbody>
</table>

It is discovered that judge D had taken a bribe to force down the total score: his mark is removed as invalid giving a revised total score of +7.

Multiplication is then introduced as a unary operation by extending the process of removing scores of 'crooked' judges. If, for instance, judges B and C have their scores discounted, we have to 'take away' two scores of +2:

\[- (2 \times +2) = -(4)\]

A score of -4 has to be taken away from +6 leaving +2. Had the judges B and C in question awarded negative scores then their removal would have increased the total: we would have taken away two scores of -2:

\[- (2 \times -2) = +(4)\]

and a score of +4 would be added to the total.

Although decisions have to be made as to whether in fact a positive score is labelled (+) or left simply as a corresponding natural-number symbol and recognising that possible confusion of the use of the minus sign (-) for 'take away' with the label (-) for a negative score can arise, the student does emphasize that the explanation 'works' and that he is able to "convince pupils of the validity of the rule of signs obtained" as a result of the exercise.
2.15 **Group Structure Under Multiplication Denied**

If we now return to our rule for multiplication on the 'cross-totting' number-families, the rule can be tested for the remaining properties of respectability. As for addition so with multiplication: we can test for the respectability of the 'cross-totting' number-family system under multiplication: commutativity has been assumed, as also has the distributivity of multiplication over addition, in the search for the rule; associativity exists. We can establish also the existence of a unique *multiplicative identity* as the 
\[1,0\] 'cross-totting' number family. But we are denied the existence of unique *multiplicative inverses* and hence of the pattern of group structure which would have enabled the operation of division as inverse multiplication. This problem is considered further in Chapter 3. But first it is appropriate to consider the geometrical representation of the multiplication rule for the 'imaginary' negativeness.

2.16 **Graphical Representation of Multiplication**

As for addition and subtraction, the rule for multiplication may be represented geometrically as a graph quite simply using the simple 'cross-totting' number-family names: *something, 0* and *0, something*. Using the extended natural-number line as axes at right angles, the binary operation of multiplication for the four possible combinations of positive/negative may be illustrated in Fig 2.8:
The teacher will see that in each case, using similar triangles OIX and OYP we have:

\[
\frac{OX}{OI} = \frac{OP}{OY}
\]

where vectors \( \overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OI}, \overrightarrow{OP} \) are used to represent the 'cross-totting' number-families \( x,0 \), \( y,0 \), \( 1,0 \), \( xy,0 \) or their negative partners.

Whence:

\[
\overrightarrow{OX}.\overrightarrow{OY} = \overrightarrow{OP}.\overrightarrow{OI}
\]

\[
\text{ie: } \overrightarrow{OX}.\overrightarrow{OY} = \overrightarrow{OP}
\]

and vector \( \overrightarrow{OP} \) is seen to represent the product* of vectors \( \overrightarrow{OX}, \overrightarrow{OY} \) in magnitude and sign. When translated into terms of the 'cross-totting' number-families which the vectors represent we have the geometrical representation of the rule of signs:

* The product of vectors thus illustrated must not be confused with other products (inner product, cross product) of vectors in general
A consultant of the Nuffield Mathematics Teaching project, the late Donald Mansfield and author of the pioneer texts [10], indicated that there appeared to be no graphical representation of the multiplication rule for the 'cross-totting' number-families of numbers in terms of representative member-pairs. The following was prepared by the author and was in fact similar to that produced independently in Donald Mansfield's guide to the television school broadcast series Maths Today in about 1965.

Consider again the example \( \begin{array}{cccc} 2 & 0 & \times & \begin{array}{cccc} 3 & 0 \end{array} \end{array} \) of §2.12 which will now be illustrated, using representative member-pairs \((3,1), (5,2),\) as: \((3,1) \times (5,2).\) We take as axes, a pair of perpendicular natural-number lines against which the first and second number elements of the number-pairs are plotted.

Note that: 
\[
(3,1) \times (5,2) = (3,1) \times [(5,0) + (0,2)] \\
= [(3,1) \times (5,0)] + [(3,1) \times (0,2)]
\]
using the distributivity of multiplication over addition already established, or rather agreed upon.

(i) First we represent \((3,1) \times (5,0)\) using repeated addition as \((15,5)\)

(ii) Next we represent \((3,1) \times (0,2)\) using repeated addition (reversed) as \((2,6)\).

The product is obtained as the sum of these two enlargements using the parallelogram law of addition of §2.9 and on page 32 (see Fig 2.9).
The author would stress again the value of the use that is made of the geometrical representation of the 'imaginary' for the pupils – of algebraic concepts, in this case the directed whole-number positives and negatives as vectors, and thus laying the foundations for mathematics that is to come whilst teaching a useful topic of the lower-secondary curriculum.

2.17 Conclusion: A Look Ahead to the Rational-Number System

As far as the author is aware, only one other modern project director, besides the Nuffield Project, Raymond Heritage of the Shropshire Mathematics Experiment introduces the directed whole-numbers as equivalence classes of ordered differences of natural numbers. But he stops short at the definition and then proceeds with a more or less traditional presentation of the operations on the directed numbers themselves.
The reader is urged to consider carefully, in the work so far discussed, the way in which the language of modern algebra is used to describe the pattern of a mathematical group structure and to illustrate the properties geometrically in the presentation of a perfectly normal mathematical topic in which the pupils are invited to create a mathematical system to answer the age-worn problem: 5 from 3 you cannot do.

In the next chapter, the same approach is adopted to introduce pupils to rational numbers as equivalence classes of fractions. The same language is used to describe the same patterns, or rather structures, in mathematics and to represent those patterns, or abstract ideas, geometrically in the presentation of an elementary topic which all too often in the minds of the secondary teachers has been spoiled by teachers of the lower age-groups.
3.1 Elementary Fractions - The Pitfalls

The term rational number is seldom used in elementary mathematics courses in schools. Rational numbers are commonly called fractions. Mathematically speaking a fraction is just one of many possible representations of a rational number. An answer \( \frac{4}{10} \) (say) to a problem is often indicated as wrong by a cross (x) placed against it by the teacher. The pupil will surely be confused. It is, in fact, mathematically correct but given in a form not usually accepted due to a lack of experience with equivalent fractions.

A pupil's knowledge of fractions, on entering the secondary phase of his mathematical education, may be more hazy than the teacher has been prepared to admit. At the primary level, a fraction will have been associated almost entirely with 'cheeseboard' sharing experiences or with measurement and a rough approximation to measurement. But 6-sixths, for example, may not always convey the idea of a whole; and, in measurement, only the simplest of fractions:

\[
\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \text{ (possibly), } \frac{1}{10}, \frac{1}{100} \text{ (possibly not)},
\]

will have been used with very little understanding of the meaning of these terms and symbols.
The addition of fractions is another source of possible confusion for the pupil who often has no understanding whatsoever of the part played by the lowest common multiple in determining the lowest common denominator. And, in both the addition and multiplication of mixed numbers, experience of commutativity and associativity and less insistence on an artificial formal layout would help to avoid confusions between, say,

\[
\frac{5}{8} + \frac{2}{7} = 5 \frac{35}{56} + 16
\]

which is, of course, correct; and

\[
\frac{5}{8} \times \frac{2}{7} = \frac{610}{56}
\]

which is absolute rubbish. The processes involved are rarely understood by any but the more able pupil and must appear as imaginary artifices in the minds of the majority of the pupils.

The use of the word 'of' is often misunderstood. We have all, as teachers, had pupils who insist that 'to find the half of something' the of means divide: \( \frac{1}{2} \) of \( \frac{5}{8} \), for example, may often be translated into: \( \frac{1}{2} \div \frac{5}{8} \) ! The pupil very often thinks: 'divide by 2' but cannot put down on paper what is in his mind: the rule 'invert and multiply' for division has to be reversed and applied backwards. How can we expect him to translate:

\[
\frac{1}{2} \text{ of } \frac{5}{8} \text{ to } \frac{1}{2} \times \frac{5}{8}
\]

from the idea of: \( \frac{5}{8} \div 2 \) as \( \frac{5}{8} \times \frac{1}{2} = \frac{1}{2} \times \frac{5}{8} \)! Commutativity of multiplication, too, is involved somewhere in the mental process.

For these, and other reasons, the author breaks with the customary sequential development of elementary mathematics in schools and, at secondary level, deals with the arithmetic of fractions as rational numbers after the arithmetic of the
The directed whole-numbers of Chapter 2. This procedure has the added advantage: she can avoid having 'her thunder stolen' by treating early experiences of the same concepts in a different way and in one which is often unrecognisable to the pupils, at least until many of the mathematical properties have been re-established.

In the 'cross-totting' number-pair approach to the integers in Chapter 2 the 'cross-totting' number-families were not recognised in the film sequence as the 'plus-and-minus numbers' until the number-families were actually given their traditional positive-and-negative names - even though the class teacher had left a large wall-chart of the extended natural-number line pinned beside the blackboard and clearly visible in the film!

Flexibility in the order of presentation of mathematical concepts is however essential [5.1] and the author, when required to do so, is not averse to varying the sequence as in the audio-tape recorded lesson described later in this chapter (§3.12) in which fractions are taught before the directed-whole numbers.

3.2 The Need to Extend the Integer-Number System

The author urges that pupils should be encouraged to recognise pattern in mathematics, to represent that pattern graphically and geometrically, to describe it in the symbolic language of algebra, and whenever possible to generalise and extend it to more sophisticated patterns. This has been the aim in the work so far. The pattern which underlies the structure of the system of natural numbers under the operations of addition and multiplication lacked the existence of a unique additive identity element and corresponding pairs of unique additive inverses which denied a definition of subtraction. The system was extended [Chapter
2) to that of the integers which illustrates the pattern of a group structure under addition. A unique additive identity element, zero, and corresponding unique pairs of additive inverses exist. Subtraction is defined as inverse addition within the system. The integer system forms a group under addition. Incidentally, addition is commutative and the group is termed a commutative group.

It was seen that the binary operation of multiplication could be defined within the system. The system does not however form a group under multiplication. It lacks the existence of unique pairs of multiplicative inverses corresponding to the multiplicative identity, the unit element $1$. The operation of division as inverse multiplication cannot be defined within the integer system. An attempt must be made to extend the integer-number system to a system in which division can be performed.

The elementary fractions experienced in the primary years provided the clue to a possible solution. When used for measurement exercises they are the names of points which are placed between the natural-number representations on the natural-number line. Another look at the now extended natural-number line (§2.7) will reveal 'gaps' which need to be filled. The rationals now to be defined for the extension of the integer-number system will be used as names of points to fill the gaps in the integer-number line to form its extension to the rational-number line. The scene is again set for a mathematical game to enable pupils to construct the rationals as ordered pairs of integer-number families under a 'cross-product' equivalence relation.

This time the arena is the universal set $E$ of ordered pairs $(x,y)$ of integers (cf Chapter 2, pp 20-21). A relation $R$ is defined on $E$: 

$$(a,b) R (x,y) \iff a \times y = b \times x \text{ for all } (a,b), (x,y) \in E.$$
The relation R is again an equivalence relation: the commutative, associative and cancellation properties of the integers under multiplication ensure that the relation R is:

(i) reflexive: \((a,b) R (a,b)\) since \(a \times b = b \times a\)

(ii) symmetric: \((a,b) R (x,y) \Rightarrow a \times y = b \times x\)  
\[\Rightarrow b \times x = a \times y\]
\[\Rightarrow x \times b = y \times a\]
\[\Rightarrow (x,y) R (a,b)\]

(iii) transitive: for \((c,d)\) also in E we have:  
\((a,b) R (x,y)\) and \((x,y) R (c,d)\)
\[\Rightarrow a \times y = b \times x\) and \(x \times d = y \times c\)
\[\Rightarrow a \times y \times x \times d = b \times x \times y \times c\]
\[\Rightarrow a \times y \times x \times d = b \times x \times y \times c\]
\[\Rightarrow a \times d = b \times c\]
\[\Rightarrow (a,b) R (c,d)\]

The equivalence relation R again partitions the universal set E into equivalence classes: disjoint sets of ordered pairs of integers. These sets of ordered pairs of integers are the pieces in the game of 'cross-products', to be manipulated according to rules chosen by the pupils for performing mathematical operations in a new number system.

3.3 The Construction of the Rationals as Equivalence Classes of Ordered Ratios of Directed Whole-Numbers

The method of presentation and teaching style follow the lines similar to that described in Chapter 2 for the construction and development of the integer-number system. The reader is referred to a dissertation prepared by Graham John Carlile in part requirement for the degree of Master of Arts of the University of Hull under the author's
supervision* [11], and entitled 'The Rational Extension of the Integer Number Line - An Alternative Approach to the Teaching of Fractions in Schools'. Mr Carlile acknowledges that the author made available "some of her unpublished seminar material on rational numbers" and that "the work on rational numbers...... was taught to children in Mathematics Group IX at Springfield Middle School in Grimsby....". The work is "recommended for use with children aged 12 and 13 although the writer (Mr Carlile) has used it successfully with able middle school children of 11". In the 1960's the Nuffield Mathematics Teaching Project recommended that directed numbers should be taught before fractions [6.3, 6.4] (see page 51). Mr Carlile demonstrates that "the rationals form a more complex (imaginary?!) construction than the integers and consequently are best taught after the integers".

With his class of pupils Mr Carlile starts from everyday situations with examples of ordered pairs of positive integers, for example, milk deliveries daily:

- at my house: 1 pint per day
- at my neighbour's house: 3 pints per day.

During one week this can be represented as ordered pairs:

(1,3); (2,6); (3,9); (4,12); (5,15); (6,18); (7,21).

An example in the school timetable could be the lessons of Music per week to the lessons of English: for example (2,10) and for successive weeks this could be recorded as:

(2,10); (4,20); (6,30); (8,40); (10,50)....

Children should be encouraged to notice a relationship between member elements of the pairs themselves; i.e. for the number-pair (a,b), b = 5a. This relationship can be compared with the ordered differences for integers as ordered pairs of naturals (a,b), (c,d) in which a + d = b + c.

* and has allowed free quotation and use of pupils' work included as an Exhibit
Many more examples using physical objects such as toys, blocks, etc, ..., are described in the Nuffield Mathematics Teaching Project [6.4] and enable pupils to construct the cross-product families as ordered ratios of integers. Much practice is needed by the pupils in listing families of ordered pairs of equivalences starting from concrete situations. Mr Carlile develops the work further with the use of Cuisenaire rods [12] which

"vary in length from 1 cm to 10 cm and also in colour:

<table>
<thead>
<tr>
<th>Length</th>
<th>Colour</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 cm</td>
<td>white</td>
</tr>
<tr>
<td>2 cm</td>
<td>red</td>
</tr>
<tr>
<td>3 cm</td>
<td>green</td>
</tr>
<tr>
<td>4 cm</td>
<td>pink</td>
</tr>
<tr>
<td>5 cm</td>
<td>yellow</td>
</tr>
<tr>
<td>6 cm</td>
<td>dark green</td>
</tr>
<tr>
<td>7 cm</td>
<td>black</td>
</tr>
<tr>
<td>8 cm</td>
<td>brown</td>
</tr>
<tr>
<td>9 cm</td>
<td>blue</td>
</tr>
<tr>
<td>10 cm</td>
<td>orange</td>
</tr>
</tbody>
</table>

Consider, for example, the pair of rods (dark green, blue) and use white as a measuring rod. This would give the ordered pair (6,9):

\[(\text{dark green, blue}) \rightarrow (6,9)\]

**Fig 3.1**

If light green is used as the measure, we have the ordered pair (2,3):

\[(\text{dark green, blue}) \rightarrow (2,3)\]

**Fig 3.2**

Children quickly appreciate that many ordered pairs can be generated from a given pair of rods as illustrated in the children's work reproduced below.
We took 2 rods, a dark green one and a blue one. We placed them against some white rods and we came up with (6,9). We changed the white rods for brown rods and we get the ordered pair (48,72). Here are some other ordered pairs other people got with their colours.

Yellow (5), gave us (30,45)
Pink (4), gave us (24,36)
Dark green (6), gave us (36,54).

This work soon leads on to the idea that each number-pair can be written in the form \((2 \times \square, 3 \times \square)\) where \(\square\) represents the numerical value of any rod length. Cuisenaire rods are a useful tool to make this point as seen in the child's work below:

\[(\text{Blue, yellow})\text{ is the ordered pair } (9,5)\]

\[\begin{align*}
(9 \text{ reds, 5 reds}) &= (9 \times 2, 5 \times 2) = (18,10) \\
(9 \text{ light greens, 5 light greens}) &= (9 \times 3, 5 \times 3) = (27,15) \\
(9 \text{ pinks, 5 pinks}) &= (9 \times 4, 5 \times 4) = (36,20) \\
(9 \text{ yellows, 5 yellows}) &= (9 \times 5, 5 \times 5) = (45,25) \\
(9 \text{ dark greens, 5 dark greens}) &= (9 \times 6, 5 \times 6) = (54,30) \\
(9 \text{ blacks, 5 blacks}) &= (9 \times 7, 5 \times 7) = (63,35) \\
(9 \text{ browns, 5 browns}) &= (9 \times 8, 5 \times 8) = (72,40) \\
(9 \text{ blues, 5 blues}) &= (9 \times 9, 5 \times 9) = (81,45) \\
(9 \text{ oranges, 5 oranges}) &= (9 \times 10, 5 \times 10) = (90,50)
\end{align*}\]

Clearly, 'larger' members can be found by using larger multiples of 9 and 5; for example:

\[\begin{align*}
(9 \times 17, 5 \times 17) &= (153,85) \\
(9 \times 20, 5 \times 20) &= (180,100)
\end{align*}\]
In order to find a simpler member of a given family we can factorise each component and then ignore the factors which the two numbers have in common [6.4], for example:

\[(12, 18) = (3 \times 4, 3 \times 6) \rightarrow (4, 6)\]
\[(4, 6) = (2 \times 2, 2 \times 3) \rightarrow (2, 3)\]
\[(24, 30) = (2 \times 2 \times 2 \times 3, 2 \times 3 \times 5) \rightarrow (4, 5)\]

Throughout these exercises answers can be checked for equivalence by using the cross-product test."

The technique of course gives 'useful' experience of 'cancelling-down' and may be used as an alternative way of constructing the new-number 'cross-product' families.

As mentioned on page 55 the author's own unpublished material based on the work of the Nuffield Mathematics Teaching Project [6.4] and published after her own trials and experiences when working with the Shell Centre for Mathematical Education] was used by Mr Carlile. She now continues with her own development in the form of a game of cross-products which enables pupils to experience the concept of a rational number before these numbers are used to label points which fill the gaps on the integer-number line.

3.4 The Game of 'Cross-Products'

The pupils are given a rule which enables them to construct families of ordered pairs of integers under the equivalence relation R defined on page 53. For the pupils, R is replaced by 'cross-multiplies'. This form of words gives them a rule which enables them, in general:

(i) to find as many integer-number pairs \((x, y)\) as they please which 'belong to' the same family as a given integer-number pair \((a, b)\), except for \(x = 0\) OR \(y = 0\).
(ii) to ascertain that:
(a) any given \((x, y)\) can be put into a definite and unique 'cross product' family, except for \(x = 0\) OR \(y = 0\)

(b) any given \((x, y)\) cannot belong to more than one family except \(x = 0 = y\)

A given:
\[ \begin{array}{c}
\text{cross-multiplies: } \begin{array}{c}
\times \\
\times
\end{array}
\end{array} \]

\[ \begin{array}{c}
a \\
b
\end{array} \]

if and only if \(a \times y = b \times x\)

For any given non-zero integers \(a, b\) and any integers \(x, y\):

\[(x, y) \mathcal{R} (a, b) \Rightarrow x \times b = y \times a\]

The only possible solutions are given by:

\[x = k \times a, \ y = k \times b\] for all integral \(k \neq 0\)

giving: (i) as many \((x, y) = (ka, kb) \mathcal{R} (a, b),\) \(k \neq 0,\) as we please. Conversely, by taking \(k = 1,\) given \((x, y)\) we can find a unique \((a, b)\) such that:

\[\Leftrightarrow (x, y) \mathcal{R} (a, b) \Rightarrow x \times b = y \times a \Rightarrow x = ka, \ y = kb\]

giving \(x = a, \ y = b\) (where \(k = 1\)) as the simplest of all possible \((ka, kb)\)

Exceptionally when \(k = 0,\) \(x = 0 = y\) for all \((a, b).\) Case (ii) is contradicted and \((x, y) = (0, 0)\) belongs to every family \((a, b):\)

(a) no unique \((a, b)\) exists

(b) \((0, 0)\) belongs to infinitely many families

Incidentally, for given integers \(a = 0, \ b \neq 0,\) and any integers \(x, y,\) we have:

\[(x, y) \mathcal{R} (0, b) \Rightarrow x \times b = y \times 0 \Rightarrow x \times b = 0 \Rightarrow x = 0 \ \text{OR} \ b = 0\]
If \( b = 0 \): \((x,y) \, R \, (0,0)\) for all \((x,y)\) and \((x,y)\) belongs to every family of the universe.

If \( x = 0 \): \((0,y) \, R \, (0,b)\) and \( y = kb \) for all \( k \neq 0 \); we can again find as many \((0,y)\) as we please belonging to \((0,b)\).

Similarly for given \( a \neq 0, b = 0 \), we can find as many \((x,0)\) as we please belonging to \((a,0)\).

The pupils are again handling the idea of a partitioning of a universe under an equivalence relation. In the universal set of all ordered pairs of integers the equivalence relation defined as: \((a,b) \, R \, (x,y) \iff a \times y = b \times x\) determines a partition provided we exclude the element \((0,0)\) (see page 64).

As was seen in Chapter 2 the teacher will know that the equivalence relation which dictates the rule for our new game of 'cross-products' automatically ensures the partitioning of the universe of ordered pairs of integers.

The pupils can

(i) find as many pairs as they please belonging to a given family \((a,b)\) simply by multiplying both integer components of the pair successively by 2, 3, 4, ...

The family consists of representative member pairs: 
\((a,b); (2a,2b); (3a,3b); (4a,4b); \ldots\)

and is an example of 'cancelling-up' in work on fractions. Conversely, by cancelling-down, pupils can find the smallest representative member of the family to be used to name that family.

The pupils, given an ordered pair of integers, should

(ii) be challenged to find the family to which it belongs. They will see that this can be achieved by searching for common divisors in order to express \((a,b)\) as
(ka', kb'). To find the smallest member pair which is to name the family to which (a,b) belongs, they need the largest common divisor. There can be one, and only one, such value of k; and (a,b) can belong to one, and only one, 'cross-product' family.

Teachers will wish to compare the process with that in the game of 'cross-tots' (Chapter 2) used to introduce the integers. There, the smallest representative member-pair could always be written either in the form (x, 0) or in the form (0, y). We could always find an 'x' or a 'y' to fit the 'cross-totting' rule; but there was never more than one 'x' or more than one 'y' and never an 'x' and a 'y', except when x = 0 = y giving again only one family - the 'zero' of the system. Attention is drawn again here to the special cases in which x = 0, y = 0 for the 'cross-product' families to be discussed more fully in §§3.5, 3.6).

3.5 Graphical Representation

The 'cross-product' families may now be represented graphically* in various ways, as illustrated by the following examples of work of pupils in Mr Carlile's class shown overleaf.

It will be observed that the 'cross-product' family \(1, 3\), for example, is represented as the point of concurrence of a family of lines representing the member-pairs of the family. If the diagram is extended to include more families of 'cross-product' number-pairs, it will be found that all points representing the 'cross-product' number families will lie on the vertical line through the origin on the two parallel axes (Figures 3.3, 3.4).

---

* A Rationals Board has been produced by ESA Ltd, Harlow, Essex, but it is restricted to pairs of natural numbers.
Fig 3.3

![Diagram 3.3](image1)

Fig 3.4

![Diagram 3.4](image2)
The following diagram extends the representation further to include 'cross-product' number-families in which one component is positive, one negative (Fig 3.5).

The reader will find an interesting exploration of these graphical representations in Peter Vincent's article 'A Fraction Diagram Investigation' [13].

Figure 3.6 uses axes OX, OY at right angles to each other and will be found in most school texts on fractions; see for example [4.1]. Here the 'cross-product' number-families appear as sets of collinear points which represent the

* The axes were not named on the pupil's work
representative member-pairs. It will be seen that, if produced, all the lines would pass through the origin. The teacher is urged to take every opportunity in the discussion of these graphical representations to consider with the pupils the special nature of the 'cross-product' number-families \( x, 0 \), \( 0, y \), \( 0, 0 \) discussed on pages 59-60.

It was observed by one little group of these 12 year-old pupils that the line representing the \( \lfloor 1,1 \rfloor \) 'cross-product' number-family separated the families \( \lfloor x,y \rfloor \) in which \( x \) is greater than \( y \) from those in which \( x \) is less than \( y \). They then asked if one set of number-families was in any sense larger than the other set. They had 'discovered' an ordering of the 'cross-product' number families (§3.8).
By plotting the second number-component on the y-axis against the first number-component on the x-axis we do have a steadily decreasing ordering of the cross-product number-families in the conventional negative clockwise sense of rotation.

\[ \begin{array}{c}
(a, b) \preceq (x, y) \text{ according as } a \times y \preceq b \times x
\end{array} \]

(See page 73). Mr Carlile has suggested that his pupils should adopt the conventional representation of an ordered pair of numbers \((x, y)\) in a cartesian frame of reference. The author, however, would choose to plot the first number-component on the y-axis against the second number-component on the x-axis. The cross-product number-family \([a, b]\), later to be recognised as the rational number \(\frac{a}{b}\), is thus represented by a line with gradient \(\frac{a}{b}\) and is a valuable step in the direction of further development towards the idea of a derivative. She develops this representation more extensively in §3.8 on page 70. But first she returns to the arithmetic of the cross-product number-families.

3.6 The Multiplication of the 'Cross-Product' Number-Families

The definition of a 'rule' of operation on the 'cross-product' number-families proceeds on much the same lines as for the addition of the 'cross-totting' number-families of Chapter 2, §2.8, page 31. In the present case, however, it is multiplication that lends itself to the 'obvious' rule:

\[ \begin{array}{c}
[a, b] \times [c, d] = (a \times c, b \times d)
\end{array} \]

for combining together the two given 'cross-product' number-families \([a, b], [c, d]\).

Pupils are again encouraged to satisfy themselves that the rule is 'sensible' in the sense that it gives a unique product number-family answer. The teacher must satisfy himself that multiplication is well-defined, compatible or
consistent, with the equivalence relation on which the 'cross-product' number-families are constructed (see Mr Carlile's Chapter 5[11]). One of his pupils produced:

We are now checking that our rule is "sensible"

<table>
<thead>
<tr>
<th>1,2</th>
<th>⊗</th>
<th>1,3</th>
<th>=</th>
<th>1,6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,4</td>
<td>⊗</td>
<td>2,6</td>
<td>=</td>
<td>4,24</td>
</tr>
<tr>
<td>3,6</td>
<td>⊗</td>
<td>3,9</td>
<td>=</td>
<td>9,54</td>
</tr>
<tr>
<td>4,8</td>
<td>⊗</td>
<td>4,12</td>
<td>=</td>
<td>16,96</td>
</tr>
<tr>
<td>5,10</td>
<td>⊗</td>
<td>5,15</td>
<td>=</td>
<td>25,150</td>
</tr>
<tr>
<td>6,12</td>
<td>⊗</td>
<td>6,18</td>
<td>=</td>
<td>36,216</td>
</tr>
<tr>
<td>7,14</td>
<td>⊗</td>
<td>7,21</td>
<td>=</td>
<td>49,296</td>
</tr>
</tbody>
</table>

We find that our rule is sensible because whenever number pairs we chose from the same family, they all come to \( 1,6 \).

The pupils are then encouraged to test the multiplication rule for 'respectability': it is seen to be commutative and associative; there exist a unique identity cross-product number-family \( 1,1 \) for multiplication and corresponding unique inverse pairs of cross-product number-families \( x,y \), \( y,x \):

\[
(x,y) \otimes (y,x) = (x \times y, y \times x) \in 1,1
\]

except in the cases \( x = 0 \) OR \( y = 0 \) OR \( x = 0 = y \) (see pp59-60).

Pupils who may have met: the concept of additive and multiplicative identities in modular arithmetics as 'clock arithmetic' [5.2] or, in this day and age of digital clocks and watches, as 'perpetual calendars' (See Exhibit); the additive identity in the integer-number system under addition (Chapter 2); and even, perhaps, the identity element in systems of geometrical transformations here to be considered
later in Chapter 6; will normally have little difficulty
in 'discovering' the multiplicative identity for the
'cross-product' number-families under their chosen rule
for multiplication and in verifying that it is unique.
One of Mr Carlile's pupils writes [11]:

b. We are now trying to find the identity

e.g.

\[
\begin{array}{ccc}
3.5 \times 1.1 &=& 3.5 \\
6.2 \times 1.1 &=& 6.2 \\
7.9 \times 1.1 &=& 7.9 \\
8.3 \times 1.1 &=& 8.3 \\
\end{array}
\]

We find that \[1.1\] is the identity, as anything
multiplied by that will stay the same.

The search for multiplicative inverse pairs of 'cross-product'
number-families may present a problem. For a given \[a, b\]
we require a 'cross-product' number-family \[x, y\] such that:

\[a, b \times x, y = 1, 1\]

ie \[a \times x \times 1 = b \times y \times 1\]

One 'obvious' solution is given by \[x = b, y = a\]. This
suggests that \[b, a\] is a multiplicative inverse of \[a, b\].
Is it the only one? The teacher will need to know that the
only other solutions to: \[a \times x \times 1 = b \times y \times 1\] in \[x, y\]
are given by:

\[x = k \times b, \quad y = k \times a\]

for all \(k \neq 0\). Note that all ordered pairs of integers \((kb, ka)\)
where \(k \neq 0\) belong to the 'cross-product' number-family \[b, a\]
which is the unique inverse of \[a, b\]. In the particular
case when \(k = 0\), \((kb, ka) = (0,0)\) which we know to be a
member of every 'cross-product' number-family and must be
excluded from any 'respectable' system under multiplication if the inverse is to be unique (see pages 59-60).

We have now another example of a mathematical group structure. The system of 'cross-product' number-families \( x, y \neq 0, 0 \) forms a group under multiplication. It is now possible to define the operation of division on the 'cross-product' number-families as inverse multiplication in an exactly similar way to the definition of subtraction as inverse addition on the integer-number system. We do, however, have to exclude division by \( 0, 0 \) since \( 0, 0 \) has no unique multiplicative inverse.

3.7 Pupils' Comments

All the work so far contained in this chapter is admirably described in Mr Carlile's dissertation on "The Rational Extension of the Integral Number Line" with excellent examples of his pupils' work. The dissertation is catalogued in the Brynmor Jones Library of the University of Hull. The group of able 12 year-old pupils were asked to comment on various aspects of the work which was based on the Nuffield Mathematics Teaching Project [6.4]. Most pupils "appreciated the freedom to investigate their own ideas" and the following conversation was recorded:

Teacher: Did you find any of the work too easy or too difficult?

Glenn: I thought at first it was quite difficult, but I gradually got used to it and it got easier

Nicola: I didn't find it too easy or too difficult. It just needed thinking about carefully and sensibly

Jonathan: Although it was reasonably difficult, I am now finding it very useful

Teacher: What things did you like or dislike about the work?
Simon: I disliked having to put boxes around the families. It slowed you down.

Glenn: I thought that the work on cross-product families was very interesting. The only bad thing about it was that it went on a bit long and got slightly boring. But then we found out that we could change the pairs into fractions, \( \frac{1}{2} \times \frac{3}{4} = \frac{3}{8} \) and then it got more interesting again.

David: What I liked about it was that we could investigate other branches and rules if we wanted.

Teacher: Has the work helped your understanding of anything else in Mathematics?

Jonathan: It has helped me to understand multiplying fractions because if I can't do it in fraction form, I can change it into a number pair. It has also helped me understand positive and negative fractions.

Teacher: Did you prefer to work without a textbook? Why?

Susan: The work was different from textbook work a lot. For example in textbooks you get lots of sums, then you read on a bit further and the rule is written for you, if there is one. In the work we did on this subject with Mr Carlile we had chance to test different rules and understand why we used whichever rule we chose.

Jonathan: It was more easy to understand than from the book because it has been explained in more detail than from the book.

Glenn: It is very different from working from maths textbooks because in textbooks we can only do a certain amount in a certain order, but we can do this in our own time and as much as we want in any order. I find it much more interesting working without a textbook.

In order to overcome, for Glenn, the part of the work that was "slightly boring", it is advocated that the results could be satisfactorily achieved, and more expeditiously, if the pupils were to be arranged in small groups, each group undertaking one aspect of the work. For example, after the construction of some of the cross-product number-families, these could be plotted in different ways by different groups of pupils; those working more quickly could be asked to produce more 'cross-product' number-families for graphical
representation. The whole class could investigate the properties of group structure of the system under an operation, the pupils again working in small groups, each group testing one property only. The results could then be collected for the benefit of the class as a whole in plenary discussion.

3.8 The Ordering of the 'Cross-Product' Number-Families

Recently in 1985-86 the author has developed the work further. She returns now to the graphical representation of the 'cross-product' number-families discussed in §3.5 with classes of good-average and low-ability 11 to 12 year-old third year pupils in a junior high school, assisted by students on initial and in-service courses of teacher training.

The 'cross-product' number-families were represented graphically using axes OX, OY at right angles to each other as in Fig 3.6 on page 64. But this time it is the first number-component on the y-axis that is plotted against the second number-component on the x-axis, the alternative representation suggested on page 65 (Fig 3.7).

An attempt is made to use the graph as suggested by John Hunter and Martyn Cundy [14] to construct a chart of equivalent fractions. It is the intention here to use the chart in two ways:

(i) to fill in the 'gaps' in the integer-number line referred to in §3.2 on page 53 and to give 'meaningful' number names to the missing points

(ii) to simplify the rule for the addition of fractions.

The exercise formed the basis of the practical work for a group of in-service teacher trainees in their course for a higher degree.
The method of construction of the fraction chart, as suggested by Hunter and Cundy is difficult for both pupils and teachers. The following procedure was tried. First select a vertical through one of the second component-numbers on the x-axis. Let it meet the \( \frac{1}{1} \) 'cross-product' number-family line in P. This vertical segment will represent the unit of the system. Notice that each 'cross-product' number-family line divides that unit segment in proportion to the 'cross-product' number-family, which the teacher will recognise as the rational number which it represents.
Consider, for example, the vertical segment PM in Fig 3.8 through the second component-number 12. P represents the member-pair (12,12) and PM is, in length, a representative of the 'cross-product' number-family which is the unit of the system. It will be seen that the 'cross-product' number-family lines:

\[
\begin{align*}
(1,12) &; (1,6) &; (1,4) &; (1,2) &; (2,3) &; (3,4) &; (5,6) &; (1,1) \\
\end{align*}
\]

divide MP into the ratios:

\[
1:12; \ 1:6; \ 1:4; \ 1:3; \ 1:2; \ 2:3; \ 3:4; \ 5:6; \ 12:12
\]
at points:

\[
P_0; \ P_1; \ P_2; \ P_3; \ P_4; \ P_5; \ P_6; \ P_7; \ P;
\]

which represent the corresponding member-pairs:

\[
(1,12); \ (2,12); \ (3,12); \ (4,12); \ (6,12); \ (8,12); \ (9,12); \ (10,12); \ (12,12)
\]
respectively.
In this way it can be seen that the segment PM of selected unit length – in this case 12 integer units on the original y-axis scale – has been divided into 12 equal parts at its points of intersection with the 'cross-product' number-family lines:

\[ \frac{1}{12}; \frac{1}{6}; \frac{1}{4}; \ldots; \frac{11}{12} \]

The gap in the integer-line segment MP, for M representing (0,12) and P representing (12,12), has been filled by equally-spaced points which represent the corresponding member-pairs of these 'cross-product' number-family lines; namely:

\[ M = (0,12); (1,12); (2,12); (3,12); \ldots; (11,12); (12,12) = P \]

respectively. The segment has been filled with an ordered sequence of points which may be said to represent the 'cross-product' number-families and can be so 'named'. The construction would have been a very worth-while follow-up to the pupils' comments indicated on page 64.

Pupils have already remarked that it is possible to order the rational number families (page 64). They now experience a further graphical representation of the order relation. It may not be too difficult to invite them to algebraically fill the gaps and to insert a rational-number family between two such families. If we look again at a section of Fig 3.3 the rational-number families \( \frac{1}{1}, \frac{3}{4}\), \( \frac{1}{4}, \frac{5}{3}\) are seen as points of concurrence of lines joining components of member-pairs on a pair of parallel axes (Fig 3.9). The reader may like to insert the representative points for the \( \frac{1}{1}, \frac{1}{1}\) family and also for families \( a, b\) with a and/or b negative.
We see that: \( +1, \frac{3}{15} < \frac{4}{15}, \frac{5}{15} \) because \( +1 \times \frac{5}{15} < \frac{3}{15} \times \frac{4}{15} \).

It is required to choose \( p, q \) so that:
\[
+1, \frac{3}{15} < p, q < \frac{4}{15}, \frac{5}{15} \cdot
\]

By selecting appropriate member-pairs: say \( (\frac{5}{15}, \frac{15}{12}) \), \( (\frac{12}{11}, \frac{15}{10}) \) respectively for \( +1, \frac{3}{15} \), \( \frac{4}{15}, \frac{5}{15} \) we can choose \( p, q \) so that:
\[
(\frac{5}{15}, \frac{15}{12}) < (p, q) < (\frac{12}{11}, \frac{15}{10}) \cdot
\]

ie \( +5 \times q < +15 \times p \) and \( p \times +15 < q \times +12 \)

ie \( +5q < +15p < +12q \)

'Obvious' solutions are provided by \( q = \frac{15}{12} \) giving:
\[
+75 < +15p < +180
\]

and \( p \) takes the values: \( p = \frac{6}{12}, +7, \frac{8}{12}, +9, \frac{10}{12}, +11. \)

The rational-number families: \( +6, \frac{15}{12} ; +7, \frac{15}{12} ; +8, \frac{15}{12} ; +9, \frac{15}{12} ; +10, \frac{15}{12} ; +11, \frac{15}{12} \) lie between \( +1, \frac{3}{15} \) and \( +4, \frac{5}{15} \).

They can, of course, be represented graphically in Fig 3.10. For a given number of interpolations, an appropriate (equal) second-number component must be chosen. For an extension of this work the reader is referred to §3.13.
3.9 The Extension of the Integer-Number Line

We return again to the segment MP in Fig 3.8. The process is repeated step-by-step. The 'gaps' on MP can be filled by points of intersection with as many as we like of the 'cross-product' number-family lines. The corresponding points of division are given the names of their 'cross-product' number-families and projected in the direction of the axis $OX$, giving a construction for the extension of the integer-number line to the rational-number line. The work is based on a seminar with advanced course students. One member of the group, Mr Christopher Chipperton*, produced the projections (Fig 3.11 using the $(8,8)$ vertical representation of the unit integer-line segment.

Mr Chipperton then prepared a chart (Fig 3.12) which illustrates the complete 'fill-in' of all the gaps in the selected unit by points representing the 'cross-product' number-families. These he has now re-named using their simplest rational-number names as fractions. The extended integer-number line now appears as the rational-number line. The teacher will see that the unit $\left[\frac{1}{0}\right]$ of the integer-number line now corresponds to the unit $\left[\frac{1}{1}\right]$ of the rational-number line. The construction may be performed on the complete integer-number line. He was 'fired with enthusiasm' and went on to prepare a large demonstration equivalent fraction chart for use later with his own classes (see Exhibit).

* Mr Chipperton was on secondment as a non-graduate teacher of mathematics at the Ashton Voluntary-Aided Middle School, Dunstable
The author's own Fig 3.13 was constructed from Fig 3.11 and was used to reinforce the idea of equivalent fractions for a class of good-average third-year pupils at Brunswick Junior High School, Hull. A sequence of lessons was given in response to an appeal for a "demonstration of a method for teaching the addition of fractions in a meaningful way". The teaching time-schedule was very tight. There was no time to lay appropriate foundations for the author's preferred style of teaching and use of language. 'Fractions' had to be done before 'directed whole numbers'.

There now follows an outline of a lesson sequence as presented to that particular class. It is felt that the language and notation that had to be used in this instance is readily adaptable to the language and notation of the game of cross-products as advocated in this chapter.

3.10 A Lesson Sequence - Equivalences

The pupils first constructed their own charts of equivalent fractions. Blank formats (Fig 3.14) were distributed. The reader will see that the 'easy' aliquot parts: \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12} \) and their multiples have been indicated in their respective horizontal strips by heavy black lines. The 'more difficult' aliquot parts: \( \frac{1}{5}, \frac{1}{7}, \frac{1}{8}, \frac{1}{10}, \frac{1}{11} \) and their multiples, whose denominators do not divide exactly into (in this case) 12, have not been so indicated in their respective horizontal strips.

The pupils first cover the aliquot parts with coloured pieces of gummed paper, or they colour them using felt-tipped pens. The coloured pieces are labelled: \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \) as shown (Fig 3.15).

* Now available for microcomputer [S4.2]
They then look for equivalences and fill in the chart, thus:

(i) Start with the 1st strip coloured and labelled '1 UNIT'

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Equivalent to 1 unit?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Half</td>
<td>Two</td>
</tr>
<tr>
<td>Thirds</td>
<td>Three</td>
</tr>
<tr>
<td>Quarters</td>
<td>Four</td>
</tr>
<tr>
<td>Sixths</td>
<td>Six</td>
</tr>
<tr>
<td>Twelfths</td>
<td>Twelve</td>
</tr>
</tbody>
</table>

On the same line as the 1 UNIT strip, the pupils write:

\[ \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{6}{6} = \frac{12}{12} \]

(ii) Look at the 2nd strip in which one part is coloured and labelled \( \frac{1}{2} \). Place a ruler against the heavy black line at the end of the coloured piece labelled \( \frac{1}{2} \). Follow the ruler down until it meets the next heavy black line (as indicated by the arrow-heads in Fig 3.16):

in the 4th strip:

How many quarters are equivalent to 1-half? Two

in the 6th strip:

How many sixths are equivalent to 1-half? Three

in the 12th strip:

How many twelfths are equivalent to 1-half? Six

On the same line as the piece coloured and labelled \( \frac{1}{2} \), the pupils write:

\[ \frac{2}{4} = \frac{3}{6} = \frac{6}{12} \]

(iii) Look at the 3rd strip in which one part is coloured and labelled \( \frac{1}{3} \). Place the ruler and follow it down as before until it meets the heavy black lines:
in the 6th strip:
How many sixths are equivalent to 1-third? Two

in the 12th strip:
How many twelfths are equivalent to 1-third? Four

On the same line as the piece coloured and labelled \( \frac{1}{3} \), the pupils write:
\[
\frac{2}{6} = \frac{4}{12}
\]

(iv) Repeat for the 4th and 6th strips.

The charts now appear as in Fig 3.16.

Pupils then tabulate their results and are encouraged to fill in the gaps and to continue, if possible, the following table:

<table>
<thead>
<tr>
<th>Fraction</th>
<th>( \frac{1}{2} )</th>
<th>( \frac{1}{3} )</th>
<th>( \frac{1}{4} )</th>
<th>( \frac{1}{6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 UNIT</td>
<td>( \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{6}{6} = \frac{6}{6} )</td>
<td>( \frac{2}{6} = \frac{3}{6} = \frac{4}{12} = \frac{6}{12} )</td>
<td>( \frac{3}{12} = \frac{4}{12} = \frac{6}{12} )</td>
<td>( \frac{2}{12} = \frac{4}{12} = \frac{6}{12} )</td>
</tr>
</tbody>
</table>

The pupils are told that it is difficult to divide the intervening strips equally into the appropriate numbers: 5, 7, 8, 9, 10, 11 of parts. These strips have been left blank for the time being. The teacher is referred to §3.13 on page 100 for an application of the diagonal-scale rule.

The equivalent fraction chart is now used as a teaching-aid for the addition of fractions.
3.11 Addition of Fractions Using Equivalences

The author has had little success with 'persuading' pupils to 'discover' a 'sensible' and 'respectable' rule for addition on the 'cross-product' number-families in the way in which she was able to explore the corresponding rule for multiplication in the integer number system. If we look at the natural-number line extended first to the integer-number line and then to the rational-number line:

We see that the one-to-one correspondences between the naturals and the positive integers and between the naturals and the negative integers is extended. There is a one-to-one correspondence between the integers and the rationals, the $\text{something} , +1$'s. For consistency with results for the natural- and the integer-number systems under addition (§2.8 of Chapter 2) it is now suggested that there should be isomorphism under addition between the integers and the rationals, the $\text{something} , +1$'s. This would give, for all integers $p$ and $q$, the correspondence:

\[
\text{Integer} \quad + \quad \text{Integer} \quad = \quad (\text{Integer} + \text{Integer})
\]

\[
\text{with: Rational} \quad \oplus \quad \text{Rational} \quad = \quad (\text{Rational} + \text{Rational})
\]
The obvious suggested rule: \[ \frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d} \] for the addition of the rational-number families, produced by most pupils, is immediately contradicted.

The work at Brunswick Junior High School was going well. The author had already been asked to repeat the work so far achieved with a parallel low-ability third year class. The original plea for assistance was for a "meaningful rule for the addition of fractions" which did not rely on LCM's and lowest common denominators and the confusing formal presentation of the working with the associated parrot-fashion form of words 'this into that times the number on top'.

It was decided to abandon the 'choice of rule' and the 'cross-product' number-family routine and at this stage for the development of the rationals to translate the 'cross-product' number-family names to their traditional fraction names. It must be remembered that pupils have had a wealth of experience of a rational number as an equivalence class of equivalent fractions; of cancelling-up as well as cancelling-down in the construction of those classes and in finding their simplest names; of representing geometrically what must often appear to them as 'irrational/imaginary' numbers; and of 'discovering' a 'sensible' and 'respectable' rule for multiplication. The work proceeded as follows.

The pupils were invited to consider a universal set consisting of a basket of 12 fruits:

<table>
<thead>
<tr>
<th>A-pples</th>
<th>B-ananas</th>
<th>C-herries</th>
<th>D-ates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
   Pupil: \[
   \frac{1}{12} \quad \frac{2}{12} \quad \frac{3}{12} \quad \frac{6}{12}
   \]

(ii) What fraction of the universe is: not A?
   Pupil: \[
   \frac{11}{12}
   \]

   How did you get it?
   Pupil: 1-whole \(-\frac{1}{12}\)

   which = ? \[
   \frac{11}{12}
   \]

   Could you get it in any other way?
   Pupil: \[
   B + C + D
   \]

   which = ? \[
   \frac{2}{12} + \frac{3}{12} + \frac{6}{12}
   \]

   = ? \[
   \frac{11}{12}
   \]

2 (i) What are the fractions using simplest names?
   \[
   \frac{1}{12} \quad \frac{1}{6} \quad \frac{1}{4} \quad \frac{1}{2}
   \]

(ii) What fraction of the universe is: not A?
   Pupil: \[
   \frac{11}{12}
   \]

   How did you get it?
   Pupil: 1-whole \(-\frac{1}{12}\)

   which = ? \[
   \frac{11}{12}
   \]

   Could you get it in any other way?
   Pupil: \[
   B + C + D
   \]

   which = ? \[
   \frac{1}{6} + \frac{1}{4} + \frac{1}{2}
   \]

   What do you expect the answer to be? \[
   \frac{11}{12}
   \]

   Can you suggest a rule for addition?

3 The work is repeated for the addition and subtraction of all possible combinations of the fractions involved:

(i) \('not\ B'; \ 'not\ C'; \ 'not\ D'\)
(ii) 'either A or B' and all other pairs
(iii) 'A and B' and all other pairs
(iv) 'neither A nor B' and all other pairs
(v) 'A or B or C' and all other triples
(vi) 'not A nor B nor C' and all other triples
(vii) all of the fruit
(viii) none of the fruit

The pupils should be encouraged to look for independent checks by getting their answers, whenever possible, in more than one way. They should be encouraged to compare their results at each stage. In this way they will be experiencing the meaning of the logical connectives and equivalents, learning to express themselves clearly and concisely by using mathematical language correctly. The attention of the teacher is drawn also to the way in which the questions can be phrased in a form to give experience of the notion of probability: for example in (say) 3(ii):

'What is the chance of selecting 'either an apple or a banana'?

The pupils can then be encouraged to use the equivalent fraction charts to:

4 (i) Find equivalent fractions with the same denominators for: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{12}$

(ii) add them in pairs, triples, sets of 4, and 5.

The opportunity exists to extend the work to a consideration of fractions on the unit segment $[0,1]$ to $[16,16]$ of the rational number families and/or to mixed numbers.
The response from the pupils was good. The headmaster and the staff were enthusiastic and asked for a repeat performance with the parallel low-ability third-year class (see page 85) of not only the preparation of the equivalent fraction chart but also of the addition of fractions without mention of the lowest common denominator.

In order to overcome prejudices against presentation by a specialist mathematician, it was decided that the work should be undertaken by an initial trainee, Mr Malcolm Green, an economics graduate offering mathematics as a subsidiary teaching subject. There now follows an edited transcript of the audio-tape recording of his lesson on equivalent fractions [S2].

3.12 An Audio-Tape Recorded Lesson - Equivalent Fractions

Mr Green started with a quick-firing sequence of questions to find out what, if anything, the pupils could remember about fractions:

When the class was asked: "What is a fraction?"
Lyn replied: "a shape". Steven was able to improve upon that answer with: "three-quarters, or a quarter, or a half, or something like that".

Mr Green: "Good. That is an example of a fraction, but can anyone tell me what a fraction is?"

Kim tried, but he could not provide anything more precise.
Mr Green continued: "If we had a big round cheesecake and I want to give everybody a piece of the cheesecake, how many pieces would I need? Teresa was able to reply: "18". Why 18? "For 18 people" Teresa volunteered.

Mr Green continued: "For everybody to have an equal piece of the cheesecake I would have to cut the big cheesecake into 18 pieces. Does anybody know what we call the fraction that each person has of the cheesecake?"
Noel: "a quarter"

Mr Green: "No. More than (!) a quarter. Remember that we have to cut it up into 18 pieces"

Sally: "An eighteenth"

Mr Green: "Very good, Sally. For everybody to have an equal piece we cut it up into 18 equal pieces and everybody has one-eightheenth of the cheese-cake"

Mr Green (encouraged by his success with this low-ability class): "This time we have a long chocolate cake - one of those long chocolate logs - and I want to give an equal piece to each of the boys. How many pieces do I have to cut it into?"

Grieg: "Seven"

Mr Green: Why seven?"

Grieg: "Because there are seven boys"

Mr Green: "And what do we call the fraction each boy has of the whole chocolate cake?"

Lisa: "One-seventh"

Mr Green (still further encouraged): "This time we have a large fruit cake for the girls. How many pieces do we need for everybody to have an equal piece?"

Doug: "Eleven"

Mr Green: "And what is the fraction called?"

Janice: "One-eleventh"

Unfortunately, there was no cheese or chocolate or fruit cake. But, not entirely fortuitously, there was a chart on the wall. The chart had been marked out with some heavy, continuous, black lines (___) and some dotted red lines (.....) (Fig 3.14).
Mr Green (covering the 1st strip of the chart completely with an equal strip of blue paper): "Does anybody know what we call the strip that I've covered?"

Nobody did!

Mr Green: "We call this strip 'one whole' (writes '1 WHOLE' on the blue strip). We haven't cut it into any pieces, so we call the cheesecake 'one whole'"

Mr Green continues (pointing to the second strip of the chart): "How many parts is the whole divided into?"

Graham: "Two"

Mr Green (covering one of the two parts again with an equal piece of blue paper): "How many of the two parts have I covered?"

Billy-Jo: "One"

Mr Green: "What name do I give to the part that I've covered?"

Pupil: "One-half"

Mr Green writes \( \frac{1}{2} \) on the blue strip and the process is repeated for the third strip of the chart.

Mr Green (pointing to the 3rd strip): "How many parts is the whole strip divided into?"

Dora: "Three"

Mr Green: "The whole is split into three parts (covers one of them). What fraction of the whole have I covered?"

Sally: "One-third"

Mr Green writes \( \frac{1}{3} \) on the blue strip, points to the next strip, and invites the pupils to use the heavy black lines.

He continues: "How many parts is the whole divided into?"

Vicky: "Four"
Mr Green (covering one part with an equal piece of blue paper): "What fraction do we call one of those parts?"

Nick: "One-fourth"

Mr Green appears to be pressing home his point now with:

"And the next?" Five.
"So this is called (covering one of them) . . . ?
One-fifth
and writes $\frac{1}{5}$
"The next . . . ?" Six.
"We cover one part and call it . . . ? One-sixth."

He writes $\frac{1}{6}$.

Recapping, Mr Green, pointing to the 2nd strip, reminds pupils to think how many parts the whole is split into and how many of those parts are covered. He asks:

"What is this one?" Paul: "one-half"
"And the next?" Sarah: "one-third"
"And the next?" Paul: "one-fourth"
"The next one?" Lyn: "one-fifth"
"And the next one?" Sally: "one-sixth"

Then (pointing to the bottom line): "How many is the whole split into?"
and warning: "Be careful not to mis-count this one!"

The author thinks a thoughtless continuation of pattern prompted the response from:

Teresa: "one-seventh"

Mr Green (sympathetically): "No. It is difficult to count them all"

Jack: "Twelve"

Mr Green: "Yes there are twelve; and (covering one of them) what do we call it?"

Lisa: "One-twelfth"
The pupils are then each given a blank chart and asked to colour in one of the fractions in each strip, and to write in its name as for the fractions covered in 'blue' on the wall chart. The lesson develops.

Mr Green starts again from a blank chart and re-covers one half and two quarters.

Mr Green: "I have covered one half and two quarters. Is there anything special about these fractions?"

Noel: "A half of a quarter"

Mr Green (puzzled): "You have covered the half of the whole quarter? - The quarters have covered the half?"

Noel: "You have covered the half of the four quarters (Noel is obviously thinking correctly, but finds it difficult to express his thoughts in words)

Mr Green: "Right - we have! But what can you say about the half and the two quarters?"

Billy-Jo: "The half is the same as the two quarters"

Mr Green: "Right - very good, Billy-Jo. They equal each other. We have a very special name for fractions like this. We say that they are 'EQUIVALENT' (writing that word on the board and checking that the two quarters do in fact fit exactly into the space of the one half)

Mr Green (covering another fraction with a piece of blue paper): "What fraction have I covered?"

Pupil: "A sixth"

Mr Green (covering another sixth): "And what fraction have I covered now?"
Sally: "Two-sixths"

Mr Green: "What now?"

Andrew: "A half"

Mr Green: "Well - yes!. But how many sixths have I covered?"

Andrew: "Three-sixths"

Mr Green: "Is there anything special about the fractions covered on the chart?"

Pupil: "They are all the same"

Mr Green: "What is the special word we use for fractions which are the same?"

Sarah: "They are equivalent"

Mr Green: "Very good, Sarah. So three-sixths is equivalent to ....?"

Billy-Jo: "A half"

Mr Green: "What else is it equivalent to?"

Dawn: "Two-quarters"

Mr Green: "Good. So we know that a half is equivalent to two-quarters and it is also equivalent to three-sixths. What fraction (covering another one) have I just covered?"

Lyn: "One-twelfth"

Mr Green: "How many twelfths would I have to cover for it to be equivalent to the half, the two-quarters, and the three-sixths? - Teresa - Don't tell me the answer but do you think you know? Come to the board and put them on the chart"
Mr Green encourages everybody to think: "How many twelfths should Teresa place on the chart so that they will be equivalent to the half, two-quarters and three-sixths? How many has she got so far?"

Nicky: "Three"

Mr Green: "Is that enough?"

Pupil: "No. Two more"

Mr Green: "How many is that?"

Billy-Jo: "Five. One more to go."

Mr Green: "Why only six? Why put six on?"

Teresa: "Half of twelve"

Mr Green: "They are...?"

Teresa: "Equivalent"

Mr Green: "Well done Teresa. Teresa has put six-twelfths on the chart and we say that they are equivalent to one-half. What other fractions are equivalent to one-half?"

Marie: "Two-quarters"

Mr Green: "We know a half is equivalent to two-quarters. What else?"

Steven: "Three-sixths"

Dawn: "Six-twelfths"

The pupils were then asked to colour in all those parts of strips which represented fractions equivalent to one-half, preferably in the same colour (as they had used for the corresponding aliquot part). A lot of 'purposeful' conversation was going on at the same time:
Mr Green (to one pupil): "Colour in two-quarters. You have one quarter coloured in, so I want you to colour in another quarter so that you will have two-quarters which is equivalent to one-half" - "Same colour?" - "Yes"

(to Sally): "How many sixths will you need? - "Three altogether" - "Yes. Well done, Sally. So what do you write there (pointing to the space on Sally's chart)?"

(to another pupil): "Good; another question. You have to write a quarter in there and you have to colour in two more sixths. That's it. Yes; two more of those. Yes; so write one-sixth there"

(to Lyn): "You have to write a quarter in there and

(to Steve): "Right; a quarter in each of those"

Mr Green returns to the development of the main theme of the lesson again, this time to work out fractions that are equivalent to one-third. Attention is drawn to the dotted RED lines (.....) on the wall-chart.

Mr Green: "Look at the red dotted lines. We all know what equivalent means. We have one-third on the chart (covered with a piece of blue paper). We want to find another fraction which is equivalent to one-third"

(to Paul): "Don't tell me the answer, Paul. Do you think you know? Come to the board. Everybody look at the chart and see if what you thought is correct. So (handing blue-tak to Paul) you have got fractions to stick on the board. Put down how many sixths you think you need. Which is the one-sixth column (following the dotted RED line to meet the next heavy BLACK line). We have got one-sixth here. How
many sixths do we need for it to be equivalent to one-third? You have put one on. (Pupil picks up another piece of blue paper). Try it on the chart and see if it is correct"

(to class): "How many sixths has Paul put on the chart?"

Lyn: "Two"

Mr Green: "He has put two-sixths"

(to Paul): "Is the third equivalent to two-sixths?"  
- "Yes" - "They are equivalent - well done, Paul"

(to class): "Is that it? Can anyone think of another example that would be equivalent?"

(to Lisa): "Would you like to try? We have got one-third and we have got two-sixths" (Lisa tries): "It has to be exact otherwise they wouldn't be equivalent. That is too big"

(to class): "Does somebody else want to come to the front to help Lisa?" (to Graham): "Do you want to help? How many ninths are on the board?"

(to Dawn): "Is that enough?" - "No" - (then): "Is it OK? Well done! Lisa and Graham, how many ninths are on the board?" - Vicky calls out: "Three" - "There are three ninths. Can anybody tell me which fraction is equivalent to three-ninths? It's on the board already" (Paul replies: "one-third") "So we know that three-ninths is equivalent to one-third"

(to Dawn): "Any other? What else is equivalent to one-third and to three-ninths? - already on the board" - (Dawn replies quietly and hesitantly "er-one-um...") "No" - and then
"two-sixths"). "Two-sixths is good. Is there any other?" (This time Billy-Jo and Lyn come to the front to see if they can find another fraction which is equivalent to one-third and two-sixths and three-ninths)

Mr Green: "So what do they put on the board?" - "One-twelfth". "Is that enough?" - "No" - (they continue to put twelfths on the board). "How many twelfths is that?"

Sally: "Two-twelfths"

Billy-Jo: "Four"

Mr Green: "Four what?"

Billy-Jo: "Four-twelfths"

Mr Green: "Is that right? - Good. Well done, Sally and Billy-Jo"

(to Janice): "What is the four-twelfths equivalent to?" - "One-third" "Good. Anything else?" - (Kim): "three-ninths" "Any other?"

(to Maria): "How many ninths?" - "Three-ninths" "What else?" - (to Lyn - the other Lyn): "See if you can get it. We have had one-third; we have had three-ninths; we have had four-twelfths. Which one haven't we had?" - "Two-sixths" - "Very good"

The pupils now, at the bottom of their sheets, write in the equivalents that they have found from the construction of the equivalent fraction chart.

Mr Green: "Can anyone tell me what is equivalent to one-half?"

The pupils look for the one-half that is coloured in, follow the dotted RED line down their chart to find its equivalent at the next heavy vertical BLACK line at: "two-sixths", came the response.
Mr Green: "No; not two-sixths"

Andrew: "Three-sixths"

Mr Green: "Andrew; can you tell Steven why three-sixths is equivalent to one-half because he isn't really sure"

Andrew: "Because it's a half. Because if you add another three (sixths) on it will be a whole"

Mr Green (not wholly satisfied): "But why is a half equivalent to three-sixths?"

Andrew: "Because three is a half of six"

Mr Green: "Does that help you, Steven?"

Steven looks on his sheet and checks that, by colouring in three-sixths he finds that it is equivalent to one-half. He can stick three-sixths into the space of one-half.

Mr Green: "Are there any other equivalents?"

Paul: "Two-fourths"

Mr Green: "Two-quarters: so we have got one-half which is equivalent to three-sixths which is equivalent to two-quarters; (after a pause) there isn't another one, is there?"

Lisa: "Six-twelfths"

Mr Green: "Very good"

At this point the author noticed that there was a gap on the chart: a part that hadn't been covered with an appropriate piece of blue paper. She interjected - she often does if she feels that a student teacher will not thereby be embarrassed and that the children will see her as part of a teaching team and if she feels she can demonstrate a teaching point. In this
particular lesson, the exercise was being thoroughly enjoyed by the teaching team consisting of the class teacher, two audio-recorder operators, in addition to the author herself. Just as the bell was sounding, she asked:

"Can anyone find yet another equivalent to one-half? One which hasn't yet been mentioned?"

As she had expected, there were a number of mere guesses from a few restless lower-ability pupils:

Vicky: "Two-fourths"

Mr Green (taking up the point suggested very amicably): "Which one have I missed out?"

(to Noel): "Not two-eighths"

(to Neil): "Not six-eighths"

The author (on the side): "They have given six-eighths. How many eighths are equivalent to one-half?"

Andrew: "Four-eighths"

This, for the author, and I feel sure for the other teaching team members, was an excellent finish to a very exciting and stimulating lesson for very slow learning pupils. The accompanying cassettes will bear this out more effectively than can be done by an emphasis here of the many excellent teaching techniques employed.

The author wishes to record her confidence in many of the young probationers starting out on a career in teaching mathematics. In this case, the presenter is a student offering mathematics as a subsidiary teaching subject, an economics specialist who taught mathematics on teaching practice in what the author regards as the toughest, most open-ended, classroom situation in the City of Hull.
It is with very great regret to the author that Mr Green was unable to give the follow-up lesson on the addition of fractions using the equivalent fraction chart. That lesson was presented by herself. An audio-tape recording [S2] of the lesson was made and is included in the submission of this thesis. An edited transcript* has not been included in the belief that presentations of 'unfamiliar and dubious' methods are only really accepted if the successes of the younger, inexperienced and non-specialist mathematician can be 'transported'.

3.13 A Note on the Diagonal Scale

The author was first introduced to the diagonal scale in early childhood. Before ever starting to learn mathematics in terms of algebra and geometry, she was given a fitted pencil case containing a 6-inch wooden ruler graduated in tenths of an inch. The first inch was further graduated diagonally, thus

![Fig 3.17](image)

She was unable to find anyone who could (or would) explain its use to her. At school she was in fact told not to use it in favour of an essential 12-inch wooden ruler. Even when taught the ruler-and-compass construction for dividing a line segment into a given number n of equal parts, the practical application of the constructed diagonal scale for measuring in the corresponding fractional n-th parts of the unit was not explained. It is a modification of the diagonal

* An unedited transcript is available as an Exhibit
scale that is used for the fraction chart (Fig 3.11) in §3.8. To divide the chosen unit into (say) 7 equal parts, rational-number lines are drawn through equivalent representative points \((1,7), (2,7), (3,7), \ldots, (7,7)\) respectively and produced to meet the selected unit segment in 7 equally spaced points of division (Fig 3.18(i), (ii)).

The work at Brunswick Junior High School was again repeated in the following year by two undergraduates in the third year of the BSc Integrated (Honours) degree course at the University of Hull. They were assisted in their practical classroom experience by non-specialist mathematics teachers studying for a higher degree in mathematics education. The results of this work are available as Exhibits.

It is thought that C V Durell includes a description of the diagonal scale rule in one of his early texts in school geometry and it is pleasing to note that William Wynn Willson refers to it [21] albeit almost as an afterthought in the conclusion of his text.
CHAPTER 4

SYMMETRY

4.1 What is Symmetry?

If pupils have enjoyed mathematics at school - and many do nowadays particularly in their primary years - much of the interest and excitement will have been prompted by an awareness of 'pattern' that exists in mathematics. In the previous chapters the author has described the pattern that must exist and be recognised in any number system if that system is to serve a useful purpose in the solution of everyday problems. It was also seen how that pattern may be illustrated geometrically. In this chapter the author will look at mathematical pattern in geometry in which it is more familiarly recognisable.

Many of the inherently beautiful shapes, forms and curves found in the world in which we live may be described in terms of symmetry. Symmetry has been defined variously as: "beauty of form" [15] and "harmony" [16]. Nuttall also includes in his definitions: "the union and conformity of the members of a work to the whole".

We may compare this conformity of form that exists in the harmony of, say, music with, Chambers' "the exact correspondence of parts on either side of a straight line (or plane) or about a centre (or axis)", in mathematical symmetry. More precisely, in mathematics, a symmetry of a figure is the result of a transformation which leaves the figure as a whole in the same position, although the positions of its constituent
parts: its vertices, sides, angles, ..., etc, may be changed amongst themselves. Such a transformation is called a *symmetry transformation* or *isometry*.

The letter 'A', for example, remains unchanged in position if we 'turn it over' about the dotted line drawn through the vertex (Fig 4.1), although the feet will be seen to have changed places with each other. In fact, the whole of the left-hand part and the whole of the right-hand part will have changed places with each other. There is an exact correspondence of the parts of the letter 'A' on either side of the dotted line. The letter 'A' may be regarded as being 'its own reflection' in a double-sided mirror placed along the dotted line, or *axis* as it is usually called. There is "an exact correspondence", as Chambers puts it, "of parts on either side of a straight line". The letter 'A' is said to have *axial symmetry* or *reflectional symmetry* as we prefer to call it.

There is another kind of symmetry that exists in mathematics. Consider, for example, the letter 'S'. Imagine, if you will, that a pin is inserted through the centre (Fig 4.2). A
rotation in the plane about the pin through an angle of 180° brings the letter 'S' on to itself again and leaves it unchanged in position, although the left-hand bend and the right-hand bend have changed places with each other. There is an exact correspondence of parts of the letter 'S' about the centre pin. The letter 'S' is said to have rotational symmetry about the centre.

The reader is invited to suggest ways of finding pairs of corresponding points between the object positions and the reflectional and rotational image positions respectively of the letters 'A' and 'S' (Fig 4.3 (i), (ii)).

A consideration of the letter 'H' shows reflectional symmetry about 'vertical' and 'horizontal' (mirror) axes intersecting in the centre 'O' which is a centre also of rotational symmetry of 180° (Fig 4.4).

In mathematics 2-dimensional figures may thus be looked at in two ways when considering the concept of symmetry:
(i) reflectional, or axial, symmetry in which it is possible to identify a fixed (mirror) line, or axis of symmetry

(ii) rotational symmetry in which it is possible to identify an angle of rotation and a fixed point which is the centre of that rotation.

In 3-dimensional geometry the axis of reflectional symmetry becomes a plane of symmetry and the centre of rotational symmetry becomes an axis of symmetry.

4.2 Rigidness? or Rigidity?

In Chapter 1 the author has indicated how slow was the transition from a formal Euclidean development in the teaching of geometry in the nineteenth century to the Stage A - B - C approach advocated in 1909 by the Board of Education [2]. Teachers were reluctant to move away from the dogma of the systematic and formal development of logical reasoning and adopt a more flexible and practical approach. The formal presentation of the Euclidean proofs of even elementary propositions were still demanded of the pupils. The congruence of two triangles, for example, depends on one of four sets of criteria:

(i) two sides of one respectively equal to two sides of the other and the angles included between the equal sides equal

(ii) one side and the two adjacent angles respectively equal in the two triangles

(iii) three sides of the triangles respectively equal and, exceptionally in the case of two right-angled triangles

(iv) the hypotenuses and one other pair of corresponding sides equal
in which the equal right angles are not included between the equal sides as in (i). These still required proof by application of "the questionable principle of superposition" [17] until the late 1930's.

How many students on initial teacher-training courses, the author asks, today know these four sets of conditions for the congruence of triangles? and how many know that logically, in Euclid's sequential development, the formal verification of (iii) depends on the proposition that the base-angles of an isosceles triangle are equal? and hence that (iii) cannot be used in a formal proof of the isosceles triangle theorem? A construction line which bisects the vertical angle of the isosceles triangle is all right; the same construction line defined as the join of the vertex to the mid-point of the base of the isosceles triangle results in a 'circular argument' and mathematically is not all right [3].

Without the formal sequential development, there is no way in which the average pupil can be persuaded that their reasoning is mathematically un-sound. The majority of pupils, in the author's experience, appeal to common-sense symmetry about the altitude through the vertex of the isosceles triangle. This, too, was mathematically un-respectable until the advent of so-called 'transformation' geometry of the 1960's in schools.

It is intended in this work to adopt the view that "in the modern treatment of his (Euclid's) geometry, it is usual to recognise the primitive concept point and the two primitive relations of intermediacy (the idea that one point may be between two others) and congruence (the idea that the distance between two points may be equal to the distance between two other points, or that two line segments may have the same length)" [17]. Euclid's 'principle of superposition', [17] according to Coxeter, "is nowadays replaced by a further explicit axiom such as ... 'the rigidity of a triangle with a tail'".

'Rigidity' in this mechanical sense of "resistance to change of form" [16] offers a viable foundation for geometry at the secondary level of the mathematics curriculum in schools; 'rigidness' in the sense of an unbending insistence on mathematical rigour denies the restoration of geometry to its rightful place in school mathematics and rejects the results of many enjoyable and valuable experiences of the primary years. As teachers of mathematics we must, at all age/ability levels, give training in mathematical thought and logical deduction. An attempt must be made to give the pupils some idea of a mathematical proof. But, in the words of John Hunter and Martyn Cundy: "... the word 'proof' should not imply too formal a meaning, especially in the earlier years" [14]. These words relate to results and facts concerning pattern in number work. It is equally true that the idea of geometrical proof must also be geared to the age/ability levels of the pupils. Is it not mathematically respectable if pupils are able to explain in their own words the common-sense (logical) consequences of certain given situations (premises)? Should we not accept as 'proof' at their level of mathematical understanding, statements in words of one syllable, in their own language and demonstrated with the help of concrete apparatus and materials, in the form of conclusions based on their own convictions of previous experience?

4.3 Euclid Transformed: the Rigid Motions of the Plane

We are concerned here with the rigid motion of the 2-dimensional plane in which the distance between points is preserved. Any such movement is regarded as a geometrical transformation of the plane onto itself in the sense that the object plane consisting of the set of all points in the plane corresponds to the image plane consisting of a permutation of that set of points. In other words:

- every object point corresponds to a unique image point
- every image point corresponds to a unique object point.
A 'distance-preserving' transformation is called an isometry (see §4.5) and results in a symmetry of a set of points in the object plane to its image. A symmetry is usually regarded as synonymous with a congruence (but see page 129). The study of geometry in schools, based on the isometries of a 'modern' mathematics curriculum, can again be divided into three stages. Stage A consists of an exploration of the isometries in 2-dimensions: the pupils attempt first to arrive at a meaning of a translation, a rotation, a reflection. The question arises: "Are these the only isometries of the plane?"

Quite young children appear to enjoy the experiences of 'motion' geometry. The development, since World War II, of plastics and educational technology have made it possible for children to do and to see the mathematics that they are expected to learn instead of merely to hear the teacher present it with 'chalk and talk'. The author is referring, of course, to the motto of the Nuffield Mathematics Teaching Project:

I hear and I forget
I see and I remember
I do and I understand

For a possible progression to Stage B pupils will need a working definition of these three basic isometries: translation, rotation and reflection, and to be familiar with some of their properties which can be discovered from the definitions. A possible Stage A approach for upper-junior/ lower-secondary pupils is outlined in §4.4.

Stage B deals with the combination of isometries under a binary law of composition 'followed-by'. The result of the operation is usually called the product of the two component isometries, although I M Yaglom prefers the term sum [18]. For further development and applications in a Stage C, two
key theorems are required: (i) gives criteria for the unique determination of the isometry required for a known symmetry result; (ii) asserts that every isometry may be expressed as the product of at most three reflections. A discussion of these two theorems and an investigation of the pattern that underlies the system of isometries explored in Stage A will be found in §§4.5, 4.6, 4.7.

The further developments to a Stage C and to sixth-form work will be considered in Chapters 6, 7 in Part II, and extension to 3-dimensional geometry will be dealt with in Chapter 10 Part III.

In mathematics, the logical process of deduction consists of the definition of a set of elements; the fundamental properties that dictate their definition; and the choice of a set of axioms - self-evident truths - from which deductions can be made in the form of theorems or propositions.

I M Yaglom [18] identifies two ways in which a definition of a mathematical element may be given. A descriptive, formal definition as a statement of its properties gives no guarantee of the existence of the element which is defined: we could well be heading for a mathematical system in which we are operating on a set of purely 'imaginary' elements which admit of no practical application whatsoever in the real world. Existence has, however, for the members of at least one school of mathematical thought, the 'Intuitionists', dictated the need for the actual construction of the elements before they can be said to be defined [19]. It is this latter form of definition that is more suited to the minds of younger pupils in the schools.
4.4 *Stage A*

(1) **Movement**

A mathematical game of reasoning is prompted. The arena again is set. This time it is the pupils who will define a set of elements which are to be the symmetry transformations of a 2-dimensional plane. Rules of play are to be selected for operating on those elements. These rules are to be investigated for their mathematical 'sensibleness' and 'respectability' depending on the expectations of experiences of geometrical pattern of the earlier primary years. A winning strategy of logical deduction or mathematical proof will give results which are the theorems and propositions so familiar in a traditional course of geometry. These and other possible investigations are, the author submits, absolutely essential to the realistic development of mathematics for both user and specialist (see Chapter 8 in Part III).

An attempt is made, in Stage A, to convince the pupils of the existence of the isometries by their actual construction. They, the pupils, must be encouraged to 'discover' and to 'describe' the movements that will ensure that an image set-of-points is a symmetry of an object set-of-points. The existence of such a movement must be seen to depend on its fundamental distance-preserving property. In this section, the author deals with translation, rotation and reflection, in that order, although that order is not necessarily the order in which the pupils will 'discover' them; nor is it the order dictated by pedagogical considerations as seen in §4.6 at the end of this chapter.

The pupils are given an object in position P and an image in position P'. It is to be noted that, at this stage, the object and its image can be 'concrete' physical objects and need not be abstract mathematical entities. The pupils are then invited to describe possible 'movements' that will take the object at position P to its image at position P'.

* See scheme of work facing page 118
(i) A movement in a straight line from P to P' may be described as a movement through a fixed distance in a fixed direction. It may be represented by the directed line-segment as a vector \( \overrightarrow{PP'} \).

Experimental work could lead to the 'discovery' that the joins of corresponding pairs of points: \( P_1 P'_1, P_2 P'_2, P_3 P'_3, \ldots \) are equal and parallel-in-the-same-sense line-segments which represent the same defining vector.

Further investigation could lead to the 'discovery' that corresponding line-segments in the object and image sets are also equal and parallel-in-the-same-sense.

The movement is a distance-preserving (symmetry) transformation and is called a translation. It will be denoted by \( T \) (Fig 4.5).

![Figure 4.5](image)

Translation: \( \overrightarrow{PP'} \) \( \overrightarrow{P_1 P'_1} = \overrightarrow{P_2 P'_2} = \ldots \)

ie \( P_1 P_2 = P'_1 P'_2 \) and \( P_1 P_2 \parallel (in\ same\ sense) P'_1 P'_2 \)

(ii) (a) A movement along the arc of a circle from P to P' may be described as a directed angular turn about the centre of a circular arc.

Experimental work could lead to the construction for the centre of such a circular arc and a directed-angle of turn. Is the construction unique?
Further investigation could lead to the 'discovery' that:

2 corresponding object-image pairs determine a unique centre and directed-angle of turn

What happens in the case of 3 corresponding object-image pairs?

Could it be that corresponding line-segments in the object and image sets are again equal? and parallel? and in the same or opposite sense?

The movement is a distance-preserving (symmetry) transformation and is called a rotation. It will be denoted by \( R: (O, \alpha) \) where \( O \) is the centre and \( \alpha \) the directed-angle of rotation. It is customary to regard rotation in the counter-clockwise sense as positive, rotation in the clockwise sense as negative (Fig 4.6).

![Fig 4.6](image)

(b) The case in which \( \alpha = 180^\circ \) is often discussed, or 'discovered', as a special case.

Joins of corresponding points pass through the centre and are diameters of the rotation circles.

Corresponding line-segments in object and image sets are equal and parallel-in-the-opposite-sense (Fig 4.7).
(iii) A 'mirror' image may be reached by 'walking through' a mirror separating the object position P from the image position P' and may be represented by a 'mirror-line' or axis.

Experimental work could lead to the 'discovery' that the joins PP' of corresponding pairs of points are bisected at right-angles by the axis.

Further investigation could lead to the 'discovery' that corresponding line-segments in the object and image sets are equal but not parallel.

The movement is a distance preserving (symmetry) transformation and is called a \textit{reflection}. It will be denoted by M, its axis by m (Fig 4.8).

\begin{align*}
\text{Reflection in } m \quad & P_1P_2 = P'_1P'_2
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.7}
\caption{Fig 4.7}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.8}
\caption{Fig 4.8}
\end{figure}
Mathematical definition of an element with a selected property requires that, given the element, we know that the property is present; and that conversely, given the property, we know the element exists. It is important that pupils recognise the common characteristic that has occurred in each of their 'discovered' movements: corresponding line-segments of object and image sets are equal. They must then be encouraged to reverse the procedure: starting from a corresponding object-image pair of equal line-segments, can they always arrive at one or other of their 'discovered' movements? or are there any possible movements that have been omitted?

The teacher who is prompted to attempt an investigation along the lines suggested in this section will find helpful suggestions for practical work and appropriate apparatus and materials in [20]. For the necessary mathematical background knowledge in order to answer the question posed above: viz. are the three movements: translation, rotation, reflection, a complete definition by enumeration of the isometries of the plane? the teacher will need to consider mathematically a definition of the isometries based formally on their descriptive distance-preserving property. He will need a working definition of congruence. I M Yaglom [18] suggests that:

"Two geometric figures are said to be congruent if one figure, by being moved in space, can be made to coincide with the second figure so that the two figures coincide in all their parts".

The author confesses to being "a nagging critic" of I M Yaglom. She is not satisfied with this definition as a foundation on which to base a study of the isometries in the school mathematics curriculum. She does require I M Yaglom's further conciliatory statement that:
"A motion* is a geometric transformation of the plane (or of space) carrying each point \( A \) into a new point \( A' \) such that the distance between any two points \( A \) and \( B \) is equal to the distance between the points \( A' \) and \( B' \) into which they are carried."

There is a need to eliminate "the earlier, vague, notions of 'moving points', with the accompanying difficulties about time and motion which are not strictly part of pure mathematics (and hence they are difficult!)" [4.3]. We are concerned with the properties of a metric, Euclidean, space. Of course the author agrees with William Wynn Willson, and must at this stage, "avoid getting embroiled with real numbers for just a little longer". At the same time she recognises with him that "our first thought" in considering the properties that may be deduced from our set of isometries "is that they should preserve distance... but this merely begs the question" if "we have not, as yet, introduced a notion of distance". Let us not forget that "distance ... is tied up with rather subtle properties of real numbers (as Pythagoras, the Delphic Oracle, and all good circle-squarers eventually discovered)" [21, 22].

The author submits however that it is mathematically respectable for pupils to rely on their intuitive idea of 'distance between two points' or 'length of a line-segment' to determine whether or not I.M Yaglom's congruent figures coincide. They will be using nothing in their line of reasoning that will have to be discarded later as 'improper' in the interest of mathematical rigour.

Distance is not the only entity that is 'fixed' or invariant under a distance-preserving (symmetry) transformation and pupils should be encouraged to investigate further the possibility of other elements remaining 'fixed' and to look for pairs of object-image symmetry correspondences which 'coincide' with each other.

* Isometry or rigid motion
(2) Invariance

The invariance (i) of points; (ii) of lines will be considered.

(i) Are there any fixed points? Experimental activity could lead to the 'discovery' that there are:

- no fixed points: under a translation (*except in the case of zero distance moved in any direction)
- one fixed point: under a rotation, including the half-turn. The centre is the only invariant point under all rotations (*except for \( \alpha = 2k\pi, \ k \) an integer)
- infinitely many fixed points: under reflection. All points of the axis are invariant. The axis is called an invariant line, point-for-point, and must be distinguished from a fixed line considered in (ii) below

It will be seen that under a distance-preserving (symmetry) transformation there are either no fixed points, exactly one fixed point, or infinitely many fixed points.

(ii) Are there any fixed lines? Invariant lines point-for-point have been considered in (i) above and are excluded. We are concerned here with lines in which the object-point moves into an image-point in the same line but not necessarily in the same position on that line. Experimental activity could lead to the 'discovery' that there are:

* Under these isometries every point is invariant: see identity transformation (§4.6(2)).
no fixed lines: under a rotation (except \( \alpha = k\pi \), where 
\( k \) is an integer: see \( \dagger \) below)

infinitely many fixed lines: under a translation: every line when 
moved in its own direction is invariant 
but not point-for-point; sense is 

\[ \dagger \text{ under a half-turn rotation } \alpha = k\pi, \]
where \( k \) is an integer: lines through the 
centre remain fixed; every half-line from 
the centre is 'reflected in the centre' 
into its collinear half-line counterpart; 
sense is reversed 

under a reflection: every half-line 
perpendicular to the axis is reflected 
into its collinear half-line counterpart 
on the other side of the axis; sense is 
reversed.

E A Maxwell [20] draws attention to the 'obvious' lack of the 
existence of just one fixed line under any isometry 
corresponding to the existence of just one fixed point which 
occurs under a rotation. The 'discovery', if made, of such 
an obvious gap is interesting from the point of view of the 
principle of duality, already experienced (possibly) by 
pupils in their exploration of the number systems, and will 
be discussed in §4.5(2).

Investigation into other invariances should also be attempted 
wherever possible. It can be shown that under distance-
preserving (symmetry) transformations, straight lines are 
transformed into straight lines, angles are preserved in 
magnitude but not necessarily in sense-of-direction of 
measurement, parallel lines are transformed into parallel 
lines but not necessarily in the same sense-of-direction.

In Stage A we have attempted to construct a symmetrical 
universe. We do not yet know whether the set of elements,
SCHEM OF WORK

STAGE A: Definition of the Isometries by Construction - enumeration of the isometries and their fundamental properties to establish their existence

STAGE B: Descriptive (formal) Definition of an Isometry as a Distance-preserving Transformation - no guarantee of existence

STAGE C: Applications to new Objects for the Discovery of further Properties (see Chapter 6)

A(1) Movements of Object to Image

<table>
<thead>
<tr>
<th>Movement:</th>
<th>TRANSLATION</th>
<th>ROTATION</th>
<th>REFLECTION</th>
<th>*HALF-TURN</th>
</tr>
</thead>
<tbody>
<tr>
<td>is represented by:</td>
<td>through a given distance in a given direction</td>
<td>about a given point through a given directed angle</td>
<td>a mirror image as a walk through a mirror</td>
<td>about a given point through an angle of 180°</td>
</tr>
<tr>
<td>so that</td>
<td>a vector quantity</td>
<td>a centre and a directed angle</td>
<td>an axis - or mirror line</td>
<td>a centre and an angle of 180°</td>
</tr>
<tr>
<td>joins of corresponding points</td>
<td>are line segments which are equivalent representations of a vector quantity</td>
<td>subtend the angle of rotation at the centre; rays are equal</td>
<td>are bisected at right angles by the axis</td>
<td>are collinear with the centre and form a diameter of the rotation circle</td>
</tr>
<tr>
<td>and corresponding line segments</td>
<td>are equal and parallel-in-the-same-sense</td>
<td>equal and, if produced, intersect at the angle of rotation</td>
<td>equal and equally inclined to the axis</td>
<td>equal chords of the rotation circle and parallel-in-the opposite-sense</td>
</tr>
</tbody>
</table>

Pupils will have discovered by construction the definitions of Translation, Rotation, Reflection as movements of equal corresponding line segments.

Teachers will require I.M. Yaglom's definition of congruence [18]: "Two geometric figures are said to be congruent if one figure by being moved in space, can be made to coincide with the second figure so that the two figures coincide in all their parts". He further asserts that "A motion is a geometric transformation of the plane (or of space) carrying each point A into a new point A' such that the distance between any two points A and B is equal to the distance between the points A' and B' into which they are carried". Such a motion is an isometry or a rigid motion.

(ii) Invariance

<table>
<thead>
<tr>
<th>TRANSLATION</th>
<th>ROTATION</th>
<th>REFLECTION</th>
<th>*HALF-TURN</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Points</td>
<td>none</td>
<td>centre (a ≠ 2kr)</td>
<td>infinitely many points of the axis (called an invariant line point-for-point)</td>
</tr>
<tr>
<td>(b) Lines not-point-for point</td>
<td>infinitely many (all, with sense preserved)</td>
<td>none - or infinitely many rays rotated through the angle kπ</td>
<td>lines perpendicular to the axis (with sense reversed)</td>
</tr>
</tbody>
</table>

E.A. Maxwell [20] draws attention to the lack of just ONE invariant line as compared with the possibility of ONE, NONE, or INFINITELY MANY invariant points. This prompts the investigation and the discovery of a fourth movement, the glide reflection.

* The half-turn is here treated separately, but is the special case of a rotation through an angle of 180°
consisting of translation, rotation and reflection, is the 'complete set' of universal elements and we need to investigate the possibility of the existence of other symmetry transformations. Reference has been made to the sense-of-direction of movement:

a translation is a movement in a directed straight line

a rotation is a directed angular movement

and to whether, in the invariance of straight lines and half-lines, the sense of direction of movement is preserved or reversed. This sense of direction of movement will be discussed further in Stage B in terms of the orientation of a geometrical figure (see page 122) when the situation consisting of three pairs of symmetry correspondences is considered.

4.5 Stage B: Fundamental Theorem I

(1) The Set of Isometries

The game continues. We need rules of play for the manipulation of the pieces of our game, the isometries. We require a geometrical operation which will dictate the way in which the isometries may be combined and we have to satisfy ourselves of the mathematical 'sensibleness' and 'respectability' of the operation. In the geometry of transformations, a binary operation may be expressed in the words 'followed-by'. This implies that one movement is performed on an object entity 'followed-by' a second movement which is performed on its image. It will be seen that the operation 'followed-by' behaves in many ways like the operation of addition or of multiplication in a number system. Hence the term 'sum' or 'product' (see page 108) for the result of the operation. The author chooses to use the term 'product'. 
In geometry, a 'strategy of play' is based on the process of logical deduction: we need rules also in the form of axioms. These are regarded as 'self-evident truths' of the system from which, by sound mathematical thought and reasoning, we can derive 'winning strategies' which enable us to enunciate theorems or propositions. The work will include the enunciation of two key theorems required for the development to a Stage C in Chapter 6 in Part II and suggestions for a partial restoration of classical geometry results to the school mathematics curriculum. Here the 'axioms' will be the elementary properties of triangles, parallelograms, etc, experienced at primary level.

If not already attempted, pupils must now be prepared to face the problem in reverse: Is it true that any given object line-segment corresponds to an equal image line-segment under a translation or a rotation or a reflection? or is there a missing link? I M Yaglom gives an excellent account of the situation and establishes the converses in the form:

(i) equal and parallel-in-the-same-sense line-segments are related by translation

(ii) equal and parallel-in-the-opposite-sense line-segments are related by half-turn rotation

(iii) equal line-segments inclined at an angle $\alpha$ are related by rotation $\alpha$

He almost gives the impression that reflection is redundant! The author now invites the following investigation and extension of the situation.

It will now be assumed that the investigations of Stage A will have satisfied the pupils, at their level of understanding, of the reasonableness of the fundamental properties of an isometry which (see pages 114, 115) moves a geometrical figure into a symmetry (coincident) position such that the distances
between corresponding pairs of object-image points are equal. It will further be assumed that pupils have satisfied themselves of the existence by construction of possible movements to ensure such a symmetry: if object-point $P$ is moved to image-position $P'$, we write $P + P'$ and we know that:

$P + P'$ under a translation given by the vector $PP'$

or: under a rotation about a centre on the mediator* of $PP'$

or: under a reflection in the mediator of $PP'$

We also know that corresponding line segments in the object-image correspondence of two geometrical figures are equal. I M Yaglom's converse results (i), (ii), (iii) above will also be assumed as verified for a pair of corresponding equal line segments.

The converse situation of establishing the existence of an isometry, which ensures the symmetry of an object geometrical figure consisting of a set of three or more points with its image, must now be tackled. The method adopted here was prompted by E H Lockwood and R H MacMillan [23] and the author attempts to overcome for readers some of the difficulties which she herself has experienced in working from that text.

Let $P, Q, R$ be a set of 3 non-collinear points of the plane and suppose, if possible, that an isometry can be found such that: $P, Q, R + P', Q', R'$ respectively and such that corresponding line-segments are equal.

We know that, in general, there is a single rotation that takes the line-segment $PQ$ to the equal line-segment $P'Q'$. The centre of that rotation is at the point of intersection $O$ of the mediators of $PP'$, $QQ'$ and that the angle of rotation is the angle $\alpha$ (say) contained by the equal segments $PQ$, $P'Q'$ (Fig 4.9(i)). The exceptional cases in which the mediators do not meet are those cases in which the mediators are either parallel (Fig 4.9(ii)) or coincident and will be discussed separately (Fig 4.9(iii)). It is to be noted that points on the mediators correspond.

* "One who mediates between parties at strife" [15]
In the general case (i) there are just 2 image positions $R'$ corresponding to the object point $R$:

since corresponding line-segments $PR, P'R'$ are equal, then $R'$ lies on the circle centre $P'$, radius $PR$

since corresponding line-segments $QR, Q'R'$ are equal, then $R'$ lies on the circle centre $Q'$, radius $QR$

A definition of the Euclidean metric ensures that these two circles intersect in 2 distinct points, $R'_1$ and $R'_2$ (say) (Fig 4.10).
It is at this point that we refer again to the sense-of-direction of movement: this time we distinguish between the two directions, counter-clockwise and clockwise, in which it is possible to describe (walk-around) a geometrical figure. In Fig 4.10 it will be seen that the orientation of the image triangles $P'Q'R'_1$, $P'Q'R'_2$ are different:

$\begin{align*}
\text{PQR} + P'Q'R'_1 & \text{ in the same sense} \\
\text{PQR} + P'Q'R'_2 & \text{ in the opposite sense}
\end{align*}$

The image $\triangle P'Q'R'_1$ is directly orientated to the object $\triangle PQR$ and is obtained as a result of a direct isometry. The image triangle $\triangle P'Q'R'_2$ is oppositely orientated to object triangle $\triangle PQR$ and is obtained as the result of an opposite isometry. Appropriate explorative activity at an appropriate stage for the pupils should have convinced them that, so far:

rotation and translation are the only direct isometries;
reflection is the only opposite isometry

that are 'defined'.

It follows that thus far we are able to state that, in the case of the 2 and only 2 possible kinds of symmetry direct or opposite available to us,

(a) $\text{PQR} + P'Q'R'_1$ under a (direct) rotation or translation

(b) $\text{PQR} + P'Q'R'_2$ under an (opposite) reflection.

(a) ensures that, in the general case (i) currently being considered, there is a unique rotation that takes any (and hence every) other point $R$ which is non-collinear with $P$ and $Q$ to $R'_1$:

*$\text{PQR} + P'Q'R'_1$ under the rotation centre $O$, angle $\alpha$

* The special case of the half-turn in which $\alpha = \pi$ is an interesting one and is included
We have a 'proof' at pupil level of understanding of the important result that the mediator of $RR'_1$ also passes through $O$ and that $\angle ROR'_1 = \alpha$; and hence of the:

**Theorem:** The mediators of the joins of corresponding pairs of points under rotation are concurrent (Fig 4.11)

This theorem is, incidentally, the justification for the method of construction of the *instantaneous centre of rotation* of rigid dynamics.

(a) ensures an answer also to the exceptional case illustrated in Fig 4.9 (ii). Rotation is denied if the mediators of $PP'$, $QQ'$ do not intersect: in that case either (ii) the mediators are parallel, or (iii) the mediators are coincident. If parallel then the corresponding line segments $PQ$, $P'Q'$ are either parallel-in-the-same-sense and (using I M Yaglom's converse (i) on page 119) $PQR \rightarrow P'Q'R'_1$ under the translation $\overrightarrow{PP'} = \overrightarrow{QQ'}$ (Fig 4.12(i)); or parallel-in-the-opposite-sense and (using I M Yaglom's converse (ii) on page 119) $PQR \rightarrow P'Q'R'_1$ under the half-turn whose centre is at the point of intersection of $PP'$, $QQ'$ (Fig 4.12(ii)).
(iii) If the mediators of PP', QQ' are coincident, the common mediator is a line of symmetry. It could have been 'coincidentally discovered' in preliminary investigations in the search for a centre and angle of rotation that points on the mediators correspond under rotation. The common mediator, in this case, becomes a line of invariant points and is the axis of a reflection:

$$PQ \rightarrow P'Q'$$ under reflection in the coincident mediators

It follows that: $$PQR \rightarrow P'Q'R'$$ under the same reflection and that $$R'$$ occupies the position of $$R_2'$$ under an opposite isometry (Fig 4.13) and satisfies (b) page 122. In this situation $$PQ$$ and $$P'Q'$$ are symmetrically placed with respect to the axis of symmetry.

$$PQR \rightarrow P'Q'R'$$ under the same reflection and that $$R'$$ occupies the position of $$R_2'$$ under an opposite isometry (Fig 4.13) and satisfies (b) page 122. In this situation $$PQ$$ and $$P'Q'$$ are symmetrically placed with respect to the axis of symmetry.
What happens when PQ, P'Q' are unsymmetrically placed with respect to an axis of reflection?! There can, of course, be no such position. Our attention is forced back to a further investigation of the one and only one remaining situation, viz the general position illustrated in Fig 4.10. in which PQR → P'Q'R' under an opposite isometry.

(2) The Missing Link

The author suggests an investigation on the following lines. Starting from the known results of translation, rotation and reflection of object to image positions, pupils are invited to relate the unsymmetrical and opposite image P'Q'R' to object PQR. They are being asked to investigate the results of combinations of isometries under the operation 'followed-by' (in short 'fb').

(i) (a) Rotation (O, α): PQR → P'Q'R'₁
Reflection in P'Q': P'Q'R'₁ → P'Q'R'₂
Pupils should be asked to identify the centre O and angle α of rotation. They are exploring the result of the combined operation:

Rotation (O, α) 'fb' Reflection in P'Q': PQR → P'Q'R'₂
as illustrated in Fig 4.14(i), on page 128.

What happens if the order of operations is reversed?

(i) (b) Reflection in PQ 'fb' Rotation (O, α): PQR → P'Q'R'₂
Pupils should note that the centre O and angle α of rotation are unchanged (Fig 4.14(i)(b))

(ii) (a) Translation P₁: PQR → P'Q*R* (say)
Reflection in the mediator m₁ of Q'Q*:
P'Q*R* → P'Q'R'₂

The pupils are investigating the result of the combined operation:
Translation \( \overrightarrow{PP} \): \( \text{fb} \) Reflection in \( m_1 \):

\[ \text{PQR} \rightarrow \text{P'Q'R}'_2 \]

(Fig 4.14(ii)(a))

What happens if the order of operations is reversed? It is to be noted that the mediator \( m_1 \) of \( Q'Q^* \) may be regarded as the bisector of the angle of rotation \( \alpha \) between the equal object-image line segments \( P'Q', P'Q^* \). Pupils are invited to investigate the results of:

(ii) (b) Reflection in the bisector \( m^* \) of angle \( \alpha \) between \( \text{PQ, P'Q'} \) 'fb' Translation \( \overrightarrow{PP'} \): \( \text{PQR} \rightarrow \text{P'Q'R}'_2 \)

(Fig 4.14(ii)(b))

Alternatively, the investigation may proceed with:

(iii) (a) Translation \( \overrightarrow{QQ'} \): \( \text{PQR} \rightarrow \text{P*QR*} \) (say)

Reflection in the mediator \( m_2 \) of \( \text{P'P*} \):

\[ \text{P*Q'R*} \rightarrow \text{P'Q'R}'_2 \]

so that:

Translation \( \overrightarrow{QQ'} \): \( \text{fb} \) Reflection in \( m_2 \):

\[ \text{PQR} \rightarrow \text{P'Q'R}'_2 \]

(Fig 4.14(iii)(a))

Reversing the operations and noting that the mediator \( m_2 \) of \( \text{P'P*} \) is again the bisector of the angle of rotation \( \alpha \) between the equal object-image line segments \( P'Q', P^*Q' \), we have:

(iii) (b) Reflection in the bisector \( m^* \) of angle \( \alpha \) between \( \text{PQ, P'Q'} \) 'fb' Translation \( \overrightarrow{QQ'} \): \( \text{PQR} \rightarrow \text{P'Q'R}'_2 \)

The position is identical with that of (ii)(b)

(Fig 4.14(ii)(b))

The investigation could lead to the 'discovery' of the existence-by-construction of the glide-reflection, an isometry resulting from a reflection in an axis and a
translation in the direction of that axis. Which axis? and in what direction? Is the combined result of the reflection and translation commutative? or does the result depend on the order in which the reflection and translation are performed?

The further exploration of invariance under the result of combining a reflection and a translation could, in fact, produce a one and only one invariant line, but not point-for-point, which is seen to be the join of mid-points of 2 pairs of corresponding points. The definition of the symmetry of the plane dictates that: the mid-points of joins of corresponding pairs of points are collinear, an important theorem which it is possible for the more able pupils to 'discover' for themselves.

The glide-reflection provides the 'missing link' suggested by E A Maxwell (see page 117). The reader is invited to compare the investigation outlined above with the use that Maxwell makes of the 'glide-box', a device which he states "helps to make some of the diagrams clearer when dealing with problems involving glide-reflections". Interestingly Maxwell starts by using the theorem of collinearity of the joins of mid-points of pairs of correspondences which is the desired and optimistic result of the endeavours suggested in the current investigation.

In fact if exercises (ii)(b) and (iii)(b) are combined and the reflection in the line joining the mid-points of PP', QQ' is 'followed by' translation in the direction of that line we have a definition-by-construction of the glide-reflection in the form of Maxwell's glide-box (Fig 4.14(iv)).
Fig 4.14
(3) Key Theorem I

In the course of the work outlined in this section, the pupils may have 'discovered' for themselves a number of important facts. They will have 'proved', by explanation to their own satisfaction and that of the teacher, results based on 'facts and figures' of previous experience. Some, if not all, may recognise the distinct possibility of the truth of the following Fundamental Theorem I:

Theorem An isometry is completely determined by two pairs of corresponding points and a statement that it is either direct or opposite.

The theorem is enunciated here in the form in which it appears in EH Lockwood and RH MacMurdie [23]. The discussion in that text the author found confusing and abstruse. It is more usual to split the theorem into two parts dealing respectively with the two, direct and opposite, isometric situations.

The author prefers the former statement of the theorem. She has, in her own teaching experiences, had very great difficulty in reconciling the formality of Euclidean congruence with a 'reason-able' and practical approach to the symmetry of a 'modern' transformation geometry. In her opinion, isometry differs in one respect from congruence in that it dictates not only 'correspondence of fit' of geometrically symmetric figures in shape and in size but it also dictates the orientation of that 'fit'. In traditional Euclidean geometry orientation was discussed as a separate issue. The chosen enunciation would appear to highlight this difference.

The emphasis in this section has been placed on the practical experience, on the part of the pupils, of the way in which the pattern of geometrical symmetry may be described. In the course of their explorations, they have been forced almost unwittingly into a consideration of the result of
combining two of the three 'elementary' isometries: translation, rotation, reflection. The 'missing link', the glide-reflection, was 'discovered' as the result of such a combination. It would appear, at first sight, that the glide-reflection, as such, is out-of-keeping with the apparently more simplistic isometries.

In the next section the author sets out to show that this is not the case. The combination of isometries under the binary law of composition followed-by will be discussed and the pattern of the resulting system investigated.

4.6 Stage B: Fundamental Theorem II

(1) The Isometries Combined

The game continues. A binary law of operation for the combination of two isometries has already suggested itself in the search for the glide-reflection. The operation followed-by implies that the two component isometries are to be performed in ordered sequence. The result is called the product.

An investigation into the products obtained as the result of one isometry 'followed-by' a second isometry has already begun: a glide-reflection is the product of translation 'followed-by' reflection. Furthermore, in this case, the operation is commutative.

It may be 'discovered' that the product of:

(i) 2 translations is a translation.

The operation corresponds to the familiar parallelogram law of addition of vectors (see page 33) and is commutative
(ii) 2 rotations with different centres is a rotation. What is its centre? its angle? Special cases in which the component directed-angles are either supplementary or add up to a complete revolution should be considered. The half-turn may be taken as a separate case. Commutativity, or otherwise, of the operations should also be discussed.

(iii) (a) 2 reflections in parallel axes is a translation. What is the representative vector? (b) 2 reflections in intersecting axes is a rotation. What is the centre? the angle? What special case arises from 2 reflections in the same axis? Are the operations commutative?

Just as the glide-reflection has already been established as the product of a translation and a reflection and is commutative, some pupils may be motivated to explore the results of combination of other unlike pairs of isometries under the operation 'followed-by'. The product of, for example:

(iv) a translation and a rotation is a rotation. What is the centre? the angle? Is the operation commutative?

The results obtained so far in this investigation may be tabulated:

<table>
<thead>
<tr>
<th>1st isometry</th>
<th>2nd isometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>f.b.</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>M</td>
<td>G</td>
</tr>
<tr>
<td>G</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 4.1
A number of questions still remain unanswered but there is a limit to the extent to which interest can be sustained!

We are now in a position, however, to explain the apparent incongruity in the definition of $G$ as a product of the more 'primitive' isometries: translation, rotation, reflection. Are not each of the 3 'primitive' isometries themselves expressible as products of 2 of the others?

It is perhaps interesting to turn the results (iii)(a), (b) of the investigation the other way around: we have:

(iii)' (a): a translation is the product of 2 reflections in parallel axes

(b): a rotation is the product of 2 reflections in intersecting axes

Now if we apply (iii)'(a) to $G$ we have:

(iii)' (c): a glide is the product of a translation and a reflection which, using (a), is the product of 3 reflections, but only if the associativity of the operation 'followed-by' is assumed.

The teacher must be aware of the possible 'discovery' by the pupils of the identity transformation which appears in part (ii) of the investigation as the combined result of a pair of rotations whose 'algebraic' directed-angle-sum is a complete revolution or, in part (iii), as the combined result of a pair of reflections in the same axis.

The author returns now to the situation whence the investigation started (Fig 4.10). The reader is asked to look again at the symmetry transformation which moves $P$, $Q$, $R$ to $P'$, $Q'$, $R'$, but this time using only reflection. The exploration could perhaps proceed on the following lines:
The isometry determined by Theorem I (page 129) ... is determined by the effect which it has on the object triangle PQR. What, may be asked, is that effect considered from the point of view of invariant points of the transformation?

(2) Invariance Again

The following cases arise and are 'exhaustive':

*Case 1:* 3 points invariant: the isometry is the identity transformation which will be denoted by I. From I(iii)(a) it is known that:

\[ I: \text{PQR} \rightarrow \text{PQR under reflection M 'followed-by' reflection M in the same axis} \]

The isometry in which 3 points are invariant may be expressed as the product of just 2 reflections.

*Case 2:* 2 points invariant: a pair of invariant points may be regarded as object-image correspondences under reflection in the line joining them (Fig 4.15).

\[ \text{P} \leftrightarrow \text{P}', \text{Q} \leftrightarrow \text{Q}' \]

\[ \text{Fig 4.15} \]

\[ \text{PQR} \rightarrow \text{PQR}' \text{ under reflection M in PQ} \]

The isometry in which 2 points are invariant may be expressed as a single reflection.
The reader is reminded that the isometries, which ensure the object-image correspondence of a pair of equal line-segments, leave just two possibilities for ensuring the correspondence of a third pair of points in symmetry position depending on whether that isometry is direct or opposite.

**Case 3:** 1 invariant point. The 2 possible cases, (a) direct, (b) opposite, are considered separately:

(a) **direct:** the correspondence is denoted by $\overrightarrow{PQR} \rightarrow \overrightarrow{PQ'R'}$. The one invariant point $P$ may be regarded as an object-image correspondence under reflection in an axis through $P$. Hence: $PQ \rightarrow PQ'$ under reflection $M_1$ in the mediator $m_1$ of $QQ'$ which passes through $P$.

and: $\overrightarrow{PQR} \rightarrow \overrightarrow{PQ'R'_1}$ under an opposite isometry $M_1$.

We look for a second movement such that $PQ'R'_1 \rightarrow PQ'R'$ in which there are 2 invariant points $P$, $Q'$.

Using case 2: $PQ'R'_1 \rightarrow PQ'R'$ under reflection $M_2$ in $PQ'$ as axis.

We write: $M_1$ f.b. $M_2$: $\overrightarrow{PQR} \rightarrow M_2(\overrightarrow{PQ'R'_1}) \rightarrow \overrightarrow{PQ'R'}$.

The direct isometry in which just 1 point is invariant may be expressed as the product of 2 reflections. (Fig. 4.14(i)).

(b) **opposite:** the correspondence is denoted by $\overrightarrow{PQR} \rightarrow \overrightarrow{PQ'R'}$ and, as in case 3(a), again:

$\overrightarrow{PQR} \rightarrow \overrightarrow{PQ'R'_1}$ under reflection $M_1$ in mediator $m_1$ of $QQ'$ which passes through $P$.

There can be only 2 image positions corresponding to the third object position $R$. One of those positions has been identified as $R'$ in case 3(a). The second and only other possible position is $R'_1$ already identified (by construction):

* A notation adopted by E A Maxwell to distinguish between direct and opposite symmetry.
\[ \overline{PQR} \rightarrow \overline{PQ'R_1'} = \overline{PQ'R'} \] under reflection \( M_1 \) in \( m_1 \).

The opposite isometry in which just 1 point is invariant may be expressed as a single reflection. (Fig 4.16(i)).

Pupils may like to convince themselves that the same given correspondence may be obtained as the result of two distinct movement combinations and are invited to start with reflection in the mediator of \( RR' \), which passes through the invariant point \( P \) instead of the mediator of \( QQ' \).

**Case 4:** 0 invariant points: Let \( PQR \rightarrow P'Q'R' \). It is known that \( P \rightarrow P' \) under reflection \( M_1 \) in the mediator \( m_1 \) of \( PP' \). Hence
\[ \overline{PQR} \rightarrow \overline{P'Q'R_1} \] under reflection \( M_1 \) in \( m_1 \).

We now look for a second movement which leaves \( P' \) fixed in its correct image position and which takes \( Q \) on to \( Q' \). Using case 3 we know that
\[ \overline{P'Q_1R_1} \rightarrow \overline{P'Q'R_2} \] under reflection \( M_2 \) in mediator \( m_2 \) of \( Q_1Q' \) through \( P' \).

So far: \( \overline{PQR} \rightarrow \overline{P'Q'R_2} \) under \( M_1 \ f.b. \ M_2 \).

Again there are 2 situations to be considered:

(a) **direct:** if it is known that \( PQR \rightarrow P'Q'R' \) under a direct isometry, then \( R_2 \) coincides with \( R' \) and we write:
\[ M_1 \ f.b. \ M_2: \ \overline{PQR} \rightarrow M_2(\overline{P'Q_1R_1}) = \overline{P'Q'R'} \]

The direct isometry in which there are no invariant points may be expressed as the product of 2 reflections.
(b) opposite: if it is known that PQR → P'Q'R' under an opposite isometry, then we look for a third movement which leaves P', Q' fixed in their correct image positions and which takes R₂ to its image position R'. Using case 2 we know that:

\[ \overrightarrow{P'Q'R_2} \rightarrow \overrightarrow{P'Q'R'} \] under reflection \( M_3 \) in \( P'Q' \).

We write:

\[ M_1 \text{ f.b. } M_2 \text{ f.b. } M_3: \overrightarrow{PQR} \rightarrow M_2 \text{ f.b. } M_3: \overrightarrow{P'Q_1R_1} \rightarrow M_3: \overrightarrow{P'Q'R_2} \rightarrow \overrightarrow{P'Q'R'} \]

The opposite isometry in which there are no invariant points may be expressed as the product of 3 reflections (Fig 4.17).

![Diagram](https://via.placeholder.com/150)

**Fig 4.17**

The four cases considered are mutually exclusive and the results are summed up in the following *Fundamental Theorem II*:

**Theorem** Every isometry is the product of at most three reflections.

The pupils will have verified the truth of the proposition for themselves in a 'proof by exhaustion' and will no doubt by this time be exhausted themselves. Is this a bad thing?!
The process of §4.5 in which an attempt is made to define all possible symmetry movements by construction on the one hand, and of the current §4.6 in which all possible combinations of reflection under 'followed-by' are considered on the other hand, are reversible. It is up to the pupils, under guidance from the teacher, to select the 'rules' of the particular 'game' to be played and to adopt a 'winning strategy' to achieve results in the form of mathematical truths.

It has been seen that the pupils have been led almost subconsciously into defining, by 'implication of the meaning' of words used, a law of operation 'followed-by'. With the Fundamental Theorem II they are now in a position to describe the geometry of the isometries algebraically.

4.7 The Pattern of a Group

The elements of the universal set have been defined as the set of isometries T, R, M, G. A binary operation 'followed-by' has been defined 'by implication'. Some of the 'products' have been recorded in the table, and/or obtained in the course of the investigations, on pages 130, 131. The table, if not already completed as a result of geometrical construction, may now be completed algebraically. For example:

\[
T \text{ f.b. } G = T \text{ f.b. } (T \text{ f.b. } M) \\
= (T \text{ f.b. } T) \text{ f.b. } M \\
= T \text{ f.b. } M \\
= G
\]

Again \[
R \text{ f.b. } M = (M_1 \text{ f.b. } M_2) \text{ f.b. } M \\
= M/G
\] (using results (i)-(iv) on pages 130, 131)

These results may now be inserted in the Table 4.1 on page 131.
The completed table is given below:

<table>
<thead>
<tr>
<th>f.b.</th>
<th>2nd isometry</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>M</td>
<td>G</td>
</tr>
<tr>
<td>G</td>
<td>M/G</td>
</tr>
</tbody>
</table>

Table 4.2

The universal set of isometries is seen to be closed under the operation 'followed-by'. Associativity of the operation has been assumed in the 'product' operation. There is an identity element (see page 133) which may be described variously as, for example, a zero translation in any direction or a rotation $\alpha = 2k\pi$ where $k$ is an integer. Every operation can be reversed and has an inverse which moves the image set back to the object position. The system is in every sense 'respectable' under the operation 'followed-by'. It has the pattern of the group structure already experienced in the work of Chapters 2 and 3 on number systems.

In Chapters 2 and 3 we were concerned with number systems which could be illustrated geometrically. Here we have a geometrical system which is being treated algebraically. The algebra of the system will be developed in Chapter 6 in Part II. But first the author requires the extension of the pattern of a group structure to the structure of a linear algebra.
PART II

THE LINEAR ALGEBRA OF A VECTOR SPACE
CHAPTER 5

EXTENDING THE PATTERN:
FROM GROUPS TO LINEAR ALGEBRA

5.1 The Need to Extend the Pattern

An attempt has been made in Part I to show how the school mathematics curriculum has been extended in recent years to take account of the existence of not just one classical algebra of number systems but of more than one algebra of systems of universal sets of non-numerical elements; of not just one Euclidean geometry but of more than one non-Euclidean geometry of a not flat but curved space. Such developments are crucial for the use of mathematics in an accurate and meaningful description of the physical phenomena of the world in which we live and for the efficient handling of the technology of the modern era.

A group structure has sufficed for the arithmetic of number systems and the geometry of Euclid. The pattern must now be extended to that of a 'modern' linear algebra. The author asks the reader to consider first the solution of a set of linear equations. The quotations, except where otherwise stated, are taken from the answer to a written question in the examination for the degree of Master of Arts by Mr Michael Davidson.
5.2 The Solution of a Set of Simultaneous Linear Equations

School texts provide many and varied methods for solving a set of simultaneous linear equations in two or more unknowns. In the earlier years of a secondary mathematics course such a pair of equations is considered in one or more of the following ways.

(1) A 'trial-and-error' or 'hit-and-miss' exploratory investigation leads to an infinity of pairs of numbers which may be found to fit one of the two given equations. These are then substituted in turn into the second of the two equations in an attempt to find one - possibly the only one? if any? which fits the given pair of equations simultaneously. The method is useful in anticipation of later work in approximation and in step-by-step or iterative processes appropriate to the use of the microcomputer.

(2) In a graphical solution the equations are represented by a graph, consisting of two straight lines, which enables an approximate solution to be read - to what degree of accuracy? provided that a solution does in fact exist. A graph illustrates vividly the geometry of the situation in which there is no solution: geometrically the lines are parallel; and in which there are infinitely many solutions: geometrically the lines are coincident.

Work with pupils on the lines of (1) and (2) above is described by Mr Davidson, a deputy headmaster and specialist mathematics teacher in a 13 to 18 comprehensive school.

"What are simultaneous equations and why do we need them? ... has to be seen to be answered before we leave the real world behind. One possible approach is given below.

"If my age is twice your age, how old am I?" John who is 13 would obviously give the answer 26. Mary who is 14 would give the answer 28. Discussion could follow and the question could be modified to:
'If a man's age is twice the child's age, how old is the child?' The problem should emerge! We need to know the child's age and without it there is a series of pairs which could be: 7 + 14, 18 + 36, 9 + 18, ..., etc. To find a particular solution we need more information. This can develop until children have grasped the fact that two pieces of information are required.

"The example I use at this stage is based on Alice Through the Looking Glass and goes as follows:

Tweedledum says to Tweedledee: 'My weight and twice your weight is 361 pounds'. Tweedledee says to Tweedledum: 'Contrariwise my weight and twice your weight is 362 pounds'.

What are their weights?

"This leads to plenty of discussion. Some remarks I have encountered are given below:

'361 and 362 are close together so their weights must be close together'
- correct conclusion, possibly incorrect reasoning

'Twice one weight and the other is nearly the same as switching round so they must be nearly the same'
- correct conclusion, possibly correct reasoning.

"A suggested answer from the first remark was about 180. This was immediately seen to be incorrect and the second person gradually modified this to 120 by explaining that 'we should be dividing by 3 if they were roughly the same weight'.

"Eventually the weights were seen to be 120 and 121 pounds. The value of the discussion was not in obtaining the correct answer but in getting the children to realise that any answer could be checked by going back to the original problem. No mention was made of equations, symbols or writing anything down on paper. Many more carefully chosen examples like this arouse the children's interest and give them confidence in their own ability.

* Susan Denton used this example in a dissertation prepared for the BSc Integrated (Honours) Mathematics/Education degree at the University of Hull (Exhibit)
"To communicate answers on paper it is necessary to use some form of language other than English - because it takes too long and is often misunderstood. Eventually some child mentions some previous work on linear equations and the natural development leads to what, as a teacher, we are expecting:

\[
\begin{align*}
    x + 2y &= 361 \\
    2x + y &= 362
\end{align*}
\]

"This is not a suitable pair of equations for solving as a first example so another much easier problem is used from which the equations (say):

\[
\begin{align*}
    x + y &= 8 \\
    x - y &= 2
\end{align*}
\]

are derived. Guesswork is encouraged and most children quickly see the solution: \(x = 5, y = 3\). No method has yet been formally mentioned. Many more examples like this can be given to build up confidence and to reinforce in the children's minds that the truth of the answer can be easily checked. This is a very important point because when the questions are more difficult and the methods more sophisticated we still need the children to be able to see that they can check by concrete methods.

"At this stage I mention the name of the topic so that it is easier to refer to in discussion. I also mention that we are 'going through the looking glass' into the world of mathematics. This might be disapproved of by other teachers but I want to show the children that mathematics is really a subject of the mind - we do not have to think of the real world. I find this meets with approval from them. This also avoids the problem of continually having to translate from problem into mathematical language - which proves difficult for many and often stops them from enjoying the mathematics."

The author found this an interesting and unexpected comment.
The examples are made gradually more difficult to justify the need of a better method of solution. Pupils will have had experience of plotting linear graphs and graphical representation is seen as a useful development at this stage. For example the solution of:

\[
\begin{align*}
  x + y &= 8 \\
  x - y &= -2
\end{align*}
\]

as \(x = 3; \ y = 5\)

may be represented as in Fig 5.1.

The solution of these two equations is likely to prove troublesome by guess-work and the ease with which they may be solved using a graph is persuasive. Most pupils will appreciate that the solution is at the point of intersection of the two lines and that it 'makes sense'. More graphical examples may be given to emphasise the connection between the graph and the equations which it represents. Exceptional cases such as

\[
\begin{align*}
  x + y &= 2 \\
  2x + y &= 6
\end{align*}
\]

and

\[
\begin{align*}
  2x + 2y &= 4 \\
  4x + 2y &= 5
\end{align*}
\]

may be included to prepare the way for more advanced considerations later. Pupils should recognise that, in the first case, the lines are 'identical' and there are 'many solutions' and that, in the second case, the lines are 'parallel' and there are 'no solutions'. The disadvantages of a graphical solution will become apparent from a few well-chosen examples, such as:
It is expected that:

\[ x = 1.3, \quad y = 2.7 \]

will be given as the solution, read from the graph (Fig 5.2). When checked against the equations it will be found 'not to be correct'. It is expected that discussion will lead eventually to: \( x = 1\frac{1}{3}, \quad y = 2\frac{2}{3} \). It is suggested that the graph should now be drawn using a larger scale on the x- and y-axes*. This will enable a more accurate reading to be made and a closer approximation than \((1.3, 2.7)\) to be given for the solution to the equations.

The search continues. Can the revised approximation be improved upon? Can a better method of solution be found when a solution cannot be obtained readily from a graph? The method of substitution is often found to be appropriate.

(3) Substitution is useful in the case when one of the given equations is in a form in which \( x \) is given explicitly in terms of \( y \) (or \( y \) in terms of \( x \)). That value of \( x \) (or \( y \)) is substituted into the second equation which may then be solved for \( y \) (or \( x \)).

The following observations are made concerning methods (1), (2) and (3). It is extremely helpful in this computer age to be able to arrive at (in say (1)) not only a 'good' approximation to the solution but also at an assessment of 'how good' is that approximation. If for no other reason,

* See also Exhibits
such an approximation enables us to check the accuracy of a solution arrived at by a more precise method of calculation or by the microcomputer [24]. Mr Davidson has (on page 143) referred to the need for "children to be able to see that they can check by concrete methods".

It is probably true to say that the majority of pupils whom we teach are visile rather than audile subjects and a graph as in (2) provides many of our pupils with a vivid picture of the problem. Many will in fact have used Pictorial Representation [6.2] in their primary schools. The reader is referred to an Exhibit* where an excellent account is given of the enlargement of the axis scales in the region of the solution to obtain a sequence of better and better approximations to the actual result.

The substitution method (3) is of limited application and useful only in the special case in which one of the equations is already in the 'explicit' form which can be reached as the second stage of a more general method of elimination now to be discussed. Mr Davidson states:

"Elimination rather than substitution is preferable for future development. I have found that it is also the most acceptable to the pupils. The idea that it is 'allowed' to multiply an equation by a scale factor without changing the meaning of the equation has already been experienced:

\[ x + y = 2 \text{ was seen (see page 144) to be the same line as: } 2x + 2y = 4 \]

Some time needs to be spent on showing that it is consistent to add equations together. The amount of time spent on this method - if any at all - depends on whether or not as a teacher you want to connect this topic with matrices and linear transformations."

The author was delighted with the implication of this last sentence and devotes the next section to the more general method of elimination.

*A dissertation by John Easton, an undergraduate preparing for the BSc Integrated (Honours) degree in Mathematics/Education at the University of Hull
5.3 The Elimination Method of Solution

The reader is invited to look closely at the traditional elimination method and analyse carefully the thought processes involved. At the same time the author sets out, in parallel, the sequence of steps involved in her - and that of many of her students - 'preferred' modern version in terms of a matrix (see pages 148, 149). The augmented matrix of the whole set of equations focuses attention on the coefficients which are the crucial known elements of the problem affecting the values of the unknowns $x$ and $y$ which have to be determined.

It is important to note that we are interested only in the numerical values of the coefficients which affect the answer to a pair of simultaneous linear equations. We work with these numbers in a single array:

\[
\begin{bmatrix}
3 & 4 & \vdots & 55 \\
5 & 2 & \vdots & 45
\end{bmatrix}
\]

which represents the pair of equations:

\[
\begin{align*}
3x + 4y &= 55 \\
5x + 2y &= 45.
\end{align*}
\]

This array is called the augmented matrix and the dotted line serves merely to remind us of the position of the 'equals sign' when transforming back into equation-form.

The traditional operations on the equations:

- multiplication by a scalar (real number) magnitude
- algebraic* (positive- or negative-) addition

now become operations on the rows of the augmented matrix. The operations are valid in view of their validity as operations on the equations which the rows of the augmented matrix represent.

* The term used by the author to denote 'addition or subtraction'
Traditional Language

Starting with:
3x + 4y = 55 (1)
5x + 2y = 45 (2)

The aim is to end up with the equations:

\[ x = \ ? \]
\[ y = \ ? \]

using only the operations of:

multiplication (or division)

and

addition (or subtraction)

how can we:

eliminate y to find x?

eliminate x to find y?

It is customary, once x (or y) is known, to substitute for x (or y) in one of the given equations to find y (or x). It is more consistent to repeat the elimination process to find the second unknown quantity.

Modern Language

\[
\begin{array}{ccc}
\text{col 1} & \text{col 2} & \text{col 3} \\
\text{row 1} & 3 & 4 & 55 \\
\text{row 2} & 5 & 2 & 45 \\
\end{array}
\]

the augmented matrix:

\[
\begin{bmatrix}
1 & 0 & \ ? \\
0 & 1 & \ ? \\
\end{bmatrix}
\]

scalar multiplication

and

algebraic addition

transform the augmented matrix to one which has a '1' in col 1, row 1; '0' in col 2, row 1; '0' in col 1, row 2; '1' in col 2, row 2 to find x and y explicitly

Notes

The augmented matrix. A dotted line is inserted to remind us of the position of the = sign.

of the set of equations:

\[ 1 \cdot x + 0 \cdot y = \ ? \]
\[ 0 \cdot x + 1 \cdot y = \ ? \]

multiplication by a positive or negative real no. - includes division as inverse multiplication

and

adding positively or negatively - includes subtraction as inverse addition
Step 1

Multiply (2) by 2 and subtract (1)

\[ 7x = 35 \]

Instead of finding \( x \) and substituting in (1) or (2), for consistency of method we:

Repeat the process to eliminate \( x \)

Step 2

Multiply (1) by 5 and (2) by 3 and subtract the results:

\[ 14y = 140 \]

giving: \( x = 5, y = 10 \) explicitly

<table>
<thead>
<tr>
<th>Multiplied row 1 by 2 (why 2?) and add the negative of row 2. Leave row 2 unchanged.</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & 6 & : & 65 \\
5 & 2 & : & 45 \\
\end{bmatrix}
\] |

<table>
<thead>
<tr>
<th>Leave row 1 unchanged. Replace row 2 by ( 5 \times \text{row 1} ) (why 5?) added to the negative of row 2.</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & 6 & : & 65 \\
0 & 28 & : & 280 \\
\end{bmatrix}
\] |

<table>
<thead>
<tr>
<th>Repeat the process to transform col 2.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leave row 1 unchanged. Replace row 2 by ( \frac{1}{28} \times \text{row 2} ) (why ( \frac{1}{28} )?)</td>
</tr>
</tbody>
</table>
| \[
\begin{bmatrix}
1 & 6 & : & 65 \\
0 & 1 & : & 10 \\
\end{bmatrix}
\] |

<table>
<thead>
<tr>
<th>Replace row 1 by row 1 added to (-6 \times \text{row 2}) (why (-6)?)</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & 0 & : & 5 \\
0 & 1 & : & 10 \\
\end{bmatrix}
\] |

<table>
<thead>
<tr>
<th>giving: ( x = 5, y = 10 ) explicitly</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>giving: ( x = 5, y = 10 ) explicitly</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>representing equations:</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ 1.x + 0.y = 5 ]</td>
</tr>
<tr>
<td>[ 0.x + 1.y = 10 ]</td>
</tr>
</tbody>
</table>

| i.e \( x = 5, y = 10 \) |
The object of the exercise is to transform the augmented matrix into **echelon form**:

\[
\begin{bmatrix}
1 & 0 & : & k_1 \\
0 & 1 & : & k_2
\end{bmatrix}
\]

to represent the equations in the simplest form from which the solutions can easily be seen:

1. \(x + 0.y = k_1\)
2. \(0.x + 1.y = k_2\)

merely by repetitive use of the same two operations which are the fundamental operations of a modern **linear algebra**.

It will be seen that the same process may be used to obtain the **inverse** of the coefficient matrix, should we need it, and this is dealt with in §5.6. If it is the solution to the set of simultaneous linear equations only that is required there is no need first to find the inverse of the coefficient matrix. Are there any other advantages to be gained from the use of the language of linear algebra in the form of the augmented matrix?

5.4 **Elimination? - or Augmented Matrix?**

It has often been difficult to get pupils to grasp the elimination process; confusion often arises as to:

- which unknown is to be eliminated?
- which equation is to be multiplied by which coefficient?
- are we to add? or to subtract? to leave an x-term and no y-term, or a y-term and no x-term?

Too much choice too soon can lead to confusion. Choice of method there must be. Pupils should be encouraged to select a method appropriate to the problem: it is better mathematically for pupils to tackle a problem in, say, three different ways
than to solve three problems using the same method for each. But once a technique for the solution to a problem has been selected, pupils will be helped if there is consistency of language in which that technique is described.

In developing the work from the left-hand side of the 'augmented matrix' and transforming firstly column 1, secondly column 2, we do not upset the 'echelon' pattern of 1's and 0's already established. The row-reduction of the 'augmented matrix' is completely general and may readily be extended systematically, column by column, at a later stage to the case of 3 simultaneous linear equations in 3 unknowns (see §5.7) and, in fact, to any number of simultaneous linear equations in any number of unknowns. Working column-by-column successively from the left-hand side we aim to transform the 'augmented matrix' to 'echelon' form:

\[
\begin{bmatrix}
1 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\
0 & 1 & 0 & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 1 & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

This is done by ensuring first a '1' in the correct row of the column being worked. Next a '0' is obtained in each and every other row of that column by 'algebraic addition' in turn of the appropriate scalar multiple of that 'unit' row.

The traditionalist teacher of mathematics using the 'method of elimination' will perhaps remember how cumbersome can be the extension to three or more dimensions in the symbolism of determinants*.

---

* It is suggested that the study of determinants should not be over-emphasized in the mathematics curriculum (see also §6.3(2) of Chapter 6).
The actual mathematics in the two developments, 'elimination' and 'augmented matrix', as set out in traditional and modern language on pages 148, 149 is identical. It is only the language and the symbolic representation of that language that differs in the two cases.

We are working here with a set of non-numerical elements which are the linear equations. The universal set is the set of all linear equations in any or all of a number of unknowns; the example discussed on pages 148, 149 consists of a pair of linear equations in just 2 unknowns and is a subset of the universal set.

The traditional form of 'solution by elimination' to find one or other or both, except in certain circumstances, of the unknowns involves the choice of 'suitable multiple(s)' of the equation(s) and the 'algebraic addition' of the resulting pair of equations. The modern method, which uses the 'augmented matrix' of the equations, involves the same operations: the 'scalar multiple(s)' of the row(s) which represent the equation(s) are 'algebraically added' together.

The mathematical activity is the same: transformation by algebraic sums of multiples of a given set of equations in traditional language on the one hand, or of rows of the 'augmented matrix' which represents it in modern language on the other hand, into either a more manageable set of equations or a simpler 'echelon' form of 'augmented matrix' which represents it.

There is another, more important, bonus afforded by the use of the row-reduction technique applied to a matrix.
5.5 The Inverse Matrix

In the early days of the recent reform of the mathematics curriculum in schools, a number of 'new' and apparently disparate topics was suggested for inclusion in the syllabus. 'Matrices' was one of them and, perhaps because of their unfamiliarity, some teachers tended to synonymise the 'matrix' with 'modern mathematics' itself! Whilst working with a team of mathematics specialist teachers in an early session of an in-service training course in 1969 the author was presented with the rather sardonic comment:

"stop beating about the bush; show us how to solve simultaneous equations using 'matrices' and then we can go back to our schools and 'do' modern mathematics"!

The reader is referred also to T J Fletcher [25] on this point.

The following example shows how it was 'done' in an early version of a project*text [4.4] and illustrates a use of matrices that is ill-advised at secondary level.

\[
\begin{align*}
\begin{bmatrix} x + y &= 8 \\
x - y &= 2 \\ y &= 8 \\
\end{bmatrix}
\end{align*}
\]

which focuses attention on the coefficients on which the values of \( x, y \) depend. Pupils, whose background foundations in mathematics have included 'shopping-lists' as matrices, will see that this is merely a special case of the matrix multiplication rule. For others it is a convenient symbolic way of displaying the given information which may be further

* The SMP Teachers Guide to §3.5 of Book 4 states: "All that is done in this section is to suggest the use of an inverse matrix is a simple extension to the use of inverse elements in the solution of equations".
abbreviated to:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= B
\]

where \( A \) and \( B \) are matrices. To solve for the unknown matrix \( \begin{bmatrix}
  x \\
  y
\end{bmatrix} \), methods for the solution of simple equations dictate that we should pre-multiply both sides of the 'matrix equation' by the inverse matrix \( A^{-1} \) of \( A \). Subject to the existence and uniqueness of \( A^{-1} \). This gives

\[
A^{-1}
\begin{bmatrix}
  \begin{bmatrix}
    x \\
    y
  \end{bmatrix}
\end{bmatrix}
= A^{-1}B
\]

It is to be noted that we have pre-multiplied both sides of the equation by \( A^{-1} \); we cannot at this stage assume that 'matrix multiplication' is commutative; in fact it is not. But the question of commutativity need not concern the reader here. Instead, if the associativity of 'matrix multiplication' is assumed, then the equation may be written:

\[
(A^{-1}A)
\begin{bmatrix}
  \begin{bmatrix}
    x \\
    y
  \end{bmatrix}
\end{bmatrix}
= A^{-1}B
\]

giving:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= A^{-1}B
\]

where \( I \) denotes the unit matrix \( \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix} \) of dimension 2.

This in turn gives

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
  -1 & -1 \\
  -1 & 1
\end{bmatrix}
\begin{bmatrix}
  8 \\
  2
\end{bmatrix}
\]

ie

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= \begin{bmatrix}
  5 \\
  3
\end{bmatrix}
\]

One teacher has commented that

"Many children find this method acceptable without really understanding the reasons behind it and having to rely on rote-learning to find the inverse matrix".

Taking for granted the associativity of 'matrix multiplication' and the existence and uniqueness of the multiplicative identity and inverses, the success of the method for the pupils depends on their ability actually to find the inverse.

Returning to a 'matrix equation' in the general form for the purpose of this discussion:

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\begin{bmatrix}x \\
y
\end{bmatrix} =
\begin{bmatrix}c_1 \\
c_2
\end{bmatrix}
\]

which represents the equations:

\begin{align*}
a_1x + b_1y &= c_1 \\
a_2x + b_2y &= c_2
\end{align*}

pupils could be asked to construct a 2 × 2 matrix:

\[
\begin{bmatrix}
p_1 & q_1 \\
p_2 & q_2
\end{bmatrix}
\]

(say), such that:

\[
\begin{bmatrix}
p_1 & q_1 \\
p_2 & q_2
\end{bmatrix}
\begin{bmatrix}a_1 & b_1 \\
a_2 & b_2
\end{bmatrix} =
\begin{bmatrix}1 & 0 \\
0 & 1
\end{bmatrix}
\]

apart from a possible scalar multiple (?!)

This would require that:

\begin{align*}
p_1a_1 + q_1a_2 &= 1 \\
p_1b_1 + q_1b_2 &= 0 \\
p_2a_1 + q_2a_2 &= 0 \\
p_2b_1 + q_2b_2 &= 1
\end{align*}

The pupils are faced with the problem of finding 4 unknowns \( p_1, p_2, q_1, q_2 \), from 4 simultaneous linear equations. They started with 2 equations in only 2 unknowns! A closer look, of course, reveals that the set of 4 equations consists of 2 pairs of equations: one pair in the 2 unknowns \( p_1, q_1 \); the other pair in the 2 unknowns \( p_2, q_2 \). Nevertheless, they are no nearer a solution to the original problem: that of
solving 2 simultaneous linear equations in 2 unknowns. Furthermore, it would appear that they have landed themselves in a circular argument!

To obtain the inverse of a $2 \times 2$ matrix: \[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\]
pupils are often taught the rule:

- interchange $a_1$ and $b_2$;
- change the signs of $a_2$ and $b_1$;
- divide the result by: $(a_1b_2 - a_2b_1)$ which is the determinant of the matrix
to give:

\[
\text{Inverse of: } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \frac{1}{(a_1b_2 - a_2b_1)} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.
\]

There is nothing wrong, in the opinion of the author, with a 'rule of thumb' provided that the pupils have a viable method of obtaining the result for themselves should they forget the rule! By a viable method is meant one which does not lead to mathematical inconsistencies and/or circular arguments. Not one of the author's students on initial teacher-training courses has, at the time of writing, been able to assert that such a technique was made available to them at school. And, she asks, "How would they extend that rule to 3 or more than 3 dimensions?"

Furthermore, many experienced teachers tell the author that they cannot teach the solution of a pair of simultaneous linear equations in 2 unknowns earlier than the fourth year of the secondary course. In former days the author 'did it' in year 2 or year 3! and was delighted when she read Mr Davidson's account of his further experiences with his 13 to 14 year-old pupils. He writes:

"There is another perfectly acceptable matrix method which does have future advantages.... and for the most able it gives a solid basis for more advanced work and investigation."
5.6 The Inverse of a Matrix by Row-Reduction

The author regards the row-reduction of a matrix to derive its inverse as essential mathematics background for teachers. It enables them to justify the 'rule-of-thumb'

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

if and when required to do so.

Work with pupils in the 13 to 14 age range indicates that few have the necessary algebraic manipulative skills at this stage to appreciate the implications. The value of row-reduction for them lies in its application to the solution of a set of simultaneous linear equations as a foundation for its further use at sixth-form level in the upward spiral development of their mathematical education.

There is an excellent account of the process by T J Fletcher [25]. But even teachers, the author feels, may find this difficult to follow. She prefers the following presentation which her students readily accept.
A sequence of operations in terms of 'sums of scalar multiples' of rows of a matrix has been applied step-by-step and simultaneously to a given matrix $A$ and the unit matrix $I$.

If the resulting 'product' operation is denoted by $\Omega$ we have:

$$\Omega A \rightarrow I \quad \text{and} \quad \Omega I \rightarrow M$$

ie $\Omega = A^{-1}$ assuming $A^{-1}$, as inverse of $A$, exists

and: $\Omega = M$ assuming the property of a multiplicative identity matrix $I$.

It follows that $M$ is an inverse of the matrix $A$. Of course it still has to be established that $M$ is the unique inverse in a respectable mathematical system under multiplication.
5.7 The Solution of 3 Simultaneous Linear Equations in 3 Unknowns

The row-reduction method of solution of a pair of linear equations in 2 unknowns and the geometrical interpretation in terms of the intersection of lines in 2 dimensions is readily extended to the solution of a set of 3 linear equations in 3 unknowns and the geometrical interpretation in terms of the intersection of planes in 3 dimensions. The work is again fully discussed by T J Fletcher [25] but the author suggests that, as in 2 dimensions, the algebraic-geometric links in 3 dimensions can be more realistically forged through a consideration of the symmetries of the Platonic solids. (See Chapter 10 and Exhibits)

5.8 Pupil Reaction

Teacher comments on the row-reduction method of solution of a pair of simultaneous linear equations include the following:

Pupils will modify the method to suit themselves.

(1) \[
\begin{array}{ccc}
1 & 3 & : 13 \\
0 & 1 & : 4 \\
\end{array}
\]
was converted back to:
\[
\begin{array}{c}
x + 3y = 13 \\
y = 4 \\
=> x = 1 \\
\end{array}
\]
by a pupil who "really preferred the algebraic (?elimination; ?substitution) method and wanted to get back to it as quickly as possible".

(2) \[
\begin{array}{ccc}
5 & 1 & : 6 \\
3 & 2 & : 5 \\
\end{array}
\]
was reduced to \[
\begin{array}{ccc}
0 & 1 & : 1 \\
1 & 0 & : 1 \\
\end{array}
\]
=> \[
\begin{array}{c}
x = 1 \\
y = 1 \\
\end{array}
\]
by a pupil who had noticed that "if it is easier to achieve \[
\begin{array}{c}
0 & 1 \\
1 & u \\
\end{array}
\]
then that is alright". But care must be taken! Would she interpret the following answer correctly?
(3) \[
\begin{bmatrix}
5 & 1 & 13 \\
3 & 2 & 12
\end{bmatrix}
\] reduced to \[
\begin{bmatrix}
0 & 1 & 3 \\
1 & 0 & 2
\end{bmatrix}
\Rightarrow y = 3, x = 2
\]
She in fact did!

5.9 Further Developments

The matrix representation of the equation lends itself to the identification by the pupils with earlier work on transformation geometry. For example

\[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
x + 2y \\
y
\end{bmatrix}
\]

which is a shear with the line \( y = 0 \) invariant, may be illustrated graphically:

A pupil should be encouraged to see that solving a pair of simultaneous linear equations is mathematically the same as performing a series of geometrical transformations. It is this interpretation of the experiences in the lower/middle secondary years that lies behind the application of row-reduction to derive the inverse of a matrix.

At all stages of development the teacher of mathematics must be aware of the help which is now afforded by suitable microcomputer programs. Much of the graphical work can be abbreviated by allowing the microcomputer to draw the graphs. Software is now available and simulated 3-D graphics allow investigation of the way in which three planes can intersect each other. Many pupils will thereby have a good
insight into the algebraic complexities without getting bogged down with the calculations. Much of the investigative work can be speeded up and pupils have the opportunity of seeing more clearly the connection between algebra and geometry.

The author returns to the parallel development of 2-dimensional algebra and geometry in Chapters 6 and 7. But first she wishes to say a little more for the teacher about the background mathematics of a 'modern' system of linear algebra.

5.10 Sums of Multiples

The solution of a set of simultaneous linear equations is seen as the result of the repeated application of the composition of a pair of operations as an 'algebraic sum of multiples': 'algebraic addition' is the binary operation of addition, or its subtractive inverse, of a group structure; 'multiplication by a scalar' is a unary operation which multiplies a single element of a universal set of linear equations.

The operation of 'algebraic addition' will now be replaced by the binary operation of 'addition' of a group structure and 'multiplication by a scalar' will be replaced by the unary operation of 'scalar multiplication' in which the scalar multiple will, for the moment, be taken as a real positive or negative number. The universal set of all linear equations has the structure of an additive group and the system is extended to that of a vector space under the unary operation of 'scalar multiplication'. The solution of a set of simultaneous linear equations has been given in terms of the language of linear algebra and illustrates the pattern of this new system. The elements of the system are called vectors.

5.11 The Pattern of a Vector Space

Various formal definitions of a vector space may be found: for examples, see T J Fletcher [25] and Nicolaus H Kuiper [26]. In the context of the current development the author prefers
that given by J A H Shepherd and C J Shepherd [27.2]:

"A vector space $V$ is a set of vectors, $V = \{0, a, b, c, \ldots \}$ which is an abelian (commutative) group under addition with zero $0$, and which is closed under multiplication by the elements* of a field $F$, $F = \{0, 1, x, y, \ldots \}$, such that for all $a, b \in V$ and $x, y \in F$:

$(i) \quad (xy)a = x(ya)$

$(ii) \quad (x + y)a = xa + ya$

$(iii) \quad x(a + b) = xa + xb$

$(iv) \quad 1a = a$

Thus an additive group structure is defined on a universal set of vector elements and a unary operation of multiplication by a scalar is defined to satisfy the following further conditions:

$(i) \quad$ associativity of scalar multiplication of a vector

$(ii) \quad$ distributivity over scalar addition of a scalar multiple of a vector

$(iii) \quad$ distributivity of scalar multiplication over vector addition

$(iv) \quad$ existence of a scalar multiplicative identity

5.12 Linear Dependence: Basis; Dimension

Closure of a vector space under 'scalar multiplication' and 'addition' means that any vector element may be expressed as 'sums of multiples' of one or more of the other vector elements: the one is said to be expressed linearly in terms of the others. Furthermore if we restrict ourselves to finite linear expressions [26] there is a certain least number of vectors which may be picked out and, when this is done, all other vectors in the set may be expressed linearly in terms

* The real numbers at this stage of the development
of these: the one is said to be *linearly dependent* on the others. Any such subset of the vectors is called a *basis* for the system. It follows that a set of basis vectors are *linearly independent* of each other.

The search for a basis for the vector space is a step-by-step process. It is convenient to consider the following questions in sequence:

1. can any given vector be expressed as a 'scalar multiple' of any **one** other given vector?  
2. If the answer is "no", then:  
   can any given vector be expressed as the 'sum of scalar multiples' of any **two** given vectors?  
3. If the answer is again "no", then:  
   can any given vector be expressed as the 'sum of scalar multiples' of any **three** given vectors?  
   
   and so on, until an answer in the affirmative is obtained. The corresponding number of vectors is the smallest number that is required to form a basis and is called the *dimension* of the vector space.

A number of relevant applications* of a vector space, suitable for discussion with pupils in the 9 to 18 age-range, have been investigated by T J Fletcher [25] and supplemented by the author in articles under the title: "Linearity - its place in the school curriculum" [28]. No apology is made for a reconsideration here of the topic of arithmetic progressions. It is to be very much regretted that sequences and series have, to a very large extent, been squeezed out of recent secondary mathematics syllabuses. Yet number patterns form a major part of many mathematical experiences of the primary years.

* These include Magic Squares, Curve Fitting, Difference Tables, Binomial Theorem, etc
The treatment of sequences and the summation of series in general, and of the arithmetic progression in particular, in terms of the language of the linear algebra of a vector space enables the elementary polygonal number patterns: Pascal's triangle of numbers, Fibonacci sequence, ..., to be generalised in the symbolic language of vectors. It lends itself readily to geometrical representation of the 'term-by-term addition' and 'scalar multiplication' operations of vector algebra in a way similar to the graphical representation of the directed whole-number 'cross-totting' families of Chapter 2. It also affords an excellent example of the extension at sixth-form level of the use of row-reduction and the avoidance of determinants in the elimination of variables from a set of linear conditions.

5.13 Arithmetic Progressions as a Vector Space

An arithmetic progression $A$ is completely determined by the first term $a$ and the common difference $d$ between consecutive terms:

$$A = \{a, a+d, a+2d, \ldots, a+(r-1)d, \ldots\}$$

The reader is asked to consider a set of vector elements:

$$V = \{0, A_1, A_2, A_3, \ldots, A_{r-1}, \ldots\}$$

where $A_r$ is the arithmetic progression whose first term is $a_r$, common difference $d_r$: $A_r$ may be represented by the column vector\* $\begin{bmatrix} a_r \\ d_r \end{bmatrix}$. Attention is drawn to the fact that each $A_r$ represents a set of numbers which are the terms in an arithmetic sequence. $0$ is the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which represents the arithmetic progression whose first term is $0$, common difference is $0$.

\* The author prefers the term 'column matrix' as the symbol for the abstract 'vector' but is in a minority.
Let $K$ be a set of scalar, real number, quantities:

$$K = \{0, 1, k_1, k_2, \ldots, k_{r-2}, \ldots \}$$

The operations of addition and scalar multiplication of a vector space are defined for the arithmetic progression vector elements as 'term-by-term' addition and scalar multiplication:

$$A_p + A_q = \begin{bmatrix} a_p \\ d_p \end{bmatrix} + \begin{bmatrix} a_q \\ d_q \end{bmatrix}$$

$$\begin{align*}
&= \{(a_p + r-1.d_p) + (a_q + r-1.d_q): 1 \leq r\} \\
&= \{(a_p + a_q) + r-1.(d_p + d_q): 1 \leq r\} \\
&= \begin{bmatrix} a_p + a_q \\ d_p + d_q \end{bmatrix}
\end{align*}$$

and:

$$kA = k \begin{bmatrix} a \\ d \end{bmatrix}$$

$$\begin{align*}
&= \{k(a + r-1.d): 1 \leq r\} \\
&= \{(ka + k.r-1.d: 1 \leq r\} \\
&= \begin{bmatrix} ka \\ kd \end{bmatrix}
\end{align*}$$

Note that the arithmetic progression vector elements abide by the familiar rules of addition and scalar multiplication of 'traditional' vectors.

It is easy to show that the universal set of arithmetic progression vector elements under 'term-by-term addition' have the structure of a group. The further properties required of the system under 'scalar multiplication' for it to constitute a vector space are also satisfied.

* Care is needed to distinguish between the rule for scalar multiplication of a vector and that for row reduction of an augmented matrix representing a set of simultaneous linear equations.
(i) \[(k_p k_q) \begin{bmatrix} a_r \\ d_r \end{bmatrix} = \begin{bmatrix} (k_p k_q) a_r \\ (k_p k_q) d_r \end{bmatrix}\]

and: \[k_p \begin{bmatrix} k_q \begin{bmatrix} a_r \\ d_r \end{bmatrix} \end{bmatrix} = k_p \begin{bmatrix} k_q a_r \\ k_q d_r \end{bmatrix} = \begin{bmatrix} k_p k_q a_r \\ k_p k_q d_r \end{bmatrix}\]

ie \[(k_p k_q) \begin{bmatrix} a_r \\ d_r \end{bmatrix} = k_p \begin{bmatrix} k_q \begin{bmatrix} a_r \\ d_r \end{bmatrix} \end{bmatrix}: \text{ associativity of scalar multiplication}\]

(ii) \[(k_p + k_q) \begin{bmatrix} a_r \\ d_r \end{bmatrix} = \begin{bmatrix} (k_p + k_q) a_r \\ (k_p + k_q) d_r \end{bmatrix}\]

and: \[k_p \begin{bmatrix} a_r \\ d_r \end{bmatrix} + k_q \begin{bmatrix} a_r \\ d_r \end{bmatrix} = \begin{bmatrix} k_p a_r + k_q a_r \\ k_p d_r + k_q d_r \end{bmatrix}\]

\[= \begin{bmatrix} (k_p a_r + k_q a_r) \\ (k_p d_r + k_q d_r) \end{bmatrix}\]

ie \[(k_p + k_q) \begin{bmatrix} a_r \\ d_r \end{bmatrix} = k_p \begin{bmatrix} a_r \\ d_r \end{bmatrix} + k_q \begin{bmatrix} a_r \\ d_r \end{bmatrix}: \text{ distributivity over scalar addition}\]

(iii) \[k \left( \begin{bmatrix} a_r \\ d_r \end{bmatrix} + \begin{bmatrix} a_s \\ d_s \end{bmatrix} \right) = k \begin{bmatrix} (a_r + a_s) \\ (d_r + d_s) \end{bmatrix}\]

\[= \begin{bmatrix} k a_r + ka_s \\ k d_r + kd_s \end{bmatrix}\]

\[= \begin{bmatrix} ka_r \\ kd_s \end{bmatrix} + \begin{bmatrix} ka_s \\ kd_s \end{bmatrix}\]
167

\[ d = k \begin{bmatrix} a_r \\ d_r \end{bmatrix} + k \begin{bmatrix} a_s \\ d_s \end{bmatrix} : \text{distributivity of scalar multiplication} \]

(iv) \[ \begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} 1.a \\ 1.d \end{bmatrix} = \begin{bmatrix} a \\ d \end{bmatrix} : \text{scalar multiplicative identity} \]

Hence the universal set of arithmetic progressions forms a vector space. We need to select a set of basis vectors for that space; but we first need to know the dimension of that space.

5.14 Arithmetic Progressions: Linear Dependence

(1) Dimension

We know that any given arithmetic progression vector can be expressed linearly as sums of scalar multiples of one or more arithmetic progression vectors. What is the least number that is required for that linear expression? A sequence of questions needs to be answered.

Is it possible to express a given arithmetic progression vector \( \begin{bmatrix} a \\ d \end{bmatrix} \) linearly in terms of one other arithmetic progression vector \( \begin{bmatrix} a_1 \\ d_1 \end{bmatrix} \)? This requires the existence of a scalar multiple \( k_1 \) such that:

\[ k_1 \begin{bmatrix} a_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} a \\ d \end{bmatrix} \]
This vector equation is equivalent to the 2 scalar linear equations:

\[ k_1 a_1 = a \]
\[ k_1 d_1 = d \]

In general no such value of \( k_1 \) exists, except in the case when \( a_1 d_1 = a d \). But this is merely the condition that the two equations are identical and not linearly independent of each other: in general any two arithmetic progression vectors are linearly independent of each other.

Is it possible to express a given arithmetic progression vector \( \mathbf{A} = \begin{bmatrix} a \\ d \end{bmatrix} \) linearly in terms of two other arithmetic progression vectors, \( \mathbf{A}_1 = \begin{bmatrix} a_1 \\ d_1 \end{bmatrix} \) and \( \mathbf{A}_2 = \begin{bmatrix} a_2 \\ d_2 \end{bmatrix} \)? This requires the existence of two scalar multiples \( k_1 \) and \( k_2 \) such that:

\[ k_1 \mathbf{A}_1 + k_2 \mathbf{A}_2 = \mathbf{A} \quad (1) \]

\[ \begin{align*}
&k_1 \begin{bmatrix} a_1 \\ d_1 \end{bmatrix} + k_2 \begin{bmatrix} a_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} a \\ d \end{bmatrix} \\
&k_1 a_1 + k_2 a_2 = a \\
&k_1 d_1 + k_2 d_2 = d \\
\end{align*} \quad (2) \]

In general it is possible to solve these two simultaneous linear equations for \( k_1 \) and \( k_2 \). The expression for \( \mathbf{A} \) of linear dependence on the two arithmetic progressions \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) is then obtained by substituting these values for \( k_1 \) and \( k_2 \) in the vector equation (1). The problem is the traditional question of finding the eliminant of \( k_1 \) and \( k_2 \) between the
vector equation (1) and the pair of conditions (2). This is usually done by using the method of determinants. The author will attempt a solution using row-reduction of the augmented-matrix.

The conditions (2) and vector equation (1) may be represented by the augmented-matrix:

\[
\begin{bmatrix}
  a_1 & a_2 & a \\
  d_1 & d_2 & d \\
  A_1 & A_2 & A
\end{bmatrix}
\]

Reduce to echelon form:

\[
\begin{align*}
  r'_1 &= r_1 - \frac{1}{a_1} \\
  r'_2 &= r_2 - r'_1 \times d_1 \\
  r'_3 &= r_3 - r'_1 \times A_1
\end{align*}
\]

provided \( a_1 \neq 0 \)

\[
\begin{align*}
  r''_1 &= r'_1 - r''_2 \times \frac{a_2}{a_1} \\
  r''_2 &= r'_2 - r''_1 \times \frac{a_1}{a_1} \\
  r''_3 &= r'_3 - r''_2 \times \frac{A_2a_1 - A_1a_2}{a_1}
\end{align*}
\]

provided \( d_2a_1 - d_1a_2 \neq 0 \).

If we now transform the augmented matrix back into the set of equations which it represents, we have:

\[
\begin{align*}
  1.k_1 + 0.k_2 &= \frac{d_2a_1 - da_2}{d_2a_1 - d_1a_2} \\
  0.k_1 + 1.k_2 &= \frac{d_1a_1 - da_2}{d_2a_1 - d_1a_2} \\
  0.k_1 + 0.k_2 &= \frac{A(d_2a_1 - d_1a_2) - A_1(d_2a_1 - da_2) - A_2(da_1 - d_1a)}{d_2a_1 - d_1a_2}
\end{align*}
\]

(Note: the element in row 3 and col. 3 is:

\[
\frac{A(d_2a_1 - d_1a_2) - A_1(d_2a_1 - da_2) - A_2(da_1 - d_1a)}{d_2a_1 - d_1a_2}
\]
giving solutions: \( k_1 = \frac{d_2a_1 - da_2}{d_2a_1 - d_1a_2} \) and \( k_2 = \frac{da_1 - d_1a}{d_2a_1 - d_1a_2} \)

and the eliminant: \( A(d_2a_1 - d_1a_2) = A_1(d_2a - da_1) + A_2(da_1 - d_1a) \)

The exceptional cases:

(i) \( a_1 = 0 \Rightarrow a = 0 \)

so that \( A = \begin{bmatrix} 0 \\ d \end{bmatrix} \) is linearly dependent on every other

\[ A_1 = \begin{bmatrix} 0 \\ d_1 \end{bmatrix} \]

(ii) \( d_2a_1 - d_1a_2 = 0 \)

is the condition that \( A_1 \) and \( A_2 \) are linearly dependent on each other. Hence we have the following:

**Theorem:**

Any arithmetic progression may be expressed linearly in terms of any two given arithmetic progressions which are linearly independent of it and of each other.

The set of all arithmetic progressions forms a vector space under 'term-by-term addition' and 'multiplication by a real number scalar'. The vector space is of dimension 2. Any given arithmetic progression can be expressed linearly in terms of any two other mutually linearly independent arithmetic progressions. Any two other such arithmetic progressions may be selected as a basis for the vector space. What is the simplest basis that can be chosen?

(2) **Choice of basis**

The simplest vector is a unit vector in a chosen direction. Is it, then, possible to express \( A = \begin{bmatrix} a \\ d \end{bmatrix} \) linearly in terms of \( A_1 = \begin{bmatrix} a_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} a_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)? This
requires that:

\[
\begin{bmatrix}
a \\
d
\end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

where:

\[
k_1 = \frac{d_2a - da_2}{d_2a_1 - d_1a_2} \quad \text{and} \quad k_2 = \frac{da_1 - d_1a}{d_2a_1 - d_1a_2}
\]

\[
\begin{align*}
&= 1.1 - 0.0 \\
&= a
\end{align*}
\]

\[
\begin{align*}
&= \frac{1.1 - 0.0}{1.1 - 0.0} \\
&= d
\end{align*}
\]

if not immediately obvious from the familiar traditional rule for equality of vectors. Hence:

\[
\begin{bmatrix}
a \\
d
\end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

which is consistent with our 'addition' and 'scalar multiplication' rules of operation of a vector space on page 165.

The use of linear algebra of a vector space lends itself readily to the summation of series. The author now gives an alternative version of the method for the summation of an arithmetic progression to that given in [28].

5.15 Summation of the Arithmetic Progression

Work with pupils at secondary level on the arithmetic progression may be linked closely with number pattern experiences of the primary years. It is expected that pupils will have enjoyed working with patterns of the so-called polygonal numbers: a sequence of numbers which can in some way be represented geometrically by a sequence of triangles, squares, ..., n-gons, ... In particular, the triangle numbers 1, 3, 6, ... will have been illustrated as triangular patterns of dots in which the base-row of each successive triangle contains one more dot than its predecessor; thus:
The sequence of triangle numbers is seen to be the successive sums of the natural-number arithmetic progression. Some pupils may already have experienced numerical examples of the summation formula:

\[ \sum_{r=1}^{n} r = \frac{n(n + 1)}{2} \]

If we refer again to the arithmetic progression:

\[ A = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

the associative and distributive laws of operation in a vector space suggest that:

\[
\sum_{i=1}^{n} \begin{bmatrix} a \\ d \end{bmatrix} = \sum_{i=1}^{n} \left\{ a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \\
= \sum_{i=1}^{n} a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{i=1}^{n} d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
= a \sum_{i=1}^{n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \sum_{i=1}^{n} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
= a \cdot n + d \cdot (n-1)n/2 \\
= \frac{n}{2} \{2a + (n-1)d\}
\]

Note that: \[ \sum_{i=1}^{n} \begin{bmatrix} a \\ d \end{bmatrix} \] is the sum to n terms of the vector \[ \begin{bmatrix} a \\ d \end{bmatrix} \] which represents the arithmetic sequence \( \{a, a+d, a+2d, \ldots, a+r-1.d, \ldots\} \) whose first term is a,

*means \[ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \ldots \] to n terms
common difference is \( d \). The reader will also see that the familiar summation formula has been obtained without reliance on a 'trick' in the form of 'reverse the order of the terms' in the summation!

5.16 The Arithmetic Progression: Graphical Representation

The scalar multiples \( k_1, k_2 \) which determine the linear dependence of the arithmetic progression \( \mathbf{A} = \begin{bmatrix} a \\ d \end{bmatrix} \) on the selected basis vectors \( \mathbf{A}_1 = \begin{bmatrix} a_1 \\ d_1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} a_2 \\ d_2 \end{bmatrix} \) may be regarded as the coordinates of a point \( \mathbf{A} \) which represents the vector \( \mathbf{A} \) in a cartesian frame of reference formed by the basis vectors \( \mathbf{A}_1, \mathbf{A}_2 \). Choose axes \( OA_1, OA_2 \) to represent the position vectors of \( \mathbf{A}_1, \mathbf{A}_2 \). The axes need not be at right-angles. The pupils here are quite naturally introduced to oblique axes of reference and the technique is required by the author in the development of the inter-connection between algebra and geometry in Chapter 6. It is convenient to choose unit basis vectors \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) so that:

\[
\mathbf{A} = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

is represented by the point \( (k_1, k_2) \) or by the position vector \( \overline{OA} \) in Figure 5.4. \((k_1, k_2)\) represents a 'family of terms in an arithmetic progression' and may be compared with a line which represents a 'family of ordered pairs of natural numbers' as a directed whole-number in §2.7 of Chapter 2.
The operation of 'term-by-term addition' as the sum of two vectors $A', A''$ is given by:

$$A' + A'' = k'_1 \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k'_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} + k''_1 \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k''_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$$

$$= (k'_1 + k''_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (k'_1 + k''_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and is represented by the point $A = (k'_1 + k''_1, k'_2 + k''_2)$ in Fig 5.5.

It will be seen that $A$ completes the parallelogram $OA_1AA_2$ and that the sum of the vectors $A', A''$ is represented by the diagonal $OA$ of the parallelogram and satisfies the familiar parallelogram law of addition of vectors.
The reader is reminded that the point $A$ represents the 'term-by-term addition' of the arithmetic progression vectors $A'$ and $A''$ and must not be confused with the summation of the arithmetic progression represented as a single vector $\vec{A}$.

Similarly, the operation of 'scalar multiplication' of a vector $\vec{A}$ by a real-number scalar $r$ is given by:

$$r\vec{A} = r \left\{ k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= r \left\{ k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} + r \left\{ k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= (rk_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (rk_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and is represented by a point $R$ on $\overrightarrow{OA}$ such that $\overrightarrow{OR} = r\overrightarrow{OA}$ (Fig 5.6).

For an elementary application of arithmetic progressions as a vector space the reader is referred to an Exhibit on the multiplication algorithm prepared by Neil Simmons, an advanced course student of the author.
5.17 Further Developments

Other topics that illustrate the pattern of a vector space have been developed as Exhibits. These will include the application of the linear algebra of vectors to the problem of finding the $n^{th}$ term of a Fibonacci sequence.

An interesting development is afforded by an article entitled "The Standard Deviation Vector" shortly to be published in Teaching Mathematics and its Applications. In this paper, mean and deviation are treated as vectors. Standard deviation is shown to have linear properties and an application of the idea of a transformation to different temperature scales leads to some linear examples and some useful non-linear counter-examples. The standard deviation itself may be regarded as a linear transformation. The distinction between regression of 'y-on-x' and of 'x-on-y' is made more obvious than in most approaches to the concept and the suggestion by the authors P K Armstrong, A C Bajpai and D N Hunt that the work promises to provide a wealth of investigational material at undergraduate level is supported by the author of this thesis. It is worth noting that the concept of a standard deviation vector is used in MIME [S3].

The author returns now to the application of linear algebra to the geometrical symmetry transformations, the isometries, of Chapter 4.
6.1 Algebra-Geometry Coordinated

In Chapter 4 the author investigated the isometries of a 2-dimensional Euclidean plane. A set of geometrical elements was combined under an operation 'followed-by' which produced a 'product' result and in many ways behaved like the multiplication of ordinary arithmetic. Geometry for pupils progressed from a Stage A to a Stage B in which the isometries were experienced as a mathematical system. The system was seen to have the pattern of a group under the operation 'followed-by'.

In Chapter 5 the reader was introduced to an algebra which goes beyond the techniques of classical algebra as the generalised arithmetic of numbers. The linear algebra of vectors operates on a universal set of elements which need not be numbers. The results of the permitted operations of a vector algebra were illustrated by working with the universal set of linear equations.

The algebra of vectors will now be developed to the level of a Stage B algebra for pupils in the middle/upper secondary years. The 2-dimensional vector space of linear equations will be used to describe the geometrical isometries of the Euclidean plane. To the permitted operations on the linear-equation elements, or on the rows of the augmented-matrix which represents them, will be added the 'obviously' permitted interchange of position of the 2 equations, or rows. The results of the operations:
(i) multiplication by a (real number) scalar
(ii) addition of a scalar multiple
(iii) interchange of 2 equations, or rows

will be interpreted in terms of the properties of the isometries. An algebraic interpretation of the geometrical operation 'followed-by' will then be discussed in terms of matrix multiplication.

6.2 Linear Equations as Transformations

A pair of linear equations:

\[
\begin{align*}
    a_1x + b_1y &= x' \\
    a_2x + b_2y &= y'
\end{align*}
\]  

(1)

may be regarded as the algebraic representation of a transformation which moves a point in the 2-dimensional Euclidean plane from an object position \((x,y)\) to an image position \((x',y')\). The equations express \(x', y'\) linearly, as sums of scalar multiples, in terms of \(x, y\). The 2 equations may be solved simultaneously to give \(x, y\) in terms of \(x', y'\) provided that:

\[a_1b_2 - a_2b_1 \neq 0\]

The solution is the algebraic representation of the transformation which takes \(x', y'\) back to \(x, y\). If \(a_1b_2 - a_2b_1 = 0\), no such solution exists and there is no corresponding inverse transformation.

Thus there would appear to be a correspondence between the geometrical one-to-one transformations of a plane onto itself (see Chapter 4, §4.3) and the solution to a pair of simultaneous linear equations:

transformations are either one-to-one onto:

ie an inverse transformation exists;

or they are not: ie no inverse exists;
simultaneous linear equations either have a solution:

\[ a_1 b_2 - a_2 b_1 \neq 0 \]
or they do not: \[ a_1 b_2 - a_2 b_1 = 0 \]

The author, in Chapter 4, restricted the discussion to one-to-one symmetry transformations of the plane onto itself which had an inverse; she now restricts the solution of a pair of simultaneous linear equations to those cases in which a solution exists: ie for which the expression \( a_1 b_2 - a_2 b_1 \) is non-zero.

We have the following cases for the solution in augmented-matrix form:

**Case 1:** \( a_1 \neq 0 \): a solution for \( x, y \) is possible and is given by

\[
\begin{bmatrix}
1 & 0 & : & x'b_2 - y'b_1 \\
\cdot & \cdot & : & a_1 b_2 - a_2 b_1 \\
0 & 1 & : & y'a_1 - x'a_2 \\
\cdot & \cdot & : & a_1 b_2 - a_2 b_1 \\
\end{bmatrix}
\]

The equations represent a pair of intersecting straight lines. *Exceptionally:* if \( a_1 b_2 - a_2 b_1 = 0 \) there is no solution.

The equations represent a pair of parallel lines.

**Case 2:** \( a_1 = 0; a_1 b_2 - a_2 b_1 \neq 0 \Rightarrow a_2 b_1 \neq 0 \): a solution for \( x, y \) is possible and is given by

\[
\begin{bmatrix}
0 & 1 & : & x' \\
\cdot & \cdot & : & b_1 \\
1 & 0 & : & y'b_1 - x'b_2 \\
\cdot & \cdot & : & a_2 b_1 \\
\end{bmatrix}
\]

The equations represent a pair of intersecting straight lines. *Exceptionally:* \( a_2 b_1 = 0 \Rightarrow \) either (i) \( a_2 = 0 \) or (ii) \( b_1 = 0 \)

* \( a_1 b_2 - a_2 b_1 \neq 0 \Rightarrow a_1, a_2 \) cannot both be zero. It will be assumed that \( a_2 \neq 0 \). If \( a_2 = 0 \), corresponding results may be obtained by the interchange of the equations and are thus given by the interchange of suffices 1, 2 and of \( x', y' \). The exceptional cases will be discussed in Chapter 7.
(i) $a_1 = 0, a_2 = 0$: the equations reduce to: $b_1 y = x'$

$b_2 y = y'$

(ii) $a_1 = 0, b_1 = 0$: the equations reduce to: $a_2 x + b_2 y = y'$

The question arises: is it possible for coefficients $a_1, b_1, a_2, b_2$ to be selected in such a way that the equations:

$$\begin{align*}
    a_1 x + b_1 y &= x' \\
    a_2 x + b_2 y &= y'
\end{align*}$$

may represent the isometries discussed in Chapter 4. What condition(s) on $a_1, a_2, b_1, b_2$ ensure a distance-preserving transformation? It will be remembered that in Chapter 4 the isometries were classified according to their invariant properties and it is from the point of view of invariance that the author now approaches the situation algebraically.

6.3 Invariance

The problem of finding the invariants in a system of linear equations is discussed in terms of eigen values and eigen vectors and is included in many of the more recent texts at sixth-form level or above. J A H Shepherd and C J Shepherd [27.2] and S L Parsonson [29] each devote the penultimate chapter of their respective texts to the situation; E A Maxwell [20] deals with the question in part III of an SMP Handbook, prefaced by A G Howson who writes:

"The book could usefully be used in the classroom in conjunction with SMP texts and is also most suitable for the purpose of teacher education..."

T J Fletcher deals with the subject in [25] but, as Johnston Anderson writes [30]:

"... most genuine applications of matrix theory are probably too difficult to be included in the 11-16 range."
and adds in a footnote:

"Much of this is discussed in T J Fletcher's 'Linear Algebra through its Applications'. This book is, however, not suitable for schools, at least not without some predigestion by the teacher".

It is the intention of the author here to attempt such a predigestion for the teacher and then to suggest appropriate experiences for pupils at the middle/upper secondary level.

It has been established (Chapter 4) that an isometry of the plane is completely determined by a set of 3 non-collinear points. An exhaustive, mutually exclusive, set of isometries was classified according to the invariant points and lines of the system:

- no invariant points: translation; glide-reflection
- 1 invariant point: rotation
- infinitely many invariant points as:
  - 1 invariant line, point-for-point
  - reflection
- infinitely many invariant lines: half-turn; reflection

If possible, let an isometry be represented by the pair of equations (1). A solution exists provided:

\[ a_1 b_2 - a_2 b_1 \neq 0 \]  \hspace{1cm} (condition I)

in which \( a_1 \) and \( a_2 \) cannot both be zero.

(1) Invariant Points:

The equations are 'obviously' satisfied by \( x = 0, y = 0 \). The origin is an invariant point of the transformation. It might then be expected that if the origin is the only invariant point the equations (1) could represent a rotation. The author returns to this consideration in §6.6 on page 190.

The question next arises: can there be more than 1 invariant point? This would require that \( \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \), \( x \) and \( y \) not both
zero, and that the equations:
\[ a_1 x + b_1 y = x \]
\[ a_2 x + b_2 y = y \]

\[ \Rightarrow \begin{cases} (a_1 - 1)x + b_1 y = 0 \\ a_2 x + (b_2 - 1)y = 0 \end{cases} \]

with augmented matrix
\[
\begin{bmatrix}
    a_1 - 1 & b_1 & \vdots & 0 \\
    a_2 & (b_2 - 1) & \vdots & 0 \\
\end{bmatrix}
\]

are consistent and give a (non-trivial) solution other than \( x = 0 = y \).

ie \( (a_1 - 1)(b_2 - 1) - a_2 b_1 = 0 \)

\[ \Rightarrow a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1 \]

\[ \Rightarrow a_1 + b_2 - 1 \neq 0 \text{ (using condition I)} \]

are necessary but not sufficient conditions for the equations (2) to determine a constant value for the ratio \( y:x \) of the coordinates of invariant points; we shall have a line of invariant points. Such a line of invariant points is often described as an invariant line, point-for-point, or a pointwise invariant line.

Since the origin is also an invariant point, we shall have determined a pointwise invariant line which passes through the origin.

The existence of such a line suggests that the equations (1) could represent a reflection with this pointwise invariant line as the axis of the reflection (see §6.5, on page 187).

(2) Discriminant? or Determinant?

It is worth noting at this point in the discussion that condition I involves the expression \( a_1 b_2 - a_2 b_1 \) which is
known as the value of the determinant \[
\begin{vmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{vmatrix}
\]
of the coefficients of x and y in equations (1).

The equivalent expression: \((a_1 + b_2 - 1)\) of condition II is simpler, has 'turned up' quite naturally in the consideration of invariance, and will be seen to simplify the discussion later (see §6.5 on page 187) of the different cases that can arise. For this reason the author has elected to avoid the use of the determinant – a practice advocated by T J Fletcher [25]. Instead she uses the expression \((a_1 + b_2 - 1)\) as an 'eigen discriminant', to coin a phrase, when discussing invariant lines. She returns now to the problem of finding the pointwise invariant line of item (1) of this section. At the same time, the more general question of the existence of invariant lines – not necessarily point–for–point – will be seen to occur coincidentally.

(3) Invariant Lines

In the more recent texts quoted at the beginning of this section (see page 180) a search is made for eigen values associated with a transformation which give the eigen vectors of invariant points of the transformation. These eigen values are determined from a characteristic equation. The author agrees with the critics that the work as presented is difficult even for the experienced teacher: it requires a complete re–think of the situation and a translation from the language of determinants to the more modern use of vector methods. Such apprehensions inhibit the willingness on the part of the teacher to impart the information to the pupils and to 'lend' to the author the pupils required for the viable testing of this particular phase of the work: at least, that is, until a very recent and exciting new development described in the final chapter in §11.2. The reader will appreciate that such pupils are approaching public examination entrance and their chances must not be prejudiced.
The following is an attempt to simplify for the teacher the more usual derivation of eigen values and eigen vectors. The work is pitched essentially at teacher-level. Relevant pupil experiences, for the development later to work based on this teacher background, will be found in §6.9 and on page 198.

Let an object point move from position \((x,y)\) along the straight line through the origin in either direction to the image position \((x', y')\) under the transformation given by equations (1) on page 178. Then:

\[
x' = \lambda x, \quad y' = \lambda y
\]

where \(\lambda\) is the scalar ratio of the distances from the origin of the object and image positions. Substituting for \(x', y'\) in equations (1), it is seen that \(\lambda\) satisfies:

\[
\begin{align*}
(a_1 - \lambda)x + b_1y &= 0 \\
a_2x + (b_2 - \lambda)y &= 0
\end{align*}
\]

A solution of equations (4) for the ratio \(y:x\) exists for \(\lambda\) satisfying:

\[
(a_1 - \lambda)(b_2 - \lambda) - a_2b_1 = 0
\]

\[
=> \lambda^2 - \lambda(a_1 + b_2) + (a_1b_2 - a_2b_1) = 0
\]

and is the gradient of an invariant line, not necessarily point-for-point, which passes through the origin. Equation (5) is called the characteristic equation.

In general, 2 invariant lines - not necessarily point-for-point - exist and correspond to the 2 values of \(\lambda\):

\[
\lambda = \frac{(a_1 + b_2) \pm \sqrt{(a_1 + b_2)^2 - 4(a_1b_2 - a_2b_1)}}{2}
\]

provided that: \((a_1 + b_2)^2 - 4(a_1b_2 - a_2b_1) > 0\) (condition III).
If it so happens that conditions I and II (see pages 181, 182) for pointwise invariant lines are satisfied, condition III reduces to:

\[(a_1 + b_2 - 2)^2 > 0\]  
(condition IV)

which is satisfied for all \(a_1, b_2\).

It would appear that:

2 real and distinct (pointwise) invariant lines exist for:

\[
\begin{align*}
\lambda_1 &= 1 \\
\lambda_2 &= a_1 + b_2 - 1
\end{align*}
\]

provided that

\[a_1b_2 - a_2b_1 = a_1 + b_2 - 1 \geq 1\]  
(condition V)

and:

2 coincident (pointwise) invariant lines exist for:

\[
\lambda_1 = 1 = \lambda_2 \quad \text{provided that} \quad a_1b_2 - a_2b_1 = a_1 + b_2 - 1 = 1
\]

A full discussion including the case for which

\[a_1b_2 - a_2b_1 \neq a_1 + b_2 - 1\]

is given in §6.6 on page 190. The author's 'eigen discriminant' is an important feature of that discussion and the value \(\lambda = 1\) is a special case of the transformation.

The gradients of the invariant lines are obtained from equations (4) as:

\[
\frac{y}{x} = \frac{\lambda - a_1}{b_1} = \frac{a_2}{\lambda - b_2} = \frac{y'}{x'}
\]

and, by eliminating \(\lambda\) between equations (4), are seen to be the roots of the equation:

\[b_1(\frac{y}{x})^2 + (a_1 - b_2) \frac{y}{x} - a_2 = 0\]  
(6)

* An alternative method for finding the gradient of an invariant line, which the author has not seen in any of the texts consulted, is given in §7.5 of Chapter 7.
\( \lambda \) is called an **eigen value** of the transformation or of the coefficient matrix \[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\] which represents it. \[
\begin{bmatrix}
\lambda x \\
\lambda y
\end{bmatrix}
\]
is the corresponding **eigen vector** which is an invariant line of the transformation passing through the origin.

If the transformations represented by equations (1) are to be identified as isometries, we must now establish the condition(s) under which distance is preserved.

### 6.4 Distance Preserved

Let 2 distinct object points \( P_1(x_1, y_1), P_2(x_2, y_2) \) move to image positions \( P'_1(x'_1, y'_1), P'_2(x'_2, y'_2) \) under the transformation given by the equations (1):

\[
\begin{align*}
a_1x + b_1y &= x' \\
a_2x + b_2y &= y'
\end{align*}
\]

(1)

Assuming for the moment that (not necessarily pointwise) invariant lines do exist, the transformation takes the form:

\[
x' = \lambda x, \quad y' = \lambda y
\]

of equations (3). Then, if distance is preserved, we have:

\[
|P_1P_2| = |P'_1P'_2|
\]

\[
\Rightarrow \quad P_1P_2^2 = P'_1P'_2^2
\]

\[
\Rightarrow \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2
\]

\[
\Rightarrow \quad (1 - \lambda^2)((x_2 - x_1)^2 + (y_2 - y_1)^2) = 0
\]

\[
\Rightarrow \quad \lambda = \pm 1 \quad \text{since } P_1, P_2 \text{ are distinct}
We know that: \( \lambda = 1 \) (twice) \( \Rightarrow a_1 + b_2 - 1 = 1 \)
\[ a_1 x + b_1 y = x' \]
\[ a_2 x + b_2 y = y' \]
\( \lambda = -1 \) (twice) \( \Rightarrow a_1 + b_2 - 1 = -1 \)
\( \lambda = -1 \) (twice) \( \Rightarrow \) no invariant lines

The position reached so far is now summarised.

6.5 The Algebraic Representation of the Isometries

The algebraic representation of the isometries by the equations:
\[ a_1 x + b_1 y = x' \]
\[ a_2 x + b_2 y = y' \]
is seen to depend on the following conditions:

I: \( a_1 b_2 - a_2 b_1 \neq 0 \): a one-to-one transformation of the plane onto itself exists
A solution of the equations gives the inverse transformation which moves \((x', y')\) back to \((x, y)\)

II: Either (1): \( a_1 b_2 - a_2 b_1 \neq a_1 + b_2 - 1 \): the origin is the only invariant point of the transformation
Is it possible for coefficients, the a's and the b's, to be selected for the representation of a rotation (§6.6)?

Or (2): \( a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1 \): there is more than one invariant point
Is it possible for coefficients, the a's and the b's, to be selected for the representation of a half-turn (§6.6) or a reflection (§6.6)?

III: Invariant points and lines can occur only for the pairs of values of:
(i) \( \lambda_1 = 1 = \lambda_2 \); (ii) \( \lambda_1 = -1 = \lambda_2 \); (iii) \( \lambda_1 = 1, \lambda_2 = -1 \)
The analysis falls into 2 categories:

Case 1: $a_1 b_2 - a_2 b_1 \neq a_1 + b_2 - 1$: we look for a rotation

Case 2: $a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1$: we look for a half-turn or a reflection

The Euclidean plane may be regarded as a 2-dimensional vector space in which the position vector \[
\begin{bmatrix} x \\ y \end{bmatrix}
\] of a point \(P(x,y)\) may be expressed as the sums of scalar multiples \(x, y\) respectively of 2 linearly independent unit basis vectors \[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}:
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

(cf the representation of an arithmetic progression in the 2-dimensional vector space of arithmetic progressions in Chapter 5, pages 170, 171).

The reader is reminded of the operations of addition and scalar multiplication of vector algebra (see Chapter 5, §5.11 page 161).

The equations (1) will be written symbolically in the form:

\[
\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}
\]

to indicate the dependence of the transformation on the coefficients, the \(a's\) and the \(b's\). The symbol \[
\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}
\]
called the matrix of the transformation.

In particular, with this notation, we have:
\[
\begin{bmatrix}
 a_1 & b_1 \\
 a_2 & b_2 \\
\end{bmatrix}
\begin{bmatrix}
 1 \\
 0 \\
\end{bmatrix}
= \begin{bmatrix}
 a_1 \\
 a_2 \\
\end{bmatrix};
\begin{bmatrix}
 a_1 & b_1 \\
 a_2 & b_2 \\
\end{bmatrix}
\begin{bmatrix}
 0 \\
 1 \\
\end{bmatrix}
= \begin{bmatrix}
 b_1 \\
 b_2 \\
\end{bmatrix}
\]

are the images of the unit basis vectors.

It will next be shown that:

the image of the sum of scalar multiples \(x\), \(y\) respectively of the unit basis vectors is the sum of the scalar multiples \(x\), \(y\) of the images of the unit basis vectors.

We have:

\[
\begin{bmatrix}
 a_1 & b_1 \\
 a_2 & b_2 \\
\end{bmatrix}\begin{bmatrix}
 x.1 \\
 y.0 \\
\end{bmatrix}
= \begin{bmatrix}
 a_1 & b_1 \\
 a_2 & b_2 \\
\end{bmatrix}\begin{bmatrix}
 x \\
 y \\
\end{bmatrix}
\]

and

\[
x\begin{bmatrix}
 a_1 & b_1 \\
 a_2 & b_2 \\
\end{bmatrix}\begin{bmatrix}
 1 \\
 0 \\
\end{bmatrix}
+ y\begin{bmatrix}
 a_1 & b_1 \\
 a_2 & b_2 \\
\end{bmatrix}\begin{bmatrix}
 0 \\
 1 \\
\end{bmatrix}
= x\begin{bmatrix}
 a_1 \\
 a_2 \\
\end{bmatrix}
+ y\begin{bmatrix}
 b_1 \\
 b_2 \\
\end{bmatrix}
\]

= \begin{bmatrix}
 a_1x + b_1y \\
 a_2x + b_2y \\
\end{bmatrix}
\]

The reader is referred to [31] for the work designed (optimistically?) for 11 to 13 year-old pupils and more appropriate for the middle/upper secondary range.

The author will now attempt a systematic representation of the isometries in terms of the coefficients, the \(a\)'s and the \(b\)'s, of equations (1).
6.6 The Transformation Matrix

Case 1: \[
\begin{cases}
    a_1 b_2 - a_2 b_1 \neq a_1 + b_2 - 1; & \text{a rotation?} \\
    \lambda_1 = \lambda_2 = \lambda
\end{cases}
\]

It has been established that if there is more than 1 fixed point in the transformation given by the equations:

\[
\begin{align*}
    a_1 x + b_1 y &= x' \\
    a_2 x + b_2 y &= y'
\end{align*}
\]

with augmented matrix

\[
\begin{bmatrix}
    a_1 & b_1 & x' \\
    a_2 & b_2 & y'
\end{bmatrix}
\]

in which \[
\begin{bmatrix}
    a_1 & b_1 \\
    a_2 & b_2
\end{bmatrix}
\]

is the matrix of the transformation

then \( a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1 \) (condition II on page 182).

Hence if \( a_1 b_2 - a_2 b_1 \neq a_1 + b_2 - 1 \) the origin is the only fixed point and is the centre of a possible rotation.

Let the position vectors of object and image points \( P, P' \) be \[
\begin{bmatrix}
    x \\
    y
\end{bmatrix}, \begin{bmatrix}
    x' \\
    y'
\end{bmatrix}
\]

respectively under a rotation \( R:(0, \alpha) \).

and let \[
\begin{bmatrix}
    1 \\
    0
\end{bmatrix}, \begin{bmatrix}
    0 \\
    1
\end{bmatrix}
\]

be unit basis vectors in each of 2 perpendicular directions.

![Diagram](image)

**Fig 6.1**

We have the unit basis vectors:

\[
\begin{bmatrix}
    1 \\
    0
\end{bmatrix} \rightarrow \begin{bmatrix}
    \cos \alpha \\
    \sin \alpha
\end{bmatrix},
\]

\[
\begin{bmatrix}
    0 \\
    1
\end{bmatrix} \rightarrow \begin{bmatrix}
    -\sin \alpha \\
    \cos \alpha
\end{bmatrix}
\]
Hence:
\[
\begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha 
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \cos \alpha - y \sin \alpha \\ x \sin \alpha + y \cos \alpha \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}
\]
and
\[
\begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha 
\end{bmatrix}
\]
is the matrix of the rotation \(R:(0, \alpha)\).

In terms of the coefficients, the \(a\)'s and the \(b\)'s, of equations (1) we have:

\[
a_1b_2 - a_2b_1 = \cos^2 \alpha + \sin^2 \alpha = 1
\]
and
\[
a_1 + b_2 - 1 = 2 \cos \alpha - 1
\]

ie \(a_1b_2 - a_2b_1 = a_1 + b_2 - 1\) if and only if \(\alpha = 2n\pi\)

where \(n\) is an integer.

But this is the special case of the identity transformation:

for \(\lambda_1 = \lambda_1 = \lambda_2\), the transformation takes the form:

\[
\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}
\]

for all \((x,y)\) under which all points and lines are invariant.

The representative equations reduce to:

\[
\begin{align*}
1.x + 0.y &= x' \\
0.x + 1.y &= y'
\end{align*}
\]

with augmented matrix
\[
\begin{bmatrix}
1 & 0 & \vdots & x' \\
0 & 1 & \vdots & y'
\end{bmatrix}
\]

in which
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
is the matrix of the transformation.

This agrees with the rotation matrix \(R:(0, \alpha)\) above with \(\alpha = 2n\pi\).

In terms of the roots of the characteristic \(\lambda\)-equation (5) on page 184 for invariant lines, not necessarily point-for-point:

\[
\begin{align*}
\lambda_1 + \lambda_2 &= a_1 + b_2 = 2 \rightarrow a_1 + b_2 - 1 = 1 \\
\lambda_1 \lambda_2 &= a_1b_2 - a_2b_1 = 1
\end{align*}
\]

consistent with the above relationships for \(\alpha = 2n\pi\).
Case 2: \[ \begin{cases} a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1 \\ \lambda_1 = 1; \; \lambda_2 = -1 \end{cases} \] : a reflection? or a half-turn?

It has been established that more than one invariant point may exist in the form of a pointwise invariant line given by:

\[ \lambda_1 = 1 \text{ of equation (5) on page 184} \]

One other invariant line, not necessarily point-for-point, may also exist and is given by:

\[ \lambda_2 = a_1 + b_2 - 1 \text{ of equation (5) on page 184} = -1 \text{ (in this case)} \]

Any pointwise invariant line could be the axis of a reflection (see page 187).

Let the position vectors of object and image points \( P, P' \) be \( \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \) respectively under a reflection \( M \) in an axis \( m \alpha \) inclined at an angle \( \alpha \) to the unit basis vector \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) (Fig 6.2).

![Fig 6.2](image)

For unit basis vectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) in perpendicular directions:

\[ M: \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos 2\alpha \\ \sin 2\alpha \end{bmatrix}; \; M: \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \sin 2\alpha \\ -\cos 2\alpha \end{bmatrix} \]
Hence: 
\[
\begin{bmatrix}
\cos 2\alpha & \sin 2\alpha \\
\sin 2\alpha & -\cos 2\alpha
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

and: 
\[
\begin{bmatrix}
\cos 2\alpha & \sin 2\alpha \\
\sin 2\alpha & -\cos 2\alpha
\end{bmatrix}
\]

is the matrix of reflection in \( m_\alpha \).

In terms of the coefficients of equations (1), we have:
\[
a_1 b_2 - a_2 b_1 = -\cos^2 2\alpha - \sin^2 2\alpha = -1
\]
\[
a_1 + b_2 - 1 = \cos 2\alpha - \cos 2\alpha - 1 = -1
\]
The condition: \( a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1 \) for more than 1 invariant point is satisfied. Furthermore, the condition:
\[
a_1 + b_2 - 1 = -1 < 1
\]
for 2 distinct invariant lines is satisfied.

We know that \( \lambda_1 = 1 \) with \( a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1 \) gives a pointwise invariant line with gradient:
\[
\frac{\lambda - a_1}{b_1} = \frac{1 - \cos 2\alpha}{\sin 2\alpha} = \tan \alpha
\]
which is the axis of the reflection as expected; and that \( \lambda_2 = a_1 + b_2 - 1 = -1 \) gives an invariant line through the origin for which 
\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
-x \\
y
\end{bmatrix}
\]

with gradient:
\[
\frac{\lambda - a_2}{b_1} = \frac{-1 - \cos 2\alpha}{\sin 2\alpha} = -\cot \alpha
\]

which is seen to be perpendicular to the axis of reflection, as expected.
In terms of the roots of the characteristic \( \lambda \)-equation (5) on page 184 for invariant lines, not necessarily point-for-point:

\[
\lambda_1 + \lambda_2 = a_1 + b_2 = 0 \Rightarrow a_1 + b_2 - 1 = -1 \\
\lambda_1 \lambda_2 = a_1 b_2 - a_2 b_1 = -1
\]

consistent with the transformation matrix \( M \) above.

All the above results agree with the geometry of Chapter 4 pages 110 to 118.

Case 3: \( a_1 b_2 - a_2 b_1 \neq a_1 + b_2 - 1 \)

\[
\lambda_1 = -1 = \lambda_2
\]

Attention is drawn to the fact that: \( \lambda_1 = -1 = \lambda_2 \) gives the transformation:

\[
\begin{bmatrix}
 x \\
 y
\end{bmatrix} \rightarrow -1 \begin{bmatrix}
 x \\
 y
\end{bmatrix}
\]

which may be described as a reflection in the origin or as a half-turn rotation about the origin.

We have the unit basis vectors: \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix} ; \)

\[
\begin{bmatrix}
 0 \\
 1
\end{bmatrix} \rightarrow \begin{bmatrix}
 0 \\
 -1
\end{bmatrix}
\]

Hence: \( \begin{bmatrix}
 -1 & 0 \\
 0 & -1
\end{bmatrix} ; \begin{bmatrix}
 x \\
 y
\end{bmatrix} \rightarrow \begin{bmatrix}
 -x \\
 -y
\end{bmatrix} \)

and: \( \begin{bmatrix}
 -1 & 0 \\
 0 & -1
\end{bmatrix} \) is the matrix of the transformation.

In terms of the coefficients of equations (1):

\[
a_1 b_2 - a_2 b_1 = 1 \\
a_1 + b_2 - 1 = -3 \neq a_1 b_2 - a_2 b_1
\]
The condition: \(a_1 b_2 - a_2 b_1 \neq a_1 + b_2 - 1\) for a rotation is satisfied; there are no invariant points other than the origin and there are no pointwise invariant lines. There can, however, be invariant lines not point-for-point under this condition and it so happens that every line through the origin under reflection (!) in the origin is invariant (see page 117 of Chapter 4).

In terms of the characteristic \(\lambda\)-equation (5) on page 184 for invariant lines, not necessarily point-for-point:

\[
\lambda_1 + \lambda_2 = a_1 + b_2 = -2 \Rightarrow a_1 + b_2 - 1 = -3
\]

\[
\lambda_1 \lambda_2 = a_1 b_2 - a_2 b_1 = +1
\]

consistent with the transformation matrix above for the half-turn.

6.7 Translation

The reader will note that the translation isométry of Chapter 4, page 111 has been omitted from the discussion of this chapter. For the translation defined by the vector

\[
\begin{bmatrix}
  t_1 \\
  t_2
\end{bmatrix}
\]

we have:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} \rightarrow \begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  x + t_1 \\
  y + t_2
\end{bmatrix}
\]

A change of origin to the point given by the position vector

\[
\begin{bmatrix}
  t_1 \\
  t_2
\end{bmatrix}
\]

by a transformation of coordinates:

\[
x = X + t_1, \quad y = Y + t_2
\]

in equations (1) gives:

\[
\begin{align*}
a_1 X + b_1 Y &= X' - a_1 t_1 - b_1 t_2 + t_1 \\
a_2 X + b_2 Y &= Y' - a_2 t_1 - b_2 t_2 + t_2
\end{align*}
\]
as the corresponding algebraic representation of a transformation in which \( x = t_1 \), \( y = t_2 \): ie the new origin \( X = 0 \), \( Y = 0 \), is an invariant point.

We now need an interpretation of the geometrical operation 'followed-by'.

### 6.8 An Algebraic Law of Composition

Consider the result of the composition of 2 isometries under the operation 'followed-by'. Algebraically, we have:

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} \rightarrow \begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

under a transformation represented by:

\[
\begin{cases}
a_1x + b_1y = x' \\
a_2x + b_2y = y'
\end{cases}
\]

OR

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

followed by:

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} \rightarrow \begin{bmatrix}
x'' \\
y''
\end{bmatrix}
\]

under a transformation represented by:

\[
\begin{cases}
a'_1x' + b'_1y' = x'' \\
a'_2x' + b'_2y' = y''
\end{cases}
\]

OR

\[
\begin{bmatrix}
a'_1 & b'_1 \\
a'_2 & b'_2
\end{bmatrix} \begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
x'' \\
y''
\end{bmatrix}
\]

The product situation may be represented as:

\[
\begin{cases}
a'_1(a_1x + b_1y) + b'_1(a_2x + b_2y) = x'' \\
a'_2(a_1x + b_1y) + b'_2(a_2x + b_2y) = y''
\end{cases}
\]

OR

\[
\begin{bmatrix}
a'_1 & b'_1 \\
a'_2 & b'_2
\end{bmatrix} \begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
x'' \\
y''
\end{bmatrix}
\]

* \( X = 0 \Rightarrow x = t_1 \), \( y = t_2 \) \( \Rightarrow x' = a_1t_1 + b_1t_2 \)
  
* \( Y = 0 \Rightarrow y = t_2 \) \( \Rightarrow y' = a_2t_1 + b_2t_2 \)
  
  \( \Rightarrow X' = a_1t_1 + b_1t_2 - t_1 \)
  
  \( Y' = a_2t_1 + b_2t_2 - t_2 \)

ie \( X = 0 \Rightarrow X' = 0 \)

\( Y = 0 \Rightarrow Y' = 0 \)
Hence: the product of the transformation matrices:
\[
\begin{bmatrix}
a_1' & b_1' \\
a_2' & b_2'
\end{bmatrix}
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\]
may be interpreted as:
\[
\begin{bmatrix}
(a_1'a_1 + b_1'a_2) & (a_1'b_1 + b_1'b_2) \\
(a_2'a_1 + b_2'a_2) & (a_2'b_1 + b_2'b_2)
\end{bmatrix}
\]
to be interpreted as: \[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\] (first) 'followed-by'
\[
\begin{bmatrix}
a_1' & b_1' \\
a_2' & b_2'
\end{bmatrix}
\]
(second). It may be verified that the process is not commutative.

In all cases of rotation, half-turn, reflection, the transformation matrix takes the form:
\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\text{ OR } \begin{bmatrix}
\cos \phi & \sin \phi \\
\sin \phi & -\cos \phi
\end{bmatrix}
\]
Any product of these with themselves or with each other results in a matrix of the same form. The justification requires the familiar addition formulae for sines and cosines. The teacher is referred to one of the recent school texts or to G Matthews [31] for a derivation of these formulae as the composition of rotations.
The universal set of isometric transformation matrices
\[
\begin{bmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{bmatrix}
\]
is closed and associative under matrix multiplication; has a unique multiplicative identity element:
\[
I = \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix},
\]
corresponding to a rotation \( \theta = 2n\pi \) where \( n \) is an integer; each has a unique multiplicative inverse ensured by the condition \( a_1b_2 - a_2b_1 \neq 0 \) for the solution of the defining equations (1) on page 178. The system forms a multiplicative group, albeit non-commutative, and is isomorphic with its geometrical analogue of Chapter 4.

The author now gives some examples of relevant pupil activity.

6.9 \textbf{Investigations on the Symmetries of the Regular Polygons}

(1) The reader is referred to T J Fletcher [?] for a delightful investigation into the symmetries of the regular polygons. This should form part of every stage B secondary geometry course. Opportunities are provided for forging strong links with primary experiences in, for example, 'clock-maths' [5.2] and for development with applications in the construction of calculating aids such as a circular slide-rule and in topics such as recurring decimals.

There is an excellent basis for an investigation on reflections described by H S M Coxeter [17] using the kaleidoscope as a model.

The author's own efforts are directed towards the extensions at sixth-form level to 3-dimensional symmetry (see Chapter 10) and teachers are urged to experiment with the technology now available to motivate an interest for the pupils in animated symmetry [S4] and [S7].
(2) Links with Stage A

It is assumed that Stage-A experiences of the fundamental isometries (see Chapter 4, page 110) will include a study of the properties of elementary rectilinear figures, including triangles and quadrilaterals. It is expected that the symmetries of these figures will have been discussed in terms of rotations and reflections. Examples may be found in many of the school texts: the reader is referred, for example, to E A Maxwell [20].

In particular the isosceles triangle ABC will be seen to have reflectional symmetry about the axis m which is the mediator of the base BC.

Following the practice in undergraduate mathematics courses in the University of Hull, we write:

\[ M: \begin{bmatrix} A \\ B \\ C \end{bmatrix} \rightarrow \begin{bmatrix} A \\ C \\ B \end{bmatrix} \]

which is read: reflection \( M \) takes object \( \triangle ABC \) to image \( \triangle ACB \).

The notation indicates clearly the symmetry correspondence between object and image and acts as a stepping-stone to the algebraic representation of the situation in terms of a transformation matrix.

The reader is again reminded (see Chapter 10, p.273 footnote) of the need to distinguish between:

positions defined in space

and:

topositions defined in the figure.
The transformation from an object to an image position may be regarded either as a change of address in which, for example, under a rotation of 240° an observer at A in ΔABC sees:

\[
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
\]

move from position \[
\begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix}
\]
to \[
\begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix}
\]
in space; or as an observer at position 1 in space he sees a hole in space:

\[
\begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix}
\]

filled first by object \[
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
\]

and then by image \[
\begin{bmatrix}
C \\
B \\
A
\end{bmatrix}
\]

The author has elected to use the second method.

(3) The Equilateral Triangle

Undoubtedly, the equilateral triangle is an obvious candidate for subsequent development. The first question to be asked is: how many symmetries can be found? The answer lies in the number of ways in which the triangle can be made to 'replace itself' in its position in space, or on the paper on which it is drawn, albeit its vertices having been re-arranged, or permuted, amongst themselves.

There are just 6 ways in which this can be done (Fig 6.4).
The 6 positions illustrated may be identified by naming the triangle in its transformed position, starting in each case with the vertex at the top of the triangle and tracing the triangle around from vertex to vertex in an anti-clockwise direction. The names that have been given to the triangle in the 6 different positions are obtained as the 6 permutations of the letters A, B, C.

Positions (1), (2), (3) are the results of the direct rotational symmetries about the circum-centre O of the triangle ABC:

(1) $R_1(0, 0^\circ): \begin{bmatrix} A \\ B \\ C \end{bmatrix} \rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix}$: the identity transformation

(2) $R_2(0, 120^\circ): \begin{bmatrix} A \\ B \\ C \end{bmatrix} \rightarrow \begin{bmatrix} C \\ A \\ B \end{bmatrix}$

(3) $R_3(0, 240^\circ): \begin{bmatrix} A \\ B \\ C \end{bmatrix} \rightarrow \begin{bmatrix} B \\ C \\ A \end{bmatrix}$

Positions (4), (5), (6) are the results of the opposite reflectional symmetries in the mediators of the triangle through its vertices:

(4) $M_1(\text{in } m_1): \begin{bmatrix} A \\ B \\ C \end{bmatrix} \rightarrow \begin{bmatrix} A \\ C \\ B \end{bmatrix}$

(5) $M_2(\text{in } m_2): \begin{bmatrix} A \\ B \\ C \end{bmatrix} \rightarrow \begin{bmatrix} B \\ A \\ C \end{bmatrix}$

(6) $M_3(\text{in } m_3): \begin{bmatrix} A \\ B \\ C \end{bmatrix} \rightarrow \begin{bmatrix} B \\ A \\ C \end{bmatrix}$

It is noted that the circum-centre O is an invariant point and will be taken as origin in the following alternative representation of the six symmetries of the equilateral triangle.

(4) Algebraic Representation

It is customary to represent the positions of the vertices by means of coordinates in a cartesian frame, usually rectangular, of reference. For example, with the circum-centre O as origin and perpendicular axes $OX, OY$ where $OY$
passes through the vertex A of the triangle ABC, we have (Fig 6.5):

\[ A (0, \frac{2\sqrt{3}}{3}); \quad B (-\frac{1}{2}, -\frac{\sqrt{3}}{3}); \quad C (\frac{1}{2}, -\frac{\sqrt{3}}{3}) \]

It may be assumed without loss of generality that the sides of the triangle are of unit length. The coordinates of the vertices are given by:

It will be appreciated that, at this stage of pupil development, the surds are cumbersome and this treatment of the situation has been considered inappropriate. The author suggests the use of unit basis vectors: 

\[ \overrightarrow{OA} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \overrightarrow{OB} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \] 

inclined at an angle of 120° (Fig 6.6(i)).

Experienced teachers on higher degree courses agree with the author that Stage-A experiences referred to at the beginning of this section would enable pupils to obtain the position vector of C as 

\[ \begin{bmatrix} -1 \\ -1 \end{bmatrix} \], 

by drawing lines through C parallel to OA, OB to meet OB, OA respectively in Q, P (Fig 6.6(ii)).
Each of the 6 symmetries of the equilateral triangle ABC may now be represented by a 2 × 2 transformation matrix of form 
\[
\begin{bmatrix}
  a_1 & b_1 \\
  a_2 & b_2 
\end{bmatrix}
\]
in which \(a_1\) is the image of the unit basis vector \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and \(b_1\) is the image of the unit basis vector \(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\). The results may be tabulated thus:

**under rotations:**

<table>
<thead>
<tr>
<th>Triangle</th>
<th>(R_1(0,0^\circ))</th>
<th>(R_2(0,120^\circ))</th>
<th>(R_3(0,240^\circ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B\ C)</td>
<td>(A\ B\ C)</td>
<td>(A\ B\ C)</td>
<td>(A\ B\ C)</td>
</tr>
<tr>
<td>(A\ B)</td>
<td>(A\ B\ C)</td>
<td>(A\ B\ C)</td>
<td>(A\ B\ C)</td>
</tr>
<tr>
<td>(C\ A)</td>
<td>(A\ B\ C)</td>
<td>(A\ B\ C)</td>
<td>(A\ B\ C)</td>
</tr>
</tbody>
</table>

It is easily verified that the transformation matrix determined by the position vectors of the images of \(A\begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(B\begin{bmatrix} 0 \\ 1 \end{bmatrix}\), automatically determines the position vector of the image of \(C\begin{bmatrix} -1 \\ -1 \end{bmatrix}\).

Similarly for the reflections:

<table>
<thead>
<tr>
<th>Triangle</th>
<th>(M_1(\text{in } m_1))</th>
<th>(M_2(\text{in } m_2))</th>
<th>(M_3(\text{in } m_3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B\ C)</td>
<td>(A\ C\ B)</td>
<td>(A\ B\ C)</td>
<td>(A\ C\ B)</td>
</tr>
<tr>
<td>(A\ B)</td>
<td>(A\ C\ B)</td>
<td>(A\ B\ C)</td>
<td>(A\ C\ B)</td>
</tr>
<tr>
<td>(C\ A)</td>
<td>(A\ C\ B)</td>
<td>(A\ B\ C)</td>
<td>(A\ C\ B)</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1\ 0\ -1)</td>
<td>(1\ -1\ 0)</td>
<td>(1\ 0\ 1)</td>
</tr>
<tr>
<td>(0\ 1\ -1)</td>
<td>(0\ -1\ 1)</td>
<td>(0\ 1\ -1)</td>
</tr>
<tr>
<td>(1\ -1)</td>
<td>(-1\ 0)</td>
<td>(0\ 1)</td>
</tr>
</tbody>
</table>
This early introduction to the pupils of an oblique frame of reference has been strongly supported by some of the author's advanced course students. It has however been suggested that, if a first example in which the basis vectors are perpendicular is preferred, the symmetries of the square should perhaps be used, the axes to be taken through the vertices and not parallel to the sides. With oblique axes at our disposal, the work may be extended to the regular n-sided polygons with convenient choices of n appropriate to the age/ability level of the pupils: n = 6, 8, 12 are obvious examples, and using results of the investigations of §6.9 (1) it may be seen that the symmetries of a regular, n-sided polygon form a group under the law of composition 'followed-by' (§6.8).

6.10 Tidying-up the Loose Ends

The algebraic representation of the isometries of Chapter 4 depend on the existence of a solution to the equations (1) on page 178. The mathematical background for teachers outlined in this chapter has been restricted to an analysis of the cases that arise under the corresponding condition:

$$a_1b_2 - a_2b_1 \neq 0$$

from the point of view of invariance. The existence of an isometry is given only by values of $\lambda = \pm 1$ of the characteristic $\lambda$-equation (5) on page 184 which dictated corresponding values of the author's 'eigen discriminant' $a_1 + b_2 - 1$ as numerically equal to 1. The transformation took the form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \text{ for } \lambda = \pm 1$$

The transformations corresponding to $\lambda \neq 1$ and the exceptional cases of a non-solution to the equations (1) under the condition: $a_1b_2 - a_2b_1 = 0$, will be discussed in Chapter 7 in terms of the similarity transformations.

The work of this chapter has been restricted to 2 dimensions. The methods and results may be extended to 3 or more
dimensions and the work may be found in many school texts, including those referred to by the author. She herself will deal with the 3-dimensional situation in Chapter 10 using a somewhat different approach.
CHAPTER 7

ENLARGEMENT

7.1 From Symmetry to Similarity

In Chapter 4 the author discussed some of the difficulties which she has experienced in a traditional approach to the teaching of congruence. She attempted to show how, for her, these difficulties could be resolved to a very large extent by an investigation of the geometrical properties of symmetrical figures in terms of the isometric distance-preserving transformations. As for congruence, so with similarity: problems in the traditional approach to the teaching of similarity can be overcome largely by a consideration of the geometrical properties of similar figures in terms of the similarity transformations.

By definition, according to Chambers [15], similarity means:

"exactly corresponding in shape without regard to size". Mathematically this is, strictly, not correct: similar figures have the same shape and are proportionate in size. In other words:

- corresponding angles are equal
- corresponding sides are proportional.

In some texts, see for example [17] and [32], similarity has complexly become 'similitude' which is a much looser and vaguer term. Again according to Chambers, 'similitude' implies mere:
"likeness; semblance; comparison; ....."

The late E H Neville, who was the sower of the author's own seeds of enthusiasm for mathematics teaching, is quoted by H S M Coxeter [17] as saying:

"I have often wondered why similitude ever got into elementary geometry..... I'm sure youngsters would be much more at ease with a pair of circles if they just had centres of 'similarity' instead of being made to imagine that some new idea was insinuating itself."

Problems arise in the elementary teaching of similarity, the author feels, because of an introduction to the concept by means of similar triangles. The similarity of triangles is an exceptional and a rather special case of the similarity of figures in general, in the sense that only one of the defining pair of necessary and sufficient conditions is required: either condition when applied to triangles implies the truth of the other. For a pair of triangles:

- corresponding angles equal \(\Rightarrow\) corresponding sides proportional
- corresponding sides proportional \(\Rightarrow\) corresponding angles equal

Pupils need to investigate the properties of pairs of figures which are selected to satisfy both conditions independently of each other before dealing with any exceptional cases. In the early stages of the development of similarity, the often-quoted counter-examples:

- a square and a rectangle which are equiangular but which do not have corresponding sides proportional and
- linkaged, equal-sided, pentagons which can be 'swung' into positions which are not equiangular

do not, in the author's experience, present pupils with the
un-expected situation of un-similar pairs of figures. They do nothing to convince the pupils that for similarity both conditions are in general necessary.

7.2 Similarity? or Enlargement?

Early experiences of a tiling pattern formed by three sets of parallel lines described by T J Fletcher [9] appear to have as their main objective the 'discovery' on the part of the pupils that equiangular triangles have their corresponding sides proportional.

E A Maxwell [20] uses translation of equiangular triangles that are similarly situated – i.e., have corresponding sides parallel – into a position of coincidence of a pair of corresponding vertices to give a concrete validation of the theorem that if, in Fig 7.1, AB is parallel to A'B' and

\[
\frac{OA}{OA'} = \frac{OB}{OB'} = \lambda \text{ (say)}
\]

then: \(\frac{AB}{A'B'} = \lambda\), a result that is fundamental to the notion of similar triangles.

T J Fletcher extends his tiling exercise and, with the use of a pantograph – an excellent linkage instrument designed to draw similar figures – the investigation of the transformation \(OP' = \lambda OP\) for a fixed point O of a plane is admirably described for older pupils [9, 17] and will not be
enlarged(!) upon here. The work is, for the author, an excellent introduction to the idea of similarity.

Similarity is, of course, being approached from the point of view of a one-to-one mapping of the plane onto itself in which angles are preserved: ie shape is preserved, size is not preserved. A dilatation is a transformation in which there is one invariant point and which preserves, or reverses, direction. It requires that object and image figures are similarly situated in the sense that object-image line-segments are parallel, but not necessarily in the same sense, to each other. Angles in the object-image correspondence are thus preserved and the object-image figures are equiangular. Attention can then be focused on the enlargement, or 'ensmallment' to coin a word which pupils enjoy using, of object to image, the aspect of a dilatation determined by the constant real-number scale-factor which relates the lengths of corresponding line-segments in object-image figures. It ensures that corresponding line-segments are proportional and hence the second requirement for similarity.

We are, of course, concerned here with the 'homothetic' figures of the traditional "Modern Geometry" of C V Durell [33] and of C Godfrey MVO and A W Siddons MA [34], figures which are similar and similarly situated:

\[ \text{a dilatation is a similarity} \]

but: not every similarity is a dilatation.

7.3 Dilatation - Central Similarity

H S M Coxeter [17] suggests a similar approach. His key theorem asserts that:

"two given parallel line-segments \(AB, A'B'\) are related by a unique dilatation \(AB \rightarrow A'B'\)."
The reader is asked to assume that two lines are parallel - ie have the same or opposite direction - in the generalised sense that they have either no points, or all points, in common with each other. The truth of Euclid's parallel axiom is also assumed: in 2-dimensions, if \( \ell \) is any line and \( P \) is any point not on \( \ell \) then there is exactly one line \( m \) which passes through \( P \) and which is parallel to \( \ell \). If \( P \) lies on \( \ell \) then \( m \) and \( \ell \) coincide.

The reader is reminded that for a dilatation corresponding line-segments are parallel and that the scale-factor of enlargement, or ensmallment, must be determined.

If then \( AB = A'B' \) so that \( AB \parallel A'B' \) in the same, or opposite, sense, the image \( C' \) of any other point \( C \) of the plane not on \( AB \) is determined uniquely as the point of intersection \( C' \) of \( A'C' \parallel AC \) and \( B'C' \parallel BC \) (Fig 7.2)

![Fig 7.2](image)

Note that the image of a point \( D \) (say) on \( AB \) is obtained as \( D' \) on \( A'B' \) such that \( C'D' \parallel CD \).

It is to be observed next that, if \( P \) and \( Q \) lie respectively on the lines joining any 2 pairs \( A, A' \) and \( B, B' \) of object-image points then:

\[
\begin{align*}
\text{P on } AA' & \Rightarrow P' \text{ on } AA' \\
\text{Q on } BB' & \Rightarrow Q' \text{ on } BB'
\end{align*}
\]
It follows that AA', BB' are invariant lines, but not necessarily point-for-point, under the transformation and in general intersect in a unique invariant point 0. The exceptional case in which AA' is parallel to BB' is of course the translation isometry in which there is no invariant point and which does not form part of this discussion (see page 116 of Chapter 4).

The object-image points P, P' are related by an equation of the form: \( OP = \lambda OP' \) where \( \lambda \) is a real-number scalar-multiple with the usual convention of sign in which positive/negative \( \lambda \) corresponds to external/internal division of PP' at 0.

Coxeter then appeals to* an elementary property of similar triangles to assert that not only is \( \lambda \) constant and independent of the position of P but that that same constant is the ratio which relates the given object-image line-segments AB, A'B'.

The same approach to similarity is suggested by T J Fletcher [9] in which he states that:

"The dynamic treatment of similarity, in which the attention is not on the figures which are similar but on the transformation of the plane which assigns to every point P an image P' such that \( OP' = k.OP \), is outlined in many .... texts ...."

The notion of symmetry can now be extended to the notion of similarity. The symmetry transformations, the isometries, of Chapter 4 were obtained algebraically, in Chapter 6, as the one-to-one onto transformations of the plane:

\[
\begin{bmatrix}
  x \\
  y \\
\end{bmatrix} \rightarrow \lambda \begin{bmatrix}
  x \\
  y \\
\end{bmatrix}
\]

* the SAS test of similarity
for a real-number scalar-multiple $\lambda = \pm 1$. The author now attempts a discussion for values of $\lambda \geq 1$ which will be seen to give the dilatations or central similarities which form a special subset of the general similarity transformation (§7.9, page 230).

7.4 The Transformation \[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} \rightarrow \lambda \begin{bmatrix}
  x \\
  y
\end{bmatrix}: \lambda \neq 1
\]

In Chapter 4 the author based her work on Kiselyov's idea of geometry as:

"the science that studies those properties of geometrical figures that are not changed by motions of the figures" [18].

The motions are the distance-preserving isometries which result in a symmetry. They constitute a universal set of geometrical elements which form a group under the operation 'followed-by'. The language of linear algebra explained in Chapter 5 enabled the author in Chapter 6 to represent algebraically the geometrical properties that exist under the isometries. William Wynn Willson [21] also sees geometry as a way into linear algebra: "points and lines are not dots and dashes but coordinates and equations".

The work is now to be extended and based on F Klein's definition of geometry as:

"the science that studies those properties of geometrical figures that are not changed by similarity transformations" [35].

ie transformations which preserve the ratio of the distances between pairs of corresponding points. The author considers again the transformation of the 2-dimensional plane, one-to-one onto itself, in which the position vectors of image-points are expressible as sums of scalar multiples of the position
vectors of the corresponding object-points. The transformation is again represented algebraically by the equations (1) of Chapter 6 page 178.

\[
\begin{align*}
    a_1x + b_1y &= x' \\
    a_2x + b_2y &= y'
\end{align*}
\]

in which the origin is an invariant point. A solution of the equation exists to give the inverse transformation if:

\[a_1b_2 - a_2b_1 \neq 0\]

Existence of the invariance of points and lines leads to the expression of the transformation in the form:

\[
\begin{bmatrix}
    x' \\
    y'
\end{bmatrix} = \lambda \begin{bmatrix}
    x \\
    y
\end{bmatrix}
\]

for a constant scalar-multiple value of \( \lambda \) which satisfies the characteristic equation (5) of Chapter 6 page 184

\[
\lambda^2 - \lambda(a_1 + b_2) + (a_1b_2 - a_2b_1) = 0
\]

The isometries are given by \( \lambda = \pm 1 \) and the following cases are identified:

(a) \( a_1b_2 - a_2b_1 = a_1 + b_2 - 1 \): there is more than one invariant point which form a pointwise invariant line given by \( \lambda = \pm 1 \).

The other value of \( \lambda = a_1 + b_2 - 1 \)

(b) \( \lambda = a_1 + b_2 - 1 = \pm 1 \) gives the identity transformation; all lines are pointwise invariant.

(c) \( \lambda = a_1 + b_2 - 1 = -1 \) gives reflection; infinitely many lines are non-pointwise invariant*.

(d) \( a_1b_2 - a_2b_1 \neq a_1 + b_2 - 1 \): the origin is the only invariant point.

\( \lambda = -1 \) (twice) gives a half-turn; every half-line corresponds to its collinear half-line counterpart.

* Half-lines perpendicular to the pointwise invariant line correspond to their collinear half-line counterparts.
It has been established on pages 184, 185 of Chapter 6 that:

\[ \lambda = \pm 1 \Rightarrow a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1 = \pm 1 \]

The author now considers the remaining cases for which a solution to the equations (1) exists:

\[ \text{ie } a_1 b_2 - a_2 b_1 \neq 0 \]

and

\[ a_1 b_2 - a_2 b_1 \neq a_1 + b_2 - 1; \quad |\lambda| \neq 1 \]

in which the origin is the only invariant point.

A transformation \( P \to P' \) is now defined so that:

\[ \text{OP}' = \lambda \cdot \text{OP} \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \]

in which the origin is an invariant point.

### 7.5 Invariance

If under the transformation: \( P_1 \to P'_1; \ P_2 \to P'_2 \)

\[ \begin{bmatrix} x'_1 \\ y'_1 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}; \quad \begin{bmatrix} x'_2 \\ y'_2 \end{bmatrix} = \lambda \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \]

then:

(1) The ratio between distances is preserved; we have:

\[ P'_1 P'_2 = (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 = \lambda^2 (x_2 - x_1)^2 + (y_2 - y_1)^2 \]

\[ = \lambda^2 P_1 P_2^2 \]

so that:

\[ P'_1 P'_2 : P_1 P_2 = \pm \lambda : 1 \]

The magnitude of the ratio between the lengths of corresponding line segments is preserved. The result has been established without appeal to traditional tests of similarity of triangles. Furthermore the numerical value of the ratio is the same as the defining ratio of the distances of corresponding points from the fixed origin of the transformation.
(2) Direction is preserved (or reversed); we have:

\[ \text{gradient of } P_1P_2 = \frac{y_2' - y_1'}{x_2' - x_1'} = \frac{\lambda(y_2 - y_1)}{\lambda(x_2 - x_1)} = \text{gradient of } P_1P_2 \]

(3) The invariance of points and lines: If the origin is the only invariant point it is known that there can be no pointwise invariant lines. Invariant lines, not point-for-point, may be obtained from values of \( \lambda \) satisfying the characteristic equation:

\[ \lambda^2 - \lambda(a_1 + b_2) + (a_1b_2 - a_2b_1) = 0 \]

In general, two such lines exist and their gradients may be found as suggested on page 185 of Chapter 6.

The attention of the reader is directed to the footnote on that page and focused on the invariance of gradient. Since, by definition of the transformation, the joins of corresponding points concur in the fixed origin, any line which is invariant, but not necessarily point-for-point, under the transformation may be represented by:

\[ y = mx \rightarrow y' = mx' \]

Substituting for \( y, y' \) in terms of \( x, x' \) in equations (1) of the transformation, the gradient \( m \) is given by:

\[ (a_1 + b_1m)x - x' = 0 \]
\[ (a_2 + b_2m)x - mx' = 0 \]

A solution for \( x : x' \) exists if:

\[ b_1m^2 + (a_1 - b_2)m - a_2 = 0 \] (6')
This may be regarded as the 'm-characteristic' equation. The reader is invited to compare this result with the gradient equation (6) on page 185 of Chapter 6.

m is real if: \((a_1 + b_2)^2 - 4(a_1b_2 - a_2b_1) \geq 0\)
and this condition is compatible with that for real values of \(\lambda\) given on page 184 of Chapter 6. The conditions for point-wise invariance are obtained exactly as for \(\lambda\).

In the current discussion, the interest lies in the fact that equation (6') is seen to be true for all m when:

\[
a_2 = 0; \quad a_1 = b_2; \quad b_1 = 0
\]
in which case every line through the origin is invariant. Furthermore this is the only solution compatible with:

\[
\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}
\]
with coincident roots \(a_1, b_2\) for \(\lambda\). The dilatation is obtained as an enlargement (or ensmallment) with a fixed centre \(O\), scale factor of dilatation \(\lambda\), with transformation matrix of the form:

\[
\begin{bmatrix}
\lambda & 0 \\
0 & \lambda
\end{bmatrix}
\]
It will be denoted by \(O(\lambda)\).

It is now suggested that the consideration of a pair of unequal circles provides a situation that is more attractive and fruitful for the pupils' study of the properties of a central similarity transformation than does the more usual traditional situation of a pair of homothetic triangles. The following investigation is suggested for a stage C secondary level of geometrical development.

7.6 The Properties of a Central Similarity

It is assumed that pupils will know from earlier experiences, with a pantograph for example, that if \(O\) is a fixed point and \(\lambda\) is a constant scale-factor, the transformation:
\[ P \rightarrow P' \Rightarrow OP' = \lambda OP : |\lambda| \neq 1 \]

is such that: \( OPP' \) is an invariant line, not pointwise, and
that:
\[
\left\{ \begin{array}{l}
P_1 \rightarrow P'_1 \\
P_2 \rightarrow P'_2
\end{array} \right\} \Rightarrow \left\{ \begin{array}{l}
P_1P_2 \parallel P'_1P'_2 \\
P'_1P'_2 = \lambda P_1P_2
\end{array} \right\}
\]

The dilatations are a sub-set of the general similarity transformations and are called central similarities in which
0 is the centre and \( \lambda \) is the constant scale-factor

\textit{coefficient} of the transformation.

The following investigation is designed to encourage pupils
to establish the similarity of a pair of unequal circles and
the consequent properties that exist under a central similarity
transformation.

1. In how many ways can a pair of unequal circles be drawn in
different positions relative to each other? (Fig 7.3)
In each case say what the respective pairs have in common [20].

(a) 4 common tangents
(b) 3 common tangents
(c) 2 common tangents
(d) 1 common tangent
(e) 0 common tangents

Fig 7.3
2 If $O$ is a fixed point outside a given circle $C$ whose centre is $A$, radius is $r$, construct the images $C'$ of the circle $C$ under dilatations with $O$ as centre of similarity and with various selected values of coefficient $\lambda$. In each case, determine:

- the image of $A$ which is the centre of circle $C$
- the image of $P$ on the circumference of $C$
- the image of a chord $PQ$ of the circle $C$
- the image of the tangent $OP$ to the circle $C$

Describe the image of the circle $C$ under the central similarity $O(\lambda)$.

3 Describe the inverse transformation, if it exists, which is such that: $C' \to C$.

4 Repeat 2 and 3 for a fixed centre of similarity $O$ inside the circle $C$ and for a fixed centre of similarity $O$ on the circle $C$.

Pupils are thus encouraged to discover that $C$ and $C'$ form an object-image pair of corresponding figures (circles) in two ways under a central similarity in which the centres $A$, $A'$ correspond, the coefficient of similarity $\lambda$ is such that: $r' = \lambda r$ where $r$, $r'$ are the radii of $C$, $C'$ respectively, and the centre $O$ of similarity divides $AA'$ externally/internally in the ratio $|\lambda| = \frac{r'}{r}$ according as $\lambda$ is positive/negative:

If $\lambda = \frac{r'}{r}$ is positive then corresponding chords are parallel in the same sense, $O$ lies outside of $AA'$ and is an exterior centre of similarity which divides $AA'$ externally in the ratio $1:\lambda$.

If $\lambda = \frac{r'}{r}$ is negative then corresponding chords are parallel in the opposite sense, $O$ lies between $A$, $A'$ and is an interior centre of similarity which divides $AA'$ internally in the ratio $1:\lambda$. 
In Fig 7.3(a), (b), (c), the exterior centre of similarity is seen to be the point of intersection of the external common tangents. In Fig 7.3 (d), the exterior centre is the point of contact of the one and only external common tangent.

In Fig 7.3(a), the interior centre of similarity is seen to be the point of intersection of the internal common tangents. In Fig 7.3(b), the interior centre is the point of contact of the one and only internal common tangent.

The reader should note that in Fig 7.3(c), (d), the interior, exterior centres of similarity respectively can be found only as the points of internal, external division of the line joining pairs of corresponding points in the ratio $\lambda = \pm 1$ respectively: for example, from a pair of parallel in the opposite or same sense line-segments, as the point of intersection of the joins of their extremities.

5 Determine the interior and exterior centres of similarity for the pairs of circles of Fig 7.3(a) - (e).

The determination of the centres of similarity could lead to the 'discovery' of a method of construction of common tangents to a pair of circles. The reader should compare suggested method(s) with the artificial need of an 'auxiliary' circle in the more usual traditional method of construction.

Exceptional cases of the transformation of Fig 7.3(a) - (e) in which the circles are equal, concentric, or coincident, should be considered.

6 2 equal circles are centrally similar in just one way*: the centre is the mid-point of AA', the coefficient is -1.

* If the coefficient is $+1$, the centre is the 'point at infinity' and the transformation is a translation which is not included in this discussion.
The transformation is the half-turn. Problems involving half-turns can thus be generalised [18] and [35] into central similarity problems $O(\lambda)$ involving the ratio $\lambda$. Lines bisected at $O$ become lines divided internally at $O$ in a given ratio.

2 concentric circles are centrally similar with centre at the common centre of the 2 circles, the coefficient is: \[ \frac{OP'}{OP} = \frac{r'}{r} \]

Fig 7.5

2 coincident circles are centrally similar by the identity transformation: the case of 2 equal concentric circles. The reader is reminded that the identity is regarded as a special case of a central similarity in which every point is invariant and the coefficient is $\lambda = 1$.

Fig 7.6

The author proceeds now with the further development of the system of transformations $OP' = \lambda OP$ for $|\lambda| \neq 1$ and the consideration of the result of the product of central similarities.

7.7 The Law of Composition 'Followed-by'

The central similarities: $P \rightarrow P'$ in which \[ \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \lambda \begin{bmatrix} x \\ y \end{bmatrix}, \]

$|\lambda| \neq 1$, may be combined as in the case of the isometries for which $|\lambda| = 1$ under the law of composition 'followed-by'. It will be shown that:

Theorem: The product of two central similarities $O_1(\lambda_1)$ and $O_2(\lambda_2)$ is a central similarity $O(\lambda_1 \lambda_2)$ in which $O$, $O_1$, $O_2$ are collinear.
(1) Algebraically we let:
\[
\begin{bmatrix} x \\ y \end{bmatrix}
\text{ be the position vector of } P \text{ relative to } O_1
\]
\[
\begin{bmatrix} X \\ Y \end{bmatrix}
\text{ be the position vector of } P \text{ relative to } O_2
\]
\[
\begin{bmatrix} h \\ k \end{bmatrix}
\text{ be the position vector of } O_2 \text{ relative to } O_1
\]

The transformation \( O_1(\lambda_1) \): \( P \rightarrow P_1 \) so that
\[
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}
\]
\[
O_2(\lambda_2) : P_1 \rightarrow P_2 \text{ so that }
\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}
\]

The product transformation \( O_1(\lambda_1) \) \(\times\) \( O_2(\lambda_2) \): \( P \rightarrow P_2 \) is such that:
\[
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - h \\ y_1 - k \end{bmatrix} = \lambda_2 \begin{bmatrix} x_1 - h \\ y_1 - k \end{bmatrix} = \lambda_2 \begin{bmatrix} \lambda_1 x - h \\ \lambda_1 y - k \end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix} x_2 - h \\ y_2 - k \end{bmatrix} = \lambda_1 \lambda_2 \begin{bmatrix} x \\ y \end{bmatrix} - \lambda_2 \begin{bmatrix} h \\ k \end{bmatrix}
\]

If now an origin \( \begin{bmatrix} H \\ K \end{bmatrix} \) relative to \( O_1 \) is chosen, so that position vectors \( \begin{bmatrix} x \\ y \end{bmatrix} \) are written \( \begin{bmatrix} \xi + H \\ \eta + K \end{bmatrix} \), we have:

\[
O_1(\lambda_1) \times O_2(\lambda_2) : P \begin{bmatrix} \xi \\ \eta \end{bmatrix} + P_2 \begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix} \text{ where: }
\]
\[
\begin{bmatrix} \xi_2 + H - h \\ \eta_2 + K - k \end{bmatrix} = \lambda_1 \lambda_2 \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \lambda_1 \lambda_2 \begin{bmatrix} H \\ K \end{bmatrix} - \lambda_2 \begin{bmatrix} h \\ k \end{bmatrix}
\]

Choosing: \( \begin{bmatrix} H \\ K \end{bmatrix} = \frac{1 - \lambda_2}{1 - \lambda_1 \lambda_2} \begin{bmatrix} h \\ k \end{bmatrix} \), the transformation is seen to be a central similarity: \( O(\lambda_1 \lambda_2) \) with coefficient \( \lambda_1 \lambda_2 \) and centre 0 on \( O_1O_2 \) such that:
The exceptional cases occur only if $|\lambda| = 1$ (see the investigation, pages 217 to 220).

(2) The further investigation of images of parallel line-segments (see page 219) gives rise to the following very satisfying results for the geometrical representation of the product of central similarities under the law of composition 'followed-by' (Fig 7.7).

Let $O_1(\lambda_1): AB \rightarrow A_1B_1$ so that:

\[
\begin{align*}
A_1A_2, B_1B_2 & \text{ meet in } O_1 \quad \Rightarrow \quad \frac{A_1B_1}{AB} = \lambda_1 \\
\frac{O_1A_1}{O_1A} & = \frac{O_1B_1}{O_1B} = \lambda_1
\end{align*}
\]

Hence:

\[
A_2B_2 | AB \quad \text{and} \quad \frac{A_2B_2}{AB} = \frac{A_1B_1}{AB} = \lambda_1 \lambda_2
\]

which defines a central similarity:

Let $O_2(\lambda_2): O_1A_1B_1 \rightarrow O_1A_2B_2$ so that:

\[
\begin{align*}
O_1O_2, A_1A_2, B_1B_2 & \text{ meet in } O_2 \quad \Rightarrow \quad \frac{A_2B_2}{A_1B_1} | B_2O_1 \quad \Rightarrow \quad \frac{B_1O_1}{O_1A_1} = \lambda_2 \\
\frac{O_2O_1}{O_2O_1} & = \frac{O_2A_2}{O_2A_1} = \frac{O_2B_2}{O_2B_1} = \lambda_2
\end{align*}
\]

Hence:

\[
A_2B_2 | AB \quad \text{and} \quad \frac{A_2B_2}{AB} = \frac{A_1B_1}{AB} = \lambda_1 \lambda_2
\]

(Fig 7.8)

\[
\frac{O_1O_2}{O_1O_2} = \frac{1 - \lambda_2}{1 - \lambda_1 \lambda_2}
\]

ie $O$ divides $O_1O_2$ in the ratio $(1 - \lambda_2):(1 - \lambda_1 \lambda_2)$. 

Fig 7.7

Fig 7.8
It is easy to deduce that:

$$\frac{OC_1}{O_1 O_2} = \frac{1 - \lambda_2}{1 - \lambda_1 \lambda_2}$$

agreeing with the result obtained algebraically in (1).

It is important to note that the process is not in general commutative. The result of the product $O_2(\lambda_2)$ fb $O_1(\lambda_1)$ would be a central similarity $O(\lambda_1 \lambda_2)$ in which:

$$\frac{OC_2}{O_1 O_2} = \frac{1 - \lambda_1}{1 - \lambda_1 \lambda_2}$$

In the following section, the author attempts to retrieve a number of results of the traditional sixth-form geometry course. Although in former years these 'theorems' were largely of intrinsic interest only to the mathematics specialist, she herself needs them for the development towards the geometrical representation of the imaginary of algebra. They are set in the form of an exercise for the reader and are considered to be well within the scope of the average mathematics pupil of the formative sixth-form years.

7.8 Properties of the Triangle

It should be obvious to the reader that all the classical results of 2-dimensional Euclidean geometry based on congruence and homothety can be derived from the analogous concepts of isometry and central similarity. Texts already referred to: H S M Coxeter [17], E A Maxwell [20], William Wynn Willson [21], Yaglom [18] and [35] and others including Nicolaas H Kuiper [26], offer excellent and often very demanding applications of these linear transformations and their products. The author restricts the exercise to the straightforward application of the defining properties of the transformations associated with invariance to obtain circle-and-centre properties of the triangle which she needs for the extension of the methods of linear algebra and its geometrical representation in the field of complex numbers. Solutions are given in Appendix A. She starts with the following:
Example: The medians of a triangle are concurrent

![Diagram of medians of a triangle](image)

Let the medians BB' CC' of triangle ABC meet in G. Join B'C' (Fig 7.9). Then:
\[
\overrightarrow{AB} = 2 \overrightarrow{AC}' \quad \text{and} \quad \overrightarrow{AC} = 2 \overrightarrow{AB}'
\]
defines the dilatation:

\[
A(2): \overrightarrow{AB}'C' \rightarrow \overrightarrow{AC} \rightarrow \overrightarrow{AC}' \rightarrow \overrightarrow{AB}
\]

Reversing the sense of B'C', the parallel line-segments \overrightarrow{C'B'}, \overrightarrow{CB} define the dilatation:

\[
G(-2): \overrightarrow{GC}'B' \rightarrow \overrightarrow{GC} \rightarrow \overrightarrow{GB}' \rightarrow \overrightarrow{GC}
\]

Similarly it may be shown that: \overrightarrow{GA} = -2 \overrightarrow{GA}', ie the medians AA', BB', CC' of triangle ABC concur at their common point G of trisection which is nearer to A', B', C'. G is called the centroid of the triangle ABC.

It is interesting to note that the problem in terms of the 'language' of this work is open-ended.

It will be seen that G may be regarded as the centre of the dilatation:

\[
G(1): \overrightarrow{ABC} \rightarrow \overrightarrow{A'B'C}
\]

Since G is an invariant point of the transformation, it follows that G coincides with its image G' and is also the centroid of the triangle A'B'C'.
Exercise

1 The triangles $AB'C'$, $BC'A'$, $CA'B'$, $A'B'C'$ are isometric to each other and centrally similar to the triangle $ABC$.

2 The mediators of the sides of the triangle $ABC$ are concurrent. The point of concurrence is seen to be the centre of a circle which passes through $A$, $B$, $C$ and is called the circum-centre of the triangle $ABC$.

3 The altitudes of the triangle $ABC$ are concurrent. The point of concurrence, $H$, is called the ortho-centre of the triangle $ABC$. It may be shown that $H$ is also the circum-centre of the triangle whose sides are the line-segments through the vertices $A$, $B$, $C$, and parallel to the opposite sides, of the triangle $ABC$. It can further be established that $A$, $B$, $C$ are the ortho-centres respectively of the triangles $HBC$, $HCA$, $HAB$.

The attention of the reader is again drawn to the open-endedness of the situation. Interesting results are obtained by identifying the images of the centroids $G,G'$, the circum-centres $0, 0'$ and the ortho-centres $H, H'$ of the triangles $ABC$, $A'B'C'$ respectively, under the various transformations that may be applied.

4 Each of the circum-centres $O'$ of the triangles $A'B'C'$

$O_1$ $AB'C'$

$O_2$ $A'BC'$

$O_3$ $A'B'C$

respectively is the ortho-centre of the triangle formed by the remaining three circum-centres.

What is the connection between the circum-centres and the altitudes/ortho-centres of the triangles formed by them? Investigate other concurrencies of the situation: eg $AO_1$, $BO_2$, $CO_3$ are concurrent. Can you, the reader, identify the point of concurrence?
5 The circum-circles of the triangles HBC, HCA, HAB are equal to each other and to the circum-circle of the triangle ABC.

Identify the circum-centre of the triangle formed by their circum-centres.

6 Problem: X is a point in the plane of the triangle ABC. \( G_1, G_2, G_3 \) are the centroids of the respective triangles XBC, XCA, XAB. Investigate the situation arising from the central similarity: \( ABC \sim G_1G_2G_3 \).
7.9 The Story So Far

The investigations of §7.6 and §7.8 have not only given the pupils insight into the definition and elementary properties of the linear transformations: \( OP' = \lambda OP \) but have also enabled them to 'discover' for themselves the existence of circles that may be associated with the triangle. These include the 9-point circle which almost never fails to excite even the least mathematical amongst the 'mathematical sixth' (see example 3 of §7.8 on page 226).

More importantly, for the author, is the experience provided for the natural and further development of geometry to the harmonic situation. In particular the reader should note:

1. The exterior and interior centres of similarity harmonically separate externally and internally - ie in the same ratio - pairs of corresponding points:

\[
\frac{O}{OP'} = \lambda \frac{OP}{OP'}; \quad \frac{O'O'}{O'P'} = -\lambda \frac{O'P'}{O'P'}
\]

0, P, O', P', form a harmonic range denoted by \( \{PP'; OO'\} \). The work is developed in Part III (see Chapter 10).

2. The Euler line \( OGHN \) is obtained in example 4 of §7.8 on page 226.

3. Constructions based on all possible products of internal and external central similarities could lead to the discovery of the theorems of Ceva and Menelaus concerning the ratios in which the sides of a triangle are respectively divided by sets of concurrent lines and of collinear points. These afford the opportunity for investigations into the Principle of Duality - already experienced by pupils in the graphical representations of equivalence classes of ordered pairs of natural numbers (see Chapter 2).
The following situation has now been reached in the discussion of a linear transformation of the 2-dimensional plane, one-to-one onto itself, represented by the equations (1):

\[a_1x + b_1y = x'\]
\[a_2x + b_2y = y'\]
or by the \(2 \times 2\) coefficient matrix: \[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\]. The origin is an invariant point of the transformation which exists, in the sense that it has a unique inverse, if and only if:

\[a_1b_2 - a_2b_1 \neq 0.\]

The following results have so far been established:

1. \[a_1b_2 - a_2b_1 = a_1 + b_2 - 1\]
   
   There is more than one invariant point which may give rise to pointwise invariant lines.

2. \[a_1b_2 - a_2b_1 \neq a_1 + b_2 - 1\]
   
   The origin is the only invariant point. There may, or may not be, non-pointwise invariant lines.

Invariant lines were seen to be given by the correspondence:

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \lambda \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

between position vectors of corresponding points where \(\lambda\) is a constant scalar-multiple satisfying the characteristic (quadratic) equation:

\[\lambda = 1\] gives pointwise invariant lines
\[\lambda \neq 1\] gives non-pointwise invariant lines

For real values of \(\lambda\), the following cases have been identified:

2. \[a_1b_2 - a_2b_1 = a_1 + b_2 - 1 \Rightarrow \lambda = 1\] or \(\lambda = a_1 + b_2 - 1\)

\(\lambda = 1\) gives the pointwise invariant line with gradient \(m\) satisfying equation (6') on page 215

\[b_1m^2 + (a_1 - b_2)m - a_2 = 0\]
\[
\lambda = a_1 + b_2 - 1 = 1 \text{ gives the identity transformation; all lines are pointwise invariant.}
\]

\[
\lambda = a_1 + b_2 - 1 = -1 \text{ gives a reflection, the axis of reflection is given by } \lambda = 1. \text{ All perpendiculars to the axis are invariant half-lines and correspond to their half-line counterparts bisected by the axis.}
\]

\[3 \quad a_1b_2 - a_2b_1 \neq a_1 + b_2 - 1 \]

There are no pointwise invariant lines. In general there are 2 values of \( \lambda \) which satisfy the characteristic equation giving rise to 2 non-pointwise invariant lines.

The author however chose to consider the rather special case in which the characteristic equation was identically satisfied for all values of \( \lambda \) and in which every line through the origin is non-pointwise invariant.

\[
\lambda \neq 1 \text{ gives the dilatation or central similarity; the centre is at the origin; the coefficient is } \lambda.
\]

Special cases are given by:

\[\lambda = 1: \text{ a rotation with centre at the origin}\]

\[\lambda = -1: \text{ the half-turn}\]

4 The transformations are given in terms of \( 2 \times 2 \) matrices formed by the scalar-multiple coefficients in the defining equations (1); we have:

\[
\text{I: identity: } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
\text{R: rotation: } \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}: \text{ centre the origin angle } \alpha
\]

\[
\text{M: reflection: } \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}: \text{ axis } y = x \tan \alpha
\]

\[
\text{S_c: central similarity: } \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}: \text{ centre the origin coefficient } \lambda
\]
Attention is drawn to the fact that the elements $a_1, b_1; a_2, b_2$, of the matrices are such that:

(a) for central similarity: $a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1$:
$$a_1 = \lambda = b_2, \quad a_2 = 0 = b_1$$

(b) for isometry as a special case of central similarity ($\lambda = \pm 1$)
$$\lambda = +1 \text{ (twice): identity: } a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1: \quad a_1 = 1 = b_2, \quad a_2 = 0 = b_1$$
$$\lambda = -1 \text{ (twice): half-turn: } a_1 b_2 - a_2 b_1 = a_1 + b_2 - 1: \quad a_1 = -1 = b_2, \quad a_2 = 0 = b_1$$

In every case the matrix takes the form:
$$\begin{bmatrix} a & b \\ \ast b & \ast a \end{bmatrix}$$

5 The operation table under the law of composition 'followed-by' may be obtained as:

<table>
<thead>
<tr>
<th>2nd operation</th>
<th>$\ast$</th>
<th>$R$</th>
<th>$M$</th>
<th>$S_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>fb</td>
<td>$I$</td>
<td>$R$</td>
<td>$M$</td>
<td>$S_c$</td>
</tr>
<tr>
<td>I</td>
<td>$I$</td>
<td>$R$</td>
<td>$M$</td>
<td>$S_c$</td>
</tr>
<tr>
<td>R</td>
<td>$R$</td>
<td>$R$</td>
<td>$M$</td>
<td>?</td>
</tr>
<tr>
<td>M</td>
<td>$M$</td>
<td>$M$</td>
<td>$I$</td>
<td>?</td>
</tr>
<tr>
<td>$S_c$</td>
<td>$S_c$</td>
<td>?</td>
<td>?</td>
<td>$S_c$</td>
</tr>
</tbody>
</table>

For group structure we need closure. What can be said about the products of $S_c$ with $R$? and with $M$?

Algebraically: \( R \, fb \, S_c = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \lambda \cos \theta & -\lambda \sin \theta \\ \lambda \sin \theta & \lambda \cos \theta \end{bmatrix} \)

and: \( S_c \, fb \, R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda \cos \theta & -\lambda \sin \theta \\ \lambda \sin \theta & \lambda \cos \theta \end{bmatrix} \)
We write $S_R$ as the commutative product of the central similarity $S_C$ and the rotation $R$ whose transformation matrix is seen to be of the form $\begin{bmatrix} a & b \\ \pm b & \pm a \end{bmatrix}$.

Similarly, we write $S_M$ as the commutative product of the central similarity $S_C$ and the reflection $M$ whose transformation matrix is:

$$\begin{bmatrix} \lambda \cos2\alpha & \lambda \sin2\alpha \\ \lambda \sin2\alpha & -\lambda \cos2\alpha \end{bmatrix}$$

which is again of the form $\begin{bmatrix} a & b \\ \pm b & \pm a \end{bmatrix}$.

Incidentally, it may be of interest to the reader to note that an external central similarity is the commutative product of the corresponding internal central similarity and a half-turn:

$$O(\kappa).H = O(\kappa) = H.O(\kappa)$$

The products $S_R$ and $S_M$ defined above are the general similarity transformations which the author develops in Part III. It is customary at this stage to establish the existence of the elementary matrices formed from the elementary row operations (see page 178 of Chapter 6) on the unit matrix. They correspond to the elementary transformations [27.2]:

- one-way stretch: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$: (k ≠ 0)
- shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$: (k ≠ 0)
- reflection in y = x: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

where $k$ is a real-number scalar-multiple. J A H Shepperd and C J Shepperd then go on to show how all the standard linear transformations of the 2-dimensional plane may be derived from the product of any, or all, of the elementary
matrices. These standard transformations may be represented by equations (1):

\[ a_1x + b_1y = x' \]
\[ a_2x + b_2y = y' \]

and cover all cases for which \( a_1b_2 - a_2b_1 \neq 0 \).

The author has elected to confine her attention to the group of general similarities defined in this chapter and will not be concerned here with the so-called shears and stretches whose matrices are of the form:

\[
\begin{bmatrix}
k_1 & 0 \\
0 & k_2
\end{bmatrix}; \quad
\begin{bmatrix}
1 & k \\
0 & 1
\end{bmatrix}; \quad
\begin{bmatrix}
1 & 0 \\
k & 1
\end{bmatrix}
\]

and which do not preserve shape.
PART III

THE GEOMETRICAL REPRESENTATION
OF THE
COMPLEX IN ALGEBRA
CHAPTER 8

THE SIXTH-FORM COURSE

8.1 'Modern' Foundations

The expression 'modern mathematics in schools' has covered two outstanding mid-twentieth century developments. One is the introduction of new concepts, often at a comparatively early age; the other is the 'learning-by-doing-and-discovery' movement. Both have much common ground. Today many teachers of mathematics, brought up on traditional methods, who still have difficulty in relating the 'new' to the 'old': the new and the old are related.

It cannot be stressed too strongly that there is only one mathematics and that what has produced the 'new' is largely a fresh and more understanding way of looking at the 'old'. There are others, especially those with little previous mathematical education, who do not see how the isolated lessons which they give are related to mathematics as a whole.

The author, in Parts I and II of this thesis, has tried to show that underlying the isolated topics we teach: directed numbers, rationals, symmetry or congruence, similarity or dilatation, ..... , is a pattern or structure which is more fundamental and more important than the tricks we used to play with fractions, negative numbers, factors, equations, ..... This structure includes a greater awareness of the stuff of which mathematics is made. The work described is intended to provide the background mathematics that is required by every informed teacher of mathematics. It is bound to go
into considerable detail which makes reading difficult even for the more sophisticated mathematician; for the non-mathematician it requires discussion, explanation and illustration which initial and in-service teacher training courses may be expected to provide. Just as every would-be painter of the human body needs a knowledge of anatomy, so every teacher of mathematics should have some knowledge of the underlying structure.

A word of caution is needed. Often teachers suppose that to induce pupils to learn mathematics by 'playing' and 'discovering' is sufficient in itself. Certainly pupils are more likely to do correctly what they understand and to achieve understanding by doing, but they all too easily forget what they do not constantly practise. The teacher of English may try not to quench the spark in a would-be poet by over-insistence on punctuation, but he fails in his duty if his pupils leave school unable to write letters which express what they have to say. The teacher of mathematics has a corresponding duty to ensure that school leavers can apply correctly what they understand and can explain intelligibly what they are doing.

The inclusion of 'new' concepts in the secondary mathematics curriculum is now more or less generally accepted. The philosophy of 'learning-by-doing-and-discovery' is less popular in its application and has been slow to move up with the pupils from primary level in the age/ability scale. It is however gaining favour with the introduction of the requirements for the examinations for the General Certificate of Secondary Education.

At sixth-form level the work must be seen as something which follows on naturally from this GCSE-type of course and its philosophy. It is important for the pupils to be able to suggest and/or attempt different lines of approach to a single problem. It is even more essential at the sixth-form stage for them to be able to show that they fully understand
a technique and that it is both applicable and relevant to the problem in hand. This dictates the continuation of a style of teaching which indicates that practical experience is useful and emphasises the effect of learning through investigation. It dictates also a change in the way in which questions are posed and in the form of answer expected.

8.2 New Teaching Styles

The author, in Parts I and II, has already given some examples of the way in which certain topics of the main secondary mathematics course may be better taught, by pupils' experience with concrete material and apparatus and/or by detailed investigation by practical experiment and group-activity/discussion, than by the traditional method of 'chalk-and-talk'. The foundations laid by such mathematical experience in the earlier years of schooling must now be reinforced and extended by similar but more sophisticated experience at sixth-form level. It does of course not only offer a very great challenge to the mathematics teacher but also brings with it a very rewarding response from the pupils - not least at the lower end of the ability range.

No drastic change of sixth-form syllabus is called for. What is required is a re-think of every single topic that we teach: the method of presentation to the pupils; the formulation of problems set; the style required for the presentation of the solution(s). The author now describes some attempts that have been made with a class of very able fifth year 'remove' pupils at an independent boys' school* of national repute. Some of the lessons have been audio-recorded and a transcript of some of the recordings is included in Appendix D and Exhibits. The following exercise was given for homework at the end of lesson 4 in the recorded lesson sequence. It is based on the theory that, if a solution to

* Ampleforth College
a problem can be found by accurate drawing and measurement, then it is always possible to find an algebraic solution to the problem.*

**Example 1**

Given the positions of a point P and a line m it is clearly possible, by means of accurate construction and measurement, to find the shortest distance from P to m. An algebraic solution requires a formula to be found for calculating the shortest distance from a given point to a given line. The following method is suggested for tackling the problem:

1. State the problem in algebraic terms, using coordinate geometry.
2. What is meant by 'shortest distance'?
3. What results or methods are known to you that might be useful?
4. Try a simple example. What makes your chosen example simple?
5. Try a harder example.
6. Use your suggestions for 3., 4., and 5., to solve the general problem.

**Example 2**

Given the positions of three points in a plane, it is possible, by means of accurate construction and appropriate measurements, to find the area of the triangle. An algebraic solution requires a formula to be found to give the area of a triangle in terms of the positions of the three vertices. The following method is suggested for tackling the problem:

* "All ornament should be based on Geometric Construction": Owen Jones: The Grammar of Ornament [23]
1 State the problem in algebraic terms, using coordinate geometry

2 How do you usually work out the area of a triangle?

3 Can you apply the same method to this problem?

4 Choose a simple example and analyse the method used

5 Apply the method to a more general example

6 Apply the method to a completely general specification of the three vertices to produce the required formula.

A full discussion of the results appears on Cassette No 4 [S10] and an unedited transcript of the recording appears in the Exhibits. Further questions in this style appear in Appendix B. These were set to the 'pilot' class at the end of their first year post O-level course. The boys were given a fortnight in which to prepare solutions. It is interesting to note that one of them commented: "Why haven't we had questions like this before, Sir?".
The actual recordings give an indication of the excitement that was aroused and the very valuable and varied contributions that were made by all — well, almost all — of the members of the class.

The author ventures to suggest that such an approach, at least with these topics, and undoubtedly with many more, is well within the scope of a first-year sixth-form of pupils of lesser calibre.

The extensive use of the microcomputer by both teacher and pupil is indispensable in the development of this style of teaching at sixth-form level.

8.3 The New Technology

Consideration must now be given to the use of the microcomputer in the mathematics curriculum. The author is here concerned primarily with its use at sixth-form level. It must of course be an integral part of earlier development in mathematics teaching and learning if a firm foundation on which to build is to be provided.

Three such uses of the microcomputer may be identified [24]:
- as a teaching aid
- as a learning aid
- as a tool for performing mathematical tasks

Currently mathematical software is being produced often by professional programmers who may not be teachers; on the other hand, the average mathematics teacher is unable to programme the software.

Many teachers when faced with new methods/materials are often inhibited by previous experience and tend to reject new ideas. Their pupils have no such inhibitions. It may be said that they will do just as much, or as little, as is expected of them. In [6.1] it is written:
"We have in the past underestimated what many of our children could do in mathematics... By widening the scope of the mathematics our eyes have been opened to some of the things they can tackle and understand."

or as Professor Bondi said [36]:

"... about children. If you never ask them to do anything they cannot do, they will never be able to do the things they can".

The 'scope' must 'be widened' to include the use of the microcomputer.

The teacher must look for situations in which a short, simple-to-run, program can genuinely assist the presentation and explanation of a specific topic:

**Example 3** All parabolae have the same shape

**Example 4** The envelope of curves of the form

\[ y = \sin ax + \cos bx \]

These have been discussed extensively in class with teachers, students and pupils [S10 and Exhibits*].

* Essays by Tracey Niblett and Jenny Ranson in preparation for the BSc Integrated (Honours) Mathematics/Education degree at the University of Hull
Many programs already exist which are of value to both teaching and learning situations. Amongst these may be mentioned: Micros in Mathematics Education [S3]; Microelectronics Education Programme [S4]; the Mathematical Association's 132 Short Programs [S5]; MEI Programs for Mathematical Computing [S6]; David Tall's Supergraph and Graphic Calculus [S7]; Glentop's 3-D Graphics Development System [S8].

It is important, too, that we do not underestimate the talent which many pupils have for writing their own programs. It has been said* that a lot of teachers are having fun writing things for very young pupils which the pupils should be writing for themselves. This could certainly be true at the more advanced sixth-form level. One of the greatest joys of teaching is to find one's own pupils achieving more than oneself: not every teacher is brilliant, but every teacher can and should recognise potential and encourage brilliance in the pupils. Official syllabuses and examinations will only encourage originality and flair if we as teachers decide what it is that we value in pupil-performance and take a determined stand not to settle for a cheaper substitute.

The reader is reminded that the author has advocated the use of linear algebra as a linking theme in the unification of the 'traditional' with the 'modern' in school mathematics. It is now to be emphasised that linear algebra lends itself readily to work with a microcomputer and much experience currently at advanced levels will inevitably move to an earlier stage of the school mathematics programme.

---

* by Anita Straker, President of the Mathematical Association 1986-7
8.4 The Sixth-Form Course

There is a need to develop strategies more appropriate to the skills that universities are looking for in their students today. For mathematics and engineering courses, students need a much better appreciation of 2- and 3-dimensional spaces and their geometries; a greater familiarity with, and understanding of, approximation techniques; an ability to handle a microcomputer, not as an expert programmer but more as a tool for investigation; a general understanding of basic structures such as the idea of a group and of a vector space. The students need to be better trained to work on their own initiative.

The sixth-form mathematics course must also be seen to be educationally sound, not only for the high flyer and potential university scholar but also for any sixth-former who is capable of following it.

No drastic change of syllabus is required. The needs of the user of mathematics and of the potential mathematics-specialist are not incompatible but go hand in hand. It is in the 'modern' linear algebra that we look for the unification. An investigation has to be made now of the way in which linearity can provide a linking theme across the sixth-form curriculum.

There is need, however, for drastic de-formalisation of teaching style at sixth-form level. Now, more than ever before, the teacher does have the opportunity to present realistically the 3-dimensional situation to the pupils. A team is needed who is interested in 3-dimensional mathematics and the computer for their own sake - for are they not both intrinsically of interest to every youngster eager to get to grips with the world in which he finds himself? The author believes that she has found such a team*.

* at Ampleforth College
An example of a lesson given on co-axial circles with the use of the microcomputer illustrates linearity and is described as lesson 3 in Appendix D, page D(ii). Example 4 on page 241 illustrates the use of a function graph-plotting program devised by the presenter of these lessons, Mr T M Vessey.

Consideration must be given, too, to the method of assessment of sixth-form mathematics and of results produced by this new teaching style. Course work can clearly produce the clues that universities and employers are seeking concerning the potential of the candidates. A way has to be found of taking account of the assessment made by the school of the results of investigations such as those described herein. Is there any way, the author ventures to suggest, of making use of modern technology in the form of audio-visual recordings - at least in high/low borderline cases?

The author turns now to the further application of linear algebra in geometry with particular reference to the geometrical representation of the complex in algebra. The reader is referred to an Exhibit in the form of a dissertation entitled "Euclid Transformed", submitted by Térésa Czernuszewicz as an undergraduate at the University of Hull.
CHAPTER 9

COMPLEX NUMBERS

9.1 The Algebra of $2 \times 2$ Matrices

In this chapter the author takes a closer look at the algebra of the $2 \times 2$ matrices discussed in Chapters 6 and 7. Can we treat them in a way similar to the way in which equivalence classes of ordered pairs of numbers, the integers of Chapter 2 and the rationals of Chapter 3, have been treated? Is it possible to assign to them a definition of equivalence and a structure with a binary law of operation similar to the rules for addition and/or multiplication for the integer-and rational-number systems?

(1) Equivalence

The matrices:

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} a'_1 & b'_1 \\ a'_2 & b'_2 \end{bmatrix}$$

may be regarded as matrices of the transformations:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{and} \quad A' \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad (1)$$

The transformations exist if and only if:

$$a_1b_2 - a_2b_1 \neq 0 \quad \text{and} \quad a'_1b'_2 - a'_2b'_1 \neq 0$$

The inverse transformations are given by:

$$A^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = A^{-1} A \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad A'^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = A'^{-1} A' \begin{bmatrix} x \\ y \end{bmatrix} \quad (2)$$
where $A'$ denotes the multiplicative inverse of $A$. The transformations will be considered as 'equivalent' in the sense that they represent the same linear transformation of the plane, one-to-one onto itself. Thus, equivalence means that equations (1) and (2) give the same unique values of $x'$, $y'$ in terms of $x$, $y$ and vice versa. Hence:

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2 \\
\end{bmatrix}
is in some sense 'equal' to \begin{bmatrix}a_1' & b_1' \\
a_2' & b_2' \\
\end{bmatrix}
\]

and\[
\begin{bmatrix}b_2 & -b_1 \\
-a_2 & a_1 \\
\end{bmatrix}
is in the same sense 'equal' to \begin{bmatrix}b_2' & -b_1' \\
-a_2' & a_1' \\
\end{bmatrix}
\]

It would be convenient if an equivalence relation could be found to satisfy these conditions simultaneously.

It is easy to see that:

\[a_1b_2 - a_2b_1 = a_1'b_2' - a_2'b_1'
\]

is such a relation and satisfies the required reflexive, symmetric and transitive properties of equivalence. The equivalence of $2 \times 2$ matrices is defined as:

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2 \\
\end{bmatrix} = \begin{bmatrix}a_1' & b_1' \\
a_2' & b_2' \\
\end{bmatrix} \iff a_1b_2 - a_2b_1 = a_1'b_2' - a_2'b_1'
\]

It follows that their inverses are equivalent:

\[
\begin{bmatrix}b_2 & -b_1 \\
-a_2 & a_1 \\
\end{bmatrix} = \begin{bmatrix}b_2' & -b_1' \\
-a_2' & a_1' \\
\end{bmatrix}
\]

The universal set of $2 \times 2$ matrices is partitioned into equivalence classes under the equivalence relation:

\[a_1b_2 - a_2b_1 = a_1'b_2' - a_2'b_1'
\]

The author uses the term 'quadruple' to denote a $2 \times 2$ matrix of the form \[
\begin{bmatrix}a_1 & b_1 \\
a_2 & b_2 \\
\end{bmatrix}\] such that $a_1b_2 - a_2b_1 \neq 0$. An
interpretation in terms of geometrical similarity will be
given in §9.7. But first we need a binary law of composition.

(2) Addition may be defined as:
\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix} + \begin{bmatrix}
a'_1 & b'_1 \\
a'_2 & b'_2
\end{bmatrix} = \begin{bmatrix}
a_1+a'_1 & b_1+b'_1 \\
a_2+a'_2 & b_2+b'_2
\end{bmatrix}
\]

The 'sum' quadruple does not, however, necessarily satisfy
the condition: \( a_1b_2 - a_2b_1 \neq 0 \) and the system of quadruples
is not necessarily closed under addition. It does not,
therefore, lend itself to the geometrical representation
being considered (see Chapter 7 page 233) and lies outside
the scope of this thesis. The author looks for an alternative
binary law of composition for the universal set of
quadruples.

(3) Multiplication

The linear transformations of the plane have been combined
gometrically under the law of operation 'followed-by'. This
has been interpreted as the 'product' of the corresponding
transformation matrices defined by:
\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix} \times \begin{bmatrix}
a'_1 & b'_1 \\
a'_2 & b'_2
\end{bmatrix} = \begin{bmatrix}
a_1a'_1+b_1a'_2 & a_1b'_1+b_1b'_2 \\
a_2a'_1+b_2a'_2 & a_2b'_1+b_2b'_2
\end{bmatrix}
\]

It has been established (see §6.8 of Chapter 6) that under
this rule of multiplication the system of quadruples forms a
multiplicative group. The unit quadruple \( \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} \)
is the
unique multiplicative identity and each element \( \begin{bmatrix}a_1 & b_1 \\ a_2 & b_2\end{bmatrix} \)
has a unique multiplicative inverse provided \( a_1b_2 - a_2b_1 \neq 0 \).
Multiplication thus defined is not, however, commutative.
It is interesting to note that:
\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix} \times \begin{bmatrix}
b_2 & -b_1 \\
-a_2 & a_1
\end{bmatrix} = \begin{bmatrix}
(a_1b_2 - a_2b_1) & 0 \\
0 & (a_1b_2 - a_2b_1)
\end{bmatrix}
\]
and it is convenient to define multiplication of a quadruple by a real-number scalar \( k \) as:
\[
k \begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix} = \begin{bmatrix}
ka_1 & kb_1 \\
ka_2 & kb_2
\end{bmatrix}
\]
which is compatible with the representation of the multiplicative inverse of \( \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \) as \( \frac{1}{(a_1b_2 - a_2b_1)} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix} \).

The system of quadruples is seen to be closed under scalar multiplication thus defined.

Compatibility of the multiplication rule for quadruples with the equivalence relation: \( a_1b_2 - a_2b_1 = a_1'b_2' - a_2'b_1' \) can now be established. The product of:
\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\]
which belongs to the class for which \( a_1b_2 - a_2b_1 = c \) (where \( c \) is constant)
and
\[
\begin{bmatrix}
a_1' & b_1' \\
a_2' & b_2'
\end{bmatrix}
\]
which belongs to the class for which \( a_1'b_2' - a_2'b_1' = c' \) (where \( c' \) is constant)
is:
\[
\begin{bmatrix}
a_1a_1' + b_1a_2' & a_1b_1' + b_1b_2' \\
a_2a_1' + a_2a_2' & a_2b_1' + a_2b_2'
\end{bmatrix}
\]
which belongs to the class for which:
\[
(a_1a_1' + b_1a_2')(a_2b_1' + b_2b_2') = (a_1b_1' + b_1b_2')(a_2a_1' + b_2a_2')
\]
contains all the terms, and no others, of:
\[
(a_1b_2 - a_2b_1)(a_1'b_2' - a_2'b_1')
\]
which has the constant value \( cc' \). The rule for multiplication is well-defined.

* Not to be confused with definition of scalar multiplication in row reduction of an augmented matrix (§5.3 of Chapter 5)
9.2 Extending the Number System - A Realistic Approach to Complex Numbers

A search for a solution to the equation: \( x + b = a \) in the set of natural numbers led to the extension of the natural-number line, in Chapter 2, to the integer-number line. A search for a solution to the equation: \( x \times b = a \) in the set of integers led to the extension of the integer-number line, in Chapter 3, to the rational-number line.

A search for a solution to the equation: \( x^2 = a \) in the set of rational numbers leads to the extension of the rational-number line, for positive values of \( a \), to the real-number line. The set of rationals together with the irrationals gives the set of reals. The construction of the reals and their geometrical representation on the extended number line involves the calculus which is, according to Lancelot Hogben [37], "the arithmetic of the infinite". The calculus is outside the scope of this work and the existence of the reals is assumed henceforward.

Here, the author is concerned with the search for a solution to the equation: \( x^2 = a \) for a negative value of \( a \). There is no real number \( x \) - rational or irrational, positive or negative - whose square is negative. In a very real sense, \( x \) is 'purely imaginary' to the older pupils in schools and, indeed, to the specialist mathematician in an exactly analogous way that a negative number is an imaginary concept for the younger pupils in schools.

It is customary in the school mathematics course to introduce these 'imaginary' numbers by one or other of the following methods:
1. Invent a number \( i \) such that \( i^2 = -1 \) and carry out the
normal algebraic operations to develop the arithmetic of \( i \).

2. Treat \( i \) as an operator which is seen to be equivalent to a
rotation through a positive right angle.

3. In the universal set of ordered pairs of real numbers define
rules for addition and multiplication in much the same way
as was done for the directed numbers in Chapter 2 and
for the rational numbers in Chapter 3.

The methods are developed in many school texts [27.1, 29] and the
reader is referred also to the Reports of the Mathematical
Association [38] for a critical review of the different
approaches. The author herself contributed to the work in
[38.2]. She now offers an alternative development based on
the algebra of quadruples. An investigation with a class
of trainee teachers led to some fascinating discoveries (see
Exhibits).

9.3 Quadruples - A Special Subset of \( 2 \times 2 \) Matrices

It has been seen that the universal set of quadruples does
not constitute a number system in that multiplication is
not commutative. There is, however, a very interesting
subset of the universe which is commutative under multipli-
cation.

If the product:

\[
\begin{bmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{bmatrix}
\times
\begin{bmatrix}
  a'_1 & b'_1 \\
  a'_2 & b'_2
\end{bmatrix}
= \begin{bmatrix}
  a_1 a'_1 + b_1 a'_2 & a_1 b'_1 + b_1 b'_2 \\
  a_2 a'_1 + b_2 a'_2 & a_2 b'_1 + b_2 b'_2
\end{bmatrix}
\]

is compared with the product:

\[
\begin{bmatrix}
  a'_1 & b'_1 \\
  a'_2 & b'_2
\end{bmatrix}
\times
\begin{bmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{bmatrix}
= \begin{bmatrix}
  a'_1 a_1 + b'_1 a_2 & a'_1 b_1 + b'_1 b_2 \\
  a'_2 a_1 + b'_2 a_2 & a'_2 b_1 + b'_2 b_2
\end{bmatrix}
\]

it is seen that commutativity requires that:
(i) \( b_1^2 a_2 = b_1^2 a_2 \)
(ii) \( a_1 b_1 + b_1 b_2 = a_1 b_2 + b_1 b_2 \)
(iii) \( a_2 a_1 + b_2 a_2 = a_2 a_1 + b_2 a_2 \)

Condition (i) is ensured if: \( b_1 = \pm a_2 \) and \( b_1 = \pm a_2 \), which reduces condition (ii) to condition (iii). The further restrictions: \( b_2 = a_1 \) and \( b_2 = a_1 \) then ensures condition (iii). We are then concerned with quadruples of the form:

either \[
\begin{bmatrix}
a & b \\
b & a
\end{bmatrix}
\] or \[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\]

The author continues with a discussion of the latter in which a, b are assumed to be positive or negative real numbers and which provide the greater interest from the point of view of the development of Part III of this thesis. The special subset of quadruples of type:

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\]

is not only commutative, but is also closed under the multiplication rule. The multiplicative identity is obtained by putting \( a = 1, b = 0 \), giving \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] which is a member of the subset. The equivalence classes are defined on the non-zero equivalence relation: \( a^2 + b^2 = \text{constant} \), in which a and b cannot both be zero - as required for their geometrical representation as central similarity transformations, and include as special cases the symmetries for which \( a^2 + b^2 = 1 \), of Chapter 6.

* \[
\begin{bmatrix}
a & b \\
b & a
\end{bmatrix}
\] transforms lines through the origin equally inclined to \( y = -x \) into lines through the origin equally inclined to \( y = x \). For \( a = \pm b \) object-image correspondences are the pairs of coincident lines \( y = \pm x, y = \mp x \) respectively.
The result obtained by putting \( a = 0, b = 1 \) is of particular interest. The element \[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\] is such that:

\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \times \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\]

\[= -1 \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

by the rule for multiplication by a real-number scalar in §9.1(3) on page 247. In a very real sense, \[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

is seen as a square root of the special quadruple: \[
-1 \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

and has properties more readily associated with \( i \) than can be obtained from the more familiar statement: \( i^2 = -1 \) of traditional algebra. In the search for a solution to \( x^2 = a \) where \( a \) is a negative real number, the author now investigates the extension of the real-number system to 'numbers' of type \[
\begin{bmatrix}
p & -q \\
q & p
\end{bmatrix}
\]
in which any real number \( a \) may be replaced by the element

\[
\begin{bmatrix}
a & 0 \\
0 & a
\end{bmatrix}
\]

if \( a \) is positive

\[
\begin{bmatrix}
-a & 0 \\
0 & -a
\end{bmatrix} \times \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\]

if \( a \) is negative

The integers of which \(-1\) is a member and the special subset of quadruples of which \[
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\]
is a member are two distinct mathematical systems. There is, however, a similarity of structure: an isomorphism which exists between them. It may be helpful at this stage to compare the number systems that have so far been constructed: the integers as ordered differences and the rationals as ordered ratios, with the special subset of quadruples.
9.4 Isomorphisms

The author has emphasised the importance of the recognition of pattern in mathematics and of the selection of a method of presentation of essential and useful topics to pupils which illustrate those patterns, or structures, that occur and keep on recurring in the development of mathematical ideas within the school mathematics curriculum.

In the course of the extensions of the system of natural numbers, first to the integers and then to the rationals, the teacher should take advantage of every opportunity to draw the attention of the pupils to the similarity in behaviour of the elements of the different universes under the respective operations. It may be seen that the natural numbers behave like the positive integers under addition and under multiplication in the sense that:

\[ + \]
\[
\text{(a positive integer) or (a positive integer) } \rightarrow \text{(a positive integer)}
\]

just as:

\[ + \]
\[
\text{(a natural number) or (a natural number) } \rightarrow \text{(a natural number)}
\]

There is an isomorphism between the positive integers on the one hand and the natural numbers on the other hand both under addition and under multiplication.

Again:

\[ \times \]
\[
\text{(a negative integer) } \times \text{(a negative integer) } \rightarrow \text{(a negative integer)}
\]

There is an isomorphism between the negative integers and the natural numbers under addition.

The counter example:

\[ \times \]
\[
\text{(a negative integer) } \times \text{(a negative integer) } \rightarrow \text{(a positive integer)}
\]

serves to enhance the understanding of the concept of isomorphism: no such isomorphism exists between the negative integers and the natural numbers under multiplication.
It will now be shown that in a similar sense the real numbers are 'embedded' in the quadruples. Consider first the ordered pair of real numbers \([p, 0]\) which represents the quadruple \([p, 0, 0, p]\). From the condition for equivalence we have:

\[
\begin{bmatrix}
p_1 & 0 \\
0 & p_1
\end{bmatrix} = \begin{bmatrix}
p_2 & 0 \\
0 & p_2
\end{bmatrix} \iff p_1^2 = p_2^2
\]

ie \(p_1 = \pm p_2\) or \(p_1 = -p_2\)

Hence there is a one-to-one correspondence between

the real numbers \(p\) and the quadruples \([p, 0, 0, p]\)

and also between:

the real numbers \(p\) and the quadruples \([-p, 0, 0, p]\)

The existence, or lack, of an isomorphism between these two sets of correspondences under multiplication may be illustrated by the following diagram:

![Diagram showing the correspondence between reals and quadruples](image)

Fig 9.1

Figure 9.1 (i) illustrates the fact that:

the image of the product of reals

= the product of corresponding quadruples

= the product of the images of the reals
There is an isomorphism between the real numbers $p$ and the quadruples $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ under multiplication. In particular it is to be noted that the integral rational number $1$ behaves like the quadruple $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ under multiplication.

Contrariwise, Fig 9.1(ii) illustrates the fact that there is no such isomorphism between the real numbers $p$ and the quadruples $\begin{bmatrix} -p & 0 \\ 0 & -p \end{bmatrix}$ under multiplication:

the image of the product of reals $p_1, p_2$

\[ \begin{bmatrix} -p_1p_2 & 0 \\ 0 & -p_1p_2 \end{bmatrix} \]

# the product of the images of $p_1, p_2$

It may be demonstrated similarly that no isomorphism exists under multiplication between the real numbers $q$ and the quadruples $\begin{bmatrix} 0 & -q \\ q & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}$.

We say that the reals are 'embedded' in the special subset of quadruples $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ and 'behave like' the subset $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ under multiplication.

A one-to-one correspondence will now be established between the special subset of quadruples $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ and the ordered pair of real numbers $p, q$ regarded as the position vector $\begin{bmatrix} p \\ q \end{bmatrix}$ of the point $(p, q)$ in the plane.

Graphical representation was helpful in the development of the integer- and rational-number systems. Each system could be specified completely by ordered pairs of numbers - natural
numbers in the case of the integers as classes of ordered differences \((p - q)\) and integers in the case of the rationals as classes of ordered ratios \(p:q\) - together with the definitions of equivalence, addition and multiplication. Each step in the development was represented geometrically in the 2-dimensional Euclidean plane. The author now considers the possibility of a geometrical representation of the system of imaginary numbers formed from the special subset of quadruples.

9.5 Graphical Representation of the Quadruples

Although two numbers were necessary to specify an integer or a rational number, it was nevertheless possible to represent them graphically as points on a number line. The integers were represented as sets of collinear points which were plotted against a pair of natural-number, half-line, axes. This led to an extension of one of those half-lines consisting of points equally spaced on either side of the zero represented by the origin. These points were labelled, or re-labelled, with the new integer-number symbols.

The integer-number line, when extended to the rational-number line, merely became 'more dense' of points which could be inserted between, or which replaced, those representing the integers. The points were labelled, or re-labelled, with the new rational-number symbols.

The rational-number line is again 'densified' still further when extended, with the help of the calculus, to include the irrationals to form the real-number line.

The special subset \[
\begin{bmatrix} p & -q \\ q & p \end{bmatrix}
\] of quadruples forms a universal set of 'numbers' that may be specified by an ordered pair of real numbers, \(p\) and \(q\). For reasons that will appear later, however, we use the column vector \[
\begin{bmatrix} p \\ q \end{bmatrix}
\], rather than the symbol \((p, q)\) to denote this ordered number-pair specification.
For a graphical representation we need a pair of real-number, whole-line, axes. These are normally, but need not be, taken at right angles. The quadruple \( \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \) may be represented by a point with coordinates \((p, q)\) in a 2-dimensional plane. More appropriately, it may be represented by a point with position vector \( \begin{pmatrix} p \\ q \end{pmatrix} \) in that plane. The real-number line is thus extended to the imaginary (special quadruple)-number plane. A rectangular cartesian framework is, of course, the foundation for the Argand diagram of traditional mathematics.

The scene is again set and the number game proceeds with an attempt to find binary laws of addition and multiplication for the special quadruple-number system compatible with the equivalence relation \( p^2 + q^2 = \text{constant} \). The reader is first reminded of examples of the rule for addition of 2-dimensional vectors.

9.6 The Linear Algebra of a 2-Dimensional Vector Space

In §2.9 of Chapter 2 the position vectors representing members of equivalence classes of ordered differences of natural numbers were combined by the traditional parallelogram law of addition to give a geometrical representation of the addition of integers. In §5.16 of Chapter 5 position vectors representing arithmetic progressions as elements in a 2-dimensional vector space were combined by the parallelogram law of addition to give a geometrical representation of the binary law of 'term-by-term addition' of arithmetic progressions. The operation of 'term-by-term multiplication by a real-number scalar' may now, in the language of the central similarity of Chapter 7, be seen geometrically as an enlargement or ensmallment from a fixed centre of a position vector by a real-number scalar coefficient of dilatation.
There are many other examples in the useful applications of mathematics to illustrate for pupils these important concepts of linear algebra. The binary law of addition of 2-dimensional vectors may now be generalised as follows.

Addition of vectors \( \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} \), \( \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} \) is defined as \( \begin{bmatrix} p \\ q \end{bmatrix} \) such that:

\[
\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} + \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_1 + p_2 \\ q_1 + q_2 \end{bmatrix}
\]

The system is 'obviously' closed under addition. Commutativity and associativity follow from the corresponding properties of the real numbers under addition. A unique additive identity exists: \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) is the zero position vector represented graphically by the origin. Every vector \( \begin{bmatrix} p \\ q \end{bmatrix} \) has a unique additive inverse \( \begin{bmatrix} -p \\ -q \end{bmatrix} \). Subtraction is defined as inverse addition. Vector addition thus defined may be represented geometrically by the parallelogram law of addition of 'traditional' vectors.

The operation of multiplication of a vector \( \begin{bmatrix} p \\ q \end{bmatrix} \) by a real-number scalar \( k \) is defined as:

\[
k \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} kp \\ kq \end{bmatrix}
\]

under which the system is closed. The operation may be represented geometrically by a central similarity (see §7.4 of Chapter 7): if \( \overrightarrow{OP_1} \) and \( \overrightarrow{OP_2} \) are the line segments through the origin representing \( \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} \), \( \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} \) then:

\[
\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = k \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} \Rightarrow \overrightarrow{OP_2} = k \overrightarrow{OP_1}
\]

with \( O, P_1, P_2 \) collinear points.
It is left as an exercise for the reader to show that the system of 2-dimensional vectors \[
\begin{bmatrix}
p \\ q
\end{bmatrix}
\] under addition and real-number scalar multiplication satisfies the conditions in §5.11 of Chapter 5 of a vector space.

9.7 An Algebra of the Special Subset of Quadruples

An isomorphism exists between the quadruples \[
\begin{bmatrix}
p \\ -q
\end{bmatrix}
\] and the 2-dimensional vector elements \[
\begin{bmatrix}
p \\ q
\end{bmatrix}
\] under the respective rules of addition and scalar multiplication. Using the symbol '++' to denote 'corresponds to', we have:

<table>
<thead>
<tr>
<th>basis vectors</th>
<th>quadruples</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & -1 \\
1 & 0
\end{bmatrix}
\] |

so that:

\[
\begin{bmatrix}
p \\ q
\end{bmatrix} = p. \[
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
\] + q. \[
\begin{bmatrix}
0 \\
1 \\
1 \\
0
\end{bmatrix}
\] ++ \[
\begin{bmatrix}
p \\ -q
\end{bmatrix} = p. \[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & -1 \\
1 & 0
\end{bmatrix}
\] + q. \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

It is easy to show that:

the sum of the images of \[
\begin{bmatrix}
p_1 \\ q_1
\end{bmatrix}
\] and \[
\begin{bmatrix}
p_2 \\ q_2
\end{bmatrix}
\]
corresponds to:

the image of the sum of \[
\begin{bmatrix}
p_1 \\ q_1
\end{bmatrix}
\] and \[
\begin{bmatrix}
p_2 \\ q_2
\end{bmatrix}
\]

The two systems are isomorphic under addition.

Similarly the two systems are isomorphic under scalar multiplication. We have:
A rule for the multiplication of vectors \[
\begin{bmatrix}
p \\
q
\end{bmatrix}
\] is defined more easily from geometrical than from algebraic considerations. The attention of the reader is now directed towards the very close links that exist between the special subset of quadruples and the geometrical symmetries and similarities of Chapters 6 and 7.

A comparison of the quadruples \[
\begin{bmatrix}
p & -q \\
q & p
\end{bmatrix}
\] with the results of §7.9 of Chapter 7 suggest the following correspondences:
<table>
<thead>
<tr>
<th>Geometrical Representation</th>
<th>Transformation Matrix</th>
<th>Special set of quadruples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetry or Similarity</td>
<td>( a^2 + b^2 \neq 0 )</td>
<td>( p^2 + q^2 \neq 0 )</td>
</tr>
</tbody>
</table>
|                           | \[
\begin{bmatrix}
a & b \\
b & a
\end{bmatrix}
\]          | \[
\begin{bmatrix}
p & q \\
q & p
\end{bmatrix}
\]          |
| (i) Symmetry               | \( a^2 + b^2 = 1 \)     | \( p^2 + q^2 = 1 \)        |
|                           | \( a^2 + b^2 = 2a - 1 \) | \( p^2 + q^2 = 2p - 1 \)   |
| Identity                  | \[
\begin{bmatrix}
a = 1 & 0 \\
b = 0 & 1
\end{bmatrix}
\]          | \[
\begin{bmatrix}
p = 1 & 0 \\
q = 0 & 1
\end{bmatrix}
\]          |
| (b) Half-turn              | \( a^2 + b^2 \neq 2a - 1 \) | \( p^2 + q^2 \neq 2p - 1 \) |
|                           | \[
\begin{bmatrix}
a = -1 & 0 \\
b = 0 & -1
\end{bmatrix}
\]          | \[
\begin{bmatrix}
p = -1 & 0 \\
q = 0 & -1
\end{bmatrix}
\]          |
| Rotation                  | \[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\]          | \[
\begin{bmatrix}
p & q \\
q & p
\end{bmatrix}
\]          |
| Reflection                | \[
\begin{bmatrix}
a & b \\
b & -a
\end{bmatrix}
\]          | \[
\begin{bmatrix}
? & ? \\
? & ?
\end{bmatrix}
\]          |
| (ii) Similarity           | \( a^2 + b^2 \geq 1 \)   | \( p^2 + q^2 \geq 2 \)    |
|                           | \( a^2 + b^2 \neq 2a - 1 \) | \( p^2 + q^2 \neq 2p - 1 \) |
| Central similarity        | \[
\begin{bmatrix}
\lambda & 0 \\
0 & \lambda
\end{bmatrix}
\]          | \[
\begin{bmatrix}
p & 0 \\
0 & p
\end{bmatrix}
\]          |
| General similarity        | \[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\]          | \[
\begin{bmatrix}
p & -q \\
q & p
\end{bmatrix}
\]          |

* See Footnote page 251
There appears to be an exact correspondence between the similarity transformations and the quadruples in all cases except reflection. These isomorphisms will be established in §9.8 on page 264. But first the author justifies the rule for multiplication of quadruples (see page 247) in terms of the general similarity transformation.

The general similarity defined on page 232 of Chapter 7 as the commutative product of a rotation and a central similarity is such that, under the transformation:

- corresponding angles are equal
- corresponding sides are proportional

in the object and image figures.

Let $\mathbf{OP}_1$, $\mathbf{OP}_2$ represent the position vectors $\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}$, $\begin{bmatrix} p_2 \\ q_2 \end{bmatrix}$ of points $P_1$, $P_2$ relative to a fixed origin $O$ in the 2-dimensional plane. The position vector $\begin{bmatrix} p \\ q \end{bmatrix}$ of the product of $\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}$ and $\begin{bmatrix} p_2 \\ q_2 \end{bmatrix}$ may be represented by $\mathbf{OP}$ constructed as follows:

With the notation of Fig 9.2, $\mathbf{OA}$ is the unit vector in the direction of the p-axis.

![Fig 9.2](image-url)
\[ R = \text{Rotation} \left(0, \theta_2\right): \ AOP_1 \rightarrow A'OP_1' \]
\[ S_c = \text{Central Similarity} \left(0, \frac{r_2}{r_1}\right): \ A'OP_1' \rightarrow P_2OP \]

Hence \( S_G = R \circ S_c \circ AOP_2 \rightarrow P_2OP \)

It follows that:
\[ \frac{\overrightarrow{OP}_1}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OP}_2} \]

ie \( \overrightarrow{OP}_1 \cdot \overrightarrow{OP}_2 = \overrightarrow{OP} \cdot \overrightarrow{OA} \)

Since \( \overrightarrow{OA} \) is a unit vector, \( \overrightarrow{OP} \) may be regarded as a geometrical representation in magnitude and direction of the 'product'\(^*\) of position vectors \( \overrightarrow{OP}_1 \) and \( \overrightarrow{OP}_2 \). We have:

- \( r = \text{magnitude of } \overrightarrow{OP} = \text{magnitude } \overrightarrow{OP}_1 \cdot \text{magnitude } \overrightarrow{OP}_2 = r_1 \cdot r_2 \)
- \( \theta = \text{direction of } \overrightarrow{OP} = \text{direction } \overrightarrow{OP}_1 + \text{direction } \overrightarrow{OP}_2 = \theta_1 + \theta_2 \)

If then the position vectors \( \begin{bmatrix} p_1 \cr q_1 \end{bmatrix}, \begin{bmatrix} p_2 \cr q_2 \end{bmatrix} \) are denoted by:

\[
\begin{bmatrix}
    r_1 \cos \theta_1 \\
    r_1 \sin \theta_1
\end{bmatrix},
\begin{bmatrix}
    r_2 \cos \theta_2 \\
    r_2 \sin \theta_2
\end{bmatrix}
\]

respectively, the product

\[
\begin{bmatrix}
    r \cos \theta \\
    r \sin \theta
\end{bmatrix}
\]

is given by:

\[
\begin{bmatrix}
    r_1 \cos \theta_1 \\
    r_1 \sin \theta_1
\end{bmatrix} \times \begin{bmatrix}
    r_2 \cos \theta_2 \\
    r_2 \sin \theta_2
\end{bmatrix} = \begin{bmatrix}
    r_1 r_2 \cos(\theta_1 + \theta_2) \\
    r_1 r_2 \sin(\theta_1 + \theta_2)
\end{bmatrix}
\]

\[ = r_1 r_2 \begin{bmatrix}
    \cos(\theta_1 + \theta_2) \\
    \sin(\theta_1 + \theta_2)
\end{bmatrix} \]

The general similarity transformation which prompted a geometrical representation in the real plane of a rule for the multiplication of 2-dimensional vectors corresponds exactly to the geometrical representation in the imaginary plane of the rule

\* The product thus defined on position vectors must not be confused with other products (inner product, cross-product) of vectors in general (see also Chapter 2, page 46)
Vector

\[
\begin{bmatrix}
  r_1 \cos \theta_1 \\
r_1 \sin \theta_1
\end{bmatrix},
\begin{bmatrix}
  r_2 \cos \theta_2 \\
r_2 \sin \theta_2
\end{bmatrix}
\]

Product

\[
 r_1 r_2 
\begin{bmatrix}
  \cos(\theta_1 + \theta_2) \\
  \sin(\theta_1 + \theta_2)
\end{bmatrix}
\]

Quadruple

\[
\begin{bmatrix}
  r_1 \cos \theta_1 - r_1 \sin \theta_1 \\
r_1 \sin \theta_1 + r_1 \cos \theta_1
\end{bmatrix},
\begin{bmatrix}
  r_2 \cos \theta_2 - r_2 \sin \theta_2 \\
r_2 \sin \theta_2 + r_2 \cos \theta_2
\end{bmatrix}
\]

Product

\[
 r_1 r_2 
\begin{bmatrix}
  (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\
  \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2
\end{bmatrix},
\begin{bmatrix}
  (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\
  \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2
\end{bmatrix}
\]

Fig 9.3
for the multiplication of quadruples $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$. Algebraically we have the correspondence as shown in Fig 9.3.

Incidentally the isomorphism of the position vectors in the real plane and the quadruples in the imaginary plane under their respective rules for multiplication establishes the well-known addition formulae for the sine and cosine of an angle-sum.

9.8 Real-Imaginary Links

The real-number system and its representative line has been extended to an 'imaginary-number' system and its representative plane. The latter may be specified completely by the special subset of quadruples. The author leaves it as an exercise for the reader to establish the exact correspondence between the system of quadruples $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ and the system of complex numbers $(p + iq)$ of classical algebra. As far as is known, S L Parsonson [29] is the only author of a recent school text who recognises the potential of this work for sixth-form pupils. He writes:

"The one-to-one correspondence between the set $C$ of complex numbers on the one hand and the set of $2 \times 2$ matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $(a, b, \in R)$ on the other is thus seen to preserve the structure of addition and multiplication."

In the author's view, it is a pity that the analogy ends there.

The perceptive reader will inevitably appreciate the exact similarity of:
(i) the special quadruple \[
\begin{bmatrix}
p & -q \\
q & p \\
\end{bmatrix}
\] (\(p^2 + q^2 = 1\)) to the rotation matrix \[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{bmatrix}
\] in the real plane.

(ii) the special quadruple \[
\begin{bmatrix}
p & 0 \\
0 & p \\
\end{bmatrix}
\] to the central similarity matrix \[
\begin{bmatrix}
\lambda & 0 \\
0 & \lambda \\
\end{bmatrix}
\] in the real plane.

In the next Chapter the work is extended to a consideration of the correspondence between the special quadruples, not as numbers, but as transformation matrices in the newly-constructed 'imaginary plane' and the transformation matrices of the real plane. The theory is covered comprehensively by the late F J Budden [39] in a delightful publication which the author considers to be difficult reading for all but the ablest of specialist mathematics teachers - a veritable mine of information and good hunting ground for investigative material for the sixth-form pupils. The author attempts an explanation of material relevant to the upward spiral development of the mathematics curriculum to sixth-form level and to the geometrical representation of the imaginary of algebra.
10.1 The Cross-Ratio of a Harmonic Range

The reader is referred to the results of the Exercise on pp 226, 227 of Chapter 7. It was established (see Fig A3(ii)) on page A(iii) of Appendix A that HNGO, the Euler Line of triangle ABC

![Diagram](image)

Fig 10.1

is such that: \( \frac{HN}{NG} = 3 \) and \( \frac{HO}{OG} = \frac{-6}{2} \Rightarrow \frac{HN}{NG} = \frac{-HO}{OG} \). We say that: \( N, O \) harmonically separates \( H, G \), and that \( H, N, G, O \) is a harmonic range.

It follows that: \( \frac{HN.GO}{HO.GN} = -1 \).

The expression on the left-hand side is called the cross-ratio of the range of points \( H, N, G, O \) and will be denoted by \( \{H,G; N,O\} \).

The reader is referred to [38.3] for an excellent account of the many and varied examples of the harmonic range, well within the scope of the average sixth-form pupil, in terms of the symmetry and similarity transformations discussed in this thesis. For example (§6.2 op cit):
(a) the internal and external bisectors of the vertical angle of a triangle divide the base internally and externally in the same ratio. The converse of this result gives the Apollonian circle locus

(b) orthogonal circles

(c) coaxal circles

(d) focus, directrix properties of conics

(e) the theorems of Ceva and Menelaus

Again (§6.4 op cit):

(f) Simpson's Line is linked to the result that the circum-circle of the triangle formed by three tangents to a parabola passes through the focus

(g) Ptolemy's theorem concerning four points on a circle is connected historically with the trigonometric addition theorems. Its extension supplies an admirable illustration of inversion which the author takes up in §10.8 on page 305

Further results (§6.5 op cit) include:

(h) the pole and polar properties of the circle which may be taken before or after those of the conics

and (§7.74 op cit):

(i) the replacement of ratio in cartesian coordinate geometry by cross-ratio in projective geometry.

The author uses the notation of cross-ratio to motivate interest in 3-dimensional geometry. But first she needs to establish the existence of just 6 distinct cross-ratio values corresponding to the 24 different arrangements of 4 points on a line.
10.2 The 'Group' of Cross-Ratios

If the points $P_1, P_2, P_3, P_4$ are referred to a collinear fixed origin $O$ by coordinates $x_1, x_2, x_3, x_4$ respectively (Fig 10.2).

![Fig 10.2](image)

then the cross-ratio $\{P_1, P_3; P_2, P_4\}$ may be represented by:

$$\frac{P_1P_3}{P_1P_4} \cdot \frac{P_3P_4}{P_3P_2} = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_1)(x_2 - x_3)} = \lambda \text{ (say)}$$

It is to be noted that the value of $\lambda$:

(a) is unchanged by the interchange of 2 pairs of suffices:

$$\{P_1, P_3; P_2, P_4\} = \{P_2, P_4; P_1, P_3\} = \{P_3, P_1; P_4, P_2\} = \{P_4, P_2; P_3, P_1\}$$

$$\lambda = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_1)(x_2 - x_3)} = \frac{(x_2 - x_1)(x_3 - x_4)}{(x_3 - x_2)(x_1 - x_4)} = \frac{(x_3 - x_4)(x_1 - x_2)}{(x_2 - x_3)(x_4 - x_1)} = \frac{(x_4 - x_3)(x_2 - x_1)}{(x_1 - x_4)(x_3 - x_2)}$$

(b) is changed to $1/\lambda$ by an interchange of either suffices 1, 3 or suffices 2, 4:

$$\{P_3, P_1; P_2, P_4\} = \{P_1, P_3; P_4, P_2\}$$

$$\frac{1}{\lambda} = \frac{(x_2 - x_3)(x_4 - x_1)}{(x_4 - x_3)(x_2 - x_1)} = \frac{(x_4 - x_3)(x_2 - x_1)}{(x_2 - x_1)(x_4 - x_3)}$$

and by repeating (a), ie the interchange of the 2 pairs of suffices 1, 2 and 3, 4 in each case, we have:

$$\{P_4, P_2; P_1, P_3\} = \{P_2, P_4; P_3, P_1\}$$

$$\frac{1}{\lambda} = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_3 - x_4)(x_1 - x_2)} = \frac{(x_3 - x_4)(x_1 - x_2)}{(x_1 - x_2)(x_3 - x_4)}$$
(c) is changed to \((1 - \lambda)\) by an interchange of either suffices 2, 3 or suffices 1, 4 giving

\[
\{P_1, P_2; P_3, P_4\} \text{ and } \{P_4, P_3; P_2, P_1\}
\]

which is:

\[
\frac{(x_3-x_1)(x_4-x_2)}{(x_4-x_1)(x_3-x_2)} \text{ and } \frac{(x_2-x_4)(x_1-x_3)}{(x_1-x_4)(x_2-x_3)}
\]

which are seen to be equal to each other and to \((1 - \lambda)\).

Again by repeating (a), i.e., the interchange of the 2 pairs of suffices 1, 2 and 3, 4 in each case, we have:

\[
\{P_1, P_2; P_3, P_4\} = \{P_2, P_1; P_4, P_3\} = \{P_4, P_3; P_2, P_1\} = \{P_3, P_4; P_1, P_2\}
\]

\[
(1-\lambda) = \frac{(x_3-x_1)(x_4-x_2)}{(x_4-x_1)(x_3-x_2)} = \frac{(x_4-x_2)(x_3-x_1)}{(x_3-x_2)(x_4-x_1)} = \frac{(x_2-x_4)(x_1-x_3)}{(x_1-x_4)(x_2-x_3)} = \frac{(x_1-x_3)(x_2-x_4)}{(x_2-x_3)(x_1-x_4)}
\]

An alternative derivation of this result is given by C O Tuckey and F J Swan [40].

(d) Repetition of the 2 operations: reciprocation and subtraction from unity results in the set of values:

\[
\lambda; 1/\lambda; (1-\lambda); 1/(1-\lambda); \lambda/(\lambda-1); (\lambda-1)/\lambda
\]

each of which occurs for 4 different arrangements of the 4 points \(P_1, P_2, P_3, P_4\) on a line. The set is closed and forms the structure of a group.

The results of this section lend themselves readily to the development of 3-dimensional geometry and the symmetries of the Platonic solids. The author has chosen to give an analysis of the symmetries of the regular tetrahedron. The work that follows has been attempted on many occasions in activity sessions with classes of sixth-form pupils at the invitation of schools in different parts of the country. The work described was attempted by the pupils with the help of Orbit Material [S9] in the form of plastic straws and plastic polyhedral joints.
More recently the author has obtained the assistance of Mr Andrew Hollins, a member of the Mathematics Department at Ampleforth College, in producing a programme for animated illustrations on the microcomputer (see Printout 10.1 on pages 281-3).

10.3 The Isometries in 3-Dimensional Space

In Chapter 4 on the isometries in the 2-dimensional plane it was seen that:

(a) a symmetry is an exact correspondence about a point or a line

(b) an isometry, or symmetry transformation, is a movement which transforms a geometrical object on to itself

The author discussed the symmetries in 2 dimensions of rotation about a point and reflection about a line

It was seen in Chapter 6 on page 204 that the symmetries of the equilateral triangle form a group under the operation 'followed-by'. Two sub-groups of the system may be identified:

the direct rotational symmetries
and the opposite reflectional symmetries

The system was seen also to be isomorphic to the 'group' of permutations of the vertices. The discussion was preceded on page 201 by a suggested investigation for the extension of these ideas to the set of regular polygons.

In this chapter the author extends the ideas to the 3-dimensional geometrical objects which are the five regular Platonic solíás. What is meant by a regular solid? A regular solid will be taken to be a 3-dimensional object bounded by plane faces which are regular polygons and in which:

* modified with the assistance of Drs G A Steigmann and C P Waddington of the University of Hull (see Exhibits) [S8]
faces are all alike
edges are all alike
vertices are all alike

Notice that the word 'side' is avoided: 'side' could be interpreted either as 'face' or as 'edge' and confusion could easily be caused in the minds of pupils at school.

What is meant by 'all alike'? Is it possible to find a movement, a symmetry transformation of a 3-dimensional space object which moves faces to faces, edges to edges, vertices to vertices?

10.4 The Regular Tetrahedron

The regular tetrahedron which consists of 4 faces which are equilateral triangles, has 6 edges and 4 vertices. The investigation of 2 dimensions is now repeated in 3 dimensions.

In how many ways can the regular tetrahedron be moved so as to fit on to itself? The number of symmetries thus to be determined is equivalent to finding the number of ways in which the vertices may be rearranged amongst themselves. The number of such arrangements, or permutations, is:

\[ 4! = 24 \]

The isometries which produce them again fall into two classes:
- the direct rotational isometries
- the opposite reflectional isometries

Can the positions resulting from these two sets of transformations be identified in terms of the initial position of the tetrahedron? If the vertices of the tetrahedron are numbered 1, 2, 3, 4, it may be assumed that, initially, the tetrahedron fits into a hole in space labelled A, B, C, D.
The correspondence between the vertices of the tetrahedron and the positions in space which they occupy will be indicated thus:

\[ 1 \ 2 \ 3 \ 4 \ \rightarrow \ A \ B \ C \ D \]

(1) The direct rotational symmetries of the regular tetrahedron may be identified geometrically as follows:

(a) Rotations about the altitudes, through the vertices, fixed* in the tetrahedron. If \( R_1, S_1, T_1 \) denote rotations about vertex 1 of 120°, 240°, 360° respectively, then:

\[ \begin{align*}
1 \ 2 \ 3 \ 4 & \rightarrow 1 \ 4 \ 2 \ 3 \ \rightarrow \ A \ B \ C \ D \\
1 \ 2 \ 3 \ 4 & \rightarrow 1 \ 3 \ 4 \ 2 \ \rightarrow \ A \ B \ C \ D \\
1 \ 2 \ 3 \ 4 & \rightarrow 1 \ 2 \ 3 \ 4 \ \rightarrow \ A \ B \ C \ D \\
\end{align*} \]

Similarly, rotations: \( R_2, S_2, T_2 \) about vertex 2

\[ \begin{align*}
R_3, S_3, T_3 \text{ about vertex 3} \\
R_4, S_4, T_4 \text{ about vertex 4} \\
\end{align*} \]

are defined.

It is to be noted that the results of rotations \( T_1, T_2, T_3, T_4 \) of 360° about vertices 1, 2, 3, 4 respectively bring the tetrahedron back to its initial 'stay-put' position. We have:

\[ T_1 = T_2 = T_3 = T_4 = \text{the Identity transformation} \]

There are, in fact, 9 distinct rotations about the vertices. (see Fig 10.3).

* A similar argument holds if the altitudes are fixed in space but the symbolic expression of the results will be different. For example, under \( R_1 \): 1234 in position ABCD moves to 1423 in position ABCD. ABCD filled by 1234 becomes ACDB filled by 1234
Fig 10.3
(b) Rotations about axes joining mid-points of pairs of opposite* edges fixed in the tetrahedron: if X, Y, Z respectively denote rotations of 180° about the axes:
Ox joining the mid-points U, U' of 1-2, 3-4
Oy joining the mid-points V, V' of 1-3, 2-4
Oz joining the mid-points W, W' of 1-4, 2-3
then:
\[
\begin{align*}
1 & 2 & 3 & 4 & \rightarrow & 2 & 1 & 4 & 3 \\
1 & 2 & 3 & 4 & \rightarrow & 3 & 4 & 1 & 2 \\
1 & 2 & 3 & 4 & \rightarrow & 4 & 3 & 2 & 1 \\
\end{align*}
\]
There are 3 such half-turns giving a total of 12 distinct rotational symmetries (see Fig 10.3).

(2) For the opposite reflectional symmetries in 3 dimensions a mirror plane of symmetry is required. It may readily be seen that a plane which passes through 2 vertices of the tetrahedron and the mid-point of the opposite edge is such a plane of symmetry. How many such planes are there? It is to be noted that, once the 2 vertices have been selected, the mid-point of the opposite edge is automatically determined. The number of ways in which 2 vertices may be selected from 4 is: \( C_2 = 6 \). There are correspondingly 6 such planes of reflectional symmetry. The corresponding reflections denoted by: \( M_1, M_2, M_3, M_4, M_5, M_6 \) are defined thus:

\[
\begin{align*}
M_1 & : 1 2 3 4 \rightarrow 1 2 4 3 \quad \text{in plane through vertices 1,2} \\
M_2 & : 1 2 3 4 \rightarrow 1 4 3 2 \quad \text{in plane through vertices 1,3} \\
M_3 & : 1 2 3 4 \rightarrow 1 3 2 4 \quad \text{in plane through vertices 1,4} \\
M_4 & : 1 2 3 4 \rightarrow 4 2 3 1 \quad \text{in plane through vertices 2,3} \\
M_5 & : 1 2 3 4 \rightarrow 3 2 1 4 \quad \text{in plane through vertices 2,4} \\
M_6 & : 1 2 3 4 \rightarrow 2 1 3 4 \quad \text{in plane through vertices 3,4} \\
\end{align*}
\]

* A similar argument holds if the axes are fixed in space but the symbolic expression of the results will be different.
What can be said about the remaining 6 opposite symmetries of the regular tetrahedron? No other planes of reflection can be found. In an early edition [4.5, 4.6] of a School Mathematics Project text the pupil is invited to combine each rotation with just 1 of the reflections defined above to produce the complete set of 12 opposite symmetries. The author dislikes the sudden switch to algebra in the search for the missing symmetries and suggests the following geometrical investigation by asking the question "When is a reflection not a reflection?" as illustrated by the (original) entrance to King's College London before its rebuilding in the 1970's.

(3) At this stage, pupils involved in the investigation have been encouraged to arrange the 24 permutations of the vertices into 6 cyclic sets of 4 permutations and to identify the 12 rotations and the 6 reflections so far determined (Table 10.1).
Table 10.1

| I : 1 2 3 4 | R₁ : 1 4 2 3 | S₁ : 1 3 4 2 |
| 4 1 2 3 | 3 1 4 2 | M₆ : 2 1 3 4 |
| Y : 3 4 1 2 | R₄ : 2 3 1 4 | S₂ : 4 2 1 3 |
| 2 3 4 1 | M₄ : 4 2 3 1 | 3 4 2 1 |
| Z : 4 3 2 1 | R₂ : 3 2 4 1 | S₃ : 2 4 3 1 |
| M₂ : 1 4 3 2 | M₃ : 1 3 2 4 | M₁ : 1 2 4 3 |
| X : 2 1 4 3 | R₃ : 4 1 3 2 | S₄ : 3 1 2 4 |
| M₅ : 3 2 1 4 | 2 4 1 3 | 4 3 1 2 |

The rotations are then linked in pairs, within each cyclic set of 4 permutations, each resulting from its partner by the interchange of 2 pairs of vertices. The corresponding movements are the symmetry transformations which are half-turns about the axes Ox, Oy, Oz fixed in the tetrahedron. The reader should note that the half-turns are from non-identity positions and are not necessarily the half-turns X, Y, Z defined in (1(b)) on page 275. They are denoted by Hₓ, Hᵧ, Hₚ, the suffices x, y, z denoting the axes Ox, Oy, Oz respectively of the half-turn rotations (Table 10.2).

The pupils are now encouraged to search for an intervening, geometrically recognisable, link within each cyclic set of 4 permutations. Is it possible to move one step forwards, or one step backwards, cyclically by (say) a quarter-turn about some identifiable axis? The vertices do move on one step cyclically in each set and a movement of two cyclic steps is represented by a half-turn. A word of warning is
Table 10.2

promoted! It must not be forgotten that we are looking for opposite symmetries and reference to a model (see Exhibits) indicates that we have a twisted movement from one position to the next which the author chooses to call 'the googly twist'. But the tetrahedron is 'turned inside-out'!

(4) It will be seen (Table 10.2) that:

a half-turn about OY: \[ 1\ 2\ 3\ 4 \rightarrow H_y \rightarrow 3\ 4\ 1\ 2 \]

What, then, does a quarter-turn about OY do to the tetrahedron? Under the quarter-turn \( Q_y \) about Oy:

\[ 1\ 2\ 3\ 4 \rightarrow Q_y \rightarrow 4'\ 1'\ 2'\ 3' \]

where the primed numbers represent the reflections of the vertices of the regular tetrahedron in the centre O.
The specialist mathematics teacher will recognise that the intervening position 4 1 2 3 is thus obtained by following the quarter-turn by the \textit{Central Inversion} which may be described as a turning of the tetrahedron inside out. Denoting the central inversion by $\Omega$ we have:

\[ *\Omega Q_y : 1 \ 2 \ 3 \ 4 \rightarrow \Omega : 4' \ 1' \ 2' \ 3' \rightarrow 4 \ 1 \ 2 \ 3 \]

It may now be verified that in each cyclic set of 4 permutations all the intervening steps may be identified as the commutative product of a quarter-turn about one or other of the axes $O_x$, $O_y$, $O_z$, and Central Inversion $\Omega$ (Table 10.3).

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
I: & $\Omega'Q_y$ & $\Omega'Q_x$ \\
\hline
& $\rightarrow 4' \ 1' \ 2' \ 3'$ & $\rightarrow 3 \ 1 \ 4 \ 2$ \\
\hline
Y: & $\Omega'Q_y$ & $\Omega'Q_y$ \\
\hline
& $\rightarrow 3 \ 4 \ 1 \ 2$ & $\rightarrow 4 \ 2 \ 3 \ 1$ \\
\hline
\end{tabular}
\caption{Table 10.3}
\end{table}

* The author has adopted the notation $\Omega Q_y$ to denote $Q_y$ 'followed-by' $\Omega$
10.5 The Platonic Solids

It has always been a problem for the mathematics teacher to motivate in all but the ablest of pupils an interest in geometry. Tiny tots live in a make-believe 3-dimensional world and are unable to grasp the significance of a 2-dimensional flat image in the form of a picture or a photograph. Older pupils in the secondary school, however, find it difficult to grasp the abstract 3-dimensional relationships: this is largely due to a dearth of suitable 3-dimensional models.

If the author is permitted to gaze into the crystal ball, the very modest attempt at the animated 2-dimensional representation of the regular tetrahedron as a microcomputer program (see Printout 10.1 on pages 281-3) prompts her to believe that her current programme in initial teacher training courses is already anticipating the 'motivation' of 3-dimensional geometry not only for its intrinsic aesthetic value to the more able pupils but also for the engineering and manufacturing applications demanded by the users of mathematics in industry.

Is it too much to hope for the further investigation of 3-dimensional space into the symmetry groups of the remaining 4 platonic solids: the cube and the octahedron; the dodecahedron and the icosahedron? (see (5) on page 293 and Exhibits).

Experience of thinking in three dimensions is often best given to pupils in the form of a game or puzzle. The following problems arising out of 3-dimensional noughts-and-crosses, or tick-tack-toe as it is sometimes called, has been used by the author on many occasions with pupils at all age/ability levels and, often, with experienced teachers. Pencil and paper are not allowed.
1*KEY100. !MRUN:M
5REM GAS
ODIM(20),V(20),Z(20),X2D(20),Y2D(20),st%(20),end%(20),li%(3)
20N ERROR IF ERR=17 GOTO700:ELSE GOTO10000
ODATA5,-500,300,0,500,300,0,0,-600,0,0,0,600,0,0,100
ODATA3,6,1,1,2,2,3,3,1,1,4,2,4,3,4,4,4,4,4,4,4,4,4,0
ODATA5,500,500,0,-500,500,0,-500,-500,0,500,-500,0,0,0,500
ODATA4,4,1,1,2,2,3,3,4,4,1,5,5,1,5,2,5,3,5,4,1,3
ODATAB,-500,500,500,500,500,500,500,500,500,500,-500,-500,500
5DATA-500,500,-500,500,500,500,500,-500,-500,-500,-500
ODATA4,8,1,1,2,2,3,3,4,4,1,5,2,6,3,7,4,8,5,6,6,7,7,8,8,8,5,1,3
ODATA5,-500,300,0,500,300,0,0,-600,0,0,0,600,0,0,0,600
ODATA3,6,1,1,2,2,3,3,1,1,4,2,4,3,4,1,5,2,5,3,5,4,5
ODATA9,-500,300,-300,500,300,-300,0,0,-600,-300,-250,150,300,250,150,300,0,-30
0,300,-250,150,600,250,150,600,0,-300,600
ODATA3,12,1,1,2,2,3,3,1,1,4,2,5,3,6,4,5,5,6,4,4,7,5,8,6,9,7,8,9,7,1,4
MODE1
2VDU23,250,8,28,42,8,8,8,8,0
3VDU23,251,8,8,8,8,42,28,8,0
4VDU23,252,0,4,2,127,2,4,0,0
5VDU23,253,0,16,32,127,32,16,0,0
6VDU23,1,0;0;0;0;
7COLOUR2:PRINTTAB(1,2)"WHICH SHAPE WOULD YOU LIKE TO DISPLAY?"
8COLOUR3:PRINT"A. CUBE"
9PRINT"B. TETRAHEDRON"
10PRINT"C. SQUARE BASED PYRAMID"
11PRINT"D. HEXAHEDRON"
12PRINT"E. 1ST EXTRA FIGURE"
13PRINT4:PRINTTAB(10,15)"MAKE YOUR CHOICE"
14PRINTTAB(1,16)"BY TAPPING THE APPROPRIATE LETTER"
15PRINT":"CONTROL KEYS":COLOUR3
16PRINT CHR$250;" and ";CHR$251;":ROTATION ABOUT HORIZONTAL AXIS"
17PRINTCHR$252;" and ";CHR$253;":ROTATION ABOUT VERTICAL AXISS"
18PRINT"*: ROTATION ABOUT AN AXIS":""PERPENDICULAR TO THE SCREEN"
19PRINT"< and > :CHANGE SIZE OF FIGURE"
20COLOUR1
21PRINT": Press ESCAPE to return to menu"
22REPEAT=GET:UNTILCH>64ANDCH<70
23IFCH=65 RESTORE 70
24IFCH=66 RESTORE 30
25IFCH=67 RESTORE 50
26IFCH=68 RESTORE 90
27IFCH=69 RESTORE 110
28MODE1
29VDU29,640;512;VDU23,1,0;0;0;0;
30*=FX4,1
31PROCass
32PROCpo
33PROCli
34dist%=5000
35dist%=500
36anst=PI/18
37REPEAT
38PROC2D
39PROCdr
40PROCup
41PROCro
42UNTIL0

1240 DEF PROC ass
1260 DIM P% 25
1270 C0T 2
1280 .clg LDA £0
1290 .dx £0
1300 LDY £&50
1310 .loop STA &3000,X
1320 INX
1330 BNE .loop
1340 INC .loop+2
1350 DEY
1360 BNE .loop
1370 LDA £&30
1380 STA .loop+2
1390 RTS
1400]
1410 ENDPROC
1430 DEF PROC po
1450 READ po7.
1470 FOR co7.=1 TO po7.
1480 READ X(co7.), Y(co7.), Z(co7.)
1490 NEXT co7.
1500 ENDPROC
1520 DEF PROC li
1540 READ li7.(1), li7.(2), li7.(3)
1545 B7.=0
1550 FOR N=1 TO li7.(N)
1560 FOR co7.=1 TO li7.(N)
1565 B7.=B7.+1
1570 READ st7.(B7.), end7.(B7.)
1580 NEXT co7.
1585 NEXT
1590 ENDPROC
1610 DEF PROC 2D
1630 FOR co7.=1 TO po7.
1640 X2D(co7.)=X(co7.)*2500/(di7.-Z(co7.))
1650 Y2D(co7.)=Y(co7.)*2500/(di7.-Z(co7.))
1660 NEXT co7.
1670 ENDPROC
1690 DEF PROC d.
1700 CALL .clg
1710 .loop STA &3000,X
1720 INX
1730 BNE .loop
1740 INC .loop+2
1750 DEY
1760 BNE .loop
1770 LDA £&30
1780 STA .loop+2
1790 RTS
1800]
1810 ENDPROC
1830 READ po%
1850 FOR co7.=1 TO po%
1860 READ X(co7.), Y(co7.), Z(co7.)
1870 NEXT co7.
1880 ENDPROC
1890 READ li7.(1), li7.(2), li7.(3)
1900 FOR N=1 TO li7.(N)
1910 FOR co7.=1 TO li7.(N)
1915 B7.=B7.+1
1920 READ st7.(B7.), end7.(B7.)
1930 NEXT co7.
1935 NEXT
1940 ENDPROC
1960 DEF PROC 2D
1980 FOR co7.=1 TO po7.
1990 X2D(co7.)=X(co7.)*2500/(di7.-Z(co7.))
2000 Y2D(co7.)=Y(co7.)*2500/(di7.-Z(co7.))
2010 NEXT co7.
2020 ENDPROC
2040 DEF PROC d.
2050 CALL .clg
2060 FOR co7.=1 TO po%
2070 FOR co7.=1 TO li7.(co7.)
2080 MOVE X2D(st7.(co7.)), Y2D(st7.(co7.))
2090 PLDT -(pa7.=3)+5, X2D(end7.(co7.)), Y2D(end7.(co7.))
2100 NEXT co7.
2110 NEXT pa7.
2120 ENDPROC
2140 CALL .clg
2150 ph=0: th=0: ps=0
2160 REPEAT
2170 Printout 10.1 (continued)
1800*FX15,0
1810K=GET
1820UNTILK=13 OR K=58 OR K=44 OR K=46 OR K=136 OR K=137 OR K=138 OR K=139
1840IF K=138 ph=anst
1850IF K=139 ph=-anst
1870IF K=136 th=anst
1880IF K=137 th=-anst
1900IF K=13 ps=anst
1910IF K=58 ps=anst
1930IF K=44 di%=di%+DIST
1940IF K=46 di%=di%-DIST
1950ENDPROC
1970DEFPROCро
1980IF ph<>0 PROCро
1990IF th<>0 PROCyro
2000IF ps<>0 PROCzro
2010ENDPROC
2030DEFPROCyro
2050Cph=COS(ph):Sph=SIN(ph)
2060FORco%=1TO po%
2070Y(Y(co%))=Z=Z(co%)
2080Y(co%)=Y*Cph-Z*Sph
2090Z(co%)=Z*Cph+Y*Sph
2100NEXTco%
2110ENDPROC
2130DEFPROCzro
2150Cth=COS(th):Sth=SIN(th)
2160FORco%=1TO po%
2170X(X(co%))=Z=Z(co%)
2180X(co%)=X*Cth-Z*Sth
2190Z(co%)=Z*Cth+X*Sth
2200NEXTco%
2210ENDPROC
2230DEFPROCzro
2250Cps=COS(ps):Sps=SIN(ps)
2260FORco%=1TO po%
2270X(X(co%))=Y=Y(co%)
2280X(co%)=X*Cps-Y*Sps
2290Y(co%)=Y*Cps+X*Sps
2300NEXTco%
2310ENDPROC
0000ON ERROR IF ERR=17:MODE7:*FX4,0
0010END
Problem 1: Imagine a solid cube made of wood. Its faces are painted red. It is then cut vertically top to bottom and horizontally, front to back and left to right, to form $3 \times 3 \times 3 = 27$ smaller cubes. How many of the smaller cubes have: 1, 2, 3, 4, 5, 6, red faces? Do any of the smaller cubes have no red faces? Repeat with: 2, 3, 4, ..., cuts in each direction and generalise to $(n-1)$ cuts in each direction.

Problem 2: Imagine a cube consisting of $3 \times 3 \times 3 = 27$ cells. How many straight lines of 3 cells can be identified? When you think you have an answer (a) say what it is; (b) explain clearly and precisely how it was obtained.

The explanation of the solution to Problem 2 depends on the ability to describe the mental image of different sets of straight lines of 3 cells. These sets fall into 3 categories:

<table>
<thead>
<tr>
<th>from:</th>
<th>to:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>vertically:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>top face</td>
<td>bottom face</td>
<td>9</td>
</tr>
<tr>
<td>horizontally:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>left face</td>
<td>right face</td>
<td>9</td>
</tr>
<tr>
<td>front face</td>
<td>back face</td>
<td>9</td>
</tr>
<tr>
<td>obliquely:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>back edge top</td>
<td>front edge bottom</td>
<td>6</td>
</tr>
<tr>
<td>front edge top</td>
<td>back edge bottom</td>
<td></td>
</tr>
<tr>
<td>left edge top</td>
<td>right edge bottom</td>
<td>6</td>
</tr>
<tr>
<td>right edge top</td>
<td>left edge bottom</td>
<td></td>
</tr>
<tr>
<td>left edge front</td>
<td>right edge back</td>
<td>6</td>
</tr>
<tr>
<td>right edge front</td>
<td>left edge back</td>
<td></td>
</tr>
</tbody>
</table>

There are, in addition, the diagonals of the cube: 4

We have: $3 \times 3^2 + 3 \times 2 \times 3 + 4 = 49$. 
The leader of the discussion has to be able to identify any omissions or duplications in the statement of the attempted solution. As in the case of Problem 1, the audience can be invited to repeat the question for a $n \times n \times n$ - celled cube for values of $n = 4, 5, 6, ...$ and to derive the general formula for the solution.

A reference has already been made by the author (see Chapters 2 and 3) to 2-dimensional duality in which parts played by points and lines are interchanged: equivalence classes of ordered pairs of numbers were represented graphically as collinear sets of points and as concurrent sets of lines.

The more able sixth-formers may now be introduced to 3-dimensional duality in which the roles of points and planes are interchanged. It will be seen that if the faces of a regular solid are replaced by their centroids and if the centroids are joined to form edges which pass through the mid-points of the edges of the regular solid, the cube and the octahedron form a pair of dual solids, as do the dodecahedron and the icosahedron. The regular tetrahedron is self-dual and its symmetries have been discussed in §10.4. The author attempts now the identification of the symmetries of the remaining regular Platonic solids.

(1) The Symmetries of the Cube

1. How Many? Unlike the tetrahedron, the vertices of the cube cannot be permuted to give the number of discrete symmetries. It must be noted that the position of the three edges which meet at any one of the eight vertices determines the positions of the remaining edges. In terms of the vertices at their extremities, other than the common vertex, the number of symmetries may be determined as follows.
Let the vertices of the cube be numbered 1, 2, 3, 4; 1', 2', 3', 4' as in Fig 10.6 corresponding respectively to positions A, B, C, D; A', B', C', D' in space. Note that the diagonals of the cube are in positions AA', BB', CC', DD', in space.

Then the position A in space may be filled by any one of the eight vertices numbered 1 to 4, 1' to 4'. Notice next that 3 edges meet at the selected vertex in position A. Hence there are 3 ways in which position B' may then be filled and so positions A and B' may be filled in: $8 \times 3 = 24$ ways. There are now only 2 remaining edges that meet at the vertex at C. Hence position C may be filled in only 2 ways and positions A, B', C may be filled in: $8 \times 3 \times 2 = 48$ ways. The remaining 5 positions are automatically filled in one and only one way giving the total number of symmetries of the cube as 48.

It is to be observed that, of the 2 ways in which position C could be filled in relation to A, B', one of these gives a direct, the other an opposite, symmetry position of the cube: the direction of rotation in the tri-rectangular frame B' (ACD) will be reversed in the 2 cases.
When at B' and looking in the directions B'A, B'C, B'D

Fig. 10.7 illustrates a right-handed directional screw rotation, and Fig. 10.7(ii) a left-handed directional screw rotation, in turning cyclically from B'A to B'C to B'D and back to B'A again. Conventionally, in a 3-dimensional frame of reference, the right-handed screw rotation is regarded as positive, the left-handed screw rotation as negative.

An alternative method of determining the number of symmetries of the cube is the consideration of the four diagonals which join the vertices 1 to 1', 2 to 2', 3 to 3', 4 to 4' [39 and 40]. These 4 diagonals may be arranged amongst themselves in 24 ways corresponding to the 24 direct symmetries of the cube. The simultaneous interchange of the ends of the diagonals in each of these 24 positions is a central inversion of the cube giving 24 corresponding opposite symmetries. These 48 positions of the diagonals correspond to the 48 arrangements of one of their extremities in the positions A, B', C, D' discussed above.

2. The Direct Rotational Symmetries of the Cube. The cube may be regarded as the regular solid formed by the regular tetrahedron 1 2 3 4 and its central inversion 1' 2' 3' 4' (see §10.4 on-page 279). Any symmetry of 1 2 3 4 determines the corresponding central inversion position of 1' 2' 3' 4' and vice-versa.
The 24 symmetries of the tetrahedron were arranged, in Table 10.3 on page 279, in 6 sets of 4 permutations of the vertices 1, 2, 3, 4. Each permutation in a single set of 4 permutations was obtained from its predecessor by a quarter-turn from one of the 12 rotational symmetries of 1 2 3 4, or its central inversion, about the axes OX, OY, OZ which are the joins of mid-points of opposite edges of 1 2 3 4 and hence also of 1' 2' 3' 4'.

If now, in that Table, alternate symmetries are replaced by the central inversion positions, we have the 24 rotational symmetries of the cube. It is to be noted that we have 6 sets of symmetry positions reached as a result of 3 quarter-turns successively from each of 6 starting positions: i.e. 18 rotational symmetries. The 6 starting positions, including the identity I, in each set are all reached by a half-turn about the axes OX, OY, OZ, making a total of 24 rotational symmetries.

It is important to note that these 24 symmetries are generated from the identity position I of 1 2 3 4. It will be seen that the rotation which takes the permutation p q r s (say) of 1, 2, 3, 4 to a permutation of p', q', r', s' simultaneously takes that permutation of p', q', r', s' to p q r s. Table 10.4 below gives the complete set of rotational symmetries of the cube as quarter-turns about the axes OX, OY, OZ and represents the same set of symmetries as its complement obtained by interchanging the vertices 1, 2, 3, 4 with the primed vertices 1', 2', 3', 4' respectively.
The opposite symmetries of the cube may be regarded as the central inversions of these 24 quarter-turn positions. Alternatively, certain planes of reflexive symmetry may be identified.

The planes through the diagonals of the cube are obviously planes of reflexive symmetry. Each diagonal plane may be identified in one of the following ways: (cf Problem 2 on page 284):

(a) a plane of symmetry of the tetrahedron, or of its central inversion, through one edge and the mid-point of the opposite edge

OR

(b) a plane through a pair of opposite edges of the cube

OR

(c) a plane through a pair of parallel diagonals of opposite faces of the cube.

In each case there are just 6 reflexive planes of the cube that can be identified. As for the tetrahedron, so for the cube: the central inversion is needed to describe the complete set of opposite symmetries of the cube.
THE PLATONIC SOLIDS

Tetrahedron

Cube

Octahedron

Dodecahedron

Icosahedron

Fig 10.8
(2) The Symmetries of the Octahedron

The octahedron is the dual of the cube, formed by replacing the 6 square faces of the cube by their centroids. The author chooses to move these centroids outwards in the directions of the axes OX, OY, OZ to meet perpendicular bisectors of the edges of the cube. The 8 vertices of the cube are thus seen (Exhibits) to be replaced by the 8 triangular faces of the solid so formed. The axes OX, OY, OZ fixed in the cube may now be regarded as axes joining the 3 pairs of opposite vertices of the octahedron. Any symmetry of the cube corresponds to its dual octahedron partner. The rotational symmetries are again quarter-turns about the axes OX, OY, OZ. The reflexive symmetries are their central inversions. There are thus a total of 48 symmetries of the octahedron.

(3) The Symmetries of the Icosahedron

The icosahedron is a regular polyhedron consisting of 20 faces which are equilateral triangles and 12 vertices each formed by the meet of 5 edges.

1. How many? As in the case of the cube, the number of symmetries of the icosahedron cannot be determined as the number of permutations of the vertices amongst themselves. The position of the 5 edges which meet at any one vertex determines the position of all the other edges of the solid. It may be seen (Fig 10.8, Exhibits) that the vertices may be regarded as filling positions in a spherical hole in space. These positions in space may be labelled A, Z at a pair of diametrically opposite poles; B, C, D, E, F; and Y, X, W, V, U; at vertices of a pair of pentagons lying in parallel planes with corresponding diametrically opposite pairs B,Y; C,X; D,W; E,V; F,U; of vertices.
If the vertices of the icosahedron are numbered correspondingly: 1, 12; 2, 3, 4, 5, 6; 11, 10, 9, 8, 7; it will be seen that the position A in space may be filled by any one of the 12 numbered vertices of the icosahedron. Notice next that there are then 5 ways in which the position B may be filled corresponding to the other extremities of the 5 edges which meet at the vertex at A. So there are just $12 \times 5$ ways in which the positions A, B may be filled. There remain just 2 ways of filling the position C, adjacent to B when positions D, E, F will be automatically filled. This gives $12 \times 5 \times 2 \times 1 = 120$ ways of filling positions A, B, C, D, E, F. The diametrically opposite positions Z, Y, X, W, V, U are then filled automatically by the correspondingly numbered vertices diametrically opposite to those vertices occupying positions A to F.

It is to be observed that, of the 2 ways in which position C may be filled in relation to A, B, one of these gives a direct, the other an opposite, symmetry position of the icosahedron: the direction of rotation in a 3-dimensional frame of reference in the 2 cases will be in opposite senses. There are 60 direct, 60 opposite symmetries of the icosahedron.

2. The Direct Rotational Symmetries of the Icosahedron:
Reference to a model constructed from a mesh of straws [S9] enables the 60 rotations to be identified:

(a) about the 6 diameters joining pairs of opposite vertices; angles of $n.72^\circ$ for $n = 1, 2, 3, 4$: 24 rotations

(b) about the 10 axes joining centroids of pairs of opposite faces; angles of $n.120^\circ$ for $n = 1, 2$: 20 rotations
(c) half-turns about axes joining mid-points of opposite edges: 15 rotations

(d) the identity rotation as a 360° turn about any axis in (a) or (b) or (c)

There are 60 rotational symmetries identified for the icosahedron.

3. The Opposite Symmetries of the Icosahedron: As for the cube, planes through opposite pairs of vertices of the icosahedron are obviously planes of reflexive symmetry. Nevertheless, as for the cube, it is the central inversion of each of the 60 direct rotational symmetries which give the complete set of 60 opposite symmetries of the icosahedron.

(4) The Symmetries of the Dodecahedron

The dodecahedron is the dual of the icosahedron, formed by replacing the 20 equilateral triangular faces by the centroids. Again the author chooses to move these centroids outwards in the direction of the lines joining opposite pairs of centroids to meet perpendicular bisectors of the edges of the icosahedron. The 12 vertices of the icosahedron are thus seen to be replaced by the 12 pentagonal faces of the solid so formed.

(5) An Investigation

The investigation of the properties of the Platonic solids was set recently to a group of initial trainee teachers. The results of their efforts appear as Exhibits. The object of the investigation was to build on the obvious enjoyment experienced by pupils in earlier years from the construction of solid models, often as Christmas decorations,
and their extensions to 'toys' in the form of rotating rings of tetrahedra, flexagons, ..., etc to be found in many recreational mathematics texts: see for example [41,42,43].

The construction of models from cartridge paper or manila card will help to reinforce the fundamental properties of 3-dimensional space [40].

The results established in this chapter may well be discovered from construction work and discussion. They depend fundamentally on the properties of the intersections of lines and planes, on the conditions for 3-dimensional tessellation, and on the truth of Euler's formula connecting the numbers of vertices $V$, faces $F$ and edges $E$ of a solid figure. The teacher requires a working knowledge of the following basic mathematics.

1. The Schlügel Diagram is the plane figure obtained by the removal of one face of the solid object and, supposing that the solid is made of elasticated material, stretching it outwards until it lies flat. A network is formed in which the nodes, branches and regions are the vertices, edges and faces respectively of the original solid. The number of regions is, of course, one fewer than the number of faces.

The network may be 'triangulated' by drawing in diagonals of a polygonal region in such a way that triangles on the boundary can be removed without altering the value of $V + F - E$ for the resulting network. The process is repeated until we are left with, in general, just a single triangle. It will be found that a triangle may be removed from a polygon on the boundary in such a way that either $E$ and $F$ are each decreased by 1 leaving $V$ unchanged or $V$ and $F$ are each decreased by 1 and $E$ is decreased by 2. In either case the value of $V + F - E$ remains unchanged. Ultimately we have, in general, just one triangle remaining with 3
vertices, 1 face, 3 edges giving $V + F - E = 1$. Hence, in the original Schlegel network, $V + F - E$ had the value 1. In the original polyhedron, if the face that was removed is now replaced, we have Euler's formula:

$$V + F - E = 2$$

The author has often used the Schlegel diagram for the application of matrices to describe a network.

Pupils must also be encouraged to satisfy themselves that there are just five simple regular solids of the type that have been discussed in this thesis.

2. The Platonic Solids. There are just 5 simple regular polyhedra - 3-dimensional geometrical figures in which there are no holes and which consist of $F$ faces which are regular polygons of side $n$ and $V$ vertices which are the meets of $m$ edges. Each edge belongs to 2 vertices and to 2 faces: we have

$$mV = 2E = nF$$

and by Euler's formula:

$$\frac{1}{m} + \frac{1}{n} - \frac{1}{2} = \frac{1}{E}$$

Obviously $m, n \geq 3$. For positive $E$, $m$ and $n$ cannot both be greater than 3. Also for positive $E$ and either $m$ (or $n$) equal to 3, $n$ (or $m$) must be less than 6. Possible integral solutions are tabulated below.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$E$</th>
<th>$V$</th>
<th>$F$</th>
<th>polyhedron</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>tetrahedron</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>12</td>
<td>8</td>
<td>6</td>
<td>cube</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>30</td>
<td>20</td>
<td>12</td>
<td>dodecahedron</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>12</td>
<td>6</td>
<td>8</td>
<td>octahedron</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>30</td>
<td>12</td>
<td>20</td>
<td>icosahedron</td>
</tr>
</tbody>
</table>

Table 10.5
There are just 5 regular polyhedra called the Platonic solids. By investigating their symmetry groups and their isomorphisms with other structures in terms of the complex cross-ratios and transformations in the complex plane, foundations will be laid for the further development to the study of the Möbius transformation of the real and complex planes. The following example has been suggested as a possible question for Ampleforth Mathematics (see Chapter 11) proposed new A-level examination and illustrates potential algebraic-geometric links.

Example*: T is a point on the unit sphere

\[ X^2 + Y^2 + Z^2 = 1 \]

whose centre is 0.

The complex plane XOY is perpendicular to OT.

If P represents the complex number \( x + iy \) and is mapped on to the sphere at \( P' \) so that \( TPP' \) is a straight line, draw a diagram showing the positions of \( T, P, O, P' \) relative to the sphere and to the real and imaginary axes of the complex plane.

Find the coordinates of \( P' \) if:
(a) \( P \) lies on the real axis
(b) \( P \) lies on the imaginary axis.

Show that, for \( P \) representing \( z = x + iy \) in general position

\[ z = x + iy \rightarrow \left\{ \frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right\} = (X, Y, Z) \]

If \( P_1', P_2' \) are the images of \( z_1 = x_1 + iy_1, \ z_2 = x_2 + iy_2 \) under this (stereographic) projection show that the length \( d(P_1', P_2') \) of the chord joining \( P_1', P_2' \) is given by

\[ d(P_1', P_2') = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)}} \frac{\sqrt{(1 + |z_2|^2)}}{\sqrt{(1 + |z_2|^2)}} \]

* Reworded from a question submitted for Ampleforth Mathematics by J V Armitage
Show that the points $\frac{1}{\sqrt{3}}, \frac{e^{2\pi i/3}}{\sqrt{3}}, \frac{e^{-2\pi i/3}}{\sqrt{3}}, \infty$ of the extended complex plane correspond to the vertices of a regular tetrahedron under stereographic projection.

The vertices of the tetrahedron are A, B, C, D, respectively. Find the (Möbius) transformation of the plane which corresponds to the 3-cycle of the group of rotational symmetries of the tetrahedron that permutes B, C, D cyclically.

Can you develop this idea with respect to a cube whose vertices lie on the surface of the sphere?

*Hint*: Consider the Möbius transformation:

$$z \rightarrow \frac{az - a^2 + ab - b^2}{z - b}$$

where a, b are suitable complex numbers.

It is suggested that the reader should develop the idea also to the remaining Platonic solids.

10.6 **Isomorphisms**

If an investigation is made of the sets of symmetries of a regular polyhedron and of its dual under the operation 'followed-by' it will be seen that the two systems have the structure of a group and are isomorphic to each other. A number of isomorphic sub-groups may also be identified within the dual systems.

To establish an isomorphism it has to be demonstrated that, for corresponding elements a, b and a', b' in the two systems, then the products

a 'followed-by' b and a' 'followed-by' b'

give a pair of corresponding elements c and c'. The
discussion of §10.5(1) on pages 285 to 289 and the results in Table 10.4 on page 289 establish the isomorphism between the symmetries of the tetrahedron and its own dual.

The set of elements (see §10.2, page 269):

\[
\lambda \quad \frac{1}{\lambda} \quad (1-\lambda) \quad \frac{1}{(1-\lambda)} \quad \frac{\lambda}{(\lambda-1)} \quad \frac{(\lambda-1)}{\lambda}
\]

generated by successive application of the operations '1-over' and '1-minus' may be put into one-to-one correspondence with the set of permutations of 3 of the range of 4 points \(P_1, P_2, P_3, P_4\) on a line, leaving one point \(P_4\) (say), fixed: denoting the points by their number suffices we have the correspondences:

\[
\begin{align*}
\lambda & \quad \frac{1}{\lambda} \quad (1-\lambda) \quad \frac{1}{(1-\lambda)} \quad \frac{\lambda}{(\lambda-1)} \quad \frac{(\lambda-1)}{\lambda} \\
(1 & 2 3 4) (3 2 1 4) (1 3 2 4) (2 3 1 4) (2 1 3 4) (3 1 2 4)
\end{align*}
\]
These, in turn, may be put into one-to-one correspondence with the symmetries, in 2-dimensions, of the equilateral triangle (see §6.9(3) of Chapter 6 on page 200):

\[
\begin{array}{cccc}
\lambda & \frac{1}{\lambda} & (1-\lambda) & \frac{1}{(1-\lambda)} \\
(1 & 2 & 3 & 4)
\end{array}
\]

(1 2 3 4) (3 2 1 4) (1 3 2 4) (2 3 1 4) (2 1 3 4) (3 1 2 4)

\[R_1 \quad M_2 \quad M_1 \quad R_2 \quad M_3 \quad R_3\]

and using unit basis vectors through the vertices as in §6.9(4) of Chapter 6 on page 201, with their 2 x 2 matrix representations:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
-1 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda & (\lambda-1) \\
(1-\lambda) & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & -1
\end{bmatrix}
\]

It is now an obvious extension to 3-dimensions to arrange the 24 cross-ratios of the 4 points \(P_1, P_2, P_3, P_4\) on a line into 4 sets of 6 cross-ratios. These may be put into one-to-one correspondence with the set of permutations of the 4 vertices of the 3-dimensional regular tetrahedron. The set forms a group which is isomorphic to the group of symmetries of the regular tetrahedron. In the completed Table 10.6 the vertices of the tetrahedron corresponding to \(P_1, P_2, P_3, P_4\) are denoted by their number suffices:
10.7 Algebraic Representation in 3-Dimensional Space

As in 2-dimensional space so in 3-dimensional space: the symmetry transformations of the regular tetrahedron may be represented by their 3 × 3 transformation matrices. The obvious choice of basis vectors is the set of unit vectors

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

in the tri-rectangular frame Ox, Oy, Oz which are the rotational axes joining the mid-points U, U'; V, V'; W, W' of pairs of opposite edges (Fig 10.9).
The position vector matrix of the vertices of the tetrahedron is given by:

\[
\begin{bmatrix}
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

The 3 x 3 matrix of a transformation is determined by the images of the unit basis vectors:

\[
\overrightarrow{OU} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \overrightarrow{OV} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \overrightarrow{OW} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

We have (Table 10.7).

The reader is invited to identify the remaining 3 x 3 transformation matrices to represent the symmetries of the regular tetrahedron. The complete solution set is given in Appendix C, Table C1.

An alternative matrix representation may be obtained, as suggested for the symmetry transformations of the equilateral triangle in Chapter 6, by taking an oblique set of basis vectors. It is easy to see that, if basis vectors

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

are taken as the position vectors of vertices number 1, 2, 3 with origin at the centre of the circumscribing sphere, then the position vector of the vertex number 4 is:

\[
\begin{bmatrix} -1 \\ -1 \end{bmatrix}
\]

(Fig 10.10).
<table>
<thead>
<tr>
<th>( I ) : ( UVW )</th>
<th>( UVW )</th>
<th>( I ) : ( (1 2 3 4) \rightarrow (1 2 3 4) )</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -1 \\
1 & 1 & -1 \\
-1 & -1 & 1
\end{bmatrix}
\] |

<table>
<thead>
<tr>
<th>( X ) : ( UVW )</th>
<th>( U V' W' )</th>
<th>( X ) : ( (1 2 3 4) \rightarrow (2 1 4 3) )</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -1 \\
-1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\] |

<table>
<thead>
<tr>
<th>( Y ) : ( UVW )</th>
<th>( U' V' W' )</th>
<th>( Y ) : ( (1 2 3 4) \rightarrow (3 4 1 2) )</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 1 & 1 \\
1 & 1 & 1 \\
-1 & 1 & 1
\end{bmatrix}
\] |

<table>
<thead>
<tr>
<th>( Z ) : ( UVW )</th>
<th>( U' V' W )</th>
<th>( Z ) : ( (1 2 3 4) \rightarrow (4 3 2 1) )</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 1 & 1 \\
-1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\] |

\[\text{Table 10.7}\]
Since the unit basis vectors are transformed to 3 of the 4 image positions required in each symmetry transformation the transformation matrices are readily written down. For the 4 transformations I, X, Y, Z, given in Table 10.1 on page 277 we have Table 10.8:
In each case the reader will see that the columns of the transformation matrix are the position vectors of the images of the unit basis vectors.

Again it is left as an exercise for the reader to derive the complete set of transformation matrices which is given in Appendix C, Table C2.

The author now invites the following alternative investigation into the set of 6 cross-ratio values in terms of complex numbers and transformations of the complex plane.
10.8 Transformations of the Complex Plane

The number system has been extended, in Part I, from the naturals to the integers, the rationals, the reals, and at each stage the development of the operational system was illustrated by points on the real number line and by diagrams in the real 2-dimensional Euclidean plane. The geometrical symmetry and similarity transformations of the real plane have been described, in Part II, in terms of the linear algebra of $2 \times 2$ matrices. The author was led, in Part III, to the further extension of the number system to the complex numbers. These have again been represented geometrically by points in the complex 2-dimensional plane which is familiarly known as the Argand diagram. The author now attempts an investigation into the elementary transformations of the complex plane.

By analogy with transformations of the real plane it may be seen that if $\omega, z, \alpha, \beta$, are complex numbers:

(a) $\omega = z + \beta$ is a translation in the complex plane given by the position vector representation $\overrightarrow{OB}$ in the Argand diagram of the complex number $\beta$ with magnitude $|\beta| = OB$ and direction $\arg \beta$ of $\overrightarrow{OB}$. The transformation corresponds to the translation: $\overrightarrow{AB} \rightarrow \overrightarrow{A'B'}$ in the real plane which may be represented by the vectors $\overrightarrow{AA'} = \overrightarrow{BB'}$ in magnitude and direction (see §4.4 of Chapter 4).

(b) $\omega = \alpha z$ is the spiral similarity in the complex plane which is the commutative product of a rotation $\arg \alpha$ and a dilatation $|\alpha|$. The transformation corresponds to the general similarity in the real plane which is the commutative product of a rotation and the dilatation: $\overrightarrow{OB'} = \lambda \overrightarrow{OP}$ (see §7.9 of Chapter 7). The spiral similarity may be represented by the duplex number:

$$\rho \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

where $\rho = |\alpha|$ and $\phi = \arg \alpha$. 


(c) Reflection in the complex plane: \( z \rightarrow \omega \) in the axis \( y = x \tan \psi \) may be seen as the commutative product of rotation of the conjugate of \( z \) through \( 2\psi \). We have

\[
\omega = \text{rotation of } \bar{z} \text{ through the angle } 2\psi \\
= r \begin{bmatrix} \cos 2\psi & -\sin 2\psi \\ \sin 2\psi & \cos 2\psi \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\
= r \begin{bmatrix} \cos(2\psi-\theta) & -\sin(2\psi-\theta) \\ \sin(2\psi-\theta) & \cos(2\psi-\theta) \end{bmatrix} \tag{Fig \ 10.11(i)}.
\]

Alternatively, the basis may be changed from \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) to \( \begin{bmatrix} \cos \psi \\ \sin \psi \end{bmatrix} \). Let \( z = z' = re^{i(\theta - \alpha)} \) referred to the new basis:

\[
\omega = z' = re^{-i(\theta - \psi)} \\
= re^{i(2\psi - \theta)} \quad \text{(as before)} \tag{Fig \ 10.11(ii)}.
\]

The author now considers the transformation:

(d) \( \omega = \frac{1}{z} \)

It is easy to see that this gives:

\[
\arg \omega = -\arg z \quad \text{and} \quad |\omega| = \frac{1}{|z|} \Rightarrow |\omega||z| = 1
\]

so that \( \omega \) may be represented in the Argand diagram as the commutative product of the inversion in the unit circle with the conjugate of \( z \).
If the teacher can be persuaded to re-introduce the geometry of harmonic ranges, inversion, ..... , as investigations at sixth-form level with apparatus and technology now at his disposal, the author is in a position to geometrically represent the 6 cross-ratio values, not as real numbers but as complex numbers in the complex plane.

10.9 The Complex Cross-Ratio

The six cross-ratio values: \( \lambda, \frac{1}{\lambda}, (1-\lambda), \frac{1}{(1-\lambda)}, \frac{\lambda}{(\lambda-1)}, \frac{(\lambda-1)}{\lambda} \), have been seen as real-number values involving the ratios of segments defined by 4 points \( P_1, P_2, P_3, P_4 \) on the real number line. The reader is asked now to regard these 6 values as complex-number values involving the ratios of segments defined by 4 points which represent the complex numbers \( z_1, z_2, z_3, z_4 \) in the complex plane.

Note first that the transformation: \( z + 1 - z \) is a half-turn about \( z = \frac{1}{2} \) (Fig 10.12).

The successive application of the transformations '1-over' and '1-minus' again generate the set of complex-number cross-ratio values. These may be represented in the complex plane by successive 'inversion of the conjugate in the unit circle' and 'half-turn about \( z = \frac{1}{2} \)'. Thus (Fig 10.13)

\[
\begin{align*}
\omega &= \lambda = \lambda_1 \quad \text{(say)} \\
\omega &= \frac{1}{\lambda} = \lambda_2 \quad \text{inversion of } \overline{\lambda} \text{ in unit circle} \\
\omega &= \frac{(\lambda-1)}{\lambda} = (1-\lambda_2) = \lambda_3 \quad \text{half-turn of } \lambda_2 \text{ about } \omega = \frac{1}{2} \\
\omega &= \frac{\lambda}{(\lambda-1)} = \frac{1}{\lambda_3} = \lambda_4 \quad \text{inversion of } \overline{\lambda_3} \text{ in unit circle} \\
\omega &= \frac{1}{(1-\lambda)} = (1-\lambda_4) = \lambda_5 \quad \text{half-turn of } \lambda_4 \text{ about } \omega = \frac{1}{2} \\
\omega &= (1-\lambda) = \frac{1}{\lambda_5} = \lambda_6 \quad \text{inversion of } \overline{\lambda_5} \text{ in unit circle} \\
\omega &= \lambda = (1-\lambda_6) = \lambda_1 \quad \text{half-turn of } \lambda_6 \text{ in } \omega = \frac{1}{2}
\end{align*}
\]
The specialist mathematics teacher will appreciate that the foundations are laid for the development of complex algebra to the Mobius transformation as the combination of complex cross-ratio values as operations.

10.10 The Game Continues......

Just as the real number cross-ratio values form a group which is isomorphic to the permutations of 4 points on a line, one of which is fixed, which in turn is isomorphic to the symmetries of the equilateral triangle in 2-dimensional space; so now the complex-number cross-ratio values may be seen as a group which is isomorphic to the permutations of 4 points on the unit circle in the real 2-dimensional plane, one of which is fixed, which in turn is isomorphic to a subset of the symmetries of the regular tetrahedron, with one vertex fixed, in 3-dimensional space.
It would appear to be a short step in the development of the geometrical representation of the complex in algebra to regard the 6 complex-number cross-ratio values, not as ratios of segments but as ratios of arcs on the unit circle and to describe them in terms of 'turns' — a term first used, it is thought, by Frank and F V Morley [44].

The work would appear to lend itself to further extension to the representation of 4 points on the unit sphere and to the corresponding cross-ratio values involving the arcs of great-circles which is the geometrical representation of quaternions as suggested by P G Tait [45] with the inscription

This is apparently a mis-quotation from Actius 1, 3, 8. If

we have the translation [22]

"... the tetractys, which contains the fount and root of eternal nature."

The tetrad was invoked by the Pythagoreans as their most binding oath.
CHAPTER 11

SUMMARY, RECOMMENDATIONS AND SUGGESTIONS FOR FURTHER RESEARCH

11.1 Linear Algebra - its Place in the School Curriculum

Early results were published in a linked series of articles [28] in 1982. The work was based largely on the ideas expressed by T J Fletcher [25] and was a natural development of Mathematics for the Middle Years, published as a series of articles [5] and edited by the author as the chairman of a sub-committee of the Teaching Committee of the Mathematical Association.

The work has continued in this thesis with applications of linear algebra in numerical work and in geometry with particular reference at sixth-form level to complex numbers with emphasis on the Geometrical Representation of the Imaginary in Algebra. All the topics described have been supplemented by practical work with teachers, students and pupils in actual classroom situations. Examples from the curriculum which illustrate 'sums of multiples' as the elementary interpretation of the use of linear algebra have been extensively developed at all stages of the 9 to 18 age range and have not been restricted to the subject matter of this thesis. Simple algebraic techniques which were the bread-and-butter of sixth-formers studying the conics some 30 years ago, and now largely ignored, when applied to the complex plane make admirable tools for the reintroduction of the geometry of the straight line and circle - the then 'Modern' Geometry texts of C V Durell [33] and of C Godfrey and A W Siddons [34].
Almost all the work described herein is supported by 'transportable' classroom activity in the form of audio- and/or video-taped recordings of actual 'live' lessons. Some of these recordings have been transcribed and appear as Appendices. Linear Algebra and its Applications to the key areas of Numerical Methods and Geometry lends itself readily to work on a microcomputer. The work has been enthusiastically received by the head of mathematics of a large independent boys' school.

11.2 Ampleforth Mathematics

Ampleforth Mathematics was inaugurated on 12 May 1987 under the directorship of Professor A C Bajpai (Loughborough University of Technology) with Professor J V Armitage (Durham University), Dr T J Fletcher (retired Staff Inspector) and the author of this thesis (Hull University) as consultants. Technical support is provided by the Audio Visual Centre of Hull University and T M Vessey, the Head of Mathematics at Ampleforth College is the coordinator of the team which includes also Dr R V Murphy (Director of Computing), A T Hollins and G Simpson (Mathematics Department), of the College.

It is recommended that

Ampleforth Mathematics is supported in its aims to:

(i) examine every aspect of the teaching of mathematics at sixth-form level

(ii) develop a new style of teaching at sixth-form level to
     (a) ensure continuity from the General Certificate of Secondary Education at 16+ and
     (b) take full advantage of the microcomputer
(iii) consider the demands from universities/polytechnics/job-training courses which dictate

(a) an increase in the knowledge of numerical methods
(b) an increase in the knowledge of 2- and 3-dimensional spaces
(c) an ability to work on one's own initiative

Exploratory work has already begun: in 1985-86 with a small group of 14-year-old post-O-level pupils and, currently, with the peer group of 1986-87 which is being prepared for a revised A-level examination in 1989.

11.3 The Syllabus

(1) 9 to 16: Many of the more recent school syllabuses remain disjointed and give expression still to a mathematics course as a step-by-step progression through a list of disparate topics. Linkages must be found to integrate these topics into a unified whole. There are two areas of study in school mathematics that today present serious problems for the teacher of mathematics and two major gaps to be filled: the places formerly occupied by geometry and algebra. The language of linear algebra is seen as eminently suitable to contribute to the closing of these gaps.

It is recommended that

a selection be made of topics in the 9 to 16 mathematics curriculum and an investigation made of strategies for presenting them in the classroom.

The linear algebra of vectors is seen literally as a 'vehicle' for teaching and handling many such topics in an informative way. Furthermore the linear algebra of a vector space is a key area of applicable mathematics at university level; the informal introduction of the ideas is both essential and relevant at the 9 to 16 stage.
16 to 18: The Ampleforth Mathematics programme dictates that a new sixth-form course is essential. Following the recent trials in the classroom and advice from the consultants and GCE A-level examiners the following proposals have been submitted for the consideration of the Oxford and Cambridge Schools Examination Board for a new examination and revised syllabuses in Mathematics and in Further Mathematics.

The proposed A-level course is designed for pupils who have shown sufficient aptitude at the General Certificate of Secondary Education examination at 16+, or equivalent level, and is considered to be suitable for a wider range of candidature than is usual for A-level candidates.

The philosophy of the GCSE must inevitably apply to future A-level courses. Not only must work at advanced level be a natural continuation of GCSE but it must also be a stepping-stone to the next stage of development at sixth-form and tertiary level and educationally worthwhile in its own right. At sixth-form level pupils must have the opportunity to

(i) show initiative in their approach to a problem
(ii) demonstrate an ability to select and evaluate methods of solution
(iii) comment on the accuracy and validity of a solution

A change is recommended in

(i) the style in which questions are set
(ii) the form of answers expected
(iii) the method of assessment of those answers

* © Ampleforth Mathematics: February 1987 (see Chapter 8 and Appendix B)
The new style of question will demand a form of assessment without any pre-determined maximum mark. It will need to be marked qualitatively rather than quantitatively. Modern procedures should enable conversions to be made to normal percentage scales and/or grade classifications for purposes of correlation.

11.4 The Microcomputer

Before the microcomputer became available in the classroom, any problem in pure mathematics, statistics or mechanics was restricted to one capable of an analytic solution. Now, by using numerical methods it is possible more freely and realistically to provide a model in the knowledge that a better approximation to reality can be achieved. Linear algebra is a language which lends itself readily to work with a microcomputer. It is suggested that it will dictate the movement of much of the current sixth-form work to an earlier stage in the school mathematics programme.

The microcomputer can promote discussion in the classroom. Examples of audio-recordings are transcribed in the text* and in Exhibits. Pupils can readily appreciate the validity of their responses and can make corrections and improvements themselves and/or encourage others to participate. It is important that we do not underestimate the talent which many pupils have for the new technology. It is not for us to be brilliant but to encourage brilliance in others. It is important, too, that that encouragement is well informed.

The microcomputer makes a visual impact which is likely to remain with a pupil whose understanding will thereby be enhanced. The microcomputer has already been recognised both as a teaching and as a learning aid with considerable potential. This potential demands for its realisation a change of teaching style now at sixth-form level.

* See §3.12 of Chapter 3 and Appendix D
It is recommended that

An investigation be made of

(i) teaching styles at sixth-form level

(ii) what use schools are making of existing software

(iii) what sort of software is required by teachers for the specific use of the microcomputer in the mathematics classroom

11.5 Further Research

The tide of events is overtaking the writing programme. Everything that is currently being done in the context of this thesis will even now be out-dated unless a move is made swiftly towards the use of the microcomputer in the mathematics classroom.

It is recommended that

an investigation be made into the key areas of numerical methods and geometry in the school mathematics programme.

The tetrahedral group of symmetries has been discussed fully in Chapter 10 and has provided some exciting activity sessions with structural materials with sixth-formers. The work has been extended by a group of students as an investigation into the Platonic solids (see Exhibits). It is felt that the study is well within the scope of a good average sixth-former with the help of animated graphics on the microcomputer. A tentative example has been prepared by A T Hollins of Ampleforth College to enhance this investigation (see page 280).
Extensions to:

(i) the partition of a sphere into 4 nodal points and the projection of the edges of the tetrahedron so formed into great circles

(ii) the operation on groups by matrices

(iii) the use of quaternions to represent a group of operations

are now, optimistically, felt by the author and the coordinator of Ampleforth Mathematics to be appropriate to Further Mathematics in the foreseeable future: these studies can now be linked by graphics on a microcomputer [S7]. The reader is referred to P M Cohn [46] and, for the geometrical link, to E G Rees [47]. There is also a delightful development in the form of investigations by R P Burn [48].

11.6 Conclusion

This exposition has attempted to show how the foundations of mathematical experiences in the early years of schooling may be re-inforced and extended; how the needs of the users of mathematics and of the potential specialist mathematicians are not incompatible but go hand-in-hand. Advanced ideas have been discussed from an elementary point of view, at a level suitable for adaptation by teachers, with a view to making the work accessible to secondary pupils and for ultimate inclusion in the school curriculum.

The originality demanded of every PhD Thesis goes beyond such exposition. Here the originality consists of new insights into ideas appropriate to senior pupils in schools and a rewriting of existing material often thought to be beyond their scope.
It has to be emphasised that opportunities for sixth-form experimentation are rare. At this early stage in the development of the ideas discussed there cannot have been any formal testing of results by quantitative analysis. The results speak for themselves. Qualitatively there is already a very high degree of viability of the proposals demonstrated by the recordings made by the technical adviser to Ampleforth Mathematics.

It is further recommended that

the work of the Ampleforth Mathematics study group be extended and interested schools be invited to join the group.
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Budden, F. J.: Introduction to Algebraic Structures: Longman
Ford, L. R.: Automorphic Functions: Chelsea Publishing Co: 1929
Mike James: Creative Animation and Graphics on the BBC Micro: Collins: 1985
Kaplansky: Linear Algebra and Geometry – A Second Course: Allyn and Bacon: 1969
Schwerdtfeger, H.: Geometry of Complex Numbers: Dover 1979 (Reprint of University of Toronto: 1962)
 Included in the submission are a number of Exhibits referred to in the text.

These may be seen at the University of Hull on application to the author.
A(i)

Solutions to Exercise on page 226

1

We have:

Translational

\[ \overrightarrow{AB} : AB'C' \rightarrow C'A'B \]
\[ \overrightarrow{AC} : AB'C' \rightarrow B'CA' \]

ie Triangles AB'C', C'A'B, B'CA' are isometric.

Half-turn about \( A'' \), the mid-point of B'C': AB'C' \( \rightarrow \) A'C'B'

ie Triangle AB'C' is isometric also to triangle A'C'B'.

Dilatation \( (A,2) \): AB'C' \( \rightarrow \) ACB

ie Triangle AB'C' is similar to triangle ACB.

It follows that the triangles AB'C', C'A'B, B'CA', A'C'B' are isometric to each other and similar to triangle ACB with scale factor 2.

2
A(ii)

Reflections $M_{OB'}$ and $M_{OC'}$ in the mediators $OB'$, $OC'$ of $AC$, $AB$ respectively are such that:

- $M_{OB'}$: $C \rightarrow A$ so that $OC = OA$
- $M_{OC'}$: $B \rightarrow A$ so that $OB = OA$

Hence $OC = OB$.

It follows that $OC$, $OB$ are images under reflection in the mediator of $BC$ which passes through $O$

ie the mediators $OA'$, $OB'$, $OC'$ of triangle $ABC$ are concurrent

Furthermore, $O$ is the centre of a circle which passes through the vertices $A$, $B$, $C$ of the triangle $ABC$. The circle is the circum-circle of triangle $ABC$ and $O$ is the circum-centre of the triangle.

Note first that the mediators $A'O$, $B'O$, $C'O$ of triangle $ABC$ are the altitudes of triangle $B'C'A'$. This means that the altitudes of triangle $A'B'C'$ are concurrent at $O$, the circum-centre of triangle $ABC$. 

Fig A3(i)
A(iii)

It has already been established (see example on page 225 of Chapter 7) that the medians of triangle ABC are concurrent at the centroid G of the triangle and that

Dilatation \((G, 2)\): \(A'B'C' \rightarrow ABC\)

It follows that, under the same dilatation, the altitudes of triangle \(A'B'C'\) have as their images the altitudes of the triangle ABC

ie the altitudes of triangle ABC are concurrent in \(H\) (say)

\(H\) is called the orthocentre of the triangle ABC. The reader should note that the orthocentre, \(H'\) (say) of the triangle \(A'B'C'\) coincides with the circum-centre of the triangle ABC (Fig A3 (ii)).

\[\text{Fig A3(ii)}\]

The following correspondences under the dilatation \((G, 2)\) should be noted:

\[O \equiv H' \oplus H \quad \text{and} \quad GH = 2.GH'\]

\[N \quad (\text{say}) \equiv O' \oplus O \quad \text{and} \quad GO = 2.GN\]

where \(N \equiv O'\) is the circum-centre of triangle \(A'B'C'\).

It is interesting to note the orthocentre \(H\) of the triangle ABC is also the circum-centre of the triangle whose sides are the lines through the vertices \(A, B, C\) and parallel to
the opposite sides BC, CA, AB respectively (Fig A3 (ii)). Just as triangle A'B'C' is, in a sense, the 'median' triangle of ABC, so this newly constructed triangle has ABC as its 'median' triangle.

It is easy to establish that: \( \text{OH} = 2 \cdot \text{ON} \). The line OGNH is called the Euler line of triangle ABC (see §10.1 of Chapter 10) and N is the nine-point centre of the triangle.

It remains to establish that A, B, C are the orthocentres of the triangles HBC, HCA, HAB respectively. It may be seen from Fig A3 (ii) that HA, CA, BA are perpendicular to the sides (produced if necessary) BC, BH, CH respectively of triangle HBC.

ie A is the orthocentre of triangle HBC.

Similarly it may be proved that B, C are the orthocentres respectively of the triangles HCA, HAB.

It is interesting to note that, under the dilatation \((G, 2)\) in which A'B'C' - ABC, G' - G

ie the centroids of the triangles coincide at the invariant centre of the dilatation.
Note first that (Fig A4(i))

Translation \((\frac{1}{4}AC)\): AC'B' \(\rightarrow\) B'A'C

so that: \(O_1 \rightarrow O_3\) and \(0_10_3\) is parallel to AC and hence to A'C'.

Similarly it may be seen that

\(0_30_2\) is parallel to CB and hence to C'B'

and \(0_20_1\) is parallel to BA and hence to B'A'.

Now, \(O_1\) and \(O'\) lie on the mediator of C'B'

ie \(O_1O'\) is the mediator of C'B' and an altitude of \(O_1O_2O_3\)

Similarly it may be seen that

\(O_2O'\) and \(O_3O'\) are altitudes of \(O_1O_2O_3\)

Hence \(O'\) is the orthocentre of the triangle \(O_1O_2O_3\).

Using example 3, \(O_1\), \(O_2\), \(O_3\) are the orthocentres respectively of the triangles \(O'0_20_3\), \(O'0_30_1\), \(O'0_10_2\).

What can be said about the circum-centres of these triangles?

![Diagram](Fig A4(ii))

Note first that ABC is the image of \(O_1O_2O_3\) under a dilatation whose scale factor is 2: corresponding line segments are parallel and ensure that \(A0_1\), \(B0_2\), \(C0_3\) are concurrent in the
centre of the dilatation. To identify that centre, consider the images of \( \text{O}' \), \( \text{O}_1 \), \( \text{O}_2 \), \( \text{O}_3 \) under the dilatation and under the half-turn which takes \( \text{O}_1\text{O}_2\text{O}_3 \) to \( \text{A}'\text{B}'\text{C}' \) (Fig A4(ii)).

<table>
<thead>
<tr>
<th>Object</th>
<th>Image under Dilatation</th>
<th>Image under Half-turn</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{O}_1\text{O}_2\text{O}_3 )</td>
<td>( \text{ABC} )</td>
<td>( \text{A}'\text{B}'\text{C}' )</td>
</tr>
<tr>
<td>orthocentre ( \text{O}' )</td>
<td>orthocentre ( \text{H} )</td>
<td>orthocentre ( \text{H}' = \text{O} )</td>
</tr>
</tbody>
</table>

The centre of the half-turn is the mid-point of \( \text{O}\text{O}' \). Since \( \text{OH} = 2\text{OO}' \) (see example 3) then \( \text{O} \) is the centre of the dilatation and \( \text{AO}_1 \), \( \text{BO}_2 \), \( \text{CO}_3 \) are concurrent in \( \text{O} \), the circum-centre of triangle \( \text{ABC} \).

Next, consider the image of \( \text{O}'\text{O}_1\text{O}_2 \) under the same dilatation.

![Diagram](image)

**Fig A4(iii)**

We have (Fig A4(iii)): \( \text{O}'\text{O}_1\text{O}_2 \rightarrow \text{HAB} \)

and orthocentre \( \text{O}_3 \) of \( \text{O}'\text{O}_1\text{O}_2 \rightarrow \text{orthocentre of HAB} \)

We know that: \( \text{OC} = 2\text{O}_3 \) so that \( \text{C} \) is the image of \( \text{O}_3 \) and \( \text{C} \) is therefore the orthocentre of \( \text{HAB} \) (as is well-known from traditional methods of proofs).

Similarly: \( \text{A} \) and \( \text{B} \) are the orthocentres of \( \text{HBC} \) and \( \text{HCA} \) respectively.
Since the mediators B'C' and A'C' of O'O_1 and O'O_2 respectively meet at C' then C' is the circum-centre of O'O_1O_2. Similarly it may be seen that A' and B' are the circum-centres of O'O_2O_3 and O'O_3O_1 respectively.

An interesting result follows from the image of C' under the dilatation of O'O_1O_2 to HAB.

![Diagram](image)

Under the dilatation (O, 2): O'O_1O_2 \rightarrow HAB

circum-centre C' of O'O_1O_2 \rightarrow circum-centre of HAB

which is at the point \( O_3 \) such that \( O_3 = 2 \cdot OC' \). This may be obtained as the image of O in the half-turn about C' (Fig A5).

Note that, under this half-turn, the image of C is C" which completes the parallelogram CAC"B; the image \( O_3 \) of O is the circum-centre of the triangle AC"B. It follows that the circle HAB passes through C" and is equal to the circum-circle of the triangle ABC.

Similarly it may be established that the circles HBC and HCA are also equal to circle ABC and to circle HAB.

The same dilatation (O, 2) is such that: the triangle formed by the circum-centres \( \Omega_1, \Omega_2, \Omega_3 \) of the triangles HBC, HCA, HAB...
respectively has as its image the triangle A'B'C'. Furthermore, the circum-centre of $\Omega_1 \Omega_2 \Omega_3$ corresponds to the circum-centre $O'$ of A'B'C'. We know that:

$$\hat{O}H \rightarrow 2\hat{O}O'$$

so that $H$ is the circum-centre of $\Omega_1 \Omega_2 \Omega_3$.

Problem 6 is left as an exercise for the reader.
New Teaching Styles - Specimen Questions

The following questions were set as an end-of-year test to the 'pilot' class of boys at the end of the first year's work in Ampleforth Mathematics. One of the boys commented: "Why haven't we had questions like this before, Sir?"

1 With reference to axes Ox and Oy the points P and Q have coordinates (4,1) and (2,5) respectively. What is the equation of the line joining them?

What other methods do you know for defining the position of a point? Use each of your selected methods to define the positions of P and Q. For each method find the equation of PQ.

R and S are 2 more points with cartesian coordinates (-2,-1) and (2,1) respectively. It is required to find the angle between the lines PQ and RS. Choose what you consider to be the most convenient form of the equation of a line for this problem and find the required angle.

Show that the point with cartesian coordinates \((2, \frac{7}{3})\) is equidistant from both of the lines PQ and RS. Hence find the equations of the bisectors of the angle between them.

A and B have coordinates (3, 4) and (2,8). T is a point such that the angle ATB = 90°. If T has coordinates \((a,b)\) find, by considering the triangle ATB, an equation connecting a and b.

What is the equation of the circle whose centre is \((\frac{1}{2},2)\) and which passes through B.

Find the equation, centre and radius of the circle through the points \((0,0)\), \((1,4)\) and \((2,2)\).
2 Find the matrix

(i) $M_{90}$ which represents a rotation of $90^\circ$ about the origin
(ii) $M_{180}$ which represents a rotation of $180^\circ$ about the origin
(iii) $M_\theta$ which represents a rotation of $\theta$ about the origin

By considering $M_\alpha$, $M_\beta$, $M_{\alpha+\beta}$ prove that

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

and find a similar formula for $\cos(\alpha + \beta)$.

Write $\sin(x + 60^\circ)$ in terms of $\sin x$ and $\cos x$ leaving surds ($\sqrt{}$'s) in your answer.

Show how $2 \sin x + 4 \cos x$ may be written in the form $r \sin(x + \alpha)$ and find $r$ and $\alpha$. Hence solve for $x$ the equation

$$2 \sin x + 4 \cos x = \sqrt{5}$$

giving all possible answers for $0^\circ \leq x \leq 720^\circ$.

If $K = 2/(1 + \sin x - 3 \cos x)$ what are the maximum and minimum values of $K$? Can you sketch the graph of $K$?

3 (i) Write down a simple $2 \times 2$ matrix and denote it by $M$. What is meant by: "$M$ represents a transformation of the plane"?

(ii) Using the matrix $M = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ find the image of the points $(1,3)$, $(1,6)$, $(1,k)$.

(iii) What is the image of the line $x = 1$ under $M$?

(iv) What can you say about the image under $M$ of any line parallel to the $y$-axis? Justify your answer.

(v) Generalise the above steps (i) to (iv) and prove that the image of any straight line under any transformation of the plane represented by a matrix is a straight line.
(vi) Prove that parallel lines are mapped onto parallel lines under any transformation of the plane.

(vii) Under a transformation $T$ the image of a square of side one unit and with one vertex at the origin is a rhombus of side one unit with one vertex at the origin. Is this possible? If so, suggest one or more matrices to represent $T$ and specify the square and the rhombus in each case; if not, explain why not.

(viii) Consider the situation in which the image rhombus does not have a vertex in common with the square and neither has a vertex at the origin. How many points and their images need to be specified in order to determine the transformation uniquely?

4 ABC is a triangle and $O$ is its circumcentre. $D$ is the foot of the perpendicular from $O$ to $BC$.

(i) Prove that angle $DOC$ equals angle $BAC$.

(ii) Find a relationship between $R$, the circumradius, $\alpha$, the angle $BAC$, and $a$, the length of $BC$.

(iii) Prove the 'sine rule' for any triangle.

(iv) What is the 'cosine rule'?

RT and SV are vertical towers with heights 25 m and 18 m respectively. Their bases $T$ and $V$ are on the same level and are 30 m apart. $A$ is a point on the same horizontal level as $T$ and $V$. The angles of elevation of $R$ and $S$ from $A$ are $21^\circ$ and $18^\circ$ respectively. Find the angle $RAS$. 
5  (i) What is meant by: \( y = \log_k x \)?

Simplify: \( y^a x^b + y^c \).

Prove that: \( \log_k x + \log_k y = \log_k (xy) \)

(ii) What do you understand by:

\[
\lim_{\delta x \to 0} \frac{1}{\delta x} \int_a^b y \, dx
\]

A diagram may help you to give a clear answer.

(iii) A function \( F(x) \) is defined as \( F: x \to \) the area under the curve \( y = \frac{1}{t} \) and between the \( t \)-axis and the lines \( t = 1 \) and \( t = x \). Use your calculator, or write a program, to obtain a reasonable estimate of:

\( F(2); \ F(4); \ F(8); \ F(3); \ F(9) \)

What is \( F(1) \)?

Bearing in mind that your answers are approximations (but good ones) what do they suggest to you about the function \( F(x) \)?

6 Investigate the following curve. Find all maximum and minimum points and points of inflexion, if they exist:

\[
y = \frac{6x^2 + 8x + 1}{2x}
\]

Sketch the curve and indicate on your diagram the results that you have obtained.

7 Newton's law of motion states that \( P = ma \) where \( P \) is the net force acting on a mass \( m \) moving with acceleration \( a \) (in Newton, kg, m/s/s units). This equation is really a vector equation. What does this mean?
Assuming that $P$ and $m$ are constant, derive the equations:

$$Ps = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \quad \text{and} \quad Pt = mv - \frac{1}{2}mu$$

by integrating the equation $P = ma$. Explain the meaning of these equations.

A bead of mass $\frac{1}{4}$ kg slides down a shute. The shute offers a constant resisting force of 1N and is 18.5 m long. The difference in height between the top and the bottom is 5 m. If the bead starts from rest what will be its speed when it reaches the bottom?

On reaching the bottom it strikes another bead of mass $\frac{1}{4}$ kg which moves off with speed $6\sqrt{2}$ m/s. What is the speed of the first bead immediately after impact? (Take $g = 10$ m/s/s)

The system of weights and strings shown in the diagram is in equilibrium. Find the tension in each string.

![Diagram](image-url)
| I = \[
1 0 0 \\
0 1 0 \\
0 0 1
\] | R₁ = \[
0 1 0 \\
0 0 1 \\
1 0 0
\] | S₁ = \[
0 0 1 \\
1 0 0 \\
0 1 0
\] | M₁ = \[
1 0 0 \\
0 0 1 \\
0 1 0
\] | Ω₊Qₓ = \[
-1 0 0 \\
0 0 1 \\
0 1 0
\] |
|---|---|---|---|---|
| X = \[
1 0 0 \\
0 -1 0 \\
0 0 -1
\] | R₂ = \[
0 -1 0 \\
0 0 1 \\
-1 0 0
\] | S₂ = \[
0 0 -1 \\
-1 0 0 \\
0 1 0
\] | M₂ = \[
0 0 1 \\
0 1 0 \\
1 0 0
\] | Ω₋Qₓ = \[
-1 0 0 \\
0 0 -1 \\
0 1 0
\] |
| Y = \[
1 0 0 \\
0 1 0 \\
0 0 -1
\] | R₃ = \[
0 -1 0 \\
0 0 -1 \\
1 0 0
\] | S₃ = \[
0 0 1 \\
-1 0 0 \\
0 -1 0
\] | M₃ = \[
0 -1 0 \\
0 0 -1 \\
1 0 0
\] | Ω₊Qᵧ = \[
0 0 -1 \\
0 -1 0 \\
1 0 0
\] |
| Z = \[
1 0 0 \\
0 -1 0 \\
0 0 -1
\] | R₄ = \[
0 1 0 \\
0 0 -1 \\
-1 0 0
\] | S₄ = \[
0 0 -1 \\
1 0 0 \\
0 -1 0
\] | M₄ = \[
0 -1 0 \\
-1 0 0 \\
0 0 1
\] | Ω₋Qᵧ = \[
0 0 1 \\
0 -1 0 \\
-1 0 0
\] |
| M₅ = \[
0 0 -1 \\
0 1 0 \\
-1 0 0
\] | Ω₊Qᶻ = \[
0 1 0 \\
-1 0 0 \\
0 0 -1
\] |
| M₆ = \[
1 0 0 \\
0 0 -1 \\
0 -1 0
\] | Ω₋Qᶻ = \[
0 -1 0 \\
1 0 0 \\
0 0 -1
\] |

**Table C1**
|   | I = \[
|   | \begin{bmatrix}
|   | 1 & 0 & 0 \\
|   | 0 & 1 & 0 \\
|   | 0 & 0 & 1 \\
|   | \end{bmatrix} | R_1 = \[
|   | \begin{bmatrix}
|   | 1 & -1 & 0 \\
|   | 0 & 1 & 1 \\
|   | 0 & -1 & 0 \\
|   | \end{bmatrix} | S_1 = \[
|   | \begin{bmatrix}
|   | 1 & 0 & -1 \\
|   | 0 & 0 & -1 \\
|   | 0 & -1 & 0 \\
|   | \end{bmatrix} | M_1 = \[
|   | \begin{bmatrix}
|   | 1 & 0 & -1 \\
|   | 0 & 1 & -1 \\
|   | 0 & 0 & -1 \\
|   | \end{bmatrix} | \Omega^* Q_x = \[
|   | \begin{bmatrix}
|   | 0 & -1 & 0 \\
|   | -1 & 0 & 0 \\
|   | 1 & 1 & 0 \\
|   | \end{bmatrix} |
|   | \[
|   | X = \[
|   | \begin{bmatrix}
|   | 0 & 1 & -1 \\
|   | 1 & 0 & -1 \\
|   | 0 & 0 & -1 \\
|   | \end{bmatrix} | R_2 = \[
|   | \begin{bmatrix}
|   | 0 & 0 & -1 \\
|   | 0 & 1 & -1 \\
|   | 1 & 0 & -1 \\
|   | \end{bmatrix} | S_2 = \[
|   | \begin{bmatrix}
|   | -1 & 0 & 1 \\
|   | -1 & 1 & 0 \\
|   | -1 & 0 & 0 \\
|   | \end{bmatrix} | M_2 = \[
|   | \begin{bmatrix}
|   | 1 & -1 & 0 \\
|   | 0 & 1 & 0 \\
|   | 0 & -1 & 1 \\
|   | \end{bmatrix} | \Omega^* Q_x = \[
|   | \begin{bmatrix}
|   | -1 & 0 & 1 \\
|   | -1 & 0 & 0 \\
|   | -1 & 1 & 0 \\
|   | \end{bmatrix} |
|   | \[
|   | Y = \[
|   | \begin{bmatrix}
|   | 0 & -1 & 1 \\
|   | 0 & -1 & 0 \\
|   | 1 & -1 & 0 \\
|   | \end{bmatrix} | R_3 = \[
|   | \begin{bmatrix}
|   | -1 & 1 & 0 \\
|   | -1 & 0 & 0 \\
|   | -1 & 1 & 0 \\
|   | \end{bmatrix} | S_3 = \[
|   | \begin{bmatrix}
|   | 0 & -1 & 0 \\
|   | 1 & -1 & 0 \\
|   | 0 & -1 & 1 \\
|   | \end{bmatrix} | M_3 = \[
|   | \begin{bmatrix}
|   | -1 & 1 & 0 \\
|   | -1 & 0 & 0 \\
|   | -1 & 0 & 1 \\
|   | \end{bmatrix} | \Omega^* Q_y = \[
|   | \begin{bmatrix}
|   | -1 & 1 & 0 \\
|   | -1 & 0 & 1 \\
|   | -1 & 0 & 0 \\
|   | \end{bmatrix} |
|   | \[
|   | Z = \[
|   | \begin{bmatrix}
|   | -1 & 0 & 0 \\
|   | -1 & 0 & 1 \\
|   | -1 & 1 & 0 \\
|   | \end{bmatrix} | R_4 = \[
|   | \begin{bmatrix}
|   | 0 & 0 & 1 \\
|   | 1 & 0 & 0 \\
|   | 0 & 1 & 0 \\
|   | \end{bmatrix} | S_4 = \[
|   | \begin{bmatrix}
|   | 0 & 1 & 0 \\
|   | 0 & 0 & 1 \\
|   | 1 & 0 & 0 \\
|   | \end{bmatrix} | M_4 = \[
|   | \begin{bmatrix}
|   | -1 & 0 & 0 \\
|   | -1 & 1 & 0 \\
|   | -1 & 0 & 1 \\
|   | \end{bmatrix} | \Omega^* Q_y = \[
|   | \begin{bmatrix}
|   | 0 & 0 & -1 \\
|   | 1 & 0 & -1 \\
|   | 0 & 1 & -1 \\
|   | \end{bmatrix} |
|   | \[
|   | M_5 = \[
|   | \begin{bmatrix}
|   | 0 & 0 & 1 \\
|   | 0 & 1 & 0 \\
|   | 1 & 0 & 0 \\
|   | \end{bmatrix} | \Omega^* Q_z = \[
|   | \begin{bmatrix}
|   | 0 & -1 & 1 \\
|   | 1 & -1 & 0 \\
|   | 0 & -1 & 0 \\
|   | \end{bmatrix} |
|   | \[
|   | M_6 = \[
|   | \begin{bmatrix}
|   | 0 & 1 & 0 \\
|   | 1 & 0 & 0 \\
|   | 0 & 0 & 1 \\
|   | \end{bmatrix} | \Omega^* Q_z = \[
|   | \begin{bmatrix}
|   | 0 & 1 & -1 \\
|   | 0 & 0 & -1 \\
|   | 1 & 0 & -1 \\
|   | \end{bmatrix} |
|   | \[
|   | \text{Table C2}
APPENDIX D
AMPLEFORTH MATHEMATICS

A 'pilot' class was formed in the summer term of 1986 and consisted of a set of 16 pupils, including 3 thirteen-year-olds, all of whom had obtained grade A in a fourth year GCE O level entry. Some of the author's suggestions for the use of the language of Linear Algebra as *sums of multiples* of vector elements in mathematics had already been tried out in live classroom situations. Emphasis at this stage of the work was on a change of teaching style at fifth-form-remove and sixth-form level and a re-think of selected topics and how to teach them in the light of technology now available for use in the mathematics classroom. The idea of linearity was to link together the topics to be taught. Some of the work then was considered for sixth-form sets in September 1986 and urgent consideration was given to the preparation of some standard micro-based investigations of suitable topics for all sixth-form sets in 1986-7.

The work has been developed further with the 1987-8 peer set of post-O-level pupils who will be the first candidates for the newly proposed A-level examination, if approved, in 1990 (see §11.3 of Chapter 11).

There now follows an edited transcript of the sound-track of one of these lessons, all of which have been audio-recorded [S10]. As far as possible, blackboard and microcomputer illustrations have been reproduced. The lesson was presented by T M Vessey (Head of Mathematics, Ampleforth College). It is perhaps not surprising that, in lessons 1 and 2, the approach was rather tentative. The presenter was obviously feeling his way gradually towards a more open-ended style of teaching with senior pupils, whereas the pupils were having to respond and actively participate in a way that is almost unknown at this level of the curriculum. In lesson 3, which took place on 19 May 1986, both class and teacher literally 'took off' as is evident from the extract from the recording:
TMV: Where do you think the right place might be? I give you no marks for guessing.

Pupil: LINE 2000

TMV: Where do you think? (laughter) My goodness, you are on form this evening! That's good stuff!!

Lesson 3: The Radical Axis

In lesson 2, the pupils had been introduced to the method for finding the common chord of a pair of intersecting circles $S_1$ and $S_2$, from the linear relationship

$$S_1 + \lambda S_2 = 0$$

for $\lambda = -1$. The problem "What happens if the circles do not intersect?" was posed and the following investigation was set for 'prep':

(i) Find the equation of the common chord (TMV: "There is one!") of the 2 circles:

$$S_1: (x - 1)^2 + (y - 4)^2 - 25 = 0$$
$$S_2: (x - 5)^2 + (y + 1)^2 - 16 = 0$$

(ii) Replace $S_2$ by the circle:

$$S_3: (x - 8)^2 + (y + 4)^2 - 16 = 0$$

and find the 'common chord' of $S_1$ and $S_3$. Where does it turn out to be?

Lesson 3 started with a lively discussion of the homework. Algebraic 'slips' were teasingly glided over:

TMV: To most people, four 4's are sixteen; to one or two of you it is 8; but it does help if you put 16!

And again, we've got the odd 10 coming in here: five 5's actually are 25!!
Several of you quite happily did that and then spoilt it because you forgot that there was a minus sign outside the brackets...

ending with the words:

So we end up with: $4x - 5y - 9 = 0$
So you all in fact know what to do but some of you made mistakes.

The following diagram appeared on the blackboard:

![Diagram of a circle and a line]

**Fig D1(i)**

This delightfully happy, light-hearted and relaxed discussion continued with "the interesting bit of 'prep'" where the pupils "found it a bit difficult". A quick revision of the evaluation of the centre and radius for the general circle:

$$S: x^2 + y^2 + 2gx + 2fy + c = 0$$
with centre \((-g, -f)\) and radius equal to \(\sqrt{g^2 + f^2 - c}\) was followed with the insertion in the diagram [Fig D1 (ii)] of \(S_3\) and of the 'common chord':

\[7x - 8y - 36 = 0\]

of \(S_1\) and \(S_3\) which, understandably few pupils had actually obtained and, the author suspects, few were expected to do so.

\[\text{Fig D1(ii)}\]
The lesson continued:

TMV: Pictorially, the problem I have set you is this: there is a circle (points to $S_1$); okay? - out of the blue; here is another circle (points to $S_3$) which doesn't even cross it, but you found the equation of a line which is supposed to be where they meet. And, had you sketched it, you would have found that it seemed to lie somewhere about there (points to the line: $7x - 8y - 36 = 0$) for some curious reason. One or two of you got as far as saying "it does also I notice seem to be at right angles to the line joining the two centres". That's a start anyway. And one or two of you made the suggestion last week that maybe it is sort of halfway between the gap here, I think somebody said, or maybe its something to do with where the tangents meet. And you could have a lot of other ideas and they may or may not have been right. The next step is fairly typical. We have a very general problem here. I don't want a particular answer. I want something which holds for all situations.

The pupils readily agreed that, if the properties of the two circles were to be examined, then the general equation of the circle in the form: $x^2 + y^2 + 2gx + 2fy + c = 0$ is very "top heavy and very difficult". So the lesson continued:

TMV: What methods could I use?
I have made it fairly clear that we wish to use coordinate geometry methods. But if you are going to do that you will need some axes. Does it matter where I put the axes?

Pupil: No!

TMV: No: it won't affect the properties of the circles. They will remain invariant. Just because I happen to draw the axes there, or somewhere else, won't affect
any relationship between those axes. So I can say, well, in that case let us try and choose somewhere which makes my algebra easy. And I'm going to say, why don't we make that line (points to the line-of-centres) the x-axis because that will mean that the centre of the circle has a y-coordinate zero which will be easier for me. And I've done that just here as you can see (draws two circles whose centres are (-3, 0) and (4, 0) with radii 2 and 3 respectively); there's nothing special about that. I've just written down a couple of circles whose centres are at (-3, 0) and (4, 0) so they lie on the x-axis. And this has got a radius of 2 and this a radius of 3. If you get your imagination working before I draw, I think you can see that they don't meet. (Draws a diagram on the blackboard; the y-axis has been left out deliberately)

Let us find out where this line is, which we are interested in; and then, with the help of 'this little chappie' here (points to a microcomputer) we can see what's going on.
The equations of the circles were obtained as:

\[(1) \quad (x + 3)^2 + y^2 - 4 = 0; \quad (2) \quad (x - 4)^2 + y^2 - 9 = 0.\]

and the line of interest has equation:

\[\{(x + 3)^2 + y^2 - 4\} - \lambda ((x - 4)^2 + y^2 - 9) = 0\]

with \(\lambda = -1\). The lesson continues:

TMV: Firstly, if I choose \(\lambda = -1\) I shall get this line. Secondly, if I choose any other value of \(\lambda\) - please note because that is what I am going to ask you to do in a moment - I shall get a circle and, fortunately for you, I can get the machine to do all that. It will work out for you where the centre and radius are... So, what about this line? We put a minus one in there, don't we, and see what happens - like that (writes up the information). I've got

\[\{x^2 + 6x + 9 + y^2 - 4\} - \{x^2 - 8x + 16 + y^2 - 9\}\]

which comes to nothing. All this disappears and - just check that I don't make a mistake -

6x and plus 8x looks to me like 14x.

There is no y-term which, on reflection, of course is correct, isn't it?

Pupil: Yes it is, Sir!

TMV: The line has got to be at right angles to the axis so it is not a surprising result: \(x = \frac{1}{7}\) (inserts the line \(x = \frac{1}{7}\) in the diagram on the blackboard (Fig D2)).

A strange sort of place because these are quite nice numbers: I think this (points to circle on blackboard) has got its centre at (4, 0). Why should there be a line down there? What is it?
The following exposition is a keynote of this new style of teaching at sixth-form level.

TMV: What I'm showing you now is, I would like to think, in due course the sort of thing you would do when someone like me says to you:

"Go away and find out what it is".

Not that you are going to sit down and say: "nobody's told me what to do". No! - just go and play at it. Now I happen to know what the answer is, but that's not the point. The point is: how do I play this?

Well! Let's just have lots of circles: see where they all are as I vary $\lambda$ and see whether it's got anything to do with this line. Okay! So the programme that I am going to run is simply going to ask me to put in values of $\lambda$ and it will draw the circle. Right! Let's do that. At least I hope it is!

The reader can see a printout of the programme on page D(ix) (Printout D1). The discussion continues:

Pupil: Do you already put those equations into the program?

TMV: Yes; those equations are already in there. I've done that. It's equation number (1) at the top (of the blackboard) [see page D(vii)] and it's drawn that circle.

A sequence of values of $\lambda$ are then fed in (Printout D1 on page D(ix)). A lot of exciting discussion took place: for example

TMV: I'll put $\lambda = 2$: 2 is a fairly innocent sort of result and it says "No". It's rejected that. Why should it do something like that?

Pupil: It's one of those circles with a negative square root.
5 REM.TMV.3
10 CLS: MODE 1
20 GCOL 0,129: CL8
30 MOVE 50,500
35 GCOL 0,3
40 PLOT 5,1250,500
50 INPUT L
60 R=((4*L-3)/(1+L))^2-(5+7*L)/(1+L)
70 IF R<0 PRINT "No circle": GOTO 50
80 R = (INT(SQR(R)*10))/10
90 C = (4*L-3)/(1+L)
100 R = R*50: C = C*50
120 MOVE 600+C+R,500
130 GCOL 0,2
140 FOR T=1 TO 360 STEP 3
150 P = T*3.142/180: X = R*COS(P): Y = R*SIN(P)
160 X = INT(X): Y = INT(Y)
170 PLOT 5,600+C+X,500+Y
180 NEXT T
190 INPUT "Again Y/N"; A$
200 IF A$="Y" GOTO 50
210 END
That's right: in fact, if you put \( \lambda = 2 \) and you go through all the working you find the radius is the square root of a negative number, which is impossible. So that says "No! You cannot have \( \lambda = 2 \)." So whilst in theory I could put \( \lambda \) equal to anything I like, in practice it produces some imaginary circles which are not really there.

When \( \lambda = 20 \) was entered:

TMV: Not concentric are they?

Pupil: No!

TMV: Does that worry you a bit?

-21 produced a result which went 'off the screen'*(Printout D5(ii)). (see p. D(xx))

It was then suggested that the pupils should tabulate the results (otherwise it is easy to get carried away with the program, with pretty pictures etc, and to lose track of what is happening) corresponding to selected values of \( \lambda \), as L(eft) circles and R(ight) circles. It was thought that a third column I(mpossibles) was needed:

<table>
<thead>
<tr>
<th>L</th>
<th>R</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 0 )</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-21</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table D1

The discussion continued:

TMV: I am now going to want you to suggest some numbers when you think you can begin to see what happens.

What is in the L(eft-hand) column?

---

*Scrolling of the screen may be prevented by changing LINE 50 of Printout D1 to read: 50 INPUT L: PRINT TAB (1,1)" (Printout D2)
5 REM.TMV.3
10CLS:MODE 1
20GCOL 0,129:CLS
30MOVE 50,500
35GCOL 0,3
40PLOT 5,1250,500
50 INPUT L:PRINT TAB(1,1);TAB(1,1);
60 R=((4*L-3)/(1+L))^2-(5+7*L)/(1+L)
70IF R<0 PRINT"No circle":GOTO 50
80R = (INT(SQR(R)*10))/10
90C = (4*L-3)/(1+L)
100R = R*50:C = C*50
120MOVE 600+C+R,500
130GCOL 0,2
140FOR T=1 TO 360 STEP 3
150P = T*3.142/180:X = R*COS(P);Y = R*SIN(P)
160X = INT(X):Y = INT(Y)
170PLOT 5,600+C+X,500+Y
180NEXT T
190INPUT "Again Y/N";A$
200 IF A$="Y" GOTO 50
210END

Printout D2
D(xii)

Well! there's only a zero at the moment, but presumably we must get some more circles over here (points to left-hand side of display...)... its pure guesswork ... you've nothing else to go on at the moment.
You are trying to build up a picture. So, over to you: you suggest something.

Pupil: Positive 21

TMV: Positive 21: you probably can't just see it, but it's just 'trinkling' around inside there and on top of the other one ($\lambda = 20$)
So that's not working: so that's the Right-hand. Right!

Pupil: Point 1

TMV: Point 1! Ah! Success, success!! Tichy little thing. It's on there, you see. It's just about hitting the centre of the very first circle, isn't it. That was point 1 - Left-hand side.
No! Plus point 1.
Sorry! Negative point 1.
No! that's on the Left as well.
I see

Pupil: Negative 9

TMV: Negative point 9. Oh well, why?
Think! It's near negative 1.
We must be getting towards that line.
We have actually got a circle but the centre is...

Pupil: It's not completed.

There follows a delightful discussion about a circle "with a huge radius" which has "just got down to the cricket pitch and come back up here (TMV points to screen) again". The exposition includes a brief description of the program which
draws the circumference of a circle in a sequence of steps subtending angles of 3 degrees at the centre (LINE 140). The boys are all eagerly contributing to expectations for:

\[
\lambda = 1.9, \ 4.9, \ .9, \ 1.9
\]

Table D1 is now extended (Table D2).

<table>
<thead>
<tr>
<th>L</th>
<th>R</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>\lambda = 0</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\lambda = -21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>21</td>
<td>1.9</td>
</tr>
<tr>
<td>-.1</td>
<td>4.9</td>
<td>.9</td>
</tr>
<tr>
<td>-.9</td>
<td>-1.9</td>
<td></td>
</tr>
</tbody>
</table>

Table D2

and the screen display appeared as on page D(xx) (Printout D5(ii)).

The pupils were next invited to do a "standard bit of research". It was suggested that the L(eft) and R(ight) circles should be indicated by their corresponding \lambda-values as points on a number line, thus:

\[
\begin{array}{ccccccc}
\text{R} & \text{R} & \text{R} & \text{L} & \text{L} & \text{L} & \text{R} \\
-3 & -2 & -1 & 0 & 1 & 2 & 3
\end{array}
\]

The sequence of suggestions for \lambda continues. Table D2 is again extended to Table D3.
D(xiv)

\[
\begin{array}{ccc}
L & R & I \\
\lambda = & 0 & 7 & 2 \\
& .1 & 20 & 1.9 \\
& -.1 & -21 & .9 \\
& -.9 & 21 & \\
& 4.9 & \\
& -1.9 & \\
\end{array}
\]

\[
\begin{array}{ccc}
.111 & 1.1 & \\
.5 & .2 & \\
.3 & 3 & \\
.4 & .14 & \\
\end{array}
\]

Table D3

and on to:

Pupil: .111

TMV: Ah - yes! \( \lambda = .111 \) (gives a circle 'trinkling' round .11)

But, here, the display started to move off the screen and circles were drawn in the wrong place. Nothing daunted and, encouraged by the 'electronic' atmosphere in the classroom, the discussion continued:

TMV: What shall we try now?

Pupil: New screen!

TMV: Yes, I'll do that (laughter)... you are on the ball tonight!!

What number next? - because we are actually winning somewhere.

What do you think I'm trying to get you to put?

What interesting numbers are we trying to get at?
Yes? You said it — at least I think you did. I was interested in the small circle. Well! — where are we going if the circle is getting smaller and closer to a point? In other words, can we head for a zero radius? What kind of a number do you think?

Pupil: .1111

The screen was cleared and with a little prompting, because time was running out for this 40-minute only lesson, for values in order of size and great excitement as circles got very small, the Table D3 was extended again to Table D4.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>L</th>
<th>R</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>.1</td>
<td>20</td>
<td>1.9</td>
<td></td>
</tr>
<tr>
<td>-.1</td>
<td>-21</td>
<td>.9</td>
<td></td>
</tr>
<tr>
<td>-.9</td>
<td>21</td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>.11</td>
<td>4.9</td>
<td>.5</td>
<td></td>
</tr>
<tr>
<td>-1.9</td>
<td>.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.111</td>
<td>10</td>
<td>.119</td>
<td></td>
</tr>
<tr>
<td>.112</td>
<td>5</td>
<td>3.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table D4**

The circle corresponding to \( \lambda = 5 \) was again seen 'trinkling' around that for \( \lambda = 4.9 \). The L(eft) and R(ight) circles were again indicated on the '\( \lambda \)' number-line:
and the following DISPLAY appeared on the screen (Printout D6(i)). (see p.D(xxi))

It was evident that there was a whole family of circles any one of which could be obtained by combining any two given circles linearly by means of a scalar multiple 'λ'. By getting the correct value λ, a point circle with zero radius could be obtained. There are in fact two such point circles. The pupils had 'discovered' the existence of the limiting points (near 3.9 and .113) of a system of coaxal circles.

Notwithstanding a further error in the program at this first attempt on the part of the presenter to design his own program and to 'experiment' with it - live - with a class in front of a microphone, every effort was made to insert, by approximating to, the radical axis by means of values of λ = -1*

It was further 'discovered' that no value of λ could be found to make the circles intersect. The L(eft) and R(ight) circles were separated by a region of I(maginary) circles. Each L(eft) circle was inside another: no two L(eft) circles intersected. Similarly, no two R(ight) circles intersected. L(eft) and R(ight) circles were separated by a region of I(maginary) circles with the radical axis somewhere in that 'imaginary' region and acting as a 'boundary' between the two sets of L(eft) and R(ight) circles.

Finally for 'prep.' the pupils were asked to consolidate the experience by writing about half a page on what the investigation had been all about and what results they thought they had achieved.

* Further modification of the program by the author, assisted by Dr C P Waddington of the University of Hull appears in Printout D4 to produce graphic displays in Printouts D5, D6
5 REM. TMV.3
6 REM. COAXAL_CIRCLES
7 REM. AM/RKT3
8 REM. AM/RT3'
9 REM. RAXAD
10CLS:MODE 1
20GCOL 0,129:CLS
30MOVE 50,500
35 GCOL 0,3
40PLOT 5,1250,500
50 INPUT L;PRINT TAB(1,1);"\$TAB(1,1);")
60 R=((4*L-3)/(1+L))^2-(5+7*L)/(1+L)
70IF R<0 PRINT"No circle";GOTO 50
80 R = (INT(SQR(R)*10))/10
90 C = (4*L-3)/(1+L)
100 R = R*50:C = C*50
120 MOVE 600+C+C,500
130 GCOL 0,2
140 FOR T=1 TO 360 STEP 3
150 P = T*3.142/180:X = R*COS(P);Y = R*SIN(P)
160 X = INT(X);Y = INT(Y)
170 PLOT 5,600+C+X,500+Y
180 NEXT T
190 INPUT. "Again ? Y/N";A$
200 IF A$="Y" GOTO 50
205 INPUT "Radical Axis, Y/N";A$
207 IF A$="Y" MOVE 600+50/7,0
208 PLOT 5,600+50/7,1000
210 END

Printout D3
Printout D4 (continued)
0
Again Y/N? Y
?2
No circle
?1.9

Again Y/N? Y
?7
Again Y/N? Y
?1.1
Again Y/N? Y
?-1
Again Y/N? Y
?-0.9
Again Y/N? Y
?-1.9
Again Y/N? Y
?1.1
No circle
?21
Again Y/N? Y
?.1
Again Y/N? Y
?-1
Again Y/N? Y
?-0.9
Again Y/N? Y
?-1.9
Again Y/N? Y
?4.9
Again Y/N? Y
?0.9
No circle
?-1.9
Again Y/N? Y
?1.1
No circle

Printout D5
No circle

Again Y/N?Y

Again Y/N?Y

Again Y/N?Y

Again Y/N?Y

Again Y/N?Y

Radical Axis Y/N?Y

Printout D6
Selected Topics

Audio recordings with microcomputer illustration have been made of lessons on the following further topics [S10]*:

(i) approximations: eg \( \sin \theta \approx \theta \) near the origin

(ii) change of scale: to determine the value of the radian unit of angular measurement

(iii) the gradient function

(iv) the shape of the cubic and extension to polynomials of higher degree

(v) periodic functions: eg \( y = a \sin bx + c \cos bx \)

(vi) all parabolas have the same shape

(vii) friction

Further Topics in Preparation

(i) difference methods

(ii) step-by-step method for the solution of first order differential equations

(iii) rounding errors

(iv) truncating errors

(v) iterative processes

(vi) linear interpolation

(vii) step-by-step method for the solution of second order differential equations

* Unedited transcripts of some of the recordings are submitted as Exhibits
D(xxiii)

(viii) group theory up to, and including, Lagrange's Theorem
(ix) geometry of 3-dimensions with particular reference to
the Platonic solids and their symmetry groups

Audio-visual recordings of some of this work are being made
during the Autumn Term 1987.