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# Optimal experimental design in structural dynamics

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**ABSTRACT:** Theoretical and computational issues arising in experimental design for model identification and parameter estimation in structural dynamics are addressed. The objective is to optimally locate sensors in a structure such that the resulting measured data are most informative for estimating the parameters of a family of mathematical model classes used for structural modeling. The information entropy, measuring the uncertainty in the parameters of a structural model class, is used as a performance measure of a sensor configuration. For a single model class, the optimal sensor location problem is formulated as an information entropy minimization problem. For model class selection and/or damage detection applications, the problem is formulated as a multi-objective optimization problem of finding the Pareto optimal sensor configurations that simultaneously minimize appropriately defined information entropy indices related to multiple model classes and/or probable damage scenarios. Asymptotic estimates for the information entropy, valid for large number of measured data, are presented that rigorously justify that the selection of the optimal experimental design can be based solely on the nominal structural model from a class, ignoring the details of the measured data that are not available in the experimental design stage. The effect of the measurement and model prediction error variances on the optimal sensor location design is examined. Finally, heuristic algorithms are proposed for constructing effective sensor configurations that are superior, in terms of accuracy and computational efficiency, to the sensor configurations provided by genetic algorithms.

**Keywords:** Structural identification, experimental design, information entropy, sensor placement, Pareto optima.

## 1 INTRODUCTION

Structural model identification using measured dynamic data has received much attention over the years because of its importance in structural model updating, health monitoring, damage detection and control. The quality of information that can be extracted from the measured data for structural identification purposes depends on the type, number and location of sensors. The objective in this work is to optimise the number and location of sensors in the structure such that the resulting measured data are most informative for estimating the parameters of a family of mathematical model classes used for structural identification and damage detection.

Information theory based approaches (e.g. Shah & Udawadia 1978, Sobczyk 1987, Kammer 1991, Kirkegaard & Brincker 1994, Udawadia 1994, Heredia-Zavoni & Esteva 1998, Heredia-Zavoni et al. 1999) have been developed to provide rational solutions to several issues encountered in the problem of selecting the optimal sensor configuration. In Shah & Udawadia (1978), Kammer (1991), Kirkegaard & Brincker (1994) and Udawadia (1994), the optimal

sensor configuration is taken as the one that maximizes some norm (determinant or trace) of the Fisher information matrix (FIM). Heredia-Zavoni & Esteva (1998), Heredia-Zavoni et al. (1999) treat the case of large model uncertainties expected in model updating. The optimal sensor configuration is chosen as the one that minimizes the expected Bayesian loss function involving the trace of the inverse of the FIM for each model.

Papadimitriou et al. (2000), introduced the information entropy norm as the measure that best corresponds to the objective of structural testing, which is to minimize the uncertainty in the model parameter estimates. Specifically, the optimal sensor configuration is selected as the one that minimizes the information entropy measure since it gives a direct measure of this uncertainty. It has been shown (Papadimitriou 2004) that, asymptotically for very large number of data, the information entropy depends on the determinant of the Fisher information matrix. An important advantage of the information entropy measure is that it allows us to make comparisons between sensor configurations involving a different number of sensors in each configuration. Fur-

thermore, it has been used to design the optimal characteristics of the excitation (e.g. amplitude and frequency content) useful in the identification of linear and strongly nonlinear models (Metallidis et al. 2003).

The optimal sensor placement strategies depend on the class of mathematical models selected to represent structural behavior as well as the model parameterization within the model class. However, it is often desirable to use the measured data for selecting the most appropriate model class from a family of alternative model classes chosen by the analyst to represent structural behavior. Such classes may be linear (modal models or finite element models), nonlinear elastic or inelastic, each one involving different number of parameters. Model class selection is also important for damage detection purposes for which the location and severity of damage are identified using a family of model classes with each model class monitoring a specific region in a structure (Sohn & Law 1997, Papadimitriou & Katafygiotis) or incorporating different mechanisms of damage. The information entropy-based optimal sensor location methodology has been extended in (Papadimitriou 2004) to design optimal sensor locations for updating multiple model classes useful for damage detection purposes.

In this work, the problem of optimally placing the sensors in the structure is revisited and the information entropy approach is used to design the optimal sensor configurations for two type of problems: (i) identification of structural model (e.g. finite element) parameters or modal model parameters (modal frequencies and modal damping ratios) based on acceleration time histories, and (ii) identification of structural model parameters based on modal data. Asymptotic estimates for the information entropy, valid for large number of measured data, are presented that rigorously justify that the selection of the optimal experimental design can be based solely on the nominal structural model from a class, ignoring the details of the measured data that are not available in the experimental design stage. Analytical expressions are developed that show the relative effect of model and measurement error on the design of the optimal sensor configuration. In particular, the expressions developed for the information entropy are specialized for the case of modal model identification using response time histories. The methodology is also extended for addressing the problem of optimally locating the sensors in the structure such that the resulting measured data are most informative for estimating the parameters of multiple model classes. Finally, a summary of heuristic algorithms is presented for constructing effective sensor configurations that are superior, in terms of accuracy and computational efficiency, to the sensor configurations provided by genetic algorithms (Bedrossian & Masri 2003, Abdullah et al. 2001, Yao et al. 1993,

Papadimitriou 2002), suitable for solving the resulting single and multi-objective optimization problems. Results on a four-span bridge structure are used to illustrate some of the theoretical developments.

## 2 STRUCTURAL IDENTIFICATION METHODOLOGY

Consider a parameterized class  $M$  of structural models (e.g. a class of finite element models or a class of modal models) chosen to describe the input-output behavior of a structure. Let  $\theta \in R^{N_\theta}$  be the vector of free parameters (physical or modal parameters) in the model class. A Bayesian statistical system identification methodology (Beck & Katafygiotis 1998, Katafygiotis et al. 1998) is used to estimate the values of the parameter set  $\theta$  and their associated uncertainties using the information provided from dynamic test data. For this, the uncertainties in the values of the structural model parameters  $\theta$  are quantified by probability density functions (PDF) that are updated using the dynamic test data. The updated PDF is then used for designing the optimal sensor configuration.

### 2.1 Identification Based on Response Time History Data

Let  $D = \{\hat{x}_j(k\Delta t) \in R^{N_0}, j = 1, \dots, N_0, k = 1, \dots, N_D\}$  be the measured sampled response time history data from a structure, consisting of acceleration, velocity or displacement response at  $N_0$  measured DOFs, where  $N_D$  is the number of the sampled data using a sampling rate  $\Delta t$ . The measured DOFs are usually referred to translational DOFs. Let also  $\{x_j(k; \theta) \in R^{N_d}, j = 1, \dots, N_d, k = 1, \dots, N_D\}$ , where  $N_d$  is the number of model degrees of freedom (DOF), be the predictions of the sampled response time histories obtained from a particular model in the model class  $M$  corresponding to a specific value of the parameter set  $\theta$ . The prediction error  $e_j(k)$  between the sampled measured response time histories and the corresponding response time histories predicted from a model, for the  $j$ th measured DOF and the  $k$ th sampled data, is given by the prediction error equation

$$e_j(k) = \hat{x}_j(k) - x_j(k; \theta) \quad (1)$$

where  $j = 1, \dots, N_0$  and  $k = 1, \dots, N_D$ . The predictions errors at different time instants are modeled by independent (identically distributed) zero-mean Gaussian variables. Specifically, the prediction error  $e_j(k)$  for the  $j$ th measured DOF is assumed to be a zero mean Gaussian variable,  $e_j(k) \sim N(0, \sigma_j^2)$  with

variance  $\sigma_j^2$ . The model prediction error is due to modeling error and measurement noise.

Applying the Bayesian system identification methodology (Papadimitriou et al. 2000, Papadimitriou 2004), assuming independence of the prediction errors  $e_j(k)$ , the updating PDF  $p(\theta, \sigma | D)$  of the parameter sets  $\theta$  and  $\sigma = (\sigma_1, \dots, \sigma_{N_0})$ , given the measured data  $D$  and the class of models  $M$ , takes the form:

$$p(\theta, \sigma | D) = \frac{\tilde{c}}{(\sqrt{2\pi})^{N_D N_0} \rho(\sigma)} \exp\left\{-\frac{N_D N}{2} J(\theta; \sigma)\right\} \pi_\theta(\theta) \pi_\sigma(\sigma) \quad (2)$$

where

$$J(\theta; \sigma, D) = \frac{1}{N_0} \sum_{j=1}^{N_0} \frac{1}{\sigma_j^2} J_j(\theta), \quad J_j(\theta) = \frac{1}{N_D} \sum_{k=1}^{N_D} [\hat{x}_j(k) - x_j(k; \theta)]^2 \quad (3)$$

is the overall weighted measure of fit between measured and model predicted response time histories for all measured DOFs,  $\rho(\sigma) = \prod \sigma_j^{N_D}$  is a scalar function of the prediction error parameter set  $\sigma$ ,  $\pi_\theta(\theta)$  and  $\pi_\sigma(\sigma)$  are the prior distribution for the parameter sets  $\theta$  and  $\sigma$ , respectively,  $N = N_0$  and  $\tilde{c}$  is a normalizing constant chosen such that the PDF in (2) integrates to one.

## 2.2 Identification Based on Modal Data

The methodology is next extended to the case where the dynamic data consist of modal data. Let  $D = \{\hat{\omega}_r^{(k)}, \hat{\phi}_r^{(k)} \in R^{N_0}, r = 1, \dots, m, k = 1, \dots, N_D\}$  be the measured modal data from a structure, consisting of modal frequencies  $\hat{\omega}_r^{(k)}$  and modeshape components  $\hat{\phi}_r^{(k)}$  at  $N_0$  measured DOFs, where  $m$  is the number of observed modes and  $N_D$  is the number of modal data sets available. Let also  $\{\omega_r, \theta, \phi_r \in R^{N_d}, r = 1, \dots, m\}$  be the predictions of the modal frequencies and modeshapes obtained for a particular value of the model parameter set  $\theta$  by solving the eigenvalue problem corresponding to the model mass and stiffness matrices.

The prediction error  $e_r^{(k)} = [e_{\omega_r}^{(k)} \ e_{\phi_r}^{(k)}]$  between the measured modal data and the corresponding modal quantities predicted by the model is given separately for the modal frequencies and the modeshapes by the prediction error equations:

$$e_{\omega_r}^{(k)} = \hat{\omega}_r^{(k)} - \omega_r(\theta) \quad \text{and} \quad e_{\phi_r}^{(k)} = \hat{\phi}_r^{(k)} - \beta_r^{(k)} L_0 \phi_r(\theta) \quad (4)$$

$r = 1, \dots, m$ , where  $e_{\omega_r}^{(k)}$  and  $e_{\phi_r}^{(k)} \in R^{N_d}$  are respectively the prediction errors for the modal frequency and modeshape components of the  $r$ -th mode,  $k = 1, \dots, N_D$ ,  $\beta_r^{(k)} = \hat{\phi}_r^{(k)T} \phi_r / \phi_r^T \phi_r$  is a normalization constant that accounts for the different scaling between the measured and the predicted modeshape,

and  $L_0$  is a  $N_0 \times N_d$  matrix of ones and zeros that maps the model DOFs to the measured degrees of freedom. The model prediction error is due to modeling error and measurement noise.

The prediction error  $e_{\omega_r}^{(k)}$  for the  $r$ -th modal frequency is assumed to be a zero mean Gaussian variable,  $e_{\omega_r}^{(k)} \sim N(0, \sigma_{\omega_r}^2 \hat{\omega}_r^{(k)2})$ , with standard deviation  $\sigma_{\omega_r} \hat{\omega}_r^{(k)}$ . The prediction error for the  $r$ -th truncated modeshape vector  $e_{\phi_r}^{(k)} \in R^{N_0}$  is also assumed to be zero mean Gaussian vector,

$$e_{\phi_r}^{(k)} \sim N(\mathbf{0}, C_{\phi_r}^{(k)}),$$

with diagonal covariance matrix

$$C_{\phi_r}^{(k)} = \sigma_{\phi_r}^2 \|\hat{\phi}_r^{(k)}\|_{N_0}^2 I \in R^{N_0 \times N_0},$$

where  $\|\hat{\phi}_r^{(k)}\|_{N_0}^2 = \|\hat{\phi}_r^{(k)}\|^2 / N_0$ ,  $\|\cdot\|$  is the usual Euclidean norm and  $I$  is the identity matrix.

Applying the Bayesian identification, assuming independence of the prediction errors  $e_{\omega_r}^{(k)}$  and  $e_{\phi_r}^{(k)}$ , the updating PDF  $p(\theta, \sigma | D)$  of the parameter sets  $\theta$  and  $\sigma = \{\sigma_{\omega_r}, \sigma_{\phi_r}, r = 1, \dots, m\}$ , given the data  $D$  and the class of models  $M$ , takes the form (2), where

$$J(\theta; \sigma) = \frac{1}{N_D N} \sum_{r=1}^m \left[ \frac{1}{\sigma_{\omega_r}^2} \sum_{k=1}^{N_D} \frac{[\omega_r(\theta) - \hat{\omega}_r^{(k)}]^2}{[\hat{\omega}_r^{(k)}]^2} + \frac{N_0}{\sigma_{\phi_r}^2} \sum_{k=1}^{N_D} \frac{\|\beta_r^{(k)} L_0 \phi_r(\theta) - \hat{\phi}_r^{(k)}\|^2}{\|\hat{\phi}_r^{(k)}\|^2} \right] \quad (5)$$

represents the weighted measure of fit between the measured modal data and the modal data predicted by a particular model within the selected model class, and  $N = m N_0 + 1$  is the number of measured data per modal set.

## 3 OPTIMAL SENSOR LOCATION BASED ON INFORMATION ENTROPY

The marginal updated PDF  $p(\theta | D)$  in (2) specifies the plausibility of each possible value of the structural model parameters. It provides a spread of the uncertainty in the structural model parameter values based on the information contained in the measured data. A unique scalar measure of the uncertainty in the estimate of the structural parameters  $\theta$  is provided by the information entropy, defined by :

$$H(\delta, D) = E_\theta [-\ln p(\theta | D)] = -\int \ln p(\theta | D) p(\theta | D) d\theta \quad (6)$$

The information entropy depends on the available data  $D \equiv D(\delta)$  and the sensor configuration vector  $\delta$ .

An asymptotic approximation of the information entropy, valid for large number of data ( $N_D N \rightarrow \infty$ ), is available (Papadimitriou 2004) which is useful in the experimental stage of designing an optimal sensor configuration. The asymptotic approximation is

obtained by substituting  $p(\boldsymbol{\theta} | D) = \int p(\boldsymbol{\theta}, \boldsymbol{\sigma} | D) d\boldsymbol{\sigma}$  and (2) into (6) and observing that the resulting integral can be re-written as Laplace-type integral which can be approximated by applying Laplace method of asymptotic approximation (Bleistein & Handelsman 1986). Specifically, it can be shown that for a large number of measured data, i.e. as  $N_D N \rightarrow \infty$ , the following asymptotic result holds for the information entropy (Papadimitriou 2004)

$$H(\boldsymbol{\delta}, D) \sim H(\boldsymbol{\delta}; \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}) = \frac{1}{2} N_{\boldsymbol{\theta}} \ln(2\pi) - \frac{1}{2} \ln[\det \mathbf{h}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}; \boldsymbol{\delta})] \quad (7)$$

where  $\hat{\boldsymbol{\theta}} \equiv \hat{\boldsymbol{\theta}}(\boldsymbol{\delta}, D) = \arg \min J(\boldsymbol{\theta}; D)$  is the optimal value of the parameter set  $\boldsymbol{\theta}$  that minimizes the measure of fit  $J(\boldsymbol{\theta}; D)$  given in (3) or (5), and  $\mathbf{h}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}; \boldsymbol{\delta})$  is an  $N_{\boldsymbol{\theta}} \times N_{\boldsymbol{\theta}}$  positive definite matrix defined and asymptotically approximated by

$$\mathbf{h}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}; \boldsymbol{\delta}) = -\nabla_{\boldsymbol{\theta}} \nabla_{\boldsymbol{\theta}}^T \ln[J(\boldsymbol{\theta}; D)]^{-N_D N} \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \sim \mathbf{Q}(\boldsymbol{\delta}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}) \quad (8)$$

as  $N_D N \rightarrow \infty$  in which  $\nabla_{\boldsymbol{\theta}} = [\partial/\partial\theta_1, \dots, \partial/\partial\theta_{N_{\boldsymbol{\theta}}}]^T$  is the usual gradient vector with respect to the parameter set  $\boldsymbol{\theta}$  and  $\hat{\boldsymbol{\sigma}}^2$  is the optimal prediction error variance.

### 3.1 Information Entropy Based on Response Time History Data

For response time history data, substituting (3) into (8) and considering the limiting case  $N_D N \rightarrow \infty$ , the resulting matrix  $\mathbf{Q}(\boldsymbol{\delta}, \boldsymbol{\theta})$  appearing in (8) simplifies to a positive semi-definite matrix of the form

$$\mathbf{Q}(\boldsymbol{\delta}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}) = \frac{N_D}{2} \sum_{j=1}^{N_d} \delta_j \frac{1}{\hat{\sigma}_j^2} \mathbf{P}^{(j)}(\hat{\boldsymbol{\theta}}) \quad (9)$$

known as the Fisher information matrix and containing the information about the values of the parameters  $\boldsymbol{\theta}$  based on the data from all measured positions specified in  $\boldsymbol{\theta}$ , while the optimal prediction error variances  $\hat{\boldsymbol{\sigma}}^2$  are given by  $\hat{\sigma}_j^2 = J_j(\hat{\boldsymbol{\theta}}; D)$ . The matrix  $\mathbf{P}^{(j)}(\boldsymbol{\theta})$  is a positive semi-definite matrix given by

$$\mathbf{P}^{(j)}(\boldsymbol{\theta}) = \sum_{k=1}^{N_D} \nabla_{\boldsymbol{\theta}} x_j(k; \boldsymbol{\theta}) \nabla_{\boldsymbol{\theta}}^T x_j(k; \boldsymbol{\theta}) \quad (10)$$

containing the information about the values of the parameters  $\boldsymbol{\theta}$  based on the data from one sensor placed at the  $j$ -th DOF.

The only dependence of the resulting asymptotic value of the information entropy (7) on the data comes implicitly through the optimal values  $\hat{\boldsymbol{\theta}} \equiv \hat{\boldsymbol{\theta}}(\boldsymbol{\delta}, D)$  and the prediction errors

$\hat{\sigma}_j^2 = J_j(\hat{\boldsymbol{\theta}}; D)$ . Consequently, the information entropy (7) is completely defined by the optimal value  $\hat{\boldsymbol{\theta}}$  of the model parameters and the optimal prediction error  $\hat{\sigma}_j^2 = J_j(\hat{\boldsymbol{\theta}}; D)$ ,  $j=1, \dots, N_0$ , expected for a set of test data, while the time history details of the measured data do not enter explicitly the formulation.

In experimental design, it is desirable to design the sensor configurations such that the resulting measured data are most informative about the parameters of the model class used to represent the structure behavior. Since the information entropy gives the amount of useful information contained in the measured data, the optimal sensor configuration is selected as the one that minimizes the information entropy (Papadimitriou et al. 2000). That is,

$$\boldsymbol{\delta}_{best} = \arg \min_{\boldsymbol{\delta}} H(\boldsymbol{\delta}; \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}) \quad (11)$$

where the minimization is constrained over the set of  $N_p$  measurable DOFs. However, in the initial stage of designing the experiment the data are not available, and thus an estimate of the optimal model parameters  $\hat{\boldsymbol{\theta}}$  and  $\hat{\boldsymbol{\sigma}}$  cannot be obtained from analysis. In practice, useful designs can be obtained by taking the optimal model parameters  $\hat{\boldsymbol{\theta}}$  and  $\hat{\boldsymbol{\sigma}}$  to have some nominal values chosen by the designer to be representative of the system. An analysis of the prediction error variance is next presented.

#### 3.1.1 Prediction Error Variance Models

In order to derive a useful expression for the prediction error variance  $\hat{\sigma}_j^2$ , we assume that the prediction error  $e_j(k)$  in (1) is due to a term,  $e_{j,meas}(k)$ , accounting for the measurement error and a term,  $e_{j,model}(k)$ , accounting for the model error, that is

$$e_j(k) = e_{j,meas}(k) + e_{j,model}(k) \quad (12)$$

Assuming independence between the measurement error and model error, the variance  $\hat{\sigma}_j^2$  of the total prediction error is given in the form

$$\hat{\sigma}_j^2 = \sigma_{j,meas}^2 + \sigma_{j,model}^2 \quad (13)$$

where  $\sigma_{j,meas}^2$  is the variance of the measurement error and  $\sigma_{j,model}^2$  is the variance of the model error. The designer has to assume values for the individual variances in (13). Such assumptions may depend on the nature of the problem analyzed. Specifically, it may be reasonable to assume that the variance of the measurement error is same for all measurements, independent of the level of response. Also, it may as well be reasonable to assume that the variance of the

model error is proportional to the average response strength given by

$$g_j(\hat{\theta}) = (1/N_D) \sum_{k=1}^{N_D} x_j^2(k; \hat{\theta}) \quad (14)$$

In particular, the assumption that the model error is proportional to strength of the response is valid when the response is insensitive to small variations in the sensor location. However, there are problems for which the response is extremely sensitive to very small variations of the measurement location. Such problems, for example, are encountered in the case of measuring strains close to a crack tip. Due to  $1/\sqrt{r}$  variation of the strain distribution, where  $r$  is the distance from the crack tip, small variations in the sensor location, due to inaccurate sensor location, may result in extremely high variations in the response close to the crack tip. In this case, small errors in the location of the sensors may result to large errors for the part of the error that is proportional to the strength of the response. Thus, the sensitivity of the measured response to sensor location may play an important role in defining the measurement error. To properly account for these variations, it is reasonable to assume that the error is a function of the sensitivity of the response to variations in the sensor positions. Usually this error and the corresponding prediction error variance may be considered to be a function of the measured response or its spatial derivatives. Adding all this errors together, one can derive the following expression for the variance of the prediction error

$$\hat{\sigma}_j^2 = J_j(\hat{\theta}; D) = s_1^2 + s_2^2 g_j(\hat{\theta}) + s_3^2 h_j(x_j(\hat{\theta})) \quad (15)$$

where the first term accounts for constant errors, independent of the response, the second term accounts for prediction errors that depend on the strength of the response, and the third term accounts for prediction errors that depend on the details of the response. In practical applications, only the first term has been conveniently used. However, for prediction errors that are due to model errors, the second term seems to be more applicable. The third term appears only in very special cases as in the crack problem mentioned above.

Using the aforementioned analysis and neglecting the third term in (15), the optimal sensor location depends on the optimal model  $\hat{\theta}$  and the values of  $s_1^2$  and  $s_2^2$  assumed for the relative size of measurement and model errors, respectively.

### 3.1.2 Design of Optimal Sensor Location for Modal Identification

Consider the case in which the response  $x_j$  at the  $j$ -th DOF is given with respect to the  $r$ -th modal coordinates  $\xi_r$  in the form

$$x_j = \sum_{r=1}^m \Phi_{jr} \xi_r \quad (16)$$

and the objective is to design the sensor configuration such that one gets the most information in order to estimate the modal coordinates  $\xi_r$ ,  $r=1, \dots, m$ . In this case the parameter set  $\theta$  in the optimal sensor configuration methodology consist of the modal coordinates  $\xi_r$ ,  $r=1, \dots, m$ . Introducing the relation (16) into (10) and noting that  $\nabla_{\theta} x_j = \Phi^{(j)}$ , where  $\Phi^{(j)}$  is the  $j$ -th row of the modeshape matrix  $\Phi$ , one gets the following relation for the information matrix:

$$\mathbf{Q}(\delta, \hat{\theta}, \hat{\sigma}) = \frac{N_D}{2} \sum_{j=1}^{N_d} \delta_j \frac{1}{\hat{\sigma}_j^2} \Phi^{(j)T} \Phi^{(j)} \quad (17)$$

which is exactly the same relation proposed in Kammer 1991, for designing the optimal sensor location. The optimal sensor locations are independent of the excitation used and provide the most information for identifying the modal coordinates.

Consider next the case for which the parameter set  $\theta = [\omega_1, \dots, \omega_m, \zeta_1, \dots, \zeta_m]$  to be identified consists of the modal characteristics such as modal frequencies  $\omega_r$  and modal damping ratios  $\zeta_r$ . In this case, the information matrix takes the form

$$\mathbf{Q}(\delta, \hat{\theta}, \hat{\sigma}) = \frac{N_D}{2} \sum_{j=1}^{N_d} \delta_j \frac{1}{\hat{\sigma}_j^2} \mathbf{H}^T \Phi^{(j)T} \Phi^{(j)} \mathbf{H} \quad (18)$$

where  $\mathbf{H} = [H_{\omega} \ H_{\zeta}]$ , with  $H_{\omega}$  and  $H_{\zeta}$  being diagonal matrices with the  $r$ -th diagonal element

$$\text{given by } [H_{\omega}]_r = \frac{\partial \xi_r}{\partial \omega_r} \equiv \alpha_r \text{ and } [H_{\zeta}]_r = \frac{\partial \xi_r}{\partial \zeta_r} \equiv \beta_r,$$

respectively. Using the modal equation

$$\ddot{\xi}_r + 2\zeta_r \omega_r \dot{\xi}_r + \omega_r^2 \xi_r = \phi_r^T \mathbf{f}(t) \quad (19)$$

that relates the modal coordinate  $\xi_r$  to the modal parameters  $\omega_r$  and  $\zeta_r$ , the sensitivity factors  $\alpha_r$  and  $\beta_r$  are obtained from the equations

$$\ddot{\alpha}_r + 2\zeta_r \omega_r \dot{\alpha}_r + \omega_r^2 \alpha_r = -2\zeta_r \dot{\xi}_r - 2\omega_r \xi_r \quad (20)$$

and

$$\ddot{\beta}_r + 2\zeta_r \omega_r \dot{\beta}_r + \omega_r^2 \beta_r = -2\omega_r \dot{\xi}_r \quad (21)$$

respectively. It is seen that in this case the information entropy and as a result design of the optimal sensor locations depend on the input exciting the structure. The optimal sensor configurations arising from the above formulation take into account also the sensitivity of the modal coordinates  $\xi_r$  to the modal parameters  $\omega_r$  and  $\zeta_r$ . In contrast, such sensitivity is not tak-

en into account in the information matrix (18) resulting the previous formulation.

### 3.2 Information Entropy Based on Modal Data

For modal data, following a similar analysis as before by substituting (5) into (8) and considering the limiting case  $N_D N \rightarrow \infty$ , the resulting matrix  $\mathbf{Q}(\delta, \theta)$  simplifies to a positive semi-definite matrix given by

$$\mathbf{Q}(\delta, \theta) = \frac{N_D}{2} \sum_{r=1}^m \left[ \frac{\nabla_{\theta} \omega_r(\theta) \nabla_{\theta}^T \omega_r(\theta)}{s_1^2 + s_2^2 \omega_r^2(\theta)} + \sum_{j=1}^{N_d} \delta_j \frac{\nabla_{\theta} L_0 \phi_{jr}(\theta) \nabla_{\theta}^T L_0 \phi_{jr}(\theta)}{s_1^2 + s_2^2 \|L_0 \phi_r(\theta)\|^2 / N_0} \right] \quad (22)$$

containing the information about the values of the model parameters  $\theta$  based on the modal data from all sensors placed in the structure. As before, in developing the expression (22), we assume that the prediction errors for the modal frequencies and the modeshape components are independent and that these errors consist of the measurement and modeling errors which are also independent so that the variance of the total error is related to the variance of the measurement error and model error through (13). Assuming that the variance of the measurement error is constant, independent of the value of the modal frequency or the values of the modeshape components, and that the variance of the model error for the modal frequency and the modeshape components are proportional to the values of the corresponding modal frequencies and the size of the modeshape component, respectively, the variance of the total errors are easily obtained and shown in the denominators of the terms in (22).

It is worth noticing that for given number of modes the first term in  $\mathbf{Q}(\delta, \theta)$  in (22) is not affected by the location of sensors. So the design of optimal sensor location, for given number of contributing modes, depends on the second term in (22).

### 3.3 Computational Algorithms

Based on the asymptotic analysis, two heuristic sequential sensor placement (SSP) algorithms, the forward (FSSP) and the backward (BSSP), were proposed (Papadimitriou 2004) for constructing predictions of the optimal and worst sensor configurations. According to FSSP, the positions of  $N_0$  sensors are computed sequentially by placing one sensor at a time in the structure at a position that result in the highest reduction in information entropy. The BSSP algorithm is used in an inverse order, starting with  $N_d$  sensors placed at all DOFs of the structure and removing successively one sensor at a time from the position that results in the smallest in-

crease in the information entropy. The computations involved in the SSP algorithms are an infinitesimal fraction of the ones involved in the exhaustive search method and can be done in realistic time, independently of the number of sensors and the number of model DOFs. It was found that for essentially the same accuracy, genetic algorithms, well-suited for solving the resulting discrete optimization problem, require significantly more computational effort than the heuristic SSP algorithms. Thus, although the SSP algorithms are not guaranteed to give the optimal solution, they were found to be effective and computationally attractive alternatives to the Gas (Bedrossian & Masri 2003, Abdullah et al. 2001, Yao et al. 1993, Papadimitriou 2002). In particular, SSP algorithms provide with minimal computational effort the variation of the lower and upper bounds of the information entropy as a function of the number of sensors. Such bounds are useful in evaluating the effectiveness of a sensor configuration as well as in guiding the cost-effective selection of the number of sensors, trading-off information provided from extra sensors with cost of instrumentation.

## 4 OPTIMAL SENSOR LOCATION FOR MULTIPLE MODEL CLASSES

The proposed optimal instrumentation depends on the chosen class of models, usually selected based on the purpose of identification which can vary from modal identification to model updating, model selection and damage detection. The design of optimal sensor configurations for providing informative measurements for multiple model classes  $M_1, \dots, M_{\mu}$  is next addressed. These classes may include a class of modal models parameterized by modal properties, classes of increasingly complex finite element models (linear and nonlinear) parameterized by material properties, and/or model classes associated with various damage patterns in the structure (Papadimitriou 2004). An optimal instrumentation should be capable of providing informative measurements for multiple classes of models  $M_1, \dots, M_{\mu}$ .

The general case is considered for which the model DOFs corresponding to a model class differ from the DOFs corresponding to another model class. For convenience, let us introduce the set  $\{1, 2, \dots, N_p\}$  of all possible DOFs associated with locations and directions in the physical structure along which measurements can be made. Let also  $\mathbf{x} \in R^{N_0}$  denote the sensor configuration vector containing  $N_0$  measured DOFs taken from the set  $\{1, 2, \dots, N_p\}$ . That is, each component in  $\mathbf{x}$  takes integer values ranging from 1 to  $N_p$ . In order to account for different number of DOFs between model classes, the mapping  $\delta_i(\mathbf{x})$  is introduced to map the ‘‘physical’’ sensor configuration vector  $\mathbf{x}$  to the model sensor configuration vector  $\delta_i$  for a particular



model class  $M_i$ . Specifically,  $\delta_i(\mathbf{x})$  is a vector of zeros and ones with the positions of ones in the vector  $\delta_i(\mathbf{x})$  denoting the DOFs of the model class  $M_i$  that correspond to the “physical” DOFs specified in the vector  $\mathbf{x}$ .

Let  $J_i(\mathbf{x}) \equiv IEI_i(\delta_i(\mathbf{x}))$  be the effectiveness of a sensor configuration  $\mathbf{x}$  for the  $i$ th model class  $M_i$ , where  $IEI_i(\delta_i)$  is the information entropy index given by

$$IEI_i(\delta_i) = \frac{H(\delta_i) - H(\delta_{i,best})}{H(\delta_{i,worst}) - H(\delta_{i,best})} \quad (23)$$

with  $H(\delta) \equiv H(\delta; \hat{\theta}, \hat{\sigma})$ ,  $\delta_{i,best}$  is the optimal sensor configuration and  $\delta_{i,worst}$  is the worst sensor configuration for the  $i$ th model class.  $\delta_{0,ref} = \delta_{i,worst}$ . In this case the values of  $IEI_i(\delta_i)$  range from zero to one. The most effective configuration corresponds to value of  $IEI_i(\delta_i)$  equal to one, while the least effective configuration corresponds to value of  $IEI_i(\delta_i)$  equal to zero. The function  $IEI_i(\delta_i(\mathbf{x}))$ , gives the dependence of the information entropy index on the monitoring locations  $\mathbf{x}$  for the model class  $M_i$ . The optimal sensor configuration for the model class  $M_i$  is selected as the one that minimizes the information entropy index  $J_i(\mathbf{x})$ . The problem of identifying the optimal sensor locations that minimize the information entropy indices for all  $\mu$  model classes is formulated as a multi-objective optimization problem stated as follows. Find the values of the discrete-valued parameter set  $\mathbf{x}$  that simultaneously minimizes the objectives (Papadimitriou 2004)

$$\mathbf{J}(\mathbf{x}) = (J_1(\mathbf{x}), J_2(\mathbf{x}), \dots, J_\mu(\mathbf{x})) \quad (24)$$

For conflicting objectives  $J_1(\mathbf{x}), \dots, J_\mu(\mathbf{x})$ , there is no single optimal solution, but rather a set of alternative solutions, which are optimal in the sense that no other solutions in the search space are superior to them when all objectives are considered. Such alternative solutions, trading-off the information entropy values for different model classes, are known in multi-objective optimization as Pareto optimal solutions. An advantage of the multi-objective identification methodology is that all admissible solutions are obtained which constitute model trade-offs in reducing the information entropies for each model class. These solutions are considered optimal in the sense that the corresponding information entropy for one model class cannot be improved without deteriorating the information entropy for another model class. The optimal points along the Pareto trade-off front provide detailed information about the effectiveness of the sensor configuration for each model class.

#### 4.1 Computational Algorithms

Genetic algorithms are well suited for performing the multi-objective optimization involving discrete variables. In particular, the Strength Pareto Evolu-

tionary Algorithm (SPEA) (Zitzler and Thiele 1999), based on genetic algorithms, can be used for solving the resulting multi-objective optimal sensor location problem (Papadimitriou 2004).

A more systematic and computationally very efficient approach for obtaining a good approximation of the Pareto optimal front and the corresponding Pareto optimal sensor configurations is to use a sequential sensor placement approach, extending the SSP algorithm to handle Pareto optimal solutions, as follows. The Pareto optimal configurations for one sensor are first computed. This is done by an exhaustive search and requires  $N_p$  vector function evaluations, where  $N_p$  is the number of measurable DOFs. The Pareto optimal configurations for  $(i+1)$  sensors are then obtained iteratively from the Pareto optimal configurations for  $i$  sensors as follows. At the  $i$  iteration, let  $P_i$  be the set of all computed Pareto points and  $\mathbf{x}_i^{(j)}$ ,  $j=1, \dots, n_i$ , be the corresponding Pareto optimal configurations involving  $i$  sensors, where  $n_i$  is the number of Pareto optimal configurations in the set  $P_i$ . For the  $j$ -th Pareto optimal configuration  $\mathbf{x}_i^{(j)}$ , a new set of all possible sensor configurations  $\mathbf{x}_{i+1}^{(k)}$ ,  $k=1, \dots, (N_p - i)$  involving  $(i+1)$  sensors are constructed such that each one contains all sensor locations involved in  $\mathbf{x}_i^{(j)}$ . This process is repeated for all  $n_i$  Pareto configurations in the set  $P_i$ , generating a set of  $(N_p - i)n_i$  sensor configurations. The  $n_{i+1}$  Pareto optimal sensor configurations for  $(i+1)$  sensors is then taken as the non-dominated solutions from the last set of  $(N_p - i)n_i$  sensor configurations. This iterative procedure is then continued for up to  $N_0$  sensors.

The total number of vector function evaluations for  $N_0$  sensors is  $\sum_{i=1}^{N_0} (N_p - i)n_i \leq N_0 N_p n_{\max}$ , where  $n_{\max} = \max(n_i)$  is the maximum number of Pareto solutions encountered for sensor configurations involving one up to  $N_0$  sensors. If the number of Pareto optimal solutions  $n_{\max}$  exceeds a given number  $N'$ , the Pareto set is pruned by means of clustering. Limiting the number of Pareto solutions is necessary since the number of the Pareto optimal solutions may become excessive. In most cases, however, the Pareto front can be adequately described by fewer points. Clustering is introduced in order to maintain a uniform distribution of solutions along the Pareto front. Without clustering, the solutions will be biased towards certain regions along the objective space, leading to an unbalanced distribution of solutions. For  $n_{\max}$  or  $N'$  small, the total number of vector function evaluations using the Pareto sequential sensor placement algorithm (PA-SSP) algorithm is infinitesimally small compared to the number of vector function evaluations  $N_s$  required in an exhaustive search method.

The PA-SSP algorithm will accurately predict the Pareto optimal sensor configurations only in the case for which the sensor locations of any Pareto optimal sensor configuration involving  $i$  sensors is a subset

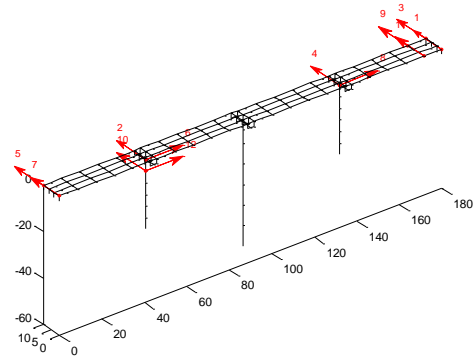
of the locations of at least one of the Pareto optimal sensor configurations involving  $(i+1)$  sensors. However, the last argument does not hold in general and the sensor configurations computed by the PA-SSP algorithm cannot be guaranteed to be the Pareto optimal ones. Numerical applications presented in the work by Papadimitriou (2004) demonstrated that the Pareto front constructed by this heuristic algorithm, in most cases examined coincide with, or is very close to, the exact Pareto front. Compared to the SPEA algorithm, the PA-SSP algorithm is preferred since it is found to maintain higher levels of accuracy with considerably less computational effort than that involved in SPEA algorithm.

## 5 ILLUSTRATIVE EXAMPLE

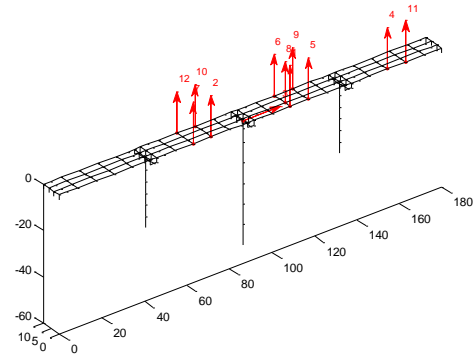
In order to demonstrate the theoretical developments and illustrate the effectiveness of the proposed algorithms the methodology is applied to the design of the optimal configuration for an array of acceleration sensors placed on the 180-meter-long 13-meter-wide four-span bridge structure, located at Kavala (Greece). The deck of the bridge, consisting of four prestressed beams supporting the 20-cm thick deck, “floats” on laminated elastomeric bearings located at the top of the three piers and the abutments. A 900-DOF finite element model of the bridge consisting of 3-d beam elements is shown in Fig. 1. The structure is parameterized using three parameters, with the first parameter modeling the stiffness of the deck, the second parameter modeling the stiffness of all bearings and the third parameter modeling the stiffness of the three columns of the bridge. For the nominal structure considered, the 1<sup>st</sup> (0.54 Hz), 3<sup>rd</sup> (0.67 Hz), 4<sup>th</sup> (1.07 Hz), 5<sup>th</sup> (1.77 Hz), 6<sup>th</sup> (2.08 Hz) and 8<sup>th</sup> (2.72 Hz) modes are transverse, the 2<sup>nd</sup> (0.58 Hz) mode is longitudinal, the 7<sup>th</sup> (2.50 Hz) is local bending mode of the central pier, and the 9<sup>th</sup> to 12<sup>th</sup> modes are closely spaced (2.80, 2.824, 2.825 and 2.84 Hz) bending modes of the deck.

The optimal sensor locations for 1-12 sensors based on modal data, for the case of model error only ( $s_1 = 0$  and  $s_2 = s$ ) are shown in Figs. 1(a) and 1(b) for 4 and 12 observable modes, respectively, while for the case of measurement error only ( $s_1 = s$  and  $s_2 = 0$ ) the optimal sensor locations are shown in Figs. 2(a) and 2(b) for 4 and 12 observable modes, respectively. It should be noted that the design of the optimal sensor configuration depends on the amount of model and measurement error. The minimum and maximum information entropy values as a function of the sensors computed by the exhaustive search method (exact method) for up to two sensors and the FSSP and BSSP algorithms are shown in Figs. 3(a) and 3(b) for 4 and 12 observable modes, respectively. The lower and upper bounds of the information entropy values, cor-

responding respectively to the optimal and worst sensor configuration, is a decreasing function of the number of sensors.

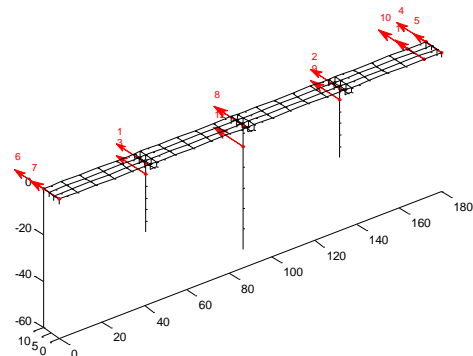


(a)

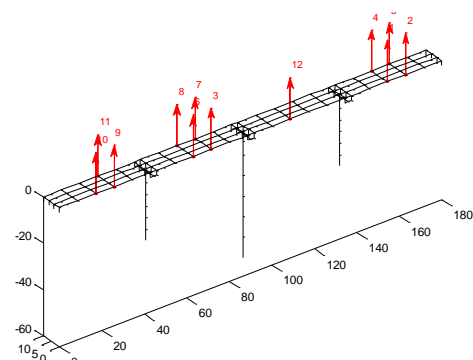


(b)

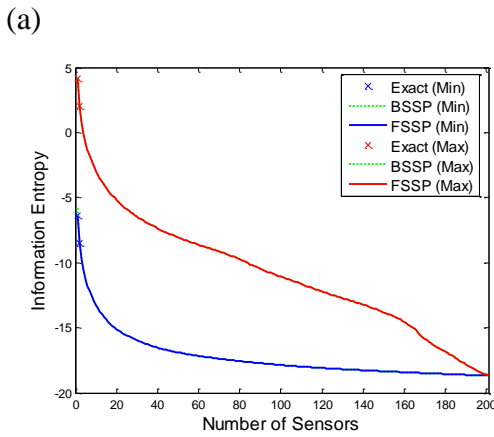
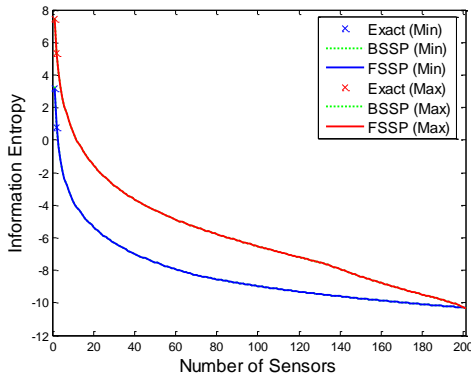
Fig. 1. Optimal locations for 1 to 12 sensors assuming model error for a) 4 and b) 12 observable modes.



(a)



(b)  
Fig. 2. Optimal locations for 12 sensors assuming measurement error for a) 4 and b) 12 observable modes.



(b)  
Fig. 3. Minimum and maximum information entropy values for a) 4 and b) 12 observable modes.

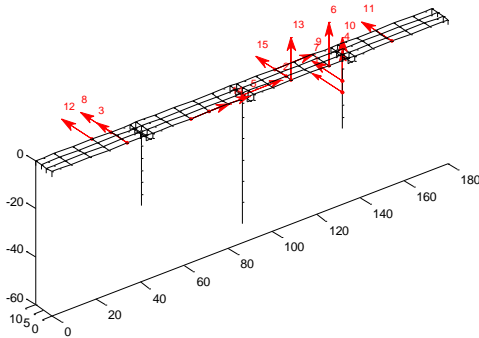


Fig. 4. Optimal locations (modal identification) for 15 sensors for 4 observable modes.

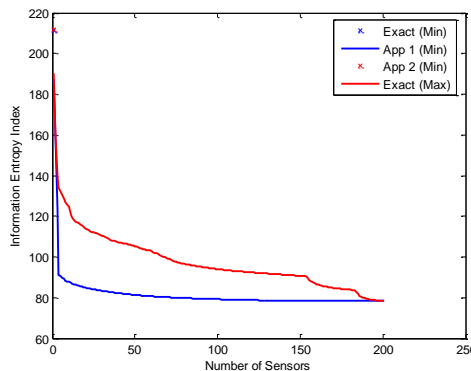


Fig. 5. Minimum and maximum information entropy values (modal identification) for 4 observable modes.

## 6 CONCLUSIONS

The methodology proposed in this study is useful for designing optimal sensor configurations that provide the most informative data for identifying the parameters of a family of model classes. Information entropy indices were introduced to measure the quality of information contained in the measured data. The optimal sensor placement problem for structural identification was presented for two types of problems: (i) identification of structural model (e.g. finite element) parameters or modal model parameters (modal frequencies and modal damping ratios) based on acceleration time histories, and (ii) identification of structural model parameters based on modal data. An asymptotic estimate, valid for large number of data, was derived and used to justify that the sensor placement design can be based solely on a nominal model, ignoring the details in the measured data. Analytical expressions and numerical results demonstrated that the design of the optimal sensor configuration also depends on model and measurement error. The optimal sensor location problem for identifying the parameters of multiple model classes is formulated as a multiple objective optimization problem. Heuristic algorithms, available for solving the optimal sensor location problem for a single or multiple model classes, are superior, in terms of accuracy and computational efficiency, to genetic algorithms suitable for solving the resulting single and multi-objective optimization problems.

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