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Trajectory tracking of autonomous helicopters using explicit nonlinear MPC augmented with disturbance observers

Cunjia Liu\textsuperscript{a,}\textsuperscript{*}, Wen-Hua Chen\textsuperscript{a}, John Andrews\textsuperscript{b}

\textsuperscript{a}Department of Aeronautical and Automotive Engineering, Loughborough University, Loughborough, LE11 3TU, UK.
\textsuperscript{b}Nottingham Transportation Engineering Centre, University of Nottingham, Nottingham, NG7 2RD, UK.

Abstract

This paper addresses robust nonlinear control of autonomous helicopters. Small-scale helicopters are very attractive for a wide range of civilian and military applications due to their unique features. However, autonomous flight of small helicopters is quite challenging because they are naturally unstable, have strong nonlinearities and couplings, and are very susceptible to wind and small structural variations.

A robust nonlinear control scheme is proposed to address these issues. It consists of a nonlinear model predictive controller (MPC) and a nonlinear disturbance observer. First, an analytic solution for the nonlinear MPC is developed based on the nominal model under the assumption that all disturbances are measurable. Then a nonlinear disturbance observer is designed to estimate the influence of external force/torque induced by wind/turbulence.

*Corresponding author

Email addresses: c.liu5@lboro.ac.uk (Cunjia Liu), w.chen@lboro.ac.uk (Wen-Hua Chen), john.andrews@nottingham.ac.uk (John Andrews)
unmodelled dynamics and variation of the helicopter dynamics. The latter is captured by the trim errors and the change of sit angles of the helicopter. Global asymptotic stability and tracking performance of the composite controller has been established through stability analysis. Flight test including hovering under wind gust and performing very challenging pirouette manoeuvre have been carried out to demonstrate the performance of the proposed control scheme.

**Keywords:** Model predictive control, Disturbance observer, Nonlinear system, helicopter, flight control

1. **Introduction**

Autonomous helicopters are versatile flying machines capable of vertical take-off and landing, hovering, flight at very low altitudes, and performing complicated manoeuvres. These properties make them suitable for a board range of applications like surveillance, broad patrol, search and rescue, etc. On the other hand, their nonlinearities and dynamic couplings pose a challenge for the controller design and attract considerable interests from academia. Many control techniques have been applied to address the autonomous flight of helicopters including classic cascaded PID control (Kim and Shim, 2003), feedback linearisation (Koo and Sastry, 1998), multivariable adaptive control (Krupadanam et al., 2002), neural network adaptive control (Johnson and Kannan, 2005), state-dependent riccati equation (SDRE) control (Bogdanov and Wan, 2007), composite nonlinear feedback control (Peng et al., 2009).

Recently, model predictive control (MPC) has been recognised as a promis-
ing method in the unmanned aerial vehicle (UAV) community (Ollero and Merino, 2004). MPC is an optimal control strategy that uses a model to predict the future behaviour of a plant over a prediction horizon. Based on these predictions, a performance index defined to penalise tracking errors or state errors is minimised with respect to the sequence of future inputs. Only the first action in the optimised control sequence is applied to the plant, and this procedure is repeatedly executed in a receding horizon fashion to continuously generate control signals. The “foresee” feature of MPC makes it a suitable control strategy for UAV applications, especially in trajectory tracking where MPC can take into account the future value of the reference to improve the performance.

The essential procedure in the implementation of MPC algorithms is to solve the formulated optimisation problem (OP). For nonlinear system, MPC technique generally requires solving an optimisation problem numerically at every sampling instant, which poses obstacles on the real-time implementation due to the heavy computational burden. Although the development of the avionics and microprocessor technology makes the online optimisation possible, the implementation of computationally demanding nonlinear MPC on small UAVs is very challenging. The associated low bandwidth and computational delay make it very difficult to meet the control requirement for systems with fast dynamics such as helicopters. Only few applications on helicopter flight control have been reported in (Kim et al., 2002; Shim et al., 2003), where the authors adopt a high-level MPC to solve the tracking problem and rely on a local linear feedback controller to compensate the high-level MPC. Moreover, the formulated nonlinear optimisation problem
has to be solved by a secondary flight computer. The extra payload and power consumption are unsuitable for a small-scale helicopter.

To avoid online optimisation, this paper introduces an explicit nonlinear MPC (ENMPC) for trajectory tracking of autonomous helicopters. By approximating the tracking error and control efforts in the receding horizon using their Taylor expansion to a specified order, an analytic solution to nonlinear MPC can be found and consequently the closed form controller can be formulated without online optimisation (Chen et al., 2003). The benefits of using this MPC algorithm are not only the elimination of the online optimisation and the associated resource, but also a higher control bandwidth, which is very important for helicopters in aggressive flight scenarios.

Apart from the control method, there are practical issues in controlling autonomous helicopters from an engineering point of view. It is known that the control performance of MPC, or other model based control technologies, heavily relies on the quality of the model. However, the model of high accuracy for a helicopter is difficult to obtain due to the complicated aerodynamic nature of the rotor system. On the other hand, due to the light-weighted structure, small-scale helicopters are more likely to be affected by wind gusts and other disturbances than their full size counterpart, and the physical parameters such as mass and inertia of moments can be easily altered by changing the payload and even its location. All these factors compromise the actual performance of the controller designed based on the nominal model.

Robust control techniques, especially $H_{\infty}$ technique, have been used in handling the parametric uncertainty and unmodelled dynamics (Marconi and Naldi, 2007; La Civita et al., 2006; Gadewadikar et al., 2008). Although
satisfactory performance has been demonstrated, robust control is known to result in conservative solutions and presents trade-offs between performance and robustness. On the other hand, adaptive control also shows promising results of controlling autonomous helicopters in the presence of uncertainties (Krupadanam et al., 2002; Johnson and Kannan, 2005). However, the controllers usually have complicated structures and very high order. Other methods to compensate the wind disturbances are also available such as (Bogdanov and Wan, 2007) where the authors provided a method of calculating the trim control by exploiting a detailed helicopter model. In (Dana-palasingam et al., 2009), the authors rely on artificial neural networks to generate trim values. However, both methods either need an estimation or direct measurement of wind conditions.

To enhance the performance of ENMPC in a complex operation environment, this paper advocates a disturbance observer based control (DOBC) approach. Disturbance observers have been applied to estimate unknown disturbances in the control process (Chen et al., 2000; Chen, 2003). As the estimation of disturbances is provided, the control system can explicitly take them into account and compensate them. The advantage of the DOBC is that it preserves the tracking and other properties of the original baseline control while being able to compensate disturbances rather than resorting to a different control strategy.

To design a DO augmented ENMPC for trajectory tracking of autonomous helicopters, two problems need to be addressed, namely, designing the non-linear disturbance observer to estimate the disturbances acting on the helicopter, integrating the disturbance information into ENMPC to compensate
their influences. To this end, another contribution of this paper lies in the synthesis of the ENMPC and DO by exploiting the helicopter model structure. The disturbances are assumed to exist in certain channels of the helicopter where the coupling terms can also be lumped into disturbance terms. In this way an ENMPC is derived under the assumption that all the disturbances are measurable and then these disturbances are replaced by their estimation provided by the proposed disturbance observers. On the other hand, the lumped disturbance terms simplify the model structure allowing the derivation of ENMPC for helicopters. The composite control framework provides a promising solution to autonomous helicopter trajectory tracking in the presence of uncertainties and disturbances. The performance of the proposed control system is tested through simulations and verified in our indoor flight testbed.

The remaining part of this chapter is organised as follows: Section 2 presents the mathematical model of small-scale helicopters with disturbances in consideration; in Section 3 the algorithm of ENMPC and its implementation on autonomous helicopters are discussed in detail; Section 4 introduces the design procedure for the nonlinear disturbance observer; stability analysis of the proposed composite controller is provided in Section 5; Section 6 provides some simulation and flight experiment results, followed by conclusions in Section 7.

2. Helicopter modelling

A helicopter is a highly nonlinear system with multiple inputs multiple outputs (MIMO) and complex internal couplings. The complete model taking
into account the flexibility of the rotors and fuselage usually results in a model of high degrees-of-freedom. The complexity of such a model would make the following system identification much more difficult. A practical way to deal with this issue is to capture the primary dynamics by a simplified model and treat the other trivial factors that affect dynamics as uncertainty or disturbances. The general dynamics of a small-scale helicopter can be captured by a six-degrees-of-freedom rigid-body model augmented with a simplified rotor dynamic model (Mettler et al., 2002; Gavrilets et al., 2001), as shown in Fig.1. In this way, the kinematics of the helicopter, i.e. the position and the orientation represented by Z-Y-X Euler angles, can be expressed as:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = R^i_b(\phi, \theta, \psi) \begin{bmatrix}
u \\
v \\
w
\end{bmatrix}^T
\]

(1)

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \phi
\\
0 & \cos \phi & -\sin \phi
\\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix} \begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

(2)

where \((x, y, z)\) describe the helicopter inertial position, \((u, v, w)\) are velocities along three body axes, \((p, q, r)\) are angular rates and \((\phi, \theta, \psi)\) are attitude angles and \(R^i_b\) is the transformation matrix from body to inertial coordinates given in (3) with short notation \(c\) for cosine and \(s\) for sine.

\[
R^i_b(\phi, \theta, \psi) =
\begin{bmatrix}
c\theta c\psi & s\phi s\theta c\psi - c\phi s\psi & c\phi s\theta c\psi + s\phi s\psi \\
c\theta s\psi & s\phi s\theta s\psi + c\phi c\psi & c\phi s\theta s\psi - s\phi c\psi \\
-s\theta & s\phi c\theta & c\phi c\theta
\end{bmatrix}
\]

(3)

The model of translational dynamics of helicopters used in this paper is modified by keeping the thrust of main rotor as a dominating force and
considering other force contributions as disturbances, such that

\[
\begin{align*}
\dot{u} &= vr - wq - g \sin \theta + d_x \\
\dot{v} &= wp - ur + g \cos \theta \sin \phi + d_y \\
\dot{w} &= uq - vp + g \cos \theta \cos \phi + T + d_z
\end{align*}
\]

where, \( T \) is the normalised main rotor thrust controlled by collective pitch \( \delta_{col} \), as \( T = g + Z_w w + Z_{col} \delta_{col} \), and \((d_x, d_y, d_z)\) are normalised force disturbances that include external wind gusts, internal couplings and unmodelled dynamics. These force disturbances directly affect the translational dynamics and result in tracking error. As force disturbances are not in the channels of control inputs, they are called “mis-matched” disturbances. This modification on one hand increases the valid range of the model compared to
simplified helicopter models for control design that neglect all other forces other than the main thrust (Koo and Sastry, 1998; Marconi and Naldi, 2007; Raptis et al., 2010). On the other hand it reduces the workload of deriving the ENMPC for helicopters as different forces are lumped into one term.

In rotation dynamics, the torques are generated by the tilting of main rotor such that

\[ \dot{p} = -qr(I_{yy} - I_{zz})/I_{xx} + L_a a + L_b b \]
\[ \dot{q} = -pr(I_{zz} - I_{xx})/I_{yy} + M_a a + M_b b \]
\[ \dot{r} = -pq(I_{xx} - I_{yy})/I_{zz} + N_r r + N_{col} \delta_{col} + N_{ped} \delta_{ped} \]  

where \( a \) and \( b \) are flapping angles to depict the tilt of the main rotor along the longitudinal and lateral axis, respectively; the other parameters in the model are the stability and control derivatives, whose values for the helicopter used in this study are obtained by system identification. The flapping angles \( a \) and \( b \) of the main rotor are originally controlled by lateral and longitudinal cyclic \( \delta_{lat} \) and \( \delta_{lon} \). Their relationship can be approximated by steady state dynamics of the main rotor (Bogdanov and Wan, 2007):

\[ a = -\tau q + A_{lat} \delta_{lat} + A_{lon} \delta_{lon} \]
\[ b = -\tau p + B_{lat} \delta_{lat} + B_{lon} \delta_{lon} \]  

Apart from force disturbances, small-scale helicopters also subject to structural uncertainties and are vulnerable to physical alterations like payload change. These factors are commonly ignored in the control design, as they can be compensated by setting control trims in the implementation. To save the trim tuning process in the real life operation, we consider trims errors in the control channel as disturbances. Thereby, combining (5) and
(6) yields

\[
\begin{align*}
\dot{p} &= -L_{pq} + L_{lat}(\delta_{lat} + d_{lat}) + L_{lon}(\delta_{lon} + d_{lon}) \\
\dot{q} &= -M_{pq} + M_{lat}(\delta_{lat} + d_{lat}) + M_{lon}(\delta_{lon} + d_{lon}) \\
\dot{r} &= -N_{pr} + N_{r}r + N_{col}\delta_{col} + N_{ped}(\delta_{ped} + d_{ped})
\end{align*}
\]

(7)

where

\[
\begin{align*}
L_{pq} &= qr(I_{gy} - I_{zz})/I_{xx} + \tau(L_{a}q + L_{b}p), \\
M_{pq} &= pr(I_{zz} - I_{xx})/I_{yy} + \tau(M_{a}q + M_{b}p), \\
N_{pq} &= pq(I_{xx} - I_{yy})/I_{zz}, \\
L_{lat} &= L_{a}A_{lat} + L_{b}B_{lat}, \\
M_{lat} &= M_{a}A_{lat} + M_{b}B_{lat}, \\
L_{lon} &= L_{a}A_{lon} + L_{b}B_{lon}, \\
M_{lon} &= M_{a}A_{lon} + M_{b}B_{lon},
\end{align*}
\]

and \(d_{lat}, d_{lon}\) and \(d_{ped}\) account for different trim errors. In addition, since they are combined into the angular dynamics and affect the angular rate directly, they can be considered as torque disturbances.

The modified helicopter model by combining (1)-(7) can be expressed by a general affine form:

\[
\begin{align*}
\dot{x} &= f(x) + g_1(x)u + g_2(x)d \\
y &= h(x)
\end{align*}
\]

(9)

where \(x = [x \ y \ z \ u \ v \ w \ p \ q \ r \ \phi \ \theta \ \psi]^T\) is the helicopter state, \(y\) is the output of the helicopter, and \(d = [d_x \ d_y \ d_z \ d_{lat} \ d_{lon} \ d_{ped}]^T\) is the lumped disturbance acting on the helicopter. In the trajectory tracking control of an autonomous helicopter, the interested outputs are the position and heading angle. Thus, \(y = [x \ y \ z \ \psi]^T\).
3. Explicit nonlinear MPC with disturbances

Trajectory tracking is the basic function required when an autonomous helicopter performs a task. To this end, we need to design a controller such that the output $y(t)$ of the helicopter (9) tracks the prescribed reference $w(t)$. In the MPC strategy, tracking control can be achieved by minimising a receding horizon performance index

$$J = \frac{1}{2} \int_0^T (\hat{y}(t + \tau) - w(t + \tau))^T Q (\hat{y}(t + \tau) - w(t + \tau)) d\tau$$

(10)

where weighting matrix $Q = \text{diag}\{q_1, q_2, q_3, q_4\}$, $q_i > 0$, $i = 1, 2, 3, 4$. Note that the hatted variables belong to the prediction time frame.

Conventional MPC algorithm requires solving of an optimisation problem at every sampling instant to obtain the control signals. To avoid the computationally intensive online optimisation, we adopt an explicit solution for the nonlinear MPC problem based on the approximation of the tracking error in the receding prediction horizon (Chen et al., 2003).

3.1. Output approximation

For a nonlinear MIMO system like the helicopter, it is well known that after differentiating the outputs for a specific number of times, the control inputs appear in the expressions. The number of times of differentiation is defined as relative degree. For the helicopter with output $y = [x\ y\ z\ \psi]'$ and the corresponding input $u = [\delta_{\text{lon}}\ \delta_{\text{lat}}\ \delta_{\text{col}}\ \delta_{\text{ped}}]'$, the relative degree is a vector, $\rho = [\rho_1\ \rho_2\ \rho_3\ \rho_4]$. If continuously differentiating the output after the control input appears, the derivatives of control input appear, where the number of the input derivatives $r$ is defined as the control order.
Since the helicopter model has different relative degrees, the control order \( r \) is first specified in the controller design. The \( i \)th output of the helicopter in the receding horizon can be approximated by its Taylor series expansion up to order \( \rho_i + r \):

\[
\hat{y}_i(t + \tau) \approx y_i(t) + \tau \dot{y}_i(t) + \cdots + \frac{\tau^{r+\rho_i}}{(r+\rho_i)!} y_i^{[r+\rho_i]}(t)
\]

\[
= \begin{bmatrix}
1 & \tau & \cdots & \frac{\tau^{r+\rho_i}}{(r+\rho_i)!}
\end{bmatrix}
\begin{bmatrix}
y_i(t) \\
\dot{y}_i(t) \\
\vdots \\
y_i^{[r+\rho_i]}(t)
\end{bmatrix}
, \quad 0 \leq \tau \leq T \quad (11)
\]

where \( i = 1, 2, 3, 4 \). In this way, the approximation of the overall output of the helicopter can be cast in a matrix form:

\[
\hat{y}(t + \tau) = \begin{bmatrix}
x(t + \tau) \\
y(t + \tau) \\
z(t + \tau) \\
\psi(t + \tau)
\end{bmatrix}
= \begin{bmatrix}
\hat{y}_1(t + \tau) \\
\hat{y}_2(t + \tau) \\
\hat{y}_3(t + \tau) \\
\hat{y}_4(t + \tau)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1, \tau, \cdots, \frac{\tau^{r+\rho_1}}{(r+\rho_1)!} \\
\vdots \\
0_{1 \times (r+\rho_1+1)} \\
& & & 1, \tau, \cdots, \frac{\tau^{r+\rho_4}}{(r+\rho_4)!}
\end{bmatrix}
\begin{bmatrix}
y_1(t) \\
\dot{y}_1(t) \\
\vdots \\
y_4^{[r+\rho_1]}(t) \\
\dot{y}_4(t) \\
\vdots \\
y_4^{[r+\rho_4]}(t)
\end{bmatrix}
\quad (12)
\]
For each channel in the output matrix, the control orders $r$ are the same and can be decided during the control design, whereas the relative degrees $\rho_i$ are different but determined by the helicopter model structure. Manipulating the output matrix (12) gives the following partition:

\[
\hat{y}(t + \tau) = \begin{bmatrix}
\bar{\tau}_1 & \cdots & 0_{1 \times \rho_4} \\
\vdots & \cdots & \vdots & \ddots & \vdots \\
0_{1 \times \rho_1} & \cdots & \bar{\tau}_4 \\
\end{bmatrix}
\begin{bmatrix}
\bar{Y}_1(t)^T \\
\bar{Y}_4(t)^T | \bar{Y}_4(t)^T \\
\end{bmatrix}^T
\]

where

\[
\bar{Y}_i = \begin{bmatrix}
y_i(t) & \dot{y}_i(t) & \cdots & y_i^{[\rho_i-1]}(t)
\end{bmatrix}^T, \quad i = 1, 2, 3, 4
\]

(14)

\[
\bar{Y}_i = \begin{bmatrix}
y_1^{[\rho_1+i-1]} & y_2^{[\rho_2+i-1]} & \cdots & y_4^{[\rho_4+i-1]}
\end{bmatrix}^T, \quad i = 1, \ldots, r + 1
\]

(15)

\[
\bar{\tau}_i = \begin{bmatrix}
1 & \tau & \cdots & \frac{\tau^{\rho_i-1}}{(\rho_i-1)!}
\end{bmatrix}, \quad i = 1, 2, 3, 4
\]

(16)

and

\[
\tilde{\tau} = \text{diag} \left\{ \frac{\tau^{\rho_1+i-1}}{(\rho_1+i-1)!}, \ldots, \frac{\tau^{\rho_4+i-1}}{(\rho_4+i-1)!} \right\}
\]

(17)

It can be observed from Eq(13) that the prediction of the helicopter output $\hat{y}(t + \tau)$, $0 \leq \tau \leq T$, in the receding horizon needs the derivatives of each output of the helicopter up to $r + \rho_i$ order at time instant $t$. Except for the output $y(t)$ itself that can be directly measured, the other derivatives have to be derived according to the helicopter model (9). During this process the control input will appear in the $\rho_i$th derivatives, where $i = 1, 2, 3, 4$.

The first derivatives can be obtained from the helicopter’s kinematics.
model:
\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3
\end{bmatrix} = \begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = \mathbf{R}_b^i \cdot \begin{bmatrix}
u \\
w
\end{bmatrix}
\]
(18)

\[
y_4 = \dot{\psi} = q \sin \phi \sec \theta + r \cos \phi \sec \theta
\]
(19)

Differentiating (18) and (19) with substitution of helicopter dynamics (1) yields the second derivatives:
\[
\begin{bmatrix}
\ddot{y}_1 \\
\ddot{y}_2 \\
\ddot{y}_3
\end{bmatrix} = \begin{bmatrix}
\ddot{x} \\
\ddot{y} \\
\ddot{z}
\end{bmatrix} = \mathbf{R}_b^i \cdot \begin{bmatrix}
d_x \\
d_y \\
T + d_z
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
g
\end{bmatrix},
\]
(20)

where \( T = Z_w w + Z_{col} \delta_{col} - g \) is the normalised main rotor thrust, and
\[
\ddot{y}_4 = \ddot{\psi} = q \frac{\cos \phi}{\cos \theta} \dot{x} + q \frac{\sin \phi \sin \theta}{\cos \theta} \dot{y} - r \frac{\sin \phi}{\cos \theta} \dot{x} + r \frac{\cos \phi \sin \theta}{\cos \theta} \dot{x} - L_{lat} \frac{\sin \phi}{\cos \theta} (\delta_{lat} + d_{lat}) + L_{lon} \frac{\sin \phi}{\cos \theta} (\delta_{lon} + d_{lon}) + N_{col} \frac{\cos \phi}{\cos \theta} \delta_{col} + N_{ped} \frac{\cos \phi}{\cos \theta} (\delta_{ped} + d_{ped})
\]
(21)

Note that although control input \( \delta_{col} \) appears in (20), the other control inputs do not, so we have to continue differentiating the first three outputs. To facilitate the derivation, we adopt the relationship \( \dot{\mathbf{R}}_b^i = \mathbf{R}_b^i \hat{\omega} \) by using skew-symmetric matrix \( \hat{\omega} \in \mathbb{R}^{3 \times 3} \):
\[
\hat{\omega} = \begin{bmatrix}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{bmatrix}.
\]
(22)

Thus, the third and fourth derivatives of the position output can be written
\[
\begin{bmatrix}
y_1^{[3]} \\
y_2^{[3]} \\
y_3^{[3]}
\end{bmatrix} =
\begin{bmatrix}
x^{[3]} \\
y^{[3]} \\
z^{[3]}
\end{bmatrix} =
\begin{bmatrix}
d_x \\
d_y \\
T + d_z
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
Z_w \dot{w} + Z_{col} \dot{\delta}_{col}
\end{bmatrix},
\]

(23)

and
\[
\begin{bmatrix}
y_1^{[4]} \\
y_2^{[4]} \\
y_3^{[4]} \\
z^{[4]}
\end{bmatrix} =
\begin{bmatrix}
x^{[4]} \\
y^{[4]} \\
z^{[4]}
\end{bmatrix} =
\begin{bmatrix}
d_x \\
d_y \\
T + d_z
\end{bmatrix} + 2\begin{bmatrix}
0 \\
0 \\
Z_w \dot{w} + Z_{col} \dot{\delta}_{col}
\end{bmatrix} +
\begin{bmatrix}
-N_r r d_y - M_{pq}(T + d_z) \\
N_r r d_y + L_{pq}(T + d_z) \\
M_{pq} d_x - L_{pq} d_y + Z_w \ddot{w}
\end{bmatrix}
\]
\[
\bar{\mathbf{A}}(\mathbf{x}, \mathbf{d}) \left[ \delta_{\text{lat}} + d_{\text{lat}} \delta_{\text{lon}} + d_{\text{lon}} \delta_{\text{col}} \ \delta_{\text{ped}} + d_{\text{ped}} \right]^T
\]

(24)

where
\[
\bar{\mathbf{A}}(\mathbf{x}, \mathbf{d}) = \mathbf{R}_b^i \begin{bmatrix}
M_{\text{lat}}(T + d_z) & M_{\text{lon}}(T + d_z) & 0 & -N_{\text{ped}} d_y \\
-L_{\text{lat}}(T + d_z) & -L_{\text{lon}}(T + d_z) & 0 & N_{\text{ped}} d_x \\
-M_{\text{lat}} d_x + L_{\text{lat}} d_y & -M_{\text{lon}} d_x + L_{\text{lon}} d_y & Z_{\text{col}} & 0
\end{bmatrix}
\]

(25)

At this stage, the control inputs explicitly appear in (24). Therefore, the vector relative degree for the helicopter is \( \rho = [4 \ 4 \ 4 \ 2] \). Note that in the formulation of (24) \( \dot{\delta}_{\text{col}} \) is the new control input, whereas \( \delta_{\text{col}} \) and \( \dot{\delta}_{\text{col}} \) are treated as the states which can be obtained by adding integrators. This procedure is known as achieving relative degree through dynamics extension Isidori (1995).

By invoking (18) -(23), we now can construct matrix \( \tilde{\mathbf{Y}}_i, \ i = 1, 2, 3, 4 \). However, in order to find the elements in \( \tilde{\mathbf{Y}}_i, \ i = 1, 2, \ldots, r + 1, \) further
manipulation is required. By combining (21) and (24) and utilizing the Lie notation Isidori (1995), we have:

\[
\dot{Y}_i = \begin{bmatrix} y_1^{[p_1]} \\ y_2^{[p_2]} \\ y_3^{[p_3]} \\ y_4^{[p_4]} \end{bmatrix} = \begin{bmatrix} x^{[4]} \\ y^{[4]} \\ z^{[4]} \\ \eta^{[2]} \end{bmatrix} = \begin{bmatrix} L_{f}^{p_1} h_1(x,d) \\ L_{f}^{p_2} h_2(x,d) \\ L_{f}^{p_3} h_3(x,d) \\ L_{f}^{p_4} h_4(x,d) \end{bmatrix} + A(x,d)\hat{u}
\]  

(26)

where \(\hat{u} = [\delta_{lat} + d_{lat} \delta_{lon} + d_{lon} \delta_{col} \delta_{ped} + d_{ped}]\); nonlinear terms \(L_{f}^{p_i} h_i(x,d)\), 
\(i = 1, 2, 3, 4\), can be found in the previous derivation, and

\[
A(x,d) = \begin{bmatrix}
L_{g_1} L_{f}^{p_1-1} h_1 & \ldots & L_{g_4} L_{f}^{p_1-1} h_{1} \\
L_{g_1} L_{f}^{p_2-1} h_2 & \ldots & L_{g_4} L_{f}^{p_2-1} h_{2} \\
\vdots & \ddots & \vdots \\
L_{g_1} L_{f}^{p_4-1} h_4(x) & \ldots & L_{g_4} L_{f}^{p_4-1} h_{4}(x)
\end{bmatrix} = \begin{bmatrix}
\tilde{A}(x,d) \\
\bar{A}(x,d)
\end{bmatrix}.
\]  

(27)

where \(\tilde{A}(x,d)\) is given in Eq.(25) and

\[
\bar{A}(x,d) = \begin{bmatrix}
L_{lat} \sin \phi \cos \theta & L_{lon} \sin \phi \cos \theta & 0 & N_{ped} \cos \phi \cos \theta
\end{bmatrix}.
\]  

(28)

Differentiating (26) with respect to time together with substitution of the system’s dynamics gives

\[
\ddot{Y}_2 = \begin{bmatrix} y_1^{[p_1+1]} \\ y_2^{[p_2+1]} \\ y_3^{[p_3+1]} \\ y_4^{[p_4+1]} \end{bmatrix} = \begin{bmatrix} L_{f}^{p_1+1} h_1(x) \\ L_{f}^{p_2+1} h_2(x) \\ L_{f}^{p_3+1} h_3(x) \\ L_{f}^{p_4+1} h_4(x) \end{bmatrix} + A(x,d)\hat{u}^{[1]} + p_1(x,\hat{u})
\]  

(29)

where \(p_1(x,\hat{u})\) is a nonlinear vector function of \(x\) and \(\hat{u}\). By repeating this procedure, the higher derivatives of the output and \(\dot{Y}_i, i = 1, 2, \ldots, r\), can be
calculated and finally we have

\[
\tilde{Y}_{r+1} = \begin{bmatrix}
y_1^{[\rho_1+r]} \\
y_2^{[\rho_2+r]} \\
y_3^{[\rho_3+r]} \\
y_4^{[\rho_4+r]}
\end{bmatrix} = \begin{bmatrix} L_f^{\rho_1+r} h_1(x) \\
L_f^{\rho_2+r} h_2(x) \\
L_f^{\rho_3+r} h_3(x) \\
L_f^{\rho_4+r} h_4(x)
\end{bmatrix} + A(x, d) \tilde{u}^{[r]} + p_r(x, \tilde{u}^{[1]}, \ldots, \tilde{u}^{[r]})
\]

(30)

So far by exploiting the helicopter model, the elements to construct \( \bar{Y} \) and \( \tilde{Y} \) in Eq.(13) are available. Therefore, the output of the helicopter in the future horizon \( y(t + \tau) \) can be expressed by its Taylor expansion in a generalized linear form with respect to the prediction time \( \tau \) and current states as shown in Eq.(13).

In the same fashion as in Eq.(13), the reference in the receding horizon \( w(t + \tau), 0 \leq \tau \leq T \) can also be approximated by:

\[
w(t+\tau) = \begin{bmatrix} w_1(t+\tau) \\
w_2(t+\tau) \\
w_3(t+\tau) \\
w_4(t+\tau)
\end{bmatrix} = T_f \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} \bar{W}_1(t)^T & \cdots & \bar{W}_4(t)^T & \tilde{W}_1(t)^T & \cdots & \tilde{W}_{r+1}(t)^T \end{bmatrix}^T
\]

(31)

where

\[
T_f = \begin{bmatrix}
\bar{\tau}_1 & \cdots & 0_{1 \times \rho_4} \\
\vdots & \ddots & \vdots \\
0_{1 \times \rho_1} & \cdots & \bar{\tau}_4
\end{bmatrix}
\]

(32)

and

\[
T_s = \begin{bmatrix}
\tilde{\tau}_1 & \cdots & \tilde{\tau}_{r+1}
\end{bmatrix}
\]

(33)

and the construction of \( \bar{W}_i(t), i = 1, 2, 3, 4 \), and \( \tilde{W}_i, i = 1, \ldots, r + 1 \), can refer to the structure of \( \bar{Y}_i(t) \) and \( \tilde{Y}_i \), respectively.
3.2. *Explicit nonlinear MPC solution*

The conventional MPC needs to solve a formulated optimisation problem to generate the control signal, where the control performance index is minimised with respect to the future control input over the prediction horizon. In this paper, after the output is approximated by its Taylor expansion, the control profile can be defined as

\[ \tilde{u}(t + \tau) = \tilde{u}(t) + \tau \tilde{u}^{[1]}(t) + \cdots + \frac{\tau^r}{r!} \tilde{u}^{[r]}(t), \quad 0 \leq \tau \leq T \]  

(34)

Thereby, the helicopter outputs depend on the control variables \( \bar{u} = \{ \tilde{u}, \tilde{u}^{[1]}, \ldots, \tilde{u}^{[r]} \} \).

Recalling the performance index (10) and the output and reference approximation (13) and (31), we have:

\[ J = \frac{1}{2} (\bar{Y}(t) - \bar{W}(t))^T \begin{bmatrix} T_1 & T_2 \\ T_2^T & T_3 \end{bmatrix} (\bar{Y}(t) - \bar{W}(t)) \]  

(35)

where

\[ \bar{Y}(t) = \begin{bmatrix} \bar{Y}_1(t)^T & \cdots & \bar{Y}_4(t)^T | \bar{Y}_1(t)^T & \cdots & \bar{Y}_r(t)^T \end{bmatrix}^T, \]  

(36)

\[ \bar{W}(t) = \begin{bmatrix} \bar{W}_1(t)^T & \cdots & \bar{W}_4(t)^T | \bar{W}_1(t)^T & \cdots & \bar{W}_{r+1}(t)^T \end{bmatrix}^T, \]  

(37)

\[ T_1 = \int_0^T T_f^T Q T_f d\tau, \]  

(38)

\[ T_2 = \int_0^T T_f^T Q T_s d\tau, \]  

(39)

and

\[ T_3 = \int_0^T T_s^T Q T_s d\tau. \]  

(40)

Therefore, instead of minimising the performance index (10) with respect to control profile \( u(t + \tau), \quad 0 < \tau < T \) directly, we can minimise the approximated
index (35) with respect to \( \bar{u} \), where the necessary condition for the optimality is given by
\[
\frac{\partial J}{\partial \bar{u}} = 0
\] (41)
After solving the nonlinear equation (41), we can obtain the optimal control variables \( \bar{u}^* \) to construct the optimal control profile defined by Eq.(34). As in MPC only the current control in the control profile is implemented, the explicit solution is \( \tilde{u}^* = \tilde{u}(t + \tau) \), for \( \tau = 0 \). The resulting controller is given by
\[
\tilde{u}^* = -A(x,d)^{-1}(KM_\rho + M_1)
\] (42)
where \( K \in \mathbb{R}^{4 \times (\rho_1 + \cdots + \rho_4)} \) is the first 4 row of the matrix \( T_3^{-1}T_2^T \in \mathbb{R}^{4(\tau + 1) \times (\rho_1 + \cdots + \rho_4)} \) where the \( ij \)th block of \( T_2 \) is of \( \rho_i \times 4 \) matrix, and all its elements are zeros except the \( i \)th column is given by
\[
\begin{bmatrix}
q_1(\rho_1, \ldots, \rho_4)^{2\rho_i + j - 1} & \cdots & q_4(\rho_1, \ldots, \rho_4)^{2\rho_i + j - 1}
\end{bmatrix}^T
\] (43)
for \( i = 1, 2, 3, 4 \) and \( j = 1, 2, \ldots, r + 1 \), and \( ij \)th block of \( T_3 \) is given by
\[
\text{diag}\left\{q_1(\rho_1, \ldots, \rho_4)^{2\rho_i + j - 1}, \ldots, q_4(\rho_1, \ldots, \rho_4)^{2\rho_i + j - 1}\right\}
\] (44)
for \( i, j = 1, 2, \ldots, r + 1 \); the matrix \( M_\rho \in \mathbb{R}^{\rho_1 + \cdots + \rho_4} \) and matrix \( M_i \in \mathbb{R}^4 \) are defined as:
\[
M_\rho = \begin{bmatrix}
\tilde{Y}_1(t)^T \\
\vdots \\
\tilde{Y}_4(t)^T
\end{bmatrix} - \begin{bmatrix}
\tilde{W}_1(t)^T \\
\vdots \\
\tilde{W}_4(t)^T
\end{bmatrix}
\] (45)
and
\[
M_i = \begin{bmatrix}
L_1^{\rho_1 + i - 1}h_1(t) \\
L_2^{\rho_1 + i - 1}h_2(t) \\
\vdots \\
L_4^{\rho_1 + i - 1}h_4(t)
\end{bmatrix} - \tilde{W}_i(t)^T, i = 1, 2, \ldots, r + 1.
\] (46)
The detailed derivation and closed-loop stability can refer to (Chen et al., 2003). The overall controller structure is shown in Fig.2.

![ENMPC structure](image)

**Figure 2: ENMPC structure**

The system has a trivial zero dynamics as $\rho_1 + \rho_2 + \rho_3 + \rho_4 = 14$, which is the order of the helicopter dynamics plus the dynamic extension. If the disturbance terms are set to zero, the controller is equivalent to that designed using the nominal model. The information of disturbances are hold in the controller to eliminate their influences.

In order to implement the above control strategy, the disturbances must be available which is unrealistic for helicopter flight. Next section will introduce a nonlinear disturbance observer to estimate these unavailable disturbances.
4. Disturbance observer based control

4.1. Disturbance observer

For a system like a small-scale helicopter, precisely modelling its dynamics or directly measuring the disturbances acting on it is very difficult. However, the disturbance observer technique provides an alternative way to estimate them. In this section, we introduce a nonlinear disturbance observer to estimate the lumped unknown disturbances $d$ in the general form of helicopter model (9). The disturbance observer is given by (Chen, 2004),

$$\begin{align*}
\hat{d} &= z + p(x) \\
\dot{z} &= -l(x)g_2(x)z - l(x)(g_2(x)p(x) + f(x) + g_1(x)u)
\end{align*}$$

(47)

where $\hat{d} = [\hat{d}_x \quad \hat{d}_y \quad \hat{d}_z \quad \hat{d}_{lat} \quad \hat{d}_{lon} \quad \hat{d}_{ped}]$ is the estimation of disturbances; $z$ is the internal state of the nonlinear observer, $p(x)$ is a nonlinear function to be designed, and $l(x)$ is the nonlinear observer gain given by

$$l(x) = \frac{\partial p(x)}{\partial x}$$

(48)

In this observer, the estimation error is defined as $e_d = d - \hat{d}$. Under the assumption that the disturbance is slowly varying compared to the observer dynamics and by combining Eq.(47)-Eq.(48) and Eq.(9), it can be shown that the estimation error has the following property:

$$\begin{align*}
\dot{e}_d &= \dot{d} - \dot{\hat{d}} \\
&= \dot{z} - \frac{\partial p(x)}{\partial x} \dot{x} \\
&= -l(x)g_2(x)e_d
\end{align*}$$

(49)

Therefore, $\hat{d}(t)$ approaches $d(t)$ exponentially if $p(x)$ is chosen such that Eq.(49) is globally exponentially stable for all $x \in \mathbb{R}^n$. 

21
The design of a disturbance observer essentially is to choose an appropriate gain $l(x)$ and associated $p(x)$ such that the convergence of estimation error is guaranteed. Thereby, there exist a considerable degree of freedom. Since the disturbance input matrix $g_2(x)$ for the helicopter model is a constant matrix as:

$$
g_2(x) = \begin{bmatrix}
0_{3\times3} & 0_{3\times3} \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
L_{lat} & L_{lon} & 0 \\
0_{3\times3} & M_{lat} & M_{lon} \\
0 & 0 & N_{ped} \\
0_{3\times3} & 0_{3\times3}
\end{bmatrix}, \quad (50)
$$

we can choose $l(x)$ as a constant matrix such that all the eigenvalues of matrix $-l(x)g_2(x)$ have negative real part. Next, integrating $l(x)$ with respect to the helicopter state $x$ yields $p(x) = l(x)x$. The observer gain matrix $l(x)$ corresponding to $g_2$ is designed in the form:

$$
l(x) = \begin{bmatrix}
0_{3\times3} & L_1 & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & L_2 & 0_{3\times3}
\end{bmatrix}, \quad (51)
$$

where matrix $L_1 = \text{diag}\{l_1, l_2, l_3\}$ and

$$
L_2 = \text{diag}\{l_4, l_5, l_6\} \begin{bmatrix}
L_{lat} & L_{lon} & 0 \\
M_{lat} & M_{lon} & 0 \\
0 & 0 & N_{ped}
\end{bmatrix}^{-1}, \quad (52)
$$
for $l_i > 0$, $i = 1, \ldots, 6$. Thereby, $-l(x)g(x) = -\text{diag}\{l_1, \ldots, l_6\}$. Form the above analysis, it can be seen that the convergence of the disturbance observer is guaranteed regardless of the helicopter state $x$.

### 4.2. Composite controller

External force and torque disturbances generated by wind, turbulences and other factors coupled with modelling errors and uncertainties may significantly degrade the helicopter tracking performance, may even cause instability unless their influence has been properly taken into account in the system design. It can be noted that in the previous derivation of ENMPC, the lumped disturbances appear in the control law. Therefore, once the disturbance observer provides the estimation of disturbances, ENMPC controller takes into account the disturbances by replacing the disturbance by their estimation and achieves desired tracking performance. Let $d = [d_f \ d_e]$, $d_f = [d_x \ d_y \ d_z]$ and $d_e = [d_{lat} \ d_{lon} \ d_{ped}]$. The composite controller law using the estimated disturbances is given in

$$\hat{u} = -A(x, \hat{d}_f)^{-1}(K\hat{M}_p + \hat{M}_1)$$  \hspace{1cm} (53)

where, the hatted variables denote the estimated values. If we consider trim errors in the helicopter dynamics, the overall control is in

$$u = \hat{u} - \hat{u}_0$$  \hspace{1cm} (54)

where $\hat{u}_0 = [\hat{d}_{lat} \ \hat{d}_{lon} \ 0 \ \hat{d}_{ped}]^T$ is the control trim error estimated by the disturbance observer. The composite controller structure is illustrated in Fig.3.
5. Stability analysis

The stabilities of the ENMPC and the disturbance observer are guaranteed in their design procedures outlined in Section 3 and 4, respectively. However, the stability of closed-loop system still needs to be examined, because the true disturbances are replaced by their estimation in the composite controller (54), and there are interactions among the ENMPC, the disturbance observer and the helicopter dynamics.

The closed-loop dynamics under the composite control law can be examined by applying Eq.(54) into helicopter model (9). Since the resulting system is too complicated, we define a new coordinates to describe the closed-loop system. First, let position tracking error defined as:

\[ z^0_p = [x - w_1 \ y - w_2 \ z - w_3]^T \]  

(55)

then it first derivative can be defined as:

\[ z^1_p = z^1_p = [\dot{x} - \dot{w}_1 \ \dot{y} - \dot{w}_2 \ \dot{z} - \dot{w}_3]^T \]  

(56)
where the expressions of $\dot{x}$, $\dot{y}$ and $\dot{z}$ are given in Eq.(18). Since the real disturbances are replaced by their estimations in the closed-loop system, we define the next state as:

$$z_p^2 = R_b^i \begin{bmatrix} \dot{d}_x \\ \dot{d}_y \\ T + \dot{d}_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} - \begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \end{bmatrix}, \quad (57)$$

Invoking Eq.(18) and (20) gives $z_p^1 = z_p^2 + R_b^i \cdot e_{df}$. Similarly, $z_p^3$ is defined as:

$$z_p^3 = R_b^i \dot{\omega} \begin{bmatrix} \dot{d}_x \\ \dot{d}_y \\ T + \dot{d}_z \end{bmatrix} + R_b^i \begin{bmatrix} 0 \\ 0 \\ Z_w \dot{w} + Z_{col} \dot{\delta}_{col} \end{bmatrix} - \begin{bmatrix} \dddot{w}_1 \\ \dddot{w}_2 \\ \dddot{w}_3 \end{bmatrix} \quad (58)$$

From Eq.(57) and (58) and recalling observer dynamics (49), it can be observed that

$$\dot{z}_p^2 = z_p^3 - R_b^i \dot{\omega}_{df} = z_p^3 + R_b^i L_1 e_{df} \quad (59)$$

Repeat this procedure, $z_p^4$ is defined from Eq.(24) by using estimated disturbances, such that

$$\dot{z}_p^3 = z_p^4 + R_b^i \dot{\omega} \cdot L_1 e_{df} \quad (60)$$

In addition, the heading tracking error and its derivatives are defined as $z_\psi^0 = \psi - w_4$, $z_\dot{\psi}^1 = \dot{\psi} - \dot{w}_4$, where $\dot{\psi}$ is given in Eq.(19) and $z_\ddot{\psi}^2 = \ddot{\psi} - \ddot{w}_4$, where $\ddot{\psi}$ is provided in Eq.(21).
Finally, by invoking Eq.(26) and the definitions of $z_p^4$ and $z_\psi^2$, we have

$$
\begin{bmatrix}
  z_p^4 \\
  z_\psi^2
\end{bmatrix} = \hat{M}_1 + A(\mathbf{x}, \mathbf{d}_f)(\mathbf{u} + \mathbf{u}_0) = \hat{M}_1 + A(\mathbf{x}, \mathbf{d}_f)(-A(\mathbf{x}, \mathbf{d}_f)^{-1}(K\hat{M}_p + \hat{M}_1) - \hat{\mathbf{u}}_0 + \mathbf{u}_0)
$$

$$
= -K\hat{M}_p + A(\mathbf{x}, \mathbf{d}_f)e_{u_0}
$$

where, $e_{u_0} = \mathbf{u}_0 - \hat{\mathbf{u}}_0$ and $K$ has the form:

$$
K = \begin{bmatrix}
  k_{11} \cdots k_{14} & 0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 2} \\
  0_{1 \times 4} & k_{21} \cdots k_{24} & 0_{1 \times 4} & 0_{1 \times 2} \\
  0_{1 \times 4} & 0_{1 \times 4} & k_{31} \cdots k_{34} & 0_{1 \times 2} \\
  0_{1 \times 4} & 0_{1 \times 4} & 0_{1 \times 4} & k_{41} \cdots k_{42}
\end{bmatrix}
$$

By recalling the definition of $\hat{M}_p$ in Eq.(45), Eq.(61) can be written as:

$$
\begin{bmatrix}
  z_p^3 \\
  z_\psi^1
\end{bmatrix} = \begin{bmatrix}
  K_1 z_p^0 + K_2 z_p^1 + K_3 z_p^2 + K_4 z_p^3 \\
  k_41 z_\psi^0 + k_42 z_\psi^1
\end{bmatrix} + \begin{bmatrix}
  \mathbf{R}_{\mathbf{d}_f}^i \mathbf{\hat{\omega}} \cdot \mathbf{L}_1 e_{d_f} \\
  0
\end{bmatrix} + A(\mathbf{x}, \mathbf{d}_f)e_{u_0}
$$

where $K_i = \text{diag}(k_{1i}, k_{2i}, k_{3i})$, for $i = 1, 2, 3, 4$.

Summarizing Eq.(55)-Eq.(63) gives a linear form of the closed-loop system:

$$
\begin{bmatrix}
  z_p^0 \\
  z_p^1 \\
  z_p^2 \\
  z_p^3 \\
  z_\psi^0 \\
  z_\psi^1
\end{bmatrix} = \begin{bmatrix}
  0_{3 \times 3} & \mathbf{I}_3 & 0_{3 \times 3} & 0_{3 \times 3} & 0 \\
  0_{3 \times 3} & 0_{3 \times 3} & \mathbf{I}_3 & 0_{3 \times 3} & 0 \\
  0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & \mathbf{I}_3 & 0 \\
  \mathbf{K}_1 & \mathbf{K}_2 & \mathbf{K}_3 & \mathbf{K}_4 & 0 \\
  0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 1 \\
  0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & k_{41} \mathbf{k}_{41}
\end{bmatrix} + \begin{bmatrix}
  0_{3 \times 1} \\
  \epsilon_1 \\
  \epsilon_2 \\
  \epsilon_3 \\
  \epsilon_4 \\
  \epsilon_5
\end{bmatrix}
$$

\[26\]
or, compactly
\[
\dot{z} = A_z z + \epsilon
\]  
(65)
where \( \epsilon_1 = R \hat{b} \cdot e_{d_f}, \epsilon_2 = R_l \hat{L} \cdot e_{d_f}, \epsilon_3 = R_{\hat{b}} \hat{\omega} \cdot L_1 e_{d_f} + \overline{A}(x, \hat{d}_f)e_{u_0} \) and \( \epsilon_5 = A(x, \hat{d}_f)e_{u_0} \). All these terms depend on the helicopter states and estimation errors \( e_d \).

System (64) can be classified as a cascade system in the new coordinates:
\[
\begin{align*}
\dot{z} &= f_1(z) + \epsilon(x, e_d) e_d \\
\dot{e}_d &= f_2(e_d)
\end{align*}
\]  
(66)
where the upper system is Eq(64) and the lower system is the observer dynamics (49).

When estimation errors are zeros, the upper system \( \dot{z} = f_1(z) \) reduces to a linear system \( \dot{z} = A_z z \). Its globally asymptotic stability can be guaranteed by proper choosing MPC gain \( K \) such that \( A_z \) is Hurwitz. In this case, it can be achieved by setting control order \( r > 2 \) (Chen et al., 2003). On the other hand, the globally asymptotic stability of the lower system is guaranteed during the design of disturbance observer by letting \( L > 0 \). Therefore, the closed-loop system under the composite control law is at least locally asymptotic stable according to Isidori (1995). We can extend the above result further by introducing the following lemma.

**Lemma 1.** (Panteley and Loria, 1998) If assumptions A1-A3 below are satisfied then the cascaded system (66) is globally uniformly asymptotically stable.

**A 1.** The system \( \dot{z} = f_1(z) \) is globally uniformly asymptotically stable with a Lyapunov function \( V(z), V : \mathbb{R}^n \to \mathbb{R} \) positive definite (that is \( V(0) = 0 \) and
\( V(z) > 0 \) for all \( z \neq 0 \) and proper which satisfies

\[
\begin{align*}
\left\| \frac{\partial V}{\partial z} \right\| \|z\| & \leq c_1 V(x), \forall \|z\| \geq \eta
\end{align*}
\]

(67)

where \( c_1 > 0 \) and \( \eta > 0 \).

A 2. The function \( \epsilon(x, e_d) \) satisfies

\[
\|\epsilon(x, e_d)\| \leq \alpha_1(||e_d||) + \alpha_2(||e_d||) \|z\|
\]

(68)

where \( \alpha_1, \alpha_2 : \mathbb{R} \to \mathbb{R} \) are continuous.

A 3. Equation \( \dot{e}_d = f_2(e_d) \) is globally uniformly asymptotically stable and for all \( t \geq t_0 \)

\[
\int_{t_0}^{\infty} ||e_d(t)|| \, dt \leq \beta(||e_d(t_0)||)
\]

(69)

where function \( \beta \) is a class \( K \) function.

The rigorous proof of lemma 1 is given in (Panteley and Loria, 1998). The basic idea is first to show that the upper system of cascade system does not escape to infinite in finite time and are bounded for \( t > t_0 \) with the condition that the input vector \( \epsilon(x, e_d) \) grows linearly and at the fastest in the state \( z \). Then it needs to show that as \( t \to \infty \), estimation error \( e_d \to 0 \) and \( z \to 0 \) due to the global asymptotic stability of \( \dot{z} = f_1(z) \).

Theorem 1. Given that the reference trajectory \( w \), its first \( \rho_i \) derivatives, and disturbance \( d \) are bounded, the closed-loop system (61) under the composite control is globally asymptotically stable.

Proof. By using Lemma 1, for closed-loop system (61) in the cascade form (66) if assumptions A1-A3 are satisfied, the proof will then be completed.
First, A1 is satisfied due to the fact that \( \dot{z} = f_1(z) = A_z z \) and \( A_z \) is Hurwitz. Then, we investigate the boundness on \( \epsilon(x, e_d) \) in terms of \( \|z\| \) and \( \|e_d\| \). From their definitions, we have

\[
\begin{align*}
\|\epsilon_1\| & \leq \|e_d\| \\
\|\epsilon_2\| & \leq \|L_1\| \|e_d\| \\
\|\epsilon_3\| & \leq \|\hat{\omega}\| \|L_1\| \|e_d\| + \|\bar{A}(\cdot, \cdot)\| \|e_d\| \\
\|\epsilon_5\| & \leq \|\bar{A}(\cdot, \cdot)\| \|e_d\|
\end{align*}
\]

(70) (71) (72) (73)

The skew-matrix \( \hat{\omega} \) can be seen consisted of nominal state decided by the reference command and the error state, i.e. \( \hat{\omega} = \hat{\omega}_c + \hat{\omega}_e \). The former one is bounded as the bounded command, and the latter one is bounded by tracking error \( \|z\| \). Therefore, there exist two constant \( b_1 > 0 \) and \( b_2 > 0 \), such that \( \|\hat{\omega}\| \leq b_1 + b_2 \|z\| \). Moreover, \( \|\bar{A}(\cdot, \cdot)\| \) linearly depends on \( \hat{d} \) and state \( T \). Due to \( d \) is bounded and disturbance observer is globally exponentially stable, \( \hat{d} \) is also bounded. On the other hand, \( T \) is bounded according to Eq(57). Hence, we have \( \|\bar{A}(\cdot, \cdot)\| \leq b_3 + b_4 \|z\| \), for some \( b_3 > 0 \) and \( b_4 > 0 \). Then the bound on \( \epsilon_3 \) can be write as \( \|\epsilon_3\| \leq \beta_1 \|e_d\| + \beta_2 \|e_d\| \|z\| \), for some \( \beta_1 > 0 \) and \( \beta_2 > 0 \). At last, following the same fashion \( \epsilon_5 \leq b_5 \|e_d\| \), for some \( b_5 > 0 \) if pitch angle \( \theta \neq \pm \pi/2 \). Combining bounds on \( \|\epsilon_i\|, i = 1, \ldots, 5 \) gives

\[
\begin{align*}
\|\epsilon\| & \leq \|\epsilon_1\| + \cdots + \|\epsilon_5\| \\
& \leq \gamma_1 \|e_d\| + \gamma_2 \|e_d\| \|z\|
\end{align*}
\]

(74)

where \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \). Thus, A2 is satisfied.

Finally, as lower system \( \dot{e}_d = f_2(e_d) \) is globally exponentially stable, A3 is satisfied. \( \square \)
6. Simulation and experiment

The simulation and experiment are based on a T rex-250 miniature helicopter which is a 200-sized helicopter with a main rotor diameter of 460mm and a trail rotor diameter of 108mm. The miniaturized size and 3D aerobatic ability make it well-suited for indoor flight test. Moreover, T rex-250 has a collective pitch rotor and well designed Bell-Hiller stabilizer mechanism that makes it represent most of widely used small-scale helicopters.

Numerical simulations are first carried out to investigate the proposed control framework. In the simulation, it is assumed that there are 20% uncertainties on the model parameters. Furthermore, there is a constant wind disturbance with speed of \(5\, \text{m/s}\) acting on the helicopter. The helicopter is required to track an eight-shape trajectory with and without the compensation of DOBC. The tracking results are shown in Fig. 4 and the corresponding cyclic controls are given in Fig. 5.

It can be seen from the simulation that the ENMPC is able to deal with uncertainties and achieve satisfactory tracking, but it cannot compensate the steady state error mainly caused by the wind disturbance. In contrast, the action of DOBC taking into account the disturbances from both external and internal sources eliminates the tracking error and smooths the control signals.

Several flight tests have been designed to investigate the control performance of the proposed scheme on real helicopter. The first test presented here is a hovering and perturbation test. The helicopter was required to take-off and hover at the origin at height of 0.5m. A wind perturbation was then applied on the helicopter by posing a fan in front of the helicopter (see
Figure 4: eight-shape tracking
Figure 5: tracking controls
Figure 6: Hovering and disturbance test

The test result is given in Fig.7. In the test, the helicopter was first under the control of ENMPC to perform take-off and hovering. It can be seen that the ENMPC stabilised the helicopter but with a steady state error due to the mismatch between the model used for ENMPC design and the real helicopter dynamics. After 25 seconds, the disturbance observer switched on and the composite controller took action to bring the helicopter to the setpoint. After 60 seconds, the fan was turned on to generate the wind gust. The average wind speed is about 3m/s, which is significant strong for our test helicopter with a small dimension. This can be observed from the attitude history in Fig.10, where the magnitude of pitch and roll angles
of the helicopter dramatically surged after wind gust is applied. However, the composite controller exhibited an excellent performance under the with the wind gust and maintained the helicopter position very well. The force disturbances estimated by disturbance observer are given in Fig.8, and the control signals are illustrated in Fig.9.

It is also interesting to see where the disturbances come from without external wind gust, and this will also explain why ENMPC based on the nominal model cannot achieve asymptotic tracking if the helicopter is not trimmed properly. By recalling helicopter dynamics model (1) and consider-
ing steady-state model, we have

\begin{align*}
0 &= -g \sin \theta_0 + d_x \\
0 &= g \cos \theta_0 \sin \phi_0 + d_y
\end{align*}

(75)

where $\phi_0$ and $\theta_0$ are the trim attitude depending on the particular helicopter. The trim attitude may be attributed to asymmetrical structure and model uncertainties. Their values are small and usually can be ignored in the theoretical analysis, but they do affect the actual control performance as they project vertical lift to longitudinal and lateral directions. This phenomena can be further explained by carefully examining the measurement from the flight test. Observing the attitude measurement in Fig.10 reveals the average roll and pitch angles are about 0.01rad, which contribute 0.1$m/s^2$ and $-0.1m/s^2$ to $d_x$ and $d_y$ according to Eq.(75), respectively. The estimated $d_x$,
Figure 9: Control signals
from observer is very close to our rough calculation, whereas the estimated $d_y$ is smaller than what we expected. This is because that the tail rotor also generates lateral thrust that has not been taken into account in the nominal model. The above quantitative analysis gives a good confidence on the proposed disturbance observer.

![Helicopter attitude](image)

Figure 10: Helicopter attitude

Unlike the conventional MPC being restricted to a low control bandwidth, the high bandwidth that ENMPC can reach makes it a suitable candidate for controlling helicopter to perform acrobatic manoeuvres. In the second flight test, the helicopter was controlled to perform a pirouette manoeuvre, in which helicopter started from the hovering position and flew along a straight line while pirouetting at a yaw rate of 120 deg/sec. This is an extremely
challenging flight pattern, because the lateral and longitudinal controls are strongly coupled by the rotation, and they have to coordinate with each other to achieve a straight progress. Besides, the varying position of the tail rotor with respect to the progress direction poses severe disturbances on the forward flight. The result from the flight test is shown in Fig.11 and the control signals are provided in Fig.12. It can be seen that the helicopter under the control of ENMPC executed the manoeuvre with a high quality.

![Helicopter trajectory and Reference trajectory](image-url)

**Figure 11: pirouette manoeuvre results**
7. Conclusions

This paper describes a composite control framework for trajectory tracking of autonomous helicopters. The nonlinear tracking control is achieved by an explicit MPC algorithm, which eliminates the computational intensive online optimisation in the traditional MPC. On the implementation side, the introducing of disturbance observer solves the difficulties of applying model based control technique into the practical environment. The design of ENMPC provides a seamless way of integrating the disturbance information. On the other hand the robustness and disturbance attenuation of the controller are enhanced by the nonlinear disturbance observer.

Simulation and experiment results show promising performance of the combination of ENMPC and DO. Apart from the reliable tracking that the
proposed controller guarantees, it also has the ability of estimating the helicopter trim condition during the flight which helps controller to deal with the variation of the helicopter status like payload changing and component upgrades.

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