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Absolute vs. relative formulations of the moving oscillator problem

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Abstract

In the general framework of the bridge-vehicle dynamic interaction, the so-called “moving oscillator” problem is revisited in order to provide a deeper insight into some analytical and physical aspects not specifically analyzed in previous investigations. Without lack of generality, the case of a stream of moving oscillators crossing a simply supported beam with arbitrary time law is considered. The formulations in terms of both absolute and relative displacements of the moving oscillators are critically reviewed and compared, and alternative sets of differential equations with time-dependent coefficients are derived. The study enlightens, both theoretically and numerically, that impulsive contributions to the dynamic response appear in the relative displacement formulation at the time instants in which each vehicle enters or exits the bridge. It is demonstrated that such contributions, somehow “hidden” in the absolute displacement formulation, may have a significant influence on the vibration of the moving oscillators, and thus cannot be a priori neglected in the analysis. It is also shown that the analytical and computational difficulties associated with these additional impulses make preferable the use of the absolute displacement formulation. Far from being
restricted to the case of simply supported beams, these findings are valid for any type of bridge structure which induces a discontinuity in the slope of the road profile experienced by the vehicles.

**Keywords:** Moving loads; Moving oscillators; Bridge-vehicle dynamic interaction; Dirac’s delta functions; Bending vibration of flexible beams; Modal equations of motion.

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1 Introduction

A vast literature has been devoted to the dynamic analysis of distributed-parameter systems, such as beams and cables, crossed by one or more subsystems (see, e.g. Tzou and Bergman, 1998; Frýba, 1999). Indeed, this topic is of great interest in many engineering applications, such as the design of bridges, railway tracks, cableways, etc. It has long been recognized that the passage of a vehicle may induce significant impact (or dynamic amplification) effects in the supporting structure. For this reason, in the last decades several studies have focused on the problem of bridge-vehicle dynamic interaction (see, e.g. Yang et al., 2004). The simplest vehicle model adopted in the literature with the aim of investigating this phenomenon is the so-called “moving oscillator” model. This type of moving subsystem is characterized by finite coupling stiffness and damping between a single lumped mass and the supporting structure, and allows to take into account the main vibrational properties of the vehicle; thus, it turns out to be more realistic than the well-known “moving force” and “moving mass” models. Furthermore, multiple moving oscillators at fixed relative distance can be used to simulate the
behaviour of complex vehicles without resorting to more sophisticated models, with many degrees of freedom.

This paper revisits the moving oscillator problem, extensively studied in the literature, with the intention of highlighting some theoretical and practical features which, to the authors’ best knowledge, have not been addressed in previous investigations. First, in the vast majority of contributions the problem is formulated in terms of absolute displacements of the moving oscillator (Chatterjee et al., 1994; Pesterev and Bergman, 1997a, 1997b, 1998; Muscolino et al., 2001; Chen et al., 2002; Pesterev et al., 2003; Chang et al., 2006; Stancoiu et al., 2008), while comparatively few authors make use of relative displacements (Yang et al., 2000; Biondi and Muscolino, 2005; Muscolino and Palmeri, 2007; Muscolino et al., 2007a, 2007b). However, advantages and disadvantages related to the use of the two formulations seem to have never been discussed in the literature through a comprehensive comparison. Furthermore, the attention is usually focused on the time interval in which a single vehicle crosses the bridge, while just few studies concerning multiple moving oscillators are available in the literature (see, e.g. Cheng et al., 1999; Pesterev et al., 2001; Biondi and Muscolino, 2005). In particular, as far as the authors know, the dynamic effects arising when a vehicle crosses the supports of the bridge have not been specifically investigated by other researchers.

In the present study, with the intent of providing a deep insight into the aforementioned aspects, the bridge-vehicle dynamic interaction is scrutinized by considering a simply supported Bernoulli-Euler beam vibrating under a stream of moving oscillators. Both the formulations in terms of absolute and relative displacements are reviewed and compared. In particular, it is analytically demonstrated and numerically confirmed that impulsive terms arise in the relative displacement formulation at the time instants in which each vehicle enters or exits the bridge (Muscolino et al., 2007b); that is, Dirac’s delta functions ideally centred at bridge supports appear in the governing equations. It is worth emphasizing that such terms, which stem from the so-called convective acceleration, do not arise when absolute displacements of the
oscillators are considered, being in some sense “hidden” as the Dirac’s delta functions enlightened by Makris (1997) in a completely different context. As a matter of fact, numerical results prove that in the relative displacement formulation, especially when many oscillators are dealt with, these impulsive terms may have a significant influence on the vibration of the moving oscillators, and therefore cannot be a priori neglected in the dynamic analysis. Classical mode superposition technique is applied to get an approximate solution of the problem, formulated either in terms of absolute or relative displacements. In both cases, a set of coupled ordinary differential equations with time-dependent coefficients is obtained. More precisely, when the absolute displacements of the moving oscillators are used, the mass matrix in the modal space is a constant diagonal matrix, while stiffness and damping matrices are time-dependent ones; furthermore, the stiffness matrix turns out to be non symmetric. In the relative displacement formulation, on the contrary, all the matrices are time-dependent ones; moreover, only the mass matrix is symmetric, while impulsive terms arise in the stiffness matrix at the entrance and exit times of each moving oscillator. Interestingly, simple transformations of coordinates provided in Appendix allow to pass from the matrices in the space of absolute displacements to the corresponding ones in the space of relative displacements, and vice versa.

For validation purposes, a simply supported beam carrying a pair of moving oscillators is analyzed by applying the two formulations derived in the paper. Numerical results in terms of beam’s transverse displacements and beam-oscillator interaction forces are presented and discussed, focusing the attention on the effects of the impulses in the relative displacement formulation. It has to be emphasized that these results are not restricted to the case of simply supported beams only, being valid for any type of bridge structure where a discontinuity appears in the slope of the road profile experienced by the vehicles. In all these situations the analytical and computational burden associated with the additional impulses arising in the relative displacement formulation make preferable the one in terms of absolute displacements of the moving subsystems.
2 Simply supported bridge carrying multiple moving oscillators

Without lack of generality, let us consider a single-span simply supported bridge, initially at rest, crossed by \( n \) vehicles moving from left to right (Fig. 1). The bridge structure is treated as a homogeneous Bernoulli-Euler beam of length \( l_b \), mass density \( \rho_b \), modulus of elasticity \( E_b \), and cross section with area \( A_b \) and second moment \( J_b \). The approaches are regarded as rigid soil (but a flexible model could be also used). The \( i \)-th vehicle is modeled as a moving oscillator with mass \( m_{v,i} \), elastic stiffness \( k_{v,i} \) and viscous damping \( c_{v,i} \). The moving subsystems are supposed to be in permanent contact with the road surface along bridge and approaches.

Assuming that the only dynamic loads are those due to the \( n \) moving vehicles, the bridge vibration is governed by the partial differential equation:

\[
\rho_b A_b \frac{\partial^2 u_b(z,t)}{\partial t^2} + E_b J_b \frac{\partial^4 u_b(z,t)}{\partial z^4} + D_b(z,t) = f^{(s)}_b(z) + \sum_{i=1}^{n} \chi_b(z_{v,i}(t)) f_{v,i}(t) \delta(z-z_{v,i}(t)),
\]

where \( t \) and \( z \) denote the time and the spatial coordinate measured along the axis of the beam, respectively; \( u_b(z,t) \) is the field of the beam’s transverse displacements, positive if downward; \( f^{(s)}_b(z) \) is the field of the transverse static forces distributed along the beam (Fig. 2 top), which in practice may be the effects of gravitational dead and live loads, and of prestressing systems as well; \( D_b(z,t) \) is the internal damping force per unit length of the beam; \( f_{v,i}(t) \) is the point force transmitted by the \( i \)-th traveling oscillator (Fig. 2 centre), which is located at the instantaneous position \( z_{v,i}(t) \); \( \chi_b(z) \) denotes the so-called window function for the supporting beam (Fig. 2 bottom), given by:

\[
\chi_b(z) = \mathcal{U}(z) - \mathcal{U}(z - l_b);
\]

\( \mathcal{U}(z) \) being the unit-step function, so defined: \( \mathcal{U}(z) = 0 \) when \( z < 0 \); \( \mathcal{U}(z) = 1/2 \) at \( z = 0 \); \( \mathcal{U}(z) = 1 \) when \( z > 0 \); and \( \delta(z) = \partial \mathcal{U}(z)/\partial z \) is the Dirac’s delta function, symmetric with respect to \( z = 0 \). It is recalled that the attendant boundary conditions for the simply supported
beam modeling the bridge read: \[ u_b(z,t) \big|_{z=0} = u_b(z,t) \big|_{z=l_b} = 0 \] and \[ \frac{\partial^2 u_b(z,t)}{\partial z^2} \big|_{z=0} = \frac{\partial^2 u_b(z,t)}{\partial z^2} \big|_{z=l_b} = 0. \]

Clearly, when all the oscillators are outside the bridge, the last term in Eq. (1) vanishes. Conversely, when the oscillators move along the bridge, additional excitations arise as a consequence of bridge-vehicle dynamic interaction, which is realized by the \( n \) concentrated forces \( f_{v,j}(t) \) appearing in the right-hand side of Eq. (1). Such forces depend on the response of both the supporting beam and the moving oscillators, thus making the governing equations coupled. It is worth emphasizing that the interaction force \( f_{v,j}(t) \) takes different forms depending on whether the motion of the oscillators is described in terms of absolute or relative displacements, as will be outlined next in detail.

An approximate solution of the problem under consideration can be derived by applying the classical modal analysis. Accordingly, the field of transverse displacements \( u_b(z,t) \) of the simply supported beam can be approximated as:

\[
 u_b(z,t) = \sum_{j=1}^{n_b} \phi_{b,j}(z)q_{b,j}(t) = \Phi_b(z)q_b(t), \tag{3}
\]

where:

\[
 \Phi_b(z) = \left\{ \phi_{b,1}(z), \phi_{b,2}(z), \ldots, \phi_{b,n_b}(z) \right\}^T,
\]

\[
 q_b(t) = \left\{ q_{b,1}(t), q_{b,2}(t), \ldots, q_{b,n_b}(t) \right\}^T, \tag{4a,b}
\]

are the vectors collecting the first \( n_b \) modal shapes of the simply supported beam, \( \phi_{b,j}(z) = \sqrt{2/(\rho_b A_b l_b)} \sin(j \pi z/l_b) \), orthonormal with respect to the mass per unit length \( \rho_b A_b \), and the associated \( n_b \) modal coordinates, \( q_{b,j}(t) \), respectively.

Substituting Eq. (3) into Eq. (1), pre-multiplying both sides by \( \Phi_b(z) \), and integrating with respect to \( z \) between 0 and \( l_b \), the following set of \( n_b \) ordinary differential equations with time-independent coefficients is obtained:
\begin{equation}
\ddot{q}_b(t) + \Xi_b \dot{q}_b(t) + \Omega_b^2 q_b(t) = p_b^{(s)} + \sum_{i=1}^{n_b} \chi_b \left(z_{\nu_i}(t)\right) f_{\nu_i}(t) \phi_b \left(z_{\nu_i}(t)\right),
\end{equation}

where the over-dot denotes total derivative with respect to time $t$ and:

\[
\Omega_b = \text{Diag} \left[ \omega_{b,1}, \omega_{b,2}, \ldots, \omega_{b,n_b} \right],
\]
\[
\Xi_b = 2\zeta_b \Omega_b; \quad \chi_b = \int_{0}^{L} \phi_b(z) f_b^{(s)}(z) \, dz.
\]

In the previous equations, $\text{Diag} \left[ \cdot \right]$ means diagonal matrix, $\omega_{b,j} = (j \pi / l_b)^2 \sqrt{E_b J_b / (\rho_b A_b)}$ denotes the $j$-th undamped circular frequency of the beam, and $\zeta_b$ is the modal damping ratio, herein assumed constant in all vibration modes.

In the next sections, the expressions for the interaction forces either in terms of absolute or relative displacements of the moving oscillators will be deduced. Then, in both cases the equations governing bridge-vehicle dynamics in the modal space will be derived.

### 3 Equations of motion in terms of absolute displacements

The equation of motion of the $i$-th traveling oscillator, in terms of absolute displacements, can be written as:

\begin{equation}
m_{\nu_i} \dddot{u}_{\nu_i}^{(s)}(t) = -c_{\nu_i} \left[ u_{\nu_i}^{(s)}(t) - \dddot{u}_{m_i}(t) \right] - k_{\nu_i} \left[ u_{\nu_i}^{(s)}(t) - u_{m_i}(t) \right],
\end{equation}

where $u_{\nu_i}^{(s)}(t)$ and $u_{m_i}(t)$ are the absolute displacements of the lumped mass $m_{\nu_i}$ and of the ideal point wheel, respectively (see Fig. 3). The interaction force, $f_{\nu_i}(t)$, which depends on both the reaction of the spring-dashpot system and the weight of the $i$-th oscillator, can be expressed as (see Fig. 3):

\begin{equation}
f_{\nu_i}(t) = m_{\nu_i} g + k_{\nu_i} \left[ u_{\nu_i}^{(s)}(t) - u_{m_i}(t) \right] + c_{\nu_i} \left[ \dddot{u}_{\nu_i}^{(s)}(t) - \dddot{u}_{m_i}(t) \right] = m_{\nu_i} \left[ g - \dddot{u}_{\nu_i}^{(s)}(t) \right],
\end{equation}

where $g$ is the acceleration of gravity, and Eq. (7) has been taken into account.
Assuming that there is no loss of contact between vehicles and road surface, the compatibility between bridge and vehicle vibrations is ensured if the relationship:

\[ u_{w,i}(t) = \left[ \chi_b(z) u_b(z,t) \right]_{z=z_{v_i}(t)} = \chi_b(z_{v_i}(t)) u_b(z_{v_i}(t),t) \]  \hspace{1cm} (9)

holds at the instantaneous position \( z_{v_i}(t) \) of each oscillator. Taking into account Eq. (9), the total time derivative of the absolute displacement of the wheel \( u_{w,i}(t) \), appearing in Eq. (8), is given by:

\[
\dot{u}_{w,i}(t) = \left[ \frac{d}{dt} \left( \chi_b(z) u_b(z,t) \right) \right]_{z=z_{v_i}(t)} = \frac{\partial}{\partial t} \left[ \chi_b(z) u_b(z,t) \right]_{z=z_{v_i}(t)} + \dot{z}_{v_i}(t) \frac{\partial}{\partial z} \left[ \chi_b(z) u_b(z,t) \right]_{z=z_{v_i}(t)} = \chi_b(z_{v_i}(t)) \left[ \frac{\partial u_b(z,t)}{\partial t} \right]_{z=z_{v_i}(t)} + \dot{z}_{v_i}(t) \frac{\partial u_b(z,t)}{\partial z} \right]_{z=z_{v_i}(t)},
\]  \hspace{1cm} (10)

where the geometric boundary conditions for the simply supported beam have been taken into account. Upon substitution of Eq. (3) into Eqs. (9) and (10), we get the kinematical relationships:

\[
\begin{align*}
\dot{u}_{w,i}(t) &= \chi_b(z_{v_i}(t)) \Phi_{v_i}^T(z_{v_i}(t)) \Phi_q(t) \\
&= \mathbf{a}_{\Phi_b}(t) \Phi_q(t); \\
\dot{u}_{w,i}(t) &= \chi_b(z_{v_i}(t)) \left[ \Phi_{v_i}^T(z_{v_i}(t)) \Phi_q(t) + \dot{z}_{v_i}(t) \phi_k^T(z_{v_i}(t)) \Phi_q(t) \right] \\
&= \mathbf{a}_{\Phi_b}(t) \Phi_q(t) + \Phi_{vb}(t) \Phi_q(t),
\end{align*}
\]  \hspace{1cm} (11a,b)

where the prime means total derivative with respect to \( z \), that is:

\[
\Phi'(z_{v_i}(t)) = \frac{d\Phi(z)}{dz} \bigg|_{z=z_{v_i}(t)} = \left\{ \phi_{b,1}'(z), \quad \phi_{b,2}'(z), \quad \ldots, \quad \phi_{b,nb}'(z) \right\}^T, 
\]  \hspace{1cm} (12)

with \( \phi_{b,j}'(z) = j \pi \sqrt{2/\left( \rho_{b} A_{b} l_{b}^3 \right)} \cos \left( j \pi z / l_{b} \right) \), and where the time-dependent vectors \( \mathbf{a}_{\Phi_b}(t) \) and \( \Phi_{vb}(t) \) are so defined:
Substituting Eqs. (11a,b) into Eq. (7), the equation of motion of the $i$-th moving oscillator takes the form:

$$m_{v,i} \dddot{u}_{v,i}^{(a)}(t) + c_{v,i} \dddot{u}_{v,i}^{(a)}(t) + k_{v,i} u_{v,i}^{(a)}(t) = \left[ c_{v,i} \mathbf{b}_{v,i}^T(t) + k_{v,i} \mathbf{a}_{v,i}^T(t) \right] \mathbf{q}_b(t) + c_{v,i} \mathbf{a}_{v,i}^T(t) \mathbf{q}_b(t).$$

(14)

Then, the equations governing the motion of the $n_v$ oscillators can be posed in a compact matrix form as:

$$\mathbf{M}_v \dddot{\mathbf{u}}_v(t) + \mathbf{C}_v \dddot{\mathbf{u}}_v(t) - \mathbf{C}_{vb}(t) \dddot{\mathbf{q}}_b(t) + \mathbf{K}_v \mathbf{u}_v^{(a)}(t) - \left[ \mathbf{K}_{vb}(t) + \mathbf{L}_{vb}(t) \right] \mathbf{q}_b(t) = \mathbf{0},$$

(15)

where the array of the absolute displacements of the moving oscillators is $\mathbf{u}_v^{(a)}(t) = \left\{ u_{v,1}^{(a)}(t), u_{v,2}^{(a)}(t), \ldots, u_{v,n_v}^{(a)}(t) \right\}^T$, and where:

$$\mathbf{M}_v = \text{Diag}\left[ m_{v,1}, m_{v,2}, \ldots, m_{v,n_v} \right];$$
$$\mathbf{C}_v = \text{Diag}\left[ c_{v,1}, c_{v,2}, \ldots, c_{v,n_v} \right];$$
$$\mathbf{C}_{vb}(t) = \mathbf{C}_v \mathbf{X}_v(t) \Phi_{vb}^T(t);$$
$$\mathbf{K}_v = \text{Diag}\left[ k_{v,1}, k_{v,2}, \ldots, k_{v,n_v} \right];$$
$$\mathbf{K}_{vb}(t) = \mathbf{K}_v \mathbf{X}_v(t) \Phi_{vb}^T(t);$$
$$\mathbf{L}_{vb}(t) = \mathbf{C}_v \mathbf{X}_v(t) \dot{\mathbf{Z}}_v(t) \Phi_{vb}^T(t),$$

(16a-f)

with:

$$\mathbf{X}_v(t) = \text{Diag}\left[ \mathbf{X}_b\left( z_{v,1}(t) \right), \mathbf{X}_b\left( z_{v,2}(t) \right), \ldots, \mathbf{X}_b\left( z_{v,n_v}(t) \right) \right];$$
$$\dot{\mathbf{Z}}_v(t) = \text{Diag}\left[ \dot{z}_{v,1}(t), \dot{z}_{v,2}(t), \ldots, \dot{z}_{v,n_v}(t) \right];$$
$$\Phi_{vb}(t) = \left[ \Phi_b\left( z_{v,1}(t) \right), \Phi_b\left( z_{v,2}(t) \right), \ldots, \Phi_b\left( z_{v,n_v}(t) \right) \right];$$
$$\Phi_{vb}'(t) = \left[ \Phi_b'\left( z_{v,1}(t) \right), \Phi_b'\left( z_{v,2}(t) \right), \ldots, \Phi_b'\left( z_{v,n_v}(t) \right) \right].$$

(17a-d)

Upon substitution of Eqs. (11a,b) into Eq. (8), moreover, the expression of the interaction force $f_{v,i}(t)$, in terms of beam’s modal coordinates is obtained:
\[ f_{v,i}(t) = m_{v,i} \ddot{u}_{v,i}(t) + k_{v,i} \left[ u_{v,i}(t) - a_{v,ib}^T(t)q_b(t) \right] + c_{v,i} \left[ \dot{u}_{v,i}(t) - a_{v,ib}^T(t)\dot{q}_b(t) - b_{v,ib}^T(t)q_b(t) \right]. \]  \hspace{1cm} (18)

Finally, substituting Eq. (18) into Eq. (5), the equation of motion of the bridge in the modal space becomes:

\[ \ddot{q}_b(t) + \left[ \Xi_b + \Delta C_b(t) \right] \ddot{q}_b(t) - C_{vb}^T(t) \dddot{u}_v(t) + \left[ \Omega_v^2 + \Delta K_b(t) \right] q_b(t) - K_{vb}^T(t)u_v(t) = p_b(t), \]  \hspace{1cm} (19)

where:

\[
\Delta C_b(t) = \Phi_{vb}(t)C_vX_v(t)\Phi_{vb}^T(t) = \Phi_{vb}(t)C_{vb}(t);
\]

\[
\Delta K_b(t) = \Phi_{vb}(t)\left[ K_vX_v(t)\Phi_{vb}^T(t) + C_vX_v(t)\dot{Z}_v(t)\Phi_{vb}^T(t) \right] = \Phi_{vb}(t)[K_{vb}(t) + L_{vb}(t)]; \hspace{1cm} (20a-d)
\]

\[ p_b(t) = p_b^{(v)} + p_b^{(a)}(t); \quad p_b^{(v)}(t) = g\Phi_{vb}(t)M_vX_v(t)\tau_v. \]

Interestingly, \( \Delta C_b(t) \) and \( \Delta K_b(t) \) in Eq. (19) may be viewed, respectively, as modifications of the damping and stiffness matrices in the modal space due to the passage of the moving oscillators, while \( \tau_v = \{1, 1, \ldots, 1\}^T \) in the last of Eqs. (20) is the \( n_v \)-dimensional incidence vector of the moving oscillators. Eqs. (15) and (19) can be rewritten in compact form as:

\[ M_a \ddot{u}(t) + C_a(t)\dddot{u}(t) + K_a(t)u(t) = p(t), \]  \hspace{1cm} (21)

where the subscript “a” stands for “absolute displacements” and where:

\[
M_a = \begin{bmatrix} M_v & 0 \\ 0 & I_{n_b} \end{bmatrix};
\]

\[
C_a(t) = \begin{bmatrix} C_v & -C_{vb}(t) \\ -C_{vb}^T(t) & \Xi_b + \Delta C_b(t) \end{bmatrix};
\]

\[
K_a(t) = \begin{bmatrix} K_v & -K_{vb}(t) - L_{vb}(t) \\ -K_{vb}^T(t) & \Omega_v^2 + \Delta K_b(t) \end{bmatrix}; \hspace{1cm} (22a-e)
\]

\[
u(t) = \begin{bmatrix} u_v(t) \\ q_b(t) \end{bmatrix}; \quad p(t) = \begin{bmatrix} 0_{n_v \times 1} \\ p_b(t) \end{bmatrix}, \]
the symbols \( \mathbf{I}_n \) and \( \mathbf{0}_{n \times m} \) being the identity matrix of size \( n \) and the zero matrix of dimensions \((n \times m)\), respectively. It is worth noting that, while the mass matrix \( \mathbf{M}_a \) is diagonal and constant, the matrices \( \mathbf{C}_a(t) \) and \( \mathbf{K}_a(t) \) are sparse and time-dependent ones. Furthermore, the stiffness matrix \( \mathbf{K}_a(t) \) turns out to be non-symmetric because of the submatrix \( \mathbf{L}_{vb}(t) \), whose elements are proportional to the horizontal speed of the moving oscillators.

From a numerical point of view, it is advantageous to make all the entries of the matrices appearing in Eq. (21) have the same physical dimensions. To this aim, the position

\[
u^s_v(t) = \mathbf{M}_v^{1/2} \mathbf{q}^s_v(t)
\]

is introduced, so that Eq. (21) can be rewritten as:

\[
\ddot{\mathbf{q}}(t) + \mathbf{c}(t) \dot{\mathbf{q}}(t) + \mathbf{k}(t) \mathbf{q}(t) = \mathbf{p}(t),
\]

where:

\[
\mathbf{c}(t) = \begin{bmatrix} \Xi_v & -\mathbf{M}_v^{1/2} \mathbf{C}_{vb}(t) \\ -\mathbf{C}_{vb}(t) \mathbf{M}_v^{1/2} & \Xi_v + \Delta \mathbf{C}_v(t) \end{bmatrix},
\]

\[
\mathbf{k}(t) = \begin{bmatrix} \Omega_v^2 & -\mathbf{M}_v^{1/2} \left[ \mathbf{K}_{vb}(t) - \mathbf{L}_{vb}(t) \right] \\ -\mathbf{K}_{vb}(t) \mathbf{M}_v^{1/2} & \Omega_v^2 + \Delta \mathbf{K}_v(t) \end{bmatrix},
\]

\[
\mathbf{q}(t) = \begin{bmatrix} \mathbf{q}^s_v(t) \\ \mathbf{q}_b(t) \end{bmatrix},
\]

in which:

\[
\Omega_v^2 = \text{Diag} \left[ \frac{k_{v,1}}{m_{v,1}}, \frac{k_{v,2}}{m_{v,2}}, \ldots, \frac{k_{v,n}}{m_{v,n}} \right] = \text{Diag} \left[ \omega_{v,1}^2, \omega_{v,2}^2, \ldots, \omega_{v,n}^2 \right];
\]

\[
\Xi_v = \text{Diag} \left[ \frac{\zeta_{v,1}}{m_{v,1}}, \frac{\zeta_{v,2}}{m_{v,2}}, \ldots, \frac{\zeta_{v,n}}{m_{v,n}} \right] = 2 \text{Diag} \left[ \zeta_{v,1} \omega_{v,1}, \zeta_{v,2} \omega_{v,2}, \ldots, \zeta_{v,n} \omega_{v,n} \right],
\]

\[
\omega_{v,i} = \sqrt{\frac{k_{v,i}}{m_{v,i}}} \text{ and } \zeta_{v,i} = \frac{c_{v,i}}{2 \sqrt{m_{v,i} k_{v,i}}} \text{ being undamped natural circular frequency and viscous damping ratio of the } i\text{-th moving oscillator, respectively.}
\]

Equation (23) can be solved with the help of any suitable step-by-step algorithm, e.g.
the Newmark’s $\beta$-method, to get the response of both the bridge and the vehicles. The initial conditions at the time instant $t_0$ in which the first moving oscillator approaches the beam can be assumed to be:

$$q(t_0) = \begin{bmatrix} \mathbf{0}_{n_x \times 1} \\ \mathbf{0}_{n_x \times 1} \end{bmatrix}; \quad \dot{q}(t_0) = \begin{bmatrix} \mathbf{0}_{n_x \times 1} \\ \mathbf{0}_{n_x \times 1} \end{bmatrix},$$

(26a,b)

where $q^{(s)}_b = \mathbf{\Omega}_b^{-2} p^{(s)}_b$ is the array of the modal coordinates of the beam associated with the static distributed load $f_b^{(s)}(z)$.

### 4 Equations of motion in terms of relative displacements

The equation of motion of the $i$-th traveling oscillator in terms of relative displacements can be written as (analogous to Eq. (7)):

$$m_{v,i} \ddot{u}^{(a)}_{v,i}(t) + c_{v,i} \dot{u}_{v,i}(t) + k_{v,i} u_{v,i}(t) = m_{v,i} g,$$

(27)

where $\ddot{u}^{(a)}_{v,i}(t)$ and $u_{v,i}(t)$ are, respectively, the absolute and relative displacements of the mass $m_{v,i}$, both including the static contribution $u^{(s)}_{v,i} = m_{v,i} g / k_{v,i}$ (see Fig. 3).

If there is no loss of contact between vehicle and road surface, the following relationship holds at the instantaneous position $z_{v,i}(t)$ of each oscillator (see Fig. 3):

$$\ddot{u}^{(s)}_{v,i}(t) = u^{(a)}_{v,i}(t) + u^{(s)}_{v,i} = u_{w,i}(t) + u_{v,i}(t) = \left[ \mathcal{X}_b(z) u_b(z,t) \right]_{z=z_{v,i}(t)} + u_{v,i}(t),$$

(28)

where use has been made of Eq. (9).

The interaction force $f_{v,i}(t)$, concentrated at the abscissa $z_{v,i}(t)$, coincides with the reaction of the spring-dashpot system. Thus, it can be expressed as (analogous to Eq. (8)):

$$f_{v,i}(t) = k_{v,i} u_{v,i}(t) + c_{v,i} \dot{u}_{v,i}(t) = m_{v,i} \left[ g - \ddot{u}^{(s)}_{v,i}(t) \right].$$

(29)

Taking into account that $\ddot{u}^{(s)}_{v,i}(t)$ (see Eq. (28)) depends on time $t$ both directly and
through the instantaneous position \( z_{ij}(t) \), it can be readily verified that the absolute acceleration \( \ddot{u}_{ij}^{(a)}(t) = \ddot{u}_{ij}^{(a)}(t) \) of the mass \( m_{ij} \), appearing in Eq. (29), is given by:

\[
\ddot{u}_{ij}^{(a)}(t) = A_{ij} \left[ \chi_b(z) u_b(z,t) \right]_{z=z_{ij}(t)} + \ddot{u}_{ij}(t) \tag{30}
\]

where \( A_{ij}[\bullet] \) denotes the so-called convective acceleration operator for the \( i \)-th moving oscillator:

\[
A_{ij}[\bullet] = \frac{d^2(\bullet)}{dt^2} = \frac{\partial^2 (\bullet)}{\partial t^2} + 2 \ddot{z}_{ij}(t) \frac{\partial^2 (\bullet)}{\partial z \partial t} + \dddot{z}_{ij}(t) \frac{\partial (\bullet)}{\partial z} + \dddot{z}_{ij}(t) \frac{\partial^2 (\bullet)}{\partial z^2}. \tag{31}
\]

As a result, the transverse acceleration experienced by the \( i \)-th moving oscillator when in contact with the supporting beam depends also on its longitudinal position \( z_{ij}(t) \), velocity \( \dot{z}_{ij}(t) \) and acceleration \( \ddot{z}_{ij}(t) \).

By taking into account the geometric boundary conditions for the simply supported bridge, and recalling also the properties of the derivatives of the Dirac’s delta function (Bracewell, 1999), the first- and second-order derivatives of \( \chi_b(z) u_b(z,t) \) with respect to the abscissa \( z \), required by the operator \( A_{ij}[\bullet] \) (Eq. (31)), are given by:

\[
\begin{align*}
\frac{\partial \left[ \chi_b(z) u_b(z,t) \right]}{\partial z} &= \chi_b(z) \frac{\partial u_b(z,t)}{\partial z}; \\
\frac{\partial^2 \left[ \chi_b(z) u_b(z,t) \right]}{\partial z^2} &= \chi_b(z) \frac{\partial^2 u_b(z,t)}{\partial z^2} + \frac{\partial u_b(z,t)}{\partial z} \left[ \delta(z) - \delta(z-l_b) \right].
\end{align*}
\tag{32a,b}
\]

Then, the first term in the right-hand side of Eq. (30) takes the form:

\[
A_{ij} \left[ \chi_b(z) u_b(z,t) \right]_{z=z_{ij}(t)} = \left\{ \chi_b(z) \left[ \frac{\partial^2 u_b(z,t)}{\partial t^2} + 2 \ddot{z}_{ij}(t) \frac{\partial^2 u_b(z,t)}{\partial z \partial t} + \dddot{z}_{ij}(t) \frac{\partial u_b(z,t)}{\partial z} \right. \right. \\
+ \left. \right. \left. \dddot{z}_{ij}(t) \frac{\partial^2 u_b(z,t)}{\partial z^2} \right] + \dddot{z}_{ij}(t) \frac{\partial u_b(z,t)}{\partial z} \left[ \delta(z) - \delta(z-l_b) \right] \right\}_{z=z_{ij}(t)} \tag{33}
\]

What Eqs. (32a,b) reveal is that when \( z_{ij}(t) = 0 \) or \( z_{ij}(t) = l_b \), i.e. when the \( i \)-th moving
oscillator enters or exits the beam, the slope of the road profile felt by its ideal point wheel (Eq. (32a)) undergoes a sudden jump, whereas the curvature (Eq. (32b)) exhibits an impulse centered at either the left or right support of the beam (Fig. 2).

Introducing Eq. (30) into Eq. (27), the equation of motion of the \( i \)-th oscillator reads:

\[
m_{v, i} \ddot{v}_{i, j}(t) + c_{v, i} \dot{v}_{i, j}(t) + k_{v, i} u_{i, j}(t) = m_{v, i} \left\{ g - A_{v, i} \left[ \chi_b(z) u_b(z, t) \right] \right\}_{z = z_{v, i}(t)}.
\]

(34)

In a similar way, substitution of Eq. (30) into Eq. (29) yields the following expression of the \( i \)-th dynamic interaction force:

\[
f_{v, i}(t) = m_{v, i} \left\{ g - A_{v, i} \left[ \chi_b(z) u_b(z, t) \right] \right\}_{z = z_{v, i}(t)} - \ddot{v}_{i, j}(t).
\]

(35)

It can be seen that when the \( i \)-th oscillator moves on the left (or right) rigid approach, i.e. \( z_{v, i}(t) < 0 \) (or \( z_{v, i}(t) > l_b \)), Eqs. (1) and (34) are not coupled since \( \chi_b(z) u_b(z, t) = 0 \) and consequently the convective acceleration is zero. Conversely, when the \( i \)-th vehicle crosses the left (or right) support of the bridge, i.e. \( z_{v, i}(t) = 0 \) (or \( z_{v, i}(t) = l_b \) and \( \dot{z}_{v, i}(t) > 0 \), it is subject to an additional excitation due to a sudden change in the slope of the road profile. More precisely, when \( z_{v, i}(t) = 0 \) (or \( z_{v, i}(t) = l_b \)), besides the convective acceleration terms associated with the slope \( \partial \left[ \chi_b(z) u_b(z, t) \right] / \partial z \) (Eq. (32a)), an impulsive excitation (see Eq. (33)) relating to the curvature \( \partial^2 \left[ \chi_b(z) u_b(z, t) \right] / \partial z^2 \) (Eq. (32b)) arises in both Eqs. (1) and (34). To the authors’ best knowledge, such term is not taken into account by classical formulations, which commonly focus just on the time interval in which the vehicle travels along the bridge, in so neglecting the effects due to the passage from the approach, modeled as rigid soil, to the beam (and vice versa).

Substituting Eq. (3) into Eq. (34), the equation of motion of the \( i \)-th moving oscillator takes the form (analogous to Eq. (14)):

\[
m_{v, i} \ddot{u}_{i, j}(t) + \tilde{m}_{v, i}^T(t) \ddot{q}_{i, b}(t) + c_{v, i} \dot{u}_{i, j}(t) + \tilde{c}_{v, i}^T(t) \dot{q}_{i, b}(t) + k_{v, i} u_{i, j}(t) + \tilde{k}_{v, i}(t) q_{i, b}(t) = m_{v, i} g,
\]

(36)
where:

\[
\begin{align*}
\mathbf{m}_{vb,i}(t) &= m_{vb,i} \chi_b \left( z_{vb,i}(t) \right) \dot{\phi}_b \left( z_{vb,i}(t) \right) = m_{vb,i} \mathbf{a}_{vb,i}(t); \\
\mathbf{c}_{vb,i}(t) &= 2 m_{vb,i} \ddot{z}_{vb,i}(t) \chi_b \left( z_{vb,i}(t) \right) \dot{\phi}_b' \left( z_{vb,i}(t) \right) = 2 m_{vb,i} \mathbf{b}_{vb,i}(t); \\
\mathbf{k}_{vb,i}(t) &= m_{vb,i} \chi_b \left( z_{vb,i}(t) \right) \left[ \dddot{z}_{vb,i}(t) \dot{\phi}_b' \left( z_{vb,i}(t) \right) + \dddot{z}_{vb,i}(t) \dot{\phi}_b' \left( z_{vb,i}(t) \right) \right] \\
&\quad + m_{vb,i} \dddot{z}_{vb,i}(t) \left[ \dot{\phi}_b' \left( 0 \right) \delta \left( z_{vb,i}(t) \right) - \dot{\phi}_b' \left( l_b \right) \delta \left( z_{vb,i}(t) - l_b \right) \right],
\end{align*}
\]

(37a-c)

in which:

\[
\phi_b'' \left( z_{vb,i}(t) \right) = \frac{d^2 \phi_b(z)}{dz^2} \bigg|_{z=z_{vb,i}(t)} = \left\{ \phi_{b,1}(z), \phi_{b,2}(z), \ldots, \phi_{b,n_b}(z) \right\}^T \bigg|_{z=z_{vb,i}(t)},
\]

(38)

with \( \phi_{b,j}(z) = -j^2 \pi^2 \sqrt{2} \left( \rho' A_b l_b \right) \sin \left( j \pi z / l_b \right) \).

Then, the equations governing the motion of the \( n_v \) oscillators can be written in a compact matrix form as (analogously to Eq. (15)):

\[
\mathbf{M}_v \ddot{\mathbf{u}}_v(t) + \mathbf{M}_v \dddot{\mathbf{q}}_b(t) + \mathbf{C}_v \dddot{\mathbf{u}}_v(t) + \mathbf{C}_v \dddot{\mathbf{q}}_b(t) + \mathbf{K}_v \mathbf{u}_v(t) + \mathbf{K}_v \mathbf{q}_b(t) = \mathbf{M}_v \tau_v \mathbf{g},
\]

(39)

where the array of the absolute displacements of the moving oscillators is \( \mathbf{u}_v(t) = \{ u_{v,1}(t), u_{v,2}(t), \ldots, u_{v,n_v}(t) \}^T \), and where:

\[
\begin{align*}
\mathbf{M}_{vb}(t) &= \mathbf{M}_v \mathbf{X}_v(t) \mathbf{\Phi}_b^T(t); \\
\mathbf{C}_{vb}(t) &= 2 \mathbf{M}_v \mathbf{X}_v(t) \mathbf{\dot{Z}}_v(t) \mathbf{\Phi}_b^T(t); \\
\mathbf{K}_{vb}(t) &= \mathbf{M}_v \mathbf{X}_v(t) \left[ \mathbf{\dot{Z}}_v(t) \mathbf{\Phi}_b^T(t) + \mathbf{\dddot{Z}}_v(t) \mathbf{\Phi}_b^T(t) \right] \\
&\quad + \mathbf{M}_v \mathbf{\dot{Z}}_v(t) \left[ \Delta_{v,0}(t) \tau_v \phi_b'(0) - \Delta_{v,1}(t) \tau_v \phi_b'(l_b) \right],
\end{align*}
\]

(40a-c)

with:
By applying to Eq. (35) the coordinate transformation of Eq. (3), the expression of the interaction force in terms of beam’s modal coordinates is obtained (analogous to Eq. (18)):

\[
f_{v,t}(t) = m_{v,j} \ddot{g} - \ddot{m}_{vb}(t) \dot{q}_b(t) - \ddot{c}_{vb}(t) \ddot{q}_b(t) - \ddot{k}_{vb}(t) q_b(t) - m_{v,j} \ddot{u}_{v,j}(t),
\]

where use has been made of Eqs. (37a-c). Finally, substituting Eq. (42) into Eq. (5), the set of \( n_b \) ordinary differential equations with time-dependent coefficients governing the beam’s modal coordinates (Eq. (5)) takes the form (analogous to Eq. (19)):

\[
\begin{bmatrix}
I_{v,j} + \Delta \tilde{M}_b(t)
\end{bmatrix} \dddot{q}_b(t) + \Delta \tilde{C}_b(t) \dddot{q}_b(t) + \Delta \tilde{K}_b(t) q_b(t) = \dddot{p}_b(t),
\]

where:

\[
\Delta \tilde{M}_b(t) = \Phi_{vb}(t) M_v X_v(t) \Phi_{vb}^T(t) = \Phi_{vb}(t) \tilde{M}_b(t);
\]

\[
\Delta \tilde{C}_b(t) = 2 \Phi_{vb}(t) M_v X_v(t) \dot{Z}_v(t) \Phi_{vb}^T(t) = \Phi_{vb}(t) \tilde{C}_b(t);
\]

\[
\Delta \tilde{K}_b(t) = \Phi_{vb}(t) M_v X_v(t) \left[ \dddot{Z}_v(t) \Phi_{vb}^T(t) + \dddot{\hat{Z}}_v(t) \Phi_{vb}^T(t) \right] + \Phi_{vb}(t) \tilde{Z}_v(t) \left[ \Delta_{v,j}(t) \tau_b \Phi_b^T(l_b) - \Delta_{v,j}(t) \tau_b \Phi_b^T(l_b) \right] = \Phi_{vb}(t) \tilde{K}_b(t);
\]

\[
\dddot{p}_b(t) = \dddot{p}_b(t) = \dddot{p}_b(t) + \dddot{p}_b(t).
\]

Interestingly, \( \Delta \tilde{M}_b(t) \), \( \Delta \tilde{C}_b(t) \) and \( \Delta \tilde{K}_b(t) \) may be viewed, respectively, as modifications of the mass, damping and stiffness matrices in the modal space due to the passage of the moving oscillators.

Analogously to Eq. (21), Eqs. (36) and (43) can be rewritten in compact form as:

\[
\tilde{M}_v(t) \dddot{u}(t) + \tilde{C}_v(t) \dddot{u}(t) + \tilde{K}_v(t) \dddot{u}(t) = \dddot{p}(t),
\]
where the subscript “r” stands for “relative displacements” and where:

\[
\begin{align*}
\tilde{M}_r(t) &= \begin{bmatrix} M_v & \tilde{M}_{vb}(t) \\ \tilde{M}_{vb}(t) & I_{n_b} + \Delta M_b(t) \end{bmatrix} ; \\
\tilde{C}_r(t) &= \begin{bmatrix} C_v & \tilde{C}_{vb}(t) \\ \tilde{C}_{vb}(t) & \Xi_b + \Delta \tilde{C}_b(t) \end{bmatrix} ; \\
\tilde{K}_r(t) &= \begin{bmatrix} K_v & \tilde{K}_{vb}(t) \\ \tilde{K}_{vb}(t) & \Omega_b^2 + \Delta \tilde{K}_b(t) \end{bmatrix} ; \\
\tilde{u}(t) &= \begin{bmatrix} u_v(t) \\ q_b(t) \end{bmatrix} ; \quad \tilde{p}(t) = \begin{bmatrix} M_v \tau_v \xi \\ p_b(t) \end{bmatrix}.
\end{align*}
\] 

(46a-e)

As opposed to the formulation in terms of absolute displacements, the mass matrix \(\tilde{M}_r(t)\) is not diagonal and depends on time. Furthermore, both the damping and stiffness matrices, \(\tilde{C}_r(t)\) and \(\tilde{K}_r(t)\), are not symmetric. Finally, impulsive terms arise in the stiffness matrix \(\tilde{K}_r(t)\) at the time instants in which each oscillator enters and exits the simply supported bridge.

As mentioned in the previous section, from a numerical point of view, it is preferable to homogenise the physical dimensions of the elements of the matrices appearing in the coupled equations of motion. To do this, the position \(u_v(t) = M_v^{-1/2} q_v(t)\) is made, so that Eq. (45) can be rewritten as (analogous to Eq. (23)):

\[
\ddot{m}(t) \ddot{q}(t) + \dot{c}(t) \dot{q}(t) + \tilde{K}(t) \ddot{q}(t) = \tilde{p}(t),
\]

(47)

where:
\[ \dot{\mathbf{m}}(t) = \begin{bmatrix} I_b & M_{vb}^{1/2} \\ M_{vb}^{1/2} & I_b + \Delta M_b(t) \end{bmatrix} ; \]
\[ \ddot{\mathbf{c}}(t) = \begin{bmatrix} \Xi & M_{vb}^{1/2} \dot{\mathbf{C}}_{vb}(t) \\ 0_{n_b \times n_b} & \Xi + \Delta \mathbf{C}_b(t) \end{bmatrix} ; \]
\[ \ddot{\mathbf{k}}(t) = \begin{bmatrix} \Omega^2 & M_{vb}^{1/2} \ddot{\mathbf{K}}_{vb}(t) \\ 0_{n_b \times n_b} & \Omega^2 + \Delta \mathbf{K}_b(t) \end{bmatrix} ; \]
\[ \ddot{\mathbf{q}}(t) = \begin{bmatrix} q_v(t) \\ q_b(t) \end{bmatrix} . \]

(48a-d)

The corresponding initial conditions at the time instant \( t_0 \) immediately before the first moving oscillator reaches the beam can be defined as (analogous to Eqs. (26a,b)):

\[ \ddot{\mathbf{q}}(t_0) = \begin{bmatrix} \Omega^2 M_{vb}^{1/2} \tau_{vb} \mathbf{g} \\ \mathbf{q}_b^{(0)} \end{bmatrix} ; \quad \ddot{\mathbf{q}}(t_0) = \begin{bmatrix} 0_{n_b \times 1} \\ 0_{n_b \times 1} \end{bmatrix} . \]

(49a,b)

The equations of motion (47) can be integrated by means of a suitable step-by-step algorithm, paying special attention to the impulsive terms appearing in the stiffness matrix \( \ddot{\mathbf{k}}(t) \) at the entrance and exit times of each oscillator.

5 Numerical applications

For validation and comparison purposes, the alternative formulations presented in the previous sections have been applied to study the transverse vibration experienced by an homogeneous simply supported beam subjected to a pair of moving oscillators \( (n_v = 2) \). The following mechanical parameters have been selected for the supporting beam: \( l_b = 27.5 \text{ m}, \rho_b = 2.500 \text{ kg/m}^3, E_b = 35.0 \times 10^6 \text{ kN/m}^2, A_b = 0.954 \text{ m}^2, J_b = 0.355 \text{ m}^4 \) and \( \zeta_b = 0.020 \).

The first moving oscillator, having lumped inertia \( m_{v_1} = 9,840 \text{ kg}, \) elastic stiffness \( k_{v_1} = 7.07 \times 10^3 \text{ kN/m} \) and viscous damping coefficient \( c_{v_1} = 52.8 \text{ kN s/m} \), enters the beam at the time instant \( t = 0.000 \text{s} \) with constant speed \( \dot{z}_{v_1}(t) = 26.1 \text{ m/s} \) \( (= 94.0 \text{ km/h}) \). The second
moving oscillator, characterized by \( m_{v,2} = 3,280 \text{ kg} \), \( k_{v,2} = 6.54 \times 10^3 \text{ kN/m} \) and \( c_{v,2} = 14.6 \text{ kN s/m} \), enters the beam at \( t = 0.527 \text{ s} \) with \( \dot{z}_{v,2}(t) = 39.1 \text{ m/s} \) (\( = 141 \text{ km/h} \)). It is assumed that the only static load acting on the supporting beam is its self weight, \( f_b(t) = \rho_b A_b g = 23.4 \text{ kN/m} \). The first five modal shapes are retained in the analyses (\( n_b = 5 \)).

If total mass of the beam, \( m_b = \rho_b A_b l_b = 65,600 \text{ kg} \), first period of vibration of the beam, \( T_{b,1} = 2\pi / \omega_{b,1} = 0.211 \text{ s} \), and acceleration of gravity, \( g = 9.81 \text{ m/s}^2 \), are assumed as independent dimensional parameters, beam and oscillators are fully defined by the following dimensionless quantities: flexural stiffness \( E_b J_b / \left( m_b g^3 T_{b,1}^4 \right) = 101,000 \), length \( l_b / \left( g T_{b,1}^2 \right) = 63.0 \) and viscous damping ratio \( \zeta_b = 0.020 \) for the beam; mass ratio \( m_{v,1} / m_b = 0.150 \), frequency ratio \( \omega_{v,1} T_{b,1} / (2\pi) = 0.900 \), viscous damping ratio \( \zeta_{v,1} = 0.100 \) and speed \( \dot{z}_{v,1} / \left( g T_{b,1} \right) = 12.6 \) for the first moving oscillator; \( m_{v,2} / m_b = 0.050 \), \( \omega_{v,2} T_{b,1} / (2\pi) = 1.500 \), \( \zeta_{v,2} = 0.050 \) and \( \dot{z}_{v,2} / \left( g T_{b,1} \right) = 18.9 \) for the second moving oscillator (lighter and faster).

In a first stage, the formulation in terms of absolute displacements (Section 3) has been applied. The unconditionally stable Newmark’s \( \beta \)-method, with coefficients \( \beta = 1/4 \) and \( \gamma = 1/2 \) (constant average acceleration), has been used in order to solve the equations of motion in terms of generalized displacements \( q(t) \) for the moving oscillators and \( q_b(t) \) for the supporting beam. The time step has been chosen as \( \Delta t = \pi / \left( 10 \omega_{b,1} \right) = 0.000422 \text{ s} \), which allows representing the contribution of the higher mode of vibration retained in the analysis. Once the numerical integration has been performed in the time interval \(-0.1 \text{s} \leq t \leq 2.0 \text{s} \), all the statical and kinematical quantities of interest can be computed starting from the knowledge of \( q(t) \) and \( q_b(t) \), e.g. the beam’s deflection at the midspan position is given by \( u_b \left( l_b / 2, t \right) = \Phi_b \left( l_b / 2 \right) q_b(t) \), while the interaction force between the beam and the \( i \)-th oscillator is given by:


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\[ f_{v,i}(t) = m_{v,i} \ddot{v} + k_{v,i} \left( \frac{q_{v,i}(t)}{\sqrt{m_{v,i}}} - a_b^T (z_{v,i}(t)) \mathbf{q}_b(t) \right) \]
\[ + c_{v,i} \left( \frac{\ddot{q}_{v,i}(t)}{\sqrt{m_{v,i}}} - a_b^T (z_{v,i}(t)) \mathbf{q}_b(t) - b_b^T (z_{v,i}(t)) \mathbf{q}_b(t) \right), \]

where \( q_{v,i}(t) \) and \( \dot{q}_{v,i}(t) \) are the \( i \)-th elements of the vectors \( \mathbf{q}_v(t) \) and \( \mathbf{q}_v(t) \), respectively. The time histories so obtained are depicted with solid thick lines in Figures 4 and 5.

In a second stage, the formulation in terms of relative displacements (Section 4) has been applied. With respect to the first one, this second formulation proves to be more cumbersome, mainly because of the impulses arising each time in which a moving oscillator enters \( (z_{v,i}(t) = 0) \) or exits \( (z_{v,i}(t) = l_b) \) the supporting beam. Moreover, the second-order derivatives of the modal shapes of the beam, \( \phi_{v,i}^{\prime \prime}(z) \), as well as the horizontal accelerations of the moving oscillators, \( \ddot{z}_{v,i}(t) \), are required in the analysis.

Since the Newmark’s \( \beta \)-method, like other standard numerical schemes, does not allow to include directly the impulses, the Dirac’s delta function has been approximated in the form of a Gaussian Probability Density Function (PDF), with zero mean and standard deviation \( \sigma \ll l_b \):

\[
\delta_{\sigma}(z) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left[ -\frac{1}{2} \left( \frac{z}{\sigma} \right)^2 \right].
\]

It is worth noting that, in order to get accurate results, the standard deviation \( \sigma \) has to be properly selected. On the one hand, this parameter should be small enough, since the generalized function \( \delta(z) \) can be theoretically defined as the limit of the PDF \( \delta_{\sigma}(z) \) when the standard deviation \( \sigma \) goes to zero; on the other hand, the PDF \( \delta_{\sigma}(z) \) can be accurately represented in the numerical solution only if the parameter \( \sigma \) is greater than the distance covered in the time step \( \Delta t \) by the fastest moving oscillator entering or exiting the supporting beam. In Figure 6, three approximate impulse functions \( \delta_{\sigma}(z) \), for three different values of the dimensionless ratio \( \sigma/l_b = 0.04, 0.02 \) and 0.008, are depicted in the same horizontal
scale as the sketch of the simply supported bridge. From a qualitative comparison it emerges that for \( \sigma/l_b = 0.008 \) the Gaussian PDF \( \delta_{\sigma}(z) \) provides a good approximation of the Dirac’s delta function, while for \( \sigma/l_b = 0.04 \) the Gaussian PDF is spread over an excessively large interval.

For each of the ratios \( \sigma/l_b \) specified above, the dynamic response of the beam-oscillator coupled system has been numerically evaluated in terms of midspan deflection of the beam, \( u_b(l_b/2,t) = \Phi_b^T(l_b/2)q_b(t) \), and beam-oscillator interaction forces, \( f_{v,i}(t) = \left[ k_{v,i} \ddot{q}_{v,i}(t) + c_{v,i} \dot{q}_{v,i}(t) \right]/\sqrt{m_{v,i}} \), \( \ddot{q}_{v,i}(t) \) and \( \dot{q}_{v,i}(t) \) being the \( i \)-th elements of the vectors \( \ddot{q}_v(t) \) and \( \dot{q}_v(t) \), respectively. The time histories so obtained are compared in Figures 4 and 5 with those provided by the formulation in terms of absolute displacements, where different symbols denote different ratios \( \sigma/l_b \). As expected, the results obtained with the smallest value of the ratio \( \sigma/l_b \) are in good agreement with those provided by the formulation in terms of absolute displacements of the moving oscillators, while the accuracy reduces as the ratio \( \sigma/l_b \) increases. This tendency is specifically highlighted in the enlargements depicted on the right-hand side of Figures 4 and 5. For this specific example, moreover, one can see that the peak response of the beam is slightly underestimated by the formulation in terms of relative displacements (Fig. 4b), while the response of the moving oscillators is slightly anticipated (Fig. 5d). These differences, however, are absolutely negligible from an engineering point of view. For the sake of completeness, in Figures 4 and 5 the time histories obtained by neglecting the impulses in the formulation in terms of relative displacements of the moving oscillators are also depicted (dashed thick lines). In the latter case the bridge-vehicle interaction forces, which are strictly related to the absolute acceleration experienced by the moving oscillators (Eqs. (8) and (29)), and hence to the comfort of the passengers, are hugely inaccurate (Fig. 5), and this is only due to the neglected impulses. Minor discrepancies also exist in the midspan deflection of the beam (Fig. 4). These numerical results clearly demonstrate that (i) the impulses theoretically deduced in Section 4 really exist, and that (ii) their effects cannot be neglected when the formulation in terms of relative displacements is applied.
6 Conclusions

The “moving oscillator” problem, extensively adopted in the literature to simulate bridge-vehicle dynamic interaction, has been reviewed in this paper, with the aim of providing a deeper insight into some theoretical and physical issues not specifically addressed by other investigators. Without lack of generality, the case of a simply supported bridge Bernoulli-Euler beam carrying multiple moving oscillators has been considered. For comparison purposes, the governing equations have been derived both in terms of relative and absolute displacements of the oscillators. It has been shown that impulsive terms, i.e. Dirac’s delta functions, appear in the relative displacement formulation when a vehicle enters or exits the beam. To the authors’ knowledge, presence and relevance of such terms have been not revealed in the past, probably because the attention in most of the previous studies has been mainly focused on the time interval in which the vehicle travels on the bridge, rather than on the dynamic effects arising when the vehicle crosses the supports of the bridge. Numerical results have demonstrated that in the context of the relative displacement formulation these impulsive terms, in some sense “hidden” in the absolute displacement formulation, cannot be neglected, since they prove to have a significant influence on the response of the moving oscillators. Furthermore, it has been shown that these impulses make the relative displacement formulation more cumbersome, both theoretically and practically, than the one in terms of absolute displacements. On the other hand, the formulation of the problem in terms of relative displacements of the moving oscillators might be properly exploited to cope with more complicated models of vehicles. Specifically, in this case it appears more advantageous to define first the mass, damping and stiffness matrices in terms of relative displacements resorting to the substructure approach. Then, in order to perform the dynamic analysis, the corresponding matrices in the space of absolute displacements could be derived by using an appropriate extension of the transformations of coordinates provided in Appendix for a stream of moving oscillators.
Appendix

In this Appendix, the coordinate transformations useful to obtain the equations of motion of the combined system (supporting bridge-moving oscillators) in terms of absolute displacements starting from the formulation in terms of relative displacements, and vice versa, are derived. To this aim, it is recalled that (see Fig. 3 and Eq. (28)):

\[ u_v(t) = u_v^{(a)}(t) - u_w(t) + u_v^{(s)}(t), \]  

(A.1)

where \( u_v(t) \) and \( u_v^{(a)}(t) \) are the \( n_v \)-dimensional vectors collecting the relative displacements \( u_{v,i}(t) \) and the absolute displacements \( u_{v,i}^{(a)}(t) \) of the moving oscillators, respectively with and without the static contributions \( u_{v,i} = M_{v,j} g/K_{v,j} \); while \( u_w(t) \) and \( u_v^{(s)}(t) \) are the vectors listing the displacements of the ideal point wheels and the static displacements of the suspended masses, respectively. These vectors can be written as follows:

\[ u_w(t) = M_v^{-1} \tilde{M}_v q_v(t); \]  

(A.2a)

\[ u_v^{(s)}(t) = K_v^{-1} M_v \tau g. \]  

(A.2b)

Taking into account that

\[ u_v^{(a)}(t) = M_v^{-1/2} q_v^{(a)}(t) \] and \( u_v(t) = M_v^{1/2} q_v(t) \), after very simple algebra, the following coordinate transformation is obtained:

\[ \ddot{q}(t) = \Gamma(t) q(t) + \left[ M_v^{1/2} \Omega_v^{-2} \tau_v g \right] \]  

(A.3)

being:

\[ \Gamma(t) = \begin{bmatrix} I_{n_v} & -M_v^{-1/2} \tilde{M}_v(t) \\ 0_{n_v \times n_v} & I_{n_v} \end{bmatrix}; \]  

(A.4a)

\[ \ddot{q}(t) = \begin{bmatrix} q_v(t) \\ q_b(t) \end{bmatrix}; \quad q(t) = \begin{bmatrix} q_v^{(a)}(t) \\ q_b(t) \end{bmatrix}. \]  

(A.4b-c)
Then, the first- and second-order time derivatives of the vector $\ddot{q}(t)$ defined in Eq. (A.3) are given by:

$$
\dot{q}(t) = \Gamma(t)\dot{q}(t) + \dot{\Gamma}(t)q(t);
\ddot{q}(t) = \Gamma(t)\ddot{q}(t) + 2\dot{\Gamma}(t)\dot{q}(t) + \dot{\Gamma}(t)q(t)
$$

where:

$$
\dot{\Gamma}(t) = \begin{bmatrix}
0_{n_{v}\times n_{v}} & -\frac{1}{2}M_v^{-1/2}\tilde{C}_{vb}(t) \\
0_{n_{v}\times n_{v}} & 0_{n_{v}\times n_{v}}
\end{bmatrix}; \\
\ddot{\Gamma}(t) = \begin{bmatrix}
0_{n_{v}\times n_{v}} & -M_v^{-1/2}\tilde{K}_{vb}(t) \\
0_{n_{v}\times n_{v}} & 0_{n_{v}\times n_{v}}
\end{bmatrix}.
$$

(A.5a,b)

In order to derive the equations of motion (23) in terms of absolute displacements from those in terms of relative displacements, substitute Eq. (A.3) into Eq. (46) and premultiply the resulting equation by $\Gamma^T(t)$. Taking into account Eqs. (A.5a,b), and observing that the following relationships hold:

$$
\Gamma^T(t)\ddot{m}(t)\Gamma(t) = I_n; \\
\Gamma^T(t)\ddot{c}(t)\Gamma(t) + 2\dot{\ddot{m}}(t)\dot{\Gamma}(t) = \epsilon(t); \\
\Gamma^T(t)\ddot{k}(t)\Gamma(t) + \dddot{k}(t)\Gamma(t) + \ddot{\ddot{m}}(t)\dot{\Gamma}(t) = k(t); \\
\Gamma^T(t)\dddot{p}(t) - \dddot{k}(t)\begin{bmatrix}
\Omega^2 M_v^{1/2} \tau_v g \\
0_{n_{v}\times 1}
\end{bmatrix} = p(t),
$$

(A.7a-d)

Eq. (23) can be easily derived.

In a similar way, Eq. (46) can be readily deduced from the absolute displacement formulation by introducing the following coordinate transformation into Eq. (23)

$$
q(t) = \tilde{F}(t)\ddot{q}(t) - \begin{bmatrix}
M_v^{1/2}\Omega^2 \tau_v g \\
0_{n_{v}\times 1}
\end{bmatrix}
$$

(A.8)
where

\[
\dot{\Gamma}(t) = \Gamma^{-1}(t) = \begin{bmatrix}
I_{n_v} & M^{-1/2}_v \dot{M}_{vb}(t) \\
0_{n_v \times n_v} & I_{n_v}
\end{bmatrix}
\]  \hspace{1cm} (A.9)

is a time-dependent transformation matrix analogous to \( \Gamma(t) \).

**References**


FIGURE CAPTIONS

**Figure 1.** Simply supported bridge crossed by a stream of moving oscillators.

**Figure 2.** Schematic view of the forces acting on the simply supported bridge crossed by the $i$th moving oscillator.

**Figure 3.** Sketch of the generic moving oscillator (a); static displacement $u_{v,i}^{(s)}$ of the mass $m_{v,i}$ (b); relative displacement $u_{v,i}(t)$ of the mass $m_{v,i}$, including the static contribution (c); displacement $u_{w,i}(t)$ of the ideal point wheel, and absolute displacements, $\bar{u}_{v,i}^{(a)}(t)$ and $u_{v,i}^{(a)}(t)$, of the mass $m_{v,i}$, with or without the static contribution (d).

**Figure 4.** Time history of beam deflection at mid-span: (a) solutions provided by the absolute and relative displacement formulations (with and without impulses); (b) enlargement showing the comparison between the responses obtained modeling the impulses as Gaussian PDFs with different values of the ratio $\sigma/l_b$.

**Figure 5.** Time histories of the interaction forces obtained by applying the absolute and relative displacement formulations (with and without impulses): (a) force transmitted by the first moving oscillator; (b) force transmitted by the second moving oscillator; (c, d) enlargements showing the comparison between the solutions obtained modeling the impulses as Gaussian PDFs with different values of the ratio $\sigma/l_b$.

**Figure 6.** Impulse function approximated as a Gaussian PDF for three different values of the ratio $\sigma/l_b$. 
FIGURE 1

Simply supported bridge crossed by a stream of moving oscillators.
FIGURE 2

Schematic view of the forces acting on the simply supported bridge crossed by the $i$-th moving oscillator.
FIGURE 3

Sketch of the generic moving oscillator (a); static displacement $u^{(s)}_{v,j}$ of the mass $m_{v,j}$ (b); relative displacement $u_{v,j}(t)$ of the mass $m_{v,j}$, including the static contribution (c); displacement $u_{w,j}(t)$ of the ideal point wheel, and absolute displacements, $\overline{u}_{v,j}(t)$ and $u^{(a)}_{v,j}(t)$, of the mass $m_{v,j}$, with or without the static contribution (d).
FIGURE 4

Time history of beam deflection at mid-span: (a) solutions provided by the absolute and relative displacement formulations (with and without impulses); (b) enlargement showing the comparison between the responses obtained modeling the impulses as Gaussian PDFs with different values of the ratio $\sigma/l_b$. 
FIGURE 5

Time histories of the interaction forces obtained by applying the absolute and relative displacement formulations (with and without impulses): (a) force transmitted by the first moving oscillator; (b) force transmitted by the second moving oscillator; (c, d) enlargements showing the comparison between the solutions obtained modeling the impulses as Gaussian PDFs with different values of the ratio $\sigma / l_b$. 
FIGURE 6

Impulse function approximated as a Gaussian PDF for three different values of the ratio $\sigma/l_b$. 

\begin{itemize}
\item $\sigma/l_b = 0.04$
\item $\sigma/l_b = 0.02$
\item $\sigma/l_b = 0.008$
\end{itemize}