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OPTIMAL POSITIONING OF A LOAD SUSPENDED FROM A
STATION-KEEPING HELICOPTER

BY

L. TSITSILONIS, B.Sc., M.A.I.A.A., S.M.R.Ae.S.

A Doctoral Thesis
Submitted in Partial Fulfilment of the Requirements
for the Award of
the Degree of Ph.D. of the Loughborough University of Technology
November, 1981.

Department of Transport Technology

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Αφερωμένο

στοὺς γονεῖς μου
DECLARATION

This thesis is a record of research work carried out by the author in the Department of Transport Technology of Loughborough University of Technology and represents the independent work of the author; the work of others has been referenced where appropriate.

The author also certifies that neither this thesis nor the original work contained herein has been submitted to any other institution for a degree.

L. TSITSILOMIS.
ACKNOWLEDGEMENTS

The author wishes to express his deep gratitude and appreciation to his parents Ioannis and Irene Tsitsilonis for their encouragement and support throughout his education.

He also wishes to thank his supervisor, Dr. D. McLean for his guidance and inspiration at every stage of this research work.
CONTROLLING THE POSITION AND ATTITUDE OF A HELICOPTER HOVERING IN THE PRESENCE OF ATMOSPHERIC TURBULENCE IS A DIFFICULT TASK WHICH DEMANDS CONSIDERABLE PILOT WORK-LOAD WHICH BECOMES EVEN MORE DIFFICULT WHEN A LOAD IS SUSPENDED FROM THE HELICOPTER, BECAUSE THE OSCILLATIONS OF THE LOAD AGGRAVATE THE SITUATION. TASKS THAT REQUIRE A SUSPENDED LOAD TO BE KEPT FIXED RELATIVE TO A POINT IN SPACE, WHILE THE HELICOPTER REMAINS AT HOVER, ARE EXTREMELY DIFFICULT TO ACHIEVE.

Several load-positioning systems exist but provide inadequate solutions to the problem. A brief account of such systems and their limitations is given before describing the automatic hovering control system proposed in this thesis. It causes appropriate motion of the helicopter to achieve the desired stationarity of the load. The techniques of modern control theory were employed to design this optimal controller. Digital simulation was used for testing the response of the resulting optimal system.

The mathematical model of two connected rigid bodies moving in space (representing the helicopter and suspended load) is described in detail. Several combinations of cable length-load weight were chosen and in each case the response of the closed-loop system was investigated. It was found that considerable reduction of the oscillations of the load can be achieved when suitable cable arrangements are used. The use of winch control of lateral displacement of the load also improves the lateral response of the entire system. An augmented mathematical model
was used which included both the dynamics of the control actuators and
the models representing atmospheric turbulence and sensor noise. Since
many of the state variables of the system cannot be physically measured,
it is obvious that only limited information on the state of the system
would be available for processing by such a controller. Therefore two
solutions to the problem were considered:

(i) the use of a state estimator to provide to the controller
the lost feedback information;

and (ii) the use of an output regulator which takes into account
the fact that limited feedback information is available.

The responses of the closed-loop systems using each of these solutions
were investigated and compared. The numerical problems encountered in
this design are analysed and some means of overcoming them are suggested.

Finally, the best combination of cable arrangement and controller
is described with reference to several important factors such as system
simplicity and performance.
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CHAPTER 1

GENERAL INTRODUCTION
1.1 DISCUSSION OF THE PROBLEM

In its present form a helicopter is a vehicle which is capable of several distinctive modes of flight which make it possible for it to be used in an extensive range of operational roles. For helicopters used in missions such as air-sea rescue, anti-submarine warfare (ASW), or precision placement of externally-carried cargo (Figure 1.1), the operational requirements are such that a large amount of the total flying time is spent at hover (i.e. flying takes place at near zero forward speed). Such a flying task is almost impossible under conditions of poor visibility, i.e. when Instrument Flight Rules (IFR) are in effect, without the aid of a Stability Augmentation System (SAS), since helicopters have no inherent aerodynamic stability at hover and usually have extremely poor handling characteristics for low-speed flight. Thus, problems of station-keeping become even greater when a helicopter with a hanging load of considerable mass (such as a cargo container, or an army vehicle, etc.) which may be as much as 50% of the mass of the helicopter itself, must be positioned.

Such a hovering situation is further aggravated when operating in gusty weather conditions. The positioning of a load under severe weather conditions is a technical challenge which has not yet been met adequately, for the oscillation of the load increases the instability of the combined helicopter/load system to such a degree that often at present pilots must either drop the cargo or pull out of the hover mode.

Hovering with a suspended cargo under either Visual Flight Rules (VFR) or IFR has proved to be affected by the position of the pilot with
Atmospheric Disturbance

Main Rotor

Tail Rotor

SIKORSKY S-61
Mass: 7000Kg
$U_0 = 0.3m/s$

Cable Angle

Standard size shipping container (2.4x2.4x6.1m)

Load Displacement

DESIRED HOVER POINT

FIGURE 1.1: Schematic Representation of the Problem of Precision Placement of a Suspended Load
respect to the body-fixed axes of the helicopter. As the load-carrying
capability of helicopters increases, their airframe grows and the
pilot's kinaesthetic environment changes. The cockpit is located
further away from the centre of gravity of the aircraft and although
this shift places the pilot in a better position to view the load,
especially when long load cables are used, it will cause difficult
piloting cues. The cues a pilot receives can be classified as either
'good', 'different', or 'bad'. The 'different' type requires only
pilot training, but the 'bad' type requires aircraft re-design.
The forward shift of the pilot, further from the centre of gravity of
the aircraft, causes him to receive the 'different' type of cues which
it is expected he will have time to learn to fly with. Undoubtedly,
at first, his perception of the new motion cues will be inhibited, and
may be associated with partial disorientation but, with training, these
defects should be overcome, permitting flight with the modified cues.
The position of a pilot with respect to the longitudinal (roll) axis, OX,
of the helicopter (i.e., height above or below the centre of gravity)
requires special consideration. The pilot will find himself receiving
'bad' cues if he is below the roll axis. Here a right roll of the
aircraft will give him a sense of left translation. This situation
would be critical in IFR conditions, and even in VFR conditions, when the
pilot senses one motion but sees another. This roll location is not
critical only for large crane-type helicopters but for all helicopters.
(Szustak and Jenney [1971]).
Apart from the unwanted motion due to instability, the damage, or injuries to persons, which an oscillating load can possibly cause must be considered since the load in most heavy-lift helicopters is wholly out of the view of the pilot.
1.2 PREVIOUS STUDIES AND RELATED RESULTS

The problem of carrying an underslung load is broadly similar to that of a body towed beneath an aircraft, which was encountered in the earliest stages of aviation development. Initial investigations involved the use of fin surfaces (Taylor, [1915]). Then Glauert [1930], in deriving the shape of a towing wire by neglecting its weight and considering it to be inextensible and attached at the centre of gravity of the body, investigated the conditions for the stability of the towed body. Phillips [1944] took the cable form to be one-quarter of a sine-wave. An effective drag force due to the cable was assumed to act at the body at its centre of gravity. He derived equations for lateral stability but referred to Glauert for treatment of longitudinal stability.

Söhne [1953], who was investigating lateral stability, was probably not aware of the work by Glauert and Phillips because his tow-line was straight and not attached at the centre of gravity. A translation of his paper appeared in 1956 (Söhne [1956]). Etkin and Macworth [1963] investigated the lateral stability of a cylindrical body for the case when the tow-line is attached at its centre of gravity. Reid [1967] derived the linearised differential equations of motion of a towed point-mass and used the characteristic equations obtained to test for longitudinal and lateral stability.

Like the aircraft towed-body case, the earliest attempts at using helicopters to carry underslung loads involved the use of single suspension cables. Lucassen and Steck [1965] carried out theoretical investigations into the study of a hovering helicopter with the load
suspended from a fixed hook. Wolkovitch and Johnston [1965] completed a comprehensive study of the dynamic stability of a helicopter carrying an underslung load for a single point suspension with the main emphasis being the automatic control and stabilisation of a helicopter. Gabel and Wilson [1968] gave results of some early experimental work and flight test experiences which helped to identify the major problem areas associated with the carriage of external underslung loads. They proposed some ways of overcoming these problems. A frequent observation in these early experiments was that the principal aerodynamic instability was an oscillatory, yawing motion of the underslung load. The next logical step, therefore, was to provide some form of yaw motion restraint for the load. Two methods tried were the use of drag chutes and the re-employment of fins, but it was found that neither was very satisfactory, chiefly on account of the operational difficulties encountered.

The most successful approach has been the suspension of the load from two or more points on the helicopter. Consequently multipoint suspension systems are mentioned increasingly in the literature concerned with suspended loads. Abzug [1970] derived the equations of motion for a helicopter carrying an underslung load a two-point, parallel cable suspension but without including the aerodynamic reactions of the load. After deriving the equations of the complete system Abzug concentrated on the closed-loop feedback control of the yawing instability of the load. By about 1970 the multi-point suspension had been accepted as a practical system and new, heavy-lift helicopters were being designed with the multi-hook capability. Szustak and Jenney [1971] examined some controllability criteria of
large crane helicopters and the effect of the suspension arrangements and pilot position (with respect to the body-fixed axes of the helicopter) upon the stability of the load. It was pointed out by Tencer and Cosgrave [1971] that the increasing use of containers for shipping, (starting from about the mid-fifties) and the subsequent adoption by the U.S. Department of Defence of a standard sized container of 6.1x2.44x2.44m (20x8x8ft) meant that this type of container would be the most important load type. The use of the container, coupled with the capability of the helicopter to carry loads externally, means that in military and civil environments the helicopter-rectangular load system assures a prime importance.

The corresponding theoretical development of the helicopter-suspended load models is given by Dukes [1972] who published the results of analytical studies of a helicopter carrying a sling-load near hover. Asseo and Whitbeck [1973], however, proposed stabilising the load by using winches in the helicopter as active controllers, and by utilising a three-point longitudinally- and laterally-displaced suspension arrangement. Also Prabhakar [1976] derived the equations of motion of a helicopter with a rectangular container suspended by a Sheldon's Double 'V' arrangement. On the experimental side Sheldon and Pryor [1973] have published reports on the previous aspects of carrying loads externally, giving the results of wind tunnel studies at RMCS.†

A point to be noted is that most of the theoretical investigations cited so far have ignored the aerodynamics of the suspended loads and there remains a lack of experimental aerodynamic data. Poli and

† Royal Military College of Science
Cromack [1973] carried out some theoretical investigations of the single-point suspension, which included the steady-state aerodynamics of the load. Liu [1973] commented on the aerodynamic characteristics of several bluff bodies (which are often transported by a helicopter as external cargo) and stressed that the available methods for the derivation of the aerodynamic coefficients of those bodies were not accurate enough and that any published values were therefore suspect.

It was Simpson and Flower [1978] who observed in detail the airflow around a standard size container by performing wind tunnel tests; they pointed out the extreme non-linearities present. They also calculated (based on experimental data) the aerodynamic coefficients of the container.

American interest has shifted increasingly to the automatic stabilisation of underslung loads. For example, one of the conclusions drawn by Liu [1973] was that automatic stabilisation with control inputs into the helicopter Stability Augmentation System (SAS) was the best solution for any load instabilities. The stability of a two degree-of-freedom model of the slung load was examined analytically and a flight simulator was used to study the combined helicopter-load stability. Knoer [1974] investigated the techniques for determining position and motion of helicopter loads as a preliminary to automatic load stabilisation.
1.3 AUTOMATIC CONTROL SYSTEMS FOR HELICOPTERS

It is well known (Simons [1976]) that in flight the pilot work load can be greatly reduced by using an Automatic Flight Control System (AFCS) with appropriate feedback signals to augment the stability of the several modes of motion of an aircraft.

Since 1954 some helicopter types have been equipped with a wide variety of Automatic Approach and Hover Systems. The design aims of such systems were always the same. It was intended to produce a system that was:

- reliable, repeatable and smoothly-operating;
- economical to produce and to maintain;
- easy to engage and to disengage;
- fail-safe (the most important requirement).

It was also intended that such systems should improve the handling qualities of the vehicle and would reduce the pilot work load by damping out any oscillations of a slung load. (Tsitsilonis and McLean, [1981]).

Some of these types of systems in current use in helicopters are listed below:

1. Automatic Hovering for ASW sonar coupling,
2. Automatic Approach and Hover with hover trim control,
3. Automatic transition to Hover over a terrain.

A block diagram representing a typical control system for a helicopter is shown in Figure 1.2. The system requires motion sensors
FIGURE 1.2: AFCS for a Large Crane Helicopter

FIGURE 1.3: Schematic of Precision Hover System
to produce feedback signals as inputs to the controller of the AFCS. These feedback signals are added to any trim inputs and manoeuvre command inputs from the pilot. The resultant summation signal is fed via appropriate electrical transducers to the electronic mixing unit which is designed to ensure that the control actions are appropriately distributed to the control actuators for both the main and the tail rotors (Figure 1.3).

Progress in the development of these systems since 1954 has been steady and substantial, but there still remains a need for further improvement and refinement. These areas of particular advantage for development include:

(i) improving the handling qualities of the vehicle;
(ii) reducing the work load of the pilot;
and (iii) stabilising any external suspended load.

Control system designers have attempted to solve these problems in several ways. Of the most promising solutions three features are noteworthy:

(i) the number of motion sensors used has been increased to provide more complete feedback information for the AFCS;
(ii) integrated displays have been used to assist the pilot to obtain as much instantaneous information on the state of the helicopter as possible as quickly as possible. Such types of display do assist in reducing the time delay of the pilot's response and in improving the accuracy of any
of his control actions to correct a particular disturbance;

(iii) optimal control theory has been used in the design of APCS (Murphy and Narendra [1969], Prasad et al [1980]).

Since 1970 some precision hover systems have been available (Herzog, [1972]) which use new and sophisticated motion sensing devices such as a pulsed, sine-modulated, laser-ranging beam tracking system. Although the results obtained are good, it appears that such systems still require considerable pilot effort. They are also heavy and expensive (with their cost including, not only initial purchase, but also maintenance and spares, and equipment changes).

For the flight situation, when the helicopter is carrying a suspended load, information on the position of the load with respect to the vehicle is usually obtained by means of a load sensor. Load sensors have been much developed and have been tried in flight. Most aircraft sensors are not directly adaptable to load use because they give absolute measurements rather than measurements relative to the helicopter. Rate sensors are generally usable; however, further development of position and cable angle sensors is required. One possible technique is described by Gupta and Bryson [1976]. The load sensor is mounted on the floor of the helicopter and provides a measure of the cable angle (Figure 1.4). Elastic transducers are fitted at right angles to each other so that a signal representing the cable angle can be obtained for both longitudinal and lateral deflections. This cable angle signal provides all the information required to determine the exact position of the load below the helicopter. However, this position is a function of the length of the cable.
Ring elastically mounted in plane of helicopter floor

Cable to constant-tension winch

HELICOPTER CENTRE LINE

Floor of Helicopter
Hole in the floor of Helicopter

Cable to Load

FIGURE 1.4: Representation of a Cable Angle Sensor
FIGURE 1.5: The Princeton Integrated Display
At present the principal visual indicators in general use in military helicopters are the Artificial Horizon, the Direction Compass, the Altimeter, the Airspeed Indicator, the Slip Angle Indicator, and the Vertical Speed Indicator. The steady development of display techniques, however, has led to considerable research, and experimental flight trials, involving the use of multimode cathode-ray tube displays, to provide an integrating display of all the necessary flight data. For example, the Princeton Integrated Display, represented in Figure 1.5, presents information on the position of the helicopter to which it is fitted, relative to some target, to its velocity vector, to its altitude, and to its attitude angles.

It is generally accepted in this field that satisfactory and consistent performance of any complex flight mission can be achieved by the pilot only with the combined use of integrated displays and an effective AFCS (Keane and Milelli, [1971]).

In the past only separate control channels, such as roll, pitch and yaw, and modes such as height-hold for ASW missions, were automatically controlled, because the designs were conceived in terms of conventional control theory, which provides its most satisfactory results for single-input, single-output (SISO) linear, time-invariant systems.

Once optimal control theory was developed, the design of AFCS for multi-input, multi-output (MIMO) systems was possible. Such methods were applied to the design of Automatic Hovering Control Systems (AHCS) using low-order mathematical representations of the motion of a helicopter and by assuming instantaneous rotor tilting, Murphy and
Narendra [1969] were the first to design such an AHCS, which also employed a state estimator. Then, Crossley and Porter [1975] improved the already existing AHCS by adding to it a disturbance rejection controller, which allowed Gupta and Bryson [1976] to design a similar feedback controller by using a low-order mathematical model of a helicopter with a suspended load. The position and velocity of the load (for a particular cable length) was computed by using a cable angle sensor as shown in Figure 1.4. The digital flight control system designed by Broussard et al [1979] (for a helicopter without a suspended load) used only the information provided by certain channels which were considered important to the stability of the system. State estimators and proportional and integral control logic comprised their digital control system which provided control of attitude and velocity command modes and which adapted to varying flight conditions by means of gain-scheduling.

The use of low-order mathematical models and the control of certain channels only, was mainly due to the requirement that the installed AFCS had to be simple because any increase in the complexity of control systems has a direct effect on the overall weight, and hence on the payload, of the helicopter. Therefore designers tried in the past to distribute the necessary flight control performance economically†, but appropriately, between the pilot and the AFCS.

The recent advances in micro-computer technology have made possible

†The economic aspect of the AHCS is beyond the scope of this work. However, Beltrano [1980] has made a parametric study of the costs involved in helicopter systems which may provide an estimate of the cost of an AHCS.
the construction of powerful and relatively cheap and light on-board computers, which allow the control of considerably more motion variables. Such high order controllers were considered in this work. The aspect of using a particular suspension arrangement in conjunction with an appropriate controller was examined in order to achieve complete hands-off autostabilisation. Since most helicopters are not provided with position sensors of an external load, it was reasonable to assume that none of the motion variables of the load was measurable.

Two methods were considered for stabilising and positioning a helicopter with a suspended load, particularly in the presence of atmospheric turbulence.

In Chapter 2 the construction of some mathematical models of a helicopter/load system are discussed in detail.

It is necessary to assume that the helicopter is operating in a turbulent environment. Therefore, some appropriate mathematical model representing atmospheric disturbances had to be chosen (Chapter 3). The relevant theory which governs the optimal AHCS is presented in Chapter 4.

The dynamic response of the helicopter/load system, using an optimal AHCS, was then thoroughly investigated and the results and a discussion of these presented in Chapters 5 and 6. Some of the problems encountered in the computation of the AHCS are discussed in Chapter 7 and several solutions for these problems are suggested.

Finally, the conclusions which can be drawn from the present investigation are presented in the final chapter.
CHAPTER 2

MATHEMATICAL REPRESENTATION OF A HELICOPTER

WITH A SUSPENDED LOAD
2.1 INTRODUCTION

The dynamic behaviour of a helicopter-load system is governed by the corresponding equations of motion. However, the complexity of this dynamical representation is such that for analytical purposes a much simplified mathematical model is usually considered.

By ignoring the structural deformation of both the helicopter and load, the simplified system is represented by two rigid bodies moving through space and connected by a suspension arrangement. The helicopter is usually taken to be the primary body with the six degrees-of-freedom: three translational and three rotational. The suspended load is taken to be a secondary body attached to the helicopter by the suspension which allows the load to move relative to the helicopter in accordance with the constraints which arise for different suspension arrangements.

Further simplifications of the theoretical model can be made by ignoring the mass of the cables and any aerodynamic effects caused by them, as well as the dynamics of the rotor.

Thus the model obtained for the helicopter-load system is essentially that of two rigid inter-connected bodies moving through the atmosphere.

The problems associated with the dynamics of multiple point systems have received considerable attention since 1950, as a result of studies of attitude control of artificial satellites. These generally have a large number of parts which move relative to the main body.
One of the earliest attempts was that of Roberson [1960], who considered the motion of a multi-body system, the equations obtained being referred to the instantaneous mass centre of the system. The mass centre of the system itself varied as the various parts moved relative one to another, thus making these equations difficult to use.

Another form of the equations of motion of a multi-body system was derived by Abzug [1969], referring the equations to axes fixed in the mass centre of the main body. The equations were later modified (Abzug, [1970]) to apply specifically to the case of a helicopter carrying a load on a two cable parallel string arrangement. Asseo and Whitbeck [1973] derived the equations of motion of a similar system but in terms of the generalised coordinates associated with a three-point suspension system in which the cable length was continuously adjusted by actively-controlled winches. Finally, Prabhakar [1976] derived the equations of motion for a helicopter/load system for a particular four cable tandem suspension arrangement.

From a study of all the available sets of equations of motion, those derived by Asseo and Whitbeck were judged to be the most convenient for use in this research, being easily adaptable to the helicopter and load data available.

By making further appropriate assumptions the equations can be modified to represent either one or two parallel cable suspension systems. In every case the equations of motion of the helicopter and load were derived using their own body-fixed axis systems and then the constraint equations representing the specific suspension
geometry were added. This approach inevitably yields a set of non-linear differential equations which completely describe (as far as the stated assumptions allow) the 12 degrees-of-freedom of the helicopter and load. For the work reported here, the non-linear equations were linearised about a nominal, trimmed, flight condition (at constant airspeed) for which the attitude angles with respect to earth-fixed axes of the roll, pitch and yaw motions of the helicopter and load were zero.

The equations presented in Appendix A were developed (from the general equations derived by Asseo and Whitbeck) for use in the particular case of a near-hovering helicopter with a uniformly-loaded, squared block, suspended from the helicopter by one, two or three parallel cables. A lateral-displacement winch control was also assumed to be available for use in certain cases. From the stability derivatives of the helicopter shown in Table A1 it is evident that, at hover, the derivatives such as $X_v$, $Y_v$, $L_v$, and $M_v$ have relatively high values which means that there is a strong coupling between the longitudinal and lateral motions of the aircraft. However, the coupling terms of the motion of the load can be assumed to be negligible at this particular flight condition.

The mathematical model of the helicopter load system, considered in this chapter, includes the dynamics of the control actuators and the motion of the helicopter with its load is also assumed to be disturbed by atmospheric turbulence.
2.2 THE SUBJECT HELICOPTER

The dynamics of a hovering helicopter can be represented by the set of linearised equations shown in Table 2.1. These equations are based on body-fixed axes through its centre of gravity and they may be applied to any single rotor helicopter. For this study the S-61 was chosen as a representative example of the external-load-carrying class of helicopter.

The S-61 is a multi-role helicopter, with all-weather flying capability, which is manufactured in the USA by Sikorsky and in the UK by Westland Helicopters under licence. Another reason for choosing this type of helicopter was that most of the data required for the mathematical model of this helicopter was available in the literature. The stability derivatives for the type S-61A helicopter at hover were available from Gupta and Bryson [1973] and the modified derivatives for the S-61D at hover were available from Murphy and Narendra [1969]. By examining the stability derivatives of both types of the S-61 helicopter shown in Table A1 it can be inferred that there exists strong coupling between the longitudinal and lateral motions of a hovering helicopter. Nevertheless, the stability derivatives for a hovering S-61D, used in Prabhakar [1976], show that no coupling exists between the longitudinal and lateral modes (see Table A1 where the numerical values of the longitudinal and lateral stability derivatives employed by Prabhakar [1976] are presented in normalised form: the force derivatives have been divided by the aircraft mass and the moment derivatives have been divided by the corresponding
moment of inertia). Such a result makes the mathematical model of Prabhakar over-simplified and unrealistic for this flight condition. From all the available sets of stability derivatives, those corresponding to the most powerful type (the S-61D), i.e. those from Murphy and Narendra, were used in this study.

The small perturbation longitudinal and lateral equations of motion for a helicopter are shown in Table 2.1 upon which the mathematical model, ATHOS, of the S-61 helicopter (Appendix A) is based. These equations can be written in state variable form as follows:

\[
\dot{x}_v = A x_v + B u
\]  

(2.1)

where \( x_v \in \mathbb{R}^n \) is the state vector of the helicopter;

\( u \in \mathbb{R}^m \) is the control vector of the helicopter;

and \( A_s \) and \( B_s \) are the coefficient and control matrices, of appropriate dimension.

The definition of the state vector, \( x_v \), and of the control vector, \( u \), are given in Table 2.2.
\[ \dot{u}_v = x_{uv}u + x_{uv}v + x_{p}p_v - g\theta_v + (x_{q-v} - W_o)q_v + x_{Lq}L + x_{Cq}C \]

\[ \dot{w}_v = z_{uv}u + z_{uv}v + z_{w}w + z_{p}p_v + (z_{q-v} + U_o)q_v + z_{Cq}C \]

\[ \dot{\phi}_v = q_v \]

\[ \dot{\theta}_v = q_v \]

\[ \dot{\psi}_v = p_v \]

\[ \dot{r}_v = L_{uv}u + L_{uv}v + L_{w}w + L_{p}p_v + L_{q}q_v + L_{r}r_v + L_{Cq}C \]

\[ \dot{\psi}_v = r_v \]

\[ \dot{z}_v = N_{uv}u + N_{uv}v + N_{w}w + N_{p}p_v + N_{q}q_v + N_{r}r_v + N_{Cq}C \]

**TABLE 2.1:** Generalised Equations of Motion of a Hovering Helicopter

(The pinned derivatives are analysed in Appendix C)
State vector, $x_v$

\[
\begin{bmatrix}
  u_v \\
  w_v \\
  \theta_v \\
  q_v \\
  v_v \\
  \phi_v \\
  p_v \\
  \psi_v \\
  r_v
\end{bmatrix}
\]
- longitudinal velocity
- vertical velocity
- pitch angle
- pitch angular rate
- lateral velocity
- roll angle
- roll angular rate
- yaw angle
- yaw angular rate

Control vector, $u$

\[
\begin{bmatrix}
  \delta_L \\
  \delta_R \\
  \delta_C \\
  \delta_T
\end{bmatrix}
\]
- longitudinal cyclic control
- lateral cyclic control
- main rotor pitch control
- tail rotor pitch control

TABLE 2.2: The Definition of the State and Control Vectors for the Mathematical Model ATHOS
2.3 THE INFLUENCE OF ROTOR DYNAMICS ON THE MOTION OF A HELICOPTER

The main rotor of the S-61 has fully articulated blades. Such rotors are much more responsive to vertical gust loads than the body of the aircraft, due to the relative flexibility of blades and they react in such a way as to reduce (at least temporarily) the response of the fuselage from the impact of the gust. Thus, while the blades are responding to the gust, the fuselage has time to build up vertical velocity which, in turn, reduces the effective velocity experienced by the rotor. In forward flight, the articulated rotor is very nearly insensitive to in-plane gusts, whereas at hover such gusts considerably affect the response of the rotor. Also, high-intensity turbulence, generated by the rotor itself, can cause severe, random, aerodynamic loads.

For such rotors, the greatest amplitudes of the blade response occur in the low frequency range. These low frequency flapping modes consist mostly of in-plane motion of all the blade elements.

In the case of rotorcraft little has been reported of any studies of the response of a rotor in turbulence. This is probably because the flapping motion is a non-stationary process, the governing linear differential equations having time-varying coefficients. (Wan et al [1972], Judd and Newman [1977]). Moreover, the flapping moment produced by gust is a complex function of both the forward speed and gust wavelength, as a ratio of the radius of the rotor.
Consequently, any realistic study of the response of the rotor blades is very complicated and it demands particular care. Although the rotor dynamics can be described by some appropriate set of equations, the mathematical model of the helicopter would then become very difficult to use, since each blade possesses several degrees of freedom. Such complicated models are only used in special studies of helicopter motion e.g. in the study of the phenomenon of air resonance or for gust alleviation studies. Several simplifying assumptions have been introduced, mainly in the original work of Hohenemser et al [1974]. There the rotor was regarded as responding instantaneously to both airspeed and blade deflection rates. At hover, the changes of torque on the rotor are invariably small, which permits the assumption of constant rotor speed. Furthermore, if the mass centre of the rotor, which is usually located directly under the hub of the rotor, is considered, its displacement due to tilting the rotor may be assumed to be negligible. As a consequence, the centre of mass of the vehicle is essentially fixed in the fuselage.

Several simplified mathematical models involving the above assumptions, and representing rotor/body coupling in a linear flight dynamics analysis, have appeared in the literature (Hall and Bryson, [1973], Hohenemser and Yin [1974], Nagabhushan [1978]). In the simplified model by Hall and Bryson, the rotor is modelled as an axially-symmetric, spinning 'rigid' wing which can be tilted, but not displaced, with respect to the fuselage. Biggers [1974] presented a constant coefficient approximation by transforming the flapping equation into a non-rotating coordinate frame.
However, even employing these simplifications, the resulting equations are complex. For stability and control investigations the motion of the individual blades is not required and it is generally sufficient to consider only the behaviour of the helicopter as a whole, by assuming that it is a single, rigid body.
2.4 A DISCUSSION ON COMMON SUSPENSION ARRANGEMENTS

One of the most common and important considerations in transporting freight by air, particularly for military operations, concerns the use of helicopters to carry cargoes which are suspended from the aircraft. The cargo may be suspended directly, or within a container, by means of strap and lifting hook arrangements.

In such ways, loads, whose bulk and accessibility preclude transport by air by any other means, can be transported successfully. By such arrangements it is possible to reduce turn-round times, an important factor in most civil and military applications.

Inevitably, however, there are limitations of this structurally simple external mode of freight transportation, principally those associated with the aerodynamic forces which arise when there is motion of the air relative to the suspended body. Such forces can lead to static or dynamic instability of the suspended load, and eventually, of the load-carrying helicopter. In the worst cases severe instabilities have led to the loss of freight; generally they lead to handling problems of varying degrees of severity. Most of these problems are associated with the poor aerodynamic characteristics of the suspended load and the poor, inherent, dutch roll characteristics of most helicopters.

Several suspension arrangements are commonly used, employing up to four cable-attachment points, thus sacrificing speed of operation for load stability. Some of these arrangements are shown in Figure 2.1.
FIGURE 2.1: Cable Suspension Arrangements
III. THREE POINT SUSPENSION

IV. FOUR POINT SUSPENSION

FIGURE 2.1: Continued
The single-point load suspension is the most common type, because it is the simplest: only one hook is required on the aircraft. Compact, heavy, dense loads are the easiest type to transport by means of a single-point suspension (Figure 2.1(a), (b) and (c)). Examples of such loads are armoured vehicles, trucks, and artillery equipment. The major disadvantage of the single-point suspension is its lack of yaw restraint, particularly for long loads. Oscillatory sideslip motion can be induced by yawing motion, depending upon the length of the cable amongst other factors. When the single-point suspension is used freight operations are restricted to Visual Flight Rules (VFR) conditions. Attempts to fly suspended loads in Instrument Flight Rules (IFR) conditions have caused severe problems with load stability, the instrument and motion cues being too dissimilar to those of VFR (or even conventional IFR) flight for the pilot to maintain effective control of the suspended load.

The two-point longitudinal suspension arrangement (Figure 2.1(d) and (e)) provides restraint of yawing motion. If the cables are well separated the container will be maintained near its position of minimum drag. However, this position is dynamically unstable in yaw and the resulting oscillatory motion couples strongly with sideslip under certain conditions. To avoid negative yaw damping, which characterises the minimum drag position, the load can be suspended with a nose-down attitude. In practice such an arrangement is prone to flutter in combined sideslip and yaw motions. The type of flutter mode obtained therefore depends, inter alia, on the uniformity of the mass of the suspended body. For uniformly-loaded bodies, suspended by short cables,
the longitudinally-displaced system exhibits good yaw restraint control and load attitude. According to Boeing/VERTOL, low density aerodynamically unstable loads, such as cargo containers are most stable when flown at slightly nose-down attitudes.

An extension of the two-point suspension involving four cables, known as the Sheldon's 'double V' configuration, was investigated by Prabhakar [1976]. Wind tunnel tests showed that the yaw motion of a rigid body suspended by this configuration is very much restricted. In practice, a serious problem may arise: lateral gusts might give rise to a non-linear yaw mode in which pairs of opposite cables become slack alternatively.

Utilisation of the three or four point suspension arrangements is very low compared to the single point and two point suspension. This is so due to (a) the requirement for making centre of gravity computations in order to maintain the in-flight stability and (b) the time required to make a three or four point hook-up. Aside from these problems, it has been found that in general loads carried by three or four points are stable permitting higher airspeeds than single point suspension systems. By using three cables the roll attitude of the load is heavily constrained but the yaw oscillations can be still quite pronounced under certain conditions.

In this research study only three suspension arrangements were examined:

(a) a single cable suspension; (Figure 2.1(a)),
(b) two-point suspension, using two parallel cables;
(Figure 2.1(d)),
and (c) three-point suspension, using three parallel cables.
(Figure 2.1(f)).
In each case it was assumed for undisturbed motion that the centre of gravity (c.g.) of the load was exactly below the c.g. of the helicopter.
2.5 THE CHOICE OF AN APPROPRIATE SUSPENDED LOAD

Several types of load can be transported by helicopter. The use of such a method of transportation for military purposes such as transportation of armoured vehicles, ordnances, ammunition, supplies, etc. and for non-military tasks, such as transporting provisions in cargo containers, or transporting construction materials for the erection of bridges and other structures, etc., is common-place.

Because external loads come in a wide variety of sizes and shapes cargo-handling systems are usually designed in terms of a few representative loads which are considered standard. The 2.4m×2.4m×6.0m (8×8×20ft) box is one of those standards (Szustak and Jenney, [1971]).

In this study the standard size cargo container was considered as a representative external load and it was used in the simulation. The subject load, particularly when empty or only partially full, if suspended from a helicopter, experiences significant aerodynamic forces and moments which create severe control problems. The aerodynamic analysis of the types of suspension mentioned presents considerable difficulties since the cargo container is not streamlined for good aerodynamic performance. The airflow around such bluff bodies is so unstable that the container is stalled at any angle of attack. Results of wind tunnel tests, with a dynamic model oscillating in yaw, indicate a moment hysteresis which results in a strong tendency to stall flutter (Szustak and Jenney, [1971]).

Several reports are available on the aerodynamics of cargo
containers of which those by Simpson and Flower [1976] and Poli and Cromack [1973] are of particular interest. From the aerodynamic data the stability derivatives of the container can be easily evaluated (Appendix B). Some information on the aerodynamic and stability derivatives may also be obtained from the Engineering Science Data Sheets on Aerodynamics.*

The accuracy of the aerodynamic coefficients obtained by aerodynamic methods, basically designed to treat streamlined shapes, becomes suspect if used to analyse bluff bodies. Liu [1973], however, uses a particular method to analyse the aerodynamic coefficients of a suspended load by considering two components. The first component is based upon slender body calculations and the second component is obtained by using a theory of viscous cross-flow. The result of adding together these two components represents the complete aerodynamic coefficient.

* Report No. 71016 on Pedestal Bases, Engineering Science Data Unit, Aerodynamics (Bodies), Subseries Volume 46, Nov. 1978.
2.6 MATHEMATICAL MODEL OF A HELICOPTER-SUSPENDED LOAD SYSTEM

The equations of motion are derived by applying Newton's Laws of motion to the helicopter and load in turn. The helicopter is considered to be a free body with the six spatial degrees of freedom. The load is constrained to move in accordance with the limitations imposed by the suspension arrangement. The effect of the load on the helicopter (and vice versa) is transmitted by the suspension. Such effects are added as additional, external forces and moments, to the equation of motion. These forces and moments are known as interaction forces and moments respectively. Other external forces and moments consist of the aerodynamic and gravitational terms. The derivation of the aerodynamic terms requires a detailed knowledge of both the body aerodynamics and the effect of suspension. The resulting four vector equations are equivalent to 12 scalar equations when resolved on orthogonal body axes systems of the helicopter and load (Prabahakar [1976]).

An orthogonal body axes system has its origin fixed at the centre of gravity of the body. The axes system corresponding to the helicopter has its x-axis along the longitudinal centre line of the vehicle, its y-axis normal to the plane of symmetry and its z-axis in the plane of symmetry. It is the system to which to refer velocities, accelerations and stability and control parameters because the orientation of the pilot with respect to this frame is fixed. Its main advantage in motion calculations is that moments of inertia of the vehicle are constant. A similar axes system corresponding to the load has its origin at the centre of gravity of the suspended body.
Several sets of equations representing the dynamics of a helicopter-load system can be found in the literature. The equations derived by Prabhakar [1976] follow a common approach: all the variables of both the suspended load and helicopter were referred to the body-fixed axes of the helicopter. The type of suspension employed was Sheldon's double 'V' arrangement (Figure 2.1(e)), which reduces the number of degrees of freedom of the load from six to three. This reduction leads to redundancy in the equations of motion; however, the redundancy may be eliminated by combining certain equations which effectively eliminates the interactive forces. Asseo and Whitbeck [1973] derived a more general set of equations which can be used for suspension arrangements involving either one, or two, or three parallel cables, and which may involve adopting the use of winch controls. In these three suspension arrangements the helicopter has the usual six spatial degrees of freedom: three translational and three rotational. When the load is suspended by one cable it has two translational and three rotational degrees of freedom since its vertical motion is restrained since the cable is assumed to be rigid and inextensible. A two parallel cable arrangement allows the load to have two translational and two rotational degrees since a constraint is imposed on the pitch attitude of the load in addition to the vertical motion. Two translational and one rotational degrees of freedom are allowed by a three parallel cable arrangement which constrains the roll attitude, in addition to the pitch and vertical motions. The approach used by Asseo and Whitbeck was to the problem of the derivation of the generalised set of equations of motion.
to derive the motion of the helicopter and load in their own body-fixed coordinates which results in 12 equations of motion. The constraint equations which represent the specific suspension geometry are then added to the equations of motion to result in a non-linear set of equations which completely describes the helicopter and load motion in space in terms of the helicopter interactive forces, that is, in terms of the cable tensions. Elastic cable and vibrating string modes of the system were considered to be second-order effects and were omitted from the dynamic representation. The set of non-linear equations was then linearised about a level flight condition, at constant airspeed, in which both the load and the helicopter had zero roll, pitch and yaw attitudes with respect to earth-fixed axes. The main assumptions used in the linearisation of the equations were the following:

(i) tension variations during transient conditions were small so that the cable remained stretched out at all times;

(ii) each winch controller was equipped with a fast-acting tension regulator which controlled cable tension in response to some tension reference signal;

and (iii) each cable was assumed to be rigid and massless and in which tension was transmitted along the centre line of the cable.

The equations of motion of the helicopter-load system used in this

†It is reported that the full derivation can be found in the Report No. AK-5069-J-1 by Asseo and Erickson, Cornell Aeronautical Laboratory which is not available in the open literature, however.
study are based on the generalised equations derived by Asseo and Whitbeck. The reason for this choice was the generality and completeness of this particular model.

In this research the flight condition of interest was hover, by which it was taken to mean that the airspeed of the helicopter/load system was less than 1 m/sec. In hover the longitudinal and lateral motions of the helicopter are strongly coupled. These coupling terms had therefore to be added to the equations of motion derived by Asseo and Whitbeck [1973] to allow the model to be used for this particular flight condition. The longitudinal and lateral modes of the load were considered to be uncoupled, however.

The transformed equations of motion for either one, or two, or three parallel cable suspensions are given in Appendix C. A careful examination of these equations shows two important features (which characterise the helicopter/load system) are included in the mathematical model:

(i) there exists a coupling between the roll and yaw motions of the helicopter, shown by such terms as \( N_v + \frac{I_{xzv}}{I_{zv}} \) and \( L_v + \frac{I_{xzv}}{I_{xv}} \). This coupling is due to the asymmetry of the helicopter with respect to the vertical plane, defined by the X and Z body-fixed axes.

and (ii) The load inertia affects the control activity of the helicopter: the higher the inertia of the suspended load, the less effective the control action becomes.

The importance of this term will be discussed in Chapters 5 and 6.
2.7 HELICOPTER CONTROLS AND THEIR ACTUATOR DYNAMICS

A realistic mathematical representation of any helicopter should also include the dynamics of those elements which actuate its controls. Most helicopters have five controls:

- longitudinal cyclic;
- lateral cyclic;
- main rotor collective pitch;
- tail rotor collective pitch;
- engine power control.

At hover, however, the speed of the engines is usually constant and any changes required in the magnitude of the rotor thrust vector of the helicopter can be achieved by using the main rotor collective pitch control. Thus, the engine power control did not need to be included in the mathematical model.

Longitudinal cyclic and lateral cyclic controls correspond to tilting the rotor disc along the longitudinal plane or lateral plane, respectively, thereby changing the direction of the thrust vector of the rotor. To generate the required thrust on any shaft-driven machine a torque must be applied to the rotor shaft to overcome the resistive torque which is created in a helicopter by aerodynamic drag of the blades. This torque must be countered to prevent the helicopter from spinning around in a direction opposite to that of the rotor's rotation. Thus, a helicopter uses a second rotating wing, a tail rotor, to provide this necessary antitorque control. However, such tail rotors rotate in the vertical plane so that a sideways force may be
generated by changing the pitch of the tail-rotor blades, which affects the heading of the aircraft. To balance this sidewards force the thrust of the main rotor has to be tilted slightly by using the lateral cyclic pitch control. The actuating elements of the helicopter are those system components which drive the prime movers (i.e. engines, control surfaces). The command signal, either from the pilot or from some on-board flight controller, activates usually an electrohydraulic linear actuator which moves the control valve of the hydraulic actuator and the flow from the hydraulic power supply causes a force to be generated to act on the piston of the actuator which finally moves the control surface via mechanical linkages. The actuating elements depend primarily on the power required to drive the control surface: the larger the surface the more power is required, so that the motion of the helicopter can be controlled easily and rapidly with precision. This imposes a serious constraint on the life of the actuating element.

An actuating element can be classified as a first or second order system depending on its dynamic response characteristics. However, an actuator with high power output and negligible leakage, can be approximated to a second order system (Greensite [1970], Schwarzenbach and Gill [1978]) to represent the dynamic characteristics of the actuating elements. The transfer function of such a system can be written as

\[
\frac{\delta_O(s)}{\delta_I(s)} = \frac{1}{K_A s^2 + \left(\frac{MV}{K_B A^2}\right) s + \left(\frac{\mu V}{K_B A^2}\right)}
\]

where \( V \): volume of trapped fluid

\( \mu \): friction coefficient
A: effective ram area

\( K_B \): bulk modulus of the fluid

M: total mass being moved.

This second order approximation is adequate for most systems (Schwarzenbach and Gill [1978]) to represent the dynamic characteristics of their actuating elements. If the total mass to be moved (i.e. the control surface) is small then a small power actuator can be used, for which the first order representation (2.3) is a good approximation

\[
\frac{\delta_0(s)}{\delta_1(s)} = \frac{\frac{1}{A}}{\frac{-\mu v}{K_B A} s + 1}
\]

Such low power actuators are used to drive small control surfaces (e.g. tail rotor pitch control). High power actuators are needed to drive longitudinal and lateral cyclic pitch control as well as main rotor collective pitch control, because the actuators have to move large control surfaces. The range of frequencies over which the actuating elements operate is such that their natural frequencies do not interfere with the rigid-body motion and the natural frequencies of the structural modes of the aircraft. Since the actual natural frequencies were not known, the values shown in Table 2.3 were used in the simulation. The natural frequencies and damping ratios were based on previous experience. The following models were used to represent the actuating elements for the control surfaces considered.
Helicopter Actuating Element Dynamics

Longitudinal cyclic pitch control

\[
\frac{\delta_{L_0}(s)}{\delta_{L_1}(s)} = \frac{4000}{s^2 + 40s + 4000}
\]  

(2.4)

Lateral cyclic pitch control

\[
\frac{\delta_{R_0}(s)}{\delta_{R_1}(s)} = \frac{4000}{s^2 + 40s + 4000}
\]  

(2.5)

Main rotor collective

\[
\frac{\delta_{C_0}(s)}{\delta_{C_1}(s)} = \frac{3597}{s^2 + 65s + 3597}
\]  

(2.6)

Tail rotor collective

\[
\frac{\delta_{T_0}(s)}{\delta_{T_1}(s)} = \frac{10}{s + 10}
\]  

(2.7)

<table>
<thead>
<tr>
<th>Actuator</th>
<th>Damping ratio, $\xi$</th>
<th>Natural frequency, $\omega_n$ rad/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal cyclic</td>
<td>0.32</td>
<td>63</td>
</tr>
<tr>
<td>Lateral cyclic</td>
<td>0.32</td>
<td>63</td>
</tr>
<tr>
<td>Main rotor collective</td>
<td>0.55</td>
<td>60</td>
</tr>
</tbody>
</table>

TABLE 2.3: Data for Actuating Elements
### TABLE 2.4: Rate and Travel Limits of the Actuating Elements

The block diagram representation of the actuating elements is shown in Figure 2.2 and a list of their rate and travel limits is given in Table 2.4 based on the information available in Prabhakar [1976].

<table>
<thead>
<tr>
<th>Control</th>
<th>Rate limits (rad/sec²)</th>
<th>Travel limits (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal cyclic</td>
<td>-2.5 &lt; ( \dot{\delta}_L ) &lt; 2.5</td>
<td>-0.454 &lt; ( \delta_L ) &lt; 0.454</td>
</tr>
<tr>
<td>Lateral cyclic</td>
<td>-2.5 &lt; ( \dot{\delta}_R ) &lt; 2.5</td>
<td>-0.279 &lt; ( \delta_R ) &lt; 0.279</td>
</tr>
<tr>
<td>Main rotor collective</td>
<td>-2.5 &lt; ( \dot{\delta}_c ) &lt; 2.5</td>
<td>-0.25 &lt; ( \delta_c ) &lt; 0.25</td>
</tr>
<tr>
<td>Tail rotor collective</td>
<td>-0.419 &lt; ( \delta_T ) &lt; 0.419</td>
<td></td>
</tr>
</tbody>
</table>
FIGURE 2.2: Block Diagram Representation of First and Second Order Actuating Elements
As was mentioned earlier the oscillatory motion in yaw of a standard size container can be severe and destabilising. Nevertheless, if the lateral coordinate of one of the attachment points on the helicopter could be altered, by using a lateral displacement winch control, then these oscillations of the load could be prevented from occurring. The idea of using winch controls for stabilising the oscillatory motion of a suspended load was introduced in 1971 by Szustak and Jenney [1971] and by Asseo [1971]. In this present study it was assumed that the dynamics of the winch control can be represented by a linear first-order differential equation, the corresponding transfer function of which is given by

\[
\frac{\delta w_0(s)}{\delta w_1(s)} = \frac{7}{s+7}
\]

(2.8)

The value of the time constant has been chosen arbitrarily to be 0.14 sec, which is a representative value.
2.9 THE INFLUENCE OF ATMOSPHERIC TURBULENCE ON THE MOTION OF THE
HELICOPTER AND ITS SUSPENDED LOAD

The principal factors influencing gust response of an S-61 helicopter are described briefly below.

(a) Rotor blade response. It was mentioned earlier that the articulated blades of the rotor are much more responsive than the aircraft as a whole and that these blades react to reduce the impact of the gust on the fuselage. If, prior to a gust encounter, the rotor is operating on the verge of stall it can be expected to generate less additional lift due to the gust than if it were initially operating at a deflection angle somewhat removed from stall.

(b) Fixed wing response. The horizontal stabiliser of the S-61 generates additional gust loads. The tail plane provides positive incidence stability but it is completely ineffective in the lower half of the speed range of the helicopter. It has negligible effect, therefore, at hover.

(c) Gust characteristics. The gust profile and intensity are important factors. At hover, the penetration time of a gust front is of significance and therefore the contribution of each blade to the loading of the fuselage is considerable.

For constant collective pitch, a disturbance in forward speed would tilt the rotor disk backwards, which causes the thrust vector to tilt and thereby creates a nose-up pitching moment while at the

† A negative disturbance is assumed.
same time increasing the drag force. A lateral speed disturbance produces a sideways tilt of the rotor, thus producing a lateral force which causes the aircraft to roll. The blades of the tail-rotor experience a change of incidence and the tail rotor acts like a fin in producing favourable 'weathercock' stability. An upward (positive) velocity imposed on the helicopter increases incidence of all the blades and consequently increases the total lift. Since the lift components applied on each side of the rotor disk are not equal in magnitude, a backward tilt results which causes a nose-up pitching moment. This effect increases roughly in proportion with speed but is zero at hover (Bramwell [1976]).

It can be seen that in practice any rotary wing aircraft is a complex, aeroelastic mechanism operating in a complicated aerodynamic environment. For analytical convenience the helicopter is assumed to be a single rigid body. It is also assumed that the hovering helicopter is mainly affected by the longitudinal and lateral components of the gust.

The suspended load is affected too by gust and by the downwash produced by the rotor blades. However, since the load is positioned at hover exactly below the helicopter and the suspension cable has a considerable length, rotor effects on the suspended load may be considered to be negligible. It is reasonable to assume, however, that the load is affected by a turbulent field (separate from that affecting the helicopter). How these turbulent fields can be described mathematically is outlined in Chapter three.
2.10 REPRESENTATION OF THE HELICOPTER-LOAD SYSTEM BY ITS STATE VARIABLES

Since optimal control theory will be used for the design of AHCS it was convenient to arrange that the helicopter/load first-order, differential equations shown in Appendix C were expressed in state variable form. Thus the first derivatives of the state variables appear on the left hand side of the equation and on the right hand side appear terms containing only state variables and control variables. The highest order (32) mathematical model used in this study is referred to as GIONA (Appendix G). The first 24 elements of the state vector, \( x_s \), correspond to the vehicle and load as shown in Appendix C1. The final eight elements correspond to the actuator dynamics. The state variable equation representing the dynamics of the helicopter and load with actuators is given by

\[
\dot{x}^* = A^* x^* + B^* u
\]

where

\[
x^* = \begin{bmatrix} x_s \\ x_a \end{bmatrix}, \quad x^* \in \mathbb{R}^n
\]

\( x_s \) is the state vector of the helicopter/load system

\( x_a \) is the state vector of the actuating elements

\[
A^* = \begin{bmatrix} A_s & B_s \\ 0 & A_a \end{bmatrix}
\]

\[
B^* = \begin{bmatrix} 0 \\ B_a \end{bmatrix}
\]
The control vector is defined (see also Appendix C2) as:

\[
\mathbf{u} = \begin{bmatrix}
\delta L_c \\
\delta R_c \\
\delta T_c \\
\delta C_c \\
\delta W_c
\end{bmatrix}
\]  

where \( \mathbf{u} \in \mathbb{R}^m \).

When disturbances, \( x_d \), are considered an appropriate form for the state variable equation is

\[
x_d^* = A^*x_d^* + B^*u + G^*x_d \\
\]

where \( x_d \in \mathbb{R}^s \). As it will be shown in Chapter 3, the disturbance vector is usually obtained from an equation of the form

\[
x_d^* = A^*x_d + B^*u + G^*x_d \\
\]

where \( \eta_w \) represents a vector consisting of independent white noise components.

For the models used in this research, it was found to be convenient to combine (2.13) and (2.14) which results in (2.15), viz.

\[
\begin{bmatrix}
x_d^* \\
x_d^* \\
\end{bmatrix} = \begin{bmatrix}
A^* & G^* \\
0 & A_d^* \\
\end{bmatrix} \begin{bmatrix}
x_d^* \\
x_d^* \\
\end{bmatrix} + \begin{bmatrix}
B^* \\
0 \\
\end{bmatrix} u + \begin{bmatrix}
0 \\
B_d^* \\
\end{bmatrix} \eta_w
\]

or

\[
x_d^* = A^*x_d + B^*u + C^*\eta_w \\
\]

\( ^*_* \) The elements of matrix \( G^* \) were obtained from McRuer et al [1976] and correspond to the stability derivatives of the system associated with the forces and moments created by the turbulence on the helicopter and its load.

\( ^*_* \) If the actuator dynamics are not included in the mathematical model then \( x_d^* = u \) and \( C^* = C_c^* \) etc.
The information available for processing by an AHCS, consists of variables of the system which can be actually measured by using appropriate sensors. Therefore it is necessary to include the measurable state variables in an output vector, \( y \), which is related to the state vector, \( x \), and the control vector \( u \), as follows,

\[
x = \begin{bmatrix} x^* \\ \vdots \\ x_g \end{bmatrix}
\]

\[
A = \begin{bmatrix} A^* & C^* \\ 0 & A_g \end{bmatrix}
\]

\[
B = \begin{bmatrix} B^* \\ 0 \end{bmatrix}
\]

\[
G_g = \begin{bmatrix} 0 \\ B_g \end{bmatrix}
\]

where \( y \in \mathbb{R}^P \).

Sensor noise usually contaminates the measurement of the outputs of the system. Thus it is reasonable to define a measurement vector, \( z \), as shown by equation (2.22), viz.

\[
z = y + v
\]

where \( z \) measurement vector, \( z \in \mathbb{R}^P \)

and \( v \) is the sensor noise vector.
Another mathematical model, OLYMPUS is of order 30, because the helicopter and load vertical displacements, \( z_v \) and \( z_l \), can be ignored since no vertical disturbances are introduced in the models to affect considerably the motion along the vertical axis. The definition of the state vector, \( x_s \), for the model OLYMPUS is given in Table 2.5.

\[
\begin{align*}
\begin{bmatrix}
  x_v \\
  u_v \\
  w_v \\
  \theta_v \\
  q_v \\
  x_l \\
  u_l \\
  w_l \\
  \theta_l \\
  q_l \\
  y_v \\
  v_v \\
  \phi_v \\
  p_v \\
  y_l \\
  v_l \\
  \phi_l \\
  p_l \\
  r_l
\end{bmatrix}
\end{align*}
\]

**Longitudinal motion variables**

**Lateral motion variables**

**vehicle**

**load**

**vehicle**

**load**

**TABLE 2.5:** Definition of the State Vector, \( x_s \), of the mathematical model OLYMPUS Without Actuator Dynamics
Since it is usually assumed that the helicopter is operating in a turbulent environment, an appropriate mathematical model representing atmospheric turbulence had to be found. Such a model is analysed in the following Chapter 3.
CHAPTER 3

APPROPRIATE MATHEMATICAL REPRESENTATION OF

ATMOSPHERIC TURBULENCE
3.1 INTRODUCTION

The physical and operational characteristics of aircraft capable of V/STOL operations generally result in such aircraft being susceptible to pronounced effects of atmospheric turbulences on their flying qualities. In particular, helicopters which are capable of operating at very low, or even zero, airspeeds, are affected greatly by wind shear and fluctuating gusts. Also, the handling qualities of any helicopter flying in turbulence are determined chiefly by the inherently low levels of aerodynamic damping and speed stability of the vehicle. Thus any trajectory, or station-keeping, control task may be characterised by a non-deterministic dynamic response.

Helicopter behaviour in gusts merits special consideration which is reflected in the stringent requirements laid down by aviation authorities. These requirements must be met before an aircraft is certified as airworthy.

For the past ten years the Federal Aviation Administration (FAA) of the U.S.A. has conducted a series of experiments (Sinclair and West, [1978]) with the aim of identifying operational limits and minimum acceptable levels of flying qualities for powered lift aircraft (see Appendix D). The special significance in the identification of any interaction between aircraft dynamics and the dynamics of the atmosphere during low speed flight was the major purpose of these experiments, the results of which were based upon pilot evaluations and which proved that in general turbulence possesses a distinctly non-Gaussian velocity distribution, and exhibits intermittency and patchiness of mean duration.
of 10s. It was also found that the probability of encountering turbulence is highest at low altitudes (Chalk, et al [1969]).

According to Von Karman (Von Karman, [1937]), atmospheric turbulence may be considered to be a continuous, random physical process in three dimensional space. Any turbulent field may be characterised by its velocity decomposed into its orthogonal components.

Helicopters are primarily affected by the horizontal and lateral components of the gust: the horizontal component causes the helicopter to roll and the lateral component causes it to pitch, since the lift produced by the blades moving against the turbulence is greater than that lift produced by the blades moving with the turbulence.

In the work reported here, it was necessary to generate from computer models of atmospheric disturbances, turbulence inputs to test the simulated response of some mathematical models of a helicopter with suspended load. The vehicle and load response were then evaluated assuming a state of 'statistical equilibrium' to exist between excitation and response.

Several mathematical models of atmospheric turbulence are available in the literature. The turbulence models most commonly used by American Authorities are the continuous spectral representations of Von Karman and Dryden. Both forms have Gaussian amplitude distributions (Chalk et al [1969]).

U.K. authorities have taken an approach based on time domain representation. They define a series of static strength cases which
must be satisfied by the aircraft irrespective of the probability of their occurrence. The main reason for employing this approach is that large gusts do not conform to the assumption of a Gaussian velocity distribution which is the basis of the power spectral approach.

Predictability and monotonic composition are often viewed as unrealistic qualities of modelled turbulence. The less-than-realistic quality of the available mathematical models of atmospheric turbulence has been pointed out in several reports (Jones [1971], Jacobson and Joshi [1977] and Jewell et al [1978]). In these reports it is confirmed that different handling qualities can be obtained from non-Gaussian models of computer-generated atmospheric turbulence when compared with corresponding ratings from a Gaussian model of about the same root mean square (r.m.s.) turbulence level. According to Jewell, for example, a more satisfactory representation might be provided by using the Dryden model with a time-varying rms level exhibiting a correlated pulse-like character of the velocity of the turbulence. The few available reports on the use of non-Gaussian models of computer generated turbulence unfortunately present conflicting results with regard to the pilot-vehicle performance.

In this chapter, two mathematical descriptions of atmospheric turbulence, involving the use of either a Dryden filter or a simple time-lag filter are analysed in Section 3.2 to find an appropriate model of turbulence for use with the mathematical model of the helicopter and its suspended load. This section is also concerned with how these models may be simulated and in the final Section, 3.3, some remarks are presented on the choice of the appropriate model.
3.2 MODELLING TECHNIQUES AND DIGITAL SIMULATION OF ATMOSPHERIC TURBULENCE

Since turbulence is a random process it can be regarded as comprising a number of individual events which are unpredictable by the observer but which may be characterised by statistical methods. To describe the fundamental properties of a random process, three types of statistical measures are used:

(a) the probability density function,
(b) autocorrelation function,
and (c) power spectral density function.

To introduce turbulence effects into any flight control study requires some simplification of the mathematical form representing the turbulence mainly because of computational constraints. The simplification is based on the assumptions given in Section 3.2.1.

3.2.1 Assumptions for the Simplifying Derivation of Turbulence Models

When atmospheric turbulence has to be accounted for it is generally (Chalk, et al [1969], Saouliis [1980]) considered acceptable to neglect the interaction with the main airflow, but even with that assumption only the simplest models are possible, for the physical properties of the turbulence are identified over volumes chosen to be sufficiently small, so that these volumes are reasonably uniform. For the case of a helicopter with suspended load it can be assumed that the motions of the vehicle and its load are disturbed by two separate turbulence conditions: one relative to the body-fixed axes of the helicopter and another relative to the body-
fixed axes of the load. It is further assumed that turbulence is both homogeneous and isotropic. The assumption of homogeneity implies that the statistical properties of turbulence are the same at every point of the volume considered. Complete isotropy ensures that the statistical properties of turbulence are unaffected by any translation, or rotation, of the axes used to define the three dimensional space. It also implies that all three components of velocity have equal intensities and are uncorrelated. With complete isotropy this property extends over all wavelengths of turbulence. In practice, turbulence can be considered isotropic only over some finite range of wavelengths depending on the level of the intensities of its velocity components and nearness to the ground. Near the ground, where the helicopter is assumed to hover, the locally isotropic range of wavelengths is proportional to the height above the ground. The models described in this chapter do not take into account ground effects on turbulence for low altitude flight but they do exhibit some locally isotropic properties.

Another important assumption is that the turbulent field has a pattern 'frozen' in space; in other words the statistical characteristics of the disturbance input to a helicopter, flying through a turbulent field, are not appreciably affected by the variation of that field with time. This assumption is referred to as Taylor's hypothesis (Taylor, [1937]), which implies that the turbulence-induced response of the helicopter results only from the motion of the helicopter relative to the turbulent field. The 'frozen' field concept has been found to be valid for those cases in which the rms components of the velocity of the turbulence are small relative to the ground speed of the helicopter;
when the helicopter is at hover this turns out to be an unrealistic assumption.

The assumptions discussed above were invoked to simplify the modelling technique. Some further assumptions were then made to simplify the simulation techniques employed. These assumptions were:

(a) Turbulence is a Gaussian random process.
A Gaussian noise signal with zero mean value is considered to best represent the turbulence characteristics. The useful property of such a signal is that if it is the input to a linear system then the output from that system will have the same Gaussian nature.

(b) Turbulence is a stationary random process.
The concept of stationarity is an idealisation of the intuitive notion that the physical character of a noise signal does not change over the total time interval of observation, that is, over any given 'patch' of turbulence. This means that the probability functions of stationary processes can be defined by observing over a sufficiently long interval of time only a single random process and not an ensemble of them. Thus turbulence can be characterised easily by its power spectral density function. If the process is stationary the covariance of the noise signal does not change with time.

(c) Turbulence is an ergodic process.
Ergodicity is a useful simplifying assumption which permits one to equate ensemble averages and averages with respect to time performed on a single representative member of the ensemble.
3.2.2 The Von Karman and Dryden Filters

When using Taylor's generalised representation of turbulence (Taylor, [1937]) several forms of atmospheric turbulence may be obtained, given in terms of kinetic energy density functions, as follows:

\[
\phi_u^{(k)} = 4\sigma_i^2 L/(1+4\pi^2a^2k^2)^{2} \quad \text{n+1}
\]

\[
\phi_v^{(k)} = 2\sigma_i^2 L[1+8\pi^2a^2k^2(n+1)]/[1+4\pi^2a^2k^2]^{2} \quad \text{n+3}
\]

where the suffixes \(u\) and \(v\) represent respectively, the longitudinal and lateral velocity components of the turbulence, \(k\) is the inverse of the wavelength, \(\lambda\), \(\sigma_i^2\) is the mean square value (msv) of the \(i\)th component of the turbulence, \(L\) is a scale length, \(n\) is a shape parameter, and \(a\) is a parameter given by

\[
a = [(n-1)!/\sqrt{\pi(n-\frac{1}{2})!}]L
\]

The shape parameter, \(n\), can be assigned any value which results in a range of shapes of turbulence. If \(n\) is not a positive integer then the terms \([(n-1)!]\) and \([(n-\frac{1}{2})!]\) of equation (3.3) are gamma functions.

If \(n\) is chosen to be one half, the resulting formula is identical to the empirical model proposed by Dryden in 1938. When \(n\) is equal to 1/3 then the model is identical to the model suggested later by Von Karman [1948].

\[\text{\textsuperscript{†}A similar expression to (3.2) exists for the vertical component of the turbulence which is not used in this study.}\]
Dryden Model

From (3.3)

\[ a = L \]  \hspace{1cm} (3.4)

since \( n = 0.5 \).

Hence,

\[ \Phi_u (k) = 4\sigma_u^2 L/[1 + (2\pi n L)^2] \]  \hspace{1cm} (3.5)

and

\[ \Phi_v (k) = 2\sigma_v^2 L[1 + (2\pi n L)^2]^2 \]  \hspace{1cm} (3.6)

Since it was assumed in Section 3.2.1 that the gust field is isotropic, it follows that

\[ \Phi_v (k) = \Phi_u (k) \]  \hspace{1cm} (3.7)

Von Karman Model

When \( n = 0.333 \), then from (3.3),

\[ a = [1.3399]L \]  \hspace{1cm} (3.8)

and, hence,

\[ \Phi_u (k) = 4\sigma_u^2 L/[1 + (2\pi (1.3399L)k)^5/6] \]  \hspace{1cm} (3.9)

and

\[ \Phi_v (k) = 2\sigma_v^2 L[1 + 8/3(2\pi (1.3399L)k)^2]/[1 + (2\pi (1.3399L)k)^2]^{11/6} \]  \hspace{1cm} (3.10)

The Von Karman model more closely corresponds with the observed behaviour of turbulence (if compared to all such representations). However, the Von Karman model cannot easily be programmed for simulation in real time because of the non-integer exponent in (3.9) and (3.10).
The Dryden model is a suitable alternative which provides a power spectral density closely matching that of the Von Karman model. Some small differences occur at higher frequencies but these are of small consequence to the design procedures of this study.

The mathematical models of turbulence are most commonly described by power spectral density functions, $\phi(\omega)$. The power spectral density (PSD) of a component of turbulence, $i_g(t)$, is a real function which describes the distribution of the mean square value (m.s.v.) of $i_g(t)$ with frequency. It can be shown (Saoullis, [1980]), that

$$k = \frac{\omega}{2\pi U_0}$$

and

$$\phi(\omega) = \frac{1}{2\pi U_0} \phi(k)$$

where $U_0$ is the relative speed of the airflow, and $\omega$ is the frequency of the turbulence.

Using (3.11) and (3.12) the Dryden model can be shown to possess PSD functions as follows:

$$\phi_{u_g}(\omega) = \sigma_u^2 \frac{L_u}{\pi U_0} \left[ \frac{1}{1+(\frac{U}{U_0})^2} \right]$$

$$\phi_{v_g}(\omega) = \sigma_v^2 \frac{L_v}{\pi U_0} \left[ \frac{1}{1+3(-\frac{\omega}{U_0})^2} \right]$$

Each of these PSD functions may be obtained by passing white noise through a linear system with a particular transfer function, $G(s)$, as represented in Figure 3.1.
It can easily be shown (Truxal, [1955], McRuer et al [1973], that the spectral density, $\Phi_O(\omega)$ is given by equation (3.15), viz.

$$\Phi_O(\omega) = |G(j\omega)|^2 \Phi_{in}(\omega) \tag{3.15}$$

where

$$|G(j\omega)|^2 = G^*(j\omega)G(j\omega) \tag{3.16}$$

with the Laplace transform variable $s$ being replaced by $(j\omega)$ and the asterisk denoting the complex conjugate.

Hence, the spectral equations (3.13) and (3.14) may be considered to be equivalent to the following transfer functions:

$$G_u(s) = k_{fu} \left( \frac{1}{1+ \frac{u}{u_O} s} \right)$$

$$G_v(s) = k_{fv} \left( \frac{\sqrt{3} v}{1+ \frac{v}{v_O} s^2} \right)$$

where $k_{fu}$ and $k_{fv}$ are scaling factors given as

$$k_{fu} = \sigma_u \sqrt{\frac{2L_u}{uO}}$$

$\Phi_{in}(\omega)$ is assumed to be equal to one. (see section 3.2.4).
and

\[ k_f = \sigma_v \sqrt{\frac{L_v}{\pi U_0}} \]  

(3.20)

Zero-mean Gaussian noise signal \[ \eta_g, \sigma_g \]  

Scaling Factor \[ K_f \]  

Shaping Filter \[ G(s) \]  

Turbulence Variable \[ x \]  

Dryden Filter

**FIGURE 3.2: Block Diagram of the Dryden Filter**

### 3.2.3 Choosing Gust Intensities and Scale Lengths

**a) Gust Intensities**

Over those intervals of time and space which are of concern, the velocity field of the turbulence may be regarded as comprising a steady component of velocity with a known mean value and a fluctuation superimposed. When a helicopter penetrates turbulence the energy that it absorbs can be expressed in terms of the r.m.s. values of the velocities and the amplitude of these fluctuations. The feature that distinguishes one patch of atmospheric turbulence from another is the total turbulence intensity, \( \sigma \), which is defined as the kinetic energy of turbulence per unit mass of air. Thus,
\[ \sigma^2 = \int_0^\infty \Phi_0(k) \, dk \]  

(3.21)

\( \Phi(k) \) is the kinetic energy distribution of turbulence; the argument, \( k \), is the inverse of the wavelength, \( \lambda \), of the turbulence. In this manner turbulence is described by the frequency of appearance of a particular wavelength per unit length of travel.

Equation (3.21) is also equivalent to the definition of the rms intensity of a random signal described in terms of its power and frequency, viz.

\[ \sigma^2 = \int_0^\infty \Phi_0(\omega) \, d\omega \]  

(3.22)

or

\[ \sigma^2 = \int_0^\infty |G(j\omega)|^2 \Phi_{in}(\omega) \, d\omega \]  

(3.23)

b) Scale Lengths

Each of the analytical functions (3.1) and (3.2) used to describe the energy density of the turbulence, includes a parameter which has length as a dimension and which is proportional to the scale of the turbulence. The value of that scale may be chosen purely for analytical convenience.

The turbulence scale length can be defined as

\[ L = \int_0^\infty R_g(r) \, dr \]  

(3.24)

or

\[ L = \int_0^\infty R_g(r) \, dr \]  

(3.25)

where \( R_g \) is the loaded area of the vehicle, and \( r \) is a distance in the direction of the velocity component of the turbulence.
When the scale of turbulence is large compared with the helicopter surface being loaded the turbulence velocity may be assumed to be the same over the whole surface at any instant. The one-dimensional energy density for lateral turbulence, $\phi_v$, is then applied.

The scale lengths, $L_u$, $L_v$ and $L_w$, depend on the altitude of the aircraft according to the approximate relationships shown in Table 3.1.

<table>
<thead>
<tr>
<th>Height (h) ft.</th>
<th>$L_w$</th>
<th>$L_v$</th>
<th>$L_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h &gt; 2500$</td>
<td>2500</td>
<td>2500</td>
<td>2500</td>
</tr>
<tr>
<td>$100 &lt; h \leq 2500$</td>
<td>$h$</td>
<td>$184h^{1/3}$</td>
<td>$184h^{1/3}$</td>
</tr>
<tr>
<td>$h \leq 100$</td>
<td>100</td>
<td>$184h^{1/3}$</td>
<td>$184h^{1/3}$</td>
</tr>
<tr>
<td>$h &gt; 1750$</td>
<td>1750</td>
<td>1750</td>
<td>1750</td>
</tr>
<tr>
<td>$100 &lt; h \leq 1750$</td>
<td>$h$</td>
<td>$145h^{1/3}$</td>
<td>$145h^{1/3}$</td>
</tr>
<tr>
<td>$h \leq 100$</td>
<td>100</td>
<td>$145h^{1/3}$</td>
<td>$145h^{1/3}$</td>
</tr>
</tbody>
</table>

**TABLE 3.1:** Scale Lengths † for Clear Air Turbulence for the Von Karman and Dryden Forms of Turbulence (Ref: Chalk et al, [1969])

†Formulas are valid for Imperial units only.

Choosing Appropriate $\sigma$ and $L$

The random continuous turbulence models described in MIL-F-8785B have a single rms vertical gust specification for clear air turbulence which is valid for both Von Karman and Dryden forms, (Figure 3.3).
Once the altitude of flight is known, the rms intensity, $\sigma_w$, may be obtained from Figure 3.3 and the scale lengths $L_w$, $L_u$, and $L_v$ can be evaluated by using Table 3.1.

Two sets of equations relating the mean square intensities and the scale lengths are provided, namely:

for the Dryden model

$$\frac{\sigma^2}{L_u} = \frac{\sigma^2}{L_v} = \frac{\sigma^2}{L_w}$$

(3.26)

for the Von Karman model

$$\frac{\sigma^2}{L_u^{2/3}} = \frac{\sigma^2}{L_v^{2/3}} = \frac{\sigma^2}{L_w^{2/3}}$$

(3.27)

The gust intensities and scale lengths shown in Table 3.2 apply to this study, since it is assumed that the helicopter is hovering at low altitude (approximately 30m) in clear air turbulence.

<table>
<thead>
<tr>
<th>Gust Intensities (m/sec)</th>
<th>Dryden</th>
<th>Von Karman</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_w$</td>
<td>2.06</td>
<td>2.06</td>
</tr>
<tr>
<td>$\sigma_u$</td>
<td>5.34</td>
<td>4.21</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>5.34</td>
<td>4.21</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scale Lengths (m)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_w$</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$L_u$</td>
<td>194</td>
<td>257</td>
</tr>
<tr>
<td>$L_v$</td>
<td>194</td>
<td>257</td>
</tr>
</tbody>
</table>

**TABLE 3.2: Appropriate Scale Lengths and Intensities of the Turbulence for the Dryden and Von Karman Models**
3.2.4 The Definition and Computer Generation of White Noise

A white noise (Figure 3.4) process is Gaussian and its density function is uniquely defined by its mean and its variance. Gaussian white noise, at any time, \( t_1 \), is a random variable with infinite variance.

Consequently the correlation of white noise is zero for any value of time shift, \( \tau \), except \( \tau = 0 \); then the correlation is equal to the variance, which is defined as,

\[
\phi_{nn}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \eta'(t) \eta(t) dt
\]

\[= \text{mean square value} = \text{variance.}\]

In general, the value of the variance governs the strength of the noise process in terms of the power level at each frequency; the larger the variance the stronger the noise and the more pronounced are its effects on the system.
White noise is thus defined as a signal having uniform power density at all frequencies, that is,
\[ \phi_{\text{nn}}(\omega) = C \quad \text{for} \quad -\infty < \omega < \infty, \quad (3.29) \]
since the autocorrelation function (given by the inverse Fourier transform of a constant) is an impulse \( \delta(r=0) \).

In practice, band limited white noise (BLWN) may be used which has a constant power spectral density, \( A \), for all frequencies less than some given frequency, \( \omega_1 \), and is zero for all higher frequencies, that is,
\[
\phi_{\text{nn}}(\omega) = A \quad \text{for} \quad -\omega_1 \leq \omega \leq \omega_1 \\
0 \quad \text{for} \quad \omega > \omega_1 \quad \text{and} \quad \omega < -\omega_1 \]
(3.30)
Provided that \( \omega_1 \) is an order of magnitude larger than the bandwidth of the system subjected to it, then BLWN is an adequate approximation to true white noise.

In digital simulation a random number generator is used to produce band limited white noise. Random noise generators often employ pseudo-random binary sequences. In most digital random number generators the program generates the same sequence of random numbers each time. This was found to be very useful during the simulation because in this way the results could be compared.

3.2.5 Simulation of the Dryden Filter

Since digital simulation was employed to study the helicopter's response it was convenient to express the transfer functions in state variable form.
It can easily be shown that the state variable forms of the two turbulence components of interest are as follows:

**Horizontal Component**

\[
\dot{u}_g = -\left[\frac{U_0}{L_u}\right] u_g + \left[\sigma_{u_g} \sqrt{\frac{2U_0}{\pi L_u}}\right] \eta_u
\]  

(3.31)

**Lateral Component**

\[
\begin{bmatrix}
\dot{v}_g \\
\dot{\gamma}_g
\end{bmatrix} = \begin{bmatrix}
\frac{U_0}{L_v} & 0 \\
\sigma_{v_g} \left(\frac{U_0}{L_v}\right)^{3/2} & \frac{U_0}{L_v}
\end{bmatrix} \begin{bmatrix}
\dot{v}_g \\
\dot{\gamma}_g
\end{bmatrix} + \begin{bmatrix}
0 \\
\sigma_{v_g} \frac{3U_0}{\pi L_v}
\end{bmatrix} \eta_v
\]

(3.32)

The values of the parameters in the equations above may be obtained from Table 3.2.

For the case of a hovering helicopter, the stream velocity, \(U_0\), is very small, and consequently the coefficients \(\frac{U_0}{L_u}\) and \(\frac{U_0}{L_v}\) of the state variable equations become very small. As a result, the velocity components of the turbulence have low frequencies. This is considered as an undesirable property of the Dryden model for this particular flight condition, since it is known that the highest frequency turbulence is encountered at low altitudes (Chalk et al., [1969]) where the helicopter is assumed to hover. In addition, the values of the suggested intensities for these two components of turbulence are very much higher than the rms velocity of the hovering helicopter. Therefore, in this case, Taylor's hypothesis of a 'frozen' field is not a realistic assumption.
3.2.6 A Simple Shaping Filter

Since none of the available models of turbulence allows for scale lengths and disturbance intensities which are appropriate for this difficult phase of flight (i.e. hover), and in order to minimise the costs of computation, it was decided that a simple, time-lag, shaping filter should be used for each of the velocity components of the turbulence. This simple filter allows for appropriate intensities to be used without the dependence upon any scale lengths.

Therefore, the following transfer functions were considered:

\[
G_u(s) = k_u \frac{1}{\tau_u s + 1} \]

\[
G_v(s) = k_v \frac{1}{\tau_v s + 1} \]

where \(k_u, k_v\) are scaling factors,

and \(\tau_u, \tau_v\) are correlation times.

Once again, turbulence was considered to consist of independent Gaussian processes with zero mean values and white noise was used as an input to each of the filters. A similar block diagram (Figure 3.2) to that for the Dryden filter can be constructed, with the appropriate scaling factors and shaping filters. Thus, the block diagram representation of the two independent turbulent fields affecting the helicopter and the load can now be constructed as shown in Figure 3.5.
FIGURE 3.5: Block Diagram Representation of the Turbulence Fields Affecting the Helicopter and the Suspended Load
3.2.7 Simulation of the Simple Filter

This turbulence model was a lot easier to simulate. The transfer functions representing the two-dimensional turbulence model which are shown by equations (3.32) and (3.34) can be written in state variable form as follows:

\[
\dot{u}_g = -\frac{1}{\tau_c} u_g + \frac{k_u}{\tau_c} \eta_u \quad (3.35)
\]

\[
\dot{v}_g = -\frac{1}{\tau_c} v_g + \frac{k_v}{\tau_c} \eta_v \quad (3.36)
\]

The response of this model was investigated thoroughly. Several values were chosen for the correlation time, \( \tau_c \), the scaling factor, \( k \), and the standard deviation, \( \eta_g \), of the input noise signals, \( \eta_u \) and \( \eta_v \), and the response of the turbulence model was obtained in each case. The input to this model is, once again, independent uncorrelated white noise. The filter effectiveness is a function of the correlation time, \( \tau_c \). Figure 3.6 shows that, for very small values of the correlation time, \( \tau_c \), the break-point frequency of the filter is, of course, very high and consequently very little of the noise is filtered. If \( \tau_c \) is very large, there is excessive loss of information. Therefore it can be seen that, although the correlation time affects the sharpness of the turbulence, the standard deviation affects the level of the turbulence (Figure 3.7).

Figure 3.8 shows the effect that different noise generators have on the response of the system. Typical responses of the velocity components of the turbulence are shown in Figures 3.9 and 3.10.

It is worth noting that the results above were obtained by using a unity scaling factor. An interesting case arises when the scaling factor is equal to the correlation time, \( \tau_c \). It can then be seen (Figure 3.11)
FIGURE 3.6: The Effect of the Correlation Time Upon the Atmospheric Disturbance Changes
FIGURE 3.7(a): The Effect of the Standard Deviation Upon the Atmospheric Disturbance Changes
FIGURE 3.7(b): The Effect of the Standard Deviation Upon the Atmospheric Disturbance Changes
\[ \sigma_\eta = 3 \text{ m/sec.}, \tau_c = 1.0 \text{ sec.} \]

\[ \sigma_\eta = 3 \text{ m/sec.}, \tau_c = 3.0 \text{ sec.} \]

FIGURE 3.8: The Effect of Different Noise Generators

- Noise generator No. 1
- Noise generator No. 2
- Noise generator No. 3
\[ u_g (\text{m/sec}) \]

\[ \tau_c = 2.0 \text{ secs.}, \sigma_{\eta_u} = 1.8 \text{ m/sec.}, k_u = 1.0 \]

\[ v_g (\text{m/sec}) \]

\[ \tau_c = 2.0 \text{ secs.}, \sigma_{\eta_v} = 0.9 \text{ m/sec.}, k_v = 1.0 \]
FIGURE 3.11: The Effect of the Correlation Time Upon the Atmospheric Disturbance Changes When $k=\tau_C$
that the sharpness of the gust is not affected by the value of the correlation time, $\tau_c$. Instead, $\tau_c$ now affects the level of turbulence. It should be noticed that the response curves are linearly displaced with the correlation time.

The effect of the standard deviation in this case is similar to that obtained by the correlation time.
3.3 CONCLUDING REMARKS

In this chapter it has been demonstrated how some atmospheric turbulence models can be used by digital simulation. A pseudo-random noise generator was used to produce a sequence of numbers with a Gaussian probability distribution and zero mean value. The signal obtained from the noise generator was fed into a shaping filter after scaling by a factor, k. Two independent turbulence fields were considered: one affecting the helicopter motion and the second affecting the motion of the suspended load. Each turbulence field was assumed to consist of two velocity components. No angular velocities were considered in this work, since their relative amplitude was sufficiently small and thus they were regarded as being comparatively insignificant. Two models of turbulence were examined in detail: the Dryden filter and a simple-lag filter. The Dryden filter is considered to be the second best representation of atmospheric turbulence after the Von Karman filter. The Von Karman filter is very difficult to simulate due to its possession of non-integer exponents. The Dryden model is sufficient representation of turbulence for most flight conditions. But for the case of a helicopter at hover, even this model is not considered to be very representative, since the suggested intensities of turbulence for the Dryden filter are too high for a stationary helicopter and the frequency of the turbulence is at unrealistically low levels. Also, the fact that some components are represented by second-order linear differential equations, means that more time is needed for simulation purposes if compared to the simplified first order model. This simple time-lag model is considered to be sufficient for the present investigation where the mathematical model of
a helicopter with a suspended load is subjected to atmospheric turbulence. The intensity and sharpness of the turbulence can be adjusted as required by varying the three basic parameters of this model, namely, the correlation time, $\tau_c$, the standard deviations of input noise $\sigma_z$ and the scaling factor, $k$. 
CHAPTER 4

OPTIMAL AHCS
4.1 INTRODUCTION

The dynamics of the helicopter with a suspended load can be adequately described by a set of linear differential equations which are most conveniently expressed in state variable form. For such a dynamic system, it is relatively easy to determine a satisfactory feedback controller, particularly designed to keep the deviation of the state of the system from a reference condition within acceptable limits whilst using only acceptable control action.

Modern control theory is a body of knowledge which permits satisfactory design of control systems for multi-input, multi-output (MIMO) systems such as a helicopter with a suspended load. The problem of precision hover can be solved by considering it to be a member of the class of linear quadratic control problems, a special case of modern control theory.

Optimal control methods are particularly suitable for designing Automatic Hovering Control Systems (AHCS) to provide active control to augment the stability of the flight control system. In particular, the main objective of such control is to provide optimal, fully-automatic hover control system which both stabilises and positions the helicopter and its load by moving that helicopter about some desired hover point in such a way as to minimise any unwanted motion of the load while keeping the associated rotor deflections to within prescribed limits and reducing such rotor activity to a minimum. These features are achieved by a means of minimising a performance index, which is a single measure of the performance of the system and which is chosen to emphasise those characteristics of
the response that are considered to be specially important. When a particular performance criterion is minimised, subject to the constraints imposed by the state equation, the resulting optimal design of the AHCS is unique.

However, since many of the state variables of a system are unlikely to be easily measured, it is obvious that only limited information concerning the state of a system could be available for processing by such a feedback controller. Such a limitation on the available state information is a serious problem for which two solutions may be considered:

(i) use of a state estimator to provide the controller with as much of the lost feedback information as possible;

(ii) use of an output regulator which explicitly takes into account in its formulation the fact that only limited feedback information is available for the control law.

Usually there is a limit to the number of states which can be excluded from a feedback loop. This limit is dictated by controllability and stabilisability considerations.

In this chapter, the procedures for designing output regulator and a full-state controller with an estimator are formulated by solving the Linear Quadratic Problem (LQP) for the controller design and the Linear Quadratic Gaussian problem (LQG) for the estimator design.
4.2 PROBLEM DEFINITION

A helicopter with a suspended load can be represented as a linear, time-invariant dynamic system, a block diagram representation of which is shown in Figure 4.1. The dynamical behaviour of such a system can be described (see Chapter 2) by the set of vector differential equations (4.1), i.e.,

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x(0) \tag{4.1a}
\]

\[
\chi(t) = Cx(t) + Du(t) \tag{4.1b}
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector of the system

\( u(t) \in \mathbb{R}^m \) is the input vector of the system

\( \chi(t) \in \mathbb{R}^p \) is the output vector of the system

and \( A, B, C, D \) are of corresponding order matrices, with constant elements in the real field, \( \mathbb{R} \).

The state of the system (4.1) at any initial time, \( t_0 \), together with the input, \( u(t) \), defined for all \( t \in [t_0, \infty] \), uniquely determines the complete behaviour of the open-loop system for all \( t \geq t_0 \).

It is of interest to obtain the optimal overall performance of the helicopter-load closed-loop system. In modern control theory it is customary to adopt as a measure of performance an integral of the form

\[
J = \int_{t_0}^{T} L(x, u, t) \, dt \tag{4.2}
\]

whenever it is not important that the terminal error, that error persisting at the end of the time interval \( [t_0, T] \), should be carefully controlled. The proposed solution involves the synthesis and design
FIGURE 4.1: Block Diagram of a Linear, Time-Invariant Dynamical System
of an optimal, linear feedback control, $u^o$, for the dynamical system of Figure 4.1 for augmenting the stability and performance of the helicopter-load system. Figure 4.2 shows the structure of this solution.

At hover, the pilot control command, $u_c$, is equal to the control trim position, that is,

$$u_c = 0$$

(4.3)

One of the principal features of any optimal control problem is the choice of an appropriate performance index. Flight control systems of interest are designed so that the state vector, $\mathbf{x}(t)$, in response to any disturbance, will tend to zero as time tends to infinity. If the interval is semi-infinite then it is known (Bryson & Ho [1975]) that the resulting feedback control will not be time-varying, i.e. the feedback gains will be constant. Then the performance index can be given by (4.4), viz.

$$J = \int_0^\infty L(x, u, t)dt$$

(4.4)

If the integral of the performance index is chosen to be quadratic, viz.

$$J = \frac{1}{2} \int_0^\infty (x^TQx + u^TGu)dt$$

(4.5)

a linear solution may then be possible, where the $\frac{1}{2}$ is added for analytical convenience. In this case the problem is referred to as the Linear Quadratic Problem (LQP).
FIGURE 4.2: Structure of the Proposed Solution
4.3 THE FULL-STATE FEEDBACK CONTROLLER

The full-state feedback controller required for the optimal solution must use all the available information about the state of the helicopter-load system to provide the required control action. Since usually only a few of the state variables of the system can physically be measured to any degree of accuracy, a method for reconstructing the state variables from the available outputs of the system is also required. Such a reconstruction is referred to as an estimation of the state vector.

Two main types of state reconstruction are of interest: one is based on a deterministic design resulting in an observer (Luenberger, [1971]) and the second type is based on stochastic processes and it results in a Kalman filter (Kalman, [1960]). In this work, since the simulated response of the helicopter-load system is to be examined in the presence of simulated atmospheric turbulence and measurement noise, a Kalman filter is an acceptable result since it provides the best reconstruction of the state vector of the system in that the squares of any errors which may finally exist between the true state and its reconstruction are at their minimum values.

A block diagram representation of an AHCS is shown in Figure 4.3, where the helicopter system is shown as being disturbed by atmospheric turbulence, $n_w$, and its output, $y$, is contaminated by measurement noise, $v$. The design of this AHCS may be separated into two phases according to the separation principle. The most important feature of the separation principle is that the gain matrix of the feedback control law
FIGURE 4.3: Block Diagram Representation of a Helicopter/Load System with an AHCS
is independent of all the statistical parameters in the problem, whereas the optimal filter is independent of the matrices in the performance measure (Meditch, [1969]). The two phases of design are:

(i) the solution of the LQP (Figure 4.4) producing the design of an optimal full-state feedback controller;

and (ii) the solution of the LQG (Figure 4.5) which leads to the design of a Kalman filter.

The validity of separating the problem into these two separate phases of control and estimation is a result of the linearity of the system, the nature of the performance index, and the character of the input disturbance and measurement noise (Schultz and Melsa, [1967]).

4.3.1 The Control Problem

For this problem it will be assumed that the state vector of the system can be measured physically and is available for use in the feedback control. Therefore the appropriate gain matrix, $K^*$, for the feedback control law to be applied to the uncontrolled helicopter/load system can be obtained by minimising the quadratic performance index (4.5), viz.

$$J = \frac{1}{2} \int_0^\infty (x^T Q x + u^T G u) dt$$

where $Q$ and $G$ are weighting matrices for the state vector, $x$ and the control vector $u$ respectively, of the system. $Q$ and $G$ can take different forms but it is convenient to assume that $Q$ and $G$ are diagonal.

The problem is now to find an optimal control vector, $u$, which
FIGURE 4.4: A Simplified Block Diagram of the System with an Optimal AHCS
FIGURE 4.5: A Simplified Block Diagram of a Helicopter-Load System with AHCS Incorporating an Optimal Controller and a Kalman Filter
subject to the constraint of (4.1) will minimise the performance index (4.5), given the initial state $x(t_0) = x_0$.

The solution of the performance integral (4.5), subject to the specified constraints, involves the use of undetermined Lagrange multipliers. The modified p.i. becomes:

$$
\overline{J} = \frac{1}{2} \int_0^T \left( \dot{x}^T Q \dot{x} + u^T Q u \right) + \dot{\psi}^T [(A \dot{x} + Bu) - x] dt
$$

where $\overline{J}$ is the adjoint performance index, and $\dot{\psi}$ is the co-state vector, whose elements are the undetermined Lagrange multipliers.

A scalar function, $\mathcal{H}$, associated with the performance index, $J$, called the Hamiltonian is defined as follows:

$$
\mathcal{H} = \frac{1}{2} \dot{x}^T Q \dot{x} + u^T Q u + \dot{\psi}^T (Ax + Bu)
$$

(4.7)

By integrating by parts the last term on the right hand side of (4.6) and by taking the first variation in $J$ due to variations in the control vector, $u(t)$, only, it can be shown (Bryson and Ho, [1975]) that the co-state vector has to be chosen such that:

$$
\dot{\psi} = - \frac{\partial \mathcal{H}}{\partial x}
$$

(4.8)

and the resulting first variation of $\overline{J}$ becomes,

$$
\delta \overline{J} = \dot{\psi}^T (0) \delta x(0) + \int_0^T \frac{\partial \mathcal{H}}{\partial u} \delta u dt
$$

(4.9)

where $\dot{\psi}^T (0)$ is the gradient of $\overline{J}$ with respect to variations in the initial conditions. For an extremum, $\delta \overline{J}$ must be zero for arbitrary

+ The performance index which is a function of the adjoint states...
\[ \delta y(t); \text{this can only happen if the impulse response function } \frac{\partial J}{\partial u} \text{ is equal to zero, i.e.} \]
\[ \frac{\partial J}{\partial u} = 0, \quad 0 \leq t < \infty \quad (4.10) \]

Equations (4.8) and (4.10) are the Euler-Lagrange equations and by differentiating (4.7) with respect to \( \dot{x} \) and \( u \), (4.11) and (4.12) can be obtained, viz.
\[ \ddot{\psi} = -\frac{\partial J}{\partial \dot{x}} \]
\[ = -Q\dot{x} - A^T \dot{\psi} \quad (4.11) \]
\[ \frac{\partial J}{\partial u} = Qu + B^T \dot{\psi} \quad (4.12) \]

Therefore, from (4.10) and (4.12),
\[ u^0 = -G^{-1}B^T \dot{\psi} \quad (4.13) \]

For the inverse of the matrix, \( G \), to exist it is necessary that \( G \) be positive definite (Guillemin, [1949]).

For (4.12), and hence for (4.13) to be true (i.e. for the system to be at least locally optimal), the associated Jacobian matrix of the second variation of \( \bar{J} \) must be at least positive definite, viz.
\[ \begin{bmatrix}
\frac{\partial^2 J}{\partial \dot{x}^2} & \frac{\partial^2 J}{\partial \dot{x} \partial u} \\
\frac{\partial^2 J}{\partial u \partial \dot{x}} & \frac{\partial^2 J}{\partial u^2}
\end{bmatrix} > 0 \quad (4.14) \]

Differentiating (4.11) and (4.12) the elements of the Jacobian (4.14) can be found, i.e.
\[ \frac{\partial^2 J}{\partial \dot{x}^2} = Q \quad (4.15a) \]
\[
\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} = \mathbf{G} \quad (4.15b)
\]

and
\[
\frac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} = \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u} \partial \mathbf{x}} = 0
\quad (4.15c)
\]

Therefore, (4.14) becomes (4.16), viz.
\[
\begin{bmatrix}
\mathbf{Q} & \mathbf{O} \\
\mathbf{O} & \mathbf{G}
\end{bmatrix} > 0
\quad (4.16)
\]

Since the matrix \( \mathbf{G} \) has been selected to be positive definite to ensure that the Jacobian matrix is positive definite it is only necessary that \( \mathbf{Q} \) be positive definite.\(^\dagger\)

From (4.1a), (4.12) and (4.13) the following set of vector differential equations is obtained:
\[
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{G}^{-1}\mathbf{B}^T \psi
\quad (4.17a)
\]
\[
\dot{\psi} = -\mathbf{Q}\mathbf{x} - \mathbf{A}^T \psi
\quad (4.17b)
\]

If a new composite vector, \( \mathbf{z} \), is defined as:
\[
\mathbf{z} \triangleq \begin{bmatrix} \mathbf{x} \\ \psi \end{bmatrix}
\quad (4.18)
\]

then the set of equations (4.17) may be represented by (4.19), i.e.
\[
\dot{\mathbf{z}}(t) = \mathbf{M}\mathbf{z}(t)
\quad (4.19)
\]

where the canonical matrix, \( \mathbf{M} \), is given by
\[
\mathbf{M} = \begin{bmatrix}
\mathbf{A} & -\mathbf{B}\mathbf{G}^{-1}\mathbf{B}^T \\
-\mathbf{Q} & -\mathbf{A}^T
\end{bmatrix}
\quad (4.20)
\]

The dimension of the vector \( \mathbf{z} \) is \( 2n \). For the solution of (4.19) \( 2n \) boundary conditions are needed of which \( n \) are the elements of the

\(^\dagger\) However since the higher derivatives of \( \mathcal{H} \) are zero, this restriction can be relaxed and \( \mathbf{Q} \) can be allowed to be at least non-negative definite (Athans and Falb [1966]).
vector, \( x(0) \); the remaining \( n \) conditions are obtained from the
transversality condition,

\[ \Psi(T) = \frac{3}{2} x^T(0) \left\{ \frac{1}{2} x^T(T) S^{-1} x(T) \right\} \]  
(4.21)

In this case terminal time, \( T = \infty \) and since a terminal condition has
not been specified in the performance index,

\[ \Psi(\infty) = 0 \]  
(4.22)

Since it is of interest to design a control system such that

\[ x(\infty) = 0 \]  
(4.23)

then, it can be assumed that \( \Psi \) is related to \( x \) by (4.24)

\[ \Psi(t) = Kx(t) \]  
(4.24)

where \( K \) is the solution of an Algebraic Riccati Equation (ARE) which
has the form

\[ 0 = -Q - A^T K - KA + KBG^{-1}B^T K \]  
(4.25)

The ARE is obtained by substituting (4.24) in (4.17) and combining
the resulting differential equations in \( x \).

The solution of the A.R.E. will be positive definite if certain
conditions, described later, are met. Several methods* exist for
solving the ARE. The method proposed by Marshal and Nicholson, [1970],
which uses eigenvector decomposition (Appendix F) and is based upon the
work of Potter [1966], is superior† with regard to computational time
and it was the preferred method therefore for digital computation.

*Some of these methods are discussed in Chapter 6.
†The limitations of this method, however, are described and discussed in Chapter 6.
Finally by substituting (4.24) in (4.13), the optimal control law (4.26) may be obtained, viz.

\[ \mathbf{u}^0(t) = K^0 \mathbf{x}(t) \]  
(4.26a)

where

\[ K^0 = -G^{-1}BTK \]  
(4.26b)

The theoretical development above was first expounded, but not in the same form, by Kalman [1960], who showed that the resulting optimal control law was both unique and stabilising. If the pair \( \{A,B\} \) is controllable, then the matrix \( K \) is the unique, positive definite, symmetric solution of the ARE (4.25). However, even if the pair \( \{A,B\} \) is controllable and \( Q \) is positive semi-definite, for \( K \) to be the unique solution, the pair \( \{A,C\} \) must be observable. If the pair \( \{A,B\} \) is uncontrollable a solution is not guaranteed, especially if the uncontrollable part of the system is unstable (Wolovich, [1974]). Therefore, if a unique optimal control law is to be obtained, it is essential to ensure that the system (4.1) is controllable. This is a condition of special concern in this work.

4.3.2 The Estimation Problem

The optimal feedback control law (4.26) may be used only if it is assumed that the full state vector \( \mathbf{x} \) of the helicopter-load system is available. In practice, only a few of the state variables can be either easily or accurately measured, which means that the state vector has to be reconstructed on the basis of the variables, \( \mathbf{z} \) (Figure 4.3), which can be measured. The possibility of reconstructing the state vector from output measurements is strongly related to a basic property of a system called observability.
Some procedures have been developed (Hofmann, [1972]) for determining the least order output vector, \( \mathbf{y} \), which can be used by any filter to reconstruct the state vector.

The Kalman filter is widely used in current aerospace systems for state and stability derivative estimation (Molusis, [1972], Schmidt, [1981]). Any state estimation requires sensor signals but practical sensors are subject to a variety of errors that degrade the accuracy of estimation of the state. However some analytical techniques have been developed to determine the effect of sensor errors on the accuracy of estimation (Gupta and Hall, [1978]). Therefore, the Kalman filter is usually employed to estimate, on the basis of measurements of the noise, the values of the state variables of a system subject to stochastic input disturbances often representing atmospheric turbulence. When the noise variables are added to the system (4.1), it becomes

\[
\begin{align*}
\dot{\mathbf{x}} &= \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} + \mathbf{G}_n \mathbf{n}_w \\
\mathbf{y} &= \mathbf{C} \mathbf{x} + \mathbf{D} \mathbf{u} \\
\mathbf{z} &= \mathbf{y} + \mathbf{v}
\end{align*}
\]

where \( \mathbf{n}_w \) and \( \mathbf{v} \) are vectors representing input disturbances representing atmospheric turbulence and measurement noise, respectively, and \( \mathbf{z} \) is the measurement vector (measured output vector).

An optimal feedback control law can be obtained by solving the Linear Quadratic Gaussian (LQG) problem, but the control law which results has been proven (Athans, [1971]) to be exactly identical to that obtained as the solution of the LQP, viz.
where $u = K^0 X$ \hfill (4.26a)

\[ K^0 = -G^{-1} B^T X \] \hfill (4.26b)

The block diagram of a helicopter-load system using an AHCS with a Kalman filter is shown in Figure 4.3, from where it can be seen that the filter has a structure closely resembling that of the original system.

The overall system equations are given below:

\[ \dot{x} = A x + B u + G p \] \hfill (4.28a)

\[ z = C x + D u + v \] \hfill (4.28b)

\[ \dot{\hat{x}} = A \hat{x} + B u + K_f (z - \hat{z}) \] \hfill (4.28c)

\[ \hat{y} = C \hat{x} + D u \] \hfill (4.28d)

\[ u = K^0 \hat{x} \] \hfill (4.28e)

where $K_f$ is a constant matrix the elements of which are the so-called Kalman gains, and $\hat{x}, \hat{y}$ are the estimated state and output vectors.

Both the input disturbance, $\eta$, and the measurement noise, $v$, are assumed to be stationary and independent Gaussian processes with zero mean value. Such processes are frequently described as uncorrelated white noise, where

\[ E\{\eta(t)\eta^T(t+\tau)\} = \tilde{Q}(t-\tau) \] \hfill (4.29a)

\[ E\{v(t)v^T(t+\tau)\} = \tilde{G}(t-\tau) \] \hfill (4.29b)

where $\tilde{Q}$ and $\tilde{G}$ are covariance matrices of order $[p \times p]$ and $[n \times n]$ respectively and $\delta(t-\tau)$ is the Dirac function.
Equations (4.29) a and b correspond to the actual system, (4.28) c and d correspond to the Kalman filter and (4.28) e to the optimal full-state controller. The difference between the actual output, $z$, of the system and the estimated one is given by

$$z = z - \hat{z} \quad (4.30)$$

and is called the residual and if $\bar{r}$ is large then a greater correction is needed to improve the estimated state vector, $\hat{z}$.

From equations (4.28) it can be easily shown that the reconstruction error, $e$, satisfies the differential equation

$$\dot{e} = [A - K_f C]e + G w - K_f v \quad (4.31a)$$

where

$$e \triangleq x - \hat{x} \quad (4.31b)$$

The stability of the estimator and the asymptotic behaviour of the reconstruction error are both determined primarily by the behaviour of the matrix $[A-K_f C]$. The Kalman filter gain matrix, $K_f$, is desired to ensure that any errors which arise in the reconstructed vector, $x$, are rapidly and smoothly minimised in the presence of disturbances. Thus, as time passes, the output of the estimator will approach the state vector that must be reconstructed iff the estimator is stable. The eigenvalues of $[A-K_f C]$ are usually referred to as the poles of the estimator. For stability the poles should lie far in the left half of the complex plane.

A simplifying characteristic of the Kalman filter is that the techniques used to solve the estimation problem by determining this filter are very closely related to the techniques used to solve the
control problem. In fact it is possible to cast the problem as a so-called 'dual control problem' (Schultz and Melsa, [1967]). The solution of this estimation problem (Kalman, [1960]; Kalman, and Bucy, [1961]) consists of the definition of a linear dynamic system, similar to the original system, whose input is $z$ and whose output is $\hat{x}$, as shown in equation (4.28c), viz.

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f[z-\hat{x}]$$

where

$$K_f = \tilde{\kappa} C^{-1}$$

(4.32)

$K_f$ being the Kalman filter gains, and $\tilde{\kappa}$ is the solution of the ARE (4.34), viz.

$$O = A\tilde{\kappa} + \tilde{\kappa} A^T - \tilde{\kappa} C^{-1} C\tilde{\kappa} + G G^T$$

(4.34)

The performance measure for estimation is the mean square reconstruction error, $e$, so that

$$E = \{(x-\hat{x})^T (x-\hat{x})\}$$

(4.35)

It is not important to know the absolute magnitude of each element in $\tilde{\kappa}$ and $\tilde{\kappa}$. Since $\tilde{\kappa}$ and $\tilde{\kappa}$ determine the performance index in the dual problem, only their relative magnitudes are important. Figure 4.5 shows a simplified block diagram of the helicopter-load system with an ARCS incorporating an optimal controller and a Kalman filter. It can be shown (Bryson and Ho, [1975]) that the average performance of the overall optimally controlled system can be obtained by solving an ARE similar to (4.34) with $P$ replacing $\tilde{\kappa}$, and the closed loop matrices $(A+B K^o)$ and $(C+D K^o)$ replacing $A$ and $C$ respectively, (Table 4.1) such that

$$O = (A+B K^o) P + P (A+B K^o)^T - P (C+D K^o) C^{-1} (C+D K^o) P + G G^T$$

(4.36)
<table>
<thead>
<tr>
<th>CONTROL PROBLEM</th>
<th>ESTIMATION PROBLEM</th>
<th>PERFORMANCE PREDICTION OF CLOSED LOOP SYSTEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{x} = Ax + Bu$</td>
<td>$\dot{x} = Ax + Bu + G \hat{x}_w$</td>
<td>$\hat{x} = Ax + BK^O \hat{x} + G \hat{x}_w$</td>
</tr>
<tr>
<td>$y = Cx + Du$</td>
<td>$y = Cx + Du + v$</td>
<td>$\hat{y} = C\hat{x} + Du$</td>
</tr>
<tr>
<td>$\hat{x} = A\hat{x} + Bu + K_f [y - \hat{y}]$</td>
<td>$\hat{x} = (A+BK^O)\hat{x} + K_f [y - \hat{y}]$</td>
<td>$\hat{y} = (C+DK^O)\hat{x}$</td>
</tr>
<tr>
<td>$J = \frac{1}{2} \int_0^\infty (x^T Q x + u^T Gu) , dt$</td>
<td>$J = E {(x - \hat{x})^T (x - \hat{x})}$</td>
<td>$J = E {(x - \hat{x})^T (x - \hat{x})}$</td>
</tr>
<tr>
<td>$O = -KA - A^T K + KBG^{-1} BT - KQ$</td>
<td>$O = -A^T K + KA + KC^T Q \sim - C\sim K - B\sim GB^T g$</td>
<td>$O = -(A+BK^O)^T P - P(A+BK^O)$</td>
</tr>
<tr>
<td>$K^O = -G^{-1} B^T K$</td>
<td>$K_f = KC^T Q$</td>
<td>$O = -(A+BK^O)^T P - P(A+BK^O)$</td>
</tr>
</tbody>
</table>

**TABLE 4.1:** Dual Control Problem Formulation and Prediction of Overall System Performance
The solution, $P$, of this transformed ARE is the covariance matrix of $x$ and allows the mean square histories of the state variables of the original system (4.27) and their cross-correlation to be predicted, by using (4.37), viz.

$$E[x(t)x(t)^T] = P$$  \hspace{1cm} (4.37)

The mean square histories of the control variables, $u$, and their cross-correlations may be obtained from equation (4.38), viz.

$$E[u(t)u^T(t)] = K^OPK^T$$  \hspace{1cm} (4.38)

The principal advantage of this filter is that it may be designed to be driven only by those signals which can be easily measured and will have as their output an estimate of the full state of the system. While the Kalman filter approach provides a complete, rigorous and optimal solution to the problem, it is often not practicable as a result of several factors:

1) the characteristics of input and output are often not known in sufficient detail to justify the approach,
and
2) the approach requires the simulation of an $n$th order system.

Because of severe restrictions on computer time and memory allocation, low order Kalman filters are sometimes used with little sacrifice of performance.

Due to these disadvantages a second approach was considered for comparison: the use of an output regulator.
4.4 FORMULATION OF THE OUTPUT REGULATOR

Since only a few of the states of the system can be physically measured and since helicopter control applications often require the minimisation of measurable motion variables (such as normal accelerations) other than those which are expressed as elements of the state vector, it is obvious that a state controller would have been inappropriate for this case. These motion variables, however, can be made to form part of the output vector, $\mathbf{y}$. It can be shown that when such motion variables are of direct concern the station-keeping control problem may still be expressed in a similar way to that outlined in Section 4.3, for a performance integral, such as (4.39), is minimised subject to the constraint of equations (4.1) that $y(t)$ be the unique solution of (4.1) with any arbitrary initial condition $x(t_0)$.

$$J_0 = \frac{1}{2} \int_0^\infty (\mathbf{y}^T \mathbf{Q} \mathbf{y} + \mathbf{u}^T \mathbf{G} \mathbf{u}) dt$$

(4.39)

The matrices $\mathbf{Q}$ and $\mathbf{G}$ in (4.39) must still retain appropriate definiteness properties, but where the output has been regulated, these properties are more critically related to the controllability properties of the helicopter-load system.

As for the full-state controller, the aim of the design of the output regulator is exactly the same, that is, the regulator has to make the entire state, $\mathbf{x}(t)$, of the compensated system (4.1) rapidly seek the origin of the state space in response to any non-zero initial conditions, with relatively little oscillatory behaviour and without involving an excessive amount of control action.
It has also to be shown that the pair \( \{A, B\} \) is completely controllable and the pair \( \{A, C\} \) is completely observable. The assumption of the weighting matrix, \( G \), being symmetric and positive definite, together with the assumption of system controllability are necessary to ensure the existence, as well as the uniqueness, of the optimal control, \( u^o(t) \). The observability assumption is necessary to ensure that all \( (n) \) states of the system contribute to the final value of the performance index \( J_0 \) or, in other words, that the states, \( x(t) \), can be reconstructed from a knowledge of the outputs, \( y(t) \).

The minimisation of the outputs, \( y(t) \), can be attempted implicitly by minimising a performance index involving the states and the control inputs in the manner described in Section 4.3.

Substituting (4.1b) in (4.35) yields

\[
J_0 = \frac{1}{2} \int_{0}^{\infty} \left( (Cx + Du)^T Q (Cx + Du) + u^T Qu \right) dt
\]  
(4.40)

The associated Hamiltonian, \( H \), is expressed as:

\[
H = \frac{1}{2} \left[ (Cx + Du)^T Q (Cx + Du) + u^T Gu \right] + \dot{y}^T (Ax + Bu)
\]  
(4.41)

For \( H \) to be minimised with respect to \( u \), then

\[
\frac{\partial H}{\partial u} = D^T QCx + (G + D^T QD)u + B^T \dot{y} = 0
\]  
(4.42)

or, by solving for \( u \)

\[
u = -(G + D^T QD)^{-1} [D^T QCx + B^T \dot{y}]
\]  
(4.43)

Also,

\[
\frac{\partial H}{\partial x} = C^T QCx + C^T QDu + A^T \dot{y}
\]  
(4.44)

\[\text{In general, the existence of the inverse cannot be guaranteed; but, for most aerospace engineering problems the inverse can be easily obtained.}\]
Substituting for $u$ in (4.1a) and (4.44) using (4.43) yields the canonical equation of the optimal system, viz.

$$\begin{bmatrix} \dot{x} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}(G + D^TQD)^{-1}D^TQC & \mathbf{-B}(G + D^TQD)^{-1}B^T \\ -C^T(QD(G + D^TQD)^{-1}D^TQC) & -[\mathbf{A} - \mathbf{B}(G + D^TQD)^{-1}D^TQC] \end{bmatrix} \begin{bmatrix} x \\ \psi \end{bmatrix}$$

(4.45)
or

$$\dot{x} = \mathbf{M}x$$

(4.46)

where

$$\begin{bmatrix} x \\ \psi \end{bmatrix} = \begin{bmatrix} x \\ \psi \end{bmatrix}$$

(4.47)

and $\mathbf{M}$ is the system matrix of (4.45) of order $2n \times 2n$. The optimal solution is obtained from the solution of (4.45) with the known boundary conditions $x(t_0) = 0$ and $\psi(\infty) = 0$. As for the canonical equation of section (4.3), an explicit solution of the canonical equation (4.45) may be obtained in the form of two single-point boundary value problems using eigenvector decomposition (Appendix F). $\psi$ is found to be related to $x$ by the equation

$$\psi = \hat{K}x$$

(4.48)

where $\hat{K}$ is the positive definite solution of the algebraic Riccati equation, viz.

$$\hat{K}^\mathbf{A} + \hat{\mathbf{A}}^\mathbf{A} - \hat{\mathbf{A}}\mathbf{B}(G + D^TQD)^{-1}D^TQC + \hat{\mathbf{Q}} = 0$$

(4.49)

and where

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{B}G^{-1}D^TQC$$

(4.50)

$$\hat{\mathbf{G}} = \mathbf{G} + D^TQD$$

(4.51)

and

$$\hat{\mathbf{Q}} = C^T[Q - QDG^{-1}D^TQC]C$$

(4.52)

Substituting (4.48) in (4.44) yields

$$u = -[(G + D^TQD)^{-1}[B^T\hat{K} + D^TQC]x$$

(4.53)
or
\[ u = \hat{K}^O x \]  \hspace{1cm} (4.54)

where
\[ \hat{K}^O = - (G + D^T Q D)^{-1}[B^T X + D^T Q C] \]  \hspace{1cm} (4.55)

Figure 4.6 gives a block diagram representation of the output regulator as applied to the helicopter-load system. As for the solution of the full-state controller, the optimal control law obtained by solving the LQP is exactly the same as the one obtained by solving the LQG problem.

It is generally assumed that only the output, \( y \), is available for feedback. Thus, the state vector, \( x \), has to be reconstructed before being used by the optimal controller (4.54) by simply rearranging (4.1b) as follows:
\[ x = C^{-1}[y - Du] \]  \hspace{1cm} (4.56)

If \( C \) is a singular matrix, its inverse, \( C^{-1} \), will not exist, in which case \( x \) cannot be obtained uniquely in terms of \( y \). However, the pseudo-inverse of \( C \) may be used. If the rank of the matrix, \( C \), is equal to \( p \), then its right inverse is used, viz.
\[ C^+ = [C^T (CC^T)^{-1}] \]  \hspace{1cm} (4.57)

where the symbol \( + \) denotes the pseudo-inverse.

Hence, (4.54) can be written as
\[ u = \hat{K}^O C^+ [y - Du] \]  \hspace{1cm} (4.58)
or
\[ u = ([I] + \hat{K}^O C^+)^{-1} \hat{K}^O C^+ y \]  \hspace{1cm} (4.59)

For (4.59) to exist, the term \((I + \hat{K}^O C^+)\) has to be non-singular. Otherwise, its pseudo-inverse has to be obtained resulting in a certain loss of accuracy of state reconstruction.

\[ + \text{This is known as the Moore Pseudoinverse. A more accurate pseudoinverse can be obtained by using singular value decomposition techniques.} \]
FIGURE 4.6: Block Diagram of an AHCS Employing an Output Regulator, $K^o$, for the Helicopter-Load System
If the rank of the matrix, $C$, is equal to $m$, then the left inverse of $C$ is used, viz.

$$C^+ = [(C^T C)^{-1} C^T]$$  \hspace{1cm} (4.60)

and a similar expression to (4.59) can be obtained. For simulation purposes, it will be assumed that $x$ can always be obtained from the output vector, $y$, in which case a simplified block diagram, similar to the one in Figure 4.4 can be used with $\hat{K}^0$ being replaced by $K^0$.

Finally, it can be easily shown that, $\hat{K}^0$ of (4.54) becomes equal to $K^0$ by setting the matrix, $C$, equal to an $n \times n$ identity matrix and $D$ equal to a null matrix, viz.

$$\hat{K}^0 = -G^{-1} B^T \hat{K}$$

$$= -G^{-1} B^T K = K$$  \hspace{1cm} (4.61)

The same computer program, OUTREG, was used for solving by eigen-analysis both the LQP and the output regulator problem, by using appropriate $C$ and $D$ matrices.
4.5 THE CHOICE OF WEIGHTING ELEMENTS

The control law and system response are greatly influenced by the weighting matrices $Q$ and $G$ which are chosen to be diagonal to preserve the linearity of the performance index (4.5 and 4.41). Selection of these weighting factors is a difficult task, since no direct relationship has ever been demonstrated between the weighting factors and the optimum system response.

However, several techniques have been proposed for the selection of the weighting factors. The disadvantage of all these techniques is that they only work well for some particular systems but not for others. One such technique was developed by Ellert (DeRusso, Roy and Close [1971]) where a knowledge of the damping factor and natural frequency of the system was required making this technique complicated to use with high order systems such as the one considered in this work. In addition, the method proposed by Harvey and Stein [1978] was not considered since it places restrictions on the dimension of the output vector compared with that of the control vector which could not be met in this work. Nevertheless, the method proposed by Bryson and Ho [1975] proved to be of some value in choosing initial values for the diagonal elements of the weighting matrices $Q$ and $G$. Their method suggests that each weighting element can be determined from the expressions:

\begin{align*}
q_i &= \frac{1}{\rho_i} \left[ \frac{1}{y_i^{\max}} \right] \tag{4.62} \\
q_j &= \frac{1}{\mu_j} \left[ \frac{1}{u_j^{\max}} \right] \tag{4.63}
\end{align*}
where $p$ is the dimension of the output vector,

$m$ is the dimension of the control vector,

$T$ is the interval of time which is allowed for the system to reach steady state. The terminal time of the performance indices (4.5) and (4.41), $T$, is chosen to be infinity, but, in this case it may be assumed that steady state can be reached in any realistically long time interval (e.g. $T=10$ secs.),

$y_i \max$ is the maximum possible value which the $i^{th}$ output is allowed to attain,

and $u_j \max$ is the maximum possible value which the $j^{th}$ control is allowed to attain.

This method allowed initial values of $q_i$ and $g_j$ to be determined comparatively easily and then a trial and error technique was employed until a required helicopter-load system response, or set of closed-loop eigenvalues, was obtained. Therefore, the required mission-related flying qualities of the helicopter with the suspended load can be specified as acceptable regions of pole locations. A similar, but more systematic, approach, by Govindaraj and Rynaski [1980], employs two sequential design procedures, one computing the Riccati solution from a set of linear equations and the other computing the closed-loop eigenvectors, to determine at each step, both the pole and zero movements of the closed loop transfer function as the weighting matrices are varied. The disadvantage of this method is that it can be used effectively with low order models only and therefore it was not considered in this work.
4.6 EXAMINATION OF BASIC PROPERTIES OF THE SYSTEM

The concepts of complete state or output controllability, and observability were found to be important considerations whenever an optimal control law was to be evaluated and tested. (Sections 4.3 and 4.4).

It was shown by Larson and Dressler [1968] that complete controllability is only a sufficient, but not a necessary, condition for closed-loop system stability. If the original state description of the helicopter-load system was itself stable, then this was a necessary and sufficient condition for guaranteeing that the feedback law would stabilise the closed-loop system.

The concept of complete controllability is due mainly to Kalman [1960]. A system (4.1a) is said to be completely state controllable (or simply controllable) iff there exists an unconstrained control \( u(t) \) which transfers any initial state \( x(t_0) \) to another state \( x(t_1) \) in a finite time, \( t_1 - t_0 > 0 \). (4.1a) is completely controllable if the controllability matrix \( X \), given as:

\[
X = (B, AB, A^2 B, \ldots, A^{n-1} B)
\]  

spans the n-dimensional space, i.e.

\[
\text{rank}[X] = n \tag{4.65}
\]

Otherwise, the system (4.1a) is said to be uncontrollable (although it may be partially controllable). In a number of cases in this study it was found that some mathematical models of the helicopter with the suspended load were partially controllable. In this case it was required that additional tests be made to determine whether the system was at least stabilisable. The dynamic stability of the uncontrolled
The helicopter/load system is most easily checked by observing the signs of
eigenvalues of the coefficient matrix of the state equation (4.1a) viz.
\[ \dot{x} = Ax + Bu \]  (4.1a)

When defining controllability more explicitly one can speak about the
system (4.1) being not only state-controllable but, at the same time,
output-controllable. The system (4.1) is output controllable iff the
\[ p \times (n+m) \] composite matrix, \( Y \), viz.
\[ Y = [CB, CAB, ..., CA^{n-1} B, D] \]  (4.66)

has rank \( p \). The property of output controllability is necessary to
ensure the existence of an output regulator.

Another property that was necessary for the output regulator is
that of complete observability. The system (4.1) is said to be
completely observable iff the entire state \( x(t) \) of the system can be
determined over any finite interval \( [t_0, t_1] \) from complete knowledge of
the system input and output over the time interval \( [t_0, t_1] \) with \( t_1 > t_0 \).
This notion of observability thus implies the ability to reconstruct the
entire \( n \)-dimensional state of a system from knowledge of the output \( y(t) \) and the input, \( u(t) \), alone. The system (4.1) is completely observable
iff the composite \( n \times (n+m) \) matrix, \( Z \), viz.
\[ Z = [C^T, A^T C^T, ..., (A^T)^{n-1} C^T D^T] \]  (4.67)
is of rank \( n \).

The relationship between observability and controllability is that
the system (4.1) described by the quadruple \{A, B, C, D\} is completely
observable iff its dual system \( \{A^T, C^T, B^T, D^T\} \) is completely controllable.
and vice versa (Wolovich, [1974]).

This relationship is employed by the computer program CONOBS which was used for controllability and observability checks on the helicopter-load mathematical model.
4.7 **CONCLUDING REMARKS**

Two solutions to the helicopter-load control problem were considered for comparison:

(i) the use of an optimal full (i.e. complete) state controller with Kalman filter providing optimal state reconstruction (Section 4.3),

and (ii) the use of an output regulator (Section 4.4).

Once the basic properties of the system (i.e. controllability, observability) have been investigated by using the computer program CONOBS and appropriate weighting matrices have been chosen, the problem is relatively easy to solve. In all cases, the problem reduces to solving an ARE which can be best obtained by eigenanalysis. For this solution the computer program OUTREG can be used.
CHAPTER 5

COMPLETE-STATE LINEAR FEEDBACK CONTROLLERS

FOR SUBJECT HELICOPTER/LOAD SYSTEM
5.1 INTRODUCTION

The results of a stability investigation of the Sea King Helicopter carrying a standard size container and employing a complete (full) state feedback controller are presented in this chapter. The simulated response of the helicopter/load system was examined for test cases in which either the helicopter/load was assumed to divert from trimmed flight conditions by some small initial values or it was disturbed by atmospheric turbulence. The initial conditions, (Table 5.1), were chosen so that both the longitudinal and lateral rigid-body modes of the helicopter and load would be severely excited.

Similarly atmospheric turbulence was chosen to be of two dimensions (as described in Chapter 3).

The cases which were examined are listed in Table G.2. Three new mathematical models were developed, viz:

- ATHOS
- GIONA
- OLYMPUS

The mathematical model ATHOS is of 9th order and it represents the dynamics of a Sea King helicopter (Chapter 2). There is an alternative model based on ATHOS which has as additional elements of its state vector the displacements, \(x, y, z\). These represent respectively horizontal, lateral and vertical displacements along the body-fixed axes of the helicopter. The mathematical model GIONA is of order 24 and corresponds to the dynamics of the Sea King helicopter with a suspended load. Both ATHOS and GIONA were simulated digitally using the language SLAM, which
<table>
<thead>
<tr>
<th></th>
<th>CASE A</th>
<th>CASE B</th>
<th>CASE C</th>
<th>CASE D</th>
<th>CASE E</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Uncontrolled</strong></td>
<td>Yes</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Controlled</strong></td>
<td></td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td><strong>Deterministic</strong></td>
<td>Yes</td>
<td></td>
<td>Yes</td>
<td></td>
<td>Yes</td>
</tr>
<tr>
<td><strong>Stochastic</strong></td>
<td></td>
<td></td>
<td>Yes</td>
<td></td>
<td>Yes</td>
</tr>
<tr>
<td><strong>Initial Condition</strong></td>
<td>$x_{k_0} = 0.9m$</td>
<td>$x_{k_0} = 0.9m$</td>
<td>All zero</td>
<td>All zero</td>
<td>$x_{k_0} = -0.6m$</td>
</tr>
<tr>
<td></td>
<td>$\theta_{v_0} = 0.087rad$</td>
<td>$\theta_{v_0} = 0.087rad$</td>
<td></td>
<td></td>
<td>$\theta_{v_0} = 0.087rad$</td>
</tr>
<tr>
<td><strong>Standard Deviation of Atmospheric Disturbances</strong></td>
<td>$u_0, u_\perp = 1.5m/sec$</td>
<td>$v_0, v_\perp = 0.75m/sec$</td>
<td>$u_0 = 1.5m/sec$</td>
<td>$v_0 = 0.75m/sec$</td>
<td>$u_\perp = 1.5m/sec$</td>
</tr>
<tr>
<td></td>
<td>$v_0, v_\perp = 0.75m/sec$</td>
<td>$w_0, w_\perp = 0.75m/sec$</td>
<td>$u_\perp = 1.5m/sec$</td>
<td>$v_\perp = 0.75m/sec$</td>
<td>$u_\perp = 1.5m/sec$</td>
</tr>
<tr>
<td></td>
<td>$w_0, w_\perp = 0.75m/sec$</td>
<td>$P_0, P_\perp = 0.03rad/sec$</td>
<td></td>
<td></td>
<td>$v_\perp = 0.75m/sec$</td>
</tr>
</tbody>
</table>

**TABLE 5.1: Test Cases**
stands for Simulation Language for Analogue Modelling. SLAM is a CSSL (Continuous System Simulation Language) which translates the source program into FORTRAN; it was proposed by International Computers Limited (ICL) for use on its 1900 series computers. The language can accept expression-oriented and block-oriented statements.

Two disadvantages can be encountered if using SLAM:†

- the vector integration routines provided are neither effective nor efficient;
- it has a limited number of variables available to the user.

Because the vertical displacement of the vehicle and load did not alter appreciably, the mathematical model OLYMPUS was derived. OLYMPUS was similar to GIONA but with its order reduced to 22, thus reducing somewhat the execution time required for computation.

Another technique for saving computing time was used with this model by recasting it as a discrete model (Appendix H) and then solving the discrete state equation (H.11) using FORTRAN.

The procedure which was followed to obtain a closed-loop response, when an optimal linear feedback control was applied to the model, is shown in Figure 5.2. The FORTRAN computer program MXDATA was developed to form those matrices of the system described by equations (2.16) and (2.21) by using as inputs the aerodynamic derivatives of the suspended load, the stability derivatives of the helicopter and the coordinates of the attachment.

†It is also a language with an interactive facility, although such a facility has not been made available to users at L.U.T. because of the high demand for cpu time that such use incurs.
points of the suspension. The FORTRAN program PLANT1 transforms the state equation (2.16) into the discrete version (H.11) and plots the response of the helicopter/load system. The ALGOL programs CONOBS and CORANK were used to investigate the controllability and observability properties of the systems. CORANK computes the rank of both the controllability matrix, $Y$, and the observability matrix, $Z$, (see Chapter 4 for details). WAYMX is an ALGOL program which provides an initial estimate of the appropriate sizes of the elements of the weighting matrices, $Q$ and $G$, of the performance index, $J$, (Chapter 4). The optimal control gains can be obtained by using OUTREG$^*$ which solves the ARE by eigenvalue decomposition (Appendix F).

KALFIL is an ALGOL program which produces the Kalman filter gains based on the closed-loop helicopter load system. PLANT2 can then be used to obtain the response of the closed-loop system when a Kalman filter is added to it.

The design philosophy which was followed is summarised in the flow chart of Figure 5.1.

$^*$Other available methods are also discussed in Chapter 7.
REAL SYSTEM

CONTROL GOAL FORMULATION

OPEN-LOOP SYSTEM DESCRIPTION

DYNAMICS MEASUREMENTS DISTURBANCES

SOPHISTICATED HIGH ORDER MODEL

SIMPLIFIED LOW ORDER MODEL

EXAMINATION OF BASIC PROPERTIES

CONTROLLABILITY OBSERVABILITY

CONTROLLER OBSERVER

CLOSED-LOOP DYNAMICS (LOW-ORDER)

CLOSED-LOOP DYNAMICS (HIGH ORDER)

SYSTEM AND PERFORMANCE ANALYSIS

FIGURE 5.1: Control System Design Philosophy
FIGURE 5.2: General Procedure Followed to Obtain the Response of OLYMPUS
5.2 DYNAMIC STABILITY OF THE SEA KING HELICOPTER

Before examining the stability of a helicopter/load system it is worth examining the stability of the helicopter by itself so that the effect of the load on the response of the helicopter can subsequently be investigated more easily by comparing the response of the helicopter with that of the helicopter/slung load system.

The motion of the Sea King helicopter is analysed in this section. Considering the simpler case of solely the equations of rigid-body motion of the helicopter (Table 2.1), the longitudinal modes of motion are generally found to be:

(a) oscillatory forward velocity, \( u_v \), associated with a pair of complex, conjugate roots. The oscillation is generally of low frequency and mildly unstable.

(b) a subsidence in the vertical velocity, \( w_v \), associated with a real negative root given approximately by \( \lambda = \frac{-z}{w_v} \).

(c) a subsidence in the pitch angle, \( \theta_v \), associated with another real negative root.

The principal modes of lateral motion are generally found to be:

(a) a Dutch Roll oscillation involving changes in roll, \( \phi_v \), yaw, \( \psi_v \), and lateral velocity, \( v_v \). It is usually a short-period oscillation which at hover is usually unstable. However, it can become damped or even more unstable as the forward speed is increased, depending on the helicopter configuration.

(b) a yaw subsidence mode associated with a real negative root which at hover is equal to \( N_{\psi} \). This mode is damping of the
yaw angle (or, alternatively, a subsidence mode in yaw rate).

(c) a roll subsidence mode which involves damping of the roll rate.

The modes described above are the modes of motion of a single rotor helicopter. The modes (and the corresponding mode shapes) of the Sea King helicopter do not differ much from the general case.

By examining Table 5.2 for the longitudinal modes of the helicopter, it can be seen that there is a low natural frequency (0.327 Hz), unstable (time to double amplitude 0.24 secs.) oscillation, labelled Mode 1. The motion is predominant in the forward velocity perturbations, \( u_v \), and to a slightly smaller extent, in the pitch angle perturbation, \( \theta_v \). One of the remaining longitudinal modes is associated with a real and negative root which is approximately equal to the derivative, \( \frac{Z_v}{W_v} \) (Appendix A). This subsidence mode is labelled Mode 2 and it involves the damping of the vertical velocity, \( w_v \).

The other (Mode 3) is a damped oscillation (natural frequency 1.469 Hz and time to half amplitude 0.53 secs.). At hover, this root is associated with the pitch oscillation, the dominant degrees of freedom being the pitch angle, \( \theta_v \), and the forward velocity perturbation, \( u_v \).

It can be seen then that in hovering flight the vertical motion is uncoupled from the pitching and fore-and-aft motion. The two pairs of complex roots associated with longitudinal motion involve pitch attitude changes, \( \theta_v \), and forward speed changes, \( u_v \), only. A physical interpretation of the motion corresponding to these complex pairs is offered as follows: when a Sea King helicopter is hovering and experiences a small change in its horizontal velocity the resulting change in forward velocity, \( u_v \),
<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Comment/Description</th>
<th>State Variables Involved</th>
<th>Frequency $\omega_n$</th>
<th>Time to $\frac{1}{2}(2)$ Amplitude</th>
<th>Damping Factor $\xi$</th>
<th>Period of Oscillation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>OPEN LOOP MODES</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.079±0.3176</td>
<td>forward velocity oscillation $u_r(\theta)$</td>
<td></td>
<td>0.327</td>
<td>8.77 (Double)</td>
<td>-0.24</td>
<td>19.78</td>
</tr>
<tr>
<td>2</td>
<td>-0.324</td>
<td>vertical damping, $\lambda Z_w$ $w$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-1.314±0.6558</td>
<td>pitch oscillation $\theta_r(u)$</td>
<td></td>
<td>1.469</td>
<td>0.53 (half)</td>
<td>0.89</td>
<td>9.58</td>
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<tr>
<td>4</td>
<td>-8.468x10^-12</td>
<td>heading angle $\psi$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-0.03907±0.4168</td>
<td>lateral oscillation $\phi_r(v,\psi)$</td>
<td></td>
<td>0.419</td>
<td>17.74 (half)</td>
<td>0.09</td>
<td>15.07</td>
</tr>
<tr>
<td>6</td>
<td>-0.305</td>
<td>yaw rate convergence, $\lambda \omega N_z \tau_v$ $\psi$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CLOSED LOOP MODES</td>
<td></td>
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<td></td>
</tr>
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<td>0.87</td>
<td>2.51</td>
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</tr>
<tr>
<td>3</td>
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<td></td>
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<td>0.89</td>
<td>2.39</td>
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<tr>
<td>5</td>
<td>-4.0117±2.1369</td>
<td></td>
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<td>0.17</td>
<td>0.88</td>
<td>2.94</td>
</tr>
<tr>
<td>6</td>
<td>-0.0359</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 5.2:** Characteristic Modes of the Sea King Helicopter
FIGURE 5.3: Disturbed Longitudinal Motion of a Single Rotor Helicopter
causes the rotor to tilt backwards thus exerting a nose-up pitching moment on the helicopter. A nose-up attitude then begins to develop and the backward component of the rotor thrust decelerates the helicopter until the change in its forward motion is annulled. At this point (Figure 5.3.b) the rotor disc tilts and the rotor moment vanishes but the nose-up attitude remains so that backward motion begins to cause the rotor to tilt forwards thus exerting a nose-down pitching moment (Figure 5.3.c). As a result, a nose down attitude is attained (Figure 5.3.d) which in turn imparts to the helicopter forward acceleration and thereby restores it to the initial situation (Figure 5.3.a). The cycle then begins again. Since, the real part of one of the complex roots is positive (Table 5.2) the motion is unstable; consequently the amplitude increases, as can be seen in Figures 5.4.a and 5.4.b.

One of the lateral modes (Mode 4) is characterised by a zero root corresponding to the helicopter's heading (yaw) degree of freedom, which confirms the fact that the motion of the basic helicopter in hovering flight is not affected by any displacements in the yaw angle. Two of the roots form a complex conjugate pair giving an oscillation (Mode 5). The real part of the roots has a small negative value at hover giving slight stability. The response in this mode at hover is in the roll angle $\phi_v$ and to a smaller extent, in the sideslip velocity $v_v$ and the yaw angle $\psi$.

The final mode (Mode 6) is a slow subsidence in the yaw rate (yaw damping root). At hover the root is given approximately by $\lambda = N_\psi$. The coupling between the yawing and rolling motions of the Sea King helicopter is due to the position of the tail rotor. This coupling is represented
FIGURE 5.4(a): Simulated Stick-Fixed Response of the Sea King Helicopter Hovering in a Turbulent Environment
FIGURE 5.4(b): Simulated Stick-Fixed Response of the Sea King Helicopter Hovering in Calm Air When Its Pitch Angle was Initially Disturbed by 0.087 rad.
by the derivatives $L_v$ and $N_v$. However, it is usually assumed (Table A.1) that $L_v$ is negligible since the tail rotor shaft is close to the roll-axis of the helicopter.

Although, all the lateral modes of motion of the helicopter are stable (since they are associated with roots having negative real parts), the overall stick-fixed response of the helicopter is unstable (as can be seen from Figure 5.4). This is due to the instability of one of the longitudinal modes (Table 5.2) and due to the strong coupling between the longitudinal and lateral modes of the hovering helicopter (Chapter 2). All helicopters display similar unstable response at hover, but the Sea King is considered to have good handling characteristics when compared with other helicopters of its class.

When a linear state feedback controller was included in the helicopter model, all the eigenvalues of the closed-loop system (shown in Table 5.2) had negative real parts which is an indication that the response of the closed-loop system is stable. From Table 5.2 it can also be seen that the damping of all the oscillatory modes and their frequency had increased considerably.

It is worth noting that the near-zero eigenvalue was moved well into the left half plane, thus removing the neutral stability of Mode 4.

The time response of the closed-loop system is shown in Figure 5.5, where the longitudinal cyclic control, $\delta_L$, is mainly used to stabilise effectively the helicopter in about 4 seconds.
FIGURE 5.5: Closed Loop Response of the Sea King Helicopter When Its Pitch Angle Was Initially Disturbed by 0.087 rad.
5.3 DYNAMIC STABILITY OF THE LOAD

The stability analysis of the load alone can be carried out by assuming that the helicopter motion is unperturbed by any disturbances and unaffected by the presence of the load. The equations for the complete helicopter-load system (Appendix C) were not separable into longitudinal and lateral sets because of terms strongly coupling the helicopter's longitudinal/lateral motion. By ignoring all the helicopter terms, however, the resulting equations for the dynamics of the load can be separated into two mutually independent sets.

Although Prabhakar [1976], reported that the aerodynamic derivatives associated with the oscillatory modes of the rectangular container were strongly influenced by the incidence of the load, it was decided to assume in this study that no aerodynamic derivative changes appreciably with the incidence of the load since only hovering motion was of interest.

Longitudinal Motion of the Load

The longitudinal mode is the fore-and-aft motion of the suspended load. The largest contribution to the damping terms associated with the characteristic equations is the aerodynamic drag of the load, but at hover this force is very small.

Lateral Motion of the Load

The solution of the lateral equations of motion yield two pairs of complex conjugate roots which are associated with two oscillatory modes - one corresponding to the yaw motion and the other to the lateral "pendulum"
mode (see Figure 5.6). The most significant result is that the complex roots associated with the yaw motion have positive real parts which means that the amplitude of the oscillation increases with time. The load motion is most unstable whenever a suspension arrangement involving more than one cable, with cables parallel to each other, is used.

Thus, in this research, parallel cable arrangements were deliberately employed to demonstrate the stabilising effects of the optimal feedback controller. In the case of multi-cable suspension, the frequency of the yaw oscillation is strongly influenced by the distance between the attachment points of the cables with the helicopter, because the yaw stiffness increases as the spread is increased (Prabhakar [1976]). The lateral "pendulum" mode is damped very lightly since the real parts of the associated complex conjugate roots are negative but very close to zero.

In practice, some of the physical factors which could affect the motion of the load are hysteresis, energy losses in the cables, slacking of the cables, or even aerodynamic interference between the load and the cables.
FIGURE 5.6: Eigenvalues of the Open Loop System of the Load Suspended by a Single Cable of Length 7.6m When the Helicopter is Assumed to be Stationary
5.4 INVESTIGATION OF THE EIGENVALUES OF THE OPEN-LOOP HELICOPTER/LOAD SYSTEM

By means of a computer program, MXDATA, it was possible to obtain eigenvalues of the open loop system for a number of cases using different cable lengths and slung-load masses.

Three suspension arrangements were also investigated. The results of these investigations are presented in Figures 5.7 to 5.12 in which only some of the dominant eigenvalues are presented. Figures 5.7 to 5.9 show how the open-loop eigenvalues vary when either one to two or three cables are used with different cable lengths. From each figure it can be seen that for a particular slung load, the frequency of the system decreases as the cable length increases. From the same three figures it can also be inferred that the heavier the load, the higher the frequency of oscillations of some of the modes\(^\dagger\) of the helicopter load system. The same results can also be drawn by observing Figures 5.10 to 5.12 in which each plot corresponds to the change of the eigenvalues of the open-loop system while the mass of the load is kept constant.

From the plots it can be seen that several positive eigenvalues exist which means that in these cases the open-loop system must be unstable. Indeed, some of the modes of this load motion correspond to the modes of motion of the helicopter (as affected by the presence of the suspended load) of which roll divergence is the most important and characteristic mode for any rotary wing aircraft.

\(^\dagger\)Due to the complexity and coupling of the helicopter/load system the modes of motion cannot be easily identified.
In the case of three parallel cables being used for load suspension, there is only some small change in the dynamic performance of the system for different cable lengths. Consequently, when such a suspension arrangement is used, the only effective method of reducing the oscillations would be simply to reduce the weight of the load. The unstable modes of the helicopter/load system with a three cable arrangement are due chiefly to the coupling of the roll divergence modes of the helicopter and load. Thus from a comparison of the eigenvalues obtained for each suspension arrangement it can be inferred that the best stability is probably achieved with a two parallel cable arrangement.

These plots are useful for choosing the appropriate load mass and cable weight which results in the lowest frequency of the system, thereby causing less degradation of the overall dynamic response of the helicopter/load system. A good estimate of the required cable length and load mass can be obtained by superimposing the plots corresponding to the same cable arrangement. Of course, these plots can be used in the contrary manner, that is, they may be used to find the frequency of the helicopter/load system for some particular arrangement of cable length and load mass. Such predictions can be useful to helicopter pilot by providing him with some preflight notion of the likely dynamic response of the helicopter with its suspended load. Such information can be specially useful when flying has to take place in severe weather conditions by allowing the pilot to assess more fully the likely performance limits to be imposed on the mission.
FIGURE 5.7: Variation of Some Dominant Eigenvalues of the Open-Loop Helicopter/Load System When a Particular Slung Load is Suspended from the Helicopter by a Single Cable of Length Varying from 7.6m to 30.5m

* Although this man is well over the external load capability of the S-61 helicopter, it was considered only for the purpose of completing these charts.
FIGURE 5.8: Variation of Some Dominant Eigenvalues of the Open-Loop Helicopter/Load System When a Particular Load is Suspended from the Helicopter by Two Parallel Cables of Length Varying from 7.6m to 30.5m
FIGURE 5.9: Variation of Some Dominant Eigenvalues of the Open-Loop Helicopter/Load System When a Particular Load is Suspended From the Helicopter by Three Parallel Cables of Length Varying from 7.6m to 30.5m
FIGURE 5.10: Variation of Some Dominant Eigenvalues of the Open-Loop Helicopter/Load System When Loads of Mass Between 4 and 9100Kg are Suspended from the Helicopter by a Single Cable of a Particular Length
FIGURE 5.11: Variation of Some Dominant Eigenvalues of the Open-Loop Helicopter/Load System When Loads of Mass Between 4 and 9100Kg are suspended from the Helicopter by Two Cables of a Particular Length
FIGURE 5.12: Variation of Some Dominant Eigenvalues of the Open-Loop Helicopter/Load System When Loads of Mass Between 4 and 9100Kg are Suspended From the Helicopter by Three Cables of a Particular Length
5.5 CLOSED-LOOP RESPONSE

A number of cases were examined, but for the purposes of comparison within this dissertation only some of them are presented in Tables 5.3 and 5.4. The significance of the case titles is explained in Appendix G. The results obtained from the helicopter/load system when only four helicopter controls were used are presented in Table 5.3.

All the points enumerated in the following section refer to a standard response provided by a helicopter with a 2267 Kg load suspended from a single cable of length either 7.6m (SD401) or 15.2m (SD404).

(a) The overall response of the system is satisfactory but the control activity required to stabilise the system is too great. Indeed the required control displacements of both the longitudinal cyclic and tail rotor collective controls are greater than the value permitted in the specifications appropriate to this helicopter type.

(b) The response of the helicopter became worse when the longer cable was used, whereas the response of the load (apart from its lateral displacement, $y_L$) improved. The variable most severely affected was the yaw angle of the helicopter.

When the load was suspended by two parallel cables of length 7.6m (SD410), the overall system response was much improved, compared to that of the single cable suspension arrangement. The best improvement was in longitudinal motion. This result was expected because the two-cable suspension restrains changes in the pitch attitude of the load. However, since pitching motion of the load is strongly coupled to the pitching
<table>
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<tr>
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<th>SD410</th>
<th>SD419</th>
<th>MILSPEC-P8330</th>
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**TABLE 5.3:** RMS values of the variables of OLYMPUS obtained when the system was subjected to Test Case E (Table 5.1) for 8 secs. while using only 4 helicopter controls.
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<th>$X_v$</th>
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<td>$\delta_w$</td>
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<td>0.4731</td>
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**TABLE 5.4:** RMS values of the variables of OLYMPUS obtained when the system was subjected to Test Case E (Table 5.1) for 8 secs. while using all five controls.
motion of the helicopter, the values of the variables, \( \theta_L \) and \( q_L' \), were increased when the two-cable suspension was used. These increases were the only penalty paid for the improved performance of the system. Closer inspection of the results of SD41O and SD401 suggests that:

(a) the control activity\(^\dagger\) of both the longitudinal cyclic, \( \delta_L \), and the collective pitch, \( \delta_C \), of the main rotor has been reduced by nearly 90%. In this situation, to avoid the mechanical limits \( \delta_L \) would require 11.5% of control authority\(^*\) although \( \delta_C \) is within the 10% control authority permitted in the flying qualities and handling specifications (Appendix E). There is a lot of movement of the lateral controls, \( \delta_R \) and \( \delta_T \).

(b) the yaw attitude \((\psi_v, r_v)\) of the helicopter has not been much affected by the two cable arrangement but variables associated with rolling motion \( \phi_v \) and \( p_v \) of the helicopter and particularly the variables \( \psi_L \) and \( r_L \) of the load associated with the yawing motion have been considerably affected. Obviously such effects are the costs which must be paid to improve the system's longitudinal performance. When the three-cable arrangement was used for the same load, suspended by cables of length 7.6m (SD419) an improvement over both the single (SD401) and two cable (SD41O) arrangements was achieved in both longitudinal and lateral modes. The corresponding control activity was within the helicopter specification and the following control authorities were obtained:

---

\(^\dagger\)In this work, control activity is defined as the r.m.s. value of the control deflection.

\(^*\)Control authority is defined as the allowable percentage of the maximum control deflection.
\[ \delta_L = 14\% \]
\[ \delta_R = 9\% \]
\[ \delta_T = 30\% \]
\[ \delta_C = 1\% \]

High control activity of the tail rotor, \( \delta_T \), was characteristic of all three suspension arrangements. This level of activity implied that a heavier weighting of changes in the yaw attitude of the helicopter was required in the performance index to obtain a feedback control which resulted in less oscillatory yawing motion thereby improving the overall lateral response of the system.

In general it can be concluded that:

. for a given load, reducing the cable length will improve the dynamic performance of the helicopter/load combination because, with a shorter cable, the dynamic response of the load to any motion of the helicopter is much more rapid and requires less control activity.

. the two-cable arrangement considerably improves the dynamic response of the longitudinal motion of the helicopter and its load. In consequence however, the response of the attitudes and rates of the lateral rotational motion, particularly the yaw attitude of the load, is made worse.

. the use of three-parallel cables improves the response of the system for both longitudinal and lateral motion.

The yaw oscillations of the load and, hence, the lateral oscillations
of the helicopter, itself may also be reduced by the use of a winch controlling lateral displacement. Such a control causes one of the attachment points to move parallel to the lateral axis of the helicopter.

The 2267 Kg load was assumed to be suspended by either one (SD501) or two (SD510) or three cables (SD519) at 7.6m below the helicopter. The results of this investigation are presented in Table 5.4, from which it can be seen that:

- the longitudinal response of the system has not been much improved when compared with the corresponding cases of Table 5.3.
- the lateral response of the helicopter has been improved by the addition of the fifth control.
- both the lateral displacement, $y_L'$, and velocity, $v_L'$, of the load were decreased in each cable arrangement.
- the roll attitude of the load was improved only when either one or two cables were used, whereas, its yaw angle, $\psi_L'$, was increased in every case which indicates that the winch control can improve the yaw attitude of the helicopter only (especially when two cables are used), since a sudden displacement of the winch can induce yawing oscillations of the load.

A clear advantage of using a winch control is the reduction of activity of the tail rotor. In the best case (i.e. SD519) $\delta_T$ used only 26% of its maximum permitted travel. From an examination of the cases considered in this section (see Tables 5.3 and 5.4) it can be stated that:

- load positioning can be achieved while keeping the helicopter motion within limits laid down in the Military specifications as used by the United States Department of Defense.
changes in the roll and yaw attitudes of both helicopter and load should be strongly penalised in the performance index which is minimised by the optimal feedback control.

a change from a single cable suspension to a two cable arrangement results in a reduction of the overall control activity by 27% (Table 5.5) when four controls are used.

a change from a single cable suspension arrangement (SD401) to three cables (SD419) results in a reduction of control activity of 88%. A similar reduction can be achieved when the suspension arrangement is changed from two (SD510) to three cables (SD519) with the winch control in operation.

changing from a single suspension cable (SD501) to a two cable (SD510) system results in an improvement of control activity by about 70% when a winch control, $\delta_w$, is used. A change to a three cable arrangement (SD519) results in an improvement of nearly 84%. A two-cable suspension arrangement with five controls in operation (SD510) is better by 41% than a three cable arrangement with only four controls (SD419) (see Table 5.5). Further, when a three cable arrangement is used, it does not much matter whether the winch control is used or not.

In summary, some of the methods of improving the response of the system for a given load are:

- Reduction of the cable length thereby increasing the frequency of the oscillation and hence making the load more responsive to helicopter movements.
- Strongly penalising any changes in attitude angles and rates (especially those of the load).
- The use of three cables and a winch control.
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**TABLE 5.5:** Overall Control Activity Improvement as a Rough Measure of System Performance

*mean percentage value.
These results are presented to simply demonstrate what effects

- the cable length
- the cable arrangement
- the use of a lateral control for the load

have upon the overall dynamic response of the system. From the results some indication of the appropriate values needed for elements in the state weighting matrix, $Q$, of equation (4.5) was obtained. However, it must be emphasised that some of the results obtained did not comply with the military specifications (Appendix D and Table 5.3).
5.6 INCLUSION OF ACTUATOR DYNAMICS IN THE MATHEMATICAL MODEL

The mathematical models of actuating elements, which were discussed in Sections 2.7 and 2.8, were appropriately added to the models ATHOS, GIONA and OLYMPUS to obtain a more comprehensive representation of both the helicopter and the helicopter/load system.

The simulated responses of the control actuators to step input functions are shown in Figure 5.13. The longitudinal and lateral cyclic controls, $\delta_L$ and $\delta_R$, have the same characteristic form but due to the higher step input introduced in the longitudinal control the mechanical limits came into operation, thereby restricting the peak overshoot. In this case the effect of the limiter on the piston travel has no observable effect on the frequency of the actuator but it has reduced the time needed to reach the steady state. The actuator of the main rotor collective control, $\delta_C$, was subjected to the same step input as that for the lateral cyclic control, $\delta_R$. Since the damping ratio of the actuator of the collective control, $\delta_C$, is greater than that of the actuator of $\delta_R$, although the natural frequency of the actuator of $\delta_C$ is less than that of the actuator of $\delta_R$, both time-to-half-amplitude and the peak overshoot have been reduced.

Figure 5.14a shows the closed-loop response of ATHOS when the limits of the controls of the Sea King helicopter were included in the mathematical model. Figure 5.14b shows the time response of ATHOS when the actuator dynamics, without any control deflection limits were included in the model. Finally, both the actuator dynamics and the deflection limits of the controls of the helicopter were included in ATHOS and the response of this system was obtained (Figure 5.14c). By examining the time responses

* In this particular case, the limits were set at 25% of the maximum travel of the controls.
FIGURE 5.13: The Response of the Actuating Elements to Step Inputs
FIGURE 5.14: The Closed-Loop Response of ATHOS (from an initial pitch angle 0.087 rad) When
(a) Control Deflection Limiters, (b) Control Actuator Dynamics Without Limiters and
(c) Control Actuator Dynamics With Limiters Were Included in the Model
of ATHOS of Figure 5.14 it can be seen that although the same linear optimal controller, which was designed for the 9th order ATHOS model (Section 5.2), was used, the response in each of the three cases was a stable one. Thus a linear controller was able to stabilise even the non-linear ATHOS (which includes control deflection limiters). In each case considered the response of the system (Figure 5.14) was similar, the only difference being in the response of the longitudinal cyclic control, \( \delta_L \). The lateral motion variables were not significantly affected since only a longitudinal motion variable was disturbed initially (i.e. \( \theta_{V_0} = 0.087 \) rad). In cases (a) and (c) of Figure 5.14, it should be noted that \( \delta_L \) has reached its upper limit of travel, yet the stability of the helicopter was unaffected and the steady state was reached in approximately 4 seconds.

When the actuators were included in the mathematical models of GIONA and OLYMPUS it was found that the costs of computing were greatly increased because of the larger number of integrations required. The computing time required per run was very large and, in terms of University allocations, soon became prohibitive. This problem was circumvented somewhat by evaluating the state transition equation (see Appendix H). The discrete version of the state equation (2.16) is equation (H.11). It has to be emphasised that by expressing this problem in this fashion non-linearities cannot be incorporated into the system (see Appendix H) and therefore the limiters of the control actuators were not considered. The response checks obtained by using OLYMPUS (with actuator dynamics included) are shown in Tables 5.6 and 5.7. The results of Table 5.6 should be compared with those of Table 5.4. All the results in Table 5.6 were obtained for a single cable (SD501), two cable (SD510) and three cable (SD519) suspension arrangement.
<table>
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<tr>
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<td>$x_v$</td>
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<td>$u_v$</td>
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<td>m/sec.</td>
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<td>$w_v$</td>
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<td>0.03504</td>
<td>0.01268</td>
<td>m/sec.</td>
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<tr>
<td>$\theta_v$</td>
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<td>0.04148</td>
<td>rad.</td>
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<tr>
<td>$q_v$</td>
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<td>0.1736</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w_\ell$</td>
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<td>$q_\ell$</td>
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<td>0.2563</td>
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<td>0.03049</td>
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<td>0.4956</td>
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<td>$\delta_T$</td>
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<td>0.009185</td>
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<td>m/sec^2</td>
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**TABLE 5.6:** RMS values of the variables of the augmented model OLYMPUS (which includes actuator dynamics) obtained when the system was subjected to Test Case E (Table 5.1) for 8 secs. while using all five controls.
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<td>1.663</td>
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<td>$u_L$</td>
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<td>$v_L$</td>
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<td>rad/sec.</td>
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<td>$\delta_L$</td>
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<td>$\delta_w$</td>
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<td>$a_x$</td>
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<td>$a_y$</td>
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<td>2.057</td>
<td>2.057</td>
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<td>1.637</td>
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<tr>
<td>$v_{q_L}$</td>
<td>1.637</td>
<td>1.637</td>
<td>1.637</td>
<td>m/sec</td>
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</table>

**TABLE 5.7:** RMS values of the variables of the augmented model OLYMPUS (which includes actuator dynamics) when the system was subjected to Test Case D (Table 5.1) for 8 secs. while using all five controls.
with a winch controlling lateral displacement. Comparison of the results in Tables 5.4 and 5.6 shows that including actuator dynamics badly affects the response of the helicopter/load system, due to the time lag imposed on the response of the controls of the helicopter by their actuator dynamics. In the presence of such time delays any control deflection required by the system cannot be obtained instantaneously thereby causing the helicopter control of the load oscillations to be even less effective. Based on the results of Section 5.5 (see Table 5.3), it is probable that the use of an even shorter cable than 7.6m would have resulted in a more rapid response of the load to any control action of the helicopter, thereby improving the performance of the system.

Table 5.7 shows the simulated response of the helicopter/load system (with control actuator dynamics included in the mathematical model when hovering in conditions of atmospheric turbulence (Case D of Table 5.1)). Here it was assumed that the helicopter is disturbed by a two-dimensional turbulence (see Chapter 3) resulting in a rms air velocity of 13 Km/h and the load is disturbed by a turbulence of rms velocity 9.5 Km/h. Only the pitch rates of both the vehicle and its load are higher than expected: the system stability, however, maintained within the limits laid down in MIL.SPEC.-F8330. From these results it was considered that the initial conditions chosen for the earlier investigations (see Tables 5.3, 5.4 and 5.6) were particularly severe which resulted in the poor performance of the system.

Figures 5.16 and 5.15 show typical responses of the helicopter/load system employing a two-cable suspension with a winch control (SDA 510)

* The significance of the difference between these two velocities is merely to stress the fact that the turbulence affecting the helicopter has different characteristics than that affecting its suspended load.
and a three-cable suspension without a winch control (SDA 419) respectively.

From these figures it can be seen that:

. both system responses are stable and similar to each other,
. the winch control causes a considerable reduction of the overall (especially of the tail rotor) control activity while improving the yaw rate of the helicopter.
FIGURE 5.15(a): Typical Response of OLYMPUS With Four Controls and a Three Cable Suspension Arrangement (SDA 419) When Subjected to Test Case E of Table 5.1.
FIGURE 5.15(b)
FIGURE 5.16(a): Typical Response of OLYMPUS with Five Controls and a Two Cable Suspension Arrangement (SDA 510) When Subjected to Test Case E of Table 5.1
FIGURE 5.16(b)
5.7 THE INCLUSION OF A KALMAN FILTER IN AHCS

The results presented in the previous sections of this chapter in relation to the optimal control of OLYMPUS were based on the assumption that the Kalman filter estimated the state variables with complete accuracy. Of course, this is never achieved practically because the Kalman gains are obtained on the basis of using measurements and the noise associated with the sensors is not known perfectly. Furthermore, a particular Kalman filter is based on complete knowledge of the characteristics of atmospheric turbulence which is simply not available.

Also, since the implementation of a Kalman filter involves simulation of the mathematical model of the helicopter/load system, the round-off error associated with the computation of the state estimators increases, especially when high order models are involved. The computation of the Kalman gains was carried out by means of a computer program, KALFIL, and the response of the system was obtained by using PLANT2 (see Table 5.2). Both KALFIL and PLANT2 required the use of the full (i.e. complete) state linear feedback controller. PLANT2 forms the state transition equation (see Appendix H), associated with the augmented system shown in Figure 4.3.

The equations which describe the dynamics of this system are listed below:

\[ \dot{x} = Ax + Bu + G n \]  
\[ \hat{x} = A\hat{x} + Bu + K_f [z - \hat{z}] \]

The equation numbers refer to the appropriate chapter where all the variables have been defined.
\[ y = Cx + Du \] (4.27b)
\[ z = y + v \] (4.27c)
\[ \hat{y} = C\hat{x} + Du \] (4.28d)
\[ u = K\hat{x} \] (4.28e)

By substituting (4.28e) into (4.27a), (4.27b) and (4.28d), the equations become,

\[ \dot{x} = Ax + BK\hat{x} + Gn_w \] (5.1)
\[ \dot{\hat{x}} = A\hat{x} + BK\hat{x} + K_f[z - \hat{y}] \] (5.2)
\[ y = Cx + DK\hat{x} \] (5.3)
\[ z = y + v \] (5.4)
\[ \hat{y} = (C + DK)\hat{x} \] (5.5)

It can also be shown that by appropriate substitution the equations above become,

\[ \dot{x} = Ax + BK\hat{x} + Gn_w \] (5.6)
\[ \dot{\hat{x}} = (A-K_fC+BK)\hat{x} + K_fCx + v \] (5.7)

By combining equations (5.6) and (5.7),

\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{bmatrix}
= 
\begin{bmatrix}
A & BK \\
K_fC & (A-K_fC+BK)
\end{bmatrix}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix}
+ 
\begin{bmatrix}
O \\
I
\end{bmatrix}
v
+ 
\begin{bmatrix}
G \\
0
\end{bmatrix}
\begin{bmatrix}
n_w
\end{bmatrix}
\] (5.8)

Equation (5.8) has the same form as equation (H.11) and therefore, it can be expressed in terms of the state transition matrix, as shown in Appendix H.

\( v \) represents the sensor noise associated with the measurable
The available helicopter sensors were assumed to consist of:

1. three accelerometers used to determine the vehicle accelerations $a_x^v$, $a_y^v$, $a_z^v$.
2. three angular rate gyros to measure the angular rates $p_v^v$, $q_v^v$, and $r_v^v$ of the helicopter about the body-fixed axes, $OX$, $OY$ and $OZ$ respectively.
3. a two-degree-of freedom vertical gyro which measures the roll, $\phi$, and pitch angle, $\theta$, where as the yaw angle, $\psi$, is obtained from a gyro-magnetic compass (Broussard et al [1979]).
4. the three velocities of the helicopter may be obtainable by measuring dynamic pressure with pilot static tubes aligned with the body-fixed axes of the helicopter. However, for velocities less than 25.74 m/sec velocity measurements obtained by using a pilot static tube are usually inaccurate as rotor downwash and low dynamic pressure disrupt the measurement at low speeds (Broussard et al [1979]).

It was assumed that such measurements were not available. Thus the output and measurement vectors, $y$ and $z$ respectively were assumed to be of dimension 9. The standard deviations of the measurement noise associated with helicopter sensors are given in Table 5.8.

Due to the high order of equation (5.8), PLANT2 was not easy to use because of both the excessive core space and the time for computation needed. The expected values of the state variables at time $T=0$ were more easily obtainable from KALFIL since they appeared as the diagonal elements of the solution of the ARE, $P$, associated with the computation.
<table>
<thead>
<tr>
<th>SENSOR</th>
<th>STANDARD DEVIATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Rate Gyro</td>
<td>0.0029 rad/sec.</td>
</tr>
<tr>
<td>Roll and Pitch Attitude Gyro</td>
<td>0.0029 rad.</td>
</tr>
<tr>
<td>Heading Gyro</td>
<td>0.0175 rad.</td>
</tr>
<tr>
<td>Accelerometer</td>
<td>0.61 m/sec²</td>
</tr>
<tr>
<td>Calibrated Airspeed</td>
<td>0.515 m/sec.</td>
</tr>
<tr>
<td>Sideslip</td>
<td>0.0087 rad.</td>
</tr>
<tr>
<td>Barometric Altimeter</td>
<td>7.62 m</td>
</tr>
</tbody>
</table>

**TABLE 5.8: Measurement Noise Characteristics of Helicopter Sensors (Broussard et al [1979])**
<table>
<thead>
<tr>
<th>OUTREG</th>
<th>KALFIL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(without a Kalman filter)</td>
<td>(with a Kalman filter)</td>
</tr>
<tr>
<td>-108.5131</td>
<td>-108.5615</td>
</tr>
<tr>
<td>-56.064±j102.53</td>
<td>-56.045±j102.5324</td>
</tr>
<tr>
<td>-35.637±j68.362</td>
<td>-34.980±j69.75</td>
</tr>
<tr>
<td>-23.694</td>
<td></td>
</tr>
<tr>
<td>-7.857±j2.2914</td>
<td>-10.998±j4.6358</td>
</tr>
<tr>
<td>-6.9165</td>
<td>-6.8439</td>
</tr>
<tr>
<td>-1.843±j4.483</td>
<td>-2.559±j2.847</td>
</tr>
<tr>
<td>-3.224±j0.0552</td>
<td>-31.130±j21.6016</td>
</tr>
<tr>
<td>-0.347±j2.8914</td>
<td>-0.410±j0.395</td>
</tr>
<tr>
<td>-1.729±j0.0315</td>
<td>-3.2165</td>
</tr>
<tr>
<td></td>
<td>-1.7334</td>
</tr>
<tr>
<td>-1.028±j1.1217</td>
<td>-1.034±j0.904</td>
</tr>
<tr>
<td>-0.5695±j1.2613</td>
<td>-0.078±j1.4066</td>
</tr>
<tr>
<td>-1.0156</td>
<td>-1.0018±j0.00076</td>
</tr>
<tr>
<td>-1.0001</td>
<td></td>
</tr>
<tr>
<td>-0.9999</td>
<td>-0.1781</td>
</tr>
<tr>
<td>-0.011±j0.011</td>
<td>-0.05975</td>
</tr>
<tr>
<td></td>
<td>-0.01189</td>
</tr>
<tr>
<td>-0.00134</td>
<td>-0.0005</td>
</tr>
</tbody>
</table>

**TABLE 5.9:** Typical Closed-Loop System Eigenvalues of OLYMPUS with a Single Cable Suspension
of the gains of the Kalman filter. (Table 4.1). Some information about the closed-loop stability of the system was also obtained by examining the eigenvalues (see Table 5.9). The results of using a simpler but equally efficient controller, the output regulator, are presented in the following Chapter 6.
CHAPTER 6

OUTPUT REGULATORS FOR SUBJECT HELICOPTER/LOAD SYSTEM
6.1 INTRODUCTION

This chapter deals with the use of an output regulator to stabilise the Sea King Helicopter with the standard-sized container, of 2267 Kg, suspended from it. Again the investigation reported here concentrated on the effects of using three different suspension arrangements, when either four or five controls were employed. The augmented model included the dynamics of the control actuators and of the model of atmospheric turbulence.

At hover, for this helicopter/load system there are only nine outputs: the attitude angles and their associated rates about the three axes of motion of the helicopter and also the three translational accelerations.

Helicopter velocities are not usually considered as outputs of this system, since the velocity cannot be measured accurately at hover because pilot-static measurements are very much affected by the downwash from the rotor. Notwithstanding this particular restriction, the closed-loop response of the system was investigated for the situations when the feedback signals of the output regulator such as forward and lateral velocity were seriously contaminated.
6.2 CLOSED-LOOP RESPONSE OF THE HELICOPTER/LOAD SYSTEM

In Chapter 4, the output regulator was designed on the basis of limited information being available about the state-variables of the helicopter/load system, because only a few of the state variables can be measured. It is supposed that there is a limited number of outputs from the system which can be measured and which are required by the output regulator to stabilise the helicopter/load system. Since this limit cannot be known in advance, it was decided, to start this investigation from a situation where the response of the system was stable, to find the minimum number of outputs required to stabilise the closed-loop system when an output regulator is to be employed. In Chapter 5, it was found that the best stability situation was achieved when the winch control was included in the model OLYMPUS. The measurable output variables were assumed to be the following:

- three translational velocities of the helicopter: $u_v$, $v_v$, $w_v$
- three attitude angles of the helicopter: $\phi_v$, $\theta_v$, $\psi_v$
- three angular rates of the helicopter: $p_v$, $q_v$, $r_v$
- three translational accelerations of the helicopter: $a_x$, $a_y$, $a_z$

The attitude of the helicopter is usually measured by using gyros. Strapped-down accelerometers detect translational accelerations with respect to the body-fixed axes of the helicopter. The translational velocities of the helicopter are usually measured by using pilot-static tubes aligned with the three body-fixed axes of the helicopter (although this instrument is known to be unreliable at low speeds).

The r.m.s. values of several system variables are shown in Table 6.1
<table>
<thead>
<tr>
<th></th>
<th>OD501</th>
<th>OD510</th>
<th>OD519</th>
<th>UNITS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_x$</td>
<td>3.455</td>
<td>0.19</td>
<td>0.1547</td>
<td>m</td>
</tr>
<tr>
<td>$u_x$</td>
<td>0.7992</td>
<td>0.04932</td>
<td>0.037</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w_x$</td>
<td>0.06223</td>
<td>0.02065</td>
<td>0.01944</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\theta_x$</td>
<td>0.1679</td>
<td>0.05359</td>
<td>0.05709</td>
<td>rad.</td>
</tr>
<tr>
<td>$q_x$</td>
<td>0.449</td>
<td>0.109</td>
<td>0.1195</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$x_y$</td>
<td>3.582</td>
<td>0.2978</td>
<td>0.2136</td>
<td>m</td>
</tr>
<tr>
<td>$u_y$</td>
<td>1.821</td>
<td>0.2001</td>
<td>0.198</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w_y$</td>
<td>0.06247</td>
<td>0.01419</td>
<td>0.02501</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\theta_y$</td>
<td>0.1451</td>
<td>0.1416</td>
<td>0.1519</td>
<td>rad.</td>
</tr>
<tr>
<td>$q_y$</td>
<td>0.1703</td>
<td>0.2076</td>
<td>0.2213</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$x_z$</td>
<td>1.246</td>
<td>0.08396</td>
<td>0.04611</td>
<td>m</td>
</tr>
<tr>
<td>$u_z$</td>
<td>0.3604</td>
<td>0.09782</td>
<td>0.07352</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w_z$</td>
<td>0.07071</td>
<td>0.01515</td>
<td>0.02088</td>
<td>rad.</td>
</tr>
<tr>
<td>$\theta_z$</td>
<td>0.1986</td>
<td>0.02256</td>
<td>0.0734</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$q_z$</td>
<td>1.74</td>
<td>0.1649</td>
<td>0.1843</td>
<td>rad.</td>
</tr>
<tr>
<td>$r_x$</td>
<td>1.231</td>
<td>0.1115</td>
<td>0.1701</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$y_x$</td>
<td>1.24</td>
<td>0.3132</td>
<td>0.5204</td>
<td>m</td>
</tr>
<tr>
<td>$y_y$</td>
<td>0.8289</td>
<td>0.3897</td>
<td>0.5249</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$y_z$</td>
<td>0.07546</td>
<td>0.03376</td>
<td>0.1768</td>
<td>rad.</td>
</tr>
<tr>
<td>$p_x$</td>
<td>0.09796</td>
<td>0.03842</td>
<td>0.1516</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$p_y$</td>
<td>0.008009</td>
<td>0.7868</td>
<td>0.03422</td>
<td>rad.</td>
</tr>
<tr>
<td>$p_z$</td>
<td>0.001823</td>
<td>0.2277</td>
<td>0.01884</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$\delta_x$</td>
<td>0.2102</td>
<td>0.08115</td>
<td>0.08618</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_y$</td>
<td>0.05152</td>
<td>0.01065</td>
<td>0.02899</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_z$</td>
<td>0.3142</td>
<td>0.03093</td>
<td>0.04221</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_C$</td>
<td>0.0008223</td>
<td>0.0001763</td>
<td>0.000152</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_w$</td>
<td>2.329</td>
<td>0.9962</td>
<td>0.1278</td>
<td>m</td>
</tr>
<tr>
<td>$a_{x}$</td>
<td>0.8932</td>
<td>0.05839</td>
<td>0.0664</td>
<td>m/sec^2</td>
</tr>
<tr>
<td>$a_{y}$</td>
<td>0.04096</td>
<td>0.008201</td>
<td>0.007806</td>
<td>m/sec^2</td>
</tr>
<tr>
<td>$a_{z}$</td>
<td>0.1017</td>
<td>0.1371</td>
<td>0.1849</td>
<td>m/sec^2</td>
</tr>
</tbody>
</table>

**TABLE 6.1:** RMS values of selected variables of the mathematical model OLYMPUS corresponding to Test Case E of Table 5.1.
These values were obtained from tests in which the helicopter and its load had some initial attitude and displacements (see Test Case E of Table 5.1). The titles identify the particular case (as explained in Appendix G). From the table it can be seen that the best longitudinal and lateral performance was obtained when the three-cable arrangement (OD519) is used. The worst response was obtained when a single-cable arrangement (OD501) was used, in which case the longitudinal and lateral displacements of both the vehicle and load were higher than usual (i.e. $x_v, x_t = 3.4m$ and $y_v, y_t = 1.2m$) and the yaw attitude and angular rate of the vehicle were unacceptable ($\psi_v = 1.7rad$ and $r_v = 1.2rad/sec.$). The lateral displacement winch control was highly active in this situation.

Also the forward acceleration, $a_x$, of the helicopter was about $0.9m/sec^2$ which was acceptable. With the two or three cable arrangement the performance of the system improved considerably. In OD510 the penalty paid for improving the overall response was an increase of the yaw attitude (i.e. $\psi, r$) of the load, whereas in OD519, the penalty was an increase of the pitch attitude ($\phi, p_t$) of the load. In these two cases the control activity remained within the military specifications (see Appendix D).

Comparing the response obtained when an output regulator was used (Table 6.1) with that when a complete state feedback controller was used (Table 5.4) it can be seen that the response of OLYMPUS with the output regulator is slightly worse than that with the state controller, (even though full state feedback is employed in each case). However for the output regulator weighting penalties are applied only to the output variables, with an attendant loss of control effectiveness. If the responses are examined closely, it can be seen that lateral response of the helicopter in OD510 is superior to that of SD510 from which it can be inferred that,
by some appropriate choice of the weighting matrices in equation (4.39) one can achieve a reasonable closed-loop response even though only 12 variables were assumed to be measurable.

6.2.1 The Effects of Including Actuator Dynamics in OLYMPUS

Tables 6.2 and 6.3 show the response obtained from the mathematical model OLYMPUS with the dynamics of the control actuators included. With a complete state controller (Chapter 5) the inclusion of such actuator dynamics slightly degrades the performance of the system but reduces the control activity. However by comparing OD50l with ODA50l it can be seen that the lateral response of the helicopter was improved considerably when the actuator dynamics were included. Also, use of a two-cable suspension improved the overall performance of the system, but considerably increased the load's yaw angle.

The simulated response of the helicopter/load system flying in turbulence, (Table 6.3), is reasonable and within the military specifications, even though it was an output regulator that was used.
<table>
<thead>
<tr>
<th>Variable</th>
<th>ODA501</th>
<th>ODA510</th>
<th>UNITS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_v$</td>
<td>4.277</td>
<td>0.0481</td>
<td>m</td>
</tr>
<tr>
<td>$u_v$</td>
<td>0.9076</td>
<td>0.0159</td>
<td>m/sec</td>
</tr>
<tr>
<td>$w_v$</td>
<td>0.05739</td>
<td>0.0165</td>
<td>m/sec</td>
</tr>
<tr>
<td>$\theta_v$</td>
<td>0.124</td>
<td>0.05431</td>
<td>rad.</td>
</tr>
<tr>
<td>$q_v$</td>
<td>0.2808</td>
<td>0.1099</td>
<td>rad/sec</td>
</tr>
<tr>
<td>$x_l$</td>
<td>4.29</td>
<td>0.6501</td>
<td>m</td>
</tr>
<tr>
<td>$u_l$</td>
<td>1.906</td>
<td>0.2022</td>
<td>m/sec</td>
</tr>
<tr>
<td>$w_l$</td>
<td>0.0695</td>
<td>0.0255</td>
<td>m/sec</td>
</tr>
<tr>
<td>$\theta_l$</td>
<td>0.1509</td>
<td>0.1477</td>
<td>rad.</td>
</tr>
<tr>
<td>$q_l$</td>
<td>0.171</td>
<td>0.2162</td>
<td>rad/sec</td>
</tr>
<tr>
<td>$\gamma_v$</td>
<td>0.8159</td>
<td>0.1799</td>
<td>m</td>
</tr>
<tr>
<td>$\phi_v$</td>
<td>0.3073</td>
<td>0.1154</td>
<td>m/sec</td>
</tr>
<tr>
<td>$\psi_v$</td>
<td>0.07393</td>
<td>0.01763</td>
<td>rad.</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>0.1438</td>
<td>0.03178</td>
<td>rad/sec</td>
</tr>
<tr>
<td>$r_v$</td>
<td>0.5975</td>
<td>0.03711</td>
<td>rad/sec</td>
</tr>
<tr>
<td>$\gamma_l$</td>
<td>1.441</td>
<td>0.4038</td>
<td>m</td>
</tr>
<tr>
<td>$\phi_l$</td>
<td>1.754</td>
<td>0.462</td>
<td>m/sec</td>
</tr>
<tr>
<td>$\psi_l$</td>
<td>0.16</td>
<td>0.034</td>
<td>rad.</td>
</tr>
<tr>
<td>$\rho_l$</td>
<td>0.2084</td>
<td>0.03609</td>
<td>rad/sec</td>
</tr>
<tr>
<td>$r_l$</td>
<td>0.01277</td>
<td>1.062</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_L$</td>
<td>0.003014</td>
<td>0.2735</td>
<td>rad/sec</td>
</tr>
<tr>
<td>$\delta_R$</td>
<td>0.1361</td>
<td>0.08222</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_T$</td>
<td>0.02688</td>
<td>0.01005</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_C$</td>
<td>0.1943</td>
<td>0.01768</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_w$</td>
<td>0.0006872</td>
<td>0.0001514</td>
<td>rad.</td>
</tr>
<tr>
<td>$a_x$</td>
<td>2.603</td>
<td>1.512</td>
<td>m</td>
</tr>
<tr>
<td>$a_y$</td>
<td>0.8445</td>
<td>0.04692</td>
<td>m/sec^2</td>
</tr>
<tr>
<td>$a_z$</td>
<td>0.035</td>
<td>0.008481</td>
<td>m/sec^2</td>
</tr>
<tr>
<td></td>
<td>0.7098</td>
<td>0.08767</td>
<td>m/sec^2</td>
</tr>
</tbody>
</table>

**TABLE 6.2**: RMS values of selected variables of the mathematical model OLYMPUS with actuator dynamics corresponding to Test Case E of Table 5.1.
<table>
<thead>
<tr>
<th></th>
<th>OSA501</th>
<th>OSA510</th>
<th>OSA519</th>
<th>UNITS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_v$</td>
<td>0.3369</td>
<td>0.04042</td>
<td>0.06683</td>
<td>m</td>
</tr>
<tr>
<td>$u_v$</td>
<td>0.5154</td>
<td>0.4951</td>
<td>0.4962</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w_v$</td>
<td>0.4875</td>
<td>0.487</td>
<td>0.4869</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\theta_v$</td>
<td>0.006133</td>
<td>0.003111</td>
<td>0.003234</td>
<td>rad.</td>
</tr>
<tr>
<td>$q_v$</td>
<td>0.007828</td>
<td>0.003738</td>
<td>0.004359</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$x_r$</td>
<td>1.1086</td>
<td>0.1222</td>
<td>0.268</td>
<td>m</td>
</tr>
<tr>
<td>$u_r$</td>
<td>0.1047</td>
<td>0.05334</td>
<td>0.00766</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w_r$</td>
<td>0.006449</td>
<td>0.001855</td>
<td>0.004422</td>
<td>rad.</td>
</tr>
<tr>
<td>$\theta_r$</td>
<td>0.004533</td>
<td>0.02646</td>
<td>0.04463</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$q_r$</td>
<td>0.006338</td>
<td>0.03309</td>
<td>0.04139</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$y_v$</td>
<td>0.1061</td>
<td>0.1974</td>
<td>0.02694</td>
<td>m</td>
</tr>
<tr>
<td>$v_v$</td>
<td>0.5042</td>
<td>0.5035</td>
<td>0.507</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\phi_v$</td>
<td>0.01271</td>
<td>0.004152</td>
<td>0.005234</td>
<td>rad.</td>
</tr>
<tr>
<td>$p_v$</td>
<td>0.004611</td>
<td>0.003024</td>
<td>0.003023</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$\psi_v$</td>
<td>0.05559</td>
<td>0.05484</td>
<td>0.1249</td>
<td>rad.</td>
</tr>
<tr>
<td>$r_v$</td>
<td>0.001617</td>
<td>0.01595</td>
<td>0.03466</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$y_r$</td>
<td>0.2398</td>
<td>0.2173</td>
<td>0.1816</td>
<td>m</td>
</tr>
<tr>
<td>$v_r$</td>
<td>0.06509</td>
<td>0.08041</td>
<td>0.05843</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\phi_r$</td>
<td>0.01887</td>
<td>0.004347</td>
<td>0.151</td>
<td>rad.</td>
</tr>
<tr>
<td>$p_r$</td>
<td>0.005086</td>
<td>0.001498</td>
<td>0.04515</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$\psi_r$</td>
<td>0.001749</td>
<td>0.1468</td>
<td>0.1281</td>
<td>rad.</td>
</tr>
<tr>
<td>$r_r$</td>
<td>0.0007973</td>
<td>0.06492</td>
<td>0.07045</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$\delta_L$</td>
<td>0.00286</td>
<td>0.004214</td>
<td>0.003795</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_R$</td>
<td>0.004298</td>
<td>0.005292</td>
<td>0.005167</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_T$</td>
<td>0.000133</td>
<td>0.001445</td>
<td>0.009249</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_C$</td>
<td>0.0000517</td>
<td>0.00001963</td>
<td>0.00003661</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_w$</td>
<td>0.4682</td>
<td>0.1493</td>
<td>0.4478</td>
<td>m</td>
</tr>
<tr>
<td>$a_x$</td>
<td>0.7143</td>
<td>0.7191</td>
<td>0.7187</td>
<td>m/sec^2</td>
</tr>
<tr>
<td>$a_y$</td>
<td>0.6702</td>
<td>0.6701</td>
<td>0.6701</td>
<td>m/sec^2</td>
</tr>
<tr>
<td>$a_z$</td>
<td>0.6448</td>
<td>0.6429</td>
<td>0.6466</td>
<td>m/sec^2</td>
</tr>
<tr>
<td>$u_{g_v}$</td>
<td>2.643</td>
<td>2.643</td>
<td>2.643</td>
<td>m/sec</td>
</tr>
<tr>
<td>$v_{g_v}$</td>
<td>2.305</td>
<td>2.305</td>
<td>2.305</td>
<td>m/sec</td>
</tr>
<tr>
<td>$u_{g_r}$</td>
<td>2.057</td>
<td>2.057</td>
<td>2.057</td>
<td>m/sec</td>
</tr>
<tr>
<td>$v_{g_r}$</td>
<td>1.637</td>
<td>1.637</td>
<td>1.637</td>
<td>m/sec</td>
</tr>
</tbody>
</table>

**TABLE 6.3:** RMS values of selected variables of the mathematical model OLYMPUS with actuator dynamics as affected by simulated atmospheric turbulence (Case D of Table 5.1.)
6.3 OUTPUT REGULATOR LIMITATIONS

It was pointed out in the previous chapter (and it can also be deduced from Table 6.3) that the most difficult of all those cases, from stability considerations, examined, was that of using four controls with a single-cable suspension. This particular case was chosen for the qualitative assessment of the helicopter/load response using an output regulator to confirm its effectiveness. Table 6.4 shows the r.m.s. values of all the variables of the helicopter/load system. By comparing OSA401 with OSA501 (of Table 6.2) it can be seen that, even though four controls were available, the helicopter/load system could still be stabilised although there was some slight degradation of performance. The time response of OSA401 is shown in Figure 6.1. From that figure it is evident that the output regulator increases only the yaw angle, $\Psi$. Such an increase in yaw angle brings the helicopter into wind providing optimum balancing of all the forces and moments of the helicopter/load system while keeping the control activity as low as possible. In the same figure, it can also be seen that the sensor noise of the measurable variables is much higher for measured velocities and accelerations than for attitude angles of the helicopter system. This is due to the fact that rate gyros are in general less noisy than pilot-tube/airspeed indicator arrangement and linear accelerometers (Broussard, Berry et al, [1979]).

But, helicopters are normally equipped with one pilot-static tube to measure forward velocity of the helicopter, $u_v$. Thus the assumption which was made in Section 6.2 that all three helicopter velocities are available for measurement does not hold. However, when vertical velocity,
<table>
<thead>
<tr>
<th></th>
<th>OSA401 (12 outputs)</th>
<th>OSA401* (9 outputs)</th>
<th>OSA401** (9 outputs)</th>
<th>OSA501 (9 outputs)</th>
<th>UNITS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0.3759</td>
<td>0.5443</td>
<td>0.5522</td>
<td>0.5858</td>
<td>m</td>
</tr>
<tr>
<td>$u$</td>
<td>0.5199</td>
<td>0.4934</td>
<td>0.4934</td>
<td>0.4934</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w$</td>
<td>0.4898</td>
<td>0.4871</td>
<td>0.4871</td>
<td>0.4871</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta x$</td>
<td>0.006169</td>
<td>0.005648</td>
<td>0.005803</td>
<td>0.006707</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta u$</td>
<td>0.007809</td>
<td>0.007315</td>
<td>0.007381</td>
<td>0.007205</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$y$</td>
<td>1.216</td>
<td>1.728</td>
<td>1.745</td>
<td>1.835</td>
<td>m</td>
</tr>
<tr>
<td>$v$</td>
<td>0.1202</td>
<td>0.2004</td>
<td>0.2045</td>
<td>0.221</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w$</td>
<td>0.03098</td>
<td>0.6803</td>
<td>0.07825</td>
<td>0.0181</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\delta y$</td>
<td>0.004519</td>
<td>0.005158</td>
<td>0.00529</td>
<td>0.00577</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta v$</td>
<td>0.006284</td>
<td>0.005598</td>
<td>0.005615</td>
<td>0.005567</td>
<td>rad/sec.</td>
</tr>
</tbody>
</table>

$\delta L$, $\delta R$, $\delta T$, $\delta C$, $\delta w$:

<table>
<thead>
<tr>
<th></th>
<th>OSA401 (12 outputs)</th>
<th>OSA401* (9 outputs)</th>
<th>OSA401** (9 outputs)</th>
<th>OSA501 (9 outputs)</th>
<th>UNITS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>3.813</td>
<td>3.785</td>
<td>3.785</td>
<td>1.499</td>
<td>m</td>
</tr>
<tr>
<td>$v$</td>
<td>2.699</td>
<td>2.669</td>
<td>2.669</td>
<td>1.314</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w$</td>
<td>0.1426</td>
<td>0.1399</td>
<td>0.1399</td>
<td>0.08366</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta x$</td>
<td>0.006031</td>
<td>0.04682</td>
<td>0.04542</td>
<td>0.02838</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$\delta u$</td>
<td>0.00234</td>
<td>0.001636</td>
<td>0.00164</td>
<td>0.001465</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta w$</td>
<td>0.001054</td>
<td>0.000523</td>
<td>0.0005249</td>
<td>0.000549</td>
<td>rad/sec.</td>
</tr>
</tbody>
</table>

Control weighting matrix $G = \text{diag}(1.0,1.5,2.5,2.0)$

Control weighting matrix $G = \text{diag}(0.5,0.5,2.5,2.0)$

TABLE 6.4: Comparison of Responses of OLYMPUS When a Single Cable is Used
FIGURE 6.1(a): Typical Response of OLYMPUS With Four Controls and a Single Cable Suspension Arrangement (OSA 401) Which Employs an Output Regulator Based on the Information Provided by 12 Outputs (Test Case D of Table 5.1)
FIGURE 6.1(b): Continued
FIGURE 6.1(c): Continued
\( w_v \) was assumed to be available as an output, the output regulator (which was computed on the basis of 11 outputs) could still stabilise the helicopter/load system without any substantial degradation of dynamic performance. But if the lateral velocity was also not included (thus reducing the number of outputs to 10) the closed-loop system, based upon the 10-output regulator, became unstable. Any measurements of speed at hover are inaccurate, due to the inhomogeneous flow around the pitot-static tube which is caused by the downwash created by the main rotor; it is sensible therefore not to include the forward velocity, \( u_v \), as an output of the system. The output regulator, based on the nine remaining outputs of the system, was still able to stabilise the helicopter/load system, but with a mildly-desstabilising effect on the lateral displacement of the helicopter, \( y_v \), and load, \( y_{l} \), and lateral velocity of the load, \( v_{l} \) (see Table 6.4 and Figure 6.2).

The weighting elements corresponding to the longitudinal and lateral cyclic controls were then reduced and a new regulator was derived which would allow for more control action. But the presence of the control actuator dynamics in the system did not permit this and no substantial change to the response of the system (Table 6.4) was observed. However, the use of the lateral displacement winch control improved considerably the lateral response of the system as can be inferred from Tables 6.4 and 6.5 and Figure 6.3.

When lateral velocity of the helicopter, \( v_v \), was included in the output variables, a new output regulator was then derived based on the information contained in the 10 outputs viz. \( \phi_v, \rho_v, \theta_v, q_v, \psi_v, r_v, a_x, a_y, a_z \) and \( v_v \). The response corresponding to this new regulator was unstable.
FIGURE 6.2(a): Typical Response of OLYMPUS With Four Controls and a Single Cable Suspension Arrangement (OSA 401) Which Employs an Output Regulator Based on the Information Provided by 9 Outputs (Test Case D of Table 5.1)
FIGURE 6.2(b): Continued
FIGURE 6.3: Comparison of Responses of OLYMPUS Without and With a Winch Control When an Output Regulator was Employed Based on the Information Provided by 9 Outputs (Test Case D of Table 5.1)
By examining Table 6.5 it can be seen that whenever either the forward, $u_v$, or lateral, $v_v$, velocity of the helicopter is considered to be one of the outputs, the associated response of the system was unstable, probably due to the strong coupling between the longitudinal and lateral modes of the hovering Sea King helicopter. From the same table it can also be inferred that the forward and lateral velocities of the helicopter are important variables and should be included in the output vector if an overall stable response of the helicopter/load system is to be obtained.
<table>
<thead>
<tr>
<th>CASE TITLE</th>
<th>OUTPUT VECTOR</th>
<th>NUMBER OF OUTPUTS</th>
<th>OVERALL RESPONSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>OSA401</td>
<td>$\mathbf{\gamma}^T = { u_v, v_v, w_v, \phi_v, \rho_v, \theta_v, q_v, \psi_v, r_v, a_x, a_y, a_z }$</td>
<td>12</td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{\gamma}^T = { u_v, v_v, \phi_v, \rho_v, \theta_v, q_v, \psi_v, r_v, a_z }$</td>
<td>11</td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{\gamma}^T = { u_v, \phi_v, \rho_v, \theta_v, q_v, r_v, a_x, a_y, a_z }$</td>
<td>10</td>
<td>Unstable</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{\gamma}^T = { v_v, \phi_v, \rho_v, \theta_v, q_v, r_v, a_x, a_y, a_z }$</td>
<td>10</td>
<td>Unstable</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{\gamma}^T = { \phi_v, \rho_v, \theta_v, q_v, r_v, a_x, a_y, a_z }$</td>
<td>9</td>
<td>Stable with a mildly destabilising effect on $\mathbf{y}_x, \mathbf{y}_y, \mathbf{y}_z$</td>
</tr>
<tr>
<td>OSA501</td>
<td>$\mathbf{\gamma}^T = { u_v, v_v, w_v, \phi_v, \rho_v, \theta_v, q_v, \psi_v, r_v, a_x, a_y, a_z }$</td>
<td>12</td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{\gamma}^T = { \phi_v, \rho_v, \theta_v, q_v, r_v, a_x, a_y, a_z }$</td>
<td>9</td>
<td>Stable</td>
</tr>
</tbody>
</table>

**TABLE 6.5: Output Vector Definition**
6.3.1 The Effects of Sensor and Actuator Failures on the Dynamic Response of the Closed-Loop AHCS

It has been shown earlier in this dissertation that an output regulator can stabilise the helicopter/load system. Before considering the use of the regulator on the helicopter it is necessary to study whether this solution to the stability problem is feasible, practical, economical (Beltramo [1980]) and, of course, safe to use. Such a complete study would be complicated and extensive and time consuming and was beyond the scope and time scale of this work. However, a number of elementary safety aspects were investigated including:

- the effect on the overall response of the system of using a failed sensor (which produces continuously a zero measurement of the variable being measured).
- the effect of a faulty control actuator which does not respond to signals from the AHCS but remains at its trimmed position.

The faults can normally be simulated by setting the corresponding feedback variable to zero. However, due to the structure of the transition equation, (H.11), such faults had to be simulated by setting to zero a column or row of the matrix $K^o$ corresponding to the particular variable. The response of the system depends very much on the accuracy with which the output regulator gains were obtained. When certain gains were set to zero to represent a fault in the system some accuracy was lost and the response obtained depended on the importance of the particular variable to the hover phase of flight.

Three investigations were carried out in which it was assumed that faults had occurred in the following components of the AHCS:
speed sensors
rate gyros
control actuators

It was found that when the forward velocity signal was lost the helicopter/load became unstable in about 7 secs. from the time the vehicle started penetrating the turbulence (see Table 6.6).

When the lateral velocity signal was lost the system was driven to instability in about 0.2 secs. The loss of the signals connected with both the longitudinal and lateral velocities leads to instability in about 4 secs.

Loss of only the vertical velocity signal did not substantially affect the stable response of the closed-loop system.

When a rate gyro failure occurred in the system, a stable response was obtained (see Table 6.7).

When the actuator for the longitudinal cyclic control, \( \delta_L \), was made inoperative the response of the system slowly diverged (Table 6.8). An interesting response was obtained when the actuator associated with the lateral cyclic control, \( \delta_R \), was considered to be inactive. As may be seen from Table 6.8 and Figure 6.4, the helicopter was yawing at an increasing rate thus keeping the translational velocities and displacements of the system at low values. This situation occurred because, at hover, the rotor is normally tilted laterally so that a component of the thrust vector counteracts the side force produced by the tail rotor. Now that the lateral control is inoperative, the force equilibrium is no longer
<table>
<thead>
<tr>
<th>Variable Affected by a Failed Sensor</th>
<th>Forward Velocity</th>
<th>Lateral Velocity</th>
<th>Vertical Velocity</th>
<th>Forward and Lateral Velocities</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_v$</td>
<td>1.452</td>
<td></td>
<td>0.376</td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>$u_v$</td>
<td>0.4934</td>
<td></td>
<td>0.5198</td>
<td></td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w_v$</td>
<td>0.6901</td>
<td></td>
<td>0.4871</td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>$q_v$</td>
<td>1.405</td>
<td></td>
<td>0.006137</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$q_v$</td>
<td>0.3985</td>
<td></td>
<td>0.00775</td>
<td></td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$x_{\ddot{r}}$</td>
<td>0.8025</td>
<td></td>
<td>1.218</td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>$u_{\ddot{r}}$</td>
<td>0.9815</td>
<td></td>
<td>0.1203</td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>$w_{\ddot{r}}$</td>
<td>0.63743</td>
<td></td>
<td>0.0003584</td>
<td></td>
<td>m/sec.</td>
</tr>
<tr>
<td>$q_{\ddot{r}}$</td>
<td>0.05751</td>
<td></td>
<td>0.004543</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$q_{\ddot{r}}$</td>
<td>0.2599</td>
<td></td>
<td>0.006319</td>
<td></td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$y_v$</td>
<td>0.5957</td>
<td></td>
<td>0.7129</td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>$v_v$</td>
<td>0.6649</td>
<td></td>
<td>0.6278</td>
<td></td>
<td>m/sec.</td>
</tr>
<tr>
<td>$p_v$</td>
<td>0.2126</td>
<td></td>
<td>0.008824</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$r_v$</td>
<td>0.6295</td>
<td></td>
<td>0.004774</td>
<td></td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$q_v$</td>
<td>0.3857</td>
<td></td>
<td>0.3737</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$q_v$</td>
<td>0.3906</td>
<td></td>
<td>0.1094</td>
<td></td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$y_{\ddot{r}}$</td>
<td>0.6974</td>
<td></td>
<td>0.686</td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>$v_{\ddot{r}}$</td>
<td>0.3498</td>
<td></td>
<td>0.3263</td>
<td></td>
<td>m/sec.</td>
</tr>
<tr>
<td>$p_{\ddot{r}}$</td>
<td>0.03855</td>
<td></td>
<td>0.006109</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$r_{\ddot{r}}$</td>
<td>0.07723</td>
<td></td>
<td>0.003441</td>
<td></td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$q_{\ddot{r}}$</td>
<td>0.002345</td>
<td></td>
<td>0.002338</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$q_{\ddot{r}}$</td>
<td>0.001051</td>
<td></td>
<td>0.001052</td>
<td></td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$\delta_{L}$</td>
<td>1.525</td>
<td></td>
<td>0.002708</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{R}$</td>
<td>0.28132</td>
<td></td>
<td>0.005785</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{C}$</td>
<td>0.1057</td>
<td></td>
<td>0.007626</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{W}$</td>
<td>0.0007</td>
<td></td>
<td>0.0002</td>
<td></td>
<td>rad.</td>
</tr>
<tr>
<td>$a_{x_v}$</td>
<td>0</td>
<td></td>
<td>0</td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>$a_{y_v}$</td>
<td>16.62</td>
<td></td>
<td>0.714</td>
<td></td>
<td>m/sec²</td>
</tr>
<tr>
<td>$a_{x_v}$</td>
<td>0.678</td>
<td></td>
<td>0.6702</td>
<td></td>
<td>m/sec²</td>
</tr>
<tr>
<td>$a_{z_v}$</td>
<td>0.6722</td>
<td></td>
<td>0.6435</td>
<td></td>
<td>m/sec²</td>
</tr>
</tbody>
</table>

**TABLE 6.6:** The Effect on the Response of the Closed-Loop System With a Single Cable Suspension (OSA 401) of a Failed Speed Sensor
<table>
<thead>
<tr>
<th>Variables Affected by a Failed Sensor</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll Rate</td>
<td>Pitch Rate</td>
</tr>
<tr>
<td>$x_u$</td>
<td>0.3746</td>
</tr>
<tr>
<td>$u_x$</td>
<td>0.5197</td>
</tr>
<tr>
<td>$w_{\theta}$</td>
<td>0.4871</td>
</tr>
<tr>
<td>$\theta_{\theta}$</td>
<td>0.006246</td>
</tr>
<tr>
<td>$\psi_{\psi}$</td>
<td>0.007989</td>
</tr>
</tbody>
</table>

| $x_y$                               | 1.212    | 1.213    | 1.229    | m        |
| $u_y$                               | 0.1197   | 0.1198   | 0.1209   | m/sec.   |
| $w_{\psi}$                          | 0.000356 | 0.0003564| 0.00037  | m/sec.   |
| $\psi_{\psi}$                       | 0.004517 | 0.004517 | 0.00455  | rad/sec. |
| $\psi_{\psi}$                       | 0.006288 | 0.006291 | 0.00623  | rad/sec. |

| $y_u$                               | 0.7292   | 0.7127   | 1.04     | m        |
| $v_x$                               | 0.6301   | 0.6272   | 0.771    | m/sec.   |
| $w_{\theta}$                        | 0.008772 | 0.008829 | 0.01663  | rad/sec. |
| $\theta_{\theta}$                   | 0.003038 | 0.004791 | 0.006186 | rad/sec. |
| $\psi_{\psi}$                       | 0.3773   | 0.3737   | 0.8469   | rad/sec. |
| $\psi_{\psi}$                       | 0.1103   | 0.1094   | 0.003253 | rad/sec. |

| $y_v$                               | 0.7048   | 0.6859   | 0.9648   | m        |
| $v_x$                               | 0.3314   | 0.3263   | 0.5219   | m/sec.   |
| $w_{\theta}$                        | 0.0062   | 0.006109 | 0.01028  | rad.     |
| $\theta_{\theta}$                   | 0.003675 | 0.00344  | 0.00383  | rad/sec. |
| $\psi_{\psi}$                       | 0.00233  | 0.002338 | 0.002269 | rad.     |
| $\psi_{\psi}$                       | 0.001048 | 0.001052 | 0.0009923| rad/sec. |

| $\delta_{ux}$                       | 0.00289  | 0.00277  | 0.002653 | rad.     |
| $\delta_{u_y}$                      | 0.00578  | 0.00578  | 0.006674 | rad.     |
| $\delta_{u_z}$                      | 0.0078   | 0.0076   | 0.02624  | rad.     |
| $\delta_{w}$                        | 0.0002   | 0.0002   | 0.0002   | rad.     |
| $\delta_{w}$                        | 0        | 0        | 0        | m        |

| $a_x$                               | 0.714    | 0.714    | 0.714    | m/sec²   |
| $a_y$                               | 0.6702   | 0.6702   | 0.6702   | m/sec²   |
| $a_z$                               | 0.6434   | 0.6435   | 0.6566   | m/sec²   |

**TABLE 6.7:** The Effect on the Response of the Closed-Loop System With a Single Cable Suspension (OSA 401) of a Failed Rate Gyro
<table>
<thead>
<tr>
<th>Variable Affected by a Failed Actuator</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Longitudinal Cyclic Pitch Control</strong></td>
<td></td>
</tr>
<tr>
<td>$x_v$ 0.4363 0.4483 0.3315 0.3749</td>
<td>m</td>
</tr>
<tr>
<td>$u_v$ 0.929 0.5339 0.514 0.5198</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$w_v$ 0.5078 0.4871 0.4871 0.4871</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\theta_v$ 0.4173 0.007796 0.005949 0.007911</td>
<td>rad.</td>
</tr>
<tr>
<td>$q_v$ 0.7064 0.007011 0.007911 0.007811</td>
<td>rad/sec.</td>
</tr>
<tr>
<td><strong>Lateral Cyclic Pitch Control</strong></td>
<td></td>
</tr>
<tr>
<td>$\gamma_v$ 12.6 1.387 1.09 1.21</td>
<td>m</td>
</tr>
<tr>
<td>$\phi_v$ 6.685 0.1525 0.1036 0.1198</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\psi_v$ 0.006041 0.0007112 0.0002742 0.0003564</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\dot{\gamma}_v$ 10.55 0.004945 0.00472 0.00452</td>
<td>rad.</td>
</tr>
<tr>
<td>$\dot{\phi}_v$ 18.11 0.006105 0.006581 0.006291</td>
<td>rad/sec.</td>
</tr>
<tr>
<td><strong>Tail Rotor Collective Pitch Control</strong></td>
<td></td>
</tr>
<tr>
<td>$\delta_{\phi_{R}}$ 4.772 0.3874 1.873 0.7127</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\delta_{\phi_{R}}$ 8.881 0.6402 1.064 0.6278</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\delta_{\phi_{R}}$ 1.383 0.07133 0.03128 0.008829</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{\phi_{R}}$ 2.384 0.02243 0.01342 0.001791</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$\delta_{\phi_{R}}$ 2.233 3.633 0.006026 0.3737</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{\phi_{R}}$ 3.391 2.068 0.004371 0.1094</td>
<td>rad/sec.</td>
</tr>
<tr>
<td><strong>Main Rotor Collective Pitch Control</strong></td>
<td></td>
</tr>
<tr>
<td>$\delta_{\phi_{C}}$ 0.7573 0.7133 1.7222 0.6859</td>
<td>m</td>
</tr>
<tr>
<td>$\delta_{\phi_{C}}$ 1.904 0.4029 0.9266 0.3263</td>
<td>m/sec.</td>
</tr>
<tr>
<td>$\delta_{\phi_{C}}$ 0.6279 0.0433 0.03123 0.006109</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{\phi_{C}}$ 1.081 0.01577 0.01509 0.00344</td>
<td>rad/sec.</td>
</tr>
<tr>
<td>$\delta_{\phi_{C}}$ 0.002483 0.002334 0.002048 0.002338</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{\phi_{C}}$ 0.00128 0.001048 0.000287 0.001052</td>
<td>rad/sec.</td>
</tr>
<tr>
<td><strong>Longitudinal Control Pitch</strong></td>
<td></td>
</tr>
<tr>
<td>$\delta_{L}$ 0 0.003727 0.00265 0.002771</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{L}$ 0.539 0 0.006553 0.005783</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{L}$ 0.895 0.1942 0 0.007629</td>
<td>rad.</td>
</tr>
<tr>
<td>$\delta_{L}$ 0.0007 0.0004 0.0002 0</td>
<td>rad.</td>
</tr>
<tr>
<td><strong>Lateral Control Pitch</strong></td>
<td></td>
</tr>
<tr>
<td>$\delta_{w}$ 0 0 0 0</td>
<td>m</td>
</tr>
<tr>
<td><strong>Vertical Acceleration</strong></td>
<td></td>
</tr>
<tr>
<td>$a_{x_v}$ 1.301 0.7122 0.7145 0.714</td>
<td>m/sec^2</td>
</tr>
<tr>
<td>$a_{y_v}$ 0.6714 0.6702 0.6702 0.6702</td>
<td>m/sec^2</td>
</tr>
<tr>
<td>$a_{z_v}$ 3.206 0.734 0.6416 0.6435</td>
<td>m/sec^2</td>
</tr>
</tbody>
</table>

**TABLE 6.8:** The Effect on the Response of the Closed-Loop System With a Single Cable Suspension (OSA 401) of a Faulty Actuator
FIGURE 6.4(a): Typical Response of OLYMPUS With Four Controls and a Single Cable Suspension (OSA 401) as Affected by a Failure of the Lateral Cyclic Control, $\delta_R$, Actuator
maintained and the regulator then tries to stabilise the system by using only its remaining control, of the tail rotor, $\delta_T$. The greater the pitch angle of the blades of the tail rotor, the greater the force created by the rotor and, consequently the moment about the centre of gravity of the helicopter which is created by this force, is increased. Thus, since the force cannot be balanced, the system eventually becomes unstable.

Therefore, from the investigations described in this chapter, it can be inferred that when using an output regulator, the factors upon which the dynamic response of the closed-loop system principally depends are:

. the number of outputs being directly controlled;
. the importance in hover of certain output signals;
. the degree of coupling between output variables;
. the computational accuracy with which the feedback gains of the output regulator were derived.

The difficulties associated with the computation of the optimal feedback control gains are discussed in Chapter 7.
CHAPTER 7

NUMERICAL PROBLEMS ENCOUNTERED IN THE DETERMINATION OF OPTIMAL CONTROL LAWS
7.1 INTRODUCTION

The determination of suitable control laws and the design of optimal filters requires the solution of a Riccati equation. A number of different approaches to the solution of the Riccati equation exists (Anderson and Moore [1971]). The Algebraic Riccati equation (ARE) can be solved by iterative processes or by computing the eigenvectors of a canonical matrix, as shown by Potter [1966]. Potter's method was chosen, because of its accuracy, guarantee of convergence, and also, due to the fact that a solution of the ARE can be computed faster by using this method rather than by any iterative process.

However, even Potter's method has its limitations, some of which were encountered during the course of this work. When the optimal feedback gains were derived for ATHOS (or, for any other low-order model of the helicopter) by solving the LQP, a solution of the ARE could always be obtained for any choice of weighting matrices, Q and G. When models of higher order, such as the helicopter/load models GIONA and OLYMPUS, were used, it was found that a solution of the ARE was not always possible, due to the existence of two or more near-zero eigenvalues. These small eigenvalues cannot be computed accurately due to round-off errors, and are usually considered equal depending on the accuracy of the particular computer used. Thus the problems associated with the existence of repeated roots were also encountered. Changing the magnitude of the elements of the weighting matrices Q and G, did not give any clues on the predictability of occurrence of the near-zero roots. It was also found that neither the change of the suspension arrangement used, nor a change of the cable length and mass of the load.
affected the occurrence of the near-zero roots in a predictable manner.

Since the near-zero roots could not be avoided, it was necessary to investigate the problem further to obtain an acceptable solution of the ARE. Analysis has shown that although the problem occurs because of the existence of the near-zero roots, it could be characterised by the nature of the canonical matrix, in particular its sparsity, ill-conditioning and its high order. These greatly affect the accuracy of the results (Golub and Wilkinson [1976]) and depend upon the particular computer used. All digital computation undertaken in the course of this research was carried out on the ICL 1904S at the University of Technology, Loughborough.

In this chapter the problems created by near-zero eigenvalues are analysed and several methods for bypassing the problem or for securing solutions are proposed.
7.2 DESCRIPTION OF THE COMPUTATIONAL DIFFICULTIES

Because linear, optimal, control theory was used to obtain a suitable flight controller, two particular features of the helicopter/load dynamics were of special significance viz:

(i) the lack (in certain cases) of complete state controllability of the helicopter/load system (Appendix I); and
(ii) the existence of several zero, or near-zero roots.

Both features are closely related and both cause certain computational difficulties.

The eigenvalues of the uncontrolled helicopter (without any suspended load) are listed in Table 7.1. The corresponding mathematical model involved only the helicopter's angular displacements and rates and its translational velocities. Note from the table that there exists a very small, but negative, real root. However, the basic helicopter was completely controllable. In the same table are listed the eigenvalues of the helicopter (without load) when the optimal feedback control was applied. No computational difficulty was experienced in obtaining the optimal feedback control law for this case.

The mathematical model for the same 'load-less' helicopter was expanded to include the translational displacements. The eigenvalues corresponding to this model are listed in Table 7.2. As a result of expansion the mathematical model of the helicopter was now uncontrollable (in all three translational displacements) and has three small, real, characteristic roots, two of which were determined, as positive, by the
numerical algorithm first used. Thus the helicopter was unstable in translational displacement. An unstable, complex conjugate pair is also evident from the table. However, it was always possible to obtain for this augmented model a suitable optimal feedback control law for any choice of Q and G, provided only that Q was at least non-negative definite and G was positive definite, which is required by the linear quadratic optimal theory.

However, when the helicopter dynamics were modified, by including the suspended load, (for the mathematical model GIONA), the eigenvalues listed in Table 7.3 were obtained. In the uncontrolled (open-loop) case there were 14 real roots and 5 complex conjugate pairs. Of the real roots, seven were positive; of the complex pairs, two had positive real parts. The helicopter/load combination is clearly unstable; it was also found to be uncontrollable in two states, namely the vertical displacements of both the helicopter and its load. Nevertheless, an optimal feedback control law could be found, and the eigenvalues associated with the optimal closed-loop system are listed in the same table. Table 7.4 shows the open and closed-loop eigenvalues of the mathematical model OLYMPUS. It can be seen that both models have some very small closed-loop eigenvalues. These small eigenvalues created problems in the computation of the gains of an optimal feedback control law. Such a control law could only be determined for particular choices of Q and G, for it was found that many other choices of Q and G did not result in acceptable control laws from the program being used.
### TABLE 7.1: Characteristic Roots of the Helicopter Model ATHOS

<table>
<thead>
<tr>
<th>Open-Loop</th>
<th>Closed-Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.314 \pm j0.6558$</td>
<td>$-5.108 \pm j2.629$</td>
</tr>
<tr>
<td>$-0.03907 \pm j0.4168$</td>
<td>$-4.0117 \pm j2.1369$</td>
</tr>
<tr>
<td>$0.079 \pm j0.3176$</td>
<td>$-4.303 \pm j2.4976$</td>
</tr>
<tr>
<td>$-0.305$</td>
<td>$-0.0359$</td>
</tr>
<tr>
<td>$-0.324$</td>
<td>$-0.0753$</td>
</tr>
<tr>
<td>$-8.468 \times 10^{-12}$</td>
<td>$-0.829$</td>
</tr>
</tbody>
</table>

(9th order system*)

### TABLE 7.2: Characteristic Roots of the Helicopter Model ATHOS

<table>
<thead>
<tr>
<th>Open-Loop</th>
<th>Closed-Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.0$</td>
<td>$-4.6285 \pm j2.705$</td>
</tr>
<tr>
<td>$-1.314 \pm j0.6558$</td>
<td>$-3.915 \pm j2.4715$</td>
</tr>
<tr>
<td>$-0.03907 \pm j0.4168$</td>
<td>$-3.673 \pm j2.074$</td>
</tr>
<tr>
<td>$0.079 \pm j0.3167$</td>
<td>$-0.717 \pm j0.4914$</td>
</tr>
<tr>
<td>$-0.3055$</td>
<td>$-0.1998 \pm j0.2015$</td>
</tr>
<tr>
<td>$-0.3241$</td>
<td>$-0.1245 \pm j0.1135$</td>
</tr>
<tr>
<td>$-5.59 \times 10^{-10}$</td>
<td></td>
</tr>
<tr>
<td>$2.886 \times 10^{-12}$</td>
<td></td>
</tr>
<tr>
<td>$3.82 \times 10^{-12}$</td>
<td></td>
</tr>
</tbody>
</table>

(12th order system**) 

* System completely controllable

** Three state variables of the system are uncontrollable $(x_v, y_v, z_v)$. 

(24th order system*)

<table>
<thead>
<tr>
<th>Open-Loop</th>
<th>Closed-Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.397</td>
<td>-10.921</td>
</tr>
<tr>
<td>-3.397</td>
<td>-4.501±j0.8963</td>
</tr>
<tr>
<td>1.791</td>
<td>-3.642</td>
</tr>
<tr>
<td>-1.795</td>
<td>-2.925±j0.7235</td>
</tr>
<tr>
<td>-1.234±j0.961</td>
<td>-2.870±j0.2102</td>
</tr>
<tr>
<td>0.055±j1.389</td>
<td>-2.227±j1.244</td>
</tr>
<tr>
<td>-0.063±j1.2185</td>
<td>-1.173±j2.070</td>
</tr>
<tr>
<td>-0.6134</td>
<td>-0.5415±j1.375</td>
</tr>
<tr>
<td>0.3593</td>
<td>-0.6843±j0.7509</td>
</tr>
<tr>
<td>0.1004±j0.3325</td>
<td>-1.083</td>
</tr>
<tr>
<td>-0.3069</td>
<td>-0.4684</td>
</tr>
<tr>
<td>-0.2477</td>
<td>-0.6768</td>
</tr>
<tr>
<td>-3.5x10^-9±j2.95x10^-9</td>
<td>-3.5x10^-9±j2.95x10^-9</td>
</tr>
<tr>
<td>7.02x10^-9</td>
<td>0.6042</td>
</tr>
<tr>
<td>2.87x10^-11</td>
<td>-0.0009±j0.0009</td>
</tr>
<tr>
<td>1.542x10^-13</td>
<td>-0.0003±j0.0003</td>
</tr>
<tr>
<td>-3.4001x10^-14</td>
<td></td>
</tr>
<tr>
<td>7.059x10^-13</td>
<td></td>
</tr>
<tr>
<td>-3.5x10^-13</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 7.3: Characteristic Roots of the Helicopter/Load Model GIONA

*Two state variables (i.e., z_y and z_y) are uncontrollable.
<table>
<thead>
<tr>
<th>Open-Loop</th>
<th>Closed-Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.402</td>
<td>-25.275</td>
</tr>
<tr>
<td>-3.403</td>
<td>-6.2485±j0.315</td>
</tr>
<tr>
<td>1.757</td>
<td>-1.496±j3.722</td>
</tr>
<tr>
<td>-1.964±j0.615</td>
<td>-3.398±j0.0272</td>
</tr>
<tr>
<td>-0.1811±j1.5637</td>
<td>-0.4876±j1.9418</td>
</tr>
<tr>
<td>-0.00465±j1.4013</td>
<td>-0.7222±j1.324</td>
</tr>
<tr>
<td>0.5776</td>
<td>-0.9705±j0.8161</td>
</tr>
<tr>
<td>0.0074±j0.4294</td>
<td>-0.8645±j0.4502</td>
</tr>
<tr>
<td>-0.547</td>
<td>-1.0059</td>
</tr>
<tr>
<td>-0.305</td>
<td>-0.997</td>
</tr>
<tr>
<td>-0.245</td>
<td>-0.1125±j0.3762</td>
</tr>
<tr>
<td>-0.0302</td>
<td>-3.447×10⁻⁴±j3.43×10⁻⁴</td>
</tr>
<tr>
<td>-2.733×10⁻⁹±j1.419×10⁻⁸</td>
<td>-5.857×10⁻⁵</td>
</tr>
<tr>
<td>-8.666×10⁻⁹</td>
<td></td>
</tr>
<tr>
<td>7.0167×10⁻⁹</td>
<td></td>
</tr>
<tr>
<td>4.813×10⁻¹⁰</td>
<td></td>
</tr>
<tr>
<td>1.444×10⁻¹¹</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 7.4:** Characteristic Roots of the Helicopter/Load Model OLIMBUS

*System completely controllable.*
7.2.1 Identification of the Problem Area

In Chapter 4, it was shown, that for the helicopter/load system described by the set of equations (4.1), viz.

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  
(4.1a)

\[ y(t) = Cx(t) + Du(t) \]  
(4.1b)

and satisfying the controllability and observability requirements an optimal feedback control law exists which is unique for a particular choice of a quadratic performance index.

The derivation of this control law involved the solution of the ARE

\[ \hat{A}^T \hat{K} + \hat{K} \hat{A} - \hat{K} B \hat{G}^{-1} B^T \hat{K} + \hat{Q} = 0 \]  
(4.50)

which was shown to be a \( n \times n \) matrix \( \hat{K} \). \( \hat{K} \) can be found (Appendix F) in terms of the eigenvectors of the canonical matrix, \( M \), of the order \( 2n \times 2n \) shown below:

\[
M = \begin{bmatrix}
\hat{A} & -\hat{B} \hat{G}^{-1} B^T \\
-\hat{Q} & -\hat{A}^T
\end{bmatrix}
\]  
(7.1)

The canonical matrix, \( M \), is formed from the coefficient matrices of the Riccati equation (4.50), where \( \hat{A}, \hat{G} \) and \( \hat{Q} \) are given by equations (4.51), (4.52) and (4.53) which, for convenience, are rewritten below:

\[ \hat{A} = A - \hat{B} \hat{G}^{-1} B^T \hat{Q} \]  
(4.51)

\[ \hat{G} = G + D^T \hat{Q} D \]  
(4.52)

and

\[ \hat{Q} = C^T [Q - \hat{Q} \hat{G}^{-1} D^T \hat{Q}] C \]  
(4.53)

and where \( Q \) and \( G \) are the state and control weighting matrices.

The eigenvalues and eigenvectors of the canonical matrix were
computed using standard NAG\textsuperscript{†} routines, which were included in the computer program OUTREG. There are many iterative methods for solving the eigenvalue problem, of which the QR transformation by Francis' fast method is the most reliable, advanced and relatively accurate technique. The QR algorithm is most effective when applied to a matrix in the upper Hessenberg (or symmetric and of band) form. The triangularisation of the Hessenberg matrix can be done by several methods such as those of Given's or Householder's reduction method. In this work, the canonical matrix was reduced to upper Hessenberg form by using real, stabilised, elementary similarity transformations with extended precision accumulation of inner product (Wilkinson and Reinch [1971]). The eigenvalues and eigenvectors of the Hessenberg matrix were then calculated using the Francis' QR algorithm. Although the QR algorithm is not the fastest available algorithm (since it takes $4n^2$ multiplications in one complete step compared, for example, to $n^2$ of the LR algorithm as explained by Wilkinson [1965]) it is its guarantee of convergence that makes it more attractive to the user.

The diagonal elements of the Hessenberg matrix are related to the eigenvalues of the original canonical matrix. The eigenvectors of the canonical matrix can be obtained by multiplying the eigenvectors of the Hessenberg matrix with the transformation matrix. The modal matrix, $U$, is then constructed from the eigenvectors of the original matrix. Partitioning the modal matrix, as required by Potter's method (see equation (F.5)) needs the separation of the eigen-

vectors corresponding to the actual system, from those eigenvectors which correspond to its adjoint system, since the eigenvalues of the actual system have negative real parts as shown in Figure 7.1. This can be done by observing the sign of the real part of the array which has as elements the eigenvalues of the canonical matrix, \( M \). The position of the eigenvalue in this array determines the position of the corresponding eigenvector in the modal matrix. Thus, the columns of the modal matrix, \( U \), can be repositioned so that \( U_{11} \) and \( U_{12} \) contain the real and imaginary parts of the eigenvectors of the actual system, and, \( U_{21} \) and \( U_{22} \) the real and imaginary parts of the eigenvectors of the adjoint system (see Appendix F). The accuracy of computation of the eigenvalues was checked by using equation (7.2), viz.

\[
\sum_{i=1}^{2n} \lambda_i = \text{tr}(M) \tag{7.2}
\]

If the eigenvalues are correct, then the sum of the diagonal elements of the canonical matrix, \( M \), (i.e. the trace of the matrix) must be equal to the sum of its eigenvalues (De Russo et al [1965]). When (7.2) was valid, then the accuracy of computation of the eigenvectors was checked by using equation (7.3), viz.

\[
M u_{\lambda_i} = \lambda_i u_{\lambda_i} \tag{7.3}
\]

where \( u_{\lambda_i} \) is the eigenvector corresponding to the \( i^{th} \) eigenvalue, \( \lambda_i \).

It was found that for most choices of \( Q \) and \( G \), the smallest computed eigenvalues (and, consequently, their corresponding eigenvectors) were incorrect and the number of eigenvalues with positive real parts did not equal the number of eigenvalues with negative real parts (Table 7.5).
FIGURE 7.1: The Dominant Eigenvalues of the Open-Loop and Closed-Loop Systems as Obtained from OUTREG
This problem was only encountered with high order models, irrespective of the fulfillment of controllability (and observability) requirements. Thus further analysis was needed to find means to overcome this problem.
7.3 ANALYSIS OF THE DIFFICULTIES

There are a number of factors which affect the efficiency of the computation of the eigenvalues and which have to be taken into account if an assessment of the computational difficulties is to be made. Some of the main factors are the following:

- the order of the canonical matrix with respect to storage available in the digital computer;
- the form of the canonical matrix, i.e. symmetrical, diagonal, sparse;
- the condition of the matrix: i.e. whether it is well- or ill-conditioned;
- the numerical accuracy of the computation which depends on the accuracy of the elements of the canonical matrix, and upon the sensitivity of the problem to round-off errors.

7.3.1 The Effect of Singularity

In this case, the obvious reason for the occurrence of the difficulties was the relative sizes of the eigenvalues, and consequently, of the associated eigenvectors. The highest, absolute, real part of the eigenvalues was greater than 10.0 and the lowest very close to zero. So, the eigenvectors, which consist of the columns of the modal matrix, U, can no longer be independent, which they have to be, if the inverse of one of the submatrices required for the solution is to be obtained.

Greatly complicating the situation is the fact that the sharp
mathematical distinction between singular and non-singular matrices exists only in the mathematicians' ideal world of real numbers. When operations upon matrices are done with rounded arithmetic the distinction becomes fuzzy. Thus, certain non-singular matrices may be made singular as a result of the perturbations introduced by round-off (Forsyth and Moler [1970]). This means that, in practice, the canonical matrix, M, with near-zero eigenvalues, will require special treatment if the lowest eigenvalues (and, consequently, the eigenvectors) are to be computed with a higher degree of accuracy.

7.3.2 The Effect of the Order of the System

It was also noticed that these difficulties occurred only with the high-order mathematical models. The dimension of the state vectors of these models was already high (i.e. greater than 22) and the order of the canonical matrix, the eigenvalues of which were required, was higher still (i.e. greater than 44), thus increasing the number of calculations which had to be done to obtain the eigenvalues and eigenvectors, with an immediate effect on the order of the cumulative round-off errors. (Palmer and Morris [1980]).

7.3.3 The Effect of the Form and Condition of the Canonical Matrix

Two other important sources of computational errors are due to sparsity and ill-conditioning of the canonical matrix. The phenomenon of ill-conditioning in matrices was known a long time ago, by Gauss (Brameller [1976]). However, studies of the quantitative
characteristics of this problem and its effect on application of control theory appeared only during the past decade. This problem of ill-conditioning is apparent mainly with iterative techniques (in this case, the QR algorithm). To determine numerically a measure of ill-conditioning is a long process called sensitivity analysis and it is simpler in practice to recognise the effect by certain symptoms of the problem. The facts that some of the diagonal elements of the canonical matrix were small compared with some off-diagonal terms, and that the canonical matrix was sparse† were the first indications of the system being ill-conditioned.

Ill-conditioned equations can be created because of either of the reasons given below:

First, the physical nature of the system may be such that its mathematical representation results in an ill-conditioned form. In this case the equations describing its behaviour are ill-conditioned irrespective of the method chosen to model the system. There is little that can be done to improve the situation, except to ensure that the largest elements are used as pivots and that the calculations are carried out with sufficient word length to minimise rounding-off errors.

Secondly, the system itself may be well-conditioned but the modelling method may create a set of ill-conditioned equations. In such a case it is usually possible to re-formulate the equations so that conditioning is improved† (Brameller [1976]).

†In this work, a matrix is considered as sparse if 60% or more of its elements are zero.
The results obtained when using the coefficient matrix of an ill-conditioned system are highly sensitive to small changes in the numerical quantities involved and the significance of rounding-off errors can be magnified out of all proportion, so that the ill-conditioned matrix may turn out to be nearly singular if its elements are not given with sufficient accuracy.

A problem of this kind is very difficult to handle because in general a solution does not exist (since the determinant would be zero). Thus only an approximate solution can be obtained (Miller [1974]).

A typical set of eigenvalues which clearly illustrates the nature of the problem is shown in Table 7.5. The largest eigenvalues have been determined very accurately for both the actual system and its adjoint. The two smallest eigenvalues are obviously wrong because each system must have equal number of eigenvalues (in this case, 22) corresponding to the order of the actual system. As explained earlier, the existence of a near-zero eigenvalue and the accumulation of rounding errors due to both the ill-condition and high order of the canonical matrix affects the accuracy of the computation to such an extent that the two smallest eigenvalues become a complex conjugate pair instead of two real values with opposite signs.

A similar problem was also observed by Golub and Wilkinson [1976] who reported their work in an effort to stimulate research in this area. According to their report, the maximum inaccuracy concerning the computation of the smallest eigenvalues occurs when the order of the matrix used is very high and its subspace containing these small eigenvalues is of low dimension.
<table>
<thead>
<tr>
<th></th>
<th>ACTUAL SYSTEM</th>
<th>ADJOINT SYSTEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>-92.1785</td>
<td>92.1785</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>-15.09738</td>
<td>15.09738</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>-14.82052</td>
<td>14.82052</td>
</tr>
<tr>
<td>$\lambda_4, \lambda_5$</td>
<td>-1.935125±j4.120425</td>
<td>1.935125±j4.120425</td>
</tr>
<tr>
<td>$\lambda_6$</td>
<td>-3.429454</td>
<td>3.429454</td>
</tr>
<tr>
<td>$\lambda_7, \lambda_8$</td>
<td>-3.199315±j0.08525184</td>
<td>3.199215±j0.08525184</td>
</tr>
<tr>
<td>$\lambda_9, \lambda_{10}$</td>
<td>-0.4432501±j1.919005</td>
<td>0.4432501±j1.919005</td>
</tr>
<tr>
<td>$\lambda_{11}, \lambda_{12}$</td>
<td>-0.6530755±j0.7752097</td>
<td>0.6530755±j0.7752097</td>
</tr>
<tr>
<td>$\lambda_{13}, \lambda_{14}$</td>
<td>-0.6973331±j0.73604</td>
<td>0.6973331±j0.73604</td>
</tr>
<tr>
<td>$\lambda_{15}$</td>
<td>-1.081068</td>
<td>1.081068</td>
</tr>
<tr>
<td>$\lambda_{16}$</td>
<td>-1.000553</td>
<td>1.000553</td>
</tr>
<tr>
<td>$\lambda_{17}$</td>
<td>-0.9999747</td>
<td>0.9999747</td>
</tr>
<tr>
<td>$\lambda_{18}, \lambda_{19}$</td>
<td>-0.09024752±j0.412797</td>
<td>0.09024752±j0.412797</td>
</tr>
<tr>
<td>$\lambda_{20}, \lambda_{21}$</td>
<td>-0.09961732±j0.2515</td>
<td>0.09961732±j0.2515</td>
</tr>
<tr>
<td>computational inaccuracies $\lambda_{22}$</td>
<td>-3.43789×10^{-10}+j1.144903×10^{-4}</td>
<td>-3.43789×10^{-10}-j1.144903×10^{-4}</td>
</tr>
<tr>
<td>$\lambda_{22}$</td>
<td>-3.43789×10^{-10}+j1.144903×10^{-4}</td>
<td>-3.43789×10^{-10}-j1.144903×10^{-4}</td>
</tr>
<tr>
<td>$\lambda_{23}$</td>
<td>-3.43789×10^{-10}-j1.144903×10^{-4}</td>
<td>-3.43789×10^{-10}+j1.144903×10^{-4}</td>
</tr>
</tbody>
</table>

**TABLE 7.5:** Typical Eigenvalues of the Actual System and its Adjoint as Computed by Using the 44×44 Canonical Matrix, M, of the 22\textsuperscript{nd} Order Model, OLYMPUS by Using a Single Cable Suspension Arrangement
No reliable solution to this problem exists. Therefore, an account of alternative methods for solving the ARE is given and a proposed trial-and-error technique is discussed in the next section.
7.4 PROPOSED METHODS WHICH OVERCOME THE DIFFICULTIES

This section deals with the methods available to overcome similar difficulties to those discussed earlier. These methods can be classified as:

- those which improve Potter's solution of the ARE
- those which solve the ARE without the use of eigenvectors.

The first class includes as a means of improving the conditioning of the matrix the technique of balancing the canonical matrix prior to determining the eigenvalues and eigenvectors. The number of iterations needed to calculate the eigenvalues can be reduced considerably by shifting the eigenvalues of the canonical matrix, thereby reducing the round-off error accumulated in the determination of each eigenvalue. However, the method of prescribed stability can also be applied to Potter's method to eliminate the difficulties discussed so far. The latter class includes some well-established methods, which have fallen out of use, and some recent methods which do not depend on obtaining the eigenvectors of the canonical matrix. The well-established methods, in general, solve the Riccati equation by iteration, whereas the most recent methods use Schur vector decomposition instead of using eigenvectors as in Potter's method.

7.4.1 CLASS I Methods

(a) Prescribed Stability

This method is used whenever it is desirable for the eigenvalues of a closed-loop system to have prescribed values. Thus the method of
prescribed stability (Anderson and Moore [1971]) could be used to ensure that the near-zero eigenvalues of the helicopter/slung load problem were shifted to some prescribed value in the left half plane. The determination of appropriate feedback gains is usually termed the pole-positioning problem. In its most general form, the problem can be so formulated that the closed-loop eigenvalues lie within a certain prescribed region of the complex plane, usually in the left of the plane, instead of having specific values,

\[ R_{\lambda} = -\sigma, \]  

(7.4)

where \( \sigma > 0 \) is the required degree of stability of the system. Such a requirement means that the non-zero initial states of the system will decay at least as fast as \( e^{-\sigma t} \).

The state variable equations of the system then become,

\[
\begin{align*}
\dot{\tilde{x}}_p &= [A+I\sigma]\tilde{x}_p + B\tilde{u}_p, \\
\hat{\tilde{x}}_p &= C\tilde{x}_p + D\tilde{u}_p
\end{align*}
\]

(7.5)

(7.6)

where \( \tilde{x}_p, \tilde{u}_p \) and \( \hat{\tilde{x}}_p \) are the transformed vectors of the system, given by

\[
\begin{align*}
\tilde{x}_p &= e^{\sigma t} x \\
\tilde{u}_p &= e^{\sigma t} u \\
\hat{\tilde{x}}_p &= e^{\sigma t} \hat{x}
\end{align*}
\]

(7.7)

(7.8)

(7.9)

It can be proved that the control law has the same structure for both the original and transformed models, (Anderson and Moore [1971]) that is,

\[
\tilde{u}_p = -G^{-1}B^TK\hat{\tilde{x}}_p
\]

(7.10)

The fact that the degree of stability of the closed-loop system is
at least $\sigma$, is proved by rearranging equation (7.7), so that
\[ x(t) = e^{-\sigma t} x(t) \] (7.11)
which clearly shows that if the transformed system is asymptotically stable, then $x(t)$ goes to zero as $t$ goes to infinity at least as fast as $e^{-\sigma t}$ tends to zero.

$\sigma$ can take any positive value. For $\sigma = 0$ then the transformation vectors $\tilde{x}, \tilde{u}$ are identical to the original vectors $x, u$ and the performance index of the transformed system is equal to that of the original system (Figure 7.2). It has been noted by Anderson and Moore, that it is possible to construct a performance index, with $\sigma$ being equal to zero, such that the control law resulting from the minimisation of that performance index is the same as that obtained when $\sigma$ is non-zero. In other words there are sets of pairs of weighting matrices $Q$ and $G$ such that the associated regulator problem with $\sigma = 0$ leads to a closed-loop system with a degree of stability, $\sigma$. However, since it does not appear possible to give an explicit relationship for obtaining these matrices, the LQP has to be solved with a non-zero $\sigma$. In this research, the computer program, OUTREG, was modified to solve the LQP with a prescribed degree of stability.

It was found, that when using this method the close-to-zero roots of the canonical matrix were always separated since they were shifted to either the left half or right half of the complex plane. But it was also found that the optimal feedback control law derived from a prescribed stability method could not always guarantee the stability of the system since it depends upon the choice of $\sigma$ and the weighting
(a) The transformed system with a degree of stability at least \( \sigma \)

\[
\begin{align*}
B & \quad + \quad \Sigma \quad \frac{\dot{x}_p}{x_p} \quad \int \quad x_p \quad C \quad \frac{\dot{y}_p}{y_p}
\end{align*}
\]

\[
\begin{align*}
\bar{u}^o & \quad - \quad [A+\sigma I] \quad \bar{x}^o
\end{align*}
\]

(b) The original system with a degree of stability at least \( \sigma \)

\[
\begin{align*}
x & = e^{-\sigma t}x_p \\
y & = e^{-\sigma t}y_p
\end{align*}
\]

(c) Transformation equations

\[FIGURE 7.2: \text{ Prescribed Degree of Stability}\]
matrices, \( Q \) and \( G \). For large values of \( \sigma \) (i.e. \( \sigma > 0.5 \)), the eigenvalues of the helicopter/load system were concentrated about the specified value of \( \sigma \), as shown in Figure 7.3, in which case the problems associated with the accuracy of the computation of multiple eigenvalues may be again encountered, and the associated optimal control gains might not then be computed accurately.

According to the theory which governs prescribed stability, no closed-loop eigenvalue should exist with a lower value than \( \sigma \). However, by examining the closed-loop eigenvalues determined from the problem it can be seen (Figure 7.3) that this is untrue in this case. This result is partly due to the high order of the system which has caused an accumulation of round-off errors, thereby affecting the accuracy of the computation of the smallest eigenvalues. This result illustrates once more that the difficulties being experienced were purely of a numerical character.

Table 7.6 shows some typical eigenvalues obtained without and with prescribed stability, where \( \sigma \) was chosen to be very small (0.01). It can be seen that there are 23 negative eigenvalues, instead of the 22 obtained from OUTREG. When the prescribed stability was applied, the complex conjugate pair of near-zero eigenvalues were separated, thus forming eigenvalues, one with a positive real part and the other with a negative real part. The response of the system, with the optimal control gains obtained by using the prescribed stability method, was found to be entirely satisfactory as shown in Table 7.7.
FIGURE 7.3: The dominant Eigenvalues of the Closed-Loop System with a Degree of Stability at Least -10
<table>
<thead>
<tr>
<th>Eigenvalues* with no shift and ( \sigma = 0 )</th>
<th>Eigenvalues with prescribed stability ( \sigma = 0.01 )</th>
<th>Eigenvalues obtained by the shift method ( (a = -25) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5.5762</td>
<td>-5.5396</td>
<td>-5.5762</td>
</tr>
<tr>
<td>-2.4897±j4.0933</td>
<td>-2.4789±j4.0917</td>
<td>-2.4897±j4.0933</td>
</tr>
<tr>
<td>-3.2115±j2.266×10^{-4}</td>
<td>-3.20097</td>
<td>-3.2115±j2.266×10^{-4}</td>
</tr>
<tr>
<td>-0.129318±j2.08498</td>
<td>-3.222</td>
<td>-0.12938±j2.08498</td>
</tr>
<tr>
<td>-0.17015±j1.1914</td>
<td>-0.13329±j2.08486</td>
<td>-0.170155±j1.1914</td>
</tr>
<tr>
<td>-0.99859±j0.02278</td>
<td>-0.16686±j1.19186</td>
<td>-0.998587±j0.02278</td>
</tr>
<tr>
<td>-0.73393±j0.5102</td>
<td>-0.99876±j0.02277</td>
<td>-0.73393±j0.5102</td>
</tr>
<tr>
<td>-0.22165±j0.6852</td>
<td>-0.734±j0.50965</td>
<td>-0.22164±j0.68524</td>
</tr>
<tr>
<td>-0.17134±j0.51468</td>
<td>-0.224±j0.68594</td>
<td>-0.17133±j0.5147</td>
</tr>
<tr>
<td>-0.004649±j0.32052</td>
<td>-0.169±j0.51418</td>
<td>-0.0046±j0.32052</td>
</tr>
<tr>
<td>-1.46×10^{-12}</td>
<td>-0.01075±j0.320534</td>
<td>-5.12×10^{-9}</td>
</tr>
<tr>
<td>-2.454×10^{-6}</td>
<td>-0.01</td>
<td>-1.168×10^{-6}</td>
</tr>
<tr>
<td>-4.9045×10^{-12}±j3.127×10^{-10}</td>
<td>-0.009999</td>
<td>-1.164×10^{-9}</td>
</tr>
</tbody>
</table>

*23 negative eigenvalues instead of 22 were obtained from OUTREG.

**TABLE 7.6:** Comparison of Typical Closed-Loop Eigenvalues of the System
<table>
<thead>
<tr>
<th>Variables (rms values)</th>
<th>$\sigma = -0.01$</th>
<th>$\sigma = 0$ (shift -25 was used)†</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_v$</td>
<td>0.05206</td>
<td>0.19</td>
</tr>
<tr>
<td>$u_v$</td>
<td>0.01568</td>
<td>0.04932</td>
</tr>
<tr>
<td>$w_v$</td>
<td>0.01442</td>
<td>0.02065</td>
</tr>
<tr>
<td>$\theta_v$</td>
<td>0.05663</td>
<td>0.05359</td>
</tr>
<tr>
<td>$q_v$</td>
<td>0.1142</td>
<td>0.109</td>
</tr>
<tr>
<td>$x_l$</td>
<td>0.1981</td>
<td>0.2978</td>
</tr>
<tr>
<td>$u_l$</td>
<td>0.2056</td>
<td>0.2001</td>
</tr>
<tr>
<td>$w_l$</td>
<td>0.2739</td>
<td>0.1419</td>
</tr>
<tr>
<td>$\theta_l$</td>
<td>0.1523</td>
<td>0.1416</td>
</tr>
<tr>
<td>$q_l$</td>
<td>0.2246</td>
<td>0.2076</td>
</tr>
<tr>
<td>$y_v$</td>
<td>0.07185</td>
<td>0.08396</td>
</tr>
<tr>
<td>$v_v$</td>
<td>0.09554</td>
<td>0.09782</td>
</tr>
<tr>
<td>$\phi_v$</td>
<td>0.01509</td>
<td>0.01515</td>
</tr>
<tr>
<td>$p_v$</td>
<td>0.02263</td>
<td>0.02256</td>
</tr>
<tr>
<td>$\psi_v$</td>
<td>0.1683</td>
<td>0.1649</td>
</tr>
<tr>
<td>$r_v$</td>
<td>0.1098</td>
<td>0.1115</td>
</tr>
<tr>
<td>$y_l$</td>
<td>0.3160</td>
<td>0.3132</td>
</tr>
<tr>
<td>$v_l$</td>
<td>0.3998</td>
<td>0.3897</td>
</tr>
<tr>
<td>$\phi_l$</td>
<td>0.03555</td>
<td>0.03376</td>
</tr>
<tr>
<td>$p_l$</td>
<td>0.04026</td>
<td>0.03842</td>
</tr>
<tr>
<td>$\psi_l$</td>
<td>0.8021</td>
<td>0.7868</td>
</tr>
<tr>
<td>$r_l$</td>
<td>0.2320</td>
<td>0.2277</td>
</tr>
<tr>
<td>$\delta_L$</td>
<td>0.08537</td>
<td>0.08115</td>
</tr>
<tr>
<td>$\delta_R$</td>
<td>0.01112</td>
<td>0.01065</td>
</tr>
<tr>
<td>$\delta_T$</td>
<td>0.3114</td>
<td>0.03033</td>
</tr>
<tr>
<td>$\delta_C$</td>
<td>0.0002237</td>
<td>0.0001763</td>
</tr>
<tr>
<td>$\delta_w$</td>
<td>0.9946</td>
<td>0.9962</td>
</tr>
</tbody>
</table>

**TABLE 7.7: Response* of the System** with a Feedback Control Obtained With and Without the Use of Prescribed Stability

*Initial conditions $\theta_v = 0.087$ rad, $x_v = -0.6$m, $y_v = -0.6$m

**The closed-loop eigenvalues of the system can be found in Table 7.6.

†The shift method is described in Section 7.4.1c.
(b) Balancing

Since ill-conditioning was part of the cause of the numerical problems, some means for improving the condition of the canonical matrix was tried. The canonical matrix, $\mathbf{M}$, was ill-conditioned because the plant matrix, $\mathbf{A}$, (which is a sub-matrix of $\mathbf{M}$) was ill-conditioned. Thus, by improving the condition of the matrix, $\mathbf{A}$, an improvement of the condition of the matrix $\mathbf{M}$ will result. It is well known (Wilkinson [1965]) that the condition of a matrix can be improved by 'balancing'.

The balancing of the plant matrix, $\mathbf{A}$, was achieved by re-defining the state vector (and by using the same linear differential equations to describe the dynamics of the helicopter-load system), so that the smaller elements appeared at the top left hand corner of the plant matrix and those with the largest magnitudes appeared at the bottom right-hand corner, which was then used as a 'pivot'. However, no considerable improvement in the accuracy of determination of the eigenvalues of the canonical matrix, $\mathbf{M}$, was observed when using this technique, mainly because the transpose of the coefficient matrix, $\mathbf{A}^T$, is also a sub-matrix in the canonical matrix, and its presence destabilised the balance of the matrix, $\mathbf{M}$, by introducing a second pivot as shown in Figure 7.4. Thus balancing had to be applied directly to the canonical matrix $\mathbf{M}$.

The canonical matrix was balanced directly by using a standard NAG routine F01ATA*. By means of this routine, the norm of the real

---

*The last letter of the routine name stands for Algol. Similar subroutines are available in Fortran.*
FIGURE 7.4: The Effect on the Canonical Matrix, \( M \), of Conditioning the Coefficient Matrix, \( A \)
canonical matrix was reduced by applying diagonal similarity transformations to the matrix. The errors in the calculation of the eigenvalues of the matrix may therefore be reduced since existing eigenvalue programs usually produce results with errors proportional to the Euclidean norm of the matrix. It is reported in the NAG manual, that the subroutine FOlATA balances the matrix in a norm \( \| \cdot \|_1 \) rather than the Euclidean norm \( \| \cdot \|_E \) because fewer multiplications are required.

The canonical matrix was then reduced to upper Hessenberg form (NAG routine FOlAKA) using stabilised, elementary similarity transformations with additional precision accumulation of inner products. Since the matrix was balanced, using FOlATA, then the reduction to Hessenberg form involved operations on a submatrix of order \( L-K+1 \) which starts at the element \( A(K,K) \) and finishes at the element \( A(L,L) \). The two integers, \( K \) and \( L \), are produced by the routine FOlATA. If the matrix had not been balanced, then \( K \) is set to unity and \( L \) to \( N \) and the reduction takes place on the full matrix, \( A(1,1) \) to \( A(N,N) \). The routine is very accurate. The multipliers used in the reduction are stored in the lower triangle of the Hessenberg matrix. FOlAKA also leaves in an integer array information on row and column interchanges. Subroutine FOlAPA uses the multipliers and row and column interchanges to calculate a matrix of the accumulated transformation, \( V \), which defines the similarity reduction from the balanced matrix to upper Hessenberg form. Since the canonical matrix was balanced, the matrix, \( V \), contains values only in the submatrix of order \( L-K+1 \) from elements \( V(K,K) \) to \( V(L,L) \) where \( K \) and \( L \) are obtained from the balancing routine. If the matrix has not been balanced then \( K \) is 1 and \( L \) is \( N \). As no
arithmetic operations are carried out in the procedure the accuracy of the elements of the array $V$ must be the same as the accuracy of the multipliers produced by F01AKA.

The eigenvalues and eigenvectors of the upper Hessenberg matrix can then be found by the economical method, due to Francis, for performing the QR algorithm without using complex arithmetic (NAG routine F02AQA). This method uses double shifts of origin where the shifts, $K_1, K_2$, have either two real values or complex conjugate values. If an eigenvalue has not been determined after 10 or 20 iterations a special shift is used instead of the usual shift. If, after 30 iterations, the eigenvalue has still not been determined a failure is indicated. If the final form of the matrix, $H$, when all the eigenvalues have been found, is $T$ then we have,

$$ T = P^{-1}HP $$

(7.12)

where $P$ is the product of the transformation matrices, or if $H$ was derived from a general matrix, $A$,

$$ T = P^{-1}S^{-1}ASP $$

(7.13)

where $S$ is the product of transformation matrices that reduce $A$ to the upper Hessenberg matrix $H$. If $u_1$ is an eigenvector of $T$ then $SPu_1$ is the corresponding eigenvector of $A$. The subroutine calculates the eigenvectors of $T$ by back-substitution, then multiplies the results by the product of the transformation matrices which were calculated by the QR algorithm.

The QR algorithm is very stable. However, the accuracy of the individual eigenvalues and eigenvectors depends on their sensitivity
to small changes in the original matrix, and the multiplicity of the roots. The time taken for calculating the eigenvalues and eigenvectors is approximately proportional to $N^3$. The back transformation of the set of right-hand eigenvectors was performed by using NAG routine FOLANGUA which used the information left by FOLATA in determining the permutations and scaling factors.

Balancing improved the situation only in certain cases; in most cases it did not materially affect the results obtained.

(c) The Shift Method

When computing eigenvalues, the rate of convergence to a particular eigenvalue can be improved by shifting the eigenvalues to the LH complex plane. This method is well known (Hamming [1962], Wilkinson [1965]) and indeed, it is used by Francis in his QR algorithm, described earlier. Francis' method was not so successful in some cases of this study however. The "shift method", applied to the helicopter/load problem was based on arbitrary shifts (which were not close to any real part of the eigenvalues) to reduce their rate of convergence. By reducing the rate of convergence, the eigenvalues and eigenvectors can be computed by using fewer iterations, hence reducing the accumulation of round-off errors, which, for high order systems can affect considerably the accuracy of the smallest eigenvalues.

The rate of convergence can be expressed as:

$$r_c = \frac{|\lambda_m^2|}{|\lambda_m^1|}$$

(7.14)
where $\lambda_{m_1}$ is the largest eigenvalue and $\lambda_{m_2}$ is the second largest eigenvalue.

If the convergence is fast then, a shift value, $a$, can be applied such that

$$r_c = \frac{|\lambda_{m_2} - a|}{|\lambda_{m_1} - a|} < r_c$$  \hspace{1cm} (7.15)

The eigenvalues and eigenvectors of the canonical matrix, $M$, must satisfy the equation

$$M\mathbf{u}_i = \lambda_i \mathbf{u}_i$$  \hspace{1cm} (7.16)

where $\mathbf{u}_i$ is the $i^{th}$ eigenvector and $\lambda_i$ its corresponding eigenvalue.

The LHS of equation (7.16) is compared with the RHS of the same equation, for each eigenvalue, $\lambda_i$.

It was found that equation (7.16) was satisfied for all $\lambda_i$, apart from the conjugate pair that was close to zero.

The eigenvalues of the canonical matrix can be shifted to either the LH or RH plane by $|a|$, by forming a new matrix $[M + aI]$. The eigenvalues of this new matrix are related by (7.17) to those of the original matrix, $M$, viz.

$$\lambda^*_i = \lambda_i + a$$  \hspace{1cm} (7.17)

where $\lambda^*_i$ represents the eigenvalues of the new matrix. The real parts of $\lambda^*_i$ are shifted by $a$, whereas their imaginary parts remain unaltered and correspondingly the eigenvectors associated with $\lambda^*_i$. By shifting the eigenvalues, even the very small roots can be made comparatively larger depending upon the choice of $a$. Thus, the eigen-
values of interest can be calculated with greater accuracy and can result in the separation of the near-zero eigenvalues of the actual system from those of its adjoint system.

It was found that this method was relatively efficient provided that the shift value was chosen so that its absolute value was different from the absolute value of the real part of any of the eigenvalues of \( M \), otherwise the eigenvalues, \( \lambda^* \), were close to zero, in which case, they cannot be computed accurately.

The eigenvalues, \( \lambda_i \), of the canonical matrix, \( M \), can be found by rearranging equation (7.17), so that the shift value, \( a \), is subtracted from the real parts of the eigenvalues, \( \lambda^* \), of the matrix \([M+aI]\). Since the near-zero eigenvalues have been separated into positive and negative eigenvalues, corresponding to the actual system and its adjoint, then the LHS and RHS of equation (7.16) can be calculated. It was found that the equation was valid, showing that both the eigenvalues and eigenvectors had been calculated accurately.

Once an equal number of positive and negative eigenvalues was obtained, Potter's method for solving the ARE was continued as described earlier by selecting the eigenvectors corresponding to the eigenvalues with negative real parts and by then inverting the complex modal matrix to obtain the solution, \( K \), of the ARE, viz.

\[
K = u_{21}u_{11}^{-1}
\]  

(7.18)

It is worth emphasising that although the position of the eigenvalues in the eigenvalue array can be changed with the shift value (as shown

\[^+\text{See also Appendix I.}\]
in Table 7.4) the solution of the ARE is not affected as long as the correspondence between each eigenvalue and eigenvector is not destroyed.

This fact can be proved as follows:

Let P be a permutation matrix; such matrices are always non-singular, then the following equations exist:

\[
\tilde{U}_{21} = U_{21}P \\
\tilde{U}_{11} = U_{11}P
\]  

(7.19)

where \( U_{21}, U_{11} \) are sub-matrices obtained by partitioning the modal matrix, \( U \),

\( \tilde{U}_{21}, \tilde{U}_{11} \) are sub-matrices of the modal matrix, \( \tilde{U} \), obtained when the shift method is applied.

They correspond to the different ordering of eigenvalues described by P.

If the eigenvalues have been ordered in exactly the same way, then P would have been an identity matrix. If the eigenvalues are ordered differently, then, assuming that

\[
U = [u_1, u_2, u_3, \ldots, u_n] \quad (7.20)
\]

and

\[
\tilde{U} = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \ldots, \tilde{u}_n] = [u_2, u_1, u_3, \ldots, u_n] \quad (7.21)
\]

where \( u_1 \) represents the eigenvectors.

The matrix, \( P \), becomes

\[
P = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\end{bmatrix}
\]  

(7.22)
It was shown in Chapter 4 that the solution of the ARE is

\[ K = U_{21} U_{11}^{-1} \]  
(7.23)

or when the shift method is applied,

\[ \tilde{K} = \tilde{U}_{21} \tilde{U}_{11}^{-1} \]  
(7.24)

By substituting (7.19) in (7.24),

\[ \tilde{K} = (U_{21} P) (U_{11} P)^{-1} \]  
(7.25)

from which

\[ \tilde{K} = U_{21} PP^{-1} U_{11}^{-1} \]  
(7.26)

or

\[ \tilde{K} = U_{21} U_{11}^{-1} \]

\[ = K \]  
(7.27)

Thus, it has been demonstrated that the solution of the ARE is the same irrespective of the ordering of the eigenvalues, as long as the eigenvalue/eigenvector correspondence is not changed.

By using the 'eigenvalue shift method' it was found that convergence could be achieved using either positive or negative shift values, that is, by shifting the eigenvalues to either the left or right hand complex plane (see Table 7.8) because of the existence of equal positive and negative eigenvalues. Table 7.8 shows the closed-loop eigenvalues for OLYMPUS with a 3 cable suspension arrangement, but without actuator dynamics included. These were obtained from the program OUTREG.

From Table 7.8 the two eigenvalues with the largest real parts are

\[ \lambda_1 = 92.1784 \quad \text{and} \quad \lambda_2 = 15.0974 \]  
(7.28)

The rate of convergence achieved is given by the ratio

\[ r_{cp} = \frac{|\lambda_2|}{|\lambda_1|} = \frac{15.0974}{92.1784} = 0.1638 \]  
(7.29)
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<td>-1.718007±10±j0.6961732±j0.2515</td>
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N.B. The shift is subtracted from the eigenvalues.

**TABLE 7.8:** Shifting eigenvalues to either the left or the right half plane can cause separation of the near zero complex pair as long as the magnitude of the shift is not close to the absolute value of the real part of any of the eigenvalues.
The two smallest eigenvalues in the table are:
\[ \lambda_{21} = -15.0974 \text{ and } \lambda_{22} = -92.1784 \]  
(7.30)

The rate of convergence is the same, i.e.,
\[ r_c = \frac{|\lambda_{21}|}{|\lambda_{22}|} = \frac{|-15.0974|}{|92.1784|} = 0.1638 = r_p \]  
(7.31)

Shifting the eigenvalues by -10 to the left hand half plane causes
\[ r_c \] to decrease and \[ r_p \] to increase, that is,
\[ r_c = \frac{|\lambda_{21} - 10|}{|\lambda_{1} - 10|} = \frac{|5.0974|}{|82.1784|} = 0.062 \]  
(7.32)
\[ r_p = \frac{|\lambda_{22} - 10|}{|\lambda_{22} - 10|} = \frac{|-92.1784|}{|92.1784|} = 1 \]  
(7.33)

Shifting the eigenvalues by 10 to the right hand half plane causes
\[ r_c \] to increase and \[ r_p \] to decrease, that is,
\[ r_c = \frac{|\lambda_{21} + 10|}{|\lambda_{1} + 10|} = \frac{|25.0974|}{|102.1784|} = 0.2456 \]  
(7.34)
\[ r_p = \frac{|\lambda_{22} + 10|}{|\lambda_{22} + 10|} = \frac{|102.1784|}{|102.1784|} = 1 \]  
(7.35)

Thus, the overall reduction of the rate of convergence remains the same irrespective of the sign of the shift.

The same table shows that when the shift value was chosen to be -3, the complex conjugate pair which is close to zero was not determined accurately, since the eigenvalue 3.1993 ± 0.0852 becomes very small when shifted by -3. It is also worth noting that the error with which the smallest eigenvalues were computed; the shifted eigenvalues
can be obtained accurately to at least five decimal places.

Thus, the advantage of the shift method is that it is a quick and effective solution of this numerical problem without additional loss of accuracy. The method is particularly useful if there is more than one root close to zero and if they are not positioned at the end of the eigenvalue array.

7.4.2 CLASS II Methods

(a) Laub's Method for Solving the ARE

Laub's method (Laub [1979]) appeared recently in the literature. It is a variant of the classical eigenvector method of solving the ARE (Table 7.9) using an appropriate set of Schur vectors instead of the eigenvectors. Since the eigenvector approach is often unsatisfactory from a numerical point of view Laub's method was attractive since it uses the so-called Schur vectors. This method is intended primarily for the solution of dense, moderate-sized equations rather than large, sparse equations. In classical similarity theory, unitary transformations are important, since, when

$$T^H = T^{-1}$$  \hspace{1cm} (7.36)

then

$$S = T^{-1}MT$$  \hspace{1cm} (7.37)

and hence matrices which are unitarily similar are also conjunctive (Golub and Wilkinson [1976]). The fundamental result with respect to unitary similarities is that for any matrix, \( M \), there exists a unitary
<table>
<thead>
<tr>
<th>STEPS</th>
<th>METHOD</th>
<th>POTTER'S</th>
<th>LAUB'S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form the canonical matrix, M</td>
<td>Yes</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Balancing</td>
<td>$M_B = D^{-1}P^TMD$</td>
<td>EISPACK Routine: BALANK</td>
<td></td>
</tr>
<tr>
<td></td>
<td>where $M_B$ is the balanced matrix</td>
<td>Orthogonal similarity transformations are used</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$P$ is a permutation matrix</td>
<td>EISPACK Routines: ORTHES, ORTRAN</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$D$ is a diagonal</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Exact similarity transformations are used</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>NAG Routine: F01ATA/F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Form the upper Hessenberg form and accumulate the transformations</td>
<td>$H=Q^{-1}M_Q$</td>
<td>$H=Q^TM_Q$, $Q^T=Q^{-1}$</td>
<td></td>
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<tr>
<td></td>
<td>Stabilised elementary similarity transformations are used</td>
<td>Orthogonal similarity transformations are used</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NAG Routines: F01CKA/F, F01APA/F</td>
<td>EISPACK Routines: ORTHES, ORTRAN</td>
<td></td>
</tr>
<tr>
<td>Find eigenvalues</td>
<td>The eigenvalues and eigenvectors were obtained by using Francis' QR algorithm</td>
<td>The eigenvalues are obtained by Francis' QR algorithm</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NAG Routine: F02AQA/F</td>
<td>The real upper Hessenberg matrix, $M$, is reduced to a quasi-triangular matrix, $S$, by unitary similarity transformations,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$H^T = T^{-1}$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>to obtain a real Schur form, $S$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$S = T^HT = T^HQ_MQ^T$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>where $Q^T = V^{-1}M_BV$</td>
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<tr>
<td></td>
<td></td>
<td>EISPACK Routine: HQR3</td>
<td></td>
</tr>
<tr>
<td>Back-Balance</td>
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<td>Back balancing to find the Schur vectors of the original matrix, $M$. $U = PDT$.</td>
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</tr>
<tr>
<td></td>
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<td>EISPACK Routine: BALBAK</td>
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<tr>
<td>Form the solution of the ARE</td>
<td>$\kappa = U^{-1}U^{-1}$</td>
<td>Yes</td>
<td>Yes</td>
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**TABLE 7.9:** The Characteristic Steps of Potter's and Lamb's Methods for Solving the ARE
matrix, $T$, such that

$$T^HMT = S \quad (7.38)$$

where $S$ is upper triangular, with the eigenvalues of $S$ on its diagonal. This is known as the Schur canonical form. The ordering of the eigenvalues, $\lambda_i$, on the diagonal may be chosen arbitrarily.

Unitary transformations are of great significance because there is a wide range of numerically stable algorithms based upon them. When $M$ is real it may, in general, have some complex conjugate eigenvalues, and there is a single modification of Schur's result which states that when $M$ is real there is an orthogonal transformation matrix, $T$, (i.e. a real unitary $T$) such that

$$T^AT = S$$

where $S$ is almost triangular except that corresponding to each complex conjugate pair of eigenvalues $S$ has a $2 \times 2$ block on its diagonal having this complex pair as its eigenvalues. This form is referred to as the Wintree-Murnaghan canonical form (Golub and Wilkinson [1976]). It is precisely this form which is provided by the double Francis QR algorithm.

Laub's method can be performed by using routines from the EISPACK Guide (Smith et al [1976]) and Algorithm 506 (Stewart [1975]) from the collected algorithms of the A.C.M. (Laub [1979]).

The canonical matrix, $M$, is firstly balanced by using subroutine BALANK (which is similar to the NAG routine F01ATA). The upper-Hessenberg form can be obtained by using orthogonal transformations (subroutine ORTHES) which are accumulated in a matrix (subroutine ORTRAN).
Stewart has published Fortran subroutines for calculating and ordering the real Schur form, $S$, of a real upper Hessenberg matrix. Algorithm 506 has two main subroutines: HQR3 and EXCHNG. HQR3 is a Fortran subroutine which reduces a real upper Hessenberg matrix $A$ to quasi-triangular form $B$ by a unitary similarity transformation $T$.

$$S = T^HT$$

(7.39)

The diagonal of $S$ consists of $1\times1$ and $2\times2$ blocks as illustrated below:

$$S = \begin{bmatrix}
    x & x & x & x & x & x \\
    0 & x & x & x & x & x \\
    0 & x & x & x & x & x \\
    0 & 0 & 0 & x & x \\
    0 & 0 & 0 & 0 & x \\
    0 & 0 & 0 & 0 & x & x 
\end{bmatrix}$$

(7.40)

The $1\times1$ blocks contain the real eigenvalues of $H$ and the $2\times2$ blocks contain the complex eigenvalues, a conjugate pair to each block. The blocks are ordered so that the eigenvalues appear in descending order of absolute value along the diagonal. The transformation, $T$, is postmultiplied into the array, $V$, obtained from ORTRAN, which contains the accumulation of the transformations. The decomposition produced by HQR3 ensures that the eigenvalues of the final quasi-triangular matrix are ordered. This ordering makes the decomposition essentially unique which is important in this research. It should also be noted that when the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $M$ are ordered so that they appear in descending order of magnitude along the diagonal, that is $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ and if $|\lambda_1| > |\lambda_{i+1}|$ then the first $i$ columns of $T$ form an orthonormal basis for the invariant subspace corresponding to $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_i$. HQR3 requires the subroutines EXCHNG, SPLIT and QRSTEP.
EXCHNG is a fortran subroutine to interchange consecutive 1x1 and 2x2 blocks of an upper Hessenberg matrix. EXCHNG produces a unitary (orthogonal) similarity transformation, W, such that \( W^T \) has consecutive blocks of order 2, and 1, which have been exchanged along with their eigenvalues. The transformation W is postmultiplied into the matrix V. EXCHNG can be used to rearrange the blocks of the quasi-triangular matrix produced by HQR3.

The QR iterations are performed by the subroutine QRSTEP and the real eigenvalues of a 2x2 block are separated by the subroutine SPLIT. In other words SPLIT determines if the corresponding eigenvalues are real or complex.

The reduction to RSF is an intermediate step in computing eigenvectors (when using the double Francis QR algorithm) so it is maintained by Laub that the Schur approach is faster, by definition. Moreover, Laub claims that the main stumbling block with using Schur vectors was the ordering problem with the RSF but once that can be handled satisfactorily the algorithm is relatively easy to implement. The Schur vector approach derives its desirable numerical properties from the underlying QR-type process.

It is claimed that important advantages of the Schur vector approach, in addition to its reliability for engineering applications, is its speed in comparison with other methods.

The advantage over iterative methods in speed is even more significant which divergence may occur if a poorly conditioned system or a bad starting value is used. A possible disadvantage of the method
is the high computer storage requirements. Another disadvantage is the fact that for the computed solution of the ARE, \( K = T_{21}^{-1} T_{11} \), there is no guarantee of symmetry. In practice, Laub found that the deviation from symmetry was only slight for most problems but became more pronounced only when the Riccati equation was known to be 'ill-conditioned'. This phenomenon might be used advantageously if the deviation from perfect symmetry could be used to monitor conditioning: the more asymmetric matrix \( K \) became the higher the ill-conditioning of the ARE. All the computing reported by Laub was carried out on an IBM 370/168 at M.I.T. using Fortran H Extended (option=2) with double precision arithmetic. This means that the results were computed with a precision of 64 significant digits (Bartee [1977]).

The overall process is claimed to be quite stable numerically but no proof of that claim is given by Laub. No analytical results concerning the conditioning of the Riccati equation have been found yet.

By using Laub's approach it was hoped to obtain some of the claimed numerical advantages, including numerical stability and reliable performance, when relatively high order models were used. However it was found that no material advantage occurred and that Laub's method was just as prone to the same numerical difficulties. Moreover, the claimed efficiency of the method was not established for this work as may be seen from Table 7.10. It is considered that the discrepancy in performance occurred because of the computational superiority of the Algol compiler over the Fortran version used with the 1904S and because it is likely that the matrix handling routines employed

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†Since Laub's method is machine dependent, the main reason for the discrepancy in performance was attributed to the insufficient accuracy (up to 20 significant digits when double precision is used) allowed by the ICL 1904S computer.
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<tr>
<th>METHOD</th>
<th>NAME OF COMPUTER PROGRAM</th>
<th>COMPUTER LANGUAGE</th>
<th>ORDER OF SYSTEM</th>
<th>JOB TIME (mill units)</th>
<th>NUMBER OF ITERATIONS TO CONVERGE</th>
<th>REQUIRED ACCURACY</th>
<th>COMMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Potters</td>
<td>'OUTREG'</td>
<td>ALGOL 60</td>
<td>24</td>
<td>900</td>
<td>-</td>
<td>Single</td>
<td>1. It requires the computation of the eigenvalues and eigenvectors of a canonical matrix</td>
</tr>
<tr>
<td></td>
<td>'OUTREGF'</td>
<td>FORTRAN IV</td>
<td>24</td>
<td>800</td>
<td>-</td>
<td>Precision</td>
<td>2. It computes the optimal feedback gains</td>
</tr>
<tr>
<td>Laub’s</td>
<td>'AREL'</td>
<td>FORTRAN IV</td>
<td>24</td>
<td>1600</td>
<td>-</td>
<td>Double Precision</td>
<td>1. It requires the computation of eigenvalues and Schur vectors</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2. It computes the optimal feedback gains</td>
</tr>
<tr>
<td>Kleinman’s</td>
<td>'KLEIMAN'</td>
<td>ALGOL 60</td>
<td>24</td>
<td>2000</td>
<td>40</td>
<td>0.0001</td>
<td>1. An initial estimate of the feedback gains is required</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2. It does not involve the solution of the eigenvalue problem</td>
</tr>
<tr>
<td>Athans &amp; Falb’s</td>
<td>'ATHFALB'</td>
<td>ALGOL 60</td>
<td>24</td>
<td>4100</td>
<td>51</td>
<td>-0.01</td>
<td>1. It can be used in conjunction with HAMRIC which can produce an initial estimate of the solution of the ARE or MRE</td>
</tr>
<tr>
<td>Mehmng’s</td>
<td>'HAMRIC'</td>
<td>ALGOL 60</td>
<td>24</td>
<td>10000</td>
<td>24</td>
<td>-0.01</td>
<td>1. Solution of the matrix Riccati equation by using Runge-Kutta integration for first 3 steps only</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2. It computes the solution of the Riccati equation only</td>
</tr>
</tbody>
</table>

**TABLE 7.10:** Comparison of Computing Aspects Concerning Methods for Solving the Riccati Equation
(being those provided by NAG) were more efficient than the corresponding EISPACK routines used by Laub. However, as for Potter's method it was found that the numerical problems associated with near-zero eigenvalues can be avoided by using Laub's methods, and applying the 'shift' method in a similar manner to the one described earlier.

Holley and Wei developed independently and simultaneously a similar method to Laub's for solving the ARE (Holley and Wei [1980]). All the computing reported by Holley and Wei was carried out on a CDC Cyber 70/73 which has comparable accuracy to that of the IBM 370/168. Although it has been suggested (Sandell [1980]) that their method is less rigorous than that of Laub [1979], Holley [1980] believes that the difference between his method and that of Laub is merely in the technique for obtaining the ordered quasi-triangular (Schur) decomposition: Holley and Wei compute the ordered decomposition directly from the original Hamiltonian matrix with previously computed eigenvalues, rather than by using the reordering scheme employed in the HQR3 program by Stewart [1976] which was used by Laub. It is claimed by Holley and Wei that their procedure usually results in fewer orthogonal similarity transformations being performed thus improving the numerical stability and round-off error properties of the algorithm. Some discussion of the operation counts and a quantification of the round-off properties of the method is given in Wei's doctoral thesis of June, 1981.

† 64 significant digits when double precision is used
(b) **Iterative Methods for Solving a Riccati Equation**

The iterative methods are useful for solving a large number of equations which contain a high proportion of zero elements. They are very economical in computer storage since they require only the space of the original coefficients. They are also very easy to program. But they are not very suitable for ill-conditioned equations and can evaluate the solution only to within a degree of accuracy, which is less than needed to solve the problem of concern here. Also, they can, in some cases, diverge from the solution.

One such method was examined: Kleinman's [1968]. Two direct integration methods were also examined, one based on Euler's method (Athans and Falb's [1966]) and the integration algorithm proposed by Hamming [1959]. Figure 7.5 shows a flow chart representing Kleinman's iterative method for solving the ARE. Of the two integrating methods that of Athans and Falb is the faster, but is less accurate. The superiority of Kleinman's method over all iterative solutions of the ARE is shown in Table 7.10. In the same table it can also be seen that Potter's method is by far the fastest method for solving the ARE. Although the iterative methods are free of numerical problems, their inability to guarantee convergence and their excessive computational requirements make them very unattractive to a user.
From all the methods described in this section the following points can be concluded:

- The prescribed degree of stability is always an efficient method for as long as $|\sigma|$ is chosen to be very small.
- The shift method was also effective in most cases examined.
- Laub's method is claimed to be very efficient but it is machine dependent which makes it effective only when a very accurate computer is used.
- All iterative methods require a lot of time for computing the solution of the ARE which does not make them attractive to the user.
FIGURE 7.5: Flow Chart Representing Kleinman's Iterative Method for the Solution of the ARE
CHAPTER 8

CONCLUSIONS
Although helicopters are inherently less safe than fixed wing aircraft, helicopter accidents are not occasioned by catastrophic mechanical failure, but are due most commonly to pilot error. The frequent occurrence of pilot error and the many difficult problems which occur while flying a helicopter, are indicative of the much greater workload imposed upon the pilot compared to, say, a pilot flying a fixed-wing aircraft. (Durisch [1981]) 

Pilots have suggested that the workload may be the true causal factor of the many accidents. Although a helicopter is inherently an unstable inertial platform it is not an overly-difficult system to fly in calm weather conditions; it is the capacity of the pilot to deal simultaneously with the many acute handling problems which arise in less perfect weather that is the problem.

If a helicopter carries an external load suspended by cables, the workload of the pilot is increased even further because the attendant oscillatory behaviour of the suspended load causes some destabilisation of the helicopter/load system itself.

Positioning a load while the helicopter remains at hover in the presence of atmospheric turbulence is a task which pilots cannot yet adequately meet. A few methods for stabilising a hovering helicopter with a suspended load have been proposed in the literature (Chapter 1). The most outstanding methods concern the use of an Automatic Hovering Control system, the design of which is based on Optimal Control Theory. There are difficulties which arise from the use of mathematical models of high-order and since the costs of computing associated with such high

\[\text{\textsuperscript{1}}\text{DURISCH P., 'Helicopters are five times more dangerous',}\]
\[The\ Observer, Sunday 23rd August 1981, p.3.\]
order models are high, engineers prefer to use, when appropriate, low order mathematical models, by making appropriate approximations and to concentrate their efforts in controlling what they consider to be the important state variables of the helicopter/load system. In essence, in such an approach, they are trading-off computing time on the ground (by using low order and therefore less accurate models for the derivation of an optimal control law) with computing time on board the helicopter (by using several separate channels to stabilise the helicopter/load system).

Precise control of a helicopter/load system at hover is feasible only if exact knowledge of the states of the system can be obtained. Thus the most accurate model of the helicopter at this point in the flight envelope is essential before beginning to determine the control laws.

The accurate models, GIONA and OLYMPUS, include displacements, velocities, attitude angles and angular rates of the helicopter itself and of its load in addition to those dynamic variables which represent the actuator of the helicopter controls (Chapter 2 and Appendix C).

Simulation of the atmospheric conditions encountered by a helicopter at hover is not well documented, such lack being due probably to the absence of precise information about how the combinations of the downwash created by the rotor and the atmospheric turbulence affect the airflow around the hovering helicopter and its suspended load. Since there was no established mathematical model of atmospheric turbulence for this particular phase of flight, a simplified model was used perforce, (Chapter 3).

A choice of an appropriate suspension arrangement, cable length,
load mass and coordinates of the attachment points for a particular flight condition can be obtained by using the mathematical model OLYMPUS. It was found that:

- the heavier the slung load, the higher the frequency of some oscillatory modes of motion of the helicopter/load system, (Section 5.4)
- the use of a single cable suspension arrangement makes control of the system very difficult (Sections 5.5 and 6.2)
- the use of two parallel cables for load suspension with the attachment points displaced along the longitudinal axis, considerably improves the response of the system compared to a single cable arrangement (Sections 5.5 and 6.2)
- the use of three parallel cables does not much improve the response compared to a two cable arrangement (Sections 5.5 and 6.2)
- the use of a winch control improves considerably the performance of the system only when a single cable suspension is used. For two or three cable suspension arrangements there is only a slight improvement of performance (Section 5.5)
- the response of the helicopter/load system which employs two-parallel cables and a winch control (SDA 510) was very similar to that obtained when three parallel cables without a winch control (SDA 419) were used (Chapter 5)
- When a three parallel cable suspension is used the shorter the distance between the laterally displaced cables the more similar the response becomes to that obtained when using the two cable arrangement.
When the two parallel cable suspension is used, the shorter the distance between the cable attachment points the more similar the response becomes to that when a single cable suspension is used.

In the case of using cables which are not parallel to each other, the higher the spread distance between the helicopter/cable attachment points the lower the frequency of the oscillations (as reported by Prabhakar [1976]).

The above findings described in Section 5.5 can give helicopter pilots a preflight notion of the response that is expected by their helicopter with a suspended load and help them in choosing an appropriate suspension arrangement depending on the weather conditions under which flight had to take place. Also, depending on the particular mission, one has to consider whether it is worth in a particular situation to improve the performance of the system by using extra cables at the expense of increasing hook-up time of the load. As far as the use of a winch control is concerned one has to consider the costs involved in terms of installation and maintenance.

The mathematical models of the helicopter/load system were subjected to either deterministic or stochastic inputs. The disadvantage of using deterministic inputs is that if the initial disturbance does not correspond to a realistic situation (as it is usually the case due to insufficient knowledge about the response of the complex system) then the balancing of the high forces and moments which have been created as a result of the initial conditions is done by the forces and moments created by high initial control deflections. Thus, by introducing a sudden displacement
with the control deflections being zero (which is the trim value in this study) this upsets the stability of the system thus causing excessive control activity to regain stability of the system.

Two linear feedback controllers were designed:

1) an optimal complete (full) state feedback controller (Chapters 4 and 5) which is based on there being available perfect information about the state variables of both the helicopter and load. In this case the information was obtained by means of a Kalman filter. The use of Kalman filters as state estimators is well established in the aeronautical industry because of their efficiency but they have numerous disadvantages. The biggest is that the mathematical model of the helicopter dynamics has to be reconstructed within an airborne computer. Thus if an accurate high-order model is used, the associated computer hardware will become more complex (e.g. its memory size is increased). In terms of the associated software, more efficient integration routines must be used since both the time needed for computing the estimated state vector and the associated round-off errors increase considerably. The relationship between the computer cpu time per step of integration and the acceptable time interval required for updating information about the system from its sensors, must also be considered. The higher the order of the equations being simulated, the greater the cpu time, for the sampling intervals, during which information is obtained from the sensors are increased. Such an increase makes the possibility of an instability of the helicopter/load system more likely because the controller updates its information
about the system at larger intervals of time. The disadvantages mentioned previously suggest that the combination of Kalman filter/complete state feedback controller might not be the best choice for stabilising a helicopter with a suspended load since both the Kalman filter and state controller are based on accurate high-order models.

2) Using an output regulator, (Chapters 4 and 6) a proportional controller, removes the need for the mathematical model of the helicopter dynamics to be reconstructed and allows the AHCS to update its feedback information about the state variables at a higher rate. That increase results in a considerable reduction in the computing time required and a minimum amount of flight control system hardware and reduces the complexity of on-board computations. The output regulator was designed on the basis of the information provided by the outputs of the system. It was found, that the response of the system with a single cable suspension was compatible with MIL.SPEC.-F8330 when either the forward and lateral velocities of the helicopter were assumed to be measurable variables or a lateral displacement winch control was included in the system.

Some aspects of flight safety were considered (Chapter 6) when an output regulator was used to achieve the stability of the system. In particular, for the cases being considered, it was assumed that some sensor or helicopter control was inoperative. From the many responses considered it was established that the failure of one gyro or rate gyro caused a modest deviation in the optimal response of the system whereas failure of a velocity sensor had considerable
effect upon the situation particularly if the sensor was measuring the lateral velocity of the vehicle. Such a finding illustrates that one particular faulty feedback signal can affect all the feedback signals in the AHCS depending on the importance of the associated variable to the phase of flight considered.

Thus, the following conclusions can be drawn from the work described in this thesis:

- More accurate mathematical models of the helicopter/load system should be used in order to increase the accuracy of the designed feedback controller.
- Since accurate mathematical representations usually mean the use of high-order models, the on-board computations and complexity of the AHCS hardware have to be reduced by using proportional controllers such as an output regulator.
- In improving the accuracy of the model, the effects of the downwash created by the rotor on the suspended load and ground effects on the flight of the helicopter/load system have to be further investigated.
- For flying in severe weather conditions, the use of a two cable arrangement is recommended for improving the stability of the system without increasing considerably the hook-up time of the load.
- A lateral displacement winch control does not improve excessively the response of the system; thus the associated costs of fitting such a control on the helicopter should be considered.
The output regulator is very simple because it is only a proportional controller as a result of which on-board computer hardware and software is minimised. It is also a flexible controller in terms of the pilot being able to change the control law if the aircraft mission is changed.

Before considering the use of an output regulator on the helicopter it is necessary to study whether it qualifies as the appropriate solution to the helicopter/load positioning problem. This remains to be done by a feasibility study which will examine the practicality of the solution and economic, as well as some flight safety, aspects of using such a controller.


POLI, C. and CROMACK, D., "Dynamics of slung body using a single-point suspension system", Journal of Aircraft, 10(2), 1973, pp.80-86.


SOHNE, W., "Die Seitenstabilität eines geschleppten Flugzeuges", Deutches Ingenieur-Archiv. 21, 1953, pp.245-265.


TAYLOR, G.L., "The use of fin surface to stabilise a weight towed from an aeroplane", ARC Reports and Memoranda, No.184, 1915.


APPENDIX A
TECHNICAL INFORMATION AND DATA CONCERNING THE

Sikorsky Helicopter Type S-61

The S-61 is an amphibious all-weather transport and rescue helicopter, manufactured by Sikorsky in the U.S.A. It is also manufactured by Westland Helicopters Ltd. in Britain and by Agusta in Italy under licence from Sikorsky. A more powerful derivative of the S-61 series, the S-61D serves with the U.S. Navy as the SH-3D (Sea King), anti-submarine helicopter. The S-61 is powered by twin turboshaft engines. The main and tail rotors have five blades each. The main rotor is all metal, fully articulated and its blades are provided with an automatic powered folding system. Both turboshaft engines drive through free-wheel units and rotor brake to the main gearbox. The tail rotor shaft is driven through intermediate and tail gearboxes. Accessories are driven by power take-offs on tail-rotor shaft.

A.1 SYSTEMS

The S-61 has primary and auxiliary hydraulic systems for the flying controls, and a utility hydraulic system for the landing gear, and the winches for blade folding. The electrical system includes one 300A DC generator, two 20kVA, 115V AC generators and a 24V battery.

A.2 ELECTRONICS AND EQUIPMENT

The S-61 is provided with a Hamilton Standard Autostabilisation
equipment, a unit automatic transition into hover, and a Teledyne APN-130 Doppler Radar and Radar Altimeter. It also has provisions for a rescue hoist with 272Kg capacity and low-response cargo sling with an automatic, touch-down release, and the capability of supporting external loads of maximum mass 3630Kg.

A.3 PERFORMANCE FIGURES

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum level speed</td>
<td>267 km/h</td>
</tr>
<tr>
<td>Cruising speed for maximum range</td>
<td>219 km/h</td>
</tr>
<tr>
<td>Maximum rate of climb (at sea level)</td>
<td>670 m/s</td>
</tr>
<tr>
<td>Service ceiling</td>
<td>4480 m (14700 ft.)</td>
</tr>
<tr>
<td>Hovering ceiling (out of ground effect)</td>
<td>2500 m (8200 ft.)</td>
</tr>
<tr>
<td>Range with maximum fuel capacity (10% reserve)</td>
<td>1005 km</td>
</tr>
<tr>
<td>Engine speed</td>
<td>1957 rad/sec (18686 rpm)</td>
</tr>
<tr>
<td>Main rotor/Engine rpm ratio</td>
<td>1:93.43</td>
</tr>
<tr>
<td>Tail rotor/Engine rpm ratio</td>
<td>1:16.7</td>
</tr>
</tbody>
</table>

A.4 DIMENSIONS

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of main rotor</td>
<td>18.9 m</td>
</tr>
<tr>
<td>Main rotor blade chord</td>
<td>0.46 m</td>
</tr>
<tr>
<td>Diameter of tail rotor</td>
<td>3.23 m</td>
</tr>
<tr>
<td>Distance between rotor centres</td>
<td>11.10 m</td>
</tr>
<tr>
<td>Overall length</td>
<td>22.15 m</td>
</tr>
<tr>
<td>Overall height</td>
<td>5.13 m</td>
</tr>
</tbody>
</table>
A.5 AREAS

Main rotor blades (each): 4.14 m$^2$
Tail rotor blades (each): 0.22 m$^2$
Main rotor disc: 280.5 m$^2$
Tail rotor disc: 8.2 m$^2$
Horizontal stabiliser 1.86 m$^2$

A.6 MASSES

Mass empty: 4428 Kg
Normal take-off mass: 9300 Kg
Maximum take-off mass: 9750 Kg
<table>
<thead>
<tr>
<th>Helicopter type</th>
<th>S-61D (Sea King)</th>
<th>S-61A</th>
<th>Westland Sea King</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vehicle Mass</td>
<td>7000 Kg (15432 lbf)</td>
<td>7000 Kg (15432 lbf)</td>
<td>9295 Kg (2050 lbf)</td>
</tr>
<tr>
<td>c.g. location</td>
<td>0.9 m (36 in) above aircraft bottom under rotor centre line</td>
<td>0.9 m (36 in) above aircraft bottom under rotor centre line</td>
<td>1.55 m (61 in) above aircraft bottom under rotor centre line</td>
</tr>
<tr>
<td>Inertia Components</td>
<td>I_x = 20000 Kg m^2 (0.47452 X 10^6 lb-sec^2)</td>
<td>I_y = 20000 Kg m^2 (0.47452 X 10^6 lb-sec^2)</td>
<td>I_z = 21209 Kg m^2 (2.8946 X 10^6 lb-sec^2)</td>
</tr>
<tr>
<td>Rotor tip speed</td>
<td>(Assumed): 201 m/sec (660 ft/sec)</td>
<td>(Assumed): 201 m/sec (660 ft/sec)</td>
<td>201 m/sec (660 ft/sec)</td>
</tr>
<tr>
<td>Flight condition</td>
<td>Hover</td>
<td>Hover</td>
<td>Hover</td>
</tr>
</tbody>
</table>

**Comments:**
1. The derivatives are given in empirical units as found in the literature.
2. Force derivatives are normalised with the mass of the helicopter.
3. Moment derivatives are normalised with the inertia.
4. The velocity components are normalised with the steady state hovering velocity.

<table>
<thead>
<tr>
<th>Component</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>L</th>
<th>M</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>-0.016</td>
<td>-0.047</td>
<td>0</td>
<td>-0.0644 sec</td>
<td>1.61 sec</td>
<td>0</td>
</tr>
<tr>
<td>Y</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0047</td>
<td>-0.033</td>
<td>-0.0007</td>
</tr>
<tr>
<td>Z</td>
<td>-0.000011</td>
<td>0.00036</td>
<td>-0.3242</td>
<td>0.003671 sec</td>
<td>0.11592 sec</td>
<td>0</td>
</tr>
<tr>
<td>L</td>
<td>0.000405 sec^{-1}</td>
<td>-0.01425 sec^{-1}</td>
<td>-0.000253 sec^{-1}</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M</td>
<td>0.00405 sec^{-1}</td>
<td>-0.01425 sec^{-1}</td>
<td>-0.000253 sec^{-1}</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>N</td>
<td>0.000011</td>
<td>0.00036</td>
<td>-0.3242</td>
<td>0.003671 sec</td>
<td>0.11592 sec</td>
<td>0</td>
</tr>
</tbody>
</table>

**TABLE A1: Stability Derivatives of Subject Helicopter**
TECHNICAL INFORMATION AND DATA CONCERNING THE

STANDARD SIZE CARGO CONTAINER

B.1 AERODYNAMIC CONDITION PARAMETERS AND STABILITY DERIVATIVES

The standard size shipping container 6.10x2.44x2.44 m (20x8x8 ft) was used. All parameters and stability derivatives of concern correspond to the following aerodynamic condition:

- height - sea level
- trimmed angle of attack - $\alpha = 0^\circ$
- the container is uniformly loaded
- c.g. located at geometric centre of the box

\[ I_{xx} = \frac{1}{12} m_x (b^2 + a^2) \]
\[ I_{yy} = \frac{1}{12} m_y (a^2 + c^2) \]
\[ I_{zz} = \frac{1}{12} m_z (b^2 + c^2) \]

Since the container is symmetric the products of inertia are zero.

FIGURE B.1: Axes and Characteristic Dimensions of the Shipping Container
Table B.1 summarises the non-dimensional stability derivatives which were quoted in Poli and Cromack [1973] and also in Liu [1973].

<table>
<thead>
<tr>
<th>Flight condition:</th>
<th>Hover</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airspeed:</td>
<td>0.3 m/sec (1 ft/sec)</td>
</tr>
<tr>
<td>Altitude:</td>
<td>Sea level</td>
</tr>
<tr>
<td>Air density, $\rho_a$:</td>
<td>$7.647 \times 10^{-2} \text{ lb/ft}^3$</td>
</tr>
<tr>
<td>Characteristic area, $s$:</td>
<td>160 ft$^2$</td>
</tr>
<tr>
<td>Width, $b$:</td>
<td>8 ft</td>
</tr>
<tr>
<td>Characteristic length, $c$:</td>
<td>20 ft</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Longitudinal</th>
<th>Lateral</th>
</tr>
</thead>
<tbody>
<tr>
<td>($\text{Body fixed axes}$)</td>
<td></td>
</tr>
<tr>
<td>$C_L = 0$</td>
<td>$C_{\alpha} = 1.833$</td>
</tr>
<tr>
<td>$C_D = 0.915$</td>
<td>$C_{\beta} = 0.6875$</td>
</tr>
<tr>
<td>$C_{\alpha} = 1.833$</td>
<td>$C_{\beta} = 0$</td>
</tr>
<tr>
<td>$C_{\alpha} = 1.289$</td>
<td>$C_{\beta} = 0$</td>
</tr>
<tr>
<td>$C_{\alpha} = -0.6875$</td>
<td>$C_{\alpha} = 0$</td>
</tr>
<tr>
<td>$C_{\alpha} = 0$</td>
<td>$C_{\alpha} = 0$</td>
</tr>
<tr>
<td>$C_{\alpha} = 0$</td>
<td>$C_{\alpha} = 0$</td>
</tr>
</tbody>
</table>

**TABLE B.1: Non-Dimensional Stability Derivatives**
Longitudinal

\[ X_u = \frac{\rho S U}{m_u C_{D_u}} (C_L + C_D) \]
\[ Z_u = -\frac{\rho S U}{m_u C_{L_u}} \]
\[ M_u = \frac{\rho S U c}{I_y} (C_m + C_m) \]
\[ X_w = \frac{\rho S U^2}{2m_L} (C_L - C_{D_a}) \]
\[ Z_w = -\frac{\rho S U^2}{2m_L} (C_L + C_{D_a}) \]
\[ Z_w = 0 \]
\[ M_w = \frac{\rho S U c}{2I_y} C_{m_a} \]
\[ M_w = 0 \]
\[ Z_q = -\frac{\rho S U c}{4m_L} C_L q \]
\[ M_q = \frac{\rho S U c^2}{4I_y} C_{m_q} \]

Lateral

\[ Y_v = \frac{\rho S U c}{2m_L} C_{y_{\beta}} \]
\[ I_v = \frac{\rho S U^2 b}{2I_x} C_{\beta} \]
\[ N_v = \frac{\rho S U^2 b}{2I_z} C_{n_{\beta}} \]
\[ Y_p = \frac{\rho S U b}{4m_L} C_{y_p} \]
\[ I_p = \frac{\rho S U^2 b}{4I_x} C_{\beta_p} \]
\[ N_p = \frac{\rho S U b}{2I_z} C_{n_{\beta}} \]
\[ Y_r = \frac{\rho S U b}{4m_L} C_{y_r} \]
\[ I_r = \frac{\rho S U^2 b}{4I_y} C_{\beta_r} \]
\[ N_r = \frac{\rho S U^2 b}{4I_z} C_{n_{r}} \]

*See McRuer, Ashkenas and Graham [1973] and Babister [1980].

TABLE B.2: Dimensional Stability Derivatives*
APPENDIX C
MATHEMATICAL MODELS OF HELICOPTER & LOAD

C.1 DEFINITION OF STATE VARIABLES

\[
\begin{bmatrix}
    x_v \\
    y_v \\
    z_v \\
    u_v \\
    v_v \\
    w_v \\
    \phi_v \\
    \theta_v \\
    \psi_v \\
    p_v \\
    q_v \\
    r_v \\
    x_1 \\
    y_1 \\
    z_1 \\
    u_1 \\
    v_1 \\
    w_1 \\
    \phi_1 \\
    \theta_1 \\
    \psi_1 \\
    p_1 \\
    q_1 \\
    r_1
\end{bmatrix}
\]

- State variables of Helicopter
  - absolute displacements
  - velocities
  - attitude angles
  - angular rates

- State variables of Load
  - absolute displacements
  - velocities
  - attitude angles
  - angular rates

C.2 DEFINITION OF HELICOPTER CONTROLS

Cyclic Controls

1. Longitudinal $\delta_L$
2. Lateral $\delta_R$

Collective Pitch Controls

1. Main rotor $\delta_C$
2. Tail rotor $\delta_T$

Load Control

Lateral displacement winch control: $\delta_w$
C.3 Mathematical Models

The equations of motion of a helicopter with a symmetrical suspended load are presented in this section. The equations were derived by Asseo [1971] and the complete derivation is given in Asseo and Erickson [1971]. The set of equations used in this study were based on the generalised equations quoted by Asseo and Whitbeck [1973]. The coupling terms of the longitudinal-lateral motions of the helicopter are included to account for the hovering flight conditions. Three different suspension arrangements were studied, involving either one, or two, or three parallel cables. The equations corresponding to the case using a single suspension cable are presented in Section C.3.1. The equations for the two-cable arrangement are identical to those for the single cable, except those corresponding to the pitch attitude of the vehicle and its load. These equations are quoted in Section C.3.2. For the three-cable suspension case the equations of motion are similar to those for the two-cable arrangement except those corresponding to the roll and yaw equations of the vehicle and its load.
C.3.1 Equations of Motion of a Helicopter with a Suspended Load Using a Single Cable

a. Defined Constants

\[
\begin{align*}
\rho_m &= \frac{m_L}{m_v} \\
\rho_v &= \frac{I_{xz_v}}{I_{x_v}I_{z_v}} \\
\epsilon_v &= \rho_v T_l (\eta_{1x}^2 + \eta_{1y}^2) \\
\epsilon_L &= 0 \\
d_v &= \frac{\epsilon_v}{\rho_v I_{z_v}} \\
b_v &= \frac{\xi_z m_L g}{I_{x_v}} \\
b_L &= \frac{\eta_z m_L g}{I_{x_L}} \\
a_v &= \xi_z \rho_v m_L g \\
c_v &= \frac{\xi_z m_L g}{I_{y_v}} \\
c_L &= \frac{\eta_z m_L g}{I_{y_L}} \\
\eta_c &= \text{number of cables; in this case it is equal to one.}
\end{align*}
\]

N.B.1: The coordinates \( \eta_{1x}, \eta_{1y} \) of the suspension point may not be zero, in undisturbed hovering flight, if the load centre of gravity is not exactly below the centre of gravity of the helicopter.

N.B.2: Since the products of inertia, \( I_{xz_L} \), of the load is zero, the corresponding load constants \( \rho_L, d_L, a_L \) become zero and are not included in the equations of motion.
FIGURE C.1: A S-61 With Cargo Container Suspended by a Single Cable

Cable Tension

\[ T_1 = m_2 g \]
b. **Equations of Longitudinal Motion**

\[
\ddot{x}_v = u_v \\
\ddot{u}_v = - \frac{g}{Z_0} x_v + x_v u_v + x_v w_v - [(1 + \rho_m g) + \frac{\xi_0}{Z_0}] \theta_v + x_v q_v + \frac{\rho_m g}{Z_0} x_v + \frac{\rho_m n}{Z_0} \theta_v + x_v v_v + x_v p_v + x_r r_v \\
+ x_v \delta_l + x_v \delta_c \\
\ddot{z}_v = w_v \\
\dot{w}_v = \frac{1}{1 + \rho_m} u_v + \frac{1}{1 + \rho_m} w_v + \left[ \frac{1}{1 + \rho_m} q_v + q_v \right] g_v + \frac{\rho_m}{Z_0} u_v + \frac{\rho_m}{Z_0} w_v + \frac{\rho_m}{Z_0} q_v + \\
+ \frac{1}{1 + \rho_m} v_v + \frac{1}{1 + \rho_m} p_v + \frac{1}{1 + \rho_m} r_v + \frac{1}{1 + \rho_m} \delta_l + \frac{1}{1 + \rho_m} \delta_c \\
\ddot{\theta}_v = q_v \\
\dot{q}_v = - \frac{C_v}{Z_0} x_v + M_u u_v + M_w w_v - \frac{C_v}{Z_0} \theta_v + M_q q_v + \frac{C_v}{Z_0} x_v + \frac{C_v}{Z_0} \theta_v + M_v v_v + M_p p_v + M_r r_v + \\
+ M_{\delta_l} + M_{\delta_c} \ldots \\
\ddot{z}_l = u_l
\]
\[
\dot{u}_l = \frac{q}{t} x_v + g \frac{\xi}{t} \dot{v}_v - \frac{q}{t} x_l + x u_l + x w_l - g \frac{\eta}{t} \dot{\theta}_l + x q'_l + x v_v + x p + x r_r \\
\dot{\theta}_l = w_l \\
\dot{v}_v = \frac{\gamma}{1 + \rho_m} \dot{u}_v + \frac{\gamma}{1 + \rho_m} \dot{w}_v + \frac{\gamma}{1 + \rho_m} \dot{q}_v + \frac{\gamma}{1 + \rho_m} \dot{u}_l + \frac{\gamma}{1 + \rho_m} \dot{w}_l + \frac{\gamma}{1 + \rho_m} \dot{q}_l + \frac{\gamma}{1 + \rho_m} \delta_c + \frac{\gamma}{1 + \rho_m} \delta_L \\
\dot{\theta}_l = q'_l \\
\dot{q}_l = \frac{c_l}{t} x_v + \frac{c_l}{t} x_l + M u_l + M w_l - \frac{c_l}{t} (t + \eta'_l) q_l + M v_v + M p + M r_r \\
\]
c. Equations of Lateral Motion

\[ \dot{y}_v = v_v \]  

\[ \dot{v}_v = \frac{\rho_m g}{\delta_0} y_v + \frac{\rho_m g}{\delta_0} \xi_z + (1 + \rho_m) g \phi_v + \frac{y_{v_v}}{p_v} + \frac{y_{r_0}}{r_v} + \frac{y_{r_0}}{r_v} - \frac{\xi_z}{\delta_0} y + \frac{\rho_m g}{\delta_0} \eta_z \phi_v + \]  

\[ + u_{v_v} + w_{v_v} + q_{v_v} + \delta_R + \delta_T + \xi_c \delta - \frac{m_{g} g}{\eta m_{\phi_v} \xi} \delta \]  

\[ \phi_v = \psi_v \]  

\[ \dot{p}_v = \frac{b_v}{\delta_0} y_v + \frac{I_{v_x}}{I_x v} N_v \phi_v - \frac{b_v}{\delta_0} (\delta_z + \xi_z) \phi_v + (L_v + \frac{I_{v_x}}{I_x v} N_v) \phi_v - \frac{\xi_z}{\delta_0} \psi_v + (L_v + \frac{I_{v_x}}{I_x v} N_v) r_v \]  

\[ - \frac{b_v}{\delta_0} y_v + \frac{b_v}{\delta_0} \phi_v + \frac{\xi_z}{\delta_0} \psi_v + (L_u + \frac{I_{v_x}}{I_x v} N_u) u_v + (L_w + \frac{I_{v_x}}{I_x v} N_w) w_v + (L_q + \frac{I_{v_x}}{I_x v} N_q) q_v + \]  

\[ + (L + \frac{I_{v_x}}{I_x v} N) \delta_r + (L + \frac{I_{v_x}}{I_x v} N) \delta_t + (L + \frac{I_{v_x}}{I_x v} N) \delta_c + \frac{m_{g} g}{\eta m_{\phi_v} \xi} (\frac{\xi_z}{\delta_0} - \rho_v \eta_{l_x} \xi) \delta \]  

\[ \psi_v = r_v \]  

\( \text{(C.13)} \)  

\( \text{(C.14)} \)  

\( \text{(C.15)} \)  

\( \text{(C.16)} \)  

\( \text{(C.17)} \)
\begin{align*}
\dot{v}_v &= \frac{a_v}{z_0} v_v + \frac{I_{xz v}}{z_0} L_v v_v - \frac{a_v}{z_0} (L_{c} + \xi z) \phi_v + (N + \frac{I_{xz v}}{z_0} L_v) p_v - \frac{d_v}{z_0} \psi_v + (N + \frac{I_{xz v}}{z_0} L_v) r_v - \frac{a_v}{z_0} y_v + \\
&\quad + \frac{a_v}{z_0} n_z \phi_v + \frac{d_v}{z_0} \psi_v + (N_u + \frac{I_{xz v}}{z_0} L_u) u_v + (N + \frac{I_{xz v}}{z_0} L_v) w_v + (N + \frac{I_{xz v}}{z_0} L_c) q_v + \\
&\quad + \frac{I_{xz v}}{z_0} (N + \frac{I_{xz v}}{z_0}) \delta_R + (N_T + \frac{I_{xz v}}{z_0} L_T) \delta_T + (N_c + \frac{I_{xz v}}{z_0} L_c) \delta_c - \frac{m_{g}}{n_{c} \delta} \left( \frac{n_{1x}}{z_0} - \rho_{v_x} \xi \right) \delta_w
\end{align*}

\begin{align*}
\dot{y}_v &= \frac{g}{z_0} v_v - \frac{g}{z_0} \xi \phi_v - v_O - \frac{g}{z_0} y_v + \frac{g}{z_0} \phi_v + \frac{g}{z_0} n_z \phi_v + y_p p_v + y_r r_v + y_u u_v + y_w w_v + y_q q_v - \\
&\quad - \frac{g}{z_0} \delta_w
\end{align*}

\begin{align*}
\dot{p}_v &= p_v
\end{align*}

\begin{align*}
\dot{\phi}_v &= \frac{b_v}{z_0} v_v + \frac{b_v}{z_0} \xi \phi_v + \frac{b_v}{z_0} y_v + L_v \phi_v - \frac{b_v}{z_0} (L_c + n_z) \phi_v + L_p p_v + L_r r_v + L_u u_v + L_w w_v + L_q q_v + \\
&\quad + \frac{m_{g}}{n_{c} \delta_{1x}} \left( \frac{n_{1x}}{z_0} - \rho_{v_x} \xi \right) \delta_w
\end{align*}

\begin{align*}
\dot{\psi}_v &= \psi_v
\end{align*}

\begin{align*}
\dot{r}_v &= \frac{d_v}{z_0} \psi_v + N_v v_v + N_p p_v - \frac{d_v}{z_0} \psi_v + N_r r_v + N_u u_v + N_w w_v + N_q q_v + \frac{m_{g}}{n_{c} \delta_{1x}} \left( \frac{n_{1x}}{z_0} - \rho_{v_x} \xi \right) \delta_w
\end{align*}
C.3.2 Equations of Motion of a Helicopter with a Suspended Load Using Two Parallel Cables

In this case, a similar set of equations apply as for the single cable suspension arrangement with equations (C.6) and (C.12) being replaced by (C.25) and (C.26).

a. Defined Constants

\[ \rho_m = \frac{m_R}{m_v} \]
\[ \rho_v = \frac{I_{xz_v}}{I_{x_v} I_{z_v}} \]
\[ \rho_{\xi} = 0 \]

\[ \epsilon_v = \rho_v \left[ \frac{m_R g}{2} \right] \]
\[ \epsilon_{\xi} = 0 \]

\[ T_1 = T_2 = \frac{m_R g}{2} \]
\[ \eta_{1x} = 3.05m(10 \text{ ft}) \]
\[ \eta_{2x} = -3.05m(-10 \text{ ft}) \]
\[ \eta_{1y} = \eta_{2y} = 0 \]

(N.B. The above data assumes that the c.g. of the container is below the helicopter c.g. and that the cables are displaced longitudinally, Figure C.2).

\[ d_v = \frac{\epsilon_v}{\rho_v I_{z_v}} \]
\[ d_{\xi} = 0 \]

\[ b_v = \frac{\xi_R m_R g}{I_{x_v}} \]
\[ b_{\xi} = \frac{\xi_R m_R g}{I_x} \]

\[ a_v = \xi_v \rho_v m_R g \]
\[ a_{\xi} = 0 \]

\[ c_v = \frac{\xi_R m_R g}{I_{y_v}} \]
\[ c_{\xi} = \frac{\eta_{1y} m_R g}{I_{y_{\xi}}} \]

\[ D_v = \frac{I_{z_v} I_{y_v}}{I_{y_v} I_{z_v}} \]
\[ D_{\xi} = \frac{I_{y_{\xi}}}{I_{y_v} I_{z_v}} \]
FIGURE C.2: A S-61 With a Cargo Container Suspended by Two Cables

Cable Tension

\[ T_1, T_2 = \frac{m_L g}{2} \]
b. Equations of Pitch Motion of the Helicopter and Load

\[
\dot{q}_V = \frac{(D_e - 1)C_v + C_v}{\ell_0} x_V + D_M u_V + D_M w_V - \frac{D_e C_v (\ell_e + \xi_e) + C_v (\ell_e + \xi_e (1 - D_e))}{\ell_0} \varepsilon_V + D_M q_V +
\]

\[
+ \frac{C_v (1 + D_e) + D_e C_v}{\ell_0} x_e + D_M u_e + D_M w_e + \frac{\eta_e D_e C_v \eta_e - D_e C_e (\ell_e + \eta_e)}{\ell_0} \theta_e + D_M q_e + D_M V_e +
\]

\[
+ D_M P_e + D_M r_V + D_M \delta_c + D_M \delta_c
\]

(C.25)

\[
\dot{q}_h = \frac{C_h (1 - D_e) - D_e C_h}{\ell_0} x_V + D_M u_V + D_M w_V + \frac{C_h (1 - D_e) - D_e C_h (\ell_e + \xi_e)}{\ell_0} \theta_V + D_M q_V -
\]

\[
- \frac{C_h (1 - D_e) - D_e C_h}{\ell_0} x_e + D_M u_e + D_M w_e - \frac{C_h (\ell_e + \eta_e) (\ell_e - 1) + C_h \eta_e}{\ell_0} \theta_e + D_M q_e + D_M V_e +
\]

\[
+ D_M P_e + D_M r_e + D_M \delta_c + D_M \delta_c
\]

(C.26)
C.3.3 Equations of Motion of a Helicopter with a Suspended Load Using Three Parallel Cables

When three parallel cables are used (Figure C.3), the equations are similar to those for the two cable suspension except for the equations corresponding to the vehicle and load roll and yaw motions, (C.16), (C.18), (C.22) and (C.24) which are replaced by equations (C.27), (C.28), (C.29) and (C.30) respectively.

a. Defined Constants

\[ \rho_v = \frac{m_R}{m_v} \]
\[ \rho_\ell = 0 \]

\[ \epsilon_v = \rho_v \left[ T_1 (n_{1x}^2 + n_{1y}^2) + T_2 (n_{2x}^2 + n_{2y}^2) + T_3 (n_{3x}^2 + n_{3y}^2) \right] \]
\[ \epsilon_\ell = 0 \]

\[ \tau_1 = \frac{m_\ell g}{2} \quad \tau_2 = \frac{m_\ell g}{4} \quad \tau_3 = \frac{m_\ell g}{4} \]

\[ n_{1x} = 3.05m(10ft) \quad n_{2x} = -3.05m(-10ft) \quad n_{3x} = -3.05m(-10ft) \]
\[ n_{1y} = 0 \quad n_{2y} = 1.22m(4ft) \quad n_{3y} = -1.22m(-4ft) \]

(N.B. For the data above it was assumed that the c.g. of the standard size container is below the c.g. of the helicopter).

\[ d_v = \frac{\epsilon_v}{\rho_v I_{xz_v}} \]
\[ d_\ell = 0 \]

\[ b_v = \frac{\xi z m_v g}{I_{x_v}} \quad b_\ell = \frac{n_{z_\ell m_\ell g}}{I_{x_\ell}} \]

\[ a_v = \frac{\xi z m_v g}{I_{y_v}} \quad a_\ell = 0 \]

\[ c_v = \frac{\xi z m_v g}{I_{y_v}} \quad c_\ell = \frac{n_{z_\ell m_\ell g}}{I_{y_\ell}} \]
FIGURE C.3: The S-61 With the Cargo Container Suspended by Three Cables
\[ D_V = \frac{I_{Y_V}}{I_{Y_V} + I_{Y_2}} \]
\[ D_2 = \frac{I_{Y_2}}{I_{Y_V} + I_{Y_2}} \]
\[ E_V = \frac{I_{X_V}}{I_{X_V} + I_{X_2}} \]
\[ E_2 = \frac{I_{X_2}}{I_{X_V} + I_{X_2}} \]
b. Equations of the Yaw and Roll Motions of the Helicopter and Load

\[
\dot{p}_v = \left( -\frac{b_v (1-E_z)b_v \ell_0}{\ell_0} \right) \psi_v + E_v \left( L_{\psi_v} + \frac{I_{xv}}{I_{xv}} N_{\psi_v} \right) \psi_v + \left( -\frac{b_v (1-E_z)b_v \ell_0}{\ell_0} \right) \phi_v + E_v \left( L_{\phi_v} + \frac{I_{xv}}{I_{xv}} N_{\phi_v} \right) \phi_v + \left( -\frac{b_v (1-E_z)b_v \ell_0}{\ell_0} \right) \theta_v
\]

\[
+ \frac{b_v (1-E_z)b_v \ell_0}{\ell_0} \phi_v\]

\[
+ \frac{E_v \left( L_{\psi_v} + \frac{I_{xv}}{I_{xv}} N_{\psi_v} \right) \psi_v + E_v \left( L_{\phi_v} + \frac{I_{xv}}{I_{xv}} N_{\phi_v} \right) \phi_v + E_v \left( L_{\theta_v} + \frac{I_{xv}}{I_{xv}} N_{\theta_v} \right) \theta_v + E_v \left( L_{\phi_v} + \frac{I_{xv}}{I_{xv}} N_{\phi_v} \right) \phi_v + E_v \left( L_{\theta_v} + \frac{I_{xv}}{I_{xv}} N_{\theta_v} \right) \theta_v + \frac{m_g \ell_0}{\ell_0} \phi_v + m_g \ell_0 \phi_v + \frac{m_g \ell_0}{\ell_0} \phi_v + m_g \ell_0 \phi_v
\]

(C.27)
\[
\dot{v} = \left(\frac{a_v}{k_0} - \rho_v x_v E_z \left(\frac{b_{\ell} + b}{k_0}\right)\right) y_v + \left[\left(N_v + \frac{I_{xz}}{x_v} \left(L_v + \frac{I_{xz}}{x_v} \right)\right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] v_v - \frac{a_v}{k_0} \left(\xi_0 + \xi_v\right) -
\]

\[
-\rho_v x_v E_z \left(\frac{b_{\ell} + b}{k_0}\right) \phi_v + \left[\left(N_v + \frac{I_{xz}}{x_v} \left(L_v + \frac{I_{xz}}{x_v} \right)\right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] p_v - \left[\frac{d_v}{k_0} - \rho_v x_v E_z \frac{e_v}{k_0}\right] \psi_v
\]

\[
+ \left[\left(N_v + \frac{I_{xz}}{x_v} \right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] r_v - \frac{a_v}{k_0} - \rho_v x_v E_z \left(\frac{b_{\ell} + b}{k_0}\right) \right] v_v + \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] v_v +
\]

\[
+ \left[\frac{a_v}{k_0} \eta_z - \rho_v x_v E_z \left(\frac{b_{\ell} + \eta_z}{k_0}\right) \right] \psi_v + \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] \psi_v +
\]

\[
+ \left[\left(N_v + \frac{I_{xz}}{x_v} \right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] u_v + \left[\left(N_v + \frac{I_{xz}}{x_v} \right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] w_v +
\]

\[
+ \left[\left(N_v + \frac{I_{xz}}{x_v} \right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] q_v + \left[\left(N_v + \frac{I_{xz}}{x_v} \right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] \delta_v
\]

\[
+ \left[\left(N_v + \frac{I_{xz}}{x_v} \right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] \delta_t + \left[\left(N_v + \frac{I_{xz}}{x_v} \right) - \rho_v x_v E_z \left(L_v + \frac{I_{xz}}{x_v} \right) \right] \delta_c
\]

\[
- \left[\frac{m_x}{\eta} \frac{\delta_v}{k_0} - \frac{\delta_v}{k_0} \right] \delta_v
\]

\[
(c.28)
\]
\[ \dot{p}_v = \left[ \frac{-b_v E_v (b_v + b_v)}{p_0} \right] y_v + E_v (L_{v v} + \frac{I_{v x}}{x_{v v}} N_{v v}) y_v + \left[ \frac{b_v E_v (b_v + b_v)}{p_0} - E_v \right] \left( \frac{b_v E_v (b_v + b_v)}{p_0} \right) \phi_v + E_v (L_{p v} + \frac{I_{v x}}{x_{v v}} N_{p v}) p_v \]

\[ -E_v \frac{e_v}{p_0} \psi_v + E_v (\frac{I_{v x}}{x_{v v}}) r_v + \left( \frac{b_v - E_v (b_v + b_v)}{p_0} \right) y_v + E_v (L_{v v} y_v) + \left[ \frac{b_v (\delta_0 + \eta_2)}{p_0} + E_v (b_v (\delta_0 + \eta_2) + b_v) \right] \phi_v \]

\[ + L_{p v} = p_v + \frac{e_v}{p_0} \psi_v + E_v (L_{x x} y_v) + E_v (L_{c c} + \frac{I_{v x}}{x_{v v}}) \delta_c + E_v (L_{v v} + \frac{I_{v x}}{x_{v v}}) \delta_v \]

\[ + \frac{m_z}{\eta_{c 0}} (\frac{\eta_{c 0}}{x_z}) - \rho_{z 2} \delta_v \]

(C.29)
\[
i_{v} = \left[ \frac{a_{z} + \rho_{z} I_{x} E_{z} (b_{z} + b_{v})}{I_{x} E_{z}} \right] y_{v} + \rho_{z} I_{x} E_{z} (L_{v} + \frac{I_{x} E_{z}}{I_{x} E_{z}} N_{v} ) y_{v} + \left[ \frac{a_{z} \xi_{z} - \rho_{z} I_{x} E_{z} (b_{z} \xi_{z} + b_{v} (\xi_{z} + \xi_{v}))}{I_{x} E_{z}} \right] \phi_{v} + \\
+ \frac{d_{z} - \rho_{z} I_{x} E_{z}}{I_{x} E_{z}} \psi_{v} + \rho_{z} I_{x} E_{z} (L_{v} + \frac{I_{x} E_{z}}{I_{x} E_{z}} N_{v} ) r_{v} + \left( \frac{d_{z} - \rho_{z} I_{x} E_{z}}{I_{x} E_{z}} \right) y_{v} \\
+ (N_{v} - \rho_{z} I_{x} E_{z} L_{v}) y_{v} - \left[ \frac{a_{z} (\eta_{z} + \xi_{0}) - \rho_{z} I_{x} E_{z} (b_{z} (\xi_{0} + \eta_{z}) + b_{v})}{I_{x} E_{z}} \right] \psi_{v} + (N_{v} - \rho_{z} I_{x} E_{z} L_{v}) p_{v} - \left( \frac{d_{z} - \rho_{z} I_{x} E_{z}}{I_{x} E_{z}} \right) \psi_{v} \\
+ (N_{v} - \rho_{z} I_{x} E_{z} L_{v}) r_{v} + \rho_{z} I_{x} E_{z} (L_{v} + \frac{I_{x} E_{z}}{I_{x} E_{z}} N_{v} ) \delta_{R} + \rho_{z} I_{x} E_{z} (L_{v} + \frac{I_{x} E_{z}}{I_{x} E_{z}} N_{v} ) \delta_{C} + \rho_{z} I_{x} E_{z} (L_{v} + \frac{I_{x} E_{z}}{I_{x} E_{z}} N_{v} ) \delta_{T} + \frac{m_{z} g}{\eta_{c} \xi_{z}} (\rho_{z} \eta_{z}) \delta_{w} \right]
\]

(C.30)
C.3.3

In each case it is assumed that the container is uniformly loaded (i.e., the centre of gravity of the load is at the geometric centre of the container). Since the container is a regular rectangular block its products of inertia must be zero (because of its symmetry). The winch control for lateral displacement, \( \delta_w \), is optional: it may not be used in some particular helicopters.

C.4 TRANSLATIONAL ACCELERATIONS OF THE HELICOPTER

\[
\begin{align*}
    a_x &= \dot{\theta} - r \dot{v} - q \omega \\
    a_y &= \dot{\theta} - p \omega + r \dot{u} \\
    a_z &= \dot{\theta} - q \dot{u} + p \dot{v}
\end{align*}
\]
**Extract From MIL.SPEC. - F8330: Class II Helicopters**

**Flight condition:** Hover  
**Atmospheric condition:** Calm air  
**Time history required:** 2 min.

The values listed below represent never-to-exceed limits:

- **Forward velocity:** 5 m/s  
- **Lateral velocity:** 2.5 m/s  
- **Vertical velocity:** 2.5 m/s  
- **Longitudinal displacement:** 6.5 m  
- **Lateral displacement:** 6.5 m  
- **Vertical displacement:** 2 m

**Vehicle attitude angles:** 0.0175 radians (1°)  
**Vehicle attitude angular rates:** 0.0875 rad/sec. (5°/sec.)  
**Roll angular acceleration:** 0.0523 rad/sec² (3°/sec²)

*If the helicopter response is within these limits its motion may be described satisfactorily by small perturbation equations.*
STATE AND CONTROL VARIABLE CONSTRAINTS FOR THE

HELICOPTER/LOAD SYSTEM

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<th>VARIABLE</th>
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*Values are based on 10% Control Authority*


Solution of the ARE by Eigenvector Decomposition

The canonical equation (4.19), viz.

\[ \dot{z}(t) = Mz(t) \]  

(4.19)

where the canonical matrix, \( M \), is given by

\[ M = \begin{bmatrix} A & -B \cdot G^{-1} B^T \\ -Q & -A^T \end{bmatrix} \]  

(4.20)

has an explicit solution, based on the conditions of asymptotic stability, which may be obtained in the form of the solution of two single-point boundary-value problems by means of using eigenvectors. The time response of the system described by (4.19) can be defined, for distinct eigenvalues, in terms of the eigenvector components of the matrix, \( M \), as:

\[ z(t) = U e^{\lambda \tau} U^{-1} z(0) \]  

(F.1)

where

\[ \tau = t - t_0 \]  

(F.2)

with \( t_0 \) being the initial time.

The modal matrix, \( U \), of order \( 2n \times 2n \), consists of the eigenvectors of the canonical matrix, \( M \), and \( e^{\lambda \tau} \) is a diagonal matrix, with elements \( \lambda_1 \tau, \lambda_2 \tau, \ldots, \lambda_n \tau \).

The corresponding equation relating to the eigenvectors is

\[ MU = UA \]  

(F.3)

The eigenvalues of the matrix \( M \), with negative real parts, correspond to the eigenvalues of the actual system and the ones with positive real parts correspond to its adjoint system. The corresponding eigenvectors
of the actual and adjoint systems have equal absolute values. Partitioning
the eigenvalues of $\Lambda$ into two sets of $n$ eigenvalues, so that
\[ \Lambda_1 = \{\lambda_i\} \text{ for } i=1,2,\ldots,n \text{ with } \lambda_i \text{ having negative real parts,} \]
and
\[ \Lambda_2 = \{\lambda_j\} \text{ for } j=n+1,n+2,\ldots,2n \text{ with } \lambda_j \text{ having positive real parts,} \]
results in equation (F.4), viz.
\[ \Lambda = \begin{bmatrix} \Lambda_1^T & 0 \\ e_1 & \Lambda_2^T \\ 0 & e_2 \end{bmatrix} \]  
(F.4)

Similarly, partitioning all the matrices and vectors of equation (F.1)
results in
\[ \begin{bmatrix} x \\ \psi \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1^T & 0 \\ 0 & \Lambda_2^T \end{bmatrix} \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ \psi_0 \end{bmatrix} \]  
(F.5)

where $\nu_{11}, \nu_{12}, \nu_{21}$ and $\nu_{22}$ represents the partitioned matrices of $\nu$, which is the inverse\(^+\) of the modal matrix, $U$, so that
\[ \nu U = [I] \]  
(F.6)

where $[I]$ is an identity matrix of order $2n \times 2n$.

Resolving equation (F.5), gives
\[ x(t) = U_{11} e^{\Lambda_1^T (\nu_{11} x_0 + \nu_{12} \psi_0)} + U_{12} e^{\Lambda_2^T (\nu_{21} x_0 + \nu_{22} \psi_0)} \]  
(F.7)

The divergent modes, which correspond to the eigenvalues with positive
real parts contained in $\Lambda_2$, must be eliminated to satisfy the conditions
of asymptotic stability. Therefore, from (F.7),

\^The modal matrix, $U$, usually consists of both real and complex eigenvectors. If the real and imaginary parts of the eigenvectors are separated within, $U$, so that $U = [U_R^T \ U_I^T]$, the complex matrix inversion by Lanczos [1957] may be used. If the real and imaginary parts of the eigenvectors are not separated within $U$, then the method by Glandorf [1981] may be used.
\[ V_{11}x_0 + V_{22}y_0 = 0 \]  \hspace{1cm} (F.8)

or

\[ y_0 = -V_{22}^{-1}V_{21}x_0 \]  \hspace{1cm} (F.9)

Also, (F.6) can be written in the matrix partition form as follows:

\[
\begin{bmatrix}
  V_{11} & V_{12} \\
  V_{21} & V_{22}
\end{bmatrix}
\begin{bmatrix}
  U_{11} & U_{12} \\
  U_{21} & U_{22}
\end{bmatrix}
= \begin{bmatrix}
  I_n & 0 \\
  0 & I_n
\end{bmatrix}
\]  \hspace{1cm} (F.10)

where \( I_n \) is an identity matrix of order \( n \),

and from which (F.11) can be obtained, viz.

\[ V_{21}U_{11} + V_{22}U_{21} = 0 \]  \hspace{1cm} (F.11)

or

\[ -V_{22}^{-1}V_{21} = U_{21}U_{11}^{-1} \]  \hspace{1cm} (F.12)

Equation (F.9) may now be written as

\[ y_0 = -V_{22}^{-1}V_{21}x_0 = U_{21}U_{11}^{-1}x_0 \]  \hspace{1cm} (F.13)

Hence, from (F.7),

\[ x(t) = U_{11}e^{(V_{11} - V_{12}V_{22}^{-1}V_{21})t}x_0 \]

\[ x(t) = U_{11}e^{A_1t}x_0 \]  \hspace{1cm} (F.14)

Similarly, from (F.5),

\[ y = U_{21}e^{-A_1^{-1}t}x_0 = U_{21}U_{11}^{-1}x_0 \]  \hspace{1cm} (F.15)

Equations (F.14) and (F.15) define the optimal solution as two single-point boundary-value problems in terms of the partitioned eigenvectors associated with the \( n \) stable modes of the canonical matrix, \( M \).

The optimal control law to be applied, for all time, \( t \), may now be obtained directly from (4.19) and (F.15). Thus,

\[ u^0 = -G^{-1}B^T_{21}U_{21}U_{11}^{-1}x \]  \hspace{1cm} (F.16)
A standard identification technique was followed through the investigation of the mathematical model OLYMPUS. The general title format of each case is shown below:

\[ X Y a b c \]

where

- **X** denotes the type of linear feedback control being used:
  - S denotes a complete state feedback controller and
  - O denotes an output regulator

- **Y** denotes whether the investigation is based on stochastic inputs to the system (S) or deterministic (D)

- **a** denotes the number of controls being used; it is 4 when only the helicopter controls are used and 5 when the lateral displacement winch control is also included

- **bc** denotes the case number which can be obtained from Table G.2.

**N.B.** If an 'A' appears between Y and a, it denotes that the actuator dynamics have been included in the mathematical model OLYMPUS.
### Mathematical Models Used in Research Investigation

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<tr>
<th>Mathematical Model</th>
<th>Order</th>
<th>Helicopter Type</th>
<th>Suspended Load</th>
<th>Rotor Dynamics</th>
<th>Actuator Dynamics</th>
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**TABLE G.1:** Mathematical Models Used in Research Investigation
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<th>Coordinates of the attachment points (m)*</th>
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</table>

* See also Appendix C.

\[ η_{2x} = η_{3x} \]
\[ η_{2y} = η_{3y} \]

**TABLE G.2:** Cases to be Examined with the Mathematical Models GIONA and OLYMPUS
STATE TRANSITION METHOD

H.1 SOLUTION OF THE STATE EQUATION WITH STOCHASTIC INPUTS BY THE TRANSITION MATRIX METHOD

The state equation given in Chapter 4 is,
\[ \dot{x} = Ax + Bu + G_g n_w \]  
(H.1)

Let a solution of \( x \) be given by
\[ x(t) = \varphi(t-t_o)C_1(t) ; \varphi(t_o) = I \]  
(H.2)

where
\[ \frac{d\varphi(t-t_o)}{dt} = A\varphi(t-t_o) \]  
(H.3)

Differentiating (H.2) and using (H.3) gives
\[ \dot{x}(t) = Ax(t) + \varphi(t-t_o)\dot{C}_1(t) \]  
(H.4)

Comparing (H.1) and (H.4)
\[ \varphi(t-t_o)\dot{C}_1(t) = Bu + G_g n_w \]  
(H.5)

\[ C_1(t) = \int_{t_o}^{t} [\varphi^{-1}(r-t_o)(Bu + G_g n_w)]dr + C_2 \]  
(H.6)

Using (H.2) and (H.6) at \( t=t_o \)
\[ C_1(t_o) = x(t_o) = C_2 \]  
(H.7)

Substituting for \( C_2 \) of equation (H.7) into (H.6) and then substituting for \( C_1 \) in equation (H.2) gives,
\[ x(t) = \varphi(t-t_o)x(t_o) + \int_{t_o}^{t} [\varphi(t-t_o)\varphi^{-1}(r-t_o)[Bu(r) + G_g n_w(r)]]dr \]  
(H.8)

Using two well known properties of the transition matrix (Brockett [1970]) given by,

\[\phi^{-1}(\tau) = \phi(-\tau)\]  \hspace{1cm} (H.9)

\[\phi(t_2 - t_0) = \phi(t_2 - t_1) \phi(t_1 - t_0)\]  \hspace{1cm} (H.10)

Equation (H.8) can be rewritten in the form,

\[x(t) = \phi(t - t_0) x(t_0) + \int_{t_0}^{t} \phi(t - \tau) Bu(\tau) d\tau + \int_{t_0}^{t} \phi(t - \tau) G_n(\tau) d\tau\]  \hspace{1cm} (H.11)

\text{Deterministic Solution}

Equation (H.11) is then the solution of (H.1).
H.2 COMPUTER ALGORITHM FOR SOLUTION OF THE STATE EQUATION

For numerical solution of (H.11), it is usual to assume the control $u$ and the input noise, $n_w$, to be piecewise constant over the time interval, $T$, which can be made as small as desired.

Let

$$\Delta_z(T) \triangleq \int_0^T \Phi(T-t)zdt$$

(H.12)

where $z$ is some general driving matrix. If the system is regarded as being discrete, (H.11) may be re-expressed as:

$$x[(r+1)T] = \Phi[T]x[rT] + \Delta_b[T]u[rT] + \Delta_g[T]n_w[rT]$$

(H.13)

If the model was to be subjected to disturbances, then the matrix $G$, in (H.1) was set to the appropriate coefficient values; in a deterministic study, $G$ would be set to zero.

The transition matrix, $\Phi$, may be expressed as a series expansion which converges if $A$ is a stability matrix. To ensure convergence of that series in a limited number of terms, it is usual to determine $\Phi$, over a very small step size $\delta T$, where:

$$\Phi(\delta T) = I + A\delta T + \frac{A^2(\delta T)^2}{2!} + \frac{A^3(\delta T)^3}{3!} + \ldots$$

(H.14)

The series is truncated by using a stopping criterion based upon the magnitude of the relative difference between elements of the $i^{th}$ term and the $i+1^{th}$ term (Nicholson [1966]). Repeated squaring of $\Phi(\delta T)$ results in $\Phi(T)$ using a property of the transition matrix that:

$$\Phi(2\delta T) = \Phi^2(\delta T)$$

$$\Phi(4\delta T) = \Phi^2(2\delta T)$$

$$\vdots$$

$$\Phi(n\delta T) = \Phi^n(\frac{n\delta T}{2}) = \Phi(T)$$

(H.15)
Thus for a discrete interval \((T)\) of 0.1 seconds, \(\delta T\) may be chosen to be \(0.78125 \times 10^{-3}\) and \(\phi(T)\) may then be obtained in seven iterations.

From (H.12), if \(A\) is non-singular,

\[
\Delta_z(T) = A^{-1}[\phi(T) - I]z 
\]  
(H.16)

or

\[
\Delta_z(T) = \{IT + \frac{AT^2}{2!} + \frac{A^2T^3}{3!} + \ldots\}z 
\]  
(H.17)

\[
\Delta_z(T) = \{IS + \frac{A(\delta T)^2}{2!} + \frac{A^2(\delta T)^3}{3!} + \ldots\}z 
\]  
(H.18)

From (H.16)

\[
\Delta_z(\delta T) = A^{-1}[\phi(\delta T) - I]z 
\]

\[
\Delta_z(2\delta T) = A^{-1}[\phi(2\delta T) - I]z 
\]

\[
= [I + \phi(\delta T)]\Delta_z(\delta T) 
\]

\[
\Delta_z(n\delta T) = [I + \left(\frac{n\delta T}{2}\right)]\Delta_z\left(\frac{n\delta T}{2}\right) = \Delta_z(T) 
\]  
(H.19)

In the FORTRAN computer programs PLANT1C and PLANT1G, the stopping criterion used was to truncate (H.18), if the error between all corresponding elements of the \(i^{th}\) term and the \((i+1)^{th}\) term was less than 0.001. Thus for a prespecified value of \(\delta T\), \(\phi(\delta T)\) and \(\Delta_z(\delta T)\) were evaluated using (H.14) and (H.18) respectively. These calculations then allowed \(\phi(T)\) and \(\Delta(T)\) to be determined using (H.15) and (H.19) respectively. The appropriate matrices were then substituted in (H.13) and a recursive routine used to determine the state vector at each interval of time, typically every 0.1 seconds. The output vector was determined using the relationship

\[
x[(r+1)T] = Cx[(r+1)T] + Eu[(r+1)T] 
\]  
(H.20)
CONTROLLABILITY AND OBSERVABILITY INVESTIGATION

Some of the results obtained from controllability and observability investigations are presented in Table I.1, where it can be seen that the occurrence of a near zero complex pair and accuracy of computation do not depend on the controllability and observability properties. It is worth noting that:

- SD401 and SD404 are both partially controllable but the near zero eigenvalues of only SD401 were not computed accurately.
- SD413 and SD422 were completely controllable and observable and yet their corresponding closed-loop eigenvalues were not computed accurately.
- OD401 is controllable but partially observable and yet the near zero closed-loop eigenvalues did not form a complex conjugate pair.

Table I.2 shows that the choice of appropriate weighting elements for the control weighting matrix, G, affects the accuracy of computation of the eigenvalues. But choosing appropriate weights for either the controls or outputs of the system could become very laborious and time consuming because of the numerous combinations of weighting elements that might have to be considered.

However, it was found that the shift method was very efficient for over 90% of the cases examined. For these cases, the appropriate shift value was obtained with either the first or second trial values. A sign change of the positive part of the near zero eigenvalue often helped in
determining the appropriate shift value (see Table I.3).

Nevertheless, for some cases it was not possible to obtain a satisfactory solution as it can be seen from Table I.3 and Figure I.1. Hamming [1962] has suggested that by a suitable choice of the shift value the smallest eigenvalue can be converted to the largest in magnitude thus improving the accuracy of the computation of the smallest eigenvalue. According to Hamming's method once the approximate magnitude of the largest eigenvalue is known, then the shift should be equal to the negative of the magnitude of the largest eigenvalue. For SDA401, (Table I.3) the largest closed-loop eigenvalue was 47.8. When Hamming's method was applied it was found that no conclusive results could be obtained. In this case, a new control weighting matrix, G, was chosen and by applying the shift method once again a solution was obtained.
<table>
<thead>
<tr>
<th>PROGRAM TITLE</th>
<th>ORDER OF SYSTEM</th>
<th>NUMBER OF OUTPUTS</th>
<th>RANK OF THE CONTROLLABILITY MATRIX</th>
<th>RANK OF THE OBSERVABILITY MATRIX</th>
<th>NUMBER OF $^+$ NEGATIVE EIGENVALUES</th>
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</thead>
<tbody>
<tr>
<td>SD401</td>
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<td>22</td>
<td>21</td>
<td>22</td>
<td>23</td>
</tr>
<tr>
<td>SD404</td>
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<td>22</td>
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<td>22</td>
<td>22</td>
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<td>22</td>
<td>22</td>
<td>22</td>
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<td>22</td>
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</tr>
<tr>
<td>OD401</td>
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<td>12</td>
<td>19</td>
<td>22</td>
</tr>
<tr>
<td>OD404</td>
<td>22</td>
<td>12</td>
<td>12</td>
<td>19</td>
<td>23</td>
</tr>
<tr>
<td>OD419</td>
<td>22</td>
<td>12</td>
<td>12</td>
<td>22</td>
<td>23</td>
</tr>
</tbody>
</table>

$^+$ as obtained from OUTREG

**TABLE I.1:** The Existence of the Near Zero Complex Pair Occurs Irrespective of the Controllability and Observability Properties
<table>
<thead>
<tr>
<th>PROGRAM TITLE</th>
<th>ORDER OF SYSTEM</th>
<th>NUMBER OF OUTPUTS</th>
<th>OUTPUT WEIGHTING MATRIX, G</th>
<th>NUMBER OF NEGATIVE EIGENVALUES</th>
<th>NUMBER OF POSITIVE EIGENVALUES</th>
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</thead>
<tbody>
<tr>
<td>SD422</td>
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<td>22</td>
<td>( G = \text{diag}{0.25,0.5,0.25,0.5} )</td>
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<td>21</td>
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<tr>
<td></td>
<td>22</td>
<td>22</td>
<td>( G = \text{diag}{2.5,5.0,2.5,5.0} )</td>
<td>22</td>
<td>22</td>
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<tr>
<td></td>
<td>22</td>
<td>22</td>
<td>( G = \text{diag}{25.0,50.0,25.0,50.0} )</td>
<td>21</td>
<td>23</td>
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</table>

\(^\dagger\text{as obtained from OUTREG}\)

**TABLE I.2:** The Choice of an Appropriate Weighting Matrix Affects the Accuracy of Computation of the Eigenvalues
<table>
<thead>
<tr>
<th>PROGRAM TITLE</th>
<th>ORDER OF SYSTEM</th>
<th>NUMBER OF OUTPUTS</th>
<th>SHIFT VALUE</th>
<th>NUMBER OF NEGATIVE EIGENVALUES</th>
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<td>31</td>
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<td>30</td>
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<td>29</td>
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<td>47.8</td>
<td>29</td>
</tr>
</tbody>
</table>

\[as \text{obtained from OUTREG}\]

\textbf{TABLE I.3:} An Appropriate Shift Value is Not Always Easy to Obtain
Real part of the near zero complex eigenvalue

\[ x \times 10^{-5} \]

FIGURE 1.1: Difficulties in Obtaining an Appropriate Shift Value