Contact mechanics and impact dynamics of non-conforming elastic and viscoelastic semi-infinite or thin bonded layered solids

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Contact Mechanics and Impact Dynamics of Non-Conforming Elastic and Viscoelastic Semi-infinite or Thin Bonded Layered Solids

By

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Abstract

The thesis is concerned with the contact mechanics behaviour of non-conforming solids. The geometry of the solids considered gives rise to various contact configurations, from concentrated contacts with circular and elliptical configuration to those of finite line nature, as well as those of less concentrated form such as circular flat punches. The radii of curvature of mating bodies in contact or impact give rise to these various non-conforming contact configurations and affect their contact characteristics, from those considered as semi-infinite solids in accord with the classical Hertzian theory to those that deviate from it. Furthermore, layered solids have been considered, some with higher elastic modulus than that of the substrate material (such as hard protective coatings) and some with low elastic moduli, often employed as tribological coatings (such as solid lubricants). Other bonded layered solids behave in viscoelastic manner, with creep relaxation behaviour under load, and are often used to dampen structural vibration upon impact.

Analytic models have been developed for all these solids to predict their contact and impact behaviour and obtain pressure distribution, footprint shape and deformation under both elastostatic and transient dynamic conditions. Only few solutions for thin bonded layered elastic solids have been reported for elastostatic analysis. The analytical model developed in this thesis is in accord with those reported in the literature and is extended to the case of impact of balls, and employed for a number of practical applications. The elastostatic impact of a roller against a semi-infinite elastic half-space is also treated by analytic means, which has not been reported in literature.

Two and three-dimensional finite element models have been developed and compared with all the derived analytic methods, and good agreement found in all cases. The finite element approach used has been made into a generic tool for all the contact configurations, elastic and viscoelastic.

The physics of the contact mechanical problems is fully explained by analytic, numerical and supporting experimentation and agreement found between all these approaches to a high level of conformance. This level of agreement, the development of various analytical impact models for layered solids and finite line configuration, and the development of a multi-layered viscoelastic transducer with agreed numerical predictions account for the main contributions to knowledge.

There are a significant number of findings within the thesis, but the major findings relate to the protective nature of hard coatings and high modulus bonded layered solids, and the verified viscoelastic behaviour of low elastic modulus compressible thin bonded layers. Most importantly, the thesis has created a rational framework for contact/impact of solids of low contact contiguity.

Keywords:
Contact mechanics, Layered elastic solids, Viscoelastic Contacts, Impact dynamics, semi-infinite solids, Hertzian contact, Finite line contact
This thesis is dedicated to
My family
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Nomenclature:

\( a \) : Contact radius or semi-minor axis half-width
\( b \) : Semi-major axis half-width, or width of a finite element when indexed
\( d \) : Thickness of a layered solid or shell
\( [D] \) : Elasticity tensor/influence coefficient matrix
\( e \) : Dilatation strain
\( E \) : Young's modulus of elasticity
\( E_r \) : Reduced modulus defined as:
\[
E_r = \left( \frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2} \right)^{-1}
\]
\( E' \) : Reduced modulus defined as:
\[
E' = \frac{2}{\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}}
\]
\( f \) : Frequency
\( f(r) \) : Profile of undeformed contacting solid
\( F \) : Force
\( G \) : Modulus of rigidity
\( G' \) : Conductance
\( g \) : Gravitational acceleration
\( I \) : Current
\( K \) : Bulk modulus in chapter 6
\( \ell \) : Wavelength in a Fourier function
\( l \) : Length of contact
\( L \) : Length of roller, or length of an element when indexed
\( m \) : Mass
\( p \) : Contact pressure
\( q \) : Geometric progression ratio
\( r \) : Radial measurement
\( r,\theta,z \) : Polar co-ordinate set
\( R \) : Equivalent radius, except in chapter 5, where it denotes resistance
\( s \) : Separation
\( t \) : Time, or impact time
\( u \) : Displacement in the x-direction, along the length of contact
\( v \) : Displacement in the y-direction, in the lateral direction, or velocity
\( V \) : Voltage
\( W \) : Contact or impact load
\( x,y,z \) : Cartesian co-ordinate set
\( z \) : Into the depth of a solid
Nomenclature

\( \beta \) : Curvature, \( \beta = \frac{1}{R} \)

\( \gamma \) : Shear strain

\( \delta \) : Deflection

\( \varepsilon \) : Direct strain, Creep strain in chapter 6

\( \eta \) : Viscosity

\( \lambda \) : Lame’s constant: \( \frac{Ev}{(1+\nu)(1-2\nu)} \)

\( \mu \) : Lame’s constant: \( \frac{E}{2(1+\nu)} \)

\( \nu \) : Poisson’s ratio

\( \sigma \) : Direct stress

\( \tau \) : Shear stress, except in chapters 5 and 6, defining relaxation time

\( \varphi \) : Stress function

\( \Phi \) : Creep compliance function

\( \omega \) : Penetration/approach into the depth of material (in the z-direction)

Subscripts:

\( \theta \) : Refers to the position of maximum pressure, or initial conditions, or the instantaneous (elastic) value in chapter 6

\( 1,2,3 \) : Refer to bodies in contact, except for stress components where it signifies principal stresses

\( \infty \) : Denotes long-term (steady state) value

\( eq \) : States the equivalent value

\( h \) : defines hydrostatic condition

\( i,j \) : Belong to nodal positions, except for impacting bodies, i

\( \text{max} \) : Relates to the maximum value of a variable

\( r \) : Reduced or effective/equivalent

\( r,\theta,z \) : Correspond to polar co-ordinates

\( x,y,z \) : Denotes component directions

Superscripts:

\( \cdot \) : First derivative with respect to time, \( \frac{d}{dt} \)

\( \cdots \) : Second derivative with respect to time, \( \frac{d^2}{dt^2} \)

\( ' \) : Reduced value of modulus of elasticity, or curvature in y-direction

\( - \) : Hydrostatic component
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1.1- Background

Looking back at the history of civilisations, one can note that mankind has always been driven by curiosity and a tendency to explore. These characteristics opened new areas for investigation and led to an advanced technological revolution. An endless race to enhance understanding of physical phenomena is envisaged in order to drive the technological machinery. Now more than ever there are great demands put on the understanding of behaviour of matter for precise and fail-safe applications.

Physics of motion of matter make the background to all physical interactions of all matter. Banishing the concept of action-at-distance through action of waves, being electromagnetic or gravitational curvatures, today’s physics reserves a uniquely important place for fundamentals of contact and impact phenomena. These interactions, therefore, involve a type of contact regardless of the boundary of system investigation from microcosm to macrocosm, as described by Newton. At the heart of the spectrum of motion of matter resides continuum mechanics, foundation of which was established by the great discoveries of the 17th Century, mainly by the helios himself: Isaac Newton. For all the modifications and additions made to continuum mechanics to encompass the scheme of the very large and the minuita, Newtonian physics remains the cornerstone of science, upon which most of the contact and impact phenomena are described particularly for application and use of all materials in everyday life on this planet. For
example, a combination of a sliding contact in presence of friction intuitively led to the discovery of "manufactured" fire. Today the field of science dealing with friction, wear, and interaction of bodies in contact or impact is termed contact mechanics, and it forms one of the major pillars of human knowledge. It combines with the tenets of Newtonian viscous flow model for action of fluids in narrow conjunctions to form the basis of another pillar of physical sciences: tribology. The father of tribology is considered to be Leonardo da Vinci (1452-1519). He studied topics such as: friction, wear, bearing materials, plain bearings, lubrication systems, gears, screw-jacks, and rolling-element bearings. That was 150 years before Amontons' Laws of Friction (Guillaume Amontons 1663-1705), were introduced. Hidden or lost for centuries, Leonardo da Vinci's manuscripts were read in Spain a quarter of a millennium later. Other great scientist like Leonard Euler (1707-1783) and many more came to add to this field of knowledge.

The behaviour of solids in contact or under impact conditions accounts for a significant number of applications and events, not only on the Earth but in the cosmos. Fundamentals of these have been studied for centuries and in engineering applications set out by works of Boussinesq (1885) and Hertz (1881,1896). Their contributions makes use of the convenience of the Newtonian vision of a continuum, and has by and large covered more than 95% of all the contact and impact mechanical behaviour in machinery. In recent years, the use of advanced materials, engineered to reduce the untoward effects of friction, wear and fatigue of highly loaded contacts has led to observations that deviate from the classical theories and added an impetus to better understanding of contact mechanics in the realms of the minutia; behaviour of very thin solids.

At the extreme of cases the atomistic views of early 20th century was conceived to provide a universal framework for the behaviour of very small scale systems. The singular failure of quantum mechanics to venture out of the atomic domain into molecular and micro-level interactions, and its abandoning of the cause and effect principle has led to attempts being made to extend the continuum mechanics theories into the domain of micro-level interactions. This approach has met with success in the past 2 decades. The behaviour of thin elastic layers are better
understood now, and deviations from classical theories noted, a fortunate timely advance, given the growing application of bonded solids and protective coatings in many areas of technology. The same is also true, be it to a lesser extent for the viscoelastic behaviour of many modern engineering materials, where advances have been made, based upon the fundamentals of this science established by Clerk Maxwell and Lord Kelvin in the 19\textsuperscript{th} century.

1.2- Aims and Objectives

The overall aim of this thesis is to bring together the principles of solid contact mechanics from semi-infinite elastic to layered viscoelastic contacts, starting from the theories expounded by the leading lights of the subject: Boussinesq, and particularly Rudolph Heinrich Hertz, up to current experimental and numerical results and applications of the current research work.

When two surfaces are brought together, contact occurs at discrete points. The deformation that takes place in the region of these contact spots, establishes stresses opposing the applied load. The sum of the areas of all the contact spots constitutes the real area of contact for most materials with applied load.

Well known solutions exist for semi-infinite solids and incompressible thin pieces of rubber, rubber line bearings, etc, but when those contacts become layered very little literature is still available. Therefore, the field requires more basic investigation for applications with very small margins of error.

The specific objectives of the thesis are:

- To develop an overall framework to classify the mechanical behaviour of solids under contact or impacting interactions.
- Restate or develop analytical expressions across the spectrum of investigation, based upon Newtonian continuum mechanics.
- Develop numerical modelling and analysis approaches, based upon finite elements with the view to highlighting a generic "roadmap" in the
understanding of contact/impact mechanical behaviour of solids, but excluding the case of thin unbounded layers.

- Devise simple and repeatable/reproducible experiments or devices to supplant the numerical predictions and analytical evaluations with observations, with the view to consolidating the approaches undertaken.

Therefore, whilst the overall aim of the thesis is to contribute in laying a solid foundation in the contact mechanical behaviour of various solids, its specific objectives are to address engineering applications of the theories. In this vein, specific cases have been tackled in the body of the thesis, as well as reference made to some of the applications of the theory in the next section to make aware the potential readers of its significant and diversity, under the two topics of “on the Earth” and “in the space”.

The fundamental approach is based on the evolution of the highlighted theories, with incremental contributions, as is the case in any sustainable effort. Available literature is reviewed in each chapter and further developments and use of these made accordingly. The literature in contact mechanics is limited, but of sizeable volume, and the search and citation of these within the thesis is not regarded as exhaustive.

1.3- Applications of contact mechanics

(a)- In the Space

One is witnessing the beginning of a new space exploration era; a new century that is marked by the assembly of the International Space Station, the first space tourists, and other aerospace applications in unique environments, and where often exotic materials are employed.

For example, in thrusters mechanisms used for the positioning of the satellites like in changing altitude and attitude, bearings allow complex yaw, pitch and roll motions. Although on terrestrial applications liquid lubricants are used, in space
due to the very low temperatures the usual lubricants tend to freeze. This can result in the malfunction of the satellite. Therefore, bearings are dry and coated by solid lubricated layers, such as Molybdenum disulphide (MOS2), silver or gold. All these are layered elastic or viscoelastic solids. Dry lubricants can be provided as bulk material for a moving surface, transferred by rubbing from a solid made from, or containing, the dry lubricant as, for example, with self-lubricating cages. Alternatively, it can be applied to one (or both) counterface(s) in the form of a film, as with techniques such as sputter deposition, chemical vapour deposition or physical vapour deposition.

Dry lubrication materials provide low coefficients of friction; reduce wear, stick/slip motions, corrosion, etc. These films include long chain molecules such as PTFE, FEP, PFA, ETFE, and Lamella structure materials such as molybdenum disulfide, tungsten disulphide and electric furnace graphite are extensively employed. Soft metals such as Lead, Babbit, Brass, Gold and Silver, etc., can fall into the lubricant category.

The use of molybdenum disulphide dates back hundreds of years. Present technology dates back to the 1920's. D.O.D., N.A.S.A. and the Society of Tribological and Lubrication Engineers (formerly A.S.L.E.)

Molybdenum disulphide (Moly or MoS2) is another of the lamella structure materials. It is the most common natural form of molybdenum, and is extracted from ore and then purified for direct use in lubricants. This material by itself, since it has a layered structure, makes a very efficient lubricant. These layers can slide over each other at the molecular level, allowing the surfaces of Steel and other metals to move fluidly, even under severe pressures, as bearing surfaces indeed experience. Since molybdenum disulphide is of geothermal origin, it has the durability to withstand heat and pressure. This is particularly true if small amounts of sulphur are available to react with iron and provide a sulphide layer which is compatible with molybdenum sulphide in maintaining the lubricating film. Molybdenum disulphide is inert to many chemicals and will perform under a vacuum, where graphite usually fails.
Moisture vapor is unnecessary for lubrication as slip occurs on the sulphur atoms. Tests in vacuum show that friction decreases as vacuum increases. Friction decreases as load and surface speed increase. These results suggest that removing water vapor contamination decreases friction in molybdenum disulphide. In its operating range MoS₂ has superior load bearing and surface speed performance values to graphite or tungsten disulfide.

In dry oxygen free atmospheres it functions as a lubricant up to 1300°F. The oxidation products of MoS₂ are molybdenum trioxide (MoO₃) and sulphur dioxide. MoS₃ is hydroscopic and causes many of the friction problems in standard atmosphere. MoO₃ is itself a lubricant in dry atmospheres. MoO₃ is of itself not abrasive, but attracts moisture vapor contamination.

Space Lubricants are applied using radio-frequency and unbalanced magnetron sputtering rigs for Molybdenum Disulphide and Silver films and lead-ion plating facilities.

NASA pioneered the use of vacuum-deposited thin film solid lubricants. The lubricants are of two types: the low-shear metal lubricants—such as silver and lead—and the laminar-shearing compound materials—such as molybdenum disulphide (MoS₂). The low-shear metal lubricants are used in high-torque applications such as the rotating anodes in X-ray tubes. Low-shear compound materials are used in mechanical-bearing applications in vacuum and where lubricant “creep” can be a problem. Because only a very thin film is needed for lubrication, the application of the lubricant film does not result in significant changes of dimensions. Low friction coatings of metal-containing carbon (Me-C) are used to reduce wear in mechanical contact applications.

In general use of soft and low shear strength coating can have beneficial tribological applications which are not confined to space applications. Contact mechanical behaviour of these solids is very important as a field of investigation,
with a number of recent contributions in the past two decades, including for example by Matthewson (1981).

(b) On the Earth

There are enormous areas of application of contact mechanics in engineering and technological fields. Some examples include:

- Cutting tools like drills, millers, etc.
- Dies for injection-moulding of plastic
- All kind of automotive and engine parts; piston, piston ring-pack, cylinder liners, valve train system: cam-follower contact, valve to valve seat contact, etc., engine bearings.
- Forming and stamping tools
- Balls and roller bearings
- Meshing gear teeth
- Linear guides
- Medical devices such as endo-articular replacement joints: hip and knee, contact lenses, etc.

In fact contact mechanics plays a crucial role in all load bearing and transmission surfaces in machines and mechanisms.

Hard coatings are often called metallurgical coatings and are a type of tribological coating. The hard coatings are used to increase the cutting efficiency and operational life of cutting tools and to maintain the dimensional tolerances of components used in applications where wear can occur, such as injection moulds. In addition, the coatings can act as a diffusion barrier, where high temperatures are generated by motion between surfaces or corrosion protection in aggressive environments. There are various classes of hard coating materials. They include: ionicly bonded metal oxides (Al₂O₃, ZrO₂, and TiO₂), covalently bonded materials (SiC, boron carbon [B₄C], diamond, diamond-like-carbon [DLC], TiC, AlN, CrC, mixed carbide, nitride and carbonitride compound alloys, and cubic boron nitride), and some metal alloys (Cobalt Chromium Aluminum Yttrium
[CoCrAlY], NiAl, NiCrBSi). In some cases the coatings may be layered to combine properties.

Hard coatings are also used to minimize fatigue-wear, such as is found in ball bearings. Wear-resistant coatings also may be applied to surfaces, where there is a light or periodic load. For example, hard coatings are deposited on plastics to improve scratch resistance. Applications are on moulded plastic lenses and plastic airplane canopies. In some cases wear coatings, such as SiO₂ or Al₂O₃, may be applied to already hardened surfaces, such as glass, to increase the scratch resistance.

Ceramics are usually used in components whose surfaces are subjected to large stresses over highly localized contact areas. In the elastic limit such contact loadings, can produce Hertzian cracks of the shape of a truncated cone (frustum). The tensile stresses generated by the indentation cause these cracks to initiate just outside the contact circle and propagate downward and outward into the material. These Hertzian cone cracks, which are extremely deleterious to strength and tribological performance, compromise the overall structural integrity of the ceramic component.

Materials like UHMWPE-Al₂O₃ ceramic composite cast CoCrMo alloy Ti6Al4V alloy are also used in prosthesis. In the cases of artificial hip joints, for example, it has been reported that the post surgery state of stress is significantly altered due to the manner in which the load is transferred from the prosthesis to the surrounding bone. Here the load is partially transferred to the bone through the shear across the bone-cement prosthesis interface. Because of the implant stiffness characteristics, the stress around the hip prosthesis is reduced below normal physiological situation. This load transfer mechanism plays an important role in the performance and survival rate of an implant. Mid thigh pain is a common complaint of patients with hip replacement and the sources of this pain are due to mechanical factors like stress concentration, micro-fractures in the cement or micro-motion of the prosthesis relative to bone and cement.
This thesis also outlines a sandwich layered sensor, employing a thin film solid polymeric layer, which is pressure sensitive and can be employed for a variety of applications as a tactile imaging or a pressure monitoring device. The contact mechanics behaviour of the thin layered solid is studied via carefully controlled experimental indentation tests. Tests have been carried out to ascertain its viscoelastic relaxation behaviour under specific loads. As an artificial tactile device a sensor, comprising closely positioned sensing sites fabricated into a thin layered High Filled Carbon polymer, can be used to emulate human cutaneous skin behaviour.

Artificial skin sensors have a progressively large application in industry and commerce. Thin polymeric or elastomeric films are increasingly used in robotic or prosthetic devices for tactile manipulation of objects. The same thin layered solids have also been employed for production of magnetic data storage devices, or for personal security industry (for example in credit cards). However, little is understood about the contact mechanic behaviour of bonded and unbonded thin film solids, with repercussion that the service life cannot be accurately predicted.

The HFCP belongs to the same category of thin film viscoelastic solids as the magnetic tape in the cassettes, and floppy discs which are also made of polymeric or elastomeric solids coated with magnetic material and form a contact with the reading head. In such thin layers as those mentioned above the thickness of the layer is comparable to the deformation, in contrast to contacts between a roller or an indenter with an elastic half space (the thickness of which is usually much larger than the contact deflection). The latter follows the Hertzian theory, solutions for which have been available for sometime. In thin layers like the HFCP, the conventional Hertzian theory does not usually apply as highlighted for cases of elastic layers by Jaffar (1989), Barber (1990) and Naghieh (1999).

There is a dearth of experimental work, investigating the contact mechanics behaviour of thin layered solid. Some investigation have been carried out by Sladek and Fearing (1990) for footprint observation by optical interferometry or
by Pawluk (1997) who has described a similar sensor as in this thesis and has investigated indentation test results, but treating the problem as a viscoelastic half-space.

The thesis attempts to remedy this situation by investigating the contact mechanics behaviour of thin viscoelastic bonded solid layered film of low elastic modulus.

1.4- Structure of Thesis

Chapter 1 provides an introduction to the field of contact mechanics, its importance and diversity of application, as well as the overall aim and specific objectives of this thesis.

Chapter 2 highlights the fundamental basis of contact mechanics of elastic solids, the Hertzian assumptions for semi-infinite contacts and the Hertzian classical theory. The chapter also highlights deviations from the classical theory in the case of non-conforming solids, with theories for flat contact of a circular punch and the finite line contact configurations, such as that of a roller indenting a flat frictionless plane.

Chapter 3 extends the classical theory by modifying its underlying assumptions to outline analytical methodology for the case of layered elastic solids.

Both chapters 2 and 3, in addition to the analytical solutions, highlight two and three-dimensional finite element approaches to investigate non-conforming contacts of bodies of revolution, giving also specific applications to the cases of hard and soft layers.

Chapter 4 extends the contact problem to the domain of impact of solids of elliptical profiles in counterformal impact. The chapter commences with the restatement of the Hertzian impact theory and uses the same approach and the findings of the previous chapters to develop analytical expressions for impact on
Chapter 1

Introduction

layered solids, as well as for other non-conforming geometries. This chapter also provides dynamic finite element analyses of a generic nature to tackle this class of problems.

Chapter 5 describes a sandwich structured HFCP based sensor, its method fabrication as a layered solid and an experimental set up to obtain its viscoelastic contact mechanical behaviour under load.

Chapter 6 develops analytical and finite element viscoelastic contact mechanics models and reports on a number of simulations, with good agreement with the experimental findings.

Chapter 7 gives the overall conclusions reached, a critical assessment of the approaches contained within the thesis, and some suggestions for future work.
Chapter 2

Contact Mechanics: Semi-infinite Elastic Solids

2.1 - Introduction

Chapter 1 outlines the significance of contact mechanics characteristics of interacting surfaces for fundamental understanding of the behaviour of many load bearing surfaces in a large number of machines and mechanisms. In particular, the increasing use of surface coatings and layered load bearing and protective solids has been highlighted. This chapter is devoted to the mathematical formulation for contact mechanics analysis of elastic solids. The different contact characteristics of various configurations of solid surfaces are outlined. These include semi-infinite and layered solids, with different contact mechanical behaviour. The solutions provided conform to a set of global assumptions. These are:

- The contacting surfaces are considered to be non-conforming. In other words the contact geometry is counterformal, such as in ball or rolling element bearings-to-raceway grooves.
- The surfaces of solids of revolution in contact are considered to be perfectly smooth and, thus, frictionless.
- The interaction of contacting surfaces is considered to take place under isothermal conditions.
These global set of assumptions are coherent, and although not entirely practical, they yield solutions to many practical problems within quite acceptable levels of accuracy.

The formulations highlighted are generally for the cases of contacts of ellipsoids of revolution, where the classical Hertzian contact mechanics theory applies in most cases. In other cases, such as for layered solids with given layer thickness, or solids of revolution in certain non-conforming configurations such as rollers against a flat surface, the mathematical formulation deviates from the classical theory.

The analytical methods developed in this chapter provide a benchmark for comparison with the more often representative and generic numerical solutions. Only the classical circular point contact of a sphere indenting a flat smooth frictionless plane or a pair of smooth cylinders in a cross configuration, as shown in figure 2.1, conform precisely to the Hertzian contact mechanics that yield exact results. All other configurations diverge from this classical solution. In many cases, depending on the contact geometry or the surface quality and composition of the mating members, the validity of Hertzian theory as a prediction tool becomes questionable. Therefore, it is essential to briefly revisit the Hertzian theory, and in particular its underlying assumptions.

Where the use of the classical theory is deemed inappropriate, alternative mathematical treatments have been highlighted in the following chapters. In many cases, these necessarily embody simplifying assumptions in order to yield analytic solutions. More representative treatments of the same class of contact behaviour are often made, using finite element method, in which the choice of finite elements is crucial. In other words, the stress-strain behaviour of elements determines the constraining motion of mating members, which must be in-line with the physics of motion of the problem at hand.

The composition of contacting solids, coatings or layers yield physical behaviour that can diverge from purely elastic behaviour, such as viscoelasticity of elastomeric or polymeric materials or shear rate dependency of thin metallic or
oxide layers or poroelasticity of layers and partially compacted solids such as certain coatings, substrates or indeed the primates' skins. These issues, when arisen within the context and remit of this thesis have been tackled in later chapters. This chapter deals with the contact of elastic semi-infinite solids.

2.2- Classical Hertzian Contact Theory

Hertz (1881, 1896) investigated the case of deformation of elastic bodies of different curvatures. His developed theory is applicable to those contacting solids of revolution that under an applied contact load yield a point contact deformation zone. These non-conforming bodies, such as a sphere, indenting a flat plane or a pair of cylinders in cross contact configuration (see figure 2.1) make an idealised point contact when brought to touch each other. As a load is applied the point of contact grows, as the contiguous bodies’ surfaces deform. The footprint (i.e. the plan view of the deformed contact region) is circular under ideal conditions, but will deviate from a circle to an elliptical shape if the contact is subject to a relative motion of bodies if the principal radii of contact of the two mating members differ in the planes of contact (see figure 2.2). As the principal radii increase towards infinity in one plane, the resulting contact becomes one of a line contact geometry, such as a right circular cylinder resting on a perfectly flat surface.

![Figure 2.1: Circular point contact of cross-cylinder configuration](image)

In order to find a mathematical solution to the problem, Hertz (1881, 1896) had to make certain assumptions in order to simplify the problem, which would otherwise not yield an analytical solution. The following were his assumptions:
- The Hookean stress is not exceeded. This means that the deformation is within the elastic limit.
- The contacting bodies are homogeneous in construction and material properties are isotropic.
- The force is transmitted normally from the bodies to the contact surface, with no friction or other tangentially directed forces.
- The surfaces are smooth and frictionless.
- The prevalent contact condition is isothermal.
- The principal radii of curvature of solids in contact are much greater than the contact dimensions. Solids in contact, pertaining to this condition, are referred to as semi-infinite or half-space. This means that the depth of bodies at the point of contact are orders of magnitude larger than the radius of their circular contact footprint or the major and minor axes' lengths of any elliptical contact shape.
- The bodies are at rest in a state of equilibrium. It should be noted that deviations from theory occur for any relative motion of contacting surfaces. Therefore, the classical theory is applied to contact conditions that are often referred to as elastostatic.

From the above conditions, it can be observed that the classical theory pertains to idealised conditions. However, it must be noted that for a large class of problems in machines the load bearing elements' contact mechanical behaviour remarkably conform very closely to the predictions of the Hertzian theory.

On the basis of the above assumptions, Hertz (1881,1896) was able to derive general analytical expressions for surface deformation, contact dimensions and pressure distribution for pairwise contacts of bodies of any curvature. For a circular point contact the conditions are axi-symmetric about the line of centres of two solids of revolution in contact, and where the normal force acts through the contact. Therefore, Hertz(1881,1896) proposed that the pressure distribution acting over the contact area has an ellipsoidal profile, with the following mathematical form:
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\[ p(r) = p_0 \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^{\frac{3}{2}} \]  (2.1)

Where \( r \) is any radial position from the axis of symmetry, within the Hertzian circular point contact of radius \( a \). The maximum pressure occurs on the axis of symmetry as \( p_0 \). This can be evaluated, noting that the applied load is the area under the generated elastostatic pressure distribution, given by equation (2.1), thus:

\[ W = \int_{r=0}^{a} p(r)2\pi r dr = \frac{2\pi p_0 a^2}{3} \]  (2.2)

Or, simply:

\[ p_0 = \frac{3W}{2\pi a^2} \]  (2.3)

In the classical theory, any two bodies of revolution in contact is conceptualised as an equivalent body of given radii penetrating a semi-infinite flat elastic half space. The equivalent radii are found in each plane of contact as (see figure 2.2):

\[ \text{Figure 2.2: The general case of two bodies in contact} \]
For a two-dimensional analysis, applicable to the case of a sphere near a plane or crossed cylinders' contact, \( R = R_x = R_y \). Note that the classical theory also applies to many conforming contacts that conform to Hertzian assumptions, in which cases the radius of contact for members with concave contacting surfaces, the radii of contact are considered to be negative. It can be noted that in the case of non-conforming bodies the equivalent radii are less than the actual radii of solids, thus referred to as reduced radii, whereas in the case of the conforming pairs the equivalent radii are larger, and are called increased radii.

![Figure 2.3: Surface deformation of non-conforming solid](image-url)
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The deformation of the bodies in contact at any radial position \( r \) can be obtained in terms of their mutual convergence \( \omega_1 + \omega_2 \) of the two bodies in contact and the profile of the undeformed equivalent solid at that location (see figure 2.3), thus:

\[
\delta = \omega_1 + \omega_2 + \frac{r^2}{2R} \tag{2.5}
\]

The normal displacement of the two bodies (i.e. their mutual convergence), \( \omega_z = \omega_1 + \omega_2 \) can be obtained within the contact area, referred to as the Hertzian contact area for \( r \leq a \) and beyond it as: \( r > a \). It is the former region that yields contact deflection values that contribute to generation of pressure. It can be shown that, in the contacting region:

\[
\omega_z = \frac{\pi P_0}{4E_r a} \left( 2a^2 - r^2 \right) \tag{2.6}
\]

Where: \( E_r \) is known as the reduced Young’s modulus of elasticity, since it is less than the combined elasticity of the two contacting solids, and takes into account the compressibility of contacting solids, where for elastic solids: \( \nu < 0.5 \). Thus:

\[
\frac{1}{E_r} = \frac{1 - \nu^2_1}{E_1} + \frac{1 - \nu^2_2}{E_2} \tag{2.7}
\]

Now replacing for the normal displacement \( \omega_z = \omega_1 + \omega_2 \) into equation (2.5), the maximum contact deflection and then the corresponding Hertzian radius can be obtained at \( r = 0 \) as:

\[
\delta = \frac{\pi a p_o}{2E_r} \tag{2.8}
\]

\[
a = \frac{\pi p_o R}{2E_r} \tag{2.9}
\]
Now replacing for the maximum pressure, $p_0$, in terms of the contact load, $W$ from equation (2.3) into equations (2.8) and (2.9) and after some manipulation:

$$a = \left( \frac{3WR}{4E_r} \right)^{\frac{3}{2}} \quad (2.10)$$

And:

$$\delta = \frac{a}{R^2} \quad (2.11)$$

The equations (2.1), (2.3), (2.10) and (2.11) provide the analytical solution, referred to as the classical Hertzian contact theory under elastostatic conditions.

As previously mentioned, contact mechanics behaviour diverges from that of a Hertzian type contact for indentation of semi-infinite solids by bodies that form different contact footprints, such as rollers near a plane, giving rise to a long and thin contacts, often of a “dog-bone” or in other words a “dumbbell” shape. This form of contact is generically known as a finite line contact, as opposed to a theoretical infinite line contact, assumed to have no significant width. Although one may assume such a contact to be approximated by a very long elliptical contact, thus conforming to a great extent to Hertzian elastostatic configuration, this would often result in inaccurate prediction of pressures and deformation profiles for a number of reasons. Firstly, the length of the contact, being along the major axis of the elastostatic ellipse can be comparable to the principal radii of contacting solids, for example in spur gears, where the length of the contact is the same as the width of the tooth flank. This abrogates the semi-infinite assumption made in the Hertzian theory. Secondly, the edge stresses, often caused by the discontinuous nature of the contacting profile is ignored in the analysis, such as the pressure spikes generated at the sharp edges of unblended rollers, gear teeth, cam to flat follower contacts or indeed a flat punch indenting an elastic solid.
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Analytic solutions for such contacts follow the same principal approach as that of the Hertzian theory. However, in many cases an approximate solution may only be obtained through numerical analysis.

2.3- Frictionless Indentation of an Elastic Half-Space by a Circular Flat Punch

The displacements $u(x,z)$ and $\omega(x,z)$ in an elastic half-space under a pressure distribution $p(s)$ due to the penetration of a flat punch in a frictionless contact is obtained by Johnson (1985) as (see figure 2.4):

$$u(x,z) = \frac{(1-2\nu)(1+\nu)}{2E} \left\{ \int_{-b}^{x} p(s)ds - \int_{x}^{a} p(s)ds \right\} \quad (2.12)$$

And:

$$\omega(x,z) = -\frac{2(1-\nu^2)}{\pi E} \int_{-b}^{a} p(s) \ln |x-s|ds \quad (2.13)$$

Figure 2.4: A circular punch frictionless indentation of a semi-infinite elastic half-space
The above equations can be represented in the form of displacement gradients as:

\[
\frac{\partial u(x, z)}{\partial x} = -\frac{(1-2\nu)(1+\nu)}{E} p(x)
\]  

(2.14)

And:

\[
\frac{\partial \omega(x, z)}{\partial x} = -\frac{2(1-\nu^2)}{\pi E} \int_{-b}^{x} \frac{p(s)}{x-s} ds
\]  

(2.15)

To obtain an analytic solution for a frictionless indentation, one can prescribe \( \omega(x, z) \) (i.e. the mutual approach) to be constant. Thus:

\[
\frac{\partial \omega(x, z)}{\partial x} = 0
\]  

(2.16)

Now letting the origin at the centre of the loaded region: \( b = -a \) (see figure 2.5) and the width of the contact becomes \( 2a \), with sharp corners. The solution for pressure distribution becomes:

\[
p(x) = \frac{C}{\pi^2 (a^2 - x^2)^{\frac{3}{2}}}
\]  

(2.17)

Figure 2.5: Normal pressure on a semi-infinite body
The constant $C$ is obtained as: $C = \pi \int_{-a}^{a} p(x) dx = \pi W$, where $W$ is the applied load.

Thus:

$$p(x) = \frac{W}{\pi b(a^2 - x^2)^{\frac{3}{2}}}$$  \hspace{1cm} (2.18)

Note that the pressure reaches a theoretical value of infinity at the edges of the contact (i.e. $x = \pm a$, see figure 2.6). However, this is quite impractical and occurs due to the assumptions made in the above analytic solution. In this solution traction that occurs under physical conditions is ignored and the rate of deformation $\frac{\partial \sigma(x,z)}{\partial x} = 0$. This means that at the surface $\sigma_z = \sigma_x = -p$.

\[\]
2.4- Numerical Methods

When the contact conditions deviate from classical Hertzian theory, numerical solutions to obtain deformed profile and the corresponding pressure distribution are employed. The contact footprint shape is discretised into a number of elements, upon each of which a pressure distribution is assumed. For instance, for the case of a rigid circular punch indenting a semi-infinite elastic half-space, Conway et al (1966) noted that regardless of the pressure distribution, it can always be approximated by a series of pressures, which are constant over incremental lengths. If the number of such incremental lengths is large enough, the degree of approximation will be good. This is particularly important in regions of contact, where the pressure gradients are extremely large, such as edge stress discontinuity, resulting from penetration of sharp edges of a flat punch into an elastic solid, or an unblended roller into a race in rolling element bearings.

The problem of a roller indenting a semi-infinite elastic half-space is representative of many practical engineering applications, such as in rolling element bearings and cam-follower pairs. In the case of the latter, the roller follower makes contact with the cam, which can be considered as a roller with a varying radius. At any given time the contacting pair can be modelled as an equivalent cylinder indenting a semi-infinite elastic half-space. Although many solutions approximate such a contact as an infinite line contact, based upon the assumption that the width of the footprint is negligible compared to its length, it is clear that the sharp edges of the cam cause high pressures at the contact extremities that are significant and would be ignored in an infinite line contact solution. Thus, a proper two dimensional solution to the problem is required. Such solutions, for the case of dry elastostatic contact of a roller on a flat have been reported by Heydari and Gohar (1979), Hartnett (1979), Kannel (1974), Johns (1978), Johns and Gohar (1981), and extended to lubricated contact by Mostofi (1981), Mostofi and Gohar (1983), and for taper roller bearings by Rahnejat (1978) and Rahnejat and Gohar (1979). More recent solutions with improved mesh density have been reported by Kushwaha (2000) and Kushwaha et al (2002).
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for cylindrical rolling element bearings and by Kushwaha (2000), Kushwaha et al (2002a) and Kushwaha and Rahnejat (2002b) for cam-to-flat follower pairs.

The contact area for a cylindrical roller against a flat half-space yields a dumbbell-shape, sometimes referred to as a dog bone shape (see figure 2.7). The enlarged extremities of the footprint are due to the high edge stresses, generated due to the sharp edges of the roller indenting the plane.

The contact area, as shown in figure 2.7, is sub-divided to a number of rectangular elements, upon each of which a three dimensional pressure distribution is assumed; parabolic in transverse direction and isosceles in the longitudinal direction (see figure 2.8). Deflection at any point in the rectangular element due to this pressure distribution is obtained as (Johns (1978)):

\[
\omega(x, y) = \frac{(1 - \nu^2)}{\pi E} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p(x_1, y_1)}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} \, dx_1 dy_1
\]

Figure 2.7: Footprint shape of a roller indenting a semi-infinite elastic half-space (after Johns (1978))
where:

\[ p(x_i, y_i) = (1 - \frac{\frac{y_i}{c}}{c})(1 - \left[\frac{x_i}{a_i}\right]^2)^{\frac{1}{2}} \]  

(2.20)

An analytical solution for a lateral pressure elliptical distribution can be obtained, if a uniform pressure distribution is assumed over the length of the roller, the distribution being:  

\[ p(x) = p_0 \left[ 1 - \frac{x^2}{a^2} \right]^{\frac{1}{2}} \] . Replacing this into equation (2.19), with the limits of integration over the length of the contact, then the penetration at the centre of the roller can be obtained as obtained by Koshy and Gohar, 1977 as:

\[ \omega_0 = \frac{a_0 p_0}{\pi E'} \left[ \ell n \frac{2L}{a_0} + \frac{1}{2} \right] \]  

(2.21)

where:

\[ a_0 = \left( \frac{8WR}{\pi E'L} \right)^{\frac{1}{2}} \]  

(2.22)
and:

\[ p_0 = \frac{2W}{\pi a_0 L} \]  \hspace{1cm} (2.23)

To obtain the proper 3D pressure distribution rectangular elements as shown in figure 2.8 were used by Johns, 1978, and the integral in equation (2.19) is evaluated, using the pressure distribution in equation (2.20).

The deflection at any point in the contact area obtained above is due to the positive quadrant of one pressure element. The total deflection must be obtained due to all the quadrants of the pressure distribution upon all the rectangular contact, as shown in figure 2.9. With the symmetrical elemental co-ordinate system used, it follows that the total deflection at any location \((x, y)\) under the elemental pressure distribution is: \(2w(x, y)\).

The total deflection on any element is due to the pressure distribution over all the elements. Thus (Johns (1978)):

\[
[\delta] = \begin{bmatrix} \omega \end{bmatrix} [P] \hspace{1cm} (2.24)
\]

The entire formulation and solution procedure is highlighted by Johns (1978). Figure 2.10 shows the pressure distribution and the corresponding footprint shape for an unblended cylindrical roller bearing obtained by Johns (1978). Note in figure 2.10 that the numerical solution, using 59 elements along the contact area has captured finite edge pressure spikes. This is because an infinitesimal computation element is required to capture the theoretically infinite edge pressure spike.
Figure 2.9: Calculation of local contact deflection due to all elements of pressure (after Johns (1978))

Figure 2.10: Pressure distribution and footprint shape for unprofiled (straight-edge) roller (after Johns (1978))
2.8- Finite element approach

In this thesis a finite element approach is undertaken. First, the theoretical basis of isotropic linear elastic finite element approach is outlined. Then, two application areas in contact mechanics of semi-infinite solids are investigated. These are frictionless circular point contact of a ball on an elastic half-space, pertaining to the classical Hertzian theory, and two dimensional frictionless finite line contact of a roller against a flat plane. Classical analytical solution exists for the case of the former: the Hertzian contact, enabling the finite element approach to be verified. In the case of the latter, a number of solutions have been reported, which form the basis for further comparative work to be undertaken. In chapter 3, the case of contact mechanics of bonded layered solids is investigated, which deviates from the classical analytical solutions, making the use of finite element approach more pertinent.

2.8.1- Linear isotropic elasticity under static equilibrium

Static equilibrium condition is considered in the finite element analysis in this chapter, using the commercial code PATRAN. The total stress in any nodal location within the computational mesh is defined from the total elastic strain as:

\[ \{\sigma\} = [D]\{\varepsilon\} \quad (2.25) \]

where \(\{\sigma\}\) is the total "true" stress or Cauchy stress in finite-strain problems, \([D]\) is the fourth-order elasticity tensor and \(\{\varepsilon\}\) is the total elastic strain or the log strain in finite-strain problems. For linear elastic materials, the Drucker stability condition must be satisfied. This requires the tensor \([D]\) to be positive definite. For an incompressible material, the Drucker stability condition requires that the change in the stress \(\{d\sigma\}\), following from any infinitesimal change in strain, \(\{d\varepsilon\}\), satisfies the inequality:
\begin{equation}
\{d\sigma\} \{d\varepsilon\} > 0 \quad (2.26)
\end{equation}

Thus:

\begin{equation}
[D] \{d\varepsilon\} \{d\varepsilon\} > 0 \quad (2.27)
\end{equation}

Since, the material is assumed to be incompressible, one may choose any value for the hydrostatic pressure without affecting the strains. For instance, it is convenient to choose: \(p_h = \sigma_3 = d\sigma_3 = 0\), thus the stress-strain relations becomes:

\begin{equation}
\begin{bmatrix}
d\sigma_1 \\
d\sigma_2
\end{bmatrix} =
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
d\varepsilon_1 \\
d\varepsilon_2
\end{bmatrix} \quad (2.28)
\end{equation}

For material stability, it is clear that \([D]\) must be positive definite, or:

\begin{equation}
D_{11} + D_{22} > 0, \quad D_{11}D_{22} - D_{12}D_{21} > 0 \quad (2.29)
\end{equation}

If the elastic strain can become large, it is necessary to use a hyperelastic model. This, however, is not the case in the contact mechanics problems investigated in this thesis. Thus, linear elastic finite element approach is used here. Furthermore, the materials investigated are assumed to have isotropic elastic properties. Therefore, the stress-strain relations are given by the following relationship:

\begin{equation}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{xy} \\
\varepsilon_{xz} \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{xx}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} \\
-\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} \\
-\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} \\
0 & 0 & 1/G \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{xx}
\end{bmatrix} \quad (2.30)
\end{equation}

where the modulus of rigidity, \(G = \frac{E}{2(1+\nu)}\), and note that the matrix above is \([D]^{-1}\).

Note also that the Drucker stability condition is unconditionally satisfied in this
case, since for an incompressible assumption: $\nu = 0.5$, and the conditions set in (2.29) are satisfied.

### 2.8.2- The finite element models

A number of finite element models are described here. These include a ball against a semi-infinite plane and a roller against a semi-infinite plane.

In the case of the frictionless contact of a ball against a semi-infinite elastic half-space, the axi-symmetric nature of the contact is used to reduce the problem. In PATRAN one quarter of the periphery of the ball is modelled. In order to ensure continuity in the profile 200 such elements are used, in a length $\frac{\pi R}{2}$, with $R = 6\text{mm}$. The reference point for the model is the defined position of the centre of the ball, with all the elements connected to this point. Both the ball and the flat plane are considered as elastic bodies. 2D, 4-noded quadrilateral plane strain continuum mechanics elements are used to mesh the computation domain both for the ball and on and into the flat plane. A 2D analysis is sufficient, since this simulation represents the classical Hertzian theory, and therefore used for the validation of the finite element approach. The length of the computation domain is taken to be the same as the one quarter periphery of the indenting ball, in order to ensure that any tensile stresses generated in the compressible material outside the contacting domain (i.e. the Hertzian circular point contact) do not affect the evaluated contact pressure distribution. 200 elements are used to mesh the domain along the surface. The mesh is generated according to a geometric progression, as described in chapter 3, and in order to bias the nodal density towards those in the contact area, rather than those outside it. This is described in some detail in chapter 3 ($q^{-1} = \frac{L_2}{L_1} = 120$). Unlike the use of a regular mesh spacing into the bulk of the layered solid in chapter 3, here a geometric progression is employed. The reason for this is that the depth of the elastic solid is taken in such a way that ratio
This means that for the cases considered, a minimum thickness of 1.2 mm is used, necessitating 60-80 elements to be used into the depth of the elastic half-space (i.e. 60-80 rows of elements, each with 200 elements along the plane of contact, as described above). The geometric progression ratio is so chosen that the element on the surface (the outermost element) has a depth half of the innermost element into the depth of the material. Therefore: \( q^{-1} = \frac{b_n}{b_1} = 2 \), where \( b_1 \) is the depth of the first element and \( b_n \) is the depth of the innermost sub-surface element in the solid. Care is taken on case by case basis to ensure that all the quadrilateral elements are devoid of long and narrow geometry, which can lead to inaccuracies. Figure 2.11 shows a typical constructed mesh.

![The 2D finite element mesh based on geometric progression for a sphere indenting a semi-infinite elastic half-space](image)

The next finite element model is that of a roller indenting a semi-infinite elastic half-space. The counterformal concentrated contact of a roller near a flat plane leads to a long and narrow finite line contact, as described in section 2.7. To observe the edge stress concentrations at the roller extremities, it is necessary to treat the problem in three dimensions. In this case the roller can be considered as a rigid indenter, with the elastic half-space having a modulus of elasticity as that of the reduced modulus of two elastic members in contact. A mesh is constructed...
Chapter 2
Contact Mechanics: Semi-Infinite Elastic Solids

with the same observations as that described previously, this time with more elements packed at the edges of the contact using a geometric progression. A ratio of 10:1 is used as in the previous example along the contact length, with regular spacing in the width and 1:1 into the depth of the elastic half-space (see figure 2.12). The elements used for the elastic half-space are 3D solid elements. The mesh density is 150X150, the former in the axial direction and the latter in the transverse direction. There are typically 10 rows of elements into the depth of the flat plane. The axi-symmetric properties of the contact for an aligned roller is not taken into account for the generation of this finite element mesh, in order to make it suitable for studying the effect of misalignment of the roller, which takes place in practice as highlighted by Johns and Gohar (1981) and Kushwaha et al (2002b). The load is distributed along the length of an aligned roller, in a similar manner as described by Harris (1966).

Figure 2.12: A 3D mesh for a roller indenting a semi-infinite elastic solid

With all the above observations, one cannot still be sure that number of elements used would render a realistic analysis, since the analytical solution itself has many assumptions. Therefore, one needs to alter the mesh density progressively until a reasonable comparison can be made with the classical Hertzian theory, and increasing the mesh density any further does not alter the pressure values significantly. This has been carried out for a large number of cases, one case reported in Appendix I, and is applicable forms of contact.
2.9- Results and discussion

The above described models (in section 2.8) are employed to obtain finite element predictions and compare these with analytical solutions described in this thesis. The circular point contact model is used to validate the finite element approach against the classical Hertzian theory. The finite line contact of a roller against an elastic half-space is compared with the numerical results obtained by Johns (1978), Johns and Gohar (1981), Mostofi (1981) and Kushwaha (2000), all of whom have used finite difference solution of either the contact elasticity boundary integral equation or lubricated contacts, using combined elasticity and Reynolds’ equation.

2.9.1- Elastostatic analysis of circular point contact of semi-infinite solids

For this analysis a typical load per ball time history for a support angular contact ball bearing of a horizontal rotor is taken. The ball bearing contains 12 balls, each with a nominal radius of 6 mm, made of Cobalt-Chromium steel, with modulus of elasticity 210 GPa and Poisson’s ratio of 0.3. The races are also made of the same material. The load per ball varies from a minimum load of 10N in the unloaded region to a value of 250 N in the loaded region at the bottom of the bearing. This condition represents a narrow loaded bearing with the bottom 5 balls being in the loaded region at any given time. An approximate solution is obtained by elastostatic analysis of the contact through use of isotropic elastic materials in linear finite element analysis as described in section 2.8.

A number of simulations have been carried out at loads 10N, 50N, 100N, 150N, 200N and 250N. At each load static equilibrium condition is assumed, and the model described in section 2.8.2 is used. The total number of elements used was in the range 12000-16000 elements as described above. The CPU time for a typical simulation run was approximately from 30minutes up to 6 hours on a Pentium III 750 MHz machine.
Table 2.1 lists the results obtained by the simulation studies with the finite element model and their counterparts, using the classical Hertzian theory highlighted above. The percentage errors shown in columns 4 and 7 correspond to contact radius and the maximum Hertzian pressure respectively. It is clear that the contact area is over-estimated by the finite element analysis due to the size of elements. This problem still persists even when a fine geometric mesh is constructed to pack as many elements as possible within the contact area. Clearly, with finer meshes, the over-estimation will be improved. As the contact area obtained by the finite element analysis is predicted to be larger than the actual area obtained by the Hertzian theory, the corresponding maximum pressure is lower than would be expected and obtained through the classical theory. This can be observed from the results in the table. The maximum percentage error on the contact radius, $a$, is found to be about 9.8%, and the maximum error for $p_0$ is approximately 2.2%. These are deemed acceptable for the mesh density employed in this analysis.

<table>
<thead>
<tr>
<th>Load (N)</th>
<th>$a$ (mm)</th>
<th>Deviation</th>
<th>$p_0$ (GPa)</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>FEA</td>
<td>Analytical</td>
<td>FEA</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.073</td>
<td>0.0810</td>
<td>9.80</td>
<td>0.89</td>
</tr>
<tr>
<td>50</td>
<td>0.125</td>
<td>0.134</td>
<td>6.84</td>
<td>1.53</td>
</tr>
<tr>
<td>100</td>
<td>0.157</td>
<td>0.170</td>
<td>7.41</td>
<td>1.93</td>
</tr>
<tr>
<td>150</td>
<td>0.180</td>
<td>0.193</td>
<td>6.64</td>
<td>2.21</td>
</tr>
<tr>
<td>200</td>
<td>0.198</td>
<td>0.203</td>
<td>2.31</td>
<td>2.43</td>
</tr>
<tr>
<td>250</td>
<td>0.214</td>
<td>0.225</td>
<td>5.07</td>
<td>2.62</td>
</tr>
</tbody>
</table>

Table 2.1: Comparison of finite element predictions with the classical Hertzian theory

Figures 2.14 and 2.15 show 2D pressure distributions with applied loads 10N and 200N respectively. It can be seen that the finite element predictions conform quite closely to the corresponding analytical Hertzian distributions.

Figure 2.16 shows a three dimensional plot of the compressive stresses $\sigma_z$ in the semi-infinite solids in contact. Note that contact pressures are specified as
$p = -\sigma_z$ for $\sigma_z < 0$. When $\sigma_z > 0$, elastic recovery (tensile stress) is taking place beyond the edges of the Hertzian contact. Therefore, the edge of the Hertzian circle is detected, where: $p = \sigma_z = 0$.

It can be noted that the maximum pressure predicted by the finite element analysis for loads greater than 100N (and indeed those by the Hertzian theory) exceed 2 GPa. It is important to investigate the sub-surface absolute maximum shear stress values in order to ensure that they have not exceed the material endurance limit, causing inelastic deformation and lead to failure by fatigue spalling. According to the Hertzian theory, the absolute maximum shear stress occurs along the centre-line of the contact, beneath the surface at a depth of $z=0.47a$. Its magnitude is approximately $0.33p_0$. Table 2.2 lists the analytically obtained values with their counterpart finite element predictions.

![Figure 2.14: Comparison of finite element results with classical Hertzian theory for a load of 10N for steel-on-steel contact, R=6mm](image)
Figure 2.15: Comparison of finite element results with classical Hertzian theory for a load of 200N for steel-on-steel contact, R=6mm

Figure 2.16: 2D normal stress distribution in the contact of semi-infinite solids in conforming to the classical Hertzian theory at 200N (ball on plane). R=6mm. Units in MPa
Table 2.2: Comparison of maximum sub-surface shear stress between FEA results and the Hertzian theory

<table>
<thead>
<tr>
<th>Load (N)</th>
<th>max.τ_{max} (MPa)</th>
<th>max.τ_{max} (MPa)</th>
<th>% Deviation</th>
<th>z (mm) Analytical</th>
<th>z (mm) FEA</th>
<th>% Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>295</td>
<td>289</td>
<td>2.18</td>
<td>0.034</td>
<td>0.038</td>
<td>-10.87</td>
</tr>
<tr>
<td>50</td>
<td>505</td>
<td>502</td>
<td>0.62</td>
<td>0.059</td>
<td>0.063</td>
<td>6.84</td>
</tr>
<tr>
<td>100</td>
<td>636</td>
<td>630</td>
<td>0.89</td>
<td>0.074</td>
<td>0.079</td>
<td>7.41</td>
</tr>
<tr>
<td>150</td>
<td>728</td>
<td>723</td>
<td>0.72</td>
<td>0.085</td>
<td>0.090</td>
<td>6.64</td>
</tr>
<tr>
<td>200</td>
<td>801</td>
<td>799</td>
<td>0.33</td>
<td>0.093</td>
<td>0.095</td>
<td>2.31</td>
</tr>
<tr>
<td>250</td>
<td>865</td>
<td>858</td>
<td>0.76</td>
<td>0.10</td>
<td>0.011</td>
<td>-0.62</td>
</tr>
</tbody>
</table>

Note that the percentage errors between the numerical predictions and the classical theory are very small indeed.

Figure 2.17 shows the 3D maximum shear stress variation under the surface of the substrate (the semi-infinite elastic half-space) and into the body of the ball for the contact load of 200N. The maximum value is illustrated here by the contour level of 774 MPa. Comparing this value with those tabulated in table 2.2, indicates that the contour level at the maximum value is not shown in this 3D representation, which should be regarded as a qualitative indicator of the prevailing contact condition. When a quantitative or objective analysis is required, it is far better to refer to 2D sections into the depth of the material.

When a section into the depth of the material at x=0 (i.e. centre-line of the contact) is considered (see figure 2.18) the maximum value of τ_{max} = 799 MPa. The depth at which this occurs is: z=0.095 mm (see figure 2.18). The FEA results in this case follow the relations: τ_{max} = 0.329p₀ and z = 0.468a, which are very close to the Hertzian theory. The same trend is also true of other simulated conditions.

The importance of evaluation of sub-surface stress field is to estimate conditions that lead to onset of fatigue spalling or inelastic deformation in the depth of the
material of the contacting solids. The failure is usually initiated by the maximum shear stress exceeding the limit set by the Tresca criterion, where:

$$\tau_{\text{max}} = 0.33 p_0 \geq \frac{\sigma_y}{2} \quad (2.31)$$

The above results show that for the Cobalt-Chromium bearing Steel, with the yield stress: $\sigma_y \approx 1.7 \text{GPa}$ this limit is around 850 MPa. Of course under dynamic conditions the reversing (alternating) orthogonal shear stresses in the depth of the material can also initiate fatigue by cyclic action as shown by Johns-Rahnejat (1988), and the elastohydrodynamic pressure spike also plays an important role by inducing its own local sub-surface stress field (Johns-Rahnejat and Gohar (1997)).

Figure 2.17: 2D sub-surface stress fields in the contacting solids for the contact load of 200N (ball on plane). Units in MPa
Figure 2.18: Sub-surface maximum shear stress variation along the centre-line of the contact

2.9.2- Elastostatic analysis of finite line contact of semi-infinite solids

The contact of a roller on a flat plane represents a class of problems encountered in a large number of load bearing and power transmitting elements. This class of problem can best be described as concentrated counterformal finite line contact. A special case of this is the infinite line contact, where the width of the contact is
infinitesimal compared to its length, such as a very long and slender roller (e.g. a needle bearing) against a flat race. In practice, however, such an idealized condition does not exist in practice, but is often used as a first reasonable approximation in cases when the contact aspect ratio; that of contact length to its width becomes very large.

Finite line contact condition may be regarded as an evolution of the classical Hertzian theory (Timoshenko and Goodier, 1951), where the ellipticity ratio (the aspect ratio: that of semi-major half-width of the contact over the corresponding semi-minor half-width in an elliptical point contact) becomes quite large. For a classical Hertzian contact this value is unity, whereas in many ball bearing applications the value can be greater than 1 and perhaps not exceeding 10, due to combined rolling and sliding contact conditions. When the aforementioned ratio is an order of magnitude larger than those stated above, long and narrow elliptical footprints result, which may be approximated by the Hertzian theory. However, a number of problems emerge, when such an assumption is made. Firstly, in finite length contacts the edge profile of the contacting solids represent abrupt changes in the geometry of the contact (Johnson, 1985), resulting in edge stresses, commonly referred to as discontinuities. Therefore, the high stress concentrations can lead to spreading and corresponding deepening of deformation in these vicinities, yielding a dumb-bell or dog-bone shaped contact. An idealised infinite line contact analysis, or a long elliptical contact approximation would tend to under-estimate the edge stresses, which represent the worst contact conditions. More importantly, the length of contact in many cases is comparable with the radii of contacting solids, such as the contact of a roller bearing with certain length-to-diameter ratios, or a pair of meshing teeth in spur gears. This means that the contacting solids cannot be strictly considered as semi-infinite elastic half-spaces, a condition which forms the basis of the classical Hertzian theory.

As previously mentioned, finite line conditions are prevalent, such as in rolling element to raceway contacts, needle and tapered roller bearings, meshing gear teeth pairs in spur gears, cam-to-flat-tappet and cam-to-roller follower contacts, to name but a few. Solutions to these problems require solution to a boundary
integral equation, relating contact deformation to an arbitrary pressure distribution through discretisation of computation domain. Since the boundary integral equation is obtained through fundamental contact mechanics based upon the Boussinesq (1885) approach, a finite difference technique is usually employed, as in the previously mentioned reported research by Johns (1978), Johns and Gohar (1981), Rahnejat (1978), Rahnejat and Gohar (1979), Mostofi (1981), Kushwaha (2000) and others. All these solutions are based on the ability of defining the “arbitrary” pressure distribution for a given geometry by some form of approximation in the elemental discretisation, such as the method by Johns (1978) and Johns and Gohar (1981) described previously. When complex geometries are encountered the discretisation and definition of the “arbitrary” pressure distribution can become quite complex, such as for example in the contact of the dome-ended rollers and the retaining rib abutment.

For finite difference solution of contact problems as described by the aforementioned literature, the derived governing equations are solved at given nodal positions, within what is regarded as the “computational molecule”. In the finite element approach, the computation domain is sub-divided to small regions (i.e. elements), within which certain stress-strain relations may be assumed (such as constant strain rate). In this manner, one need not be able to derive and solve the governing contact mechanical equations directly. Thus, the use of finite elements is a generic approach and can deal with complex contact mechanical problems. The accuracy of the approach depends on appropriate mesh density, the choice of finite elements (i.e. stress-strain relations or the “shape functions”). For problems with less complex contacting geometry and loading arrangement, analytical and finite difference approximation can often lead to more accurate solutions, unless a very high number of finite elements are used. For more complex scenarios the use of finite elements is more appropriate, particularly when the boundary conditions are not subject to significant changes (requiring re-meshing- dynamic mesh regeneration) and where derivation of governing equations would necessitate assumptions that render over-simplification of the real problem. However, to rely on a finite element strategy, one would have to validate its findings with closed form analytical and finite difference numerical
solutions. The previous section has dealt with the former. For finite line contact problems, aligned and misaligned contact of rollers with semi-infinite elastic half-space have been studied here under the same conditions as those reported by Johns (1978), Johns and Gohar (1981) and Harris (1966).

Rollers are usually profiled at their edges in order to reduce the aforementioned edge stress concentration. This action is referred to as blending, reducing the pressure spikes under loaded contact conditions. Usual blending includes the introduction of edge radii, referred to as dub-off with centre of curvature somewhere on or near the longitudinal axis of the roller, or with a larger crown radius, with the radius of curvature usually somewhere along the vertical lateral axis of symmetry of the roller. Sometimes a crown radius is used to profile the entire axial profile of a roller. In this case the radius of curvature can be along the same axis far away from the plane of contact. These axially crowned rollers have the advantage of being self-aligning. Axial blending or edge blending of mating members is used in many load bearing surfaces, including some cam-follower pairs or in the case of some conforming contacts such as piston skirts. There are, however, many cases where edge blending is not employed, but unintentionally achieved since, for example, in the turning or grinding of rollers the tool edge has a finite sharpness, introducing unintentional blending of the edges. Therefore, infinite pressure spikes predicted by analytical solutions, as described in previous sections cannot practically be achieved. Theoretically, however, one can create axial roller geometry with sharp edges, but an infinite pressure spike cannot be obtained through numerical analysis, since a zero size mesh spacing cannot be created nor indeed integrated. The smaller the elemental size (i.e. the mesh spacing), the larger pressure spikes are obtained, but there is always a limit set by computational stability, the storage capability of the computer used and the CPU time requirement for inordinately large mesh densities.

Figure 2.10 shows the results obtained through finite difference approach by Johns (1978) for the elastostatic contact of an unblended Steel roller of length 0.0127 m and diameter 0.0127 m with a frictionless semi-infinite elastic half-space, also made of Steel. The contact load was 3684 N. The roller was considered to be
aligned in penetrating the elastic half-space. The figure shows the axial pressure distribution through the centreline of the contact footprint, also illustrated by the dog-bone shape in the same figure. 59 elements were used to cover the footprint domain (see figure 2.9).

The finite element approach is used to obtain pressure distribution and the corresponding footprint shape for the same conditions as reported by Johns (1978) (see figure 2.19). The geometric progression mesh employed is described in section 2.8.2, with number of elements being 225000 quadrilateral solid elements. The length of the computation domain was taken to be 15% longer than the roller length. The analysis time was 15-60 hours on a Pentium IV 2.2 GHz machine.

Figure 2.19: Central axial pressure distribution and the footprint shape for the elastostatic aligned contact of a roller against a semi-infinite plane at the load of 3684N, R=12.7mm, L=12.7mm
In practice an idealised sharp edge cannot exist, because of the finite width of tooling which produces a roller. This, automatically in an uncontrolled manner introduces a form of chamfer or radius. In numerical analysis a computational zero element is required to catch an infinite spike, which is not possible. However, as the element width is reduced (i.e. increasing mesh density) the magnitude of spike increases. This effect may not be real in practice as indicated above.

It was noted above that classical Hertzian theory may be used as a first approximation to predict contact conditions. This would yield a very long and narrow ellipse, not the actual dog-bone shape, depicted by figure 2.20. In this case, one may approximate the contact to a very long rectangle, in which the ratio of contact length to its width is given by: $\frac{l}{2a}$, where $l$ is roller length. This ratio is usually very small, and represents an elliptical-type contact, where the ellipticity ratio: $\varepsilon = \frac{b}{a} \rightarrow \infty$. Then, one can state: $P_0 = \frac{2W}{\pi ab}$ (which is the same as equation (2.23) above), where $b = \frac{l}{2}$. The footprint width is then given by equation (2.22) above, which for a contact of a roller on a semi-infinite elastic half-space of the same material reduces to (Timoshenko and Goodier (1951)):

$$a_0 = 1.52 \sqrt{\frac{WR}{El}} \quad (2.32)$$

And:

$$P_0 = 0.418 \sqrt{\frac{WE}{Rl}} \quad (2.33)$$

The maximum shear stress, $\tau_{\text{max}}$ and the depth in the material, $z$ at which it occurs was obtained by Belajef (1924) (see also Bidwell (1962)) as:

$$\tau_{\text{max}} = 0.304 P_0 \quad \text{and} \quad z = 0.78a_0 \quad (2.34)$$
The above analytic expressions can be used to verify the numerical predictions, noting that in this case, the analytic results are less accurate due to their underlying assumptions. When the above expressions are applied to the investigated contact conditions, the following results are obtained: 

\[ a_0 = 0.21 \text{mm} \]  

and  

\[ p_0 = 0.89 \text{ GPa}. \]  

These differ from the FEA predictions as: 

\[ a_0 = 0.16 \text{ mm} \]  

and  

\[ p_0 = 1.3 \text{ GPa}. \]  

These provide differences of 23.8% over-estimation in the contact semi-half-width and 31% under-estimation in the magnitude of centre pressure, when the classical Hertzian theory is used. These confirm the inaccuracy of the classical theory, when applied to the finite line configurations as described previously.

![An isobar plot related to the conditions in figure 2.19](image)

**Figure 2.20: An isobar plot related to the conditions in figure 2.19**

The classical theory also gives the maximum shear stress in the centre of the contact as 0.27 GPa at the depth of 0.19 mm. Figure 2.21 shows the variation of the maximum shear stress in the plane \(yz\) at the centre of the contact: \(x=0\) into the bulk of the contacting elastic solids, obtained by the finite element analysis.

Of course the absolute value of the maximum shear stress does not occur in this plane, but as a localised field beneath the pressure spike in the vicinity of the roller edge. Along the \(z\)-axis (into the depth of the plane) the maximum local value of \(\tau_{\text{max}}\) can be obtained from figure 2.21. This is shown in figure 2.22.
The maximum value in this section (at the centre of the contact) is 438 MPa (also shown in figure 2.21) at the depth of \( z = 0.19 \) mm. Noting that: \( a_0 = 0.16 \) mm, this surmises a relationship of the form: \( z = 1.1875a_0 \). Also: \( \tau_{\text{max}} = 0.337p_0 \).

![Figure 2.21: FEA prediction of maximum shear stress contours for a roller load of 3684 N, R=12.7mm. Units in MPa](image)

The sub-surface stress field is strongly affected by the edge stresses. Johns-Rahnejat (1988) and Johns-Rahnejat and Gohar (1997) have shown that the pressure spikes in the vicinity of roller edge contacts induce a sub-surface field of their own, which result in maximum shear stresses rising toward the surface, as well as having a considerably larger magnitude than those predicted by the classical Hertzian theory. These stresses can lead to local yielding of the material and growth of cracks to the surface of the substrate. To investigate these pressure spike sub-surface induced fields, the contours of \( \tau_{\text{max}} \) under the roller ends are investigated for the case reported above.
Figure 2.22: Variation of $\tau_{\text{max}}$ at $x=y=0$ relating to figure 2.21

Figure 2.23 shows the variation of $\tau_{\text{max}}$ in the axial direction of the contact. Note that the red area represents the island of maximum value of $\tau_{\text{max}}$, being 1.1 GPa, which is: $\tau_{\text{max}} = 0.85 p_0$, for $p_0 = 1.3$ GPa. This value is considerably in excess of the $\tau_{\text{max}} = 0.438$ GPa at $x = y = 0$, described previously. Incidentally, this value of $\tau_{\text{max}}$ is greater than one at $\frac{\sigma_t}{2}$, indicating that local yielding will result according to the Tresca criterion. The depth beneath the contact surface, where this absolute value of maximum shear stress occurs is: $z = 0.045$ mm, which indicates the relationship: $z = 0.28a$, which is far less than the classical value of $0.78a$ for a line contact condition.
Figure 2.23: Contours of maximum sub-surface shear stress under a roller indenting an elastic half-space (an axial distribution in the zx plane for half of the roller contact)

Figure 2.24 shows the same shear stress distribution in the lateral direction at the end of the roller contact. Again the axi-symmetric nature of the contact of an aligned roller indenting a smooth flat plane is used to display half of the distribution.

Figure 2.23: Contours of maximum sub-surface shear stress under a roller indenting an elastic half-space near the roller end (the lateral distribution in the zx plane for half of the roller contact)
The edge pressure spikes are exacerbated when the rollers are tilted due to the moment loading of rotors which they support. Johns (1978) studied the effect of roller misalignment on the contact pressure distribution and the corresponding footprint shape. A similar study was carried out by Rahnejat (1978) for the case of tapered roller bearings, who showed that contact reactions between a tapered roller with the inner and outer raceways and that with the retaining rib cause misalignment of the roller to achieve equilibrium under quasi-static conditions. Misalignment angle is usually very small, typically less than 0.1°. Figure 2.25 shows the results obtained by numerical analysis, based on finite difference technique by Johns and Gohar (1981) for the misaligned contact of a cylindrical roller of radius 3.57 mm, with a dub-off radius of 0.53 mm and a flat axial region of 6.038 mm, subjected to a normal load of 1126 N, with a misaligned angle of 0.1°. Harris (1966) had already obtained an approximate analytic solution for this condition, also shown in figure 2.25.

Figure 2.25: Misaligned contact of a roller on a flat frictionless plane
(after Johns and Gohar (1981) and Harris (1966))
The same condition is simulated here using the finite element technique described above. Figure 2.26 shows the pressure isobar plot of the 3D finite element predictions. Clearly, the pressures increase at the tilted end of the roller, indicated in the figure by the localised island of pressure spike in the vicinity of the roller tilted edge. The misalignment angle is large enough for the lifted end of the roller to lose contact with the semi-infinite elastic half-space, thus indicating a truncated contact patch.

A better quantitative picture emerges when certain cross-sections through the contact are investigated. Figure 2.27 shows the pressure distribution and the corresponding footprint shape. The pressure spike at the tilted end reaches a value of 2.6 GPa, as opposed to the predicted value of 2.45 GPa found by Johns and Gohar (1981), using a coarser mesh density and finite difference analysis, and the calculated value of 1.8 GPa obtained by Harris (1966), using Hertzian theory as an approximation. Therefore, the comparison between the two numerical results is quite good. With the FEA the active contact length is found to be 5.77 mm, whereas the finite difference solution of Johns and Gohar (1981) yields the value of 5.25 mm. The tilted end of the roller is at 3.53 mm, a mere 0.03 mm away from the FEA predicted contact edge.

Figure 2.26: Pressure isobar plot of a misaligned roller near a flat plane under the same conditions as in figure 2.25
Figure 2.27: Centre axial pressure distribution and footprint shape for the elastostatic contact of a tilted roller on a flat frictionless plane under the same conditions as in figure 2.25
Chapter 3

Contact Mechanics: Layered Elastic Solids

3.1- Introduction

The previous chapter deals with contact mechanics of bodies in contact, undergoing deformation when treated as semi-infinite elastic solids. The assumptions embodied there apply reasonably to contact of many load bearing surfaces, usually of non-conforming geometry, such as ball and rolling elements-to-raceway contacts in bearings, cam and follower contact and some gear meshing problems such as helical involute gears. For most conforming contacts the dimensions of the contact are comparable to the principal radii of mating surfaces and thus the semi-infinite assumption is deemed not appropriate. There are many such contacting geometries, where prevailing conditions do not lend themselves to the use of semi-infinite solid assumption, including in journal bearings and piston skirt-to-cylinder bore contact.

Additionally, surface modification often leads to conditions that abrogate the semi-infinite elastic condition, irrespective of the geometry of contacting bodies in their contacting region. These include coated surfaces, often used as protective wear resistant layers or other forms of bonded layers of tribological properties such as thin solid lubricant layers of low elastic moduli, including rubber lined bearings or Molybdenum Disulphide, silver or gold deposited layers on bearing elements. Chapter 1 discusses the applications of such coatings and layered bonded solids. In addition to these bonded layers, there are a host of applications, where unbonded layers and thin shells are employed in tribological applications,
such as thin shells as bushing element of engine journal bearings. Nearly all these thin elements, coating and layers behave as thin elastic solids, whose contact mechanical characteristics deviate significantly from the classical Hertzian theory or its extension to other geometrical contacts, discussed in chapter 2.

This chapter deals primarily with thin bonded elastic layers of low elastic moduli, indented by a frictionless flat circular punch or a spherical body such as a ball bearing. Analytic formulations, similar to work of few other research workers are derived, as well as finite element models for the purpose of comparison with the analytical solutions and experiments that are reported in later chapters. Additionally, the results for elastic layers have been compared with those obtained earlier with semi-infinite elastic half-spaces to highlight the significant differences. Some solutions for approximation of elastic layers as thin strip, using approximate columnar pressure induced deformation are also reported to gauge the extent of validity of this method, against more representative numerical solutions.

3.2- Thin Elastic Layered Solid Subjected to Frictionless Axisymmetric Contact

The displacements in an axisymmetric contact are given as $u, v$ and $w$. Using a cylindrical co-ordinate system, $r, \theta, z$, for the contact of a spherical body or a flat circular punch, the equations for equilibrium of a homogeneous, isotropic, elastic medium are given by Sneddon (1946) as:

$$\frac{\partial \sigma_z}{\partial r} + \frac{\partial \sigma_{\theta}}{\partial z} + \frac{\sigma_z - \sigma_{\theta}}{r} = 0$$

(3.1)

And:

$$\frac{\partial \sigma_{\theta}}{\partial z} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{\tau_{\theta z}}{r} = 0$$

(3.2)
Most elastic solids in contact are compressible, which means: $\nu \neq 0.5$. Of course in some cases, such as rubber the material under load behaves in an incompressible manner: $\nu = 0.5$.

For an incompressible solid under contact load, the stress components are:

\[
\sigma_r = (\lambda + 2\mu) \varepsilon_r + \lambda (\varepsilon_\theta + \varepsilon_z) \quad (3.3)
\]
\[
\sigma_\theta = (\lambda + 2\mu) \varepsilon_\theta + \lambda (\varepsilon_r + \varepsilon_z) \quad (3.4)
\]
\[
\sigma_z = (\lambda + 2\mu) \varepsilon_z + \lambda (\varepsilon_\theta + \varepsilon_r) \quad (3.5)
\]

And:

\[
\tau_n = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \quad (3.6)
\]

Where:

\[
\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad (3.7)
\]

And, the Lame's constants: $\lambda$ and $\mu$ are given as:

\[
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \quad (3.8)
\]

In practice, elastic layers and strips are either bonded to a substrate by suitable adhesives or coated upon it by a chemical process such as chemical or physical vapour deposition, sputtering or flash evaporation. Unbonded layers and strips are either held by friction or form a thin shell or a thin plate. In either case, their thickness, $d$, plays an important role in their contact behaviour. They are also usually assumed to be isotropic and homogeneous to make the solution to the
problem somewhat tractable. The above formulation method can then lead to an analytic solution by the application of appropriate boundary conditions.

3.3- Frictionless Indentation of a Layered Elastic Solid by a Flat-Ended Circular Punch

Equation (2.18), given by Johnson (1985) for a flat-ended circular indenter, penetrating a semi-infinite elastic solid (see figure 2.4) is for a strip of unit width. This expression was first obtained by Sadowsky (1928), giving an exact pressure distribution for the case of a layered solid with the ratio: \( \frac{a}{d} = 0 \), which means a layer thickness: \( d \to \infty \), translating to a semi-infinite elastic half-space. For the ratio of zero, Sadowsky (1928), using the same approach as in section 2.3 for a finite width strip, b showed that:

\[
p(x) = \frac{W}{\pi b (a^2 - x^2)^{1/2}}
\]

Therefore, equation (2.18) is a special case of the above expression, where \( b = 1 \) (a semi-infinite half-space analysis of unit width).

When the aforementioned ratio is increased, the contact dimension (here being the radius of the flat-ended circular punch) becomes progressively comparable to the thickness of the strip, and no longer one may assume the semi-infinite contact conditions. To solve this problem a similar approach as that employed by Johns (1978), Johns and Gohar (1981) and Heydari and Gohar (1979) all for the contact of a cylindrical roller near a semi-infinite elastic half space and that of Rahnejat (1978) and Rahnejat and Gohar (1979) for a tapered roller under the same conditions is employed. Chapter 2 highlights the numerical solution approach undertaken by these research workers, as well as a finite element method undertaken in this thesis. This approach is highlighted by Conway et al (1966).
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The contact is discretised into a number of incremental lengths and the columnar pressures over each of these rectangular elements are assumed to be constant for the case of a flat punch. Conway et al (1966) have shown that for a sufficiently large number of elements and for a slowly varying pressure gradient along the contact the degree of approximation is quite adequate. If an elastic strip of infinite length and thickness $d$ is compressed with a force $W$, uniformly distributed over a length of the strip, the resulting plane stress condition can be analysed by adopting a stress function approach, in which the stress function satisfies the following differential equation in Cartesian co-ordinates as:

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)^2 \varphi = 0
$$

(3.10)

The stresses are given as:

$$
\sigma_x = \frac{\partial^2 \varphi}{\partial z^2}, \quad \sigma_z = \frac{\partial^2 \varphi}{\partial x^2}, \quad \tau_{xz} = -\frac{\partial^2 \varphi}{\partial x \partial z}
$$

(3.11)

The stress function is written in the Fourier integral form as:

$$
\varphi = \int_{\ell=0}^{\infty} \frac{1}{\ell^2} \left( A \cosh \ell z + \ell B \sinh \ell z \right) \cos \ell x d\ell
$$

(3.12)

Now it is necessary to impose problem-specific boundary conditions. These are imposed at the horizontal contact face of the rigid flat-ended punch and the rigid substrate, beneath the strip as:

$$
\sigma_{z=\pm d} = -\frac{W}{\pi b} \int_{\ell=0}^{\infty} \frac{\sin \ell \ell}{\ell} \cos \ell x d\ell, \quad \tau_{x=z=\pm d} = 0
$$

(3.13)
Note that the length of the strip is considered to be infinite in dimension and the assumed frictionless contact necessitates no shear stress variation at the interface between the punch and the strip.

The above boundary conditions enable the evaluation of the constants $A$ and $B$ for each element of the discretised model in equation (3.12). The displacements on the horizontal boundary can then be obtained by integrating the stress-strain relations as follows:

$$\frac{E}{1-n} \frac{\partial u}{\partial x} = \sigma_x - n\sigma_z, \quad \frac{E}{1-n} \frac{\partial v}{\partial z} = \sigma_z - n\sigma_x \tag{3.14}$$

The displacements at the boundaries are finally obtained as (Conway et al (1966)):

$$u_{z=\pm d} = \frac{2W a}{\pi E} \int_{0}^{\infty} \frac{1}{a d} \frac{\sin a \ell (1-n) \sinh 2\ell d - 2\ell d (1+n)}{2\ell d + \sinh 2\ell d} \sin \ell x d(\ell d) \tag{3.15}$$

And:

$$v_{z=\pm d} = \frac{8W a}{\pi E} \int_{0}^{\infty} \frac{1}{a d} \frac{\sin a \ell \sinh^2 \ell d}{2\ell d + \sinh 2\ell d} \cos \ell x d(\ell d) \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.16}$$

Using the above equations the resultant equivalent vertical displacements for the centres of each element can be obtained as functions of columnar constant pressures, $p$, for each ratio of $\frac{a}{d}$ (Conway et al (1966)). For the ratio $\frac{a}{d} = 0$, an analytic solution was originally obtained by Muskhelishvili (1953), for a semi-infinite solid as:
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\[ p = \frac{W}{2\pi b (a^2 - x^2)^{1/2}} \frac{4}{1 + v} \cos \left( \frac{\log \left( \frac{3 - v}{1 + v} \right) - \log \left( \frac{a + x}{a - x} \right)}{2\pi} \right) \]  

(3.17)

Note that the above equation is divided by 2, since the derivation highlighted above makes use of two indenting surfaces, compressing an elastic strip of thickness 2d. In the case discussed here a thickness of \( d \) is used.

Conway et al (1966) provided solution for pressure distributions for other ratios: \( \frac{a}{d} = 0, 1, 4 \). Figure 3.1 illustrates the results of pressure distribution for the various values of \( \frac{a}{d} \) ratio, which for a value of zero corresponds to that obtained by Muskhelishvili (1953) for a semi-infinite elastic half-space. As the ratio is increased the pressure distribution becomes more uniform along the contact length, as is expected for thin strips (note that an increasing value for the ratio indicates a decreasing value for the thickness of the layer). Note also that the contact pressures increase as the strip thickness is decreased. Under a constant contact load the area under all the pressure distributions are the same. This means that the pressures at the extremities of the contact decrease for thinner layers. However, for an indenting punch with an unblended edge profile the edge pressures tend to an infinite value due to edge stress discontinuity there. This is an unreal effect since no punch has an idealised sharp edge.
Figure 3.1: Normal and shearing stress distribution under axially loaded flat-ended frictionless indenter applied to an isotropic strip (after Conway et al (1966))

It should be noted that the local deflection at the centre of each increment or computation element is not only affected by the columnar pressure acting over it, as is the case for the derived expressions reported in this section and those employed by Conway et al (1966), but by the entire contact pressure distribution. However, for materials of low elastic moduli, this assumption yields good approximate results. The assumption of column method, highlighted here can be used for any shell, strip or unbonded layer to determine the elastic deformation of the thin layer elastic solid.

3.4- Column Method for Determination of Deformation of Thin Solids of Low Elastic Moduli

When the strip or shell is of low elastic modulus one may employ the column model. The simplified theory provides an analytic solution, where the deflection at
any given location is considered to be wholly due to the column of pressure directly acting above the same locality. This simplified theory is highlighted here and can be employed as a first approximation for many applications where an unbonded elastic solid (be it a shell or a strip or tape) subjected to an arbitrary pressure distribution, and where the initial undeformed separation of the two bodies can be described in terms of a polynomial shape. However, for cases of bonded solids, and particularly with high elastic moduli, the method is inappropriate, and a generalised theory is developed in the following section.

The elastic properties of a strip or shell can be determined through its stress-strain relations, as:

\[
\{\varepsilon\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\} \quad (3.18)
\]

Where, the elasticity matrix \([D]\) can be obtained simply from the usual isotropic stress-strain relations. For the general stress-strain relations, following the method highlighted by Lekhnitskii (1963), ignoring the initial stresses and strains and taking \(z\) as the direction normal to the surface of the bonded thin-layered solid:

\[
\varepsilon_z = \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right] \quad (3.19)
\]

\[
\varepsilon_y = \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right] \quad (3.20)
\]

\[
\varepsilon_z = \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right] \quad (3.21)
\]

\[
\gamma_{xy} = \left\{ \frac{2(1 + \nu)}{E} \right\} \tau_{xy} \quad (3.22)
\]

\[
\gamma_{xz} = \frac{1}{G} \tau_{xz} \quad (3.23)
\]

\[
\gamma_{yz} = \frac{1}{G} \tau_{yz} \quad (3.24)
\]
Thus, for an isotropic material the equations (3.19)-(3.24) can be inverted and replaced into equation (3.18) to obtain the various elements of the stress tensor. The elasticity matrix provides the influence coefficients. Hence:

\[
\begin{bmatrix}
1 & \frac{v}{1-\nu} & \frac{v}{1-\nu} \\
\frac{v}{1-\nu} & 1 & \frac{v}{1-\nu} \\
\frac{v}{1-\nu} & \frac{v}{1-\nu} & 1
\end{bmatrix}
\]

The deflection component due to the generated pressures can be simplified by assuming that the local deflection at any location is affected only by the element of pressure directly acting above it. The effects of other pressure elements are thus ignored. This assumption is valid for thin shells made out of materials of low elastic moduli. Therefore, the element of the above matrix, of interest here, is the 

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1-2\nu & 0 \\
0 & 0 & 0 & 1-2\nu \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(3.25)

Therefore, for any arbitrary pressure distribution imposed upon a thin strip or shell of low elastic moduli or a bonded layer the deformation of the elastic solid can be obtained. Rahnejat (2000), Gupta (2003) and Gupta et al (2002) describe the use of the column method in respect of engine journal bearings with thin shells fixed onto the rigid supports of the engine block, where the deformation of the shell
occurs due to generated lubricant film pressures, inducing elastohydrodynamic regime of lubrication.

3.5- Generic Asymptotic Solution for Thin Bonded Layers

Asymptotic solutions for contact pressures have now been established, for example by Jaffar (1989), Barber (1990), Naghieh (1999) and Naghieh et al (1998 a,b, 1997). These solutions apply for the cases, where: \( \frac{a}{d} \gg 1 \), usually above the value of 2. To find an analytical solution, the formulation method in section 3.2 is used, and appropriate boundary conditions are applied. In the case of the literature referred to above, a circular point contact condition is employed, with the following boundary conditions:

- On the free surface at \( z=0 \):

\[
\sigma_z(r) = 0 \quad \text{for} \quad r \geq 0 \\
\tau_n(r) = 0 \quad \text{for} \quad r \geq 0 \\
\omega(r) = \delta - f(r) \quad \text{for} \quad 0 \leq r \leq a
\]

Where \( f(r) \) is the undeformed profile of the indenting solid, and \( f(r) = \frac{r^2}{2R} \) for a hemispherical-ended indenter or a ball contacting an elastic layer.

- For \( z=d \) (i.e. at the interface of the layer with the indenter):

\[
u(r) = \omega(r) = 0 \quad \text{For a bonded layer}\]

Furthermore, note that:

\[
\sigma_z(r) = -p(r) \quad \text{for} \quad 0 \leq r \leq a
\]
With the above assumption, it can be noted that for a bonded layer contact: 
\( \varepsilon_r = \varepsilon_\theta = 0 \).

The compressive strain in the layer is given by the geometry of deformation as:

\[
\varepsilon_z = \frac{\partial \omega}{\partial z} = -\frac{1}{d} \left( \frac{\delta - \frac{r^2}{2R}}{\delta} \right) \tag{3.30}
\]

Now substituting the strain components: \( \varepsilon_r, \varepsilon_\theta, \varepsilon_z \) into equation (3.5), the pressure distribution and normal stress can be obtained as:

\[
p(r) = -\sigma_z(r) = \frac{1}{d} \left( \frac{\lambda + 2\mu}{\lambda} \right) \left( \delta - \frac{r^2}{2R} \right) \tag{3.31}
\]

For the condition \( p(r=a)=0 \), the above expression yields: \( \delta = \frac{a^2}{2R} \), which can be replaced into the same equation, together with the Lame’s constants to yield the final form of the pressure distribution for layered bonded elastic solids as:

\[
p(r) = \frac{E(1-\nu)}{2R(1+\nu)(1-2\nu)} \left( \frac{a^2 - r^2}{d} \right) \tag{3.32}
\]

where, for the compressible layered solid the contact radius, \( a \) is given as:

\[
a = \left( \frac{(1-2\nu)(1+\nu)dRW}{\pi E(1-\nu)} \right)^{\frac{1}{4}} \tag{3.32a}
\]

The above equation holds for all compressible materials with \( \nu < 0.5 \) and \( \varepsilon_r + \varepsilon_\theta + \varepsilon_z = 0 \) (for an incompressible solid).
Note that if $\nu = 0.5$, for an incompressible layer, the above equation (due to Johnson (1985)) implies infinite contact pressures. This is because the constitutive equations (3.3)-(3.5) do not hold and simplify to the following form (Jaffar (1989) and Naghieh(1999):

\begin{align*}
\sigma_r &= -p_h + 2\mu \varepsilon_r \quad (3.33) \\
\sigma_\theta &= -p_h + 2\mu \varepsilon_\theta \quad (3.34) \\
\sigma_z &= -p_h + 2\mu \varepsilon_z \quad (3.35)
\end{align*}

Where $p_h$ is the hydrostatic pressure, which should be determined as a function of $r$ and $z$ as:

\begin{align*}
\frac{\partial p_h}{\partial z} &= \mu \left\{ \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} + \frac{\partial^2 \omega}{\partial z^2} \right\} \quad (3.36)
\end{align*}

And:

\begin{align*}
\frac{\partial p_h}{\partial z} &= \mu \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right\} \quad (3.37)
\end{align*}

Both Naghieh (1999) and Jaffar (1989) have proposed the horizontal displacement $u$ to be a second order polynomial function of depth, $z$, thus:

\begin{align*}
u &= c_2 z^2 + c_1 z + c_0 \quad (3.38)
\end{align*}

Where, the above constants are functions of $r$.

For thin elastic layers, it is clear that: $\frac{\partial}{\partial z} \gg \frac{\partial}{\partial r}$, thus the above equations become:
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\[
\frac{\partial p_h}{\partial z} \approx 0 \text{ or } \frac{\partial^2 \omega}{\partial z^2} \approx 0
\]  \hspace{1cm} (3.39)

And:

\[
\frac{\partial p_h}{\partial r} \approx \mu \frac{\partial^2 u}{\partial z^2}
\]  \hspace{1cm} (3.40)

This indicates that the hydrostatic pressure, \( p_h \), can be considered as a function of \( r \) only:

\[
\sigma_r = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial r} \right) \approx \mu \frac{\partial u}{\partial z} \quad \text{(using the above assumption)}
\]  \hspace{1cm} (3.41)

Now replacing from equation (3.38) into the above equation, the following solution is obtained:

\[
\sigma_r = \mu(2zc + c_1)
\]  \hspace{1cm} (3.42)

The boundary conditions to be used are \( \sigma_r = 0 \) at \( z = 0 \), which indicates that \( c_1 = 0 \). Furthermore, \( u = 0 \) at \( z = d \), which results in:

\[
u = c(z^2 - d^2)
\]  \hspace{1cm} (3.43)

Note that for an incompressible material:

\[
\varepsilon_r + \varepsilon_\theta + \varepsilon_z = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial \omega}{\partial z} = 0
\]  \hspace{1cm} (3.44)
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One can now replace for $u$ from equation (3.43) and its derivative with respect to $r$ (equation 3.45 below) into equation (3.44) and obtain equation (3.46) (noting that

$$
\varepsilon = \frac{\partial \omega}{\partial z} = -\frac{1}{d}\left(\delta - \frac{r^2}{2R}\right)
$$

for a spherical body indenting a flat).

$$
\frac{\partial u}{\partial r} = \frac{\partial c}{\partial r} \left( z^2 - d^2 \right)
$$

(3.45)

$$
\left( \frac{\partial c}{\partial r} + \frac{c}{r} \right) (z^2 - d^2) = \frac{1}{d}\left( \delta - \frac{r^2}{2R} \right)
$$

(3.46)

The solution for $c$ is obtained as:

$$
c = -\frac{3r}{4d^3} \left( \delta - \frac{r^2}{2R} \right)
$$

(3.47)

Thus:

$$
u = -\frac{3r}{4d^3} \left( \delta - \frac{r^2}{2R} \right) (z^2 - d^2)
$$

(3.48)

Replacing for $u$ in equation (3.40) and by subsequent integration, the hydrostatic pressure is obtained as a function of $r$ as:

$$
p_s = -\frac{3\mu r^2}{4d^3} \left( \delta - \frac{r^2}{2R} \right) + A
$$

(3.49)

where the constant of integration, $A$ is a function of $r$.

For a spherical body, penetrating with the depth, $\delta$ into an elastic incompressible layer, the following condition applies:
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\[ \iint_D \left( \delta - \frac{r^2}{2R} \right) r dr d\theta = 0 \]  

(3.50)

where, \( r dr \theta \) is an element of the area of the circular point contact of radius \( a \).

The above integration can be carried out as follows:

\[ \iint_D \delta r dr d\theta = \int_0^{2\pi} \frac{r^3}{2R} dr d\theta = \int_0^{2\pi} \frac{r^4}{2R} d\theta \]

\[ = \frac{1}{4} \left[ \frac{r^4}{2R} \right]_0^a \left[ \theta \right]_0^{2\pi} = \frac{\pi a^4}{4R} \]

(3.51)

The left-hand side can be integrated as:

\[ \iint_D \delta r dr d\theta = \int_0^{2\pi} \delta r dr \int_0^a d\theta = 2\pi \int_0^a \delta r dr \]

(3.52)

Now equate (3.51) and (3.52) to yield:

\[ \delta \int_0^a r dr = \delta \left[ \frac{r^2}{2} \right]_0^a = \frac{a^4}{8R} \]

(3.53)

Thus:

\[ \delta = \frac{a^2}{4R} \]

(3.53)

Note that the maximum penetration is half that obtained for compressible layered solids, as indicated above.

Now note that: \( p(r) = p_h \) for \( r < a \) for incompressible solids and \( p(r) = 0 \) at \( r = a \).

Thus, using equation (3.49) and imposing the boundary condition above:
\[ A = \frac{3\mu a^4}{32Rd^3} \]  

(3.54)

And substituting for \( A \) in equation (3.49), the pressure distribution for a bonded thin layered elastic solid indented by a sphere is obtained as:

\[ p(r) = \frac{3\mu a^4}{32Rd^3} \left(1 - \frac{r^2}{a^2}\right)^2 \]  

(3.55)

### 3.6- Results and Discussion

In this section analytical and finite element solutions for layered solids are presented for a number of cases. These include circular point contacts and finite line contacts. Some solutions for a circular flat ended punch against a layered solid is also provided later in chapter 6 for viscoelastic layered solids.

#### 3.6.1- Finite element mesh construction

The analytical solutions for circular point contact of a rigid ball indenting a layered elastic solid is undertaken, using a programme written in MATHCAD, based on the theory highlighted in section 3.5. Note that in this case \( \frac{a}{d} > 2 \). In practical terms, such conditions exist for ball to races contacts or other load bearing surfaces, where the raceway is coated with an elastic layer (also refer to chapter 1). To represent such cases, the modulus of elasticity of the bonded solid is considered to be the effective reduced modulus of the contacting bodies is given as: \( E' = \frac{2}{\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}} \). The ball is then considered to be a rigid hemispherical indenter (see figure 3.2). The layered solid or the coating is rigidly fixed to a rigid substrate. The thickness of the elastic layer is varied as desired.
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Figure 3.2: Close-up of a rigid ball indenting a layered solid attached to a rigid substrate (not meshed)

The model is created in PATRAN, where two dimensional quadrilateral plane strain continuum mechanics elements are used to discretise the layered solid. Rigid beam elements are used with defined rigid surfaces for the indenter. The length of the coating on the substrate should be long compared to the area of contact. This is necessary in order to ensure that tensile stresses (as the result of recovery of the material outside the contacting region) do not affect the calculated contact compressive stresses/pressures. The computational contact length is assumed to be $\frac{1}{4}$ of the circumference of the indenting rigid ball, thus, far greater than the thickness of the elastic coating, being typically 25-75 μm in most of the simulated cases reported here. The corresponding contact radii are typically 0.02-0.3 mm, depending on load and material parameters. The radius of the ball is taken to be 6mm. The length of contact is taken to be at least $6a$. This means that a regular mesh construction would yield a coarse mesh in the contacting region and a substantial rather unnecessary computing effort outside the domain of high pressures. In order to improve the accuracy of predictions and reduce the computation burden outside the contact area, a geometric mesh is constructed. A typical mesh consists of 200 elements along the contact length generated.
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according to a geometric progression, with 30 elements with regular spacing in the depth of the layered bonded solid, making a total of 6000 elements (see figure 3.2). Therefore, elements have a variable length and a constant depth. The geometric progression ratio, \( q \) is so chosen to render, as far as possible, rectangular elements within the contacting region. Typical value for \( q \) in this thesis was set as: \( q^{n-1} = \frac{L_n}{L_1} \), where \( n \) is the number of elements along the length of the layered solid, typically 200, \( L_n \) is the length of the last element, further-most from the contact centre and \( L_1 \) is the length of the element at the centre of the contact. This ratio is chosen to be 100, therefore, \( q=1.023 \) Long narrow elements cause computational inaccuracies as described by Zienkiewicz (1967). With the constructions, undertaken in this thesis, some elements far away from the contacting region may suffer from such a problem, which have no significance upon the results obtained. A typical element in the contacting region has a length of 1.8 \( \mu \text{m} \), and a typical depth of 1.2 \( \mu \text{m} \).

In the simulation study the contact load was varied typically from 0.1N to 250N for various layer thickness, simulating ball loads in the loaded and unloaded region of a bearing. The created models in PATRAN are transferred to ABAQUS for simulation studies. The simulations represent quasi-static analysis. This means that equilibrium conditions are assumed at a given instant, clearly not taking into account the lubricated conditions with entraining motion, which prevail in practice. It is now well established that the generated elastohydrodynamic pressures conform closely to the elastic pressure distribution, except for the inlet trail prior to the region of high contact pressures and a pressure spike or “pip” that occur in the vicinity of the outlet due to flow continuity condition (see Dowson and Higginson (1966). Therefore, the elastostatic dry contacts assumed here are representative of actual prevailing conditions.

The typical computing time was 30 minutes to 2 hours on a Pentium IV 2.5 GHz machine. The computing time varies in the stated range, mostly dependent upon the depth of the layer. Thinner coatings at higher loads result in longer computation times.
For the above stated simulation conditions, a ball bearing support for a horizontal rotor is considered, with 12 balls. The load per ball alters as it traverses into and out of the loaded region of the bearing. The bottom 5 balls constitute the loaded region. The load per ball alters from 0.1N to 250N. The steel balls in the bearing are made of cobalt-chromium steel with a modulus of elasticity of 210 GPa, and the raceways are made of steel, furnished with a ceramic coated layer (Alumina) of 0.06 mm thickness with modulus of elasticity of 400 GPa. The Poisson’s ratio for Steel is 0.3, whilst that of Alumina is 0.23.

As discussed later, it was noted that the prevailing contact conditions fall in between the two established analytical theories: that of semi-infinite contacts (represented here by the classical Hertzian theory) and that of layered elastic solids described in the previous sections. Consequently, finite element predictions, using the 2D analysis and axi-symmetric conditions may lead to results that could not be corroborated, and in extreme cases may lead to erroneous conclusions. In particular, this problem occurs in the case of ceramic coatings and is resolved by the use of 3D finite element models, where instead of 4-noded quadrilateral elements, more accurate 8-noded solid elements, embodying a much greater shape function capability is used. Clearly, the averaging process for the determination of stress at a node becomes much more accurate with consideration of 8 neighbouring nodes in the constructed 3D meshes. For such 3D analysis a segment of the problem is considered (see figure 3.3), using the symmetrical properties of the contact problem. In the case of the ball a 10% segment of its volume is employed with a mesh density of 120 by 80 elements, with the former in radial horizontal directions. The segment is one element deep. Therefore, the final mesh is 120 by 100 by 1 or 12000 elements in total. The top row of horizontal radial elements are constrained and a 1/36 th of the contact load is imposed for the analysis. In such a segment 120 elements are used along the length of the layered solid substrate, with 7 such rows of elements into its depth (see figure 3.4). The drawbacks in 3D analysis are the required computational memory and CPU time. The analyses of this detail had to be undertaken on Pentium IV 2.5 GHz machines, with typical computation times of the order of 1-6 hours, depending on contact load, layer thickness and model meshing. Therefore,
the 3D analysis was carried out to ascertain the validity of the results obtained from the 2D analyses, wherever possible.

Figure 3.3: An axi-symmetric 3D segmental model of a ball indenting a layered solid

Figure 3.4: Close up of a segment of a 3D mesh of a ball (upper construction) indenting a geometrical progression mesh constructed into the depth of a layered elastic solid (lower construction)
3.6.2 - Elastostatic contact of ball bearings indenting hard coated races

(a)- Ceramic races

Returning to the simulation conditions, for a ball contact \( R = 6 \text{mm} \), commencing from the unloaded region with a nominal load of 0.1N (representing almost complete off-loading of ball), the contact radius, \( a \) increases gradually as it progresses toward the loaded region, where it eventually suffers a load of 250N. The \( \frac{a}{d} \) ratio in the unloaded region is 0.3 and in the most-loaded position, it becomes 2.25. These values are obtained analytically, using the outlined theory in section 3.5. It can be observed that in most of the region the behaviour of the contact falls in between the two theories; one of a semi-infinite solid, described by the classical Hertzian theory and that of a layered bonded solid as highlighted in section 3.5. Here accurate evaluation of pressure distribution and sub-surface stress field necessitates the use of finite element analysis. In the loaded region, the contact mechanics characteristics follow the analytical elastic bonded layered solid theory, as can be seen later.

The results obtained from the 2D analyses are listed in table 3.1. For each value of load, the \( a/d \) ratio is calculated according to both the classical Hertzian theory and that for bonded layered solids. These are shown in columns 2 and 3 in the table (from left hand side). Column 4 lists the actual value obtained by 2D finite element analysis. Unlike table 2.1 in chapter 2, where the exact solution is found through analytical solution given by the Hertzian theory, here there is no exact solution for \( a/d \) ratios falling in between 0.25 (below which the classical Hertzian theory applies) and 2 (above which the layered solid theory provides a good approximate solution). For the conditions investigated here, the predictions by finite element analysis are likely to be more accurate than the analytical solutions, of course within the confines of the mesh density employed.
Columns 5 and 6 provide the percentage errors between the FEA predictions and the bonded layered elastic and the classical Hertzian theory (for semi-infinite assumption) respectively. Note that except for the lowest loads, the finite element results conform closely to the predictions made by the layered elastic theory. In fact remarkable agreement can be observed for both the value of \( \frac{a}{d} \) ratio and the corresponding maximum pressure between the layered elastic theory and FEA for contact loads greater than 10 N. The error between pressure values FEA with the layered and the semi-infinite analytical solutions are given in columns 10 and 11 respectively. The semi-infinite assumption leads to grossly inaccurate prediction of contact area (constantly over-estimating it for higher loads) and the maximum pressure (dramatically under-estimating). This is not surprising as the \( \frac{a}{d} \) ratio in column 3 is consistently outside the range of the Hertzian assumption, except for nearly unloaded condition, where the FEA results are likely to be incorrect, because of the very small contact radius, containing only few elements. The overall conclusions from this study indicate that for thin elastic layers, such as the case under investigation the bonded layered solid theory is applicable, and the use of classical Hertzian theory leads to grossly incorrect predictions, unless the contact load is very small, leading to insignificant local deformation of the bonded layer compared to its actual thickness. This finding makes physical sense. The
implication of the findings is of significant practical importance, indicating that the use of classical theory to predict fatigue life of coated bearings is not prudent.

As far as finite element approach is concerned, at low loads, due to small contact area and low deflection of the coated layer, much finer mesh is required to estimate contact conditions accurately. In order to achieve this, it is prudent to use 3D analysis of the contact conditions. This has a further advantage, in that for coatings of high elastic moduli, the rigid body motion of the ball is arrested through its local deformation, which requires the inclusion of ball’s elastic behaviour, abandoning its description as a rigid body. The 3D analysis would also incorporate the use of computationally more accurate 3D solid elements, embodying additional shape functions, as described above.

The 3D finite element model is described above, where the ball is made of Steel, with: \( E = 210 \text{ GPa} \), \( \nu = 0.3 \), and the race has a coating of 60 \( \mu \text{m} \) thick of Alumina with: \( E = 400 \text{ GPa} \), \( \nu = 0.23 \). Figures 3.5-3.7 show the pressure distribution through the centre of the contact for a combination of analytical and 2D and 3D FEA predictions for loads 0.1N, 100N and 250N respectively. In all the figures, the predictions by the 3D FEA is shown by red dotted distribution, that by the 2D FEA is indicated in blue, and the layered analytical solution also in blue by dashed-single dot lines. At the highest load (see figure 3.7), the 2D FEA and the analytical solution conform closest. This is expected as in the 2D FEA, the ball is considered rigid to emulate the analytical formulation, and consequently the extent of penetration is not arrested by the deformation of the ball itself.

The same is also true of the contact load 100 N in figure 3.6. Consequently, the pressures are somewhat reduced in the 3D analysis, whilst the contact area is increased in order to adhere to the same elastostatic conditions. However, at the lowest load of 0.1 N (representing an unloaded ball bearing, see figure 3.5 and table 3.1), the 2D FEA solution deviates significantly from either of the analytical models. It is, therefore, unclear as to which theory best describes the conditions here. The 3D analysis falls in between the two theories, but conforms closer to the classical Hertzian analytical solution. This confirms the arguments made.
previously that at low loads with very localised penetration, in the case, of the hard coatings the contact mechanical behaviour tends to semi-infinite behaviour.

**Figure 3.5:** Predicted pressures for a Steel ball indenting an Alumina coated race (contact load 0.1N), $d=0.06\text{mm}$, $R=6\text{mm}$

**Figure 3.6:** Predicted pressures for a Steel ball indenting an Alumina coated race (contact load 100N), $d=0.06\text{mm}$, $R=6\text{mm}$
Figure 3.7: Predicted pressures for a Steel ball indenting an Alumina coated race (contact load 250 N) d=0.06mm, R=6mm

Overall, the 3D analysis is of more generic nature, since it conforms to the analytical and 2D FEA solutions at both ends of loading spectrum; in the loaded and the unloaded regions of such bearings. None of the other analytical or finite element approaches enjoy the same degree of conformance to physical reality. However, the computational costs in time and storage requirements represent the drawbacks in 3D FEA contact analysis.

(b)- Raceway coated with Titanium Nitride (TiN)

Titanium Nitride with a modulus of elasticity of 600 GPa and Poisson’s ratio of 0.25 can be used as a hard coating in many load bearing applications. The advantage of very hard coatings introduced on the rolling and sliding surfaces by Physical Vapour Deposition (PVD) or Chemical Vapour Deposition (CVD) techniques is to guard the substrate materials from the high contact stresses. This is studied later in a more generic form. Furthermore, the wear resistant characteristics of such coatings makes them suitable for applications, where the formation of a lubricant film is not assured, such as under low speeds of entraining motion. Other applications of such coatings are in tools and particularly
under dry cutting conditions, such as in the machining of wood and wood composite, some metal matrix composites or other similar abrasive materials.

In this analysis the same configuration as in the previous case is used, except that the raceway is coated with a 10 µm thick layer of TiN, as the bonded layered solid. Therefore, for this analysis: \( d = 10 \) µm, and \( R = 6 \) mm. The elastostatic analyses results under different loads are list in table 3.2. In this table the second and third columns (from left hand side) list the contact radii obtained through use of bonded layered elastic analytical theory and FEA respectively.

<table>
<thead>
<tr>
<th>Load (N)</th>
<th>( a )-Analytical (µm)</th>
<th>( a )-FEA (µm)</th>
<th>% Deviation</th>
<th>( p ) Analytical (GPa)</th>
<th>( p ) FEA (GPa)</th>
<th>% Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.2</td>
<td>19.0</td>
<td>-6.24</td>
<td>1.56</td>
<td>1.63</td>
<td>4.14</td>
</tr>
<tr>
<td>5</td>
<td>30.2</td>
<td>29.2</td>
<td>-3.37</td>
<td>3.49</td>
<td>3.55</td>
<td>1.58</td>
</tr>
<tr>
<td>10</td>
<td>35.9</td>
<td>34.6</td>
<td>-3.74</td>
<td>4.94</td>
<td>5.00</td>
<td>1.18</td>
</tr>
<tr>
<td>20</td>
<td>42.7</td>
<td>42.0</td>
<td>-1.63</td>
<td>6.99</td>
<td>7.06</td>
<td>1.02</td>
</tr>
<tr>
<td>50</td>
<td>53.7</td>
<td>52.3</td>
<td>-2.71</td>
<td>11.05</td>
<td>11.20</td>
<td>1.35</td>
</tr>
</tbody>
</table>

Table 3.2: Analytic/FEA comparison for high modulus bonded layered solid results (TiN)

The fourth column provides the percentage error, in this case the under-estimation of contact radius by FEA, when compared with the analytical theory for bonded layered solid. Consequently, the percentage error on the maximum pressure in column 7 indicates a corresponding over-estimation by the 2D FEA. Note the very high maximum pressure generated over the minute contact area. The high degree of localisation of pressure and its non-transference (as shown later) to the substrate material forms the physical basis for the deployment of such hard coatings, which can withstand these pressures due to their high yield stresses. However, hard coatings usually suffer from bad thermal conductance, and their failure occurs due to thermal cracking and fracture. Thermoelastic stresses are not the subject of this thesis.
Figure 3.8 shows this good agreement between the developed FEA technique and the theory at the load of 50 N.

![Figure 3.8: Predicted pressures for a Steel ball indenting a TiN coated race (contact load: 50 N) d=10 μm, R=6mm](image)

3.6.3- Elastostatic contact of ball bearings indenting soft coated races

In many other applications, the use of soft coatings is preferred due to their tribological properties. One of these desired properties is their larger deformation, leading to the creation of a sufficient gap for a larger fluid film to be retained by entraining action or by entrapment. In some cases large enough pressures can cause plastic flow and reformation of the soft coating. In other words, the coating would act as a solid lubricant, such as MoS₂ (Molybdenum Disulphide) coating in some bearing applications or Gold and Silver coatings on balls, raceways or both in bearings in space applications, where a lubricant cannot be used due to very low temperatures render high lubricant viscosities causing undue traction and loss of flow capability.

A feature of soft coatings is the transfer of load/pressures to harder substrate material, leading to its survival, whilst still protecting the substrate due to its low coefficient of friction. This is particularly useful in low load and low speed
applications such as in instrument bearings and satellite thruster mechanism bearing supports of precision rotors, where in the case of the former the existence of excessive fluid can lead to lubricant churning effect, which can lead to thermal distortion and loss of accuracy due to small out-of-balance motions.

In the analyses carried out here two cases have been considered, in both of which the contacting geometry and loading is kept as those reported above, with the exceptions of using Gold or Silver as soft coatings on semi-infinite elastic half-space, being indented by a rigid ball of radius 6 mm. In both cases the thickness of the coating is taken to be 35 μm.

When Gold coating is used, $E = 79$ GPa and $\nu = 0.42$. The results obtained by 2D FEA and the analytic bonded layered solid theory are listed in table 3.3 below. There is good agreement between the two sets of results, as also indicated in figure 3.9 for the contact load of 50N. The percentage errors in columns 4 and 7 of the table refer to the deviation of the 2D FEA predictions from the analytical predictions.

<table>
<thead>
<tr>
<th>Load (N)</th>
<th>$a$-Analytical (mm)</th>
<th>$a$-FEA (mm)</th>
<th>% Deviation</th>
<th>$p$-Analytical (GPa)</th>
<th>$p$-FEA (GPa)</th>
<th>% Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0627</td>
<td>0.067</td>
<td>6.41</td>
<td>3.238</td>
<td>2.81</td>
<td>-15.2</td>
</tr>
<tr>
<td>30</td>
<td>0.0693</td>
<td>0.072</td>
<td>3.62</td>
<td>3.966</td>
<td>3.48</td>
<td>-14.0</td>
</tr>
<tr>
<td>50</td>
<td>0.0788</td>
<td>0.083</td>
<td>5.23</td>
<td>5.120</td>
<td>4.89</td>
<td>-4.71</td>
</tr>
<tr>
<td>100</td>
<td>0.0937</td>
<td>0.096</td>
<td>2.33</td>
<td>7.241</td>
<td>6.70</td>
<td>-8.08</td>
</tr>
</tbody>
</table>

Table 3.3: Analytic/FEA comparison for low modulus bonded layered solid results (Gold)
When Silver layered coating is used: $E = 83$ GPa and $\nu = 0.37$. The results obtained by 2D FEA and the analytical model are provided in table 3.4. Similar trend of good agreement between the aforementioned methods is obtained to that of Gold coated substrate. Figure 3.10 shows the close conformance of the alternative analysis for the contact load of 50 N.

<table>
<thead>
<tr>
<th>Load (N)</th>
<th>$a$-Analytical (mm)</th>
<th>$a$-FEA (mm)</th>
<th>% Deviation</th>
<th>$p$-Analytical (GPa)</th>
<th>$p$-FEA (GPa)</th>
<th>% Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0687</td>
<td>0.0707</td>
<td>2.83</td>
<td>3.304</td>
<td>3.11</td>
<td>-6.25</td>
</tr>
<tr>
<td>30</td>
<td>0.0760</td>
<td>0.0765</td>
<td>0.62</td>
<td>3.97</td>
<td>3.48</td>
<td>-13.97</td>
</tr>
<tr>
<td>50</td>
<td>0.0864</td>
<td>0.0872</td>
<td>0.94</td>
<td>4.266</td>
<td>4.11</td>
<td>-3.79</td>
</tr>
<tr>
<td>100</td>
<td>0.1027</td>
<td>0.1029</td>
<td>0.17</td>
<td>6.033</td>
<td>5.680</td>
<td>-6.21</td>
</tr>
</tbody>
</table>

Table 3.4: Analytic/FEA comparison for low modulus bonded layered solid results (Silver)
Most coatings and bonded layers can behave in a viscoelastic manner, even metallic materials. The relaxation behaviour of some is very slight compared to others, particularly in the case of metallic materials. Even materials with significant viscoelastic characteristics, such as polymers respond elastically under instantaneous loading condition. High Filled Carbon Polymer (HFCP) is used in this thesis to devise a pressure sensitive sensor to study viscoelastic relaxation behaviour of polymeric materials. These studies are outlined in later chapters. Here instantaneous elastic response of a thin HFCP layer of thickness 0.1 mm bonded to a rigid substrate and indented by a 6 mm Steel ball is studied under different contact loads. The practical loads for such a bonded solid of modulus of elasticity of 1 GPa and Poisson’s ratio of 0.4 is in the range 0-20 N. However, table 3.5 lists the analytical and 2D FEA results up to a load of 100N for completeness of investigations and for the sake of comparison with the previous reported cases.
Table 3.5: Analytic/FEA comparison for low modulus bonded layered solid results (HFCP)

The percentage errors in columns 4 and 7 indicate the deviation of the predicted 2D FEA results from the analytical bonded layered theory. Note should be taken that viscoelastic behaviour of HFCP leads to stress relaxation; indicated by a growth in the contact area and a corresponding decrease in the contact pressures. These have been discussed by Johnson (1985) and Naghieh (1999), and tackled in chapter 6.

Figure 3.11 shows the good conformance between the analytical and FEA results.

Figure 3.11: Predicted pressures for a Steel ball indenting an HFCP Layer (contact load: 50 N) d=0.1mm, R=6mm
It is interesting to note the difference between the pressures generated under the same conditions: contact configuration (e.g. counterformal point contact, as in the analyses described thus far) and contact load. Figure 3.12 shows the differences. Note that high pressures are generated in hard coatings, where the substrate material will be protected against their effect. For soft coatings and bonded layers of low elastic moduli, the generated pressures are lower at the same value of load, with more extensive contact areas. However, as evident from the results, pressures in all metallic layered solids are usually higher than those generated in semi-infinite elastic half-spaces. In the figure, the contact pressures are lower in the case of the Steel half-space. Ceramic coating exhibits an exception to this rule, as described previously, behaving rather similar to a semi-infinite solid at such a low contact load, mainly due to its thickness. The HFCP bonded layer has a very low elastic modulus, which yields a large contact area and low pressures.

Figure 3.12: Bonded layered solid behaviour in circular point contact predict by finite element analysis and compared to a semi-infinite Steel-on-Steel contact
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The results obtained for the contact radius, \( a \), for the various values of applied load and the different solids described above have been plotted to obtain figure 3.13. The curve for the Steel-on-Steel contact is for the FEA predictions under Hertzian conditions. Regression analysis of the results yields the relationship:

\[
W = 10^{14} a^{3.182} = K a^3 = C \delta^{3/2}, \delta = \frac{a^2}{R} \quad (3.56)
\]

This is of the same form as that for the Hertzian circular point contact (see equations 4.8 and 4.9, chapter 4, \( \delta = \omega \)), where \( K \) and \( C \) are function of material’s property and the reduced radius of the counterformal contact.

Regression analysis can also be carried out for the contacts of the various above mentioned coatings and bonded layered solids. These yield the following relations:

For Ceramic: \( W = 6 \times 10^{14} a^{3.288} = K a^3 = C \delta^{3/2}, \delta = \frac{a^2}{2R} \quad (3.57) \)

For Silver: \( W = 7 \times 10^{18} a^{4.255} = K a^4 = D \delta^2, \delta = \frac{a^2}{2R} \quad (3.58) \)

For Gold: \( W = 3 \times 10^{19} a^{4.339} = K a^4 = D \delta^2, \delta = \frac{a^2}{2R} \quad (3.59) \)

For TiN: \( W = 10^{18} a^{3.850} = K a^4 = D \delta^2, \delta = \frac{a^2}{2R} \quad (3.60) \)

For HFC: \( W = 10^{20} a^{4.062} = K a^4 = D \delta^2, \delta = \frac{a^2}{2R} \quad (3.61) \)
Note that the power index for $a$ in the above equations is rounded to the nearest integer number as an approximation. This shows that except for the case of ceramic, the deflection index is 2 (as also derived in chapter 4, equation 4.19) as opposed to 1.5 for the classical Hertzian theory. In the case of ceramic $0.25 < \frac{a}{d} < 2$ for most cases (see table 3.1), thus there is a transition from a semi-infinite behaviour towards layered response as the contact load increases.

![Figure 3.13: Load-contact radius relations for various elastic solids](image)

Some generic conclusions can be made with regard to the combination of layered solids-substrate material with regard to their contact mechanics behaviour. O’Sullivan and King (1988) and Peng and Bhushan (2001) studied the behaviour of such solids according to the ratio $\frac{E_1}{E_2}$, where $E_1$ denotes the modulus of elasticity of the layered solid and $E_2$ that of the substrate. They showed that as the ratio increases (i.e. hard coatings) the pressures increase and the contact area
diminishes. They suggested that this would protect the substrate material, with the sub-surface stresses in the substrate becoming less damaging and more localised. Their findings were broadly in-line with those highlighted in this thesis. Figure 3.14 shows their results, where the $a_0$ and $p_0$ relate to the contact radius and the corresponding maximum Hertzian pressure obtained by elastostatic analysis of the substrate alone, subjected to the same load in each case.

![Figure 3.14: Normalised elastostatic pressure distributions for combination of layered semi-infinite solids (substrate being an elastic half-space) (after O'Sullivan and King (1988), and Peng and Bhushan (2001))](image)

O’Sullivan and King (1988) used an analytical technique, based upon a least square fit method, and Peng and Bhushan (2001) by a quasi-Newton interpolation method. Both these approaches embody simplifying assumptions, which can be overcome with FEA. The results are also somewhat impractical, since the layered solid thickness was assumed to be about 10% of the radius of the indenting rigid sphere. In fact this condition yields a double layer contact, both acting as semi-infinite solids, thus enabling approximate analytic solutions to be undertaken.
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Here the same study is undertaken with the following input: \( R=6 \text{ mm} \), 1.2 mm thick substrate, 40 \( \mu \text{m} \) of which is a coated layer or a bonded layered solid, and contact load was kept constant at 100 N for all the various ratios: \( \frac{E_1}{E_2} \). Values of \( a_0 \) and \( p_0 \) used to normalise the results in figure 3.15 are based on a rigid ball indenting a Steel substrate as a semi-infinite elastic half-space. Figure 3.15 shows the results for the various values of the aforementioned modulus ratio.

![Figure 3.15: Normalised elastostatic pressure distributions for combinations of a thin layered solid bonded upon a semi-infinite substrate](image)

Taking Steel as the substrate material, the ratios greater than unity represent use of hard coatings, such as Alumina and TiN described previous, that show greater generated pressures with the increased modulus ratio. The modulus ratio of unity represents semi-infinite elastic half-space behaviour, whilst the ratios less than unity correspond to soft coatings or soft bonded layered solids, with increased contact domain and reduced pressures.
Study of sub-surface stress distributions into the bulk of the thin layer and that of the substrate show the physics of the problem clearer. The best measure of the state of sub-surface stress field is the Von Mises or the equivalent stress. The Von Mises stress describes the complex state of stress in a given spatial location according to the energy stored due to induced distortion. This approach was first proposed by Huber (1904) and extended by Maxwell, Hencky and Von Mises. In recognition of the proposed approach by Huber (1904) it is usually referred to as the Distortion Energy Hypothesis (DEH), but more commonly known as the Von Mises Shear Strain Energy Criterion. It defines the state of stress at any location within the bulk of the material by an equivalent stress, which can lead to yielding.

In the DEH, Huber (1904) hypothesised that the total strain energy per unit volume stored in a material could be considered as consisting of energy stored due to change in volume and that due to change in shape. This total stored energy can be stated as:

\[ \bar{U}_v = \frac{1}{2} \sigma_1 \varepsilon_1 + \frac{1}{2} \sigma_2 \varepsilon_2 + \frac{1}{2} \sigma_3 \varepsilon_3 = U_v + U_s \quad (3.62) \]

The stress-strain relations are given by equations (3.19)-(3.21), with the above conditions in equation (3.63) imposed. Substituting the results into equation (3.62) yields:

\[ \bar{U}_v = \frac{1}{2E} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \frac{\nu(2\sigma_1 \sigma_2 + 2\sigma_2 \sigma_3 + 2\sigma_3 \sigma_1)}{2E} \quad (3.64) \]

The volumetric strain energy per unit volume is based on the hydrostatic stress as:

\[ \bar{\sigma} = \frac{1}{3} \left( \sigma_1 + \sigma_2 + \sigma_3 \right) \]

Thus:
where, volumetric strain, $\bar{\varepsilon} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

Now using equation (3.62):

$$U_s = U_T - U_v = \frac{1}{2E} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \frac{\nu (2\sigma_1 \sigma_2 + 2\sigma_2 \sigma_3 + 2\sigma_3 \sigma_1)}{2E}$$
$$- \frac{1-2\nu}{6E} (\sigma_1 + \sigma_2 + \sigma_3)^2$$

(3.66)

where: $G = \frac{E}{2(1+\nu)}$.

The DEH proposes that in a given state of stress yielding will commence, when the quantity $U_s$ reaches the equivalent value in simple tension, where:

$\sigma_1 = \sigma_y, \sigma_2 = \sigma_3 = 0$, thus: $U_s = \frac{\sigma_y^2}{6G}$. Equating this relationship with equation (3.66) results in:

$$\left( \sigma_1 - \sigma_2 \right)^2 + \left( \sigma_2 - \sigma_3 \right)^2 + \left( \sigma_3 - \sigma_1 \right)^2 = 2\sigma_y^2 = \sigma_e^2$$

(3.67)
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Contact Mechanics: Layered Elastic Solids

Now for the biaxial state of stress in the 2D finite element analysis carried out here:

\[ \sigma_1 = \sigma_x, \sigma_2 = \sigma_y = 0, \sigma_3 = \sigma_z \] (3.68)

Substituting the conditions set in (3.68) into equation (3.67) yields:

\[ \sigma_z^2 = \sigma_y^2 = \sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3 \] (3.69)

Or in general for a biaxial state of stress:

\[ \sigma_z = \sqrt{\left( \sigma_x^2 + \sigma_z^2 - \sigma_x \sigma_z + 3t_{zx}^2 \right)} \] (3.70)

And onset of yielding occurs, when: \( \sigma_z = \sqrt{2} \sigma_y \).

Figures 3.16-3.20 show Von Mises (equivalent) sub-surface stress variations for the various \( \frac{E_1}{E_2} \) ratios. For \( \frac{E_1}{E_2} > 1 \) (see figures 3.16 and 3.17), the high stress region remains confined to the top layer (the hard coating), with the substrate suffering much lower stress levels. The boundary between the top layer and the substrate is marked by the jagged lines, caused by distortion at the boundary due to a mismatch in strain in solids of different elastic moduli. The finite element model, using a mesh of 2D quadrilateral elements makes no provision for interfacial boundary conditions (i.e. no artificially imposed assumptions as to particular conditions at the interface), allowing distortion to be governed by the elastic properties of the two solids: \( E, v \). The employed quadrilateral elements allow application of strain in the \( xz \) plane. The contact is assumed to be frictionless and internal frictional properties of the layered solid and the substrate, as well as their interfacial friction and adhesion are ignored. These can play a very important role in practice.
When the top layer has the same modulus of elasticity (i.e. $\frac{E_1}{E_2} = 1$) and Poisson’s ratio as the substrate, the resulting condition is that of a rigid sphere indenting a semi-infinite elastic half-space (see figure 3.18). Note that in this case no discontinuity is observed in the islands (contours) of equivalent stress. This justifies the approach undertaken in the mesh construction strategy.
When the modulus ratio is less than unity, figures 3.19 and 3.20 show that the soft bonded layered solid or coating effectively transfers the high stresses to the substrate material. These coatings are usually used for tribological rather than contact mechanical characteristics (i.e. to reduce friction or act as a solid lubricant) as previously discussed. In certain applications, the breakdown and reformation of the layered solid is desired in order to act as a lubricant. To fully quantify this effect, it is necessary to carry out thermo-elastoplastic analysis of the layered solid, which is outside the scope of this thesis, but a good basis for the extension of this work to such an analysis is laid out and such an extension is proposed for future work (see chapter 7).
Figure 3.19: Von Mises sub-surface stress variation for a soft coated substrate and the substrate for $\frac{E_1}{E_2} = 0.5$, Load = 100N

Figure 3.20: Von Mises sub-surface stress variation for a soft coated substrate and the substrate for $\frac{E_1}{E_2} = 0.25$, Load = 100N
3.6.4- Elastostatic contact of a roller indenting hard coated frictionless substrates

The finite line contact of a roller on a semi-infinite flat frictionless plane was investigated in chapter 2 for both aligned and misaligned rollers. It is interesting to study the elastostatic contact of rollers against flat planes as substrates coated with solids of high elastic modulus, similar to the cases of ball contacts described in the previous section. Finite line contact of layered elastic solids, using finite element method has not hitherto been reported in literature. It is expected that the hard coatings will protect the substrate material in much the same manner as in the case of ball bearings, described above. However, the edge pressure spikes are expected to be very high indeed and can cause failure of hard coatings by brittle fracture.

Due to the length of the roller and thinness of the coating, the number of elements required is quite large. Therefore, one would require to use all legitimate and acceptable means to reduce the required computer storage in order to undertake an analysis. For this purpose some apriori knowledge of finite line contact characteristics can be employed. It is clear, from the classical theory and the finite element results for contact of a roller against a semi-infinite elastic half-space that the width of the contact is considerably smaller than the length of the roller. Therefore, the extent of the contact is limited to a few degrees of the circumference of the roller penetrating the elastic substrate. Therefore, half the roller length with its circumference ±10° is used, resulting in a strip of the thin layered solid needing a high density mesh of 60 along half roller length with additional coarser elements beyond, with 20 elements in the lateral direction and 10 deep into the depth of the layered solid. A geometric mesh construction is used as in chapter 2 to increase the number of mesh points in the vicinity of the roller end.

To understand the physics of the contact problem an identical base model is used, where the layered solid materials are both of the same elastic modulus (in this case Cobalt Chromium Steel), rendering a semi-infinite contact. The layered solid
model as described above employs the same Steel substrate, with a 40 μm thick coating of Alumina. The modulus ratio, as described above is approximately 2 (for Steel, \(E=210\) GPa, for Alumina, \(E=400\) GPa). The roller radius is 6 mm, with its length being 12 mm. The contact load in both cases is 500N. The roller is considered to be rigid.

Figure 3.21 shows the results obtained for the cases discussed in the previous paragraph. The distribution for the Steel substrate is typical of the elastostatic contact of a rigid roller indenting a Steel substrate as a semi-infinite elastic half-space. The pressure spike at the edge of the contact and in the vicinity of the unblended roller end has a value of approximately 1 GPa. When a thin coating of ceramic (in this case Alumina) is deposited upon the Steel substrate, the contact pressure distribution is slightly reduced along the length of the contact, whilst the edge pressure spike has nearly doubled in magnitude. This finding is in-line with those for the elliptical point contact geometries described in the previous section.

![Figure 3.21: Axial pressure distributions for a roller indenting a semi-infinite elastic solid surface and a layered construction (Load = 500 N)](image-url)
Again, the protective nature of such hard coatings is demonstrated by the extent of the spread of the equivalent stress in the sub-surface layers. In the case of the coated flat plane, figure 3.23 shows confinement of the maximum Von Mises stresses in the thin bonded layer of Alumina, whilst in figure 3.22 the semi-infinite nature of the contact problem shows that the maximum equivalent stress contour occurs deeper in the bulk of the Steel substrate.

(a) Von Mises sub-surface stress distribution beneath the island of pressure spike in the isobar plot shown in (b)

(b) Isobars of pressure and the footprint shape for a roller indenting a semi-infinite Steel elastic half-space (related to figure 3.21)

Figure 3.22: A rigid roller indenting a semi-infinite elastic half-space.

R=6mm, L=12mm
(a)- Von Mises sub-surface stress distribution beneath the island of pressure spike in the isobar plot shown in (b)

(b)- Isobars of pressure and the footprint shape for a roller indenting a layered solid (related to figure 3.21)

Figure 3.23: A rigid roller indenting a layered solid $R=6\text{mm}$, $L=12\text{mm}$

The Von Mises distributions are shown in $zx$ planes into the depth of the material directly below the position of the pressure spike. Note that figures 3.22 (a) and 3.23 (a) are comparable in scale. The same is not true of the pressure isobars, which in the case of figure 3.23 (b) had to be zoomed upon to discriminate the edge pressure spike.
4.1- Introduction

Impact phenomena play a significant role in performance of many machines and mechanisms. They govern the dynamic behaviour and in many cases determine the structural integrity and thus the functional assurance of the system. One can cite many cases where impact phenomena play an influential role in system performance. They include piston slapping action against the cylinder bore or liner in all forms of engines, as described by Balakrishnan (2002), or cage collisions caused by ball or rolling element bearings impacting their retaining flanges as highlighted by Gupta (1976) and races, as simulated by Safa and Gohar (1986), Dowson and Wang (1994) and AlSamieh (2002), and AlSamieh and Rahnejat (2002). Unwanted impacts can cause significant practical problems, such as that of flywheel nodding into the clutch system during transient event of clutch engagement or disengagement, when the flywheel is approaching or separating from the mating friction disc, and is subjected to conical whirling motion due to crankshaft flexibility and out-of-balance rotation. This phenomenon, referred to in industry as clutch whoop (an onomatopoeic term referring to the sound radiation during the event) was studied extensively by Kelly (1999). Other powertrain impact problems include impact of gear teeth when not engaged and not in drive condition, referred to as rattle (see Biemann and Hagerodt, 1999, and Gnanakumarr et al, 2002). At high loads and with sudden demand in torque from coast to drive condition, caused by throttle tip-in action meshing gear teeth can impact from a free motion through gap due to backlash. This problem leads to
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local deformation energy causing wave propagation in the drivetrain system, and noise emission when such structural waves couple with natural modes of acoustic cavities, such as hollow driveshaft tubes (see Menday, 2003).

The transition from a free motion to a post impact situation is conceived through constrained multi-body dynamics, in which the unilateral constraint is envisaged as that imposed by the kinetics of impact as an impulse. This approach was initiated by Newton (1687) and was perfected later by Poisson (1842). In such an impact, the solution to the equations of motion must be undertaken in two distinct steps; firstly in free pre-impact motion to obtain the conditions prior to the infinitesimal impact time (taken by both Newton and Poisson as \( t=0 \), and secondly for the post impact dynamics. The output of the first step, acts as the initial condition for the second step, and one should note that the second step should be regarded as an initial value problem. The accelerative motion at the instant of impact gives rise to the deformation of bodies, a fact that remained unaccounted for by both Newton (1687) and Poisson (1842). This was realised later by Hertz (1881), who attributed local deformation effects to impacting ellipsoids of revolution. This “local” effect is regarded as stereocontact of finite impact duration, or: \( t \neq 0 \). Of course the observations made with respect to the finite nature of the impact time is equally valid to any geometry of impacting region, and not necessarily within the Hertzian assumptions. The assumption of a unilateral constraint holds good, and the impact force operates over the impact duration (formulated by Hertz (1881)). The problem here is the severe non-linearity that is induced by the large impact force and the change in the stiff characteristics of the system equations. Stiff characteristics refer to dynamic problems, yielding widely-split eigen values, in which the introduction of a sudden force (such as an impact) suddenly alters the system modal behaviour. The deformation of solids in the Hertzian impact is considered to be local; in the domain of the contact, which is far smaller in size than the principal radii of curvature of contacting surfaces. In fact Hertz developed his theory, now referred to as the classical Hertzian contact theory, for elastostatic concentrated contact of counterformal contiguous bodies, and not for the case of impacting solids. The theory was extended later for case of impacts.
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In the Hertzian theory the elastic wave motion in the bodies is ignored. This assumption may not be considered as gross, when the solids of revolution have a very small contact area, and the impact duration is still long enough compared to the elastic modal behaviour of the impacting solids in a “global” sense (i.e. insignificant stored energy in the solid does not give rise to wave propagation or vibration within its structure). A good example of this is the Newton’s trolley, where the acoustic output indicates that the dominant noise propagating source is accelerative in nature. Such, of course, is not the case if the spheres in the trolley were hollow. The elastic wave motion gives rise to accelerative response, followed by ringing noise. This phenomenon was of course realised by Von Helmholtz (1863) for hollow tubes, and even earlier by Daniel Bernoulli (1750), Euler (1739) and D’Alembert in 1740-1750 for strings (1747). Such elastic contacts with wave motion were indeed formulated by Saint-Venant (1883) for the general case of short finite duration impact, and termed as elastocontact.

Therefore, there are two broad classes of impact problems in engineering domain, if one is to banish the idea of infinitesimal impact durations. These classes are Hertzian and those due to Saint-Venant. Today one can categorise impact dynamics phenomena according to the time scale nature of the problem, as described by Schiehlen and Hu (2000). Those with “infinitesimal” impact time correspond to the Poisson’s-type impact, referred to unsteady multi-body dynamics, which are “visible” in the frequency range 5-100 Hz. The Hertzian-type impact, in this categorisation, corresponds to multi-body dynamics with elastostatic local behaviour, referring to the problems in the frequency range 100-1000 Hz (such as gear rattle problem 80-300 Hz). The Saint Venant’s-type impact cover the global effect of impact dynamics in the form of wave propagation, usually in the frequency range>1000 Hz, and are referred to as elastodynamics. When there is coincidence between wave motion in the solid and acoustic modes of a cavity or surrounding fluids the response can become audible.

This thesis is primarily concerned with Hertzian-type stereoccontact of finite duration time. However, the Hertzian impact dynamics theory applies only to the case of ellipsoidal solids impacting upon semi-infinite elastic half-spaces. When
the contacting geometry deviates from Hertzian assumptions (see chapter 2), new analytical solutions should be sought. Some analytical solutions are reported in this chapter. However, a generic solution cannot be proposed, based upon an analytic approach. Therefore, a finite element methodology is described and a few cases of impacts of solids undertaken.

4.2- Hertzian impact (semi-infinite solids)

In the first instance one can consider the impact of two solid balls as shown in figure 4.1. Using Newton’s second axiom, it follows that:

$$m_i \frac{dv_i}{dt} = m_i \dot{v}_i = -W, \quad i = 1, 2$$

(4.1)

where the impact force \( F \) opposes the direction of motion in the case of each ball, and by the virtue of Newton’s third axiom is the same for both the impacting bodies. Note that the case considered is for frictionless rolling motion of the balls.

Figure 4.1: Impact of bodies of revolution
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The velocity of approach is given by:

\[ \dot{\omega} = \sum_{i=1}^{2} v_i \]  \hspace{1cm} (4.2)

Thus:

\[ \dot{\omega} = \sum_{i=1}^{2} \dot{v}_i = \dot{v}_1 + \dot{v}_2 = -W \left( \frac{1}{m_1} + \frac{1}{m_2} \right) = -W \left( \frac{m_1 + m_2}{m_1 m_2} \right) \]  \hspace{1cm} (4.3)

where: \( m_{eq} = \left( \frac{m_1 m_2}{m_1 + m_2} \right) \) is the equivalent mass of a sphere impacting a semi-infinite elastic half-space.

![Figure 4.2: Surface deformation of non-conforming solid](image)

Hertz assumed that the impact time to be finite and long compared to the period of the lowest mode of vibration of impacting solid spheres, thus one may ignore their
structural vibrations. Therefore, one may consider the approach of any two points (such as P and Q in figure 4.2) on the surface of the two solids to be governed by the Hertzian elastostatic conditions.

The displacement of the solids approaching each other by the velocity given by equation (4.2) is given by $\omega$. The approach of the two points P and Q is given by:

$$\omega - (\delta_i + \delta_j) = s_i + s_j = \beta r^2$$  \hspace{1cm} (4.4)

where $\delta_i, i=1,2$ denote deformation of the spheres 1 and 2 in figure 4.2, whilst $s_i$ denote the separation (i.e. gap) of points P and Q from the common normal drawn at the point of contact, O. $\beta$ is a function of the radii of solids at the point of contact, this being the reduced radius $\frac{1}{R}$ for a pair of spheres, described in chapter 2.

If the Hertzian circular point contact radius is $a$, then:

$$\omega = \frac{\pi p_0 a}{E'}$$  \hspace{1cm} (4.5)

where: $E' = \frac{2}{\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}}$

and: $a = \frac{\pi p_0}{2E' \beta}$  \hspace{1cm} (4.6)

where, $p_0$ can be replaced from equation (2.3), chapter 2 in terms of the contact load $W$ and the Hertzian contact radius, $a$. Thus:
\[ \omega = \left( \frac{9 W^2}{4 E^\nu R} \right)^{\frac{\nu}{2}} \]  \hspace{1cm} (4.7)

and: \[ a = \left( \frac{3 WR}{2 E^\nu} \right)^{\frac{\nu}{2}} \]  \hspace{1cm} (4.8)

Equation (4.7) can be transposed to yield:

\[ W = \frac{2E^\nu \sqrt{R}}{3} \omega^{\nu/2} = C\omega^{\nu/2} \]  \hspace{1cm} (4.9)

Replacing into equation (4.3):

\[ \dot{\omega} = -\frac{C}{m_{eq}}\omega^{\nu/2} \]  \hspace{1cm} (4.10)

The solution to the above differential equation with \( \dot{\omega} = 0 \) yields the approach of the solids at the instant of maximum deflection (i.e. compression). To obtain a solution, both sides of equation (4.10) are multiplied by \( \dot{\omega} \):

\[ LHS = \ddot{\omega}\dot{\omega} = \dot{\omega} \frac{d\omega}{dt} = \frac{1}{2} \frac{d\omega^2}{dt} = RHS = -\frac{C}{m_{eq}} \omega^{\nu/2}\dot{\omega} = -\frac{C}{m_{eq}} \omega^{\nu/2} \frac{d\omega}{dt} \]  \hspace{1cm} (4.11)

Integrating both sides of the above equation yields:

\[ \frac{1}{2} (\dot{\omega}^2 - \nu^2) = -\frac{2C}{5m_{eq}} \omega^{\nu/2} \]  \hspace{1cm} (4.12)
where $v$ is the initial impact velocity. When $\dot{\omega} = 0$, as described above, the approach at maximum deflection $\omega_{\text{max}} = \delta_1 + \delta_2 = \delta_{\text{max}}$ (centre of contact, i.e. at point O in figure 4.2) is:

$$\omega_{\text{max}} = \delta_{\text{max}} = \left(\frac{5m_{\text{eq}}v^2}{4C}\right)^{\frac{1}{2}} = \left(\frac{15m_{\text{eq}}v^2}{8E'\sqrt{R}}\right)^{\frac{1}{2}} \quad (4.13)$$

Now the above equation can be substituted into equation (4.9) to find the maximum impact force which occurs at the instance of maximum compression, just prior to rebound. The semi-infinite impact radius is obtained by replacing for the maximum impact force in equation (4.8) and the maximum contact pressure is given as:

$$P_{\text{onax}} = \frac{3W_{\text{max}}}{2\pi a_{\text{max}}^2} \quad (4.14)$$

The only remaining parameter of interest is the impact time. To obtain this equation (4.12) can be re-written in the following form (Timoshenko and Goodier, 1951):

$$\left(\frac{d\omega}{dt}\right)^2 = v^2 - \frac{4C}{5m_{\text{eq}}} \omega^{\frac{3}{2}} \quad \text{or:}$$

$$dt = \left\{v^2 - \frac{4C}{5m_{\text{eq}}} \omega^{\frac{3}{2}}\right\}^{\frac{1}{2}} d\omega \quad (4.15)$$

Letting: $x = \frac{\omega}{\omega_{\text{max}}}$, the above equation can be re-written in the following form:

$$dt = \frac{\omega_{\text{max}}}{v} \left\{1 - x^{\frac{3}{2}}\right\}^{\frac{1}{2}} dx \quad (4.16)$$
Equation (4.16) can be integrated between the limits:
\[ x = 0 \to 1, \quad \omega = 0 \text{ (instant of impact), } \omega = \omega_{\text{max}} \text{ (maximum compression)} \], thus:

\[ t = \frac{2\omega_{\text{max}}}{v} \int_{0}^{1} \frac{dx}{\sqrt{1 - x^{\frac{3}{2}}}} = 2.94 \frac{\omega_{\text{max}}}{v} = 2.94 \frac{\delta_{\text{max}}}{v} \quad (4.17) \]

Therefore, for a ball of mass \( m \) impacting a semi-infinite elastic half-space:
\( m_{\text{eq}} = m \), and the impact condition can be described by the summary of the above derivations as:

\[ \delta_{\text{max}} = \left( \frac{15mv^{2}}{8E'(1-R)} \right)^{\frac{3}{2}}, \quad P_{0\text{max}} = \frac{3W_{\text{max}}}{2\pi a_{\text{max}}^{2}}, \quad t \approx 2.94 \frac{\delta_{\text{max}}}{v} \quad (4.18) \]

4.3- Non-Hertzian impact of a sphere on a layered bonded elastic solid

Now the analysis for a ball impacting a thin layered solid bonded upon a semi-infinite rigid plane follows the same derivation route, except that the penetration in equation (4.9) is not obtained from equation (4.7), which is only valid for the case of a semi-infinite elastic half-space. For the case of a compressible bonded elastic layered solid of thickness, \( d \), the deflection/penetration is given by
\[ \omega = \frac{a^{2}}{2R} \], where the contact radius, \( a \) is given by equation (3.32a) in chapter 3.
Thus:

\[ W = \frac{2\pi E'(1-\nu)^{2}R}{(1-2\nu)d} \omega^{2} = D\omega^{2} \quad (4.19) \]
since for a rigid ball: $E_i = 0$, $E' = \frac{2E_2}{1 - v^2}$.

Now replacing in equation (4.3) and following the same steps as those indicated in equation (4.11), it follows that:

\[
LHS = \dot{\omega}\dot{\omega} = \frac{d\dot{\omega}}{dt} = \frac{1}{2} \frac{d\dot{\omega}}{dt} = RHS = -\frac{D}{m} \omega^3 \dot{\omega} = -\frac{D}{m} \omega^3 \frac{d\omega}{dt}
\]

(4.20)

Thus:

\[
\frac{1}{2} (\dot{\omega}^2 - v^2) = -\frac{1}{3} \frac{D}{m} \omega^3
\]

(4.21)

Now the maximum penetration is obtained as $\omega_{\text{max}} = \delta_{\text{max}}$, when: $\dot{\omega} = 0$:

\[
\omega_{\text{max}} = \delta_{\text{max}} = \left\{ \frac{3(1 - 2v)dmv^2}{4\pi E'(1 - v)^2 R} \right\}^{\frac{1}{3}}
\]

(4.22)

Using equation (3.32), chapter 3, the maximum pressure is obtained as:

\[
P_{0\text{max}} = \frac{E(1 - \nu)a^2}{2R(1 + \nu)(1 - 2\nu)d}
\]

(4.23)

Where $a = a_0$ for instantaneous elastic response, when $p = P_{0\text{max}}$

Now in a similar manner to the previous case, equation (4.21) can be re-written in the following form:
\[
\left( \frac{d\omega}{dt} \right)^2 = v^2 - \frac{2 D}{3 m} \omega^3 \quad \text{or:}
\]
\[
d t = \left\{ v^2 - \frac{2 D}{3 m} \omega^3 \right\}^{-\frac{1}{2}} d\omega
\]

Now the same procedure can be followed as in the previous case; by letting \( x = \frac{\omega}{\omega_{\text{max}}} \), thus:

\[
t = \frac{2\omega_{\text{max}}}{v} \int_0^{1} \frac{dx}{v \sqrt{1-x^3}}
\]
\[
= \frac{2\omega_{\text{max}}}{v} \left[ \frac{1}{\sqrt{3}\sqrt{1-x^3}} \right]_{0}^{1} \left[ 2i\sqrt[3]{(-1)^{\frac{5}{6}}(x-1)\sqrt{1+x+x^2}} \right. \left. \text{Ellipticf} \left\{ \sin^{-1} \left( \frac{\sqrt[3]{(-1)^{\frac{5}{6}}-ix}}{\sqrt[3]{3}} \right), (-1)^{\frac{5}{6}} \right\} \right] \]

Note, that the first term in the evaluated integral tends to infinity as \( x \to 1 \). Therefore, the upper limit of the integral should be set at a value very close to unity (in this case at 0.999999). The above integral can now be evaluated using \( \Gamma \) functions, as shown below:

\[
t = \frac{2\delta_{\text{max}}}{v} \int_0^{0.999999} \frac{1}{\sqrt[3]{1-x^3}}
\]
\[
= \frac{\sqrt{\pi} \Gamma \left( \frac{4}{3} \right)}{\Gamma \left( \frac{5}{6} \right)} = 1.40207 + 6.67 \times 10^{-10} i
\]

(4.26)
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Note that the imaginary component is negligible in magnitude, and can thus be ignored.

Therefore, the summary of impact conditions for a rigid ball impacting a compressible bonded elastic layer is:

\[ \delta_{max} = \left( \frac{3(1-2\nu)dmv^2}{4\pi E'(1-\nu)^2R} \right)^{1/3}, P_{0_{max}} = \frac{E(1-\nu)a^2}{2R(1+\nu)(1-2\nu)d}, t = \frac{2.8\delta_{max}}{v} \]  (4.26)

4.4- Some practical applications of the theory

A small computer program was written to obtain the important parameters; impact force, maximum deflection, the corresponding maximum pressure, the footprint radius and the impact time for a number of examples. The program is written in the language C++, and all the simulation runs reported take approximately a couple of seconds to complete on a Pentium III 750 MHz machine.

4.4.1- Verification of analytical method against existing experimental results

First the impact test undertaken by Safa and Gohar (1986) is investigated, except that the analysis carried out here is for dry impact of a ball upon the glass plate. Safa and Gohar (1986) investigated the impact of a Steel ball of radius 0.0127 m (modulus of elasticity 210 GPa, Poisson's ratio = 0.3) upon a heat treated flat glass plate with a modulus of elasticity of 71 GPa, and Poisson's ratio of 0.23. The ball was released from a height of 5 mm above the glass plate, falling freely under the effect of gravity onto the glass plate, covered with a droplet of oil. The prevailing contact conditions, neglecting the effect of lubricant reaction (as assumed here) and insignificant friction comply with the assumptions of the Hertzian impact theory.
Safa and Gohar (1986) had fabricated a Manganin pressure micro-transducer on the surface of the glass plate by Radio Frequency (RF) sputtering technique. This device enabled them to measure the transient pressure variation during impact, which showed a variation in-line with the elastohydrodynamic regime of lubrication (see figure 4.3). The shape of the variation is reminiscent of the spatial variation of pressure (i.e. the pressure distribution) under rolling contact condition, with the exception that the generated pressures are considerably in excess of those generated under impact condition, which follow pure squeezing of the lubricant film.

![Graph showing measured maximum elastohydrodynamic pressure during impact](image)

Figure 4.3: Measured maximum elastohydrodynamic pressure during impact of a Steel ball upon an oily glass plate (after Safa and Gohar (1986))

The results obtained using the Hertzian impact theory indicates the following:

\[ F_{\text{max}} = 1189.1N, \quad \delta_{\text{max}} = 15.6\mu m, \quad p_{\text{max}} = 2.19\text{GPa}, \quad t = 147\mu s. \]

Comparing these results with those obtained by Safa and Gohar (1986) indicate a number of interesting findings. The maximum Hertzian pressure of 1.44 GPa is considerably lower than that under lubricated condition, obtained in the experiment to be approximately 2.4 GPa. This is an expected outcome as the lubricant reaction is responsible for a higher depth of penetration (contact deflection) than under dry impact condition. One should note that the intervening (resistive medium) under dry condition is air, which poses significantly lower resistance than a lubricant film. Therefore, the rate of deceleration of the ball in arresting its downward motion is lower under dry condition before rebound, resulting in lower impact time of 147 µs as opposed to 225 µs under lubricated condition.
Now referring to the impact categorisation outlined by Schiehlen and Hu (2000), the maximum impact frequency is: \( f = \frac{1}{t} \frac{10^6}{147} = 6802.7 \text{ Hz} \) for the dry contact condition, if one is to assume the effect in the plate follows the St Venant elastocontact model. This means that all elastodynamic modal characteristics of the plate up to and including that at the frequency, \( f \), stated above can be excited, whereas for the case of lubricated contact described by Safa and Gohar (1986) \( f = \frac{1}{t} \frac{10^6}{225} = 4444.4 \text{ Hz} \). This means that although the extend of deformation, thus the maximum stored energy in the impacting solids is higher under lubricated condition, its lower frequency content means that the elastodynamic response of the plate would have a narrower modal spectral content. This fact is exploited in many engineering applications, where noise propagation issue is regarded as the major troublesome factor, such as in piston slapping problem, where intentional offsetting of the gudgeon pin from the axis of symmetry of the piston, whilst increasing the gap and thus the initial impacting velocity, results in an increase in the impact time and hence limit the frequency content of the structural waves, propagating through the cylinder liner (see Balakrishnan, 2002). In the case of impacting helical gear teeth in vehicular transmission systems, an elliptical point contact geometry results, which may be regarded as Hertzian. The presence of a lubricant increases the impact time, reducing the spectral content of the stored strain energy, and thus the travelling structural waves to the hollow driveshaft tubes, the problem investigated by Menday (2003).

4.4.2- Impact of balls-to-races in unloaded conditions

As demonstrated by the above conditions, the impact forces are usually far in excess of contact forces carried by the load bearing elements, such as bearings, gears and cam-follower pairs. In concentrated contacts, these high impact forces can result in fatigue of contiguous solids by fatigue spalling. Therefore, substrate materials are often protected by hard surface coatings, depending on application. For example, for cutting tools these coatings include TiN, TiAlN, CrN, and others on the cutting edge. The introduction of coatings on the cutting edge (which itself
cannot have a radius of zero as commonly assumed) creates a certain amount of rounding. This is described in chapter 3. The conditions of cutting edge scissors impacting the workpiece is in fact one of repetitive impact, with much higher forces than those under elastostatic conditions, causing for example pulverisation of wood and wood composite rather than the assumed mechanics of machining by the process of chip formation. In other examples, ceramic coatings on bearings are used to protect the substrate material made of Steel from the effect of high generated pressures. This is also shown in chapter 3. The problem, however, is exacerbated under impact conditions due to the velocity of approaching solids.

(a) Semi-infinite Classical Hertzian Impact

In this section the behaviour of a number of layered bonded solids under impacting conditions is investigated, using the analytical expressions derived in the previous section.

A practical problem with ball and rolling element bearing is their assembly to shafts/rotors, with sufficient interference fitting and preloading to ensure a spread loaded region. This is not always possible, yielding narrow loaded region as shown in figure 4.4.

Figure 4.4: Circumferential loaded region in a bearing
The result of such an outcome is continual loading and unloading of rolling elements, which can culminate in their fatigue failure due to cyclic stressing, as described by Johns-Rahnejat (1988) and Johns and Gohar (1997). The emergence of internal clearances due to dynamics loads supported by bearings causes slipping/spinning and flight of rolling elements in unloaded regions, which result in collisions within the retaining cage, as well as with the inner and outer raceways. Large impact forces can result in plastic deformation of the raceway grooves, a phenomenon referred to as forced brinelling effect. In most cases, however, the impact forces are not large enough because of small gaps (clearances) between the rolling elements and the raceways. Therefore, the approaching velocity is not very high.

To investigate such conditions, a number of simulation studies are undertaken here. Initially, a ball bearing is assumed, in which both the balls and the raceways are made of Cobalt-Chromium Steel, with modulus of elasticity of 210 GPa, and Poisson’s ratio of 0.3. The equivalent (i.e. reduced) radius is assumed to be 20 mm, and the problem is reduced to that of a sphere impacting on a frictionless semi-infinite solid. Hertzian impact theory can then be assumed, in which the typical emerging clearance in the unloaded region of the ball bearing is taken to be 2 μm. The results obtained here can be used for the validation of finite element approach reported later.

These are: \( F_{\text{max}} = 13.65N, \delta_{\text{max}} = 0.54\mu m, p_{\text{max}} = 601MPa, t = 256\mu s \)

Note that without the impact, the load on the ball would have been zero. The impact velocity \( (v = 6.26mm/s) \) was assumed to be equivalent to a ball freely falling onto a plate, which is expected to be an under-estimation of what would be the actual prevailing conditions in practice.

(b)- Impact of coated balls-to-races using layered solution

In chapter 3, it has been shown that hard coatings can be used to protect the substrate material from generation of high contact pressures. These pressures can
be exaggerated under impact conditions. Indeed when a layer of hard coating is employed, the impact pressures increase in value, whilst both the depth of penetration and impact radius decrease. The substrate, however, remains immune from high pressures, as shown in chapter 3 and later in this chapter through finite element analysis. This fact has provided the reason for the use of hard coatings for protection of mating surfaces, which are prone to impact conditions, such as cam-follower pairs where valve spring surge and valve toss result in repetitive contact separation and subsequent impact of the surfaces.

Ceramic coatings are usually applied to balls or rolling elements or on the raceway grooves in applications with high contact loads to protect the substrate material with lower elastic moduli. However, the coated ceramic layers are usually quite smooth and lubricant film retention by entrapment is not as effective as in the case of Steel surfaces. Higher elastic moduli of the ceramic coatings also lead to lower contact deflection. Thus, such bearings perform much better in high speed applications.

The next simulation results were obtained for a rigid ball impacting a semi-infinite solid substrate, coated with a 40 \( \mu \text{m} \) thick layer of Alumina \((E=400\text{GPa})\). The same impact conditions as in the previous case were assumed. Clearly, the layered bonded elastic impact theory developed in the previous section was used for the analysis. The results obtained were:

\[
F_{\text{max}} = 10.03 \text{N}, \quad \delta_{\text{max}} = 0.197 \mu\text{m}, \quad p_{\text{max}} = 4.58\text{GPa}, \quad t = 243 \mu\text{s}
\]

The contact force is almost unchanged, whilst the maximum deflection is reduced by 63.5% due to the hard coating's higher modulus of elasticity. However, the contact radius is reduced from the previous Hertzian condition of 0.104 mm to the layered solid based value of 0.063 mm, with the corresponding maximum pressure rising from 601 MPa to 4.58 GPa. The hard coating, however, can withstand this high pressure, protects the substrate material as described later and reduces the impact time marginally. It is noteworthy that most hard coating fail due to fracture at higher contact/impact loads, or by wear due to loss of a lubricant film with the combination of low speeds and low loads.
4.5- Impact of a roller on a semi-infinite solid

For the case of a rigid roller impacting upon a semi-infinite elastic solid the same procedure can be followed, where as a first approximation one can use the penetration –load relationship given by equation (2.21) in chapter 2. Note that the penetration here is that at the centre of the roller to plane impact. Any vibrations due to bending or flexural modes of the roller are ignored in-line with the Hertzian impact theory. Thus:

\[ \omega_0 = \frac{a_0 p_0}{E'} \left\{ \ln \frac{2L}{a_0} + \frac{1}{2} \right\}, \text{ where: } p_0 = \frac{2W}{\pi a_0 L} \text{ and } a_0 = \left\{ \frac{8WR}{\pi E' L} \right\}^{1/2} \]

from chapter 2.

Let: \( \omega = \omega_0, a = a_0 \).

Therefore:

\[ W = \frac{\pi LE'}{2\left\{ \ln \frac{2L}{a} + \frac{1}{2} \right\}} \omega = K \omega \quad (4.27) \]

Substituting in equation (4.3) and multiplying both sides of the equation by \( \dot{\omega} \) yields:

\[ LHS = \ddot{\omega} \dot{\omega} = \dot{\omega} \frac{d}{dt} \left( \frac{1}{2} \dot{\omega}^2 \right) = \frac{1}{2} \frac{d}{dt} \dot{\omega}^2 = RHS = \frac{K}{m} \dot{\omega} = \frac{K}{m} \omega \frac{d \omega}{dt} \quad (4.28) \]

Integrating the equation above:

\[ \frac{1}{2} (\dot{\omega}^2 - v^2) = - \frac{K}{2m} \omega^2 \quad (4.29) \]
Now the maximum penetration is obtained as $\omega_{\text{max}} = \delta_{\text{max}}$, when: $\dot{\omega} = 0$:

$$\omega_{\text{max}} = \delta_{\text{max}} = \left(\frac{mv^2}{K}\right)^{\frac{1}{2}} = \left(\frac{2\left(\ln \frac{2L}{a} + \frac{1}{2}\right)mv^2}{\pi LE'}\right)^{\frac{1}{2}} \quad (4.30)$$

Equation (4.28) can be re-written as:

$$\left(\frac{d\omega}{dt}\right)^2 = v^2 - \frac{K}{m}\omega^2 \quad (4.31)$$

Thus:

$$dt = \left(v^2 - \frac{K}{m}\omega^2\right)^{\frac{1}{2}} d\omega \quad (4.32)$$

Now the same procedure can be followed as in the previous case; by letting $x = \frac{\omega}{\omega_{\text{max}}}$ and integrating to find the impact time, thus:

$$t = \frac{2\omega_{\text{max}}}{v} \int_0^1 \frac{1}{\sqrt{1-x^2}} dx \quad (4.33)$$

$$= \frac{2\omega_{\text{max}}}{v} \left[\sin^{-1}x\right]_0^1 = \frac{2\omega_{\text{max}} \pi}{2v} = 3.14\delta_{\text{max}}$$

Therefore, the summary of results for impact of a cylinder on a semi-infinite elastic half-space is:
4.6- Finite element models for impact of bodies of revolution

4.6.1- Mesh Construction

For finite element analysis of impact conditions rigid balls are modelled, impacting flat frictionless planes (substrate). The ball is meshed as can be seen in figure 4.5, but as it is considered to be rigid there is no need to afford to them any particular geometrical or physical properties. The substrate is meshed with three dimensional 8-noded solid elements with reduced integration effort. The mesh density is 40 by 40 in the x and y direction with 10 into the depth of the substrate (i.e. the z-direction). The impact time in each case is estimated, using the analytic solutions provided in this chapter. The simulation time is extended to be 1.5 times this estimated value and is undertaken in 20 steps. Thus, the space-time mesh density is: $40 \times 40 \times 10 \times 20 = 320000$, which requires a CPU time of an hour on a Pentium IV 2.2 GHz machine. In all cases a geometrical progression mesh is used to capture the high pressures generated under impact conditions.

\[
\frac{\delta_{\text{max}}}{v} = \left(\frac{2\ln \left(\frac{2L}{a_0} + \frac{1}{2}\right)mv^2}{\pi L E'}\right)^{\frac{1}{2}}
\]

\[
P_{\text{max}} = \frac{\delta_{\text{max}} E'}{a_0 \left(\ln \left(\frac{2L}{a_0} + \frac{1}{2}\right)\right)}
\]

Figure 4.5: The constructed mesh for a ball impacting a frictionless elastic solid
No distinction is made between mesh construction for a semi-infinite solid substrate and that for a layered solid. In the case of the latter the depth of the thin coating (40 μm) and its modulus of elasticity, as physical descriptions make the distinction with the solid substrate beneath it (this being Steel).

Figure 4.6 shows the finite element mesh created for the impact of a roller upon a substrate. The mesh on the substrate is based on a geometrical progression in both $x$ and $y$ directions (i.e. in the axial and lateral directions). In the former 90 elements are employed with nearly $\frac{2}{3}$ of these being in the vicinity of the roller end. In the lateral direction a reverse geometrical progression is employed, biased towards packing of the elements in the central region of the contact, as the width of the contact will typically be very small. In this case 30 elements are used with nearly 10 of these in the central region. The geometrically progressive element dimensions with increasing density towards the surface of the substrate are 10 rows in total, with half of these being very fine in dimensions. The dark areas in figure 4.6 illustrate the specially densely constructed mesh regions.

Figure 4.6: The specially configured mesh for a roller impacting a substrate
4.6.2- Impact simulations

Four cases have been considered. These are:

Impact of a 12.7 mm radius Steel ball, $E=210$ GPa, $\nu=0.3$ on a glass plate, $E=71$ GPa, $\nu=0.25$, from a height of 5mm. This condition replicates the experiment of Safa and Gohar (1986) and the numerical prediction of Dowson and Wang (1994) and Alsamieh and Rahnejat (2002), except that the simulation carried out here is under dry impact condition.

Impact of a 12.7 mm radius rigid ball upon a Steel semi-infinite substrate, from the same height as in the previous case.

Impact of the same ball from the same height on the same substrate, coated by a 40 µm thick layer of Alumina of $E=400$ GPa, $\nu=0.23$.

Impact of a 6 mm radius rigid roller of length 12 mm upon a semi-infinite Steel substrate.

(a)- A Steel ball impacting a semi-infinite glass plate

This case study is undertaken for the purpose of validation of the finite element study. The impact conditions can be compared to the Hertzian impact conditions, described previously, as well as comparison to be made with experimental observations of Safa and Gohar (1986), even though their investigations included the presence of an intervening droplet of lubricant. Wang and Dowson (1994) and AlSamieh (2002) have indicated that the contact conditions follow the Hertzian impact conditions, except for the deeper penetration of the solids due to larger contact reactions generated due to the presence of the lubricant.

The simulation commences at the instant of impact, when the initial velocity is given as: $v = \sqrt{2gh}$, with $h=5$ mm. This gives an initial velocity of 0.313 m/s.
According to the Hertzian impact theory, the impact time is obtained as: 0.147 ms (milliseconds), with: $\delta_{\text{max}} = 15.6 \mu m$, $p_{\text{0max}} = 2.19$ GPa and $a_{\text{max}} = 0.034$ mm. Under lubricated condition, Safa and Gohar (1986) found the impact time increases to 225 ms, and the maximum pressure at the instant of maximum penetration, when $v=0$ was monitored to be 2.4 GPa. AlSamieh and Rahnejat (2002) replicated the exact conditions as those relating to the spatial and temporal capability of Safa and Gohar’s (1986) miniature pressure transducer and found very close agreement.

AlSamieh and Rahnejat (2002) showed that prediction of maximum pressure can rise with ever decreasing spatial resolution of the computational mesh. However, their results found best agreement with the experimental evidence when their constructed mesh was the same as that employed in the current finite element analysis.

Figure 4.7 shows the variation of ball velocity during impact and rebound. It commences with the initial velocity of -0.313 m/s (negative value indicating downward motion into the bulk of the substrate) as obtained above, reducing gradually due to the increased contact reaction of the elastic solid and finally halting (i.e. $v=0$) at the maximum penetration, before commencing rebound, finally reaching the value of 0.313 m/s as it leaves the surface of the elastic solid (not shown in the figure) due to a purely elastic impact, and based on the principle of conservation of both momentum and energy. The time to reach maximum penetration is longer than that predicted by the Hertzian theory, be it with a very small difference. The impact time calculated by the Hertzian theory is 0.147 ms, whereas that predicted by FEA is 0.16 ms.
Figure 4.7: Ball velocity in impact and rebound

The corresponding maximum pressure time history, being at the centre of the contact in this axi-symmetric impact is shown in figure 4.8.

Figure 4.8: Maximum transient pressure variation during impact (Steel ball on dry glass plate)
The distribution here lacks the pressure spike (reminiscent of the exit pressure spike in elastohydrodynamic rolling condition) that Safa and Gohar (1986) obtained. This is because the impact conditions here are for dry solids. The shape of the distribution is similar to the Hertzian contact pressure distribution, but clearly with much higher pressures. The maximum value of pressure closely conforms to that found by Safa and Gohar (1986) at 2.25 GPa, (the lubricated experimental results being 2.4 GPa), and that of AlSamieh and Rahnejat (2002) at 2.57 GPa. The maximum pressure in this case reaches the value of 2.2 GPa. Percentage errors on maximum pressure and impact time between the FEA and the Hertzian theory are 0.46% and 8.2% respectively.

![Figure 4.9: Variation of sub-surface equivalent stress in the depth of the glass plate during impact and rebound](image)

The penetration time history of the impacting ball is best observed through the changes in the sub-surface stress field. Figure 4.9 illustrates the variation of Von
Mises stress in the glass substrate. Note that both the extent and the magnitude of the equivalent stress increase up to the moment of maximum penetration (the red contour indicating the maximum value of the equivalent stress). After that the stored strain energy has attained its maximum value and the motion of the ball is completely arrested. Rebound part of the cycle occurs in a symmetric manner, governed by the idealised conservation of energy between stored strain energy and the kinetic energy.

The maximum deflection occurs along the centre-line of the contact and into the depth of the semi-infinite solid (i.e. $x=y=0$, for values of $z$). Figure 4.10 shows the time history of this deflection. It reaches the maximum value of 16.1 μm, as opposed to 15.6 μm predicted by the classical theory, a deviation of 3.1%.

![Figure 4.10: Impact induced maximum deflection at the centre of the contact during impact and rebound](image)
(b)- A rigid ball impacting on a semi-infinite elastic half-space (Steel)

The same mesh density is used for the next two simulation studies, the purpose of which are to compare the response of semi-infinite and layered solids under identical impacting condition. The first case is the impact of a 12.7 mm rigid ball on a Steel substrate from height of 5 mm. Therefore, the initial impact velocity is the same as in the previous case (see figure 4.11). The analytical solution, based on the Hertzian impact of semi-infinite solids gives:

\[ p_{\text{max}} = 3.8 \text{ GPa}, \quad \delta_{\text{max}} = 12 \mu\text{m}, \quad t = 111 \mu\text{s}. \]

These results should be comparable with the finite element analysis findings, which gives:

\[ p_{\text{max}} = 3.2 \text{ GPa}, \quad \delta_{\text{max}} = 10.7 \mu\text{m}, \quad t = 125 \mu\text{s}. \]

Figure 4.11: Impact velocity variation for a rigid ball against a Steel substrate

Figure 4.12 shows the variation of maximum pressure during impact and rebound. The shape of the distribution is very close to Hertz, with clearly much higher pressures than those which would be found under elastostatic conditions. Using
the Hertzian elastostatic contact theory for the maximum impact load of 500 N (found using the Hertzian impact theory) for a quasi-static analysis yield a maximum pressure of 2.23 GPa. The difference between this value and that of Hertzian impact theory (3.8 GPa) and the finite element value of 3.22 GPa is in the physics of the problem. In elastostatic contacts, the stored energy is as the result of the work done by the contact force in deforming the solid, where as in the case of impact, additional strain energy is stored by the conversion of kinetic energy. Therefore, the temporal distribution of the maximum pressure in figure 4.12, although having a similar shape to the elastostatic Hertzian distribution, it differs from it in two ways. Firstly, it contains significantly higher pressures, and secondly the local slopes do not follow a single ellipsoidal profile since the instantaneous applied force is changing in magnitude.

![Figure 4.12: Maximum transient pressure variation during impact (rigid ball upon Steel substrate, semi-infinite condition)](image)

With the FEA one can obtain the deformation time history of any location within the contacting solids. The maximum deformation occurs at the centre of the contact in this simulation at all instants of time. This is shown in figure 4.13. Note that the absolute maximum deflection is at the onset of rebound, when \( v = 0 \). This value is approximately 10.7 \( \mu \)m, differing from the Hertzian impact theory by 10.3% (see figure 4.12).
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Figure 4.13: Time history of the central deflection in impact and rebound

Figure 4.14 shows the maximum shear stress variation during impact and rebound of the rigid ball upon the Steel plate. The absolute maximum value clearly takes place at the instance of maximum penetration, being 1.3 GPa. For the Cobalt Chromium Steel substrate the yield stress is in the range 1.5-1.7 GPa, making it prone to yielding by sub-surface shear stresses, exceeding 850 MPa according to Tresca criterion. This limit is exceeded for a period of approximately 0.15 ms in impact and rebound and over a very small sub-surface area, smaller than the region of high sub-surface stresses in figure 4.8 for the impact of a Steel ball upon a flat glass plate.

Figure 4.14: Transient behaviour of the maximum shear stress during impact and rebound of a rigid ball upon a Steel plate
(c)- Impact of a rigid ball upon a layered elastic solid

The impact of 12.7 mm rigid ball upon a layered solid with approximate modulus ratio $\frac{E_1}{E_2} = 2$ (Alumina coated Steel substrate) is studied as a typical case in many engineering applications, such as in ball bearings. The impact is caused by free falling of the ball upon the layered solid from a height of 5 mm, having therefore an identical initial impact velocity as the previous cases in sections (a) and (b).

The layered solid consists of a 40 μm thick layer of Alumina on a Cobalt Chromium substrate. The analytic model derived in this chapter gives the following results: $p_{\text{max}} = 26$ GPa, $\delta_{\text{max}} = 2.25$ μm, $t = 20$ μs. The corresponding results obtained by dynamic application of finite elements are: $p_{\text{max}} = 20.1$ GPa, $\delta_{\text{max}} = 1.7$ μm, $t = 15.1$ μs. The percentage differences are: 23% on the maximum pressure, 24.3% on the maximum deflection and 24.5% on the impact time. These differences are somewhat higher than those obtained for other cases reported in this thesis. The reason is that the analytic model considers the bonded elastic layer to be fabricated upon a rigid substrate, thus the effect of the deformation and stored strain energy in the Steel substrate are ignored.

The significant points to be observed by the comparison of the current analysis with those of the previous case (rigid ball impacting upon a semi-infinite Steel substrate) are the considerable reduction in the impact time due to the reduced deflection of the hard ceramic coating, and the large pressures generated in the layered solid. This means that although very large pressures are encountered their application time is very short indeed. Furthermore, in practical applications the impact height is very small and considerably lower pressures are encountered that can be withstood with hard coatings, having yield stresses 2.5-4 times that of bearing Steel. It should also be noted that the lower modulus substrate is guarded against the effect of these high pressures.
Figures 4.15 and 4.16 illustrating the corresponding variations in the maximum deflection and maximum pressure respectively.

![Graph showing deflection over time](image1)

**Figure 4.15:** Variation of maximum deflection in impact and rebound of a rigid ball upon a Steel substrate with thin layer of Alumina coating

![Graph showing pressure over time](image2)

**Figure 4.16:** Variation of maximum pressure in impact and rebound of a rigid ball upon a Steel substrate with thin layer of Alumina coating

The sub-surface stress variation during the transient event of impact; penetration and rebound provide a clearer picture of the important protective role that the hard coating layer with a high modulus of elasticity plays. This is shown by the variation of the Von Mises (equivalent stress) in figure 4.17.
In figure 4.17 the contour in red colour represents the region of maximum equivalent stress which increases in value and in area with the increasing penetration of the ball into the layered solid, reaching its maximum at $\frac{1}{2}t$ when the maximum penetration has taken place. The horizontal line in all the time frames in figure 4.17 is the demarcation line between the ceramic hard coating and the Steel substrate. It is interesting to note that this line indicates interruption in the areas of high stress, confining them to the coating layer.

(d)- A roller impacting on a semi-infinite elastic half-space (Steel)

Simulation of a Steel roller of radius 6 mm and 12 mm length impacting a semi-infinite flat Steel plate is carried out by a rigid roller dropped from a height of 5 mm under influence of gravity onto a dry plate of the equivalent modulus of elasticity, $E'$. These conditions were simulated initially using the developed analytical model, reported above, in which an iterative procedure is employed, with an initial guess, giving $W=mg$. Values of $a_0, p_0$ and $\delta_{\text{max}} = \omega_0$ are then calculated from the analytic expression provided in section 4.5 for the case of a roller impacting a semi-infinite elastic half-space. Finally, the value of load is recalculated from equation (4.27) and updated. The iterative procedure is continued until convergence is obtained for the impact force. The results obtained for the above stated conditions using equations (4.34) after 3 iterations, and are: $W_{\text{max}} = 844$ N, $p_{\text{0, max}} = 656$ MPa, $a_0 = 0.068$ mm, $\delta_{\text{max}} = 1.23 \mu$m and $t = 12 \mu$s.
Figure 4.17: Variation of Von Mises Stress during impact of a rigid ball upon a layered solid
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It is noteworthy that the analytical method is based on the determination of maximum deflection at the centre of the contact, which is less than the depth of penetration at the roller edges, where the actual maximum pressure will occur. An expression for contact load relating it to the penetration at the edge of the roller would require a numerical approximate solution.

The corresponding finite element solution provides much more instructive information, providing an estimate of the edge pressure spikes and the localised shear stresses beneath them (see figures 4.18 and 4.19).

The magnitude of pressure spike reaches the value of nearly 1 GPa, whereas the maximum value of pressure at the centre of the contact, $P_{\text{max}}$, reaches the value of nearly 600 MPa, 8.6% below its predicted value by the analytic calculations. There is more dramatic change in pressure values in the central region of the contact than its edges.

The shear stress variation under the pressure spike is shown in figure 4.19.

The shear stress maximum reaches its absolute maximum value at maximum penetration as 700 MPa, which is still within the elastic behaviour of the material according to the Tresca maximum shear stress criterion. The impact time is predicted to be 14 μs. There is a trailing edge in figure 4.19, cause by the crudeness of the time interval.

The central semi-minor axis half-width, $a_0$, is predicted to be 0.055 mm, which is 18% below that obtained analytically.
Figure 4.18: Transient axial pressure distribution in impact of a roller on a semi-infinite Steel plate
Figure 4.19: Localised shear stress variation under the pressure spike (roller impact on Steel substrate)

(e)- A roller impacting a semi-infinite elastic half-space (an Alumina plate)

Similar impact conditions are investigated, this time for a roller of same description impacting an Alumina plate (as a semi-infinite elastic half-space). The results for axial central pressure distribution comparable to those in figure 4.18 are shown in figure 4.20.

The analytic results obtained for this condition, using the aforementioned iterative techniques yielded the results:

\[ W_{\text{max}} = 900 \text{ N}, \ p_{o_{\text{max}}} = 794 \text{ MPa}, \ a_{o} = 0.064 \text{ mm}, \ \delta_{\text{max}} = 1.09 \ \mu\text{m} \text{ and } t = 10.9 \ \mu\text{s}. \]
Figure 4.20: Transient axial pressure distribution in impact of a roller on a semi-infinite Alumina (ceramic) plate

Note that as expected the maximum impact force is increased due to the harder plate material, with decreased footprint dimensions (semi-minor half-width and depth of penetration), larger maximum pressure and slightly shorter impact time.
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The corresponding finite element results are:

\[ P_{\text{max}} = 960 \text{ MPa}, \, a_0 = 0.06 \text{ mm}, \, \delta_{\text{max}} = 0.92 \text{ \(\mu\)m} \text{ and } t = 10 \text{ \(\mu\)s}. \] Close agreement is, therefore, noted between the FEA and analytic results.

The corresponding maximum shear stress variation with time beneath the pressure spike is shown in figure 4.21.

\[ \text{Figure 4.21: Localised shear stress variation under the pressure spike (roller impact on the ceramic substrate)} \]

One major difference between the roller impacts upon the Steel and Ceramic substrates is in their endurance. The maximum shear stress of 800 MPa is well below the Tresca limit for Alumina, whilst quite close to that of Steel.

Various ceramic materials are usually employed as a protective hard coatings, rather than the substrate material which would act as a semi-infinite elastic solid, as in the case reported above. However, with layered solids the generated pressures are usually higher than those for semi-infinite cases and can lead to their fracture, as already mentioned, particularly under high impact loads. The case of a roller impacting a thin bonded layer (in this case 40 \(\mu\)m in thickness) of high
elastic modulus has not been reported in literature, either with analytical or numerical predictions. However, many applications exist such as separation and subsequent impact of a flat tappet or a roller follower in valve train systems. In particular, it is interesting to investigate the possible exacerbated edge stress discontinuity effect, manifest by pressure spikes in finite line contacts, under impacting condition. Figure 4.22 shows the axial pressure distribution through the centre of the contact, containing the pressure spike, at the moment of maximum penetration. Note that the pressure spike at the edge of the roller contact has reached a value of 4.4 GPa, more than 4 times the value for the same impact height on a semi-infinite ceramic half-space.

![Central axial pressure distribution for a roller impacting on a hard layer coating of Alumina](image)

Figure 4.22: Central axial pressure distribution for a roller impacting on a hard layer coating of Alumina

The variation of this maximum pressure spike during the impact time of 12 µs is shown in figure 4.23.
Figure 4.23: Changes in the pressure spike in the vicinity of the roller end during impact on a hard layered solid
Chapter 5

Experimental investigation of viscoelastic contacts

5.1 Introduction

This chapter outlines experimental work carried out for the investigation of layered viscoelastic solids. It describes the design and development of a sandwich sensor, comprising a thin layer of a High Filled Carbon Polymer (HFCP), bonded to a PCB tape, upon which a number of sensing sites are fabricated through use of an optical etching process. The piezo-resistive HFCP layer’s conductance varies with application of pressure in a linear manner within the range of low applied loads. The high filled carbon polymers comprise many carbon based branched structures that break and reform, thus altering the resistance of the structure to passage of an electric current.

A protective layer of silicone rubber covers the HFCP layer. Pressure distribution is obtained for various applied contact loads, effected by indentation of a flat ended circular cylindrical punch. A data acquisition system is devised with a signal processing method to obtain the changes in the transient pressure distribution through stress relaxation of the bonded layered HFCP. Later, in chapter 6 results of numerical predictions, using finite element analysis have been compared with these experimental findings.
5.2- Design of a thin-film viscoelastic sensor

The viscoelastic layered sensor that has been developed uses as the sensory material; a thin Highly Filled Carbon Polymeric (HFCP) film, which is chosen due to its physical and mechanical properties. Figure 5.1 depicts the structure of the sensor. It comprises a PCB tape, upon which a miniature circuit is fabricated, as described later. The sensing sites are connected to terminals as shown in the figure. Each sensing site incorporates thin HFCP pieces, which is bonded to it, and finally covered by a protective silicone rubber layer.

![Figure 5.1: The HFCP pieces close the circuit forming a sandwich-layered sensor](image)

The main characteristic of HFCP is the change of its conductance with pressure in a linear manner, and as a result of the carbon bonds behaviour within the polymeric molecular structure. The carbon fibre is renowned for its strength and also for some other remarkable properties such as low coefficient of friction, high thermal conductance and very low coefficient of thermal expansion.
Another characteristic of the HFCP that makes it suitable for the use as a sensory material is the fact that its stress relaxation behaviour is independent of the state of deformation, and the temperature dependence of its static modulus is in good accord with that predicted by the statistical theory of elasticity. Figure 5.2 presents typical results for pure-shear and simple-tension stress relaxation by an investigation that has been carried out by Sullivan and Demery (1982), conducted at temperature T=55°C.

![Figure 5.2: Plot of normal stress difference versus strain at T=55°C for both simple tension (○) and pure shear (►) results (s₁ and s₂ are principal stresses) (after Sullivan and Demery (1982))](image)

The results in figure 5.2 show a slope of 1.090 MPa. The authors also state that this slope changes marginally to 1.021 at 30°C, which is closer to the conditions employed in the tests carried out in this thesis.

The task was to create a sandwich-layered sensor with HFCP and silicone rubber (see figure 5.1). The latter would be used to seal and protect the sensing sites from wear and tear, which can occur due to repeated application of normal and tangential contact loads. The thickness of the silicone layer that has been used on the final set of the constructed sensors is about 1-1.5 mm thick. Note that it is difficult to control the thickness of the silicone rubber owing to the application of
the silicone paste and its subsequent drying process. However, the variation in its thickness is not regarded as crucial (as described later). Two types of silicone rubber have been tested. One is a non-corrosive silicone rubber in the form of a paste, and the other is a silicone rubber compound as a liquid paste. The latter, when applied, flows under the force of gravity until it is cured, and therefore is easier to use in order to form a thin flat layer. The paste is more difficult to shape into a layer without the formation of air pockets. On the other hand, it does not evolve any corrosive volatile substance during the curing process and is therefore, suitable for use on copper plates. The tests with the sensors have been carried out with both forms of silicone rubber, yielding almost the same results. The reason for the use of the two products is because the compound is translucent and the loaded punch (i.e. the indenter) could be placed more accurately on top of the sensing sites. The paste is white and, therefore, poses greater problems in finding the right position for direct application of normal load. Because the circuit and the materials of the sensor are very sensitive, the non-corrosive silicon rubber was used for the sensors fabricated upon a thin flexible PCB tape (50 μm in thickness: FR4).

The modulus of elasticity of the silicone rubber is quite low; around 5 MPa compared with the HFCP layer, which has a modulus of elasticity of 1 GPa. This means that under the contact in the range 1-8 N, the silicone rubber deforms to the profile of the indenter, transmitting directly the applied force to the HFCP sensors. Note that for the low ratios of the top layer to the substrate or a lower layer, the pressure is almost entirely carried by the latter. This has been shown in chapter 3. Therefore, the deflection of the HFCP is much less than the silicone rubber and can withstand the load. The Poisson’s ratio for HFCP is 0.4, which behaves as a compressible layer. The relaxation time of the HFCP is approximately 0.9 sec. This means that the sensing sites adapt slowly to the application of pressure.

The next step was to design the circuit that would be the base for the sensor. This circuit would transmit the output signals to the computing elements. The main requirement was to design a sensor small enough to provide sufficient spatial
resolution in small contact areas, which are typical of load bearing surfaces, for example in rolling and contacting members in bearings. For that reason a flexible/conformable printed circuit board tape is used. The only limitation in the etching technique is that the gaps between the copper tracks have to be at a minimum distance of 0.177 mm for the available optical etching system in the University (see figure 5.3).

![Diagram of sensing elements](image)

**Figure 5.3:** Dimensions of the sensing elements (all dimensions in mm)

The sensor (see figure 5.3) consists of a cell in the centre and four at the corners of a 4x4 mm$^2$ area. With this arrangement there are a sufficient number of cells, for a certain class of contact conditions such as planar or line contacts. The presence of objects can be detected according to the sensor cells that are activated and from their output values. For example, a flat surface would activate most of the cells, while a curved surface (e.g. a sphere) may only make a partial contact with few of the sensing sites. The shape of the sensing element has little influence in the response of the HFCP, which depends on the actual area of the piece. However,
smaller and narrower sensing elements will clearly provide a better definition capturing sharp edge stresses, for example in the contact of a cylinder with a flat plane or the flat end of a punch, indenting a flat surface.

Figure 5.4: Overall size of the artificial skin sensor (all dimensions in mm)

The thickness of the printed circuit tape for the sensor is 50 µm and the copper tracks have a thickness of 17.5 µm. Considering that the HFCP is 100 µm in thickness and the silicone rubber has a thickness 1–1.5 mm, the whole thickness of the sensor is no more than 1.167–1.667 mm. The circuit is 22.66 mm long and 5.66 mm wide, developed on a piece of flexible board 30 by 7 mm in dimensions (see figure 5.5). Six tracks lead out of the sensor, where five of them are connected as the positive terminals for the sensor cells and one forms the common negative terminal (figure 5.5).
Figure 5.5: The shaded area, representing the gauge, is connected to positive terminals and the white to the common negative terminal.

The tracks have a width of 0.2 mm with the longest extending for 8.5 mm and the shortest for 6.5 mm away from the sensor. The wiring takes place at the underside of the contacting surface.

Figure 5.6: Location of the sensing sites (all dimensions in mm)
To create the main circuit for the sensor a square matrix 4x4 mm\(^2\) was divided in such a way so as to create five cells that could fit inside. This was achieved by using a second concentric square of 2x2 mm\(^2\) at an angle of 45° to the first, allowing the formation of four cells at the corners and one in the centre. The entire inner square is connected to the negative terminal and the four corners of the outer square, plus the centre circular sensing point are connected to the positive terminals as shown in figure 5.5 by the darkened areas. In this manner the pieces of HFCP close to the circuit allow the flow of the current from the positive to the negative terminal. The HFCP in this case acts as a variable resistor. Each time it is loaded its resistance \( R \) changes and, therefore, its conductance \( G' = \frac{1}{R} \) alters.

This means that the output voltage is at its maximum value, when there is no applied pressure. When the resistance drops, the conductance increases and a greater current flows.

The resistance for a 1 in\(^2\) of HFCP sheet is 50 KΩ. Therefore, it is very important to keep the size of the pieces as small as possible for the sake of the sensor's sensitivity. All the pieces of HFCP used on the sensor have the same area of 2.8 mm\(^2\), ensuring the same resistance for all the sensing elements. The other important consideration was to keep the cells in a close range in order to provide a good spatial resolution. The centre of the middle cell is located at 2.828 mm, from the edge of the sensor and 1.648 mm from the centres of the side cells, which are also 2.330 mm apart from one another (see figure 5.6).

5.3- The Etching and Bonding procedure

As mentioned in the previous paragraphs, it was necessary for the circuit of the sensing device to be quite small, so that one or a series of sensing sites could be placed in a small contact area. For this reason the circuit was first designed in AutoCAD version 14 and then printed on a flexible board, using an optical etching technique. Because the circuit cannot be etched, if the width of the tracks or the gaps between the tracks is less than 0.177 mm, any smaller configurations can only be obtained with other techniques, for example by deposition. However, in
this case the 4x4mm\(^2\) design has proved to be adequate for the flat ended punch to a plane contact that it was designed for.

The etching is a very similar procedure to the development of a photographic film. First the AutoCAD drawing is exported as an Encapsulated PostScript (*.EPS) file, so that it can be used in a desktop publishing program. There, it is handled as a picture and is sent to print on a photographic film. The printer used for this task is a high-resolution postscript printer (Linotronic 260) with a spatial resolution of 2540 dots per inch (dpi) or 1000 pixels per cm. The printer uses an infrared laser diode to print the drawings on the film and because the whole procedure is light sensitive the film and the components are all firmly closed inside the machine. The photographic film comes in rolls of 100 ft long, the printed part of which is later collected in a light protected cartridge, where it is also cut from the rest of the roll by an external control.

The film is then removed from the printer and is fed to a film processor. The film first goes through the developer bath, then through the fixer (with chemicals) bath and finally into the wash bath, before it comes out as a photo-plot on a clean film ready to be used to make the circuits.

The flexible thin film (FR4) was used for the circuit as a board with a copper coating and a photosensitive layer on top. The developed clean film with the circuit that is required to be fabricated is placed on top of the flexible board and on the side the circuit required or on both sides, if a double-sided circuit is desired. The next step is to expose the board to ultra-violet radiation for approximately 60 seconds. The image of the circuit on the clean film protects the part of the board that is directly underneath it from the UV rays, retaining the composition of the material, and thus leaving the footprint of the circuit on the exposed board.

The next step is to take the board and put it in a tray, where it will develop the image of the circuit. The tray contains a liquid that removes the photosensitive material from the board leaving only the part that was protected from the UV rays.
Then the board is ready for etching, where the unwanted copper is removed. The chemical used to perform this task is the ferric chloride. The board is placed on a conveyor and goes through the etching machine, where the ferric chloride is sprayed on the board removing the copper. The part with the photosensitive layer remains unaffected from the ferric chloride, revealing the final circuit. After drying and washing with a low temperature tin solution the fabricated sensor is ready for use.

The final work on the sensor is made under an optical microscope. This is necessary because both the circuit points and the HFCP pieces have surface areas of under 2.5 mm$^2$. The cutting of the HFCP pieces, the bonding and the soldering of the wires is carried out using magnifications from x5 to x20. The pieces were handled with a pair of forceps and carefully placed upon their precise location on the circuit. Then a thin layer of insulating tape covers and holds the pieces of HFCP in their position, preventing them from moving, when the silicone rubber is applied. The tape also prevents the silicone rubber from inserting between the HFCP and the circuit. It takes about 24 hours for the silicone rubber to cure and then the viscoelastic sensor is completed.

The final result is depicted in figure 5.7, with the scale shown.

![Figure 5.7: The sensor structure with the scale of construction given in mm](image-url)
5.4- Data acquisition and processing

Microlink is a High Speed Data Collection (HSDC) system. It runs on a computer, which is connected with the Microlink modular data acquisition system, which converts the analogue voltage inputs into digital data and stores these on a hard disk. In addition, this package can graphically display up to six waveforms in real time, and after the collection, the waveforms can be scaled, viewed, measured, and transported to other software as data files.

With the appropriate hardware, it can digitize up to 256 inputs simultaneously, convert analogue voltages to digital data at typical maximum rates of 25000 samples per second, and collect up to a total of 16 million sample (Microlink Hardware User Manual (1990)). Simple menus and tables, without any additional required programming make it easy to use. Selection of parameters, such as the number of samples, the sampling rate, the gain of each input amplifier, and scaling factor to convert the voltage to engineering units are specified. Sets of these parameters can be saved in named disk files, and reloaded when needed.

The data collection can start either from the keyboard, or by an external trigger signal, and the digitised raw data are recovered in a disk file. The input signals that are collected from each channel can be viewed as a graph on the screen, either in a completed form, or in sections for more detailed analysis. Any sections of interest can be transferred as data files for use in other programs such as spreadsheets, signal analysis programs, etc. The minimum hardware configuration that is required for the use of the HSDC package is:

- Microlink mainframe with power and control modules
- BA12D Buffered analogue to digital converter
- HSC High Speed Clock and multiplexer
- One to sixteen analogue input modules that can be any combination of:
  - PGA16 – 16 channels of programmable gain analogue voltage input
  - TC16H – 15 thermocouples and one RTD
The Microlink mainframe holds the power supply, the address decoding and internal bus control circuitry. It also contains all the circuitry that is necessary for a complete IEEE-488 bus operation and a 24-pin socket is used for the connection of the Microlink interface to the IEEE-488 bus. The latter have been designed to connect laboratory instruments to computers, where a 16 line bus is used to interconnect up to 15 separate devices each connected in parallel to the bus.

The PGA16 module of the Microlink multi-channeled data acquisition system, accepts 16 differential analogue inputs, and a 37 way D-connector gives access to them and to an earth connection.

Such a connector has been used for the interface of the sensors to the Microlink system. The pins 20 to 29, of the 37 way D-connector (see figure 5.8), are the 10 channels that the sensor cells are connected to. All the pins from 1 to 10 are connected to the pin 19 (see table 5.1), which is the ground and also to point C on the board in figure 5.10.
<table>
<thead>
<tr>
<th>Channel no</th>
<th>Positive Input Pin</th>
<th>Negative Input Pin</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>21</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>23</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>26</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>27</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>28</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>29</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>11</td>
</tr>
<tr>
<td>11</td>
<td>31</td>
<td>12</td>
</tr>
<tr>
<td>12</td>
<td>32</td>
<td>13</td>
</tr>
<tr>
<td>13</td>
<td>33</td>
<td>14</td>
</tr>
<tr>
<td>14</td>
<td>34</td>
<td>15</td>
</tr>
<tr>
<td>15</td>
<td>35</td>
<td>16</td>
</tr>
</tbody>
</table>

0V (gnd) is connected to pin 19

Table 5.1: Microlink pin to channel connections

5.5 Circuit Analysis

A voltage divider is used to connect the sensor cells to the different channels of the Microlink system. The voltage divider is one of the most widespread electronic circuit fragments, and is a circuit that a given certain voltage input uses to produce a predictable fraction of the input voltage as the output voltage (Horowitz and Hill (1994)).
Note that the output voltage is always less than (or equal to) the input voltage. The voltage divider (see figure 5.9) in its simplest form consists of a fixed resistor $R_1$, and a variable $R_2$, whilst the signal out is totally dependent on the variable resistor and directly proportional to its changes (Horowitz and Hill (1994)). The current is given as:

$$I = \frac{V_i}{R_1 + R_2} \quad (5.1)$$

and:

$$V_{out} = IR_2 = \frac{R_2}{R_1 + R_2}V_i \quad (5.2)$$

Figure 5.10 shows the circuit board that connects the artificial sensor to the Microlink module. The $R_1$, $R_3$, etc have been replaced with 100 KΩ fixed resistors and the variable $R_2$, $R_4$, etc with the sensor cells.

The board is connected with a power supply, which provides a voltage of 10V to the terminal (see figure 5.11). The lines at point B are connected to the pins of the
D-type connector, which is plugged into the module. Finally the point C is attached to pin 1-10, where as mentioned earlier, they are all connected to pin 19.

Figure 5.10: Block diagram of the circuit used to connect the sensor to the Mikrolink Data System

5.5 The Indenter

A testing device is developed in order to apply a normal load to the viscoelastic sensor through a flat-ended punch as shown in figure 5.11. Weights in the range 50gm-450gm are placed at the end of the lever arm. The design is simple, employing a bar balancing on a sharp edged block (see figure 5.11). The total length of the bar is 370 mm, with 16 and 20 mm in height and width respectively.
90 mm from the centre of the bar there is a hole, 8.4 mm in diameter drilled all the way through to fit the pin. Another hole at 180 mm away with 5 mm diameter is used to hang the applied weights. Therefore, the reaction force exerted on the sensor is:

\[ R = F + m_p g \]  \hspace{1cm} (5.3)

\[ RL = FL + m_p gl \]  \hspace{1cm} (5.4)

\[ R = \frac{FL + m_p gl}{l} = \frac{FL}{l} + m_p g \]  \hspace{1cm} (5.5)

For \( \frac{L}{l} = 2 \) and considering \( m_p \) the mass of the pin as negligible:

\[ R = 2F \]  \hspace{1cm} (5.6)

Hence, the pin transmits onto the sensor twice the force that is actually applied at the weight hanger.
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The pin is locked in position by a 4mm screw and its position can be adjusted. This is important for the tests because the flat circular surface of the pin must always approach the sensor along the line of contact normal. Therefore, the pin is adjusted in such a way that the bar remains almost horizontal, when the surface of the pin almost touches the surface of the silicone rubber.

5.6- Experimental Procedure

As it has been shown earlier, the prototype sensor consists of five sensing sites. However, in order to be able to investigate the viscoelastic behaviour of the sensor, a ten-cell configuration has been used to improve upon its spatial resolution during the experiments. This provides enough points on the x-axis to plot the pressure distribution curves. Due to symmetry along the y-axis the obtained results correspond to the numbered sensing sites in figure 5.12.

![Figure 5.12: The sensor configuration used in the indentation test](image)

The artificial sensor is connected to the voltage divider circuit, which links each sensor site to a separate channel in Microlink module and is supplied with a voltage of 10V (see figure 5.9). The data acquisition system is set at a 0.5 seconds
sampling rate; this is approximately half the relaxation time of the HFCP, reducing the chance of aliasing due to the sampling rate itself. Then the sensor is mounted onto the platen of the indentation rig and the pin is adjusted to ensure an aligned normal contact with the surface of the skin sensor.

The normal contact load in precision type grasps of small objects such as finger prehension is usually in the range 1-8N. The indentation tests that have been carried out are within that region with loads in the range 0.98-8.82 N and for duration as multiples of the HFCP relaxation time. The output voltages of the sensing elements are recorded through the activated channels, allowing readings to be taken from all the five elements simultaneously. The maximum voltage is 10V, when no load is applied. As it can be observed in table 5.2, the output voltage decreases with an increase in the applied pressure and with multiples of the HFCP relaxation time constant. Loading steps 0.98N, 1.96N, 3.92N, 5.8N and 7.85N were applied for 0.9, 1.8, 2.7, 4.5 and 9 sec. The loading tests are carried out in the intervals every 10sec, allowing the sensor to recover completely, before a new test is carried out. This process is repeated at each load for a number of times and an averaging process is carried out over the output data from each sensor cell for the same test, in order to eliminate the sources of spurious error such as electrical noise.

<table>
<thead>
<tr>
<th>Sensors (volts)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relaxation Time (n(\tau))</td>
<td>(\tau)</td>
<td>9.305</td>
<td>8.715</td>
<td>8.293</td>
<td>8.238</td>
</tr>
<tr>
<td></td>
<td>2(\tau)</td>
<td>9.02</td>
<td>8.325</td>
<td>7.73</td>
<td>7.925</td>
</tr>
<tr>
<td></td>
<td>3(\tau)</td>
<td>8.34</td>
<td>7.973</td>
<td>6.987</td>
<td>7.24</td>
</tr>
<tr>
<td></td>
<td>4(\tau)</td>
<td>7.995</td>
<td>7.52</td>
<td>6.693</td>
<td>6.78</td>
</tr>
<tr>
<td></td>
<td>5(\tau)</td>
<td>7.15</td>
<td>7.075</td>
<td>6.04</td>
<td>6.17</td>
</tr>
</tbody>
</table>

\(\tau=0.9\) sec.

Table 5.2: Decrease of output voltage with relaxation time at 3.92N
Referring to figure 5.12, it is clear that symmetrical values should be obtained for given sensing sites. However, in practice the slight misalignment of the pin due to the loading device and any small problems in flatness of the indenter can lead to the differences in the readings of the symmetrically positioned sensing sites. The silicone layer can undergo a small deflection, if it were to be assumed as an elastic half-space. The design of the lever can cause it to tilt by a small angle under load and thereby result in a misaligned contact.

The tables in appendix III are used to obtain the calibration curves for different applied loads and multiples of the relaxation time of HFCP for positioning a direct contact between the indenter and a typical sensing site. The calibration curves (see figure 5.13) are representative of the various sensing locations, assuming that there is only a small variation between the characteristic outputs of different sensors under the same load and with relaxation time.

![Figure 5.13: Pressure calibration curves for the sensor at different relaxation times](image)

Figure 5.13: Pressure calibration curves for the sensor at different relaxation times
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The cell located nearest to the centre of the experimental configuration shown in figure 5.12 as the cell 1, has been used to plot the calibration curves, with the voltage on the x-axis against the pressure along y-axis (see figure 5.13).

Figure 5.13 is a carpet plot, in which the dotted lines are pressure relaxation lines for a sensor site (in this case cell 1) for given values of load. Along the dotted lines voltage drop occurs, corresponding to relaxation of pressure. The full lines are at various values of $r$, show that at a given value of load, pressure is highest at lower multiples of relaxation time.

The highest pressures occur instantaneously (when the load is initially applied at $t=0$), corresponding to an elastic response of the layered solid. Then, the pressures are reduced due to relaxation (i.e. for $t>0$). It is quite hard to acquire the initial elastic response instantaneously at the instance of application of load, particularly for layered solids with short relaxation times as HFCP. The results, therefore, are recorded for whole multiples of relaxation time up to the point, where small changes in contact pressures are observed, corresponding to the adaptation of the artificial skin sensor to the applied pressures.

5.7- Experimental results for flat punch contact

As it has been mentioned in the previous paragraphs the voltage output is 10V, when no load is applied and drops with any change that occurs in the conductance of the HFCP after the application of load. In figure 5.13, one can observe that the relationship between pressure and voltage output is quite linear. The linear behaviour of the HFCP is also discussed by Mehdian and Rahnejat (1996).

The average pressure value in the central domain is given by $P = \frac{W}{A}$, where $W$ is the applied load and $A$ is the footprint contact area of the pin, indicating that for a contact of infinite dimensions the pressure remains almost constant there. Note that for a finite planar contact the edge stress discontinuity gives rise to the
formation of pressure spikes. This phenomenon has also been observed by Phillips and Johnson (1981).

Using the appropriate calibration lines in figure 5.13, the output voltage of cells 2 and 3 (see figure 5.12) can be converted to equivalent contact pressures. Therefore, the contact pressure distribution curves along x-axis can be plotted. These are plotted as sensor output values together with the equivalent FEA predictions in chapter 6.

Although the central pressure value is assumed to be equal to \( \frac{W}{A} \), for small and finite contact areas, this assumption may not hold true. Ideally to calibrate the sensors a constant pressure must be applied (perhaps by pneumatic means) and the voltage output monitored from all sensors. However, this procedure was not thought of at the outset of this work and can be employed in any future investigation. The results obtained using the above assumption has found to conform reasonably well with the numerical FEA predictions. This is shown later on.

The rise in pressure that can be observed at the extremity of the contact (see chapter 6, figures 6.8-6.10) is caused by the stress concentration there due to a sharp change in the axial profile of the indenter (i.e. the edge of the indenter). This effect is in-line with the classical Hertzian theory, observed for contact of unblended rollers against a semi-infinite elastic solid by various researchers (Johns and Gohar (1981), and Rahnejat and Gohar (1979)). The same is true for the contact of a flat indenter against an elastic half-space (Johnson (1985)), where an analytic solution based upon a described indentation profile yields a theoretically infinite pressure spike. Finite difference or finite element solutions show a finite pressure in the vicinity of the edge of the contact, which can be as much as several times higher than the central portion of the pressure distribution (Naghieh et al (1997), Naghieh et al (1998a), Maeno et al (1997), Maeno et al (1998a, 1998b)). Another solution for both thick layers as half-space and thin
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elastic layers for the contact of a circular punch has been reported by Conway et al (1966).

When the load is applied for time intervals close to the relaxation time, the HFCP exhibits an initial elastic response and the magnitude of the pressure spike is largest at $t = 0$. As it is expected the magnitude of the pressure spike reduces with time due to the viscoelastic relaxation characteristics of the polymer. The rate of reduction decays with an increasing multiple of relaxation time. One can note that the average central pressure throughout all the results is almost unaffected by the HFCP relaxation under the applied contact load.

Pressure distributions for the applied normal contact loads show a similar behaviour with relaxation time. Figures 6.7-6.10, in chapter 6, show this trend for the various applied loads. It is noted that relaxation is quicker at higher loads.

![Figure 5.14: Relaxation of maximum contact pressure at different loads with time](image)
The relaxation of pressure distribution with increasing contact load is shown in figure 5.14 for $t=10\tau$ in all cases. The reduction in the magnitude of pressure spike for various contact loads is shown in figure 5.14. The rate of fall is more rapid at higher loads. Note that the utilised width of the sensor at the edge of the contact is half of the width of the sensing element. This is not fine enough to resolve the narrow edge stress discontinuity.

All the experimentally obtained results in form of tables are provided in appendix III. Appendix II provides the properties of silicone rubber used.
6.1- Introduction to viscoelasticity

Materials with relatively low molecular mass and specially polymers behave in a very distinctive manner and can be classified as elastic and viscoelastic solids or viscous liquids. Elastic solids have a definite shape, which tends to change with the application of an external force, into a new equilibrium shape. Elastic solids store all the energy induced by the external forces during deformation and use this strain energy to return to their original form, when the applied forces are removed. The viscoelastic solids exhibit a time dependent behaviour in their relationship between stress and strain, recovering gradually after the applied forces cease to exist. Viscous liquids have no definite shape and can flow irreversibly under the action of external forces (Ward and Hadley (1993)). Some of the main features of viscoelasticity such as creep and relaxation time are discussed later in this chapter. Furthermore, some polymers can display a range of behaviour between elastic and viscoelastic solids, depending on temperature and duration of the applied forces. This behaviour is also explained later, for example, from experimental indentation results, using a devised sensor of layered bonded viscoelastic construction.

Viscous action of a fluid between a pair of adjacent solids is described by the Newtonian slow viscous model (Newton, 1687), in which the fluid molecules roll upon each other with dominant viscous force resisting their motion. They are considered to be hard (i.e. undeforming), spherical in shape (see figure 6.1). The stress $\sigma$ is proportional to the velocity gradient in a liquid. Therefore:
\[ \sigma = \eta \frac{\partial \nu}{\partial y} \]  

(6.1)

Where \( \nu \) is the entraining velocity, \( y \) is the direction of the velocity gradient and \( \eta \) is the fluid viscosity. When the velocity gradient is in the \( xy \) plane as shown in Figure 6.1, then:

\[ \sigma_{xy} = \eta \left( \frac{\partial \nu_x}{\partial y} + \frac{\partial \nu_y}{\partial x} \right) \]  

(6.2)

Where \( \frac{\partial \nu_x}{\partial y} \) and \( \frac{\partial \nu_y}{\partial x} \) are the velocity gradients in the \( x \) and \( y \) directions.

Substituting \( \nu_x = \frac{\partial u}{\partial t} \) and \( \nu_y = \frac{\partial v}{\partial t} \), the previous equation becomes:

\[ \sigma = \eta \left[ \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial t} \right) + \frac{\partial}{\partial x} \left( \frac{\partial v'}{\partial t} \right) \right] = \eta \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial y} + \frac{\partial v'}{\partial x} \right) \]  

(6.3)

Where, \( u \) and \( v' \) are the displacements in the \( x \) and \( y \) directions respectively.
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Also note that:

\[ \tau_{xy} = \eta \frac{\partial \gamma_{xy}}{\partial t} \]  \hspace{1cm} (6.4)

In equation (6.4), one can observe that the shear stress \( \tau_{xy} \) is directly proportional to the rate of change of shear strain with time in the case of linear viscoelasticity, whereas in the linear theory of elasticity the strains must remain small and quantitative predictions are possible, where results of changing stresses and strains are simply additive. When there is a single loading process a linear relation exists between stress and strain at any given time. In a multi-step loading of viscoelastic materials each increment of stress can be assumed to make an independent contribution to the overall strain and, therefore, can be analysed in terms of the Boltzmann superposition principle.

Figure 6.2: Deformations of (a) an elastic solid and (b) of a linear viscoelastic solid
6.2- Features of Viscoelasticity: Creep and Relaxation

6.2.1- Creep Compliance

The stress-strain relations for a linear viscoelastic material can be expressed in a number of ways; one of them is creep compliance. Creep can be defined as the time-dependent strain response to a step change in stress. In other words, creep is the phenomenon, where a body is suddenly stressed until the applied stress reaches a constant value and the body continues to deform. As it can be seen in figure 6.2, in the general case of a linear viscoelastic solid the total strain $\varepsilon$ is the sum of $\varepsilon_1$, which is the immediate elastic deformation, and $\varepsilon_2$, the delayed elastic deformation and $\varepsilon_3$, which is the Newtonian flow in the case of viscous fluids.

Due to the linear response of the material the magnitudes of $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are exactly proportional to the magnitude of the applied stress, and, therefore, the creep compliance $\Phi(t)$ can be described as:

$$\Phi(t) = \frac{\varepsilon(t)}{\sigma} = \Phi_1 + \Phi_2 + \Phi_3$$

(6.5)

where $\Phi_1$, $\Phi_2$ and $\Phi_3$ correspond to the strain components $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$. Cross-linked and crystalline polymers, as viscoelastic solids do not show the $\Phi_3$ term.

6.2.2- Stress Relaxation

The term stress relaxation can be defined as the particular response that the linear viscoelastic solids exhibit, when they are subjected to an instantaneous strain. During this process the initial stress is proportional to the applied strain, which decreases with time, at a rate characterised by the relaxation time of the material (see Figure 6.3). When assuming a linear viscoelastic behaviour, one can state that the stress relaxation modulus is $G(t) = \sigma(t)/\varepsilon$ and when the stress takes a finite
value, it relaxes with the modulus $G_r$ (Ward and Hadley (1993)). For example, the relaxation time of the HFCP

![Stress relaxation graph](image)

Figure 6.3: Stress relaxation

(High-Filled Carbon Polymer) used in this thesis is about 0.9 sec.

6.3- The Boltzmann Superposition

Boltzmann (1844-1906) has shown that creep is a function of the entire past loading history of a specimen, and that each loading step makes an independent contribution to the final deformation so that the total deformation can be obtained by the addition of all the contributions (see Figure 6.4).

![Creep behaviour of a linear viscoelastic solid](image)

Figure 6.4: Creep behaviour of a linear viscoelastic solid
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The total creep at time $t$ in figure 6.4 can be expressed as the sum of the incremental stresses at specific times. Therefore:

$$\varepsilon(t) = \Delta \sigma_1 \Phi(t-t_1) + \Delta \sigma_2 \Phi(t-t_2) + \Delta \sigma_3 \Phi(t-t_3) + \ldots$$  \hspace{1cm} (6.6)

where $\Phi = (t-t')$ is the creep compliance function. The above equation for the viscoelastic solids can be written as an integral:

$$\varepsilon(t) = \int_{-\infty}^{t} \Phi(t-t') d\sigma(t)$$  \hspace{1cm} (6.7)

In equation (6.7) one can separate out the instantaneous elastic response in terms of unrelaxed modulus $G_u$ and, therefore:

$$\varepsilon(t) = \frac{\sigma}{G_u} + \int_{0}^{t} \Phi(t-t') \frac{d\sigma(t')}{dt'} \, dt'$$  \hspace{1cm} (6.8)

where $\sigma$ is the total stress.

6.4- The Maxwell Model

The linear viscoelastic behaviour can be represented by various mechanical models, which are composed of linear springs and dashpots. The linear spring produces an instantaneous deformation proportional to the load, and the dashpot can be considered as an oil filled cylinder with a piston, undergoing a velocity that is proportional to the load at any instant. These models (see figures 6.5 and 6.6) include the Maxwell model, the Voigt model, and the Kelvin model, which are used to derive differential equations that can predict the deformation of the polymer under testing (Fung (1993)). The viscoelastic models are also used in biomechanics, because the biological tissues exhibit viscoelastic behaviour (Ward and Hadley (1993)). For instance, the Maxwell-Kelvin model has been used by Dinnar (1970) for the investigation of contact mechanics behaviour of human tissue.
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As can be seen in figure 6.5, the Maxwell model consists of a spring and a dashpot in series, and the equations describing the stress-strain relations are:

\[ \sigma_1 = E_m \varepsilon_1 \]  \hspace{1cm} (6.9)

and:

\[ \sigma_2 = \eta_m \frac{d \varepsilon_2}{dt} \]  \hspace{1cm} (6.10)

where \( \sigma, \varepsilon, \) and \( E_m \) are the stress, strain and modulus of the spring and \( \sigma_2, \eta_m \) and \( \varepsilon_2 \) are the stress, viscosity and strain in the dashpot respectively.

(a) A Maxwell body

(b) A Voigt body

(c) A Kelvin body (a standard linear solid)

Figure 6.5: Mechanical viscoelastic models (after Fung (1993))
Figure 6.6: Creep and relaxation functions of (a) Maxwell body, (b) Voigt body, (c) Kelvin body (standard linear solid) (after Fung (1993)).

The total stress of the Maxwell unit is equal to the stresses of the spring and dashpot, \( \sigma = \sigma_1 + \sigma_2 \) and the total strain \( \varepsilon \) is the sum of the strains \( \varepsilon = \varepsilon_1 + \varepsilon_2 \).

Therefore, equation (6.9) can be written as:

\[
\frac{d\sigma}{dt} = E_m \frac{d\varepsilon_1}{dt}
\]  

(6.11)

by adding equations (6.10) and (6.11), one can obtain equation (6.12) for the total stress and strain:
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\[
\frac{d\varepsilon}{dt} = \frac{1}{E_m} \frac{d\sigma}{dt} + \frac{\sigma}{\eta_m} \tag{6.12}
\]

If \( \frac{d\varepsilon}{dt} = 0 \), \( \frac{1}{E_m} \frac{d\sigma}{dt} + \frac{\sigma}{\eta_m} = 0 \), then:

\[
\frac{d\sigma}{\sigma} = -\frac{E_m}{\eta_m} dt \tag{6.13}
\]

Integrating the above for \( t = 0 \), when \( \sigma = \sigma_0 \):

\[
\sigma = \sigma_0 \exp \left( -\frac{E_m}{\eta_m} t \right) \tag{6.14}
\]

One can observe that stress decays exponentially with a time constant of \( \tau = \frac{\eta_m}{E_m} \).

Hence:

\[
\sigma = \sigma_0 \exp \left( -\frac{t}{\tau} \right) \tag{6.15}
\]

where \( \tau \) is the relaxation time.

6.5- The Kelvin model

The three element-type Kelvin model is shown in figure 6.5(c). The stress-strain relations for the Kelvin model are given as:

For the spring element: \( \sigma_i = E_n \varepsilon \) \( \tag{6.16} \)

And for the combination of spring and dashpot elements:
\[ \sigma_2 = E'_m \varepsilon + \eta_m \dot{\varepsilon} \]  \hfill (6.17)

Where \( \sigma = \sigma_1 + \sigma_2 \).

There are 5 unknowns in the above equations. Eliminating \( \sigma_1 \) and \( \sigma_2 \) between these equations:

\[ \sigma = (E'_m + E_m) \varepsilon + \eta_m \dot{\varepsilon} \]  \hfill (6.18)

The solution for this differential equation in strain is of the form:

\[ \varepsilon = \frac{\sigma_0}{E_m} \left(1 - e^{-\frac{t}{\tau}}\right) \]  \hfill (6.19)

Where, \( \sigma = \sigma_0 \) at \( t=0 \).

Therefore, for the Kelvin model at \( t=0 \), the extensional relaxation function can be given in terms of the modulus behaviour as:

\[ E(t) = E_m + E'_m e^{-\frac{t}{\tau}} \]  \hfill (6.20)

This means that at \( t=0 \), the instantaneous behaviour is given by \( E = E_m + E'_m \). In the long term; \( t \to \infty \), \( E = E_m \). Clearly, it is sensible to set: \( E_m = E'_m = \frac{1}{2} E \), this being the modulus of elasticity of the material, as also stated by Naghieh (1999).

### 6.6- Contact mechanics of semi-infinite viscoelastic solids

The viscoelastic behaviour of solids in load bearing surfaces plays an important role in many engineering applications in a wide variety of industries. In the
bearing industry rubber lining is used for some bearings (Kalker (1991), and o-rings are made of rubber, as well as for various seals. In automotive industry bump stops and bounce bumpers in vehicle suspension, as well as weather strips as sealants for closure systems are made of viscoelastic materials. The same is true of many other industries, such as aerospace, and medical appliances industry.

Prediction of contact mechanics behaviour of elastomers and polymers is based upon three types of approach.

First, analytical solutions based upon asymptotic contact mechanical behaviour of such solids (Aleksandrov (1968)), with the inclusion of linear viscous model has been undertaken by various research workers. Most analytic solutions involve linear viscoelasticity and semi-infinite assumptions for contacting solids, this being an extension of classical Hertzian theory for the cases of solids of revolution indenting a semi-infinite linear viscoelastic half-space. They include such practical applications as a hemispherical solid impacting upon a rubber bump stop in vehicle closure systems such as a tailgate or the bonnet lid. Such asymptotic solutions based upon creep or stress relaxation have been provided by Johnson (1985), Lee and Radok (1960), Yang (1966), Naghieh (1999), Hunter (1960) and Pawluk (1997). Yang (1966) provides a derivation of the general formulae for the contact problem of linear viscoelastic bodies with integral stress-strain relations and arbitrary profiles. This section provides an outline of such a derivation, for the purpose of comparison with subsequent finite element analysis carried out in this thesis.

Second, finite difference or boundary integral solution of the same problems can be undertaken, when closed form analytical solutions are difficult to obtain such as in rolling viscoelastic contacts with friction, and some with multi-layered structures (Kalker (1991), Braat and Kalker (1993)).

Third, finite element analysis can be employed to provide numerical prediction for contact conditions. Although the finite element approach leads to an approximate solution to the problem, it often provides more accurate results and closer to
realistic prevailing conditions than the analytical approaches. This is the case, since analytical solutions, as can be seen later on embody assumptions that render such an approach as tractable, thus making them very limited to simple contact geometries.

### 6.7- The Finite Element Analysis

Finite element analysis is used to investigate the viscoelastic contact behaviour of the HFCP and then to compare this with the experimental results (as described in chapter 5). In this case the HFCP layer is modelled as a Maxwell model.

For small strain linear isotropic viscoelasticity problem one seeks the solution to the following basic hereditary integral (Johnson (1985), Naghieh (1999), and Yang (1966)):

\[
\sigma(t) = \int_0^t K(t' - \tau) \frac{\partial \varepsilon(\tau)}{\partial \tau} d\tau + \int_0^t G(t' - \tau) \frac{\partial \varepsilon(\tau)}{\partial \tau} d\tau \quad (6.21)
\]

where for an isothermal solution: \( t' = t \).

A combined solution for the deviatoric and volumetric stresses, the two terms in equation (6.21) respectively, is obtained by its discretisation into suitably small finite elements.

The instantaneous moduli, bulk \( K(\tau) \) and shear \( G(\tau) \) can be defined as Prony series:

\[
K(\tau) = K_0 \left\{ 1 - \sum_{i=1}^{n_0} \beta_i \left( 1 - e^{-\frac{(\tau - \tau_i)}{\eta_i}} \right) \right\} = K_0 \left\{ 1 - \sum_{i=1}^{n_1} \beta_i \left( 1 - e^{-\frac{\tau_i}{\eta_i}} \right) \right\} \quad (6.22a)
\]

and:

\[
G(\tau) = G_0 \left\{ 1 - \sum_{i=1}^{n_0} \beta_i \left( 1 - e^{-\frac{(\tau - \tau_i)}{\eta_i}} \right) \right\} = G_0 \left\{ 1 - \sum_{i=1}^{n_1} \beta_i \left( 1 - e^{-\frac{\tau_i}{\eta_i}} \right) \right\} \quad (6.22b)
\]
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Where $K_0$ and $G_0$ represent the instantaneous or glassy bulk and shear moduli.

It is assumed that bulk and shear relaxation time behaviour of the material is identical, which translates to: $\tau_i = \tau_i^K = \tau_i^G$.

Substituting equations (6.22a) and (6.22b) into equation (6.21) and integrating for a finite increment of time (i.e. $\frac{d\varepsilon}{dt} = \Delta\varepsilon$ and $\frac{d\varepsilon}{dt} = \Delta\varepsilon$), where the strain components are assumed to vary linearly with $\tau$, then:

$$
\sigma = G_0 \left\{ 1 - \sum_{j=1}^{n} \alpha_j \varepsilon_j \right\} + K_0 \left\{ 1 - \sum_{i=1}^{n} \beta_i \varepsilon_i \right\}
$$

(6.23)

where: $\alpha_j = 1 - \frac{G_j}{G_0}$ and $\beta_i = 1 - \frac{K_i}{K_0}$.

The changes in the shear and dilatation creep strains, $\Delta\varepsilon_j$ and $\Delta\xi_j$, are obtained at the beginning of each step of time by recursive expressions:

$$
\Delta\phi_i = \frac{\tau_i}{\Delta\tau} \left( \frac{\Delta\tau}{\tau_i} + e^{\frac{\Delta\tau}{\tau_i}} - 1 \right) \Delta\phi + \left( 1 - e^{\frac{\Delta\tau}{\tau_i}} \right) (\phi_i^n - \phi_i^\varepsilon)
$$

(6.24)

where:

$$
\phi_i^n = \left( 1 - e^{\frac{-\Delta\tau}{\tau_i}} \right) \int_0^{\frac{\Delta\tau}{\tau_i}} \frac{d\phi}{dt} dt + e^{\frac{-\Delta\tau}{\tau_i}} \int_0^{\left( \frac{\tau_i - \tau_i}{} \right)} \frac{d\phi}{dt} dt + \frac{\Delta\phi}{\Delta\tau} \int_{\tau_i}^{\left( \frac{\tau_i}{} \right)} (1 - e^{\frac{-\Delta\tau}{\tau_i}}) dt
$$

(6.25)

and: $\phi \in \varepsilon, \xi$.
Note that as the quantity $\frac{\Delta \tau}{\tau_i}$ diminishes in value, the change in the strain components, given by equation (6.24) can be approximated by equation (6.26):

$$\Delta \phi_i = \frac{\Delta \tau}{\tau_i} \left( \frac{1}{2} \Delta \phi + \phi'' - \phi_i'' \right)$$  \hspace{1cm} (6.26)

The small changes in the shear and dilatation creep strains are obtained from equations (6.24) or (6.26) and the corresponding deviatoric and volumetric stresses are subsequently obtained from the two terms in equation (6.23).

In finite element analysis a number of variables are needed. These are $G_0$, $\alpha_j$, $\tau$. The procedure to obtain these commences with the use of relations between the elastic constants of the material. These are:

$$G = \frac{3KE_m}{9K - E_m}$$  \hspace{1cm} (6.27)

$$\nu = \frac{3K - (E_m + E_m')}{6K}$$  \hspace{1cm} (6.28)

And note that: $E_m = E_m' = \frac{1}{2} E$.

Also that:

$$G_0 = \frac{3K(E_m + E_m')}{9K - (E_m + E_m')}$$  \hspace{1cm} (6.29)

Therefore, $\alpha_j = 1 - \frac{G}{G_0}$ can be obtained.
The shear relaxation time is obtained in terms of the extensional relaxation time, which is:

\[ \tau = \frac{9K - (E_m + E_m')}{9K - E_m} \tau' \]  

(6.30)

For the Kelvin model both relaxation times become the same.

### 6.8- The Finite Element Models

Finite element analysis is carried out for the viscoelastic contact mechanics analysis, of the bonded layered HFCP sensor. The finite element model is made in the commercial code ANSYS, which has a viscoelastic element, based on the Maxwell model, the theoretical basis for which is described above. To show the use of an alternative viscoelastic model, a three-element Kelvin model is used for the contact mechanical analysis of point and finite line contact configurations, and the results obtained in the case of the former is compared to that of Naghieh et al (1998 a, b). Therefore, the finite element model is created in PATRAN and the analysis is carried out in ABAQUS. The theoretical basis for this approach is also highlighted in the previous sections.

The results of these analyses have been compared with the tests highlighted in chapter 5. Therefore, in the case of HFCP sensor, the contact mechanics analysis is for a right circular punch, indenting a bonded layer of the HFCP with its flat end.

### 6.8.1- Model description for a flat punch, indenting a HFCP layer

The analysis is carried out in ANSYS, which employs the formulation highlighted previously for Maxwell viscoelastic continuum mechanics elements. To conduct the FEA the following steps are necessary:
Choose an appropriate element type: This will give the correct shape functions as well as the strain behaviour within the element.

A mesh is generated in ANSYS, using the selected element type. An appropriate number of elements should be employed. This is important in order to reduce the element width in the region of the pressure spike in order to obtain a realistic value, which is theoretically infinite according to Johnson (1985).

The procedure is applied for meshing both the indenter, as well as the HFCP layer. It should be noted that the contact elements have to be defined in order to describe the problem. Once the element types have been chosen, material properties are assigned to them such as $K$, $G$, and $\tau$.

The element types used are VISCO-88, an 8-noded viscoelastic Maxwell element for the layered solid and CONTAC-26 (as the contact element) which is a 2D point-to-surface constrained element. In general a large number of elements, up to 2000 elements were used in the regular mesh, packing up to 5 elements in the area of the edge pressure spike.

6.8.2- Model description for the circular point contact

For this application, the finite element code: PATRAN is employed. The following procedure is carried out:

- First the layered solid, in this case the HFCP is meshed by 2D quadrilateral continuum mechanics elements. The number of elements is then chosen. In this case the same mesh density as that previously described in chapter 3 for a layered elastic solid is chosen.
- Then it is necessary to select the material type, in this case viscoelastic. For viscoelastic material, it is necessary to undertake a time domain analysis due to stress relaxation. The inputs required for the analysis, which is based upon the standard linear solid
model are the relaxation time, the relative shear modulus (in other words shear relaxation modulus ratio) and the corresponding bulk relaxation modulus ratio. These are described in section 6.7. It should be noted that PATRAN considers the relaxation time in shear and bulk behaviour to be identical.

- The ball is considered to be a rigid body
- To effect contact, the concept of master (rigid) and slave (deforming body) is employed. The master is the curvature of the rigid body, whilst the slave is a 2D solid edge, representing the nodal points along the surface of the bonded layer
- The mesh is now generated by PATRAN and is transferred as an output file into the ABAQUS solver
- The ABAQUS undertakes the analysis, the results of which are transferred to PATRAN for post-processing.

The time dependent viscoelastic behaviour of materials are given by the Prony series, as represented by the bulk and shear moduli in equations (6.29a) and (6.29b). These can be re-arranged as ratios: \( \frac{G(t)}{G_o} \) and \( \frac{K(t)}{K_o} \), which are referred to above as shear and bulk relaxation ratio moduli. They indicate the transient variation or time dependency of the material behaviour in shear and bulk with respect to their long term characteristics. Thus:

\[
\frac{G(t)}{G_o} = 1 - \sum_{i=1}^{n} \alpha_i \left(1 - e^{\frac{t}{\tau_i}}\right), \quad \frac{K(t)}{K_o} = 1 - \sum_{j=1}^{n} \beta_j \left(1 - e^{\frac{t}{\tau_j}}\right)
\]

Therefore, values of \( \alpha_i \) and \( \beta_j \) are required for all steps \( i \) and \( j \) time constants to describe the viscoelastic model. For a single step model, \( i=j=1 \), with \( \tau = \tau_i = \tau_j \), one needs to specify values for \( \alpha, \beta, G, K, E \) and \( \nu \) for the transient viscoelastic analysis to be carried out.
6.9- The Numerical Results and Comparison with the Experiment

6.9.1- Circular punch indenting the HFCP bonded layer

The experimental pressure profiles at $t=0$ and $t=10\tau$ are represented in figures 6.7-6.10, along with their corresponding numerical predictions. The experimental results are shown by symbols, fitted by approximate dashed lines (see the key to the figure), whilst the FEA predictions are indicated by lines only. The regions indicated on the figure correspond to the width of sensing elements.

It can be observed that the central portion of the pressure distribution (along the $x$-axis) is almost the same for both the sets. The shape of the experimental pressure distribution appears to follow that of a flat indenter pressed into a semi-infinite solid. This is likely to have resulted owing to the presence of the silicone rubber. However, the pressure values correspond to layered pressure distribution, which is at considerably higher values as shown by Naghieh (1999) and also in chapter 3 for circular contacts of layered solids as opposed to semi-infinite ones. There is a divergence in the magnitude of the pressure spikes between the experimental and the numerical results, especially for $t=0$. The reason is that there are an insufficient number of computational nodes in the FE model to pick up the actual pressure exhibited between the edges of the indenter and the polymer. Also the experimental value of the pressure is finite, in contrast with the theoretical which suggests an infinite value, due to the width of the sensing site directly beneath the pressure spike.
Another deviation for numerical solution from the experimental results is in the shape of the pressure distribution. The latter appears to follow variation which indicates a semi-infinite response but at the higher pressures which correspond to layered solids. This could have been caused by the pressure of the silicone rubber layer. Note also that the experimental results agree better with the numerical predictions at higher contact loads. This is expected as a more coherent contact is achieved experimentally at higher contact loads. At lower loads contact coherence may not be achieved well, particularly as the indenter is not perfectly flat, nor indeed the silicone rubber surface flatness cannot be assured.

Overall, the agreement between observations and the numerical results is quite good to provide sufficient confidence in the finite element approach.
Figure 6.8: Pressure distribution for a circular flat punch indenting a thin layer of HFCP, contact load 3.92 N (viscoelastic relaxation)

Figure 6.9: Pressure distribution for a circular flat punch indenting a thin layer of HFCP, contact load 5.80 N (viscoelastic relaxation)
6.9.2 - Circular contact of a rigid ball indenting the HFCP bonded layer

It would have been interesting to obtain experimental verification for the case of circular point contact for layered viscoelastic solids. However, due to the inadequate sensor spatial resolution this is not possible to achieve. Thus, finite element analysis is undertaken to obtain the deformation of a thin 0.11 mm thick layer of HFCP, when indented by a rigid ball of radius 12.7 mm under 1.3N, as has been reported by Naghieh et al (1998a), using an analytic solution for thin bonded layered viscoelastic solids. This simulation is carried out to compare the FEA with asymptotic analytic solutions.

Figure 6.11 shows the results obtained by Naghieh et al (1998a). Their analytical formulation is based upon a three element standard linear solid viscoelastic behaviour. Note that at $t=0$ the response of the layered solid corresponds to its elastic behaviour. At $t>0$, viscoelastic stress relaxation occurs gradually as
fractions of the relaxation time as shown in the figure, until $t = 10\tau_2$ after which no more change in the pressure distribution is observed and the layered solid has nearly asymptotically reached its long term shear behaviour.

The corresponding results obtained through finite element analysis are shown in figure 6.12. The figure consists of various time-shot responses. Note that with relaxation the contact area increases in size, whilst correspondingly the pressures are reduced. Since the pressure distribution is axi-symmetric, only half the distribution is shown in each case.

Good agreement is observed between the results obtained through FEA and those obtained analytically by Naghieh et al (1998a). The asymptotic growth of the contact area and the corresponding relaxation in the maximum contact pressure are shown in figures 6.13 and 6.14 respectively.

Figure 6.11: Viscoelastic stress relaxation behaviour of HFCP thin layer under load (after Naghieh et al (1998a))
Figure 6.12: Viscoelastic stress relaxation behaviour of HFCP thin layer under load predicted by FEA
Note that the nature of stress relaxation is quite similar to that of the reported experimental work for a circular punch indenting the HFCP sandwich sensor, although clearly the contact conditions are quite different. It should also be observed that after a very short period around few multiples of the relaxation time of the HFCP
(at 0.892 s) the pressure value has become asymptotic to its long term value, estimated here to be around 18 MPa.

Figure 6.15: Central axial pressure distribution for a roller indenting a thin 0.11 mm HFCP layer

Finite line contact of a roller against a thin-layered viscoelastic solid has not been reported in literature. Yet study of such contacts is very important with many applications in industry. The contact of a roller of 12.7 mm radius and 25.4 mm length against a thin 0.11 mm HFCP layer bonded to a Steel substrate is considered here, subjected to a contact load of 1.3 N. Figure 6.15 illustrates the relaxation of the axial pressure distribution from its initial instantaneous elastic
response to its long term variation, as $t \to \infty$. However, most of the adaptation nearly occurs after a few multiples of the relaxation time. Figure 6.15 shows the axial pressure distribution for the finite line contact conditions described above.

An interesting feature of the contact pressures is their lower value to those in circular point contact reported above, due to a much larger contact area. Even the edge pressure spikes do not exceed 2.5 MPa.

The sub-surface stresses are represented in figure 6.16 by the maximum shear stress contours. These indicate gradual relaxation of stress levels beneath the surface (stress levels increase from blue colour to red in the figure). Also noteworthy is the confinement of high stresses in the layer, with lower stresses passed onto the substrate. This characteristic is not common to the soft layers, and occurs here due to rapid relaxation of stresses. The discontinuity in the stress pattern indicates the boundary between the HFCP layer and the Steel substrate.

Finally, it should be noted that the HFCP yield stress is in the range 20-50 MPa, thus the material survives the reported conditions according to the Tresca yield criterion.
Figure 6.16: Maximum shear stress variation with viscoelastic relaxation
Chapter 7

Overall Conclusions and suggestions for future work

7.1- Overall conclusions and main findings

There are a number of main findings resulting from the reported research undertakings in this thesis. The underlying finding reiterates those already stated by a number of authors (Conway et al (1966), Johnson (1985), Jaffar (1989), Naghieh et al (1998 a,b)) that generated contact pressures in layered solids differ considerably from those obtained analytically or numerically based upon semi-infinite assumptions. It is found that when bonded solids of high elastic moduli are employed as, for example, protective coatings for substrates of lower elastic moduli, the generated pressures in the thin bonded layer exceeds far the magnitude of those that would be expected if the same substrate was not coated. This finding is in accord with general practice in industry, apparently arrived at through empirical means, employing hard coatings to protect the substrate material in a variety of applications. The analyses highlighted in this thesis indicate that the hard coatings, by reducing the area of contact cause localised concentrated stress fields that do not extend into the substrate material. Their high yield stress values ensures their survival under severe conditions that would cause yielding of unprotected semi-infinite elastic substrates. Many applications enjoy these protective characteristics of hard coatings, which have been mentioned in this thesis, including hard wearing coatings for cutting tools, dies or punches, as well as those applied to other load bearing surfaces such as cam-follower pairs in high
output power engines, such as the racing cars, or in bearings in high speeds, high load applications.

Another finding of the research is concerned with the already acknowledged deviations from the classical Hertzian theory for non-conforming contact of solids of revolution, where the semi-infinite contact conditions cannot be assumed, even for the contact of uncoated or unbonded materials, such as the cases of roller to raceway or cam to follower contacts. The resulting concentrated non-conforming contact in these cases can be approximated by long elliptical contacts or infinite line contact geometry, based on an extension of the classical Hertzian theory. These approximations have been found to result in misleading predicted pressure distributions, which do not take into account the edge stress concentrations, for example, for the case of a unblended roller indenting a semi-infinite elastic half-space. This class of contact mechanical problems is referred to as finite line contacts. The finite element solutions provided in this thesis arrive at the same findings, in this case, to those obtained by a number of authors in the past, such as Kannel (1974), Johns (1978), Rahnejat (1978), Mostofi and Gohar (1983) and Kushwaha (2000).

Due to the generic nature of the devised finite element models many combinations of substrates with bonded layers/coatings were studied, some of which have been reported in the thesis. They also include the use of soft layers, often used as coatings or anti-vibration (isolating) layers bonded to substrate materials, as in weatherstrips and rubber bump stop in closure systems of vehicles, such as in doors and tailgate systems. Analytical and finite element analyses carried out in the thesis agrees well with the findings of others such as Matthewson (1981). It has been shown that the soft layer coatings spread the area of contact, but being of low elastic modulus and low shear strength can breakdown under the generated pressures. The analysis, therefore, indicates that they are suitable to be used as solid lubricants, breaking down and reforming, although to verify this supposition to its conclusion one needs to undertake elastoplastic finite element analyses.
When soft bonded layers are used as vibration isolation or in some case as protective layers, viscoelastic properties of these layers are desired. Here, there exists a dearth of literature, particularly with regard to thin bonded compressible layers, aside from some major contributions in the case of circular point contact geometry by Naghieh (1999), Naghieh et al (1997, 1998 a,b). For semi-infinite conditions more literature is available, such as those reported by Braat and Kalker (1993), Johnson (1985) and Kalker (1991). Agreement has been found between the findings of this thesis and those of Naghieh (1999) in the cases investigated, as well as with some devised experimental work, reported within the body of this thesis. The main finding is in agreement with all the reported literature, in that viscoelastic solids respond elastically under instantaneous loading condition, which is then subjected to relaxation, increasing the contact area, whilst the generated pressures are reduced. This trend continues until adaptation occurs at some multiple of the relaxation time of the bonded viscoelastic solid, prior to which the edges of the contact pull in the layered solid in an elastic manner, whilst the region of high pressures is subject to relaxation. The adapted pressure distribution then tends to a shape reminiscent of a finite line contact, although the mechanism is totally different. The analysis is then extended to the case of a circular punch contact, where very good agreement is obtained with the experimental work.

The investigations of non-conforming contacts is then extended to the case of impact dynamics, resulting in much more severe conditions that are prevalent in many practical mechanisms, including ball or roller collision with inner and outer raceways in unloaded regions of bearings, a phenomenon referred to as forced brinelling effect, or valve toss in valve train systems, causing separation of cam to follower contact and subsequent rebound, as a severe impact condition. Contact problem, indeed, may be regarded as a special case of impact, where the impact duration tends to infinity and the force of impact diminishes, such as a ball resting on a semi-infinite elastic half-space, with a typical weight of tenths of Newton, as oppose to an impact force of few hundred Newtons if it was to fall under effect of gravity from a very small height of several millimetres. Analytical expressions are derived, commencing from the case of Hertzian impact (Hertz (1896), see also...
Timoshenko and Goodier (1951) and Johnson (1985) to those for a layered solid promoting circular point contact geometry and that for a finite line contact configuration of impact of a roller upon a semi-infinite elastic half-space. The Hertzian assumptions have been kept, giving in effect the impact time above the frequency of wave propagation inside the impacting solids. The results of circular point contact impact analysis using FEA have been compared with the Hertzian theory and experimental work of Safa and Gohar (1986) (this one under lubricated conditions), and those of AlSamieh and Rahnejat (2002). Good agreement is found, which is also extended to layered solids work for hard coating, and further to those of a roller impacting upon various semi-infinite solids. The protective nature of hard coatings has been shown by the investigation of sub-surface stress fields. The impact time increase as one tends to semi-infinite substrates of lower elastic moduli or low elastic modulus bonded layers. The impact time also decreases as the contact geometry deviates from one of circular point configuration to one of finite line. In all cases the impact time is quite small when any appreciable drop height (several millimetres and above) is used. A repercussion of this fact is that for soft viscoelastic layers the relaxation time of the material is much longer than the impact time (typically by two to three orders of magnitude). This indicates that the response of soft layers under impacting conditions can be fairly accurately approximated by their elastic response. Appreciable deviations will occur when the impact time tends to tenths of a second, or in other words the impact velocity is quite low, or the extent of deformation is quite high. The former case relates to an impact with diminished kinetic energy; this being the limiting case of impact, i.e. a contact. In the latter case, decreased penetration points to either a high elastic modulus substrate, which has almost an instantaneous relaxation time, or a significant increase in the area of contact (typically of three orders of magnitude at the same approach velocity). This would be similar to impact of very large light body like a balloon or a sport ball.
7.2- Contributions to knowledge

Contact mechanics is an established pillar of science. All contributions made in this field are of an evolutionary and incremental nature. The foundations of this field of investigation were laid in late 18th century by Hertz (1881) and Boussinesq (1885). There have been many incremental advances since the aforementioned pioneering works. In the case of impact dynamics the main principles have been laid by Newton (1687), Poisson (1842) and St Venant (1883). Thus, 1880s witnessed the greatest strides in the understanding of the elastic behaviour of solids under contact/impact loads.

This thesis is another incremental and consolidating step in the understanding of contact/impact of counterformal solids of revolution with elastic or viscoelastic characteristics.

The specific areas of contribution have been:

- Development of analytical techniques for determination of contact pressure distribution and penetration of layered elastic bonded solids, for a variety of contact configurations and also in-line with the works reported by Naghieh (1999), Jafar (1989) and Barber (1990). Furthermore, investigation of the behaviour of soft and hard coatings using the developed analytical expressions for practical applications and for various coatings to substrate material modulus of elasticity ratios, similar to the works reported by O’Sullivan and King (1988), Peng and Bhushan (2001).

- Development of 2D and 3D generic finite element techniques for the study of pressure distribution, footprint conditions and sub-surface stress field of a variety of non-conforming contacts of semi-infinite elastic half-spaces and layered bonded solids and coatings of low to high elastic moduli and validation of the same against a combination of analytical models and experimental results, including those of Hertz.

- Development of analytical methods for the evaluation of impact mechanics for layered bonded solids and semi-infinite roller to frictionless flat plate, not hitherto reported in literature, including estimation of impact pressure and footprint half-width and maximum central deflection, and the corresponding impact time.

- Development of a dynamic finite element technique with high mesh density for the resolution of pressure spike at edge stress discontinuity in roller to flat plate impact conditions, and the evaluation of the corresponding transient sub-surface stress field, and providing physical and mathematical proof of the protective nature of hard thin bonded elastic layered solids. Impact of rollers upon elastic solids has not been carried out before, using combined finite element and analytical techniques with very good agreement.

- Design and fabrication of a miniature viscoelastic multi-layered sandwich sensor and a corresponding viscoelastic transient contact mechanical finite element model, obtaining experimental and numerical predictions that conform closely to each other, and to the results of Naghieh (1999) and Naghieh et al (1997, 1998 a,b) for the case of layered thin viscoelastic solids in circular point contact geometry.

7.3- Critical assessment of the approach undertaken and suggestions for future work

A deliberate attempt has been made to obtain analytic and numerical solutions in this thesis that reflect practical applications of contact mechanics theory. However, both analytic and finite element models deviate from reality in a number of ways, and depending on the intended application.
The analytic models for semi-infinite elastic half-spaces are clearly based on the Hertzian theory, and as such embody its underlying assumptions, described in chapter 2. The main drawbacks of these assumptions are negligence of frictional properties in practical contact/impact problems, which affect the generated pressures and the footprint conditions; contact area and deformation of the bodies in contact. The same problem also exists in the case of layered elastic solids results in the thesis, which is particularly important, as the interface between the bonded layer and the elastic half-space is assumed to be devoid of internal stresses and not of any intermediate elastic properties, which is usually the case as diffusion of coating material takes place into a thin layer of the substrate surface. It is difficult to quantify the boundary conditions through analytical means and some prior experimentation is required to ascertain these interfacial boundary conditions and the emergent elastic moduli.

With hard coatings, high contact stresses and presence of friction cause additional thermal stresses in the layered solid. Due to the usual bad thermal conductance of these coatings, the increasing thermal stresses cause fracture of the hard coating. Isothermal solutions, such as those provided in this thesis, therefore, do not provide a realistic hypothesis for the failure of high elastic moduli hard coatings. Thermal mis-match between hard coated surfaces and the substrate material can also cause fracture of the coating. This problem can be investigated through micro-level investigation of interfacial contact mechanics, where the effect of adhesion forces between the coating and the thin layer must be included in the analysis.

In the case of soft coatings, isothermal solutions, based on linear elasticity theory of isotropic materials can indicate the onset of shearing deformation and yielding, but do not verify its actual occurrence. Therefore, an elastoplastic analysis is more appropriate.

Under impacting conditions, the high sub-surface stresses, indicated by the equivalent stress point to commencement of yielding, but due to the short duration of impact may be regarded as not catastrophic. However, many impact problems
are due to cyclic or repetitive loading and continual loading and off-loading of a
body in contact can lead to failure, the mechanism for which is likely to be
reversing orthogonal shear stresses very near to the contact/impact surface, as
indicated by Johns-Rahnejat (1988).

In the case of impacting solids, the analytical expressions make two major
assumptions. Firstly, the impacting solid is either assumed to be rigid or its local
deformation below the wave propagation limit. This means that the centre of the
indenter remains in the same location, and that the applied impact force remains
normal to the surface of the substrate, with no additional moment loading that can
cause spin or sliding of the surfaces. Secondly, the rate of deceleration of the
indenter through the deformation of the substrate is almost linear, resulting in a
parabolic transient distribution for the penetration. This assumption is nearly true
for the elastic impact, as borne by the FEA results, but should be considered as a
first reasonable assumption.

Finally, the solutions provided in this thesis are applicable to smooth load bearing
surfaces, where the apparent area of contact is the same as the true area. This,
however, is not the case in practice, where rough surfaces present a true area of
contact which is usually a percentage of its apparent idealised value. Roughness of
surfaces can be taken into account in the form of harmonic functions within FEA
analysis.

The above discussions are representative of some of the shortcomings of the
current approach when realistic representation of prevailing conditions is taken
into account, and form the basis for suggestions for future extension of the
research undertaken within this thesis.
References


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REFERENCES


REFERENCES


Microlink 1990, *Hardware user manual*, Biodata Limited, USA.


REFERENCES


REFERENCES


Pressure vs. Elements

- Maximum contact pressure versus number of elements used in the FEA models at 200N. The model becomes stable after 3000-4000 elements. Higher mesh density renders better results in contact area and subsurface stresses.
# Non-Corrosive Silicone Sealant

<table>
<thead>
<tr>
<th>Property</th>
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<tr>
<td>Colour</td>
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<tr>
<td>Relative Density</td>
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<tr>
<td>Hardness</td>
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<td>Shore</td>
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<tr>
<td>Tensile strength</td>
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<td>MPa</td>
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<tr>
<td>Elongation</td>
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<td>%</td>
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<td>Dielectric Constant</td>
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<td>At 1MHz</td>
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<tr>
<td>Dissipation Factor</td>
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<td>At 1MHz</td>
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<tr>
<td>Volume Resistivity</td>
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<td>Ω cm</td>
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<tr>
<td>Thermal Conductivity</td>
<td>0.33</td>
<td>W/m K</td>
</tr>
<tr>
<td>Cure Time (2mm thickness)</td>
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<td>Hours</td>
</tr>
<tr>
<td></td>
<td>(25°C at 50% R &amp; H)</td>
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- **Flowable Fluid Silicon Rubber Compound**

<table>
<thead>
<tr>
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<td>Dielectric Constant</td>
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<td>Dissipation Factor</td>
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<tr>
<td>Cure Time (2mm thickness)</td>
<td>24</td>
<td>Hours</td>
</tr>
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Appendix III

**Output Voltage Tables**

When no load is applied the output voltage is 10V.

Table 1:  Output voltage for normal load of 0.98 N

<table>
<thead>
<tr>
<th>Relaxation Time (nt)</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tr>
<td>$\tau$</td>
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<td>9.35</td>
<td>8.975</td>
<td>8.995</td>
<td>8.981</td>
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<td>$2\tau$</td>
<td>9.881</td>
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<td>8.768</td>
<td>8.443</td>
<td>8.415</td>
<td>8.435</td>
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<tr>
<td>$4\tau$</td>
<td>9.46</td>
<td>8.158</td>
<td>8.1</td>
<td>8.108</td>
<td>8.11</td>
</tr>
<tr>
<td>$5\tau$</td>
<td>9.105</td>
<td>7.785</td>
<td>7.763</td>
<td>7.705</td>
<td>7.743</td>
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</table>

Table 2:  Output voltage for normal load of 1.96 N

<table>
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<tr>
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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>9.73</td>
<td>9.15</td>
<td>8.705</td>
<td>8.72</td>
<td>8.708</td>
</tr>
<tr>
<td>$2\tau$</td>
<td>9.375</td>
<td>8.933</td>
<td>8.48</td>
<td>8.46</td>
<td>8.477</td>
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<td>$3\tau$</td>
<td>9.03</td>
<td>8.76</td>
<td>7.958</td>
<td>7.953</td>
<td>7.945</td>
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<tr>
<td>$4\tau$</td>
<td>8.832</td>
<td>8.275</td>
<td>7.602</td>
<td>7.512</td>
<td>7.499</td>
</tr>
<tr>
<td>$5\tau$</td>
<td>8.5</td>
<td>7.953</td>
<td>7.188</td>
<td>6.98</td>
<td>6.948</td>
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</table>
Table 3: Output voltage for normal load of 3.92 N

<table>
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<tr>
<th>Sensors (volts)</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tr>
<td>Relaxation Time (nt)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\tau)</td>
<td>9.305</td>
<td>8.715</td>
<td>8.293</td>
<td>8.238</td>
<td>8.213</td>
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<tr>
<td>(2\tau)</td>
<td>9.02</td>
<td>8.325</td>
<td>7.73</td>
<td>7.925</td>
<td>7.699</td>
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<td>(3\tau)</td>
<td>8.34</td>
<td>7.973</td>
<td>6.987</td>
<td>7.24</td>
<td>7.03</td>
</tr>
<tr>
<td>(4\tau)</td>
<td>7.995</td>
<td>7.52</td>
<td>6.693</td>
<td>6.78</td>
<td>6.75</td>
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<tr>
<td>(5\tau)</td>
<td>7.15</td>
<td>7.075</td>
<td>6.04</td>
<td>6.17</td>
<td>5.89</td>
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Table 4: Output voltage for normal load of 5.8 N

<table>
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<tr>
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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relaxation Time (nt)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\tau)</td>
<td>8.65</td>
<td>8.02</td>
<td>7.362</td>
<td>7.32</td>
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<td>(2\tau)</td>
<td>8.175</td>
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<td>(3\tau)</td>
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<td>5.908</td>
<td>6.313</td>
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<tr>
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<td>6.425</td>
<td>5.728</td>
<td>5.915</td>
<td>5.78</td>
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<tr>
<td>(5\tau)</td>
<td>6.22</td>
<td>6.13</td>
<td>5.175</td>
<td>5.288</td>
<td>5.148</td>
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Table 5: Output voltage for normal load of 8.83 N

<table>
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<th>Sensors (volts)</th>
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<td>(\tau)</td>
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<td>6.7</td>
<td>6.865</td>
<td>6.725</td>
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<td>7.35</td>
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<td>6.475</td>
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<tr>
<td>(3\tau)</td>
<td>6.408</td>
<td>6.288</td>
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<td>(4\tau)</td>
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<td>5.55</td>
<td>5.295</td>
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