Particle interaction in dilute slowly sedimenting systems

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PARTICLE INTERACTION IN
DILUTE SLOWLY SEDIMENTING SYSTEMS

by

STEPHEN KENNETH COWLAM, M.Sc., D.I.S.

A Doctoral Thesis submitted in partial
fulfilment of the requirements
for the award of

Doctor of Philosophy
of the
Loughborough University of Technology

May 1976

Supervisor: B. Scarlett, M.Sc.
Department of Chemical Engineering

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For Liz
Acknowledgements

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Mr. B. Scarlett, supervisor, for his advice
Members of the Department of Chemical Engineering, Loughborough University of Technology
Friends and colleagues for lively discussion and constructive criticism
My parents for their continued encouragement
And finally, Liz, for her patience and typing skill.
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<td>a</td>
<td>particle radius</td>
<td>L</td>
</tr>
<tr>
<td>e₀</td>
<td>distance between two sphere surfaces almost in contact</td>
<td>L</td>
</tr>
<tr>
<td>e⁽¹⁾, e⁽²⁾, e⁽³⁾</td>
<td>vectors determining unit cell of a particle array</td>
<td>L</td>
</tr>
<tr>
<td>A</td>
<td>vector area</td>
<td>L²</td>
</tr>
<tr>
<td>b</td>
<td>sphere radius (in two sphere system); distance from particle centre to vessel axis in Poiseuille flow</td>
<td>L</td>
</tr>
<tr>
<td>h⁽¹⁾, h⁽²⁾, h⁽³⁾</td>
<td>reciprocal lattice vectors (see e⁽¹⁾, e⁽²⁾, e⁽³⁾)</td>
<td>L⁻¹</td>
</tr>
<tr>
<td>c</td>
<td>volume concentration of solids</td>
<td>-</td>
</tr>
<tr>
<td>C_D</td>
<td>drag coefficient</td>
<td>-</td>
</tr>
<tr>
<td>C_f</td>
<td>frictional drag coefficient</td>
<td>-</td>
</tr>
<tr>
<td>C_m</td>
<td>mean volume concentration of solids</td>
<td>-</td>
</tr>
<tr>
<td>C_r</td>
<td>pressure drag coefficient</td>
<td>-</td>
</tr>
<tr>
<td>G</td>
<td>coupling tensor</td>
<td>L³</td>
</tr>
<tr>
<td>d</td>
<td>particle diameter</td>
<td>L</td>
</tr>
<tr>
<td>d̄</td>
<td>mean diameter of particle</td>
<td>L</td>
</tr>
<tr>
<td>D</td>
<td>vessel diameter drag force</td>
<td>L, MLT⁻²</td>
</tr>
<tr>
<td>D²</td>
<td>coupling tensor</td>
<td>L²</td>
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<tr>
<td>E</td>
<td>rate of strain tensor</td>
<td>T⁻¹</td>
</tr>
<tr>
<td>F</td>
<td>frictional force resisting particle motion</td>
<td>MLT⁻²</td>
</tr>
<tr>
<td>F_h</td>
<td>hydrodynamic force vector</td>
<td>MLT⁻²</td>
</tr>
<tr>
<td>g</td>
<td>acceleration due to gravity</td>
<td>LT⁻²</td>
</tr>
<tr>
<td>h</td>
<td>half distance between two spheres</td>
<td>L</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
<td>Unit</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------------------------</td>
<td>-------------</td>
</tr>
<tr>
<td>( H )</td>
<td>force/unit mass vector acting on particle</td>
<td>( LT^{-2} )</td>
</tr>
<tr>
<td>( \mathbf{i}, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3 )</td>
<td>unit vectors in coordinate system (0123)</td>
<td>-</td>
</tr>
<tr>
<td>( \mathbf{T} )</td>
<td>idemfactor</td>
<td>-</td>
</tr>
<tr>
<td>( k )</td>
<td>constant; average number of particles/cell</td>
<td>-</td>
</tr>
<tr>
<td>( K_{HR} )</td>
<td>Hadamard-Rybczinski correction factor</td>
<td>-</td>
</tr>
<tr>
<td>( K_{1}, K_{2} )</td>
<td>effective volume concentration of two solid species</td>
<td>( L^3 )</td>
</tr>
<tr>
<td>( \mathbf{K} )</td>
<td>resistance tensor</td>
<td>( L )</td>
</tr>
<tr>
<td>( l )</td>
<td>distance between particle centres</td>
<td>( L )</td>
</tr>
<tr>
<td>( \mathbf{L} )</td>
<td>torque vector</td>
<td>( ML^2T^{-2} )</td>
</tr>
<tr>
<td>( m )</td>
<td>particle mass</td>
<td>( M )</td>
</tr>
<tr>
<td>( m' )</td>
<td>fluid displaced particle mass</td>
<td>( M )</td>
</tr>
<tr>
<td>( n )</td>
<td>number of spheres contained in vessel at volume concentration, ( c )</td>
<td>-</td>
</tr>
<tr>
<td>( \mathbf{n} )</td>
<td>unit normal vector</td>
<td>-</td>
</tr>
<tr>
<td>( N, N_{\alpha} )</td>
<td>cell number, number of particles in volumes, ( V ) and ( A ) respectively</td>
<td>-</td>
</tr>
<tr>
<td>( p )</td>
<td>pressure field</td>
<td>( ML^2T^{-2} )</td>
</tr>
<tr>
<td>( p_k(\mathbf{x}) )</td>
<td>Poisson distribution of ( k ) particles</td>
<td>-</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>pressure drop</td>
<td>( ML^2T^{-2} )</td>
</tr>
<tr>
<td>( P(C_{\alpha}</td>
<td>x) )</td>
<td>conditional probability that a configuration of ( N ) particle centres are found in the range, ( dC_{\alpha} )</td>
</tr>
<tr>
<td>( P_n(t), P'_n(z) )</td>
<td>Legendre and first associated Legendre polynomials</td>
<td>-</td>
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<tr>
<td>( Q )</td>
<td>volumetric flow rate</td>
<td>( L^3T^{-1} )</td>
</tr>
<tr>
<td>( \mathbf{Q} )</td>
<td>logarithmic distribution function of velocities</td>
<td>-</td>
</tr>
<tr>
<td>( \mathbf{Z} )</td>
<td>position of sphere centre relative to ( \mathbf{Z}_0 )</td>
<td>( L )</td>
</tr>
<tr>
<td>( \mathbf{R} )</td>
<td>vessel radius</td>
<td>( L )</td>
</tr>
<tr>
<td>( \text{Re} )</td>
<td>Reynolds number</td>
<td>-</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td>Dimension</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>-----------</td>
</tr>
<tr>
<td>s</td>
<td>closed surface</td>
<td>L²</td>
</tr>
<tr>
<td>s²</td>
<td>logarithmic variance</td>
<td>1</td>
</tr>
<tr>
<td>s_p</td>
<td>particle surface</td>
<td>L²</td>
</tr>
<tr>
<td>s₁,s₂</td>
<td>fluid-solid drag for mass of two solid species</td>
<td>MLT⁻²</td>
</tr>
<tr>
<td>s</td>
<td>directed surface area vector</td>
<td>L²</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
<td>T</td>
</tr>
<tr>
<td>T_w</td>
<td>wall shear stress (boundary layer theory)</td>
<td>ML⁻¹T⁻²</td>
</tr>
<tr>
<td>t</td>
<td>torque vector</td>
<td>ML⁻²T⁻²</td>
</tr>
<tr>
<td>u</td>
<td>particle velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>U</td>
<td>Johne and Koglin average particle velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>Uₐₐ</td>
<td>arithmetic mean particle velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>Uₐₐₜ</td>
<td>arithmetic mean velocity measured with time as the variable</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>Uₐₐₜ</td>
<td>arithmetic mean velocity measured with time as the variable where T &gt; t</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>U_BH</td>
<td>Brenner and Happel wall corrected velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>U_c</td>
<td>average Kaye and Boardman particle velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>U_f</td>
<td>average fluid velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>Uₓₑ</td>
<td>geometric mean particle velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>Uₓₕ</td>
<td>harmonic mean particle velocity</td>
<td>LT⁻¹</td>
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<tr>
<td>Uₓₑ</td>
<td>Johnse and Koglin average particle velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>Uₓₑₑ</td>
<td>Kaye and Boardman average particle velocity</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>U_k</td>
<td>relative velocity of a cell of particles</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>U_L</td>
<td>Ladenburg's corrected velocity for wall and bottom effect</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>U_MF</td>
<td>mean fluid velocity in a parabolic flow profile</td>
<td>LT⁻¹</td>
</tr>
</tbody>
</table>
\( U_{of} \) fluid velocity at vessel axis in a parabolic flow profile \( \text{LT}^{-1} \)

\( \bar{U}_p \) mean particle velocity \( \text{LT}^{-1} \)

\( U_r \) velocity relative to Stokes' velocity for median size particle, \( \bar{U}_s \) \( \bar{U}_s \) Stokes' velocity \( \text{LT}^{-1} \)

\( U_0 \) instantaneous velocity vector \( \text{LT}^{-1} \)

\( U(x_o, C_N) \) instantaneous velocity vector of particle with centre at \( x_o \) and in set \( C_N \) \( \text{LT}^{-1} \)

\( \bar{U}_s \) mean particle velocity vector \( \text{LT}^{-1} \)

\( U_1, U_2 \) Koglin velocities in upper and lower test sections of sedimentation vessel \( \text{LT}^{-1} \)

\( \mathbf{V} \) velocity vector field \( \text{LT}^{-1} \)

\( \mathbf{V}(x, C_N) \) velocity at a point, \( x \), in the set of spheres, \( C_N \), supposing particle at \( x \) to be replaced by fluid \( \text{LT} \)

\( V \) volume fluid return flow velocity \( \text{LT}^3 \)

\( V_k \) volume of cell of \( k \) particles \( L^3 \)

\( V(x, C_N) \) velocity contribution to sphere at \( x_o \) by all other spheres in set \( C_N \) at positions \( x \) \( \text{LT}^{-1} \)

\( W(n_A) \) probability that \( n_A \) particles are found within a volume, \( A \) \( - \)

\( W(x, C_N) \) image systems of influence of all spheres in set \( C_N \) at positions \( x \) on a sphere at \( x_o \) \( \text{LT}^{-1} \)

\( x \) particle diameter \( L \)

\( x_k \) diameter of cell containing \( k \) particles \( L \)

\( x_o \) vector position of particle centre \( L \)

\( x_1, x_2, x_3 \) axial directions in \( (0x_1x_2x_3) \) coordinate system \( L \)

\( \mathbf{z} \) position vector \( L \)
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<tr>
<td>$\beta$</td>
<td>Brenner and Happel ratio of particle distance to vessel axis to cylinder radius</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Euler's constant</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Rotational velocity of a particle</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Boundary layer thickness</td>
</tr>
<tr>
<td>$\delta(x)$</td>
<td>Dirac's delta function of $x$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Porosity, or voidage</td>
</tr>
<tr>
<td>$\varepsilon_{ijk}$</td>
<td>Permutation symbol</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Vorticity</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Turbulence intensity</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Wavelength of light, mean free path of particles</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Refractive index, fluid viscosity</td>
</tr>
<tr>
<td>$\mu_k$</td>
<td>Viscosity of cell containing $k$ particles</td>
</tr>
<tr>
<td>$\mu_o$</td>
<td>Viscosity of pure liquid</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Relative viscosity, $(\frac{\mu}{\rho_s})$</td>
</tr>
<tr>
<td>$\nu_k$</td>
<td>Kinematic viscosity</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>Stress tensor</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Fluid density</td>
</tr>
<tr>
<td>$\rho_k$</td>
<td>Density of cell containing $k$ particles</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>Solid density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Spherical fluid surface standard deviation</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Stream function</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angular velocity</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Rotation tensor</td>
</tr>
</tbody>
</table>
1. Introduction

Sedimentation is a phenomenon well characterized in nature by the fallout of solids and liquids from the atmosphere, and the settling out of sediment in the oceans and inland waterways. It has been used on an industrial scale in many applications notably in the separation of solids and liquids in gravity settling tanks and in centrifugal devices for the rapid dewatering of slurries. In the laboratory it is adapted to techniques for characterizing fine particles.

Observation of the phenomenon may be complicated by electrostatic or magnetic fields, thermal gradients, eddy currents, complicated rheological properties of the sedimenting fluid or irregular particle shape and density, and particle concentration.

The effects of gravity on a sedimenting system may be investigated by eliminating other force fields and operating at constant temperature in a closed vessel. A Newtonian fluid yields the simplest fluid mechanics, as does the use of isotropic spherical particles.

This thesis is, therefore, concerned with just such a system. Earliest investigations into such a two phase system were concerned with either very low concentrations or single particles where mathematical analysis was possible yielding a solution of the equations of motion typified by Stokes' law and its modifications for different conditions, or high particle concentrations in the hindered settling region typified by observation of bulk properties and analysis from continuity considerations.
There exists a region between these two at concentrations roughly between 0.05% and 2% by volume of solids where it was observed that the average particle velocity did not decline steadily from the Stokes velocity with increasing concentration, but was in fact enhanced to a maximum value at about 1% by volume of solids, thereafter declining with further increases in concentration until the hindered settling region was entered. Such results were originally obtained by Kaye and Boardman\(^{57}\) and confirmed later by Johne\(^{54,55}\) and Koglin\(^{58,59}\). This thesis is concerned with gravity sedimentation within this concentration range.

The next section reviews the literature pertaining to previous work done on this subject covering many aspects such as concentration effects, modelling as flow past arrays, and exact solutions for two spheres. It presents the findings and interpretations due to each particular author and concludes by outlining the relationship of the literature to this work. These concluding remarks indicate the reasons for setting up experimental investigations in the manner described in the experimental section. Previous workers have measured vertical velocities of particles by measuring the time taken to pass between fixed points, but this work was carried out using time lapse photography enabling large numbers of results to be obtained from any region in the experimental vessel, demanding the use of semi-automatic data handling techniques. Both vertical and horizontal particle translations were measured. A laser Doppler
anemometer was used to obtain data for the interstitial fluid motion.

The discussion analyses the results of the experiments and presents some new facts relating to the reliability and interpretation of the results of sedimentation tests both on laboratory and industrial scales. Also given is a qualitative description of the phenomenon of sedimentation at these concentrations obtained from the analysis of experimental results, and thereafter a more quantitative description based on a force balance over a small volume of the suspension.

Finally, the author's conclusions from the work presented in this thesis are given and suggestions made for further work on particle interaction in dilute sedimenting systems.
2. Literature Survey

2.1 Sedimentation of a Dilute Suspension

The phenomenon of sedimentation concerns the motion of particles in a surrounding medium, in this case the motion of small solid particles in a liquid, settling under the influence of the force of gravity. The motion of a single particle due to gravity was formalized mathematically by Newton's second law.

\[ \frac{mdy}{dt} = mg - m'g - f \quad \text{2.1.1} \]

where \( m \) and \( m' \) are the masses of the particle and fluid displaced by the particle, respectively, \( u \), the particle's velocity, \( g \), the acceleration due to gravity and \( f \), the frictional force resisting the particle's motion. The simplest example of the motion of a particle is the axi-symmetrical flow of a solid sphere through an unbounded fluid otherwise at rest and this was first treated by Stokes and solved by him in terms of the stream function, yielding an expression for the frictional drag on a sphere of radius, \( a \), in a fluid of viscosity, \( \mu \).

\[ f = \frac{6}{3} \mu u \quad \text{2.1.2} \]

If the particle and fluid have densities \( \rho_s, \rho \) respectively eq. 2.1.1 becomes
An approximate solution of eq. 2.1.3 is obtained when \( \frac{du}{dt} = 0 \), since during gravity settling 99% of the terminal settling velocity is seen to be reached very quickly. This terminal settling velocity is termed the Stokes velocity.

When the inertia term is set to zero, rearranging eq. 2.1.3

\[
\frac{4\pi a^3 \rho}{3} \frac{du}{dt} = \frac{4\pi a^3 (\rho_s - \rho)}{3} g - \frac{C}{\mu} \frac{du}{dt}
\]

2.1.3

This solution to the second law of motion has certain limitations which were summarized by Smoluchowski(87). Strictly speaking eq. 2.1.4 is only applicable as the Reynolds number approaches zero, although it may be applied with only 10% error up to a Reynolds number of 0.2. The lower limit of applicability is reached when the size of the particle becomes significant with respect to the mean free path of the molecules, \( \lambda \), since slip may occur between molecules. The Cunningham correction to Stokes' law takes this into account.

\[
U_s = \frac{2a^2 (\rho_s - \rho) g}{\Theta \mu}
\]

2.1.4

When a sphere falls in a containing vessel Stokes' law must be modified to take into account the wall and bottom effects of the vessel. For a cylindrical container Ladenburg(65) gave the necessary corrections. For wall effect he gave
\[ U_L = \frac{U_s}{1 + 2.4 \left( \frac{d}{D} \right)} \]  
\(2.1.6\)

and for bottom effect

\[ U_L = \frac{U_s}{1 + 1.7 \left( \frac{d}{D} \right)} \]  
\(2.1.7\)

where \(d\) and \(D\) are respectively the particle and vessel diameters. The constant 2.4 in eq. \(2.1.6\) was corrected in 1921 by Faxen to 2.1.

These early attempts at solution of the equations of fluid mechanics provide essential information concerning the motion of a single particle both in an infinite fluid and a finite container, but practical situations involving a single sphere rarely occur.

Many studies of the sedimentation of particles have been made, many of them concerning the motion of small spherical particles in an incompressible Newtonian fluid under the influence of gravity since such a system is both easy to observe experimentally and more amenable to the application of the equations of fluid mechanics.

The usual experimental techniques under the conditions stated above correspond to the conditions for solving the continuity equation for incompressible flow

\[ \nabla \cdot \mathbf{v} = 0 \]  
\(2.1.8\)

and the creeping motion equations

\[ \nabla P = \mu \nabla^2 \mathbf{v} \]  
\(2.1.9\)
However, analytical solutions for these equations are only available at present for simple particle shapes, such as spheres, ellipsoids, discs and rods and only for up to two particles settling in proximity in the fluid. Computer techniques are available for solving for the motions of a number of particles in proximity but these are long and slow since a large number of boundary conditions for the dispersion must be satisfied simultaneously.

It is found that Stokes' law is obeyed by particles in a dispersion only up to very low concentrations. Beyond this limit deviations from Stokes' law are noticed. The remainder of this section describes the findings of other workers concerned with sedimentation at low particle concentrations. The further sections of this literature survey consider firstly the motion of single particles in a fluid and then the development of theoretical considerations for two particles in proximity showing how velocities can deviate from Stokes' law giving rise to a distribution of particle velocities and showing how a horizontal component of particle velocity arises. The following two sections deal with attempts to describe the changing properties of a dispersion as modifications of velocity and viscosity as functions of particle concentrations. The remaining sections deal with various mathematical models to account for deviations from Stokes' law.

A paper by Batchelor(6) summarizes the points for consideration in two-phase mechanics:

1. The random location of discrete elements of a dispersed phase is an essential feature and probability
methods are required to analyse and describe mechanical properties of the system. A number of different averaging procedures may be used and their use must be considered carefully. For instance, for a suspension of solid particles between two rigid planes in steady relative motion which is statistically homogeneous, except near the boundaries it may be shown that the ensemble-averaged stress at a point in the suspension, the observable average stress over one of the boundaries and the average of the stress over a plane surface parallel to the boundaries which cuts through fluid and solid alike, are all equal, and are not equal to the average of the stress over the fluid portion of the suspension. As another example consider gravitational sedimentation such that the Reynolds number is so small that momentum changes are negligible. The difference between the fluid pressure averaged over the horizontal bottom and the free surface pressure is equal to the average weight of a vertical column of unit cross-section containing the mixture. Some authors have supposed that the average excess pressure gradient in the liquid is \( \frac{1}{\rho_m} \) where \( \rho_m \) is the density of the mixture. However, integration of \( \nabla p \) gives the magnitude of the excess pressure as \( \frac{2}{3} \left( \rho_m - \rho \right) g \) for a dispersion of low solids concentration.

2. Statistical properties of flowing mixtures, such as average volume fraction of one component, average relative volume-flux velocity, average stress, usually vary with position. Often, however, the distance over which properties vary considerably is quite large compared to microscopic
considerations. It may be possible to find an intermediate length between the microscopic and macroscopic over which averaging may be done.

3. The principle of local relative equilibrium may be exploited. The two components of a mixture are in approximate relative equilibrium if the characteristic time for change of the relative velocity is large compared with other times characteristic of the local relative motion (such as time for viscous diffusion of vorticity over a distance comparable with the microscopic structure length) and the local average relative velocity of the two components may be estimated by equating to zero the local forces tending to make one component move relative to the other. An element may be accelerating but the point is that under certain conditions the equations for motion of one component relative to accelerating axes moving with the element reduce to the steady motion form.

The limitations to Stokes' law have already been discussed. It has often been concluded by many authors that providing the concentration of the suspension is kept low each particle falls as it would in an infinite fluid. An upper concentration limit of 2% by volume has been suggested by Orr and Dallevalle(76) but research, notably by Boardman, suggests that particle interaction above volume concentration 0.05% gives the limit to Stokes' law. As the inter-particle distance is decreased by increasing the solids concentration a point is reached where hindered settling takes place. All particles, regardless of size or shape, seem to settle at the same rate leaving a liquid-
suspension interface. Hindered settling is observed to take place at about 5% volume concentration. Smoluchowski showed theoretically that when two equisettling spherical particles move through a fluid, separated only by a few diameters, the terminal velocity of the pair exceeds Stokes velocity for either particle individually. Hall showed experimentally that the settling velocity of two equal spheres 2.6 diameters apart, equivalent to a volume concentration of 3% is 20% higher than the Stokes velocity. Kaye and Boardman (57) attempted to extend Hall's work to suspensions by dropping groups of spheres into liquid. With more than four spheres they noticed that the members of the cluster failed to maintain their positions with respect to each other and took on a complex rotational motion. In the case of four spheres released with their centres in a horizontal plane and surfaces touching, the spheres situated themselves diagonally opposite one another and rolled around the line of centres of the second pair until they touched, whereupon the second pair performed a similar movement around the line of centres of the first pair. Moreover the velocity of the group was much greater than that of a single isolated sphere. This shows that cluster formation in a suspension can result in settling velocities much higher than those predicted by Stokes' law. Kaye and Boardman continued by measuring the settling velocity, $U_c$, of 850µm marker particles in a suspension of 850µm median size glass particles in liquid paraffin. Fig. 2.1.1 shows their results where $U_s$ is the Stokes velocity.
Fig. 2.1.1 Sedimentation of 850μm median size particles.

A steep rise occurs at about 0.1% corresponding to a particle spacing of approximately 8 diameters between centres. $U_c$ is the average velocity timed at each concentration. At 0.25% the curve reaches a peak where the average settling velocity is about 50% greater than $U_s$ and above 3% a boundary was noticed forming between the suspension and the fluid. Kaye and Boardman attributed the change in settling velocity to the formation of clusters. They assumed that velocities measured as slower than Stokes velocity were due to particles being caught in a return flow current. Coefficients of variation were found and are shown in Fig. 2.1.2. There is a steady increase in variation up to 3% followed by a rapid decrease.

Fig. 2.1.1 shows the changing nature of the settling process. At concentrations less than 0.05% particles settle as though in an infinite fluid except for occasional interaction due to random movement of one particle which takes in the flow field of another particle. Here there is no significant change from the Stokes velocity.
With increasing concentration viscous interaction becomes much more apparent and the average settling velocity increases to 0.3% solids at which approximate point velocity increase is beginning to be checked by return flow. The coefficient of variation continues to rise despite a slight reduction in settling velocity and reaches a maximum value at 2-3% indicating the formation of large clusters. At concentrations above 4% effects of return flow become predominant and instead of localized return flow it diffuses through a uniform cloud of particles until at 10% the settling characteristics of dense suspensions are established.

Kaye and Boardman carried out further experiments with the same 850μm glass marker particles in suspensions with solids containing solids of median sizes 400μm and 100μm. Had the results been similar to those quoted above, then deviations of the same order of magnitude would have been expected due to the volume of particles in the suspension. But, if the effect were due to cluster formation the magnitude of the interaction would have diminished with increasing
size ratio. They found the latter to be the case. This is shown in Fig. 2.1.3. Kaye and Boardman tried to show that sedimentation can be divided into four distinct zones:

1. Free settling zone at low concentrations.
2. Region of viscous interaction in which particles settle faster than their Stokes velocity.
3. An unstable region where clusters form and return flow is irregular and localized.

4. A well-defined region at higher concentrations where hindered settling occurs.

The work of Kaye and Boardman was continued, notably by Johne\(^{(54,55)}\) and by Koglin\(^{(58,59)}\). Johne and Koglin
confirmed that the velocity of sedimentation at low concentrations was greater than $U_s$, but whereas Kaye and Boardman found the maximum velocity was only 60% higher than $U_s$, Johne found it to be about twice this figure. Errors may have arisen in Kaye's and Boardman's work, according to Koglin\(^{(58)}\). He suggests that the spread of particle size used was not in good agreement with the median sizes given, and secondly points out that Kaye and Boardman used as average velocity $U_c$, the sum of all the distances divided by the sum of all the times, whereas Johne\(^{(54,55)}\) averaged the sum of all the path-time quotients. Kaye's and Boardman's velocities were, therefore, slightly too low. Koglin shows that the lower maximum speed of fall found by Kaye and Boardman cannot be attributed to differences in the position of the distribution of sizes of spheres in suspension with respect to the test sphere since their particle size distribution differs less from the size of the test sphere than does Koglin's.

Kaye and Boardman did not consider wall effect as significant, but its importance was shown by Woodward\(^{(98)}\). Work by Koglin shows the remarkable effect of wall effect at values of $\frac{d}{D}$, ratio of particle diameter to column diameter, of 0.01, 0.005, 0.003. This is shown in Fig. 2.1.4.

Kaye and Boardman took measurements only when their test spheres were not near the cylinder wall and this could explain the differences between the values obtained by Johne and Kaye and Boardman. Koglin asserts his theory of wall effect by plotting $\frac{D}{d}$ against Koglin's average velocity $U$.
as $\bar{U}$ for vertical plane parallel walls and a cylindrical wall as in Fig. 2.1.5.

![Graph showing wall effect on settling velocity](image)

**Fig. 2.1.4** Wall effect on settling velocity.

![Graph showing mean rate of fall plotted against D for different vessel geometries](image)

**Fig. 2.1.5** Mean rate of fall plotted against $D$ for different vessel geometries.
Koglin uses, instead of Ladenburg's correction for wall effect, a corrected velocity $U_{BH}$ due to Brenner and Happel.

$$U_{BH} = \frac{U_0}{1 + f(\beta) \frac{d}{D}}$$

2.1.10

where $\beta$ is the ratio of the distance of the particle from the axis of the cylinder to that of the radius of the cylinder. The function $f(\beta)$ is an infinite series of integrals for which there is no exact solution. Brenner and Happel give approximate solutions:

$$f(\beta) = 2.1047 - 0.6377\beta^2 \quad \text{for } \beta \to 0$$

$$f(\beta) = \frac{0.75}{1 - \beta} \quad \text{for } \beta \to 1$$

from which they produce graphically an approximate solution for the entire range $0 \leq \beta \leq 1$. This is shown in Fig. 2.1.6

![Fig. 2.1.6 Theoretical rate of fall influenced by wall, $U_{BH}$, related to Stokes velocity, $U_0$, plotted against ratio of diameter of cylindrical vessel, $D$, to the particle diameter, $d$.](image)

Koglin plotted the velocity of particles over two
consecutive distances of equal length and found that the particles increased in velocity on settling. He then plotted the velocity in the upper test section, $U_1$, against the velocity in the lower test section, $U_2$, also showing the straight line between the origin and the mean value. This is shown in Fig. 2.1.7. There is a correlation between the speeds of fall in the two sections, as would be expected. Pairs of values, "fast-fast" (top right from mean value) and "slow-slow" (bottom left from mean value) occur markedly more often than "fast-slow" and "slow-fast" pairs. He suggests that the mean cluster size increases during sedimentation, and that wall effect has a greater significance than given by Kaye and Boardman.

![Fig. 2.1.7 Individual velocities measured over two test sections.](image)

Concerning the formation of clusters, Johne\(^{(54,55)}\) considered $r$ particles and $n$ equal sized liquid cells giving an average number of particles per cell $\bar{k} = \frac{r}{n}$. He considered
the number of particles in each cell to follow a Poisson distribution with \( k \) particles in each cell.

\[
    p_k(\bar{k}) = \frac{\bar{k}^k e^{-\bar{k}}}{k!}
\]

Denoting \( U_F \), the average fluid velocity, \( U_k' \), the relative velocity of a cell of particles, \( U_k \), the velocity of a cell of particles

\[
    u_k = u_k' - u_F
\]

and the average velocity is given by

\[
\begin{align*}
\bar{u} &= \frac{\sum \tilde{p}_k(\bar{k}) u_k}{\sum \tilde{p}_k(\bar{k})} = \frac{\sum \tilde{p}_k(\bar{k}) u_k'}{\sum \tilde{p}_k(\bar{k})} - U_F \\
\bar{u} &= \frac{\sum \tilde{p}_k(\bar{k}) u_k'}{\sum \tilde{p}_k(\bar{k})} \left[ \left( \sum \tilde{p}_k(\bar{k}) \right)^{-1} - V_k \right]
\end{align*}
\]

where \( V_k \) is the cell volume, since \( u_F = \frac{\sum \tilde{p}_k(\bar{k}) u_k' V_k}{\sum \tilde{p}_k(\bar{k})} \) and Johne gives a relationship between \( U_k' \) and \( k \) as

\[
    u_k' = k^m
\]

which he obtains from a plot of \( \bar{U} \) against \( c \) plotting also \( \bar{U}_g \) against the coordinate, \( \bar{k} \), as shown in Fig. 2.1.8, and gives \( m = 0.6 \).
Koglin (61) suggests a logarithmic distribution function, $Q$, of velocities $U$

$$Q = \frac{1}{s^n \sqrt{2\pi}} \int_{0}^{\infty} \frac{u}{\bar{u}_g} \exp \left[ -\frac{1}{2} \left( \ln \frac{\xi}{\bar{u}_g} \right)^2 \right] d\xi$$  \hspace{1cm} 2.1.16

where $\bar{u}_g$ is the geometric mean velocity $\bar{u}_g = \frac{1}{n} \sum_{i=1}^{n} \ln U_i$
and $s^2$ is the logarithmic variance $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\ln U_i - \ln \bar{u}_g)^2$

Koglin's values for $\bar{u}_g$ and $s^2$ against concentration, $c$, are shown in Fig. 2.1.9.

Fig. 2.1.9 Geometric mean velocity and variance against concentration.
With the two definitions of average velocity due to Johne, $\bar{U}_J$ and to Keye and Boardman, $\bar{U}_{KB}$, Koglin relates them to his geometric mean velocity.

$$\bar{U}_J = \bar{u}_g \exp\left(\frac{z^2}{2}\right)$$  \hspace{1cm} (2.1.17)

$$\bar{U}_{KB} = \bar{u}_g \exp\left(-\frac{z^2}{2}\right)$$  \hspace{1cm} (2.1.18)

Koglin(61) continued the work of Johne utilizing eq. 2.1.15 and denoting densities of a cluster, a solid particle and the suspending fluid by $\rho_k$, $\rho_s$, $\rho$ respectively.

$$\rho_k - \rho = \frac{k_x^3}{x_k^2} (\rho_s - \rho)$$  \hspace{1cm} (2.1.19)

Denoting the settling velocity of a cluster by $U_k$, Koglin obtains

$$\frac{U_k}{U_s} = \frac{\rho_k - \rho}{\rho_s - \rho} \left(\frac{x_k}{x}\right)^2$$  \hspace{1cm} (2.1.20)

and from eq. 2.1.19 it is obvious that

$$\frac{U_k}{U_s} = \frac{k_x x}{x_k}$$  \hspace{1cm} (2.1.21)

From Ladenburg's correction

$$\frac{U_k}{U} = \frac{U_k}{U_s} \left(1 + 2.1 \frac{x_k}{D}\right)$$  \hspace{1cm} (2.1.22)

where $U$ is the velocity of a single particle. Therefore
\[ \frac{U_k}{U} = \frac{k}{x_k} - 2.1 \frac{k}{D} \quad 2.1.23 \]

and from a graph of relative velocity against \( k \) for various values of \( \frac{D}{x} \), at \( \frac{D}{x} \to \infty \), \( \frac{U_k}{U_s} = \sqrt{k} \). Thus \( \frac{x_k}{x} = \sqrt{k} \).

\[ \frac{U_k}{U} = \sqrt{k} - 2.1 \frac{k}{D} \quad 2.1.24 \]

![Graph](image)

**Fig. 2.1.10** Relative velocity vs. \( k \) for various values of \( \frac{D}{x} \)

Batchelor\(^{4}\) considers a system of particles of radius, \( a \), in a Newtonian fluid of viscosity, \( \mu \), such that the mean flux across a stationary plane surface gives rise to a zero mean velocity of material. The velocity, \( U \), of a particular particle in the dispersion differs from the Stokes velocity, \( U_s \), due to the hydrodynamic interaction among the various particles and thus \( U - U_s \) is a random quantity with a non-zero mean which depends on the particle concentration. He determines the mean velocity of a particle by

\[ \bar{U} = \frac{1}{N!} \int u(z_0 c_n) P(c_n x_0) dC_n \quad 2.1.25 \]
for a sphere with centre at $x_0$ whose velocity is $U(x_0, C_N)$ where $P(C_N | x) dC_N$ is the conditional probability that a configuration of $N$ sphere centres being found in the range $dC_N$ about $C_N$ where $C_N$ refers to a realization of the set of position vectors, $x$, of the centres of $N$ spheres. Calculations for one or two spheres are feasible but not for more so Batchelor requires to consider somehow a group of one or two spheres by reduction of eq. 2:1.25. Integrating this equation over just one sphere in the configuration, $C_N$, on the grounds that the chance of two spheres being simultaneously close enough to $x_0$ to influence the velocity of the test sphere is of order $c^2$ and so negligible is invalidated by the slowness of the decrease to zero of the influence of one falling sphere on $U(x_0)$ with increasing distance from $x_0$. The procedure is to look for a quantity whose mean is exactly known from an overall condition whose value at $x_0$ has the same long-range dependence on the presence of a sphere at $x_0-x$ as the velocity of the test sphere, and once found the difference between $U$ and the mean of this quantity can be expressed as an integral like eq. 2.1.25 and can then legitimately be reduced to an integration over the location of just one sphere in the configuration $C_N$ and evaluated explicitly.

Owing to the asymptotic form of the dependence of $U(x_0, C_N)$ on the configuration, $C_N$, as the distance of the spheres in $C_N$ that are nearest to $x_0$ becomes large, the desired quantity becomes obvious. When the spheres of $C_N$ are well away from $x_0$ the sphere may be regarded as immersed in a fluid which, in the absence of that sphere, would have
approximately uniform velocity over a region of linear dimensions $2a$. Then $U(x_0, C_N)$ would be approximately equal to the sum of $U_o$ and that uniform velocity. Thus Batchelor looked at the relationship between $U(x_0, C_N)$ and the velocity distribution that would exist in the dispersion if the test sphere were replaced by fluid of viscosity, $\mu$, without change in the configuration, $C_N$. This velocity at the point $x$ is denoted by $v(x, C_N)$ and will in general be non-uniform on the spherical surface centred on $x_0$ with radius $a$. The resultant vector of the distribution of forces is $\frac{4}{3} \pi a^3 (\mu \gamma) \mathbf{g}$ which if applied to the fluid in the presence of all other rigid freely-moving spheres generates a fluid velocity on the fluid surface centred on $x_0$ bounded by $a$ which is the sum of a uniform vector and the variable quantity $-v(x)$. Temporarily neglecting the no-slip condition on the surface of spheres other than the test sphere, the unique translational velocity of the test sphere is $U_o - v$ where

$$V(x_0, C_N) = \frac{1}{4\pi a^2} \int_{A_0} v(x, C_N) \, dA$$

2.1.26

where $A_0 = 4\pi a^2$. Representing $v(x)$ as a Taylor series in $x-x_0$, integrating over $A_0$, and utilizing the slow-motion equation $\nabla^4 v = 0$

$$V(x_0, C_N) = v(x_0, C_N) + \frac{1}{6} a^2 \left[ \nabla^2 v(x, C_N) \right]_{x=x_0}$$

2.1.27

This expression was obtained by Faxen. It takes into account the velocity distribution in the fluid near $x_0$ due to the
motion of all other spheres than the test sphere, but it is incomplete because forces acting at the surface of the test sphere need to be accompanied by image systems in the spherical boundaries of all other spheres to ensure no-slip conditions are satisfied at those boundaries. A rigid sphere of radius, \( a \), at a distance, \( r \), from the test sphere will require an additional translational velocity which in turn induces a change in the velocity distribution near the test sphere. All other spheres in the dispersion will have a similar effect on the test sphere and this additional velocity, \( \mathbf{W} \), representing the image system gives rise to a new velocity expression

\[
\mathbf{U}(x_0, C_N) = U_0 + \mathbf{V}(x_0, C_N) + \mathbf{W}(x_0, C_N)
\]

2.1.28

The problem arises with the non-convergence of \( \mathbf{V} = \mathbf{V}' - \mathbf{V}'' \) where

\[
\mathbf{V}' = \frac{1}{N!} \int \mathbf{V}(x_0, C_N) P(C_N | x_0) dC_N
\]

2.1.29

\[
\mathbf{V}'' = \frac{1}{N!} \int \frac{a^2}{6} \left[ \nabla^2 \mathbf{V}(x, C_N) \right]_{x = x_0} P(C_N | x_0) dC_N
\]

2.1.30

because \( |\mathbf{V}| \) behaves as \( \frac{a}{r} \) at a distance, \( r \), from one falling sphere and \( a |\nabla^2 \mathbf{V}| \) behaves as \( \frac{a^3}{r^3} \), the latter giving a decrease to zero as \( \frac{a}{r} \to \infty \) which is too slow for convergence of the integral. Batchelor, therefore, integrates eq. 2.1.29 and eq. 2.1.30 with the aid of exact mean values involving all the spheres in the configuration, \( C_N \), eventually yielding
and the final expression for the mean velocity of a sphere in the dispersion is

$$\bar{u} = u_o + \bar{V} + \bar{V}' + \bar{W}$$  \hspace{1cm} 2.1.33

where $\bar{V}'$ and $\bar{V}''$ are given by eq. 2.1.31 and eq. 2.1.32 respectively and $\bar{W}$ is given by

$$\bar{W} = \int \mathcal{W}(x_o, x_{o^+} r) P(x_{o^+} r | x_o) dr + O(c^2)$$  \hspace{1cm} 2.1.34

These expressions can be evaluated with a knowledge of the probability density of the location of one sphere relative to a second sphere in a statistically homogeneous dispersion and the flow field due to two spheres falling in an infinite fluid. For a dilute suspension the initial condition is that

$$P(x_{o^+} r | x) = \begin{cases} n & \text{if } r > 2a \\ 0 & \text{if } r < 2a \end{cases}$$  \hspace{1cm} 2.1.35

where $n$ is the uniform mean number density of spheres which for the probability density of one sphere position relative to another becomes
yielding

\[ \bar{V}' = -n \int_{r<2a} v(x_0, r) dr \]

\[ \bar{V}'' = -n \int_{r<2a} \frac{a^2}{6} \left[ \frac{\partial^2 V}{\partial x^2} \right]_{x=x_0} dr + \frac{1}{2} c u_s \]

\[ \bar{V} = n \int_{r>2a} W(x_0, x_0 + r) dr \]

Substitution of

\[ W(x+r, x) = u(x+r, x) - u_s - v(x+r, x) - \frac{1}{6} \frac{\partial^2 V}{\partial x^2}(x+r, x) \]

from rearrangement of eq. 2.1.33 finally yields the expression:

\[ \bar{u} = u_s (1 - 0.55c) \]

This equation is correct only to the order of c and suggests that the average particle velocity declines with increasing concentration, unlike the experimental findings of Kaye and Boardman, Johne, and Koglin, described earlier.
2.2 Single Sphere Analyses

Hjelmfelt and Mockros\(^{(49)}\) considered the forces on a sphere undergoing arbitrary rectilinear acceleration in a viscous fluid using the equation first presented by Basset.

\[
\frac{\pi d^3}{6} \frac{d^2}{dt^2} \rho \frac{d}{dt} U + \frac{\pi d^3}{6} \frac{d}{dt} \rho \frac{dU}{d\xi} = (\rho_s - \rho) \frac{\pi d^3}{6} g - 3\pi d \rho \nu U - \frac{3}{2} \rho \nu \frac{d}{d\xi} \int_{0}^{t} \frac{dU}{\sqrt{t - \xi}} d\xi
\]

where \( \nu \) is the kinematic viscosity and \( t \) is time. The first term is the mass times the acceleration of the sphere and the second term accounts for the acceleration of the fluid added mass. The first term on the right hand side of the equation is the buoyant force and the second is the steady state drag for the instantaneous velocity. The final term is the Basset history integral which corrects viscous drag for the transient condition. The equation is based on Basset's equation and is thus subject to the limitations of rectilinear motion and negligible fluid convection.

The first closed form of eq. 2.2.1 utilizes a transformation to reduce the integro-differential equation to a second order ordinary differential equation. Considering an impulsive force applied at \( t = t_i \)

\[
\frac{\pi d^3}{6} \left( \rho_s \frac{d}{dt} + \frac{\rho}{2} \right) \frac{dU}{d\xi} = (\rho_s - \rho) \frac{\pi d^3}{6} g - 3\pi d \rho \nu U - \frac{3}{2} \rho \nu \frac{d}{d\xi} \int_{0}^{t} \frac{1}{\sqrt{t - \xi}} dU d\xi + k_6(t - t_i)
\]

which can be written in the convenient form
where \( \text{Re} \) can be regarded as the Stokes terminal Reynolds number or the nondimensional Stokes velocity. The solution for \( \frac{\rho}{\rho} \neq \frac{5}{3} \) is

\[
\begin{align*}
V(t) &= \mathcal{H}(t) \left[ V_t + \beta (V_t - V_0) \frac{e^{\alpha t} \text{erfc}(\alpha \sqrt{t})}{(\alpha - \beta)} - \alpha (V_t - V_0) \frac{e^{\beta t} \text{erfc}(\beta \sqrt{t})}{(\alpha - \beta)} \right] \\
&+ \frac{K(\alpha, \beta)}{(\alpha - \beta)} \mathcal{H}(t - \tau_i) \left[ \alpha e^{\alpha (t - \tau_i)} \text{erfc}(\alpha \sqrt{t - \tau_i}) - \beta e^{\beta (t - \tau_i)} \text{erfc}(\beta \sqrt{t - \tau_i}) \right]
\end{align*}
\]

where \( \alpha, \beta = \frac{3}{r \left( \frac{\rho}{\rho} + \frac{1}{2} \right)} \left[ 3 \pm \sqrt{5 - 8 \frac{\rho}{\rho}} \right] \)

where \( V_0 \) is the initial velocity, which must be steady state so that history integral is zero for \( t < 0 \). \( \mathcal{H}(\xi) \) is the step function equal to zero for negative arguments and unity for positive arguments. The equation can be rearranged to give
\[
\frac{V(\tau) - V_o}{V_t - V_o} = H(\tau) \left[ 1 + \frac{\beta}{\alpha - \beta} e^{\alpha \tau} e^{\text{erfc}(\alpha \sqrt{\tau})} - \frac{\alpha}{\alpha - \beta} e^{\beta \tau} e^{\text{erfc}(\beta \sqrt{\tau})} \right] \\
+ \frac{H(\tau - \tau_i)}{V_t - V_o} \left[ \frac{\alpha}{\alpha - \beta} e^{\alpha (\tau - \tau_i)} e^{\text{erfc}(\alpha \sqrt{\tau - \tau_i})} - \frac{\beta}{\alpha - \beta} e^{\beta (\tau - \tau_i)} e^{\text{erfc}(\beta \sqrt{\tau - \tau_i})} \right]
\]

For \( \frac{\beta}{\alpha} > \frac{5}{8} \) \( \alpha \) and \( \beta \) become complex, \( \alpha = u + \iota v = z \), \( \beta = u - \iota v = \bar{z} \)

\[
\omega(z) = e^{-z^2} \left[ 1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{-\gamma^2} d\gamma \right] \\
= R_\omega(z) + i I_\omega(z)
\]
giving

\[
\frac{V(\tau) - V_o}{V_t - V_o} = H(\tau) \left[ 1 - R_\omega((v_0i)\sqrt{\tau}) + \frac{u}{V} I_\omega((v_0i)\sqrt{\tau}) \right] \\
+ \frac{K\alpha\beta}{V_t - V_o} H(\tau - \tau_i) \left[ R_\omega((v_0i)\sqrt{\tau - \tau_i}) - \frac{u}{V} I_\omega((v_0i)\sqrt{\tau - \tau_i}) \right]
\]

The remaining condition is when \( \frac{\beta}{\alpha} = \frac{5}{8} \), when \( \alpha = \beta \) such that

\[
\frac{V(\tau) - V_o}{V_t - V_o} = H(\tau) \left[ 1 - \frac{3\sqrt{\tau}}{\sqrt{\pi}} + (32\tau - 1) e^{16\tau} e^{\text{erfc}(4\sqrt{\tau})} \right] \\
+ \frac{16K}{V_t - V_o} H(\tau - \tau_i) \left[ (32(\tau - \tau_i) + 1) e^{16(\tau - \tau_i)} e^{\text{erfc}(4\sqrt{\tau - \tau_i})} - \frac{3\sqrt{\tau - \tau_i}}{\sqrt{\pi}} \right]
\]

2.2.5

Fig. 2.2.1 shows the velocity time relationship for \( \tau < \tau_i \), i.e. before the impulse. Note that two spheres of the same
size and in the same fluid will have terminal velocities of the same magnitude if their density ratios \( \left( \frac{\rho_s}{\rho} \right) \) and \( \left( \frac{\rho_s}{\rho} \right)' \) satisfy

\[
\left( \frac{\rho_s}{\rho} \right)' = 2 - \left( \frac{\rho_s}{\rho} \right), \quad \left( \frac{\rho_s}{\rho} \right)' \leq \left( \frac{\rho_s}{\rho} \right) \leq 2
\]  

2.2.7

![Graph showing theoretical velocity-time curves for spheres falling in a viscous fluid.](image)

Fig. 2.2.1 Theoretical velocity-time curves for spheres falling in a viscous fluid.

More common analyses of spheres in unsteady motion are not based upon eq. 2.2.1 but on

\[
\frac{\pi d^3}{\rho} \left( \frac{\rho_s + \rho}{2} \right) \frac{dU}{dt} + 3 \pi d \rho \gamma U = \left( \frac{\rho_s - \rho}{\rho} \right) \frac{\pi d^3}{6} g
\]

2.2.8

or in non-dimensional form

\[
\frac{d}{d\tau} \left( \frac{V(\tau)}{V_t} \right) + \frac{18}{\left( \frac{\rho_s + \frac{1}{2}}{\rho} \right)} \left( \frac{V(\tau)}{V_t} \right) = \frac{18}{\left( \frac{\rho_s + \frac{1}{2}}{\rho} \right)}
\]

2.2.9

\[
\frac{V(\tau)}{V_t} = 1 - e^{\frac{-18 \tau}{\left( \frac{\rho_s + \frac{1}{2}}{\rho} \right)}}
\]

2.2.10

and neglecting the history and added mass terms the solution eq. 2.2.10 becomes
\[ \frac{V(t)}{V_k} = 1 - \exp\left(\frac{-18\tau}{\left(\frac{\rho_s}{\rho}\right)}\right) \]  \hspace{1cm} 2.2.11

A comparison of solutions showing exact and approximate solutions is shown for different values of \( \frac{\rho_s}{\rho} \) in Fig. 2.2.2

Fig. 2.2.2 Effect of neglecting history and added mass terms.

Swanson (92) made a much simpler analysis of the free settling sphere problem considering the Newtonian law of resistance

\[ \frac{\pi d^3}{6} (\rho_s - \rho) g = \frac{\pi d^2}{4} \frac{U^2}{2} C_D \]  \hspace{1cm} 2.2.12
where $C_D$ is a friction factor. He indicates that from experience when a solid sphere is greater than 1mm in diameter or Reynolds number is greater than $800-1000$, $C_D$ essentially becomes constant. Thus rearranging

$$U = \sqrt{\frac{4gd(\rho_s-\rho)}{3\rho C_D}}$$

Another special condition arises for particles smaller than 0.05mm or for Reynolds number less than 3, in which case $C_D = \frac{24}{Re}$ and eq. 2.2.13 is written as

$$U = \frac{gd^2(\rho_s-\rho)}{18\mu}$$

the Stokes equation for free settling. Swanson developed an equation for any value of Reynolds number which coincided with Stokes' and Newton's laws of settling at their respective areas of validity and also held in the intermediate regions. As shown in Fig. 2.2.3 the Stokes' region and Newton's law region have velocity-diameter relationships which are described by equations of the form $U = k_1d^2$ and $U^2 = k_2(d-d^*)$ respectively.

Fig. 2.2.3 Velocity-diameter plot for settling particles.
Swanson gives a modified Newton's law expression neglecting $C_D$

\[ U = \sqrt{\frac{4g(\rho_2 - \rho)(d-d^*)}{3\rho}} \]  \hspace{1cm} 2.2.15

where $d^*$ is given by $\frac{\mu}{\rho U_{Re=1}}$ and the substitution of this value into eq. 2.2.15 is valid for the region of intersection of eq. 2.2.15 with the Stokes region. To make the equation valid throughout the term $d - \frac{\mu}{\rho U_{Re=1}}$ was considered. The Newton's law term in $d^2$ was replaced by $(d - \frac{\mu}{\rho U_{Re=1}})^2$ and solution gives

\[ U = \left( d - \frac{\mu}{\rho U_{Re=1}} \right) \sqrt{\frac{4g d (\rho_2 - \rho)}{3\rho}} \]  \hspace{1cm} 2.2.16

but for $Re < 1$, $U$ becomes negative and Swanson arbitrarily changed the sign of $\frac{\mu}{\rho U_{Re=1}}$ from negative to positive. Thus

\[ U = \left( d + \frac{\mu}{\rho U_{Re=1}} \right) \sqrt{\frac{4g d (\rho_2 - \rho)}{3\rho}} \]  \hspace{1cm} 2.2.17

Swanson found by experimentation that eq. 2.2.17 was of the correct form in that the term in brackets was the ratio of particle diameter to particle diameter plus another term which he regarded as the thickness of a fluid layer. From boundary layer theory he obtained

\[ X = d \cos \phi \left( 1 + k_3 \sqrt{\frac{g}{\rho U_5}} \right) \]  \hspace{1cm} 2.2.18
where \( \nu \) is the kinematic viscosity and \( U = U_s \) since the flow in the boundary layer is laminar. Thus

\[
X = d \cos \phi \left( 1 + \frac{48 \kappa_3 \mu}{d \rho \sqrt{4g \delta (\gamma^2 \rho)}} \right) \tag{2.2.19}
\]

Defining the Newton's law velocity as

\[
U_N = \sqrt{\frac{4g \delta (\gamma^2 \rho)}{3 \rho}}
\]

the velocity of the particle is expressed as

\[
U = \frac{U_N}{\cos \phi \left( 1 + \frac{48 \kappa_3 \mu}{d \rho U_N} \right)} \tag{2.2.20}
\]

As diameter increases the term in brackets approaches unity and Newton's law is approached. As diameter decreases Stokes' law is approached.

Abraham\(^{(2)}\) determined a functional dependence of drag coefficient for a sphere on Reynolds number, by means of dimensional analysis such that the drag force, \( D \), on a sphere is in the form

\[
D = d^\alpha U^\beta \rho r^\gamma \mu^\delta \tag{2.2.21}
\]
giving the relationship

\[ D = \rho u^2 \delta \left( \frac{k}{\rho u_d} \right)^\gamma \]  \hspace{1cm} 2.2.22

where \( \gamma \) is an undetermined exponent. If drag is independent of viscosity then \( \gamma = 0 \) and eq. 2.2.22 becomes

\[ D = k_4 \rho u^2 \delta^2 \]  \hspace{1cm} 2.2.23

where \( k_4 \) is a constant depending on the shape of the body. Such is the case for large Reynolds numbers, so long as the boundary layer round the body remains laminar. Solutions of fluid dynamical equations are constituted to permit subdivision of the flow field into an external region where frictionless flow exists (\( \mu = 0 \)) and the boundary layer round the body of thickness, \( \delta \), where \( \mu \neq 0 \). Abraham quotes work by Tomotiki and McDonald who estimated that for a point \( 80^\circ \) from the forward stagnation point of a sphere of radius, \( r \)

\[ \frac{\delta}{r} = \frac{\delta_0}{(R_e)^{\frac{1}{2}}} \hspace{1cm} \delta_0 \approx 0.06 \]  \hspace{1cm} 2.2.24

Abraham pictured the boundary layer liquid as travelling with the body, and for a sphere of radius, \( r \), and boundary layer thickness, \( \delta \), the new dimension is \( R = r + \delta \). For this system drag is independent of viscosity. Therefore,

\[ D = C_0 \frac{\rho u^2}{2} \pi R^2 = C_0 \left( 1 + \frac{\delta}{r} \right)^2 \frac{\rho \pi r^3 u^2}{2} \]  \hspace{1cm} 2.2.25
Interpreting $D$ as the drag on a sphere of radius, $r$, moving at a speed, $U$, the drag coefficient for the sphere is

\[ C = C_0 \left(1 + \frac{\delta_c}{Re}\right)^2 \]

where eq. 2.2.24 is used. The parameters $C_0$ and $\delta_c$ must be obtained through empirical fitting, and Abraham notes further that for $Re \ll 1$, $C \rightarrow \frac{C_0 \delta_c}{Re}$, which is the functional dependence of Stokes law $C = \frac{24}{Re}$ for small $Re$.

Brenner (10,11) derived the Stokes resistance of an arbitrary particle considering the instantaneous translational velocity, $U_o$, and instantaneous angular velocity, $\omega$, of the particle by solution of the linearized Navier-Stokes equations and continuity equations for an arbitrary body in a Newtonian fluid at very low Reynolds number ($Re \ll 1$)

\[ \mu \nabla^2 \gamma = \nabla p \]

\[ \nabla \cdot \gamma = 0 \]

where $p$ refers to the dynamic pressure rather than the total pressure. The effects of hydrostatic pressure are temporarily neglected.

If the particle of arbitrary shape moves in an unbounded fluid at rest at infinity, the net effect of stresses set up at its surface is equivalent to a force, $F$, and a couple, $L$; these are the effects of the fluid on the particle.

Calculations require estimates of the rapidity with which velocity and stress fields are attenuated at great distances.
from the particle. Considering a translational motion, the particle experiences a hydrodynamic force

$$ F = \int_{s_p} dS \cdot \Pi $$  \hspace{1cm} 2.2.29

on its surface, $s_p$, where integration is carried out over an area an element of which, $dS$, is directed into the fluid. The stress tensor, $\Pi$, is, for an incompressible Newtonian fluid,

$$ \Pi = -I_p + \mu (\nabla \nu + \nabla \nu^t) $$  \hspace{1cm} 2.2.30

In the region of applicability of eq. 2.2.28 the relation

$$ \nabla \cdot \Pi = 0 $$  \hspace{1cm} 2.2.31

also applies at each point in the fluid. If eq. 2.2.31 be multiplied by a volume element of fluid, $dV$, and integrated over any fluid volume, $V$, it may be converted by Gauss' divergence theorem into a surface integral over the closed surface, $S$, containing the volume, $V$, in its interior.

$$ \int_V \nabla \cdot \Pi \, dV = \int_S dS \cdot \Pi = 0 $$  \hspace{1cm} 2.2.32

$V$ is chosen here to consist of the fluid bounded internally by the particle surface and externally by a spherical fluid surface, $\sigma$, of very large radius containing the particle at its centre. Thus,
Comparing eq. 2.2.29, eq. 2.2.32 and eq. 2.2.33, one obtains

\[ F = - \int_S dS \cdot \Pi \]  
\[ \text{eq. 2.2.34} \]

The solutions of eq. 2.2.27 and eq. 2.2.28 yield for a point force

\[ \gamma = - \frac{F}{\mathcal{G} \pi \mu r} - \frac{r^2}{24 \pi \mu} \nabla (F \cdot \nabla) \frac{1}{r} \]  
\[ \text{eq. 2.2.35} \]

\[ p = \frac{1}{4 \pi} \left( F \cdot \nabla \right) \frac{1}{r} \]  
\[ \text{eq. 2.2.36} \]

where \( r \) is measured from the singularity. A similar development for the fluid at great distances from a rotating particle follows from

\[ L = \int_{S_p} r \times (\Pi \cdot dS) \]  
\[ \text{eq. 2.2.37} \]

yielding

\[ \gamma = - \frac{L \times r}{3 \pi \mu r^3} + \frac{r}{2 \mu} \frac{\mathcal{F}}{\mathcal{P}} \]  
\[ \text{eq. 2.2.38} \]

If \( (\gamma', \Pi') \) and \( (\gamma'', \Pi'') \) be the velocity and stress fields corresponding to any two motions conforming to the eq. 2.2.27 and 2.2.28, the Lorentz reciprocal theorem allows that

\[ \int_S dS \cdot \Pi' \cdot \gamma' = \int_S dS \cdot \Pi'' \cdot \gamma'' \]  
\[ \text{eq. 2.2.39} \]
in which \( s \) is a closed surface bounding any fluid volume \( V \).

If the two distinct fluid motions are those induced by the steady translation of a particle through the unbounded fluid with velocities, \( U' \) and \( U'' \) respectively, as fluid adheres to the rigid particle it satisfies the boundary condition

\[
\mathbf{v} = U' \quad \text{on} \quad s_p
\]

In consequence of the primed motion the particle experiences a force

\[
F' = \int_{s_p} d\mathbf{S} \cdot \mathbf{\Pi}'
\]

with a similar expression for the double primed motion.

Substitution of eq. 2.2.40 into eq. 2.2.39 leads to

\[
F' + \int_{s} d\mathbf{S} \cdot \mathbf{\Pi}', \mathbf{v}'' = F'' + \int_{s} d\mathbf{S} \cdot \mathbf{\Pi}'', \mathbf{v}'
\]

As the integrands \( \mathbf{\Pi}', \mathbf{v}'' \) and \( \mathbf{\Pi}'', \mathbf{v}' \) are at most of \( O(r^{-3}) \) as \( r \to \infty \) but the area \( \sigma \) is only of \( O(r^2) \), the surface integrals of eq. 2.2.42 vanish. Thus,

\[
U' \cdot F' = U' \cdot F''
\]

implying that \( F \) is a linear vector function of \( U \). This may be shown by considering the body to move with successive velocities \( U_1, U_2 \) and \( (U_1 + U_2) \) and denoting \( F = f(U) \)
Adding and rearranging

\[(u_1 + u_2) \cdot \hat{f}(u_1) = u_1 \cdot \hat{f}(u_1 + u_2)\]

and

\[(u_1 + u_2) \cdot \hat{f}(u_2) = u_2 \cdot \hat{f}(u_1 + u_2)\]

As the velocities are arbitrary, the term in square brackets must be zero. Thus

\[\hat{f}(u_1 + u_2) - \hat{f}(u_1) - \hat{f}(u_2) = 0\]

This is the definition of a linear vector function and the force is related to the velocity by a second rank tensor, \(\hat{K}\), and further dimensional arguments require that the force is directly proportional to the viscosity, and general form is, therefore,

\[F = -\mu \hat{K} \cdot \hat{u} \quad \text{2.2.44}\]

\(\hat{K}\) is termed the resistance tensor and is an intrinsic property of the body depending only on its external configuration. In particular it is independent of orientation and speed of the body and of the fluid properties. It is also symmetric as is shown below. From eq. 2.2.43

\[u' \cdot \hat{K} \cdot u'' = u'' \cdot \hat{K} \cdot u'\]

If \(\hat{K}\) denotes the transpose of \(\hat{K}\)

\[u' \cdot \hat{K} \cdot u'' = u' \cdot (u'' \hat{K}) = u'' \cdot \hat{K} \cdot u'\]
Therefore
\[ \mathbf{u}' \cdot \mathbf{K} \cdot \mathbf{u}' = \mathbf{u}' \cdot \mathbf{K} \cdot \mathbf{u}' \]

\[ \mathbf{K} = \mathbf{K}^T \]

2.2.45

The general form for \( \mathbf{K} \) is given as
\[ \mathbf{K} = \mathbf{i}_1 \mathbf{i}_1' K_{11} + \mathbf{i}_2 \mathbf{i}_2' K_{22} + \mathbf{i}_3 \mathbf{i}_3' K_{33} \]
\[ + \mathbf{i}_1 \mathbf{i}_2' K_{12} + \mathbf{i}_1 \mathbf{i}_3' K_{13} \]
\[ + \mathbf{i}_2 \mathbf{i}_3' K_{21} \]
\[ + \mathbf{i}_2 \mathbf{i}_3' K_{22} + \mathbf{i}_2 \mathbf{i}_3' K_{23} \]
\[ + \mathbf{i}_3 \mathbf{i}_3' K_{31} + \mathbf{i}_3 \mathbf{i}_3' K_{32} + \mathbf{i}_3 \mathbf{i}_3' K_{33} \]

2.2.46

where \( \mathbf{i}_1, \mathbf{i}_2 \) and \( \mathbf{i}_3 \) represent unit vectors in any Cartesian coordinate system, and we have already shown that by symmetry \( K_{12} = K_{21} \), \( K_{13} = K_{31} \), and \( K_{23} = K_{32} \). It is a property of second rank tensors that rotation of the axes to give new unit vectors \( \mathbf{i}_1', \mathbf{i}_2' \) and \( \mathbf{i}_3' \) may be done such that

\[ \mathbf{K} = \mathbf{i}_1 \mathbf{i}_1' K_1 + \mathbf{i}_2 \mathbf{i}_2' K_2 + \mathbf{i}_3 \mathbf{i}_3' K_3 \]

2.2.47

and for a spherical particle of radius, \( a \), being an isotropic case, it represents a degenerate case in which the principal stresses in all directions are eigenvectors.

From Stokes' law one has \( K_1 = K_2 = K_3 = 6 \pi a \), whence

\[ \mathbf{K} = 6 \pi a \mathbf{I} \]
\[ \mathbf{F} = -6 \pi \mu a \mathbf{u} \]

which correctly indicates that the force on a particle is parallel to the velocity vector.

Eq. 2.2.44 has an analogue for the torque, \( \mathbf{T} \), required to maintain the steady rotation of an arbitrary body in an unbounded fluid at some angular velocity, \( \omega \). The no-slip condition for a rotating particle is
\[ \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \quad \text{on } S_p \quad 2.2.48 \]

and the torque is given by

\[ \mathbf{T} = \int_{S_p} \mathbf{r} \times (\mathbf{\Pi} \cdot dS) \quad 2.2.49 \]

Arguments similar to those used in developing the translation tensor may be applied to yield an expression relating the torque to the angular velocity via a positive-definite, symmetric tensor, the rotation tensor, \( \mathbf{\Omega} \).

\[ \mathbf{T} = -\mu \mathbf{\Omega} \cdot \mathbf{\omega} \quad 2.2.50 \]

These were Brenner's earliest considerations for particle motion. Now, however, we consider an arbitrary particle which exhibits anisotropy, that is, the particle motions are considered from a point which corresponds to the centre of hydrodynamic stress. For such a particle, whereas before the centre of gravity and the centre of hydrodynamic stress have been considered as coincident, they are not in the general case coincident and in such a case, measured from the centre of hydrodynamic stress, the particle makes a contribution to the force exerted due to its translational and rotational motions and to the torque from its translational and rotational motions. Thus, the force on the particle is a consequence of the translational and rotational contributions, \( \mathbf{F}_o \) and \( \mathbf{E}_o \) respectively.

\[ \mathbf{E}_o = \mathbf{F}_o + \mathbf{E}_o \quad 2.2.51 \]

where
In a similar manner the hydrodynamic torques due to the translational and rotational motions are given by

\[ \mathcal{T}_o = \mathcal{T}_o + \mathcal{T}_b \] \hspace{1cm} 2.2.54

where

\[ \mathcal{T}_o = \int_{s_p} r_o \times (dS \cdot \mathbf{n}_o) \] \hspace{1cm} 2.2.55
\[ \mathcal{T}_b = \int_{s_p} r_o \times (dS \cdot \mathbf{n}_o) \] \hspace{1cm} 2.2.56

For the translational motion applying the same analysis as in eq. 2.2.39 to eq. 2.2.50, but applying the boundary condition

\[ \nu = U_o + \omega x r \quad \text{on} \quad s_p \]
\[ \nu \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty \] \hspace{1cm} 2.2.57

one arrives at the expressions

\[ \mathcal{F}_o = -\mu \xi \cdot U_o \] \hspace{1cm} 2.2.58
\[ \mathcal{T}_o = -\mu \zeta \cdot U_o \] \hspace{1cm} 2.2.59

and for the rotational motion using the boundary conditions of eq. 2.2.57

\[ \mathcal{F}_o = -\mu \xi \cdot \omega \] \hspace{1cm} 2.2.60
\[ \mathcal{T}_b = -\mu \zeta \cdot \omega \] \hspace{1cm} 2.2.61
and it may be shown that

\[ \mathbf{D} = \mathbf{\tilde{C}} \]  \hspace{1cm} 2.2.62

where \( \mathbf{D} \) or \( \mathbf{\tilde{C}} \) are known as the coupling tensor.

Thus

\[ F_0 = -\mu \xi \cdot u_0 - \mu \xi \cdot \omega \]  \hspace{1cm} 2.2.63
\[ I_0 = -\mu \xi \cdot u_0 - \mu \xi \cdot \omega \]  \hspace{1cm} 2.2.64

Thus it can be seen that the force and torque on any body can be described accurately by eq. 2.2.63 and eq. 2.2.64 and in the particular case of a sphere \( \omega = 0 \) and \( \xi \) is dependent on \( r_0 \) and \( r_0 = 0 \) the problem reduces to that given in eq. 2.2.44.

Returning to practical solutions of problems involving one sphere, Dennis and Walker (30) obtained a semi-analytical solution solving equations by numerical methods for the steady flow past a sphere at low and moderate Reynolds numbers. They solved the simultaneous equations of stream function and vorticity in terms of the coordinates \((\xi, \Theta)\), where \( \xi = \log(r)/a \), \( a \) is the sphere radius, and \((r, \Theta)\) are polar coordinates in a plane through the axis of the sphere. They expanded the equations in series of Legendre functions of argument, \( z = \cos \Theta \) with functional coefficients in the variable \( \xi \). This gave two sets of second-order ordinary differential equations which were truncated and solved numerically.

\[ \nu_r = \frac{e^{-2\xi}}{\sin \Theta} \frac{\partial \gamma}{\partial \Theta}, \quad \nu_\Theta = \frac{e^{-2\xi}}{\sin \Theta} \frac{\partial \gamma}{\partial \xi} \]  \hspace{1cm} 2.2.65
The expression for vorticity is

\[ \epsilon \epsilon \zeta = \frac{\partial v_\theta}{\partial \epsilon} + v_\theta - \frac{\partial u_r}{\partial \theta} \]  \hspace{1cm} 2.2.66

The equations satisfied by \( \gamma \) and \( \zeta \) are

\[ \frac{\partial^2 \gamma}{\partial \epsilon^2} - \frac{\partial \gamma}{\partial \epsilon} + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \gamma}{\partial \theta} \right) + \sin \theta \epsilon^2 \zeta = 0 \]  \hspace{1cm} 2.2.67

and

\[ \frac{\partial^2 \zeta}{\partial \epsilon^2} + \frac{\partial \zeta}{\partial \epsilon} + \cot \theta \frac{\partial \zeta}{\partial \theta} + \frac{\partial^2 \zeta}{\partial \theta^2} \sin^2 \theta = \frac{(Re) \epsilon^2}{2} \left[ \frac{v_\theta}{\partial \epsilon} + \frac{v_\theta}{\partial \theta} + v_\theta \frac{\partial^2 \zeta}{\partial \theta^2} - v_\theta \cot \theta \right] \]  \hspace{1cm} 2.2.68

The boundary conditions to be satisfied are

\[ \gamma - \frac{\partial \gamma}{\partial \epsilon} = 0 \quad \text{at} \quad \epsilon = 0 \]

\[ \gamma = \frac{1}{2} \epsilon^2 \sin \theta \quad \text{as} \quad \epsilon \to \infty \]

Expansions for \( \gamma \) and \( \zeta \) are given as

\[ \gamma = e^{2n} \sum_{n=1}^{\infty} \int_{-1}^{1} P_n(t) \, dt \]  \hspace{1cm} 2.2.69

\[ \zeta = \sum_{n=1}^{\infty} \int_{0}^{\pi} \int_{-1}^{1} \left[ \frac{1}{2} \frac{v_\theta}{\partial \epsilon} \right] P_n^1(\zeta) \, d \zeta \]  \hspace{1cm} 2.2.70

where \( P_n(t) \) and \( P_n^1(\zeta) \) are, respectively, Legendre and first associated Legendre polynomials of order, \( n \). Over the range considered Dennis and Walker used as the drag coefficient

\[ C_D = \frac{D}{\pi \rho u_in \alpha^2} \]  \hspace{1cm} 2.2.71

They also calculated the separate contributions of friction and pressure drag whose coefficients they gave respectively as

\[ C_f = -\frac{4}{Re} \int_{0}^{\pi} \zeta(0, \theta) \sin^2 \theta \, d \theta \]  \hspace{1cm} 2.2.71
They found that their numerical values for drag on a sphere compared with the results of other workers up to the Reynolds number 0.1.

Bowen and Masliyah performed similar work to Dennis and Walker for a number of different axisymmetric particles, by truncating a series solution for the stream function obtaining results accurate to ±5%.

Ockendon and Evans made a numerical solution of the Navier-Stokes equations, without loss of the inertia terms, and the continuity equation for a sphere moving uniformly with velocity, \( U \), through an infinite incompressible fluid at low Reynolds number and obtained the value of the coefficient for \( 0(Re^2) \) in the expansion in terms of Reynolds number for the correction to Stokes drag. They used the method of matched asymptotic expansions with complex Fourier transforms yielding the value of the coefficient as 0.1879.

Verma, Pandey and Tripathi utilized a three-dimensional integral momentum equation for bodies of revolution to determine the boundary layer thickness, \( \delta \), round a sphere. Their calculated values vary by about 50% from experimental values. The integral momentum is given by

\[
F = 6 \pi \mu a U \left[ 1 + \frac{3}{8} \frac{Re}{5} + \frac{3}{40} \frac{Re^2 \log Re}{Re + 0.1679Re^2} \right]
\]
\[
\frac{\partial}{\partial x_1} \int_0^\delta u^2 \, dy - u_{\max} \frac{\partial}{\partial x_1} \int_0^\delta u \, dy + \frac{1}{n_k} \frac{\partial n_k}{\partial x_1} \left[ \int_0^\delta u^2 \, dy - u_{\max} \int_0^\delta u \, dy \right] = \delta u_{\max} \frac{\partial u_{\max}}{\partial x_1} - \frac{T_w g}{\rho} \quad 2.2.74
\]

where \( T_w \) is the wall shear stress and other variables are shown in Fig. 2.2.5. The expression for \( \delta \) is given as

\[
\delta = \frac{r_0 T_w g}{u' \rho} \left[ \frac{1 - 1.93 \sin \theta}{\sin \theta (0.571 - \sin \theta)(0.216 - 0.4167 \sin \theta)} \right] e^{-2.4 \sin \theta} f(\theta) \quad 2.2.75
\]

\[
f(\theta) = -0.5283 + 0.06 \cos \theta + 0.3112 \sin 2\theta - 0.0951 \cos \theta - 0.02495 \sin 4\theta + 0.0532 \cos 5\theta + 0.001027 \sin 6\theta - 0.00026 \cos 7\theta - 0.0000212 \sin 8\theta + 0.83
\]

Fig. 2.2.5 Flow past sphere due to Verma, Pandey & Tripathi.

Weinburger (95) considered the variational properties of steady fall in Stokes flow for a class of bodies whose centres of buoyancy and mass are not coincident. In particular he shows that the quasi-steady falling motion of a
particle converges to a steady one. Using expressions for force and torque given in eq. 2.2.63 and eq. 2.2.64 and the boundary conditions of eq. 2.2.57, the quasi-steady fall of the particle obeys the equations of motion

\[
\begin{align*}
\mu \left( \xi \dot{u} + \zeta \omega \right) &= mg \\
\mu \left( \zeta \dot{u} + \tilde{u} \omega \right) &= mg \times r_c \\
\frac{dg}{dt} &= \omega \times g
\end{align*}
\]

2.2.76

Solving the first two equations and substituting into the third of eq. 2.2.76 gives

\[
\mu \frac{dg}{dt} = -mg \left( \omega \cdot \zeta_{E}^{-1} \zeta \right)^{-1} \left( g \times r_c - \zeta_{E}^{-1} g \right)
\]

2.2.77

Taking a particular downward direction \( \xi_{o} \), there will be a steady fall in the direction \( \xi_{o} \) provided that

\[
r_c = \alpha g_o
\]

2.2.78

where \( r_c \) is the position of the centre of mass. The object is to find a value of \( \alpha \) to make motion stable. If \( \lambda = \xi_{o} \) where

\[
\begin{align*}
\lambda_c &= \frac{-1}{m |\tilde{g}|^2} g \times \int_{SF} \tilde{f} \times r \, dS + \alpha g \\
\int_{SF} \tilde{f} \, dS &= mg
\end{align*}
\]

Thus Weinburger shows that

\[
\frac{-m}{\mu} \left[ \omega \cdot \zeta_{E}^{-1} \zeta \right]^{-1} \zeta_{E} \zeta_{E}^{-1} g_o = \lambda g_o
\]

2.2.79
and that \( g \) converges exponentially to \( g_0 \) if \( \alpha |g| > \frac{\delta}{\gamma} \)

where \( \delta \) and \( \gamma \) are constants involved in the rearrangement of eq. 2.2.77 - 2.2.79 noting that \( |g|^2 = |g_0|^2 \). For sufficiently large \( \alpha \) (that is, when the centre of mass is sufficiently low) the steady motion with \( g = g_0 \) is the limit of falling motions except for the unstable motion with \( g = -g_0 \).
2.3 The Motion of Two Spheres

The first analytical solution of the Navier-Stokes equation for two spheres was due to Stimson and Jeffery. They considered two spheres, equal or unequal in size, moving with small constant velocities parallel to their line of centres. Their approach was to solve the equations in cylindrical polar coordinates in terms of the stream function. They obtained a correction factor, \( k \), to Stokes' law where \( k \) is the ratio of the force necessary to maintain the motion of either sphere in the presence of the other to the force which would be necessary to maintain its motion with the same velocity if the other sphere were at an infinite distance.

\[
F = 6\pi \mu a \, u \, k \tag{2.3.1}
\]

where

\[
k = \frac{2}{3} \sinh \alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \left[ 1 + \frac{4 \sinh^2 \alpha}{\cosh^2 \alpha} \right] \tag{2.3.2}
\]

The factor, \( k \), is obtained by transforming the coordinates into bipolar cylindrical coordinates, \( \xi, \eta \), where

\[
\xi = \log \frac{x + i(y + a)}{x + i(y - a)}
\]

\[
x = \frac{a_1 \sinh \eta}{\cosh \frac{3}{2} - \cos \eta}, \quad y = \frac{a_1 \sinh \xi}{\cosh \frac{3}{2} - \cos \eta}
\]

If the spheres have radii, \( a_1, a_2 \) respectively, then
where $\alpha$ and $\beta$ are particular values of $\xi$. The solution is obtained in the form of a Legendre polynomial giving rise to the series solution for $k$ given in eq. 2.3.2.

Table 2.3.1 shows typical values of $k$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Centre distance</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.128</td>
<td>0.663</td>
</tr>
<tr>
<td>1.0</td>
<td>1.543</td>
<td>0.702</td>
</tr>
<tr>
<td>1.5</td>
<td>2.352</td>
<td>0.766</td>
</tr>
<tr>
<td>2.0</td>
<td>3.762</td>
<td>0.836</td>
</tr>
<tr>
<td>2.5</td>
<td>6.132</td>
<td>0.892</td>
</tr>
<tr>
<td>3.0</td>
<td>10.068</td>
<td>0.931</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\infty$</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 2.3.1 Sphere size, distance to diameter ratio, and correction factor for two spheres falling parallel to their line of centres, after Stimson and Jeffery.

Happel and Pfeffer(45) carried out experiments at Reynolds numbers of 0.25 and 0.5 obtaining results within 2% and 3% of the values predicted by Stimson's and Jeffery's solution at a Reynolds number of zero.

Several workers have studied either theoretically or experimentally the motion of two spheres in a viscous medium at Reynolds numbers within the Stokes regime. Goldman, Cox and Brenner(41) initially studied the solution of equations of motion (Navier-Stokes and continuity) for two spheres of equal size separated by a distance, $2h$, between their centres.
perpendicular to the direction of gravity. Each has a translational velocity, \( U \), and an angular velocity, \( \omega \). Lack of translational motion in the z direction is only valid in the Stokes regime.

![Figure 2.3.1 Two spheres falling side by side.](image)

The boundary conditions appropriate to this problem are

\[
\begin{align*}
V_I &= i_u U + (i_y \times r_I) \omega \\
V_{II} &= i_u U - (i_y \times r_{II}) \omega \\
V^t &= 0 \quad as \ |r| \to \infty \quad in \ fluid
\end{align*}
\]

As indicated in Brenner's earlier papers \(^{10, 11}\) the velocities are separated into translation contributions due to sphere I, \( V^t_I \), due to sphere II, \( V^t_{II} \) and the fluid \( V^t \) and corresponding rotational components, \( V^r_I \), \( V^r_{II} \), \( V^r \). Thus

\[
\begin{align*}
V^t_I &= i_u U \quad sphere \ I \\
V^t_{II} &= i_u U \quad sphere \ II \\
V^t &= 0 \quad as \ |r| \to \infty \quad in \ fluid \\
V^r_I &= (i_y \times r_I) \omega \quad sphere \ I \\
V^r_{II} &= (i_y \times r_{II}) \omega \quad sphere \ II \\
V^r &= 0 \quad as \ |r| \to \infty \quad in \ fluid
\end{align*}
\]
The equation of motion can then be solved in cylindrical polar coordinates \((\rho, \phi, z)\) for pressure and velocity.

\[
\frac{\partial}{\partial \rho}\left(\rho \frac{p_t}{\mu}\right) = \left(\nabla^2 - \frac{1}{\rho^2}\right)v_t^t - \frac{2}{\rho} \frac{\partial v_z^t}{\partial \phi} \\
1 \frac{\partial}{\partial \phi}\left(\rho v_t^t\right) = \left(\nabla^2 - \frac{1}{\rho^2}\right)v_z^t + \frac{2}{\rho} \frac{\partial v_z^t}{\partial \phi} \\
\frac{\partial}{\partial z}\left(\rho v_t^t\right) = \nabla^2 v_z^t
\] \hspace{1cm} 2.3.8

\[
\frac{\partial v_t^t}{\partial \rho} + \frac{1}{\rho} v_t^t + \frac{1}{\rho^2} \frac{\partial v_z^t}{\partial \phi} + \frac{\partial v_z^t}{\partial z} = 0
\] \hspace{1cm} 2.3.9

where \(\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\)

with the boundary conditions

\[
\begin{align*}
V_{I, \rho}^t &= u \cos \phi \\
V_{I, \phi}^t &= -u \sin \phi \\
V_{I, z}^t &= 0
\end{align*}
\]

on sphere I \hspace{1cm} 2.3.10

\[
\begin{align*}
V_{II, \rho}^t &= u \cos \phi \\
V_{II, \phi}^t &= -u \sin \phi \\
V_{II, z}^t &= 0
\end{align*}
\]

on sphere II \hspace{1cm} 2.3.11

They derive a strict mathematical solution, involving changes of variable to ease solution, the use of Legendre polynomials with a resulting series solution. An approximate solution obtained, as a series by the method of reflections determining the non-dimensional force and torque acting on sphere I due to two translating spheres. The approximate forces and torques are at the most 1.41% and 9.5% respectively different from the exact solutions. The approximate solutions are
\[-F_1^{tx} \approx 1 - \frac{3}{4} \left( \frac{a}{2h} \right)^2 + \frac{9}{16} \left( \frac{a}{2h} \right)^2 - \frac{561}{64} \left( \frac{a}{2h} \right)^3 + \frac{465}{256} \left( \frac{a}{2h} \right)^4 - \frac{15813}{768} \left( \frac{a}{2h} \right)^5 + \frac{2}{1+ \left( \frac{a}{2h} \right)^6} \]

\[-T_1^{tx} \approx \frac{3}{4} \left( \frac{a}{2h} \right)^2 \left[ 1 - \frac{3}{4} \left( \frac{a}{2h} \right)^2 + \frac{9}{16} \left( \frac{a}{2h} \right)^2 - \frac{561}{64} \left( \frac{a}{2h} \right)^3 + \frac{465}{256} \left( \frac{a}{2h} \right)^4 \right] - \frac{2}{1+ \left( \frac{a}{2h} \right)^6} \]

assuming that the higher order terms are expressible as a geometric series. Similar solutions are obtained for the rotational motion of the spheres solving eq. 2.3.8 and 2.3.9 with the boundary conditions:

\[
\begin{align*}
\dot{V}_{r}^{I} &= \omega (z-h) \cos \phi \\
\dot{V}_{\theta}^{I} &= -\omega (z-h) \sin \phi \\
\dot{V}_{z}^{I} &= -\omega \rho \cos \phi \\
\end{align*}
\]

on sphere I \hspace{1cm} 2.3.14

\[
\begin{align*}
\dot{V}_{r}^{II} &= \omega (z+h) \cos \phi \\
\dot{V}_{\theta}^{II} &= \omega (z+h) \sin \phi \\
\dot{V}_{z}^{II} &= \omega \rho \cos \phi \\
\end{align*}
\]

on sphere II \hspace{1cm} 2.3.15

\[
\begin{align*}
-F_1^{rx} &\approx \left( \frac{a}{2h} \right)^2 \left[ 1 - \frac{3}{4} \left( \frac{a}{2h} \right)^2 \right] \\
-T_1^{rx} &\approx 1 - \frac{1}{2} \left( \frac{a}{2h} \right)^3 + \frac{5}{4} \left( \frac{a}{2h} \right)^4
\end{align*}
\]

Thus the total nondimensional force and torque exerted on each sphere are found:

\[
\begin{align*}
F_1^* &= F_1^{tx} + F_1^{rx} \omega_1^* \\
T_1^* &= T_1^{tx} + T_1^{rx} \omega_1^*
\end{align*}
\]

where \( F^* = \frac{F}{6 \pi \mu a} \), \( \omega^* = \frac{\omega}{u} \). For two spheres settling at
an arbitrary angle of attack, $\Theta$, as in Fig. 2.3.2 using the subscripts, $\|\text{ and } \perp$ for parallel and perpendicular to the line of centres velocities are calculated

$$U_\| = \frac{F_1^g}{G\mu a |F_1^g|} = \frac{F_1^g \sin \Theta}{G\mu a |F_1^g|} \tag{2.3.20}$$

$$U_\perp = \frac{F_1^g}{G\mu a |F_1^g|} = \frac{F_1^g \cos \Theta}{G\mu a |F_1^g|} \tag{2.3.21}$$

where $F_1^g$ and $F_1^x$ are Stimson and Jeffery correction factors to the Stokes' law force on each sphere, and $|F_1^g|$ and $|F_1^x|$ are their values recalculated by Goldman, Cox and Brenner. The effect of $F_1^g$ is to make each sphere translate with velocity $U_\|$ and rotate about an axis through its centre with angular velocity given by

$$|\omega| = \frac{U_\|}{a} |\omega^*| \tag{2.3.22}$$

The combined effect of the two translational motions, $U_\|$ and $U_\perp$, is to cause spheres to drift downwards and to the left with a velocity vector, $\mathbf{U}$, where the horizontal and vertical components are

$$U_\| = U_\| \cos \Theta - U_\perp \sin \Theta \tag{2.3.23}$$

$$U_\perp = U_\| \sin \Theta + U_\perp \cos \Theta \tag{2.3.24}$$

Steinberger, Pruppacher and Neiburger\(^\text{(87)}\) conducted experimental studies on spheres falling in a direction parallel to their line of centres for Reynolds numbers in the range 0.05 to 0.215. They found that the closer the
spheres were, the faster each settled. The drag coefficient for the upper sphere was smaller than that for the lower sphere, concluding that the validity of the Stokes approximation is smaller than commonly considered. A plot of the dimensionless velocity of each sphere, $\frac{U_1}{U_5}$, $\frac{U_2}{U_5}$ vs. the dimensionless distance between centres, $2\frac{h}{\delta}$ is shown in Fig. 2.3.3.

Fig. 2.3.2 Two spheres settling at an angle, $\theta$.

Fig. 2.3.3 Variation of velocity with distance between spheres.
Davis(29) provides a series solution with computation of coefficients for two unequal spheres translating and rotating in a direction perpendicular to their line of centres. The solution is essentially the same as that of Goldman, Cox and Brenner(41).

Majumder and O'Neill(68) made theoretical studies of the Stokes resistance of two equal spheres falling in contact in a linear shear field. They define the velocity, \( \mathbf{U} \), as the sum of an approach velocity of the undisturbed flow, \( \mathbf{U}_0 \) and a function of the velocity gradient, \( \nabla \mathbf{u} \)

\[
\mathbf{U} = \mathbf{U}_0 + \mathbf{r} \cdot \nabla \mathbf{u}
\]

The presence of the spheres disturbs the fluid velocity field such that it is now

\[
\mathbf{v} = \mathbf{U} + \mathbf{v}
\]

where \( \mathbf{v} \) is the velocity field for the particles which satisfy the Navier-Stokes and continuity equations.

\[
\nabla \mathbf{p} = \mu \nabla^2 \mathbf{v}
\]

\[
\nabla \cdot \mathbf{v} = 0
\]

with the boundary conditions, \( \mathbf{v} = -\mathbf{U} \) on the sphere surfaces and \( |\mathbf{v}| \to 0 \) as \( r^2 + z^2 \to \infty \) where \( r \) and \( z \) are cylindrical polar coordinates. In spherical polar coordinates \( (\rho, \chi, \Theta) \) the spherical resolutes of \( \mathbf{v} \) are

\[
\nabla \rho = \sum_{m=0}^{\infty} \nabla \rho \cos m\Theta + \sum_{m=1}^{\infty} \nabla \rho \sin m\Theta
\]
\[
V_\rho = \sum_{m=0}^{\infty} \hat{V}_\rho^m \cos m\theta + \sum_{m=1}^{\infty} \hat{V}_\rho^m \sin m\theta
\]
\[
V_\theta = \sum_{m=0}^{\infty} \hat{V}_\theta^m \cos m\theta + \sum_{m=1}^{\infty} \hat{V}_\theta^m \sin m\theta
\]
\[
F_\mu = \sum_{m=0}^{\infty} \hat{Q}_\mu^m \cos m\theta + \sum_{m=1}^{\infty} \hat{Q}_\mu^m \sin m\theta
\]
\[
\text{where } V^m, \ldots, Q^m \text{ are functions of } r \text{ and } \chi \text{ only. Thus the force and torque in Cartesian components become}
\]
\[
F_{11} = 2\pi \mu a_{\rho}^2 \int_0^\pi \left[ \frac{\partial}{\partial \rho} \left( V_\rho \sin \chi + V_\chi \cos \chi - V_\rho' \right) - \dot{Q} \sin \chi \right] \sin \chi \, d\chi
\]
\[
F_{13} = 2\pi \mu a_{\rho}^2 \int_0^\pi \left[ \frac{\partial}{\partial \rho} \left( V_\rho \cos \chi - V_\chi \sin \chi \right) - \dot{Q} \cos \chi \right] \sin \chi \, d\chi
\]
\[
T_{12} = \pi \mu a_{\rho}^2 \int_0^\pi \left( \frac{\partial}{\partial \rho} + \frac{1}{\rho} \right) \left( V_\chi - V_\rho \cos \chi \right) \sin \chi \, d\chi
\]
\[
T_{13} = \pi \mu a_{\rho}^2 \int_0^\pi \frac{\partial V_\rho}{\partial \rho} \sin^3 \chi \, d\chi
\]
\[
\text{where the integration is carried out over the whole surfaces of the spheres, and } a_{\rho} \text{ is the radius of one of the spheres. Because of the linear nature of eq. 2.3.27, } v \text{ may be written}
\]
\[
v = v^o + v^f + v^2 + v^3 + v^A + v^B + v^C
\]
\[
\text{and}
\]
\[
p = p^o + p^f + p^2 + p^3 + p^A + p^B + p^C
\]
\[
\text{and the problem can be solved as a set of simpler sub-problems. The velocity field, } v^o \text{ corresponds to a uniform translation of the spheres each with a velocity, } -u^0, \text{ and is found by the superposition of solutions for spheres in contact falling parallel and perpendicular to their line of}
\]
centres. For the axisymmetric velocity field, $v^o$

$$u^o = \frac{1}{2} \Sigma \sin \alpha \cos \alpha \cos \beta \left( r^i_r - 2z_i_z \right) - \frac{1}{2} \Sigma \sin \alpha \sin \beta r^i_{i_0}$$  \hspace{1cm} 2.3.39

where $\Sigma$ is the rate of strain dyadic for undisturbed flow and $\alpha$ and $\beta$ represent the changing orientations of two sets of Cartesian axes, one fixed in the fluid and the other moving with the spheres. The velocity-pressure pairs $(v^2, p^2)$ and $(v^3, p^3)$ give no contribution to the mechanical action of fluid on the spheres since terms, $\sin m\theta$ and $\cos m\theta$ in eq. 2.3.29 - 2.3.32 only take values $m = 0, m = 1$. The only asymmetric field pairs which contribute to the forces and torques are $(v^1, p^1), (v^A, p^A)$ and $(v^B, p^B)$. Dividing $v^1$ into polar coordinates $(r, \theta, z)$

$$a^o_{i_1} = 2\mu \sqrt{\alpha} \sin^2 \alpha \cos \beta \cos \theta$$  \hspace{1cm} 2.3.40

$$v^1_r = \frac{1}{2} \sqrt{\alpha} \sin^2 \alpha \cos \beta \left[ rQ + \frac{1}{2}(\chi + \gamma) \right] \cos \theta$$  \hspace{1cm} 2.3.41

$$v^1_\theta = \frac{1}{2} \sqrt{\alpha} \sin^2 \alpha \cos \beta (\chi - \gamma) \sin \theta$$  \hspace{1cm} 2.3.42

$$v^1_z = \sqrt{\alpha} \cos \beta (zQ - \phi) \cos \theta$$  \hspace{1cm} 2.3.43

where $\chi, \gamma, \phi$ and $Q$ are functions of $r$ and $z$ only satisfying $L^2_o \chi = L^2_r \phi = L^2_\theta Q = L^2_z \chi = 0$ where

$$L^2_m = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}$$

Solutions for $(v^A, p^A)$ and $(v^B, p^B)$ are found in a similar way as that for $(v^1, p^1)$ except the functions of $\alpha$ and $\beta$ in eq. 2.3.40 - 2.3.43 are replaced by $\sqrt{\alpha} \cos \beta$ for $(v^A, p^A)$ and $\sqrt{\alpha} \sin \beta$ for $(v^B, p^B)$. 
Batchelor and Green (5) calculated theoretically the hydrodynamic interaction of two small spheres of different sizes in a linear flow field. They considered two rigid spheres of radii, \( a \) and \( b \), on which no external force or couple acts. The flow field in the absence of the two spheres has velocity, \( \mathbf{U}(x,t) \), a linear function of position, which can be characterized instantaneously by a rate of strain tensor

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)
\]

and a rigid-body rotation with angular velocity

\[
\omega = \frac{1}{2} \nabla \times \mathbf{U}
\]

Again the Navier-Stokes and continuity equations are solved with boundary condition

\[
\mathbf{U}_i(x) \approx \varepsilon_{ij} x_j + \varepsilon_{ijk} \omega_j x_k \quad \text{as} \quad |x| \to \infty
\]

and no-slip condition \(|x - x_0| = a\) and \(|x - x_0 - \bar{r}| = b\) where \( x_0 \) is the instantaneous position of the centre of the sphere with radius, \( a \), and \( x_0 + \bar{r} \) is that of the sphere of radius, \( b \). The parameters to be determined are \( \mathbf{V} \), the translational velocity of the sphere of radius, \( b \), to that of the other sphere, the angular velocities of the spheres of radii, \( a \) and \( b \), denoted by \( \Gamma' \) and \( \Gamma'' \) respectively and also their respective force dipoles, \( S_{ij} \) and \( S''_{ij} \). The force dipole, \( S_{ij} \), for a rigid body with surface, \( \mathbf{A}_0 \), and unit outward normal, \( \mathbf{n} \), is defined as
\[ S_{ij} = \int\left(\sigma_{ik} x_j - \frac{1}{3} \delta_{ij} \sigma_{kk} x_l\right) \eta_k \, dA \] \tag{2.3.47}

where \( \sigma_{ij} \) is the hydrodynamic stress at \( x \). The quantities \( V, \Gamma', \Gamma'', S_{ij}', S_{ij}'' \) are functions only of \( r, a, b, E_{ij} \) and \( \omega \). The only instantaneous consequence of the rotation of the fluid at infinity is to superimpose a uniform angular velocity, \( \omega \), on the whole system. Consequently, relative to axes rotating with angular velocity, \( \omega \), the equations and boundary conditions to be solved are linear and homogeneous in fluid velocity, pressure and \( E_{ij} \). Thus each of the five quantities \( V - \omega x, \Gamma' - \omega, \Gamma'' - \omega, S_{ij}', S_{ij}'' \) must be linear and homogeneous in \( E_{ij} \). Thus the polar vector, \( V \), may be written

\[ V_i(x) = \epsilon_{ijk} \omega_j r_k + r_j E_{ij} - \left[ A \frac{r_i r_j}{r^2} + B(\delta_{ij} - \frac{r_i r_j}{r^2})\right] \eta_k E_{jk} \] \tag{2.3.48}

where \( r = |x| \) and \( A \) and \( B \) are functions of \( \frac{r}{a} \) and \( \frac{b}{a} \) respectively.

\[ \Gamma_L' = \omega_i + C' \epsilon_{ijk} E_{kl} \frac{r_i r_j}{r^2} \] \tag{2.3.49}

\[ \Gamma_L'' = \omega_i + C'' \epsilon_{ijk} E_{kl} \frac{r_i r_j}{r^2} \] \tag{2.3.50}

where \( C' \) and \( C'' \) are functions of \( \frac{r}{a} \) and \( \frac{b}{a} \).

\[ S_{ij}' = \frac{2}{3} r^2 \varphi_{\alpha't} \left[ E_{ij} (1 + K') + E_{kl} \left( r_i r_k \delta_{jl} + r_j r_l \delta_{ik} - r_i r_k \frac{2}{3} \delta_{ij} \right) L' \right] + E_{kl} \frac{r_i r_j}{r^2} \left( \frac{r_i r_j}{r^2} - \frac{1}{3} \delta_{ij} \right) M' \] \tag{2.3.51}

where \( K', L', M' \) are functions of \( \frac{r}{a} \) and \( \frac{b}{a} \). A similar expression exists for \( S_{ij}'' \). Their procedure is now to solve the equations of motion through various solutions of eq. 2.3.48 -
2.3.51 For cases when the spheres are far apart, close together, in a simple shearing motion, equal spheres in axisymmetric flow, in the presence of a plane wall, $b = \frac{1}{a}$ and $b \to 0$.

Wacholder and Sather (95) obtained the hydrodynamic forces and couples acting on two spheres in slow motion as functions of their relative configuration. They solved the equations of motion and plotted the components of relative velocity against the relative trajectories for different ratios of the two particle sizes $a = \frac{a_1}{a_2}$. Using spherical polar coordinates $(r, \theta, \phi)$, the nondimensional velocity of sphere 1 relative to sphere 2 is given by

$$\frac{u_{12}}{u_{io}} = \frac{u_1 - u_2}{u_{io}} = \frac{1}{r} u_{12} (r, a_1, I) \cos \theta + \frac{r}{a_2} \nu_{12} (r, a_1, I) \sin \theta$$  \hspace{1cm} 2.3.52

where the centre of sphere 1 moves in the plane of $i_r$ and $i_\theta$ with velocity $u_{io}$. $I$ is the net density $(\rho_1 - \rho)$. Denoting a dimensionless time by

$$\tau = \frac{a_1}{u_{io}} t$$  \hspace{1cm} 2.3.53

the trajectory of particle 1 relative to particle 2 is given by the equations

$$\begin{align*}
\frac{dr}{d\tau} &= -u_{12} (r, a_1, I) \cos \theta \\
\frac{d\theta}{d\tau} &= \frac{r}{\tau} \nu_{12} (r, a_1, I) \sin \theta
\end{align*}$$  \hspace{1cm} 2.3.54
Typical results at $a = 0.5$ are given in Fig. 2.3.4

![Graph showing relative velocity functions against relative trajectory at $a = 0.5$, after Wacholder and Sather.](image)

Fig. 2.3.4 Relative velocity functions against relative trajectory at $a = 0.5$, after Wacholder and Sather.

When two spheres are almost in contact solutions of the equations of motion already quoted become inaccurate. Cox(26) developed a lubrication theory for two spheres almost in contact. This involved solution of the usual
equations of motion. The surfaces, \( W \) and \( W' \), of the two spheres are shown in Fig. 2.3.5, where \( a \) is the distance between the two surfaces measured along the \( x_3 \) axis of a Cartesian system of axes \( (x_1, x_2, x_3) \), with origin \( 0 \) on \( W \). Surface \( W \) may be written, for small values of \( r = (x_1^2 - x_2^2)^{\frac{1}{2}} \)

\[
x_3 = -\left(\frac{x_1^2}{2R_1}\right) - \left(\frac{x_2^2}{2R_2}\right) + O(r^3)
\]

where \( R_1 \) and \( R_2 \) are the principal radii of curvature of the surface \( W \) at \( 0 \).

Fig. 2.3.5 The surfaces \( W, W' \), after Cox.

If the surface, \( W \), is symmetric, the value of \( x_3 \) remains unchanged if \( x_1 \) and \( x_2 \) are replaced by \(-x_1\) and \(-x_2\) respectively. In that case
If a new coordinate system is chosen \((x_1^*, x_2^*, x_3^*)\) with origin 0' on \(W'\) and the \(x_3\) axis coincident with the \(x_3^*\) axis and \(x_1^*, x_2^*\) are chosen to lie in the directions of the principal curvatures of surface \(W'\), the surface \(W'\) may be written, for small values of \(r^* = \left( x_1^{*2} - x_2^{*2} \right)^{\frac{1}{2}} \):

\[
x_3^* = \left( \frac{x_1^{*2}}{2S_1} \right) + \left( \frac{x_2^{*2}}{2S_2} \right) + O(r^{*3})
\]

where \(S_1\) and \(S_2\) are the principal radii of curvature at the surface \(W'\) at 0'. Again if surface \(W'\) is symmetric, replacing \(x_1^*\) and \(x_2^*\) by \(-x_1^*\) and \(-x_2^*\) respectively:

\[
x_3^* = \left( \frac{x_1^{*2}}{2S_1} \right) + \left( \frac{x_2^{*2}}{2S_2} \right) + O(r^{*4})
\]

If \(\phi\) is the angle between the \(x_1\) and \(x_1^*\) axes (in the positive sense in this direction):

\[
\begin{align*}
x_1^* &= x_1 \cos \phi + x_2 \sin \phi \\
x_2^* &= -x_1 \sin \phi + x_2 \cos \phi \\
x_3^* &= x_3 - a
\end{align*}
\]

then surface \(W'\) may be rewritten as:

\[
x_3 = a + x_1^2 \left( \frac{\cos^2 \phi + \sin^2 \phi}{2S_1} \right) + x_1 x_2 \left( \frac{1}{S_1} - \frac{1}{S_2} \right) \sin \phi \cos \phi \\
+ x_2^2 \left( \frac{\sin^2 \phi}{2S_1} + \frac{\cos^2 \phi}{2S_2} \right) + O(r^{*3})
\]
The velocity field \( \mathbf{v} = (v_1, v_2, v_3) \) is then written in terms of the Cartesian system \((x_1, x_2, x_3)\) and the boundary conditions are

\[
\begin{align*}
\mathbf{v} &= \mathbf{U} + \mathbf{\omega} \times \mathbf{r} & \text{on } W \\
\mathbf{r} &= (x_1, x_2, -\frac{x_1^2}{2\varepsilon_1} - \frac{x_2^2}{2\varepsilon_2} + O(r^3)) & 2.3.62
\end{align*}
\]

where \( x_3 \) is given by eq. 2.3.60. The velocity field \((\mathbf{v}, p)\) is expanded in terms of the gap separation, \(a\), in two regions, an outer region using the coordinates \((x_1, x_2, x_3)\) as independent variables and \((\mathbf{v}, p)\) as dependent variables and an inner region using \((\mathbf{\tilde{x}}_1, \mathbf{\tilde{x}}_2, \mathbf{\tilde{x}}_3)\) as independent variables and \((\mathbf{\tilde{v}}, \tilde{p})\) as dependent variables where

\[
\begin{align*}
\mathbf{\tilde{x}}_1 &= \frac{x_1}{a^2}, & \mathbf{\tilde{x}}_2 &= \frac{x_2}{a^2}, & \mathbf{\tilde{x}}_3 &= \frac{x_3}{a} & 2.3.65
\end{align*}
\]

\[
\begin{align*}
\mathbf{\tilde{v}}_1 &= a^{1-k} \mathbf{v}_1, & \mathbf{\tilde{v}}_2 &= a^{1-k} \mathbf{v}_2, & \mathbf{\tilde{v}}_3 &= a^{-k} \mathbf{v}_3, & \tilde{p} &= a^{k} \tilde{p} & 2.3.66
\end{align*}
\]

where \( k \) is a constant, as yet undefined. Making use of eq. 2.3.65, 2.3.66, the equations of motion may be converted to inner variables where

\[
\begin{align*}
\mathbf{\tilde{v}} &= \mathbf{\tilde{v}}_0 + a \mathbf{\tilde{v}}_1 + \ldots \ldots \\
\tilde{p} &= \tilde{p}_0 + a \tilde{p}_1 + \ldots \ldots
\end{align*}
\]

where

\[
\begin{align*}
\left( \mathbf{\tilde{v}}_0, \tilde{p}_0 \right) & \text{ satisfies } \mu \nabla^2 \mathbf{\tilde{v}}_0 = \nabla \tilde{p}_0 & 2.3.68 \\
\left( \mathbf{\tilde{v}}_1, \tilde{p}_1 \right) & \text{ satisfies } \mu \nabla^2 \mathbf{\tilde{v}}_1 = \nabla \tilde{p}_1 & 2.3.69
\end{align*}
\]
Relative to the inner variables of eq. 2.3.65, eq. 2.3.55 takes the form

$$\tilde{x}_3 = - \left( \frac{\tilde{x}_1}{2R_1} \right) - \left( \frac{\tilde{x}_2}{2R_2} \right) + O\left( a^{1/2} \right)$$  \hspace{1cm} 2.3.70

for the wall $W$, whilst eq. 2.3.60 for the wall $W'$ takes the form

$$\tilde{x}_3 = 1 + \tilde{x}_1^2 \left( \frac{\cos^2 \phi + \sin^2 \phi}{2S_1} \right) + \tilde{x}_1 \tilde{x}_2 \left( \frac{1}{S_1} - \frac{1}{S_2} \right) \sin \phi \cos \phi$$

$$+ \tilde{x}_2^2 \left( \frac{\sin^2 \phi + \cos^2 \phi}{2S_2} \right) + O\left( a^{1/2} \right)$$  \hspace{1cm} 2.3.71

In order to solve eq. 2.3.68 with boundary conditions on $W$ and $W'$ it is convenient to change variables to $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, where

$$\tilde{x}_3 = \frac{\tilde{x}_1}{2R_1} + \left( \frac{\tilde{x}_1}{2R_1} \right) + O\left( a^{1/2} \right)$$  \hspace{1cm} 2.3.72

so that the wall $W$ becomes

$$\tilde{x}_3 = O\left( a^{1/2} \right)$$  \hspace{1cm} 2.3.73

and wall $W'$ becomes

$$\tilde{x}_3 = h(\tilde{x}_1, \tilde{x}_2) + O\left( a^{1/2} \right)$$  \hspace{1cm} 2.3.74

where the function $h(\tilde{x}_1, \tilde{x}_2)$ is defined as

$$h(\tilde{x}_1, \tilde{x}_2) = 1 + \tilde{x}_1^2 \left( \frac{1}{2R_1} + \frac{\cos^2 \phi + \sin^2 \phi}{2S_1} \right) + \tilde{x}_1 \tilde{x}_2 \left( \frac{1}{S_1} - \frac{1}{S_2} \right) \sin \phi \cos \phi$$

$$+ \tilde{x}_2^2 \left( \frac{1}{2R_2} + \frac{\sin^2 \phi + \cos^2 \phi}{2S_2} \right)$$  \hspace{1cm} 2.3.75

With these new variables eq. 2.3.68 for $(\tilde{\phi}, \tilde{\rho})$ takes the
form

\[
\begin{align*}
\mu \frac{\partial^2 (\tilde{\nu}_0)}{\partial x_2^2} - \frac{\partial \tilde{\nu}_0}{\partial x_2} - \frac{\tilde{\omega}_1}{\tilde{\omega}} \frac{\partial \tilde{\nu}_0}{\partial x_3} &= 0 \\
\mu \frac{\partial^2 (\tilde{\nu}_0)}{\partial x_3^2} - \frac{\partial \tilde{\nu}_0}{\partial x_3} - \frac{\tilde{\omega}_2}{\tilde{\omega}} \frac{\partial \tilde{\nu}_0}{\partial x_1} &= 0 \\
\frac{\partial \tilde{\nu}_0}{\partial x_1} + \frac{\partial \tilde{\nu}_0}{\partial x_2} + \frac{\partial \tilde{\nu}_0}{\partial x_3} + \frac{\tilde{\omega}_1}{\tilde{\omega}} \frac{\partial \tilde{\nu}_0}{\partial x_3} + \frac{\tilde{\omega}_2}{\tilde{\omega}} \frac{\partial \tilde{\nu}_0}{\partial x_1} &= 0
\end{align*}
\]

2.3.76

The boundary conditions eq. 2.3.61 and eq. 2.3.63 may be written in terms of the inner variables for \( \tilde{\nu}_0 \)

\[
\begin{align*}
(\tilde{\nu}_0)_1 &= \alpha_k \left[ a^2 U_1 - a \omega_3 \tilde{x}_2 - a^3 \omega_2 \left( \frac{\tilde{x}_1^2}{2 \tilde{x}_1^2} \right) + O\left( \alpha^2 \right) \right] \\
(\tilde{\nu}_0)_2 &= \alpha_k \left[ a^2 U_2 - a \omega_3 \tilde{x}_1 - a^3 \omega_1 \left( \frac{\tilde{x}_1^2}{2 \tilde{x}_1^2} \right) + O\left( \alpha^2 \right) \right] \\
(\tilde{\nu}_0)_3 &= \alpha_k \left[ U_3 + a^2 \left( \omega_1 \tilde{x}_2 - \omega_2 \tilde{x}_1 \right) \right]
\end{align*}
\]

2.3.77

for \( W \) and a similar set exists for \( W' \). Since the creeping motion equations are linear and since \( (\tilde{\nu}, p) \) and hence \( (\tilde{\nu}_0, \tilde{p}_0) \) depend linearly on \( U, U', \omega \) and \( \omega' \), it follows that

\[
\tilde{\nu}_0 = \tilde{\nu}_0 + \tilde{\nu}_b + \tilde{\nu}_c , \quad \tilde{p}_0 = \tilde{p}_0 + \tilde{p}_b + \tilde{p}_c
\]

2.3.78

where the flow fields \((\tilde{\nu}_0, \tilde{p}_0), (\tilde{\nu}_b, \tilde{p}_b), (\tilde{\nu}_c, \tilde{p}_c)\) each individually satisfy eq. 2.3.68 these being the resulting flows from \( U_3, U'_3 \) and \( U_1, U_2, \omega_1, \omega_2, U'_1, U'_2, \omega'_1, \omega'_2 \) and \( \omega_3, \omega'_3 \) respectively. Thus the boundary conditions to be satisfied
69

are, for \((\tilde{\nu}_a, \tilde{p}_a)\)

\[
\begin{align*}
(\tilde{\nu}_a)_1 &= (\tilde{\nu}_a)_2 = 0, \quad (\tilde{\nu}_a)_3 = a^{-k}u_3 & \text{on } W \\
(\tilde{\nu}_a)_1 &= (\tilde{\nu}_a)_2 = 0, \quad (\tilde{\nu}_a)_3 = a^{-k}u_3' & \text{on } W'
\end{align*}
\]

\[2.3.79\]

for \((\tilde{\nu}_b, \tilde{p}_b)\)

\[
\begin{align*}
(\tilde{\nu}_b)_1 &= a^{-k+\frac{1}{2}}u_1 + O(a^{-k+\frac{3}{2}}) \\
(\tilde{\nu}_b)_2 &= a^{-k+\frac{1}{2}}u_2 + O(a^{-k+\frac{3}{2}}) \\
(\tilde{\nu}_b)_3 &= a^{-k+\frac{1}{2}}(\omega_1\tilde{x}_2 - \omega_2\tilde{x}_1) & \text{on } W \\
(\tilde{\nu}_b)_1 &= a^{-k+\frac{1}{2}}u_1' + O(a^{-k+\frac{3}{2}}) \\
(\tilde{\nu}_b)_2 &= a^{-k+\frac{1}{2}}u_2' + O(a^{-k+\frac{3}{2}}) \\
(\tilde{\nu}_b)_3 &= a^{-k+\frac{1}{2}}(\omega_1\tilde{x}_2 - \omega_2\tilde{x}_1) & \text{on } W'
\end{align*}
\]

\[2.3.80\]

and for \((\tilde{\nu}_c, \tilde{p}_c)\)

\[
\begin{align*}
(\tilde{\nu}_c)_1 &= a^{-k+1}(-\omega_3\tilde{x}_2) \\
(\tilde{\nu}_c)_2 &= a^{-k+1}(\omega_3\tilde{x}_1) \\
(\tilde{\nu}_c)_3 &= 0 & \text{on } W \\
(\tilde{\nu}_c)_1 &= a^{-k+1}(-\omega_3'\tilde{x}_2) \\
(\tilde{\nu}_c)_2 &= a^{-k+1}(\omega_3'\tilde{x}_1) \\
(\tilde{\nu}_c)_3 &= 0 & \text{on } W'
\end{align*}
\]

\[2.3.81\]

From these boundary conditions one must take \(k = 0\) to calculate \((\tilde{\nu}_a, \tilde{p}_a)\), \(k = \frac{1}{2}\) for \((\tilde{\nu}_b, \tilde{p}_b)\) and \(k = 1\) for \((\tilde{\nu}_c, \tilde{p}_c)\), with an error \(O(\varepsilon^2)\) in the eq. 2.3.73, 2.3.74. Three cases for solution arise. For direct approach of the two surfaces the flow field \((\tilde{\nu}_a, \tilde{p}_a)\) of the inner expansion results from \(U_3\) and \(U_3'\), where \(k = 0\). The effect of flow field \((\tilde{\nu}_b, \tilde{p}_b)\)
results from a tangential and rolling motion of the surfaces $W$ and $W'$ due to components $U_1$, $U_2$, $U'_1$, $U'_2$, $\omega_1$ and $\omega'_2$. The flow field $(\vec{v}_c, \vec{p}_c)$ results from the rotational motion of the surfaces about their mutual normal, and is produced by the components $\omega_3$ and $\omega'_3$.

Each of these three cases involves substitution of the appropriate boundary conditions and corresponding value of $k$. Considered here is only the direct approach of the two surfaces. Thus, taking $k = 0$, the boundary conditions on $\vec{v}_a$ become

$$
\begin{align*}
(\vec{v}_a)_1 &= (\vec{v}_a)_2 = 0, \\
(\vec{v}_a)_3 &= U_3 \quad \text{on } \bar{x}_3 = 0 \\
(\vec{v}_a)_1 &= (\vec{v}_a)_2 = 0, \\
(\vec{v}_a)_3 &= U'_3 \quad \text{on } \bar{x}_3 = h(\bar{x}_1, \bar{x}_2)
\end{align*}
$$

in the limit $a \to 0$. Since $(\vec{v}_a, \vec{p}_a)$ satisfies eq. 2.3.76

$$
\begin{align*}
\vec{p}_a &= \vec{p}_a(\bar{x}_1, \bar{x}_2), \\
(\vec{v}_a)_1 &= \frac{1}{2\mu} \frac{\partial \vec{p}_a}{\partial x_1} \bar{x}_3^2 + A \bar{x}_3 + C \\
(\vec{v}_a)_2 &= \frac{1}{2\mu} \frac{\partial \vec{p}_a}{\partial x_2} \bar{x}_3^2 + B \bar{x}_3 + D
\end{align*}
$$

where $A$, $B$, $C$ and $D$ are constants obtained by integrating with respect to $\bar{x}_3$ and are arbitrary functions of $\bar{x}_1$ and $\bar{x}_2$.

Thus, from eq. 2.3.82

$$
\begin{align*}
A &= -\frac{1}{2\mu} \left( \frac{\partial \vec{p}_a}{\partial x_1} \right) h(\bar{x}_1, \bar{x}_2) \\
B &= -\frac{1}{2\mu} \left( \frac{\partial \vec{p}_a}{\partial x_2} \right) h(\bar{x}_1, \bar{x}_2) \\
C &= 0 \\
D &= 0
\end{align*}
$$
Substituting and integrating with respect to $\vec{x}_3$,

$$
(\vec{v}_0)_3 = -\frac{1}{6\mu} \left( \frac{\partial^2 \vec{v}_0}{\partial x_1^2} + \frac{\partial^2 \vec{v}_0}{\partial x_2^2} \right) \vec{x}_3^2 - \frac{1}{2} \left( \frac{\partial A}{\partial x_1} + \frac{\partial B}{\partial x_2} \right) \vec{x}_3^2 - \frac{1}{2\mu} \left( \frac{\partial \vec{p}_0}{\partial x_1} + \frac{\partial \vec{p}_0}{\partial x_2} \right) \vec{x}_3^2 - \left( \frac{\partial A}{\partial x_1} + \frac{\partial B}{\partial x_2} \right) \vec{x}_3 + E
$$

2.3.85

Since $(\vec{v}_0)_3 = U_3$ on $\vec{x}_3 = 0$ it follows that $E = U_3$. Also, since $(\vec{v}_0)_3 = U_3$ on $\vec{x}_3 = h(x_1, x_2)$

$$
(U'_3 - U_3) = \frac{1}{6\mu} \left( \frac{\partial^2 \vec{v}_0}{\partial x_1^2} + \frac{\partial^2 \vec{v}_0}{\partial x_2^2} \right) h^2 - \frac{1}{2} \left( \frac{\partial A}{\partial x_1} + \frac{\partial B}{\partial x_2} \right) h^2 - \frac{1}{2\mu} \left( \frac{\partial \vec{p}_0}{\partial x_1} + \frac{\partial \vec{p}_0}{\partial x_2} \right) h - \left( \frac{\partial A}{\partial x_1} + \frac{\partial B}{\partial x_2} \right) h
$$

2.3.86

Substitution of the values of $A$ and $B$ from eq. 2.3.84 into eq. 2.3.86 yields

$$
\frac{1}{12\mu} \bar{V}^2 \vec{v}_0 + \frac{1}{4\mu} h^2 \left( \bar{V} \vec{v}_0 \cdot \bar{V} h \right) = (U'_3 - U_3)
$$

2.3.87

or alternatively

$$
\bar{V} \left( h^3 \bar{V} \vec{v}_0 \right) = 12\mu (U'_3 - U_3)
$$

2.3.88

Now $h$ may be written in the form

$$
h = 1 + \bar{x}^T \bar{x} \leq \bar{x}
$$

2.3.89

where $\bar{x}$ is the column vector $\bar{x} = (x_1 \ x_2)$ and $\bar{x}^T$ is its transpose and
Defining $\lambda_1$ and $\lambda_2$ as the eigenvalues of the matrix, $\mathbf{K}$, they are the roots of

$$\left( \mathbf{K} - \lambda \mathbf{I} \right) = 0 \quad 2.3.91$$

Thus eq. 2.3.89 transforms into

$$h = 1 + \lambda_1 \frac{\Delta_1^2}{2} + \lambda_2 \frac{\Delta_2^2}{2} \quad 2.3.92$$

which eventually gives the solution

$$\tilde{p}_a = - \left[ \frac{2 \mu (u_3^2 - u_3)}{\lambda_1 \lambda_2} \right] \frac{1}{(1 + \tilde{r}^2)^2} + O\left( a^{\frac{1}{2}} \right) \quad 2.3.93$$

The forces and torques resulting are

$$(\mathbf{F}_a)_1 = O(\ell \mu a)$$
$$(\mathbf{F}_a)_2 = O(\ell \mu a)$$
$$(\mathbf{F}_a)_3 = - \frac{3 \pi \mu (u_3^2 - u_3)}{\alpha (\lambda_1 \lambda_2)} + O\left( a^{-\frac{1}{2}} \right) \quad 2.3.94$$

$$(\mathbf{G}_a)_1 = O(\ell \mu a)$$
$$(\mathbf{G}_a)_2 = O(\ell \mu a)$$
$$(\mathbf{G}_a)_3 = - \frac{\rho (\ell \mu a) 3 \pi (u_3^2 - u_3) (1 - \frac{1}{2}) (1 - \frac{1}{2}) \sin \kappa \cos \kappa}{2 (\lambda_1 + \lambda_2) (\lambda_1 \lambda_2)^{\frac{1}{2}} (\ell \mu a) (\lambda_1 \lambda_2)} \quad 2.3.95$$

Similar methods are shown by Cox for the solution for the
other two forms of motion. The resulting expressions for all the forces and torques may be expressed in terms of the grand resistance matrix defined by Brenner (10, 11, 13).
2.4 Viscosity Effects

In 1906 and 1911 Einstein\(^{32,33}\) showed that as concentration of solid particles approaches zero, the viscosity of a suspension of spherical particles is described by

\[ \mu = \mu_o \left(1 + 2.5c \right) \quad 2.4.1 \]

where \( \mu_o \) is the viscosity of the pure liquid and \( \mu \) is the viscosity of the suspension at volume concentration, \( c \).

Hess suggested the general form

\[ \frac{\mu - \mu_o}{\mu} = \ell c \quad 2.4.2 \]

where \( \ell \) is a constant numerically greater than one, assuming a simple arrangement for particles in a fluid flowing in a capillary. Robinson suggested

\[ \frac{\mu}{\mu_o} = \frac{1+bc}{1-c_o} \quad 2.4.3 \]

where \( c_o \) is the volume of sediment obtainable from unit volume of suspension, and \( b \) is a constant which he tentatively described as a frictional coefficient. The term \( c_o \) sets the upper limiting value, the condition at which sufficient numbers of particles are in contact with other particles for the apparent suspension viscosity to approach infinity.

Various workers have found Einstein's equation to have a constant variable from 1.5 to 3.5. Orr and Blocker\(^{75}\),
therefore, suggested an equation

\[
\frac{\mu - \mu_o}{\mu} = ac^k \tag{2.4.4}
\]

where \(a\) and \(k\) are constants which depend on suspension properties. Orr and Blocker suggest plotting the function in logarithmic form and obtaining \(a\) and \(k\) from experimental data

\[
\ln \left(1 - \frac{\mu_o}{\mu} \right) = \ln a + k\ln c \tag{2.4.5}
\]

Further, after experimentation Orr and Blocker found that \(k\) is closely related to the reciprocal of the geometric standard deviation of the size distribution of the particles in suspension and that is closely related to the reciprocal of \(c_o\), the limiting concentration.

Studying sedimenting systems of methyl methacrylate and glass spheres in various liquids Oliver and Ward(74) obtained the general equation

\[
1 - \frac{1}{\mu_r} = kc + K_1 \tag{2.4.6}
\]

where the relative viscosity \(\mu_r = \frac{\mu}{\mu_0}\) for \(0.1 < c < 0.3\), and the value of \(K_1\) is either positive or negative according as the suspensions are downward or upward settling respectively. Expanding eq. 2.4.6

\[
\mu_r = (1 + K_1) + (1 + 2K_1)c + (1 + 3K_1)c^2 + \ldots \tag{2.4.7}
\]
The value of $K_t$ seems to be a characteristic of a suspension of settling spheres. For concentrations below 10% Oliver and Ward use the expression

$$
\mu_r = \frac{4}{3} K_2 C
$$

although they found values of $K_1$ to vary from 1.2 to 3.6. The magnitude of $K_1$ is closely connected to the degree of instability of the suspension and Oliver and Ward relate this to $\frac{\beta - \rho}{\rho_0}$ as in Fig. 2.4.1.

![Graph](image)

Fig. 2.4.1 Relationship between $K_1$ and $\frac{\beta - \rho}{\rho_0}$ for unstable suspension of spherical particles.

The equation of the line is $K_1 = 0.33 \left( \frac{\beta - \rho}{\rho_0} \right)$ but Oliver and Ward point out that it is not capable of extrapolation.

In general for Newtonian fluids, a settling suspension of spherical particles is characterized by a straight line plot of $\frac{1}{\mu_r}$ against concentration. The magnitude of the wall effect changes with the degree of instability of the
suspending but is insignificant in downward settling systems at concentrations less than 30%.

Oliver (73) suggests a relative velocity expression

$$\frac{u}{u_s} = \frac{1-c}{\mu_{re}}$$  \hspace{1cm} 2.4.9

where $\mu_{re}$ is the relative viscosity of a suspension, and compares this to another due to Gurel

$$\frac{u}{u_s} = \frac{(1-c)^2}{\mu_r}$$  \hspace{1cm} 2.4.10

where $\mu_r$ is a measured relative viscosity. He quotes two others, due to Hawskley and Steinour respectively.

$$\frac{u}{u_s} = (1-c)^2 \exp\left(-\frac{2.5c}{1-0.6c}\right)$$  \hspace{1cm} 2.4.11

$$\frac{u}{u_s} = (1-c)^2 10^{-1.92c}$$  \hspace{1cm} 2.4.12

Kynch (64) gave theoretical consideration to the effective viscosity of suspensions. He points out that Einstein's analysis was restricted to low solid sphere concentrations because the interaction between particles was neglected in his analysis, and that Einstein's result breaks down at solid concentrations of the order of 1%. When effective viscosity, $\frac{1}{\mu}$, is plotted against concentration a straight line is obtained such that $\mu$ becomes infinite as the concentration approaches the value for close packing.

Kynch considers the fluid motion of spherical particles as divisible into two parts; a mean velocity, $\bar{u}$, and a local
velocity round each particle which is necessary to satisfy
boundary conditions. He assumes that the mean velocity, \( \bar{u} \),
is that of a fluid of viscosity, \( \mu \). Thus the force of the
containing walls is either the viscosity multiplied by the
normal velocity gradient calculated from \( \bar{u} \), or the rate of
dissipation of energy calculated with viscosity, \( \mu \) and
velocity, \( \bar{u} \), equals the actual dissipation. For steady
viscous flow of an incompressible fluid, neglecting inertia
terms we have the Navier-Stokes and continuity equations,

\[
\mu \nabla^2 u = \nabla p \quad 2.4.13
\]
\[
\nabla \cdot u = 0 \quad 2.4.14
\]

Thus
\[
\nabla^2 p = 0 \quad 2.4.15
\]

This gives rise to a particular integral of eq. 2.4.13
\[
u = \frac{r p}{2\mu} \quad \text{where} \quad r \quad \text{is the position vector of the point}
\]
considered. The total solution is

\[
u = \frac{r p}{2\mu} + v \quad 2.4.16
\]

where \( v \) satisfies
\[
\nabla^2 v = 0 \quad 2.4.17
\]
\[
-2\mu \nabla \cdot v = 3p + r \nabla p
\]
\[
= 3p + r \frac{\partial p}{\partial r} \quad 2.4.18
\]

For a pure liquid with infinite boundaries the term in \( rp \)
in eq. 2.4.16 becomes infinite, but for boundaries with a
finite distance the velocity at the boundary is given by
\[ \dot{y} = u_b - \frac{rP}{2\mu} \]  
2.4.19

The solution of eq. 2.4.17 is

\[ y_P = \int \left( u_b - \frac{rP}{2\mu} \right) \frac{\partial G}{\partial n} \, dS \]  
2.4.20

where \( G = \frac{1}{4\pi R} \), where \( R \) is the distance near \( P \) where we consider \( y_P \). Thus if \( U_o \) is the main flow between boundaries which satisfy the boundary conditions and \( p \) is the extra pressure due to a disturbance, and \( y \) is also due to the disturbance,

\[ u = U_o + \frac{rP}{2\mu} + y \]  
2.4.21

In eq. 2.4.30 the area integral extends over the area of bodies disturbing the main flow. This is only true to a first approximation, however. The function, \( G \), is changed considerably. Using the analogy of electrostatics, it is the potential due to a unit positive charge, \( P \), when all the boundaries are earthed. These earthed particles have an induced negative charge and the effect of all these negative charges is to reduce the field and, therefore, the potential away from \( P \). Thus the effect on the velocity at \( P \) due to particles some way away from \( P \) which are shielded from it is reduced considerably. The shielding effect becomes more noticeable as the concentration rises. When particles touch the shielding of \( P \) is complete. The motion of fluid at \( P \) is calculated entirely from the motion of the
boundaries nearest to it. At a concentration of 10% the average gap between nearest particles is little more than their linear dimension. Thus any calculation which assumes that the effects of the particles on one another are additive is incorrect.

In consideration of the velocity near a sphere Kynch assumes the pressure distribution on the surface of a particle is

\[ p = A\lambda \frac{xz}{d^2} \quad 2.4.22 \]

where \( A \) is a constant, \( \lambda \) the angular vorticity, \( x \) and \( z \) two of the major Cartesian directions, and \( d \), the particle radius. Kynch obtains, assuming eq. 2.4.22, that

\[ A = -10\mu (1+c) \quad 2.4.23 \]

and

\[ G_o = \frac{1}{k} \left[ (x-x_f)^2 + (y-y_f)^2 + (z-z_f)^2 \right]^{-\frac{3}{2}} \quad 2.4.24 \]

the integral of eq. 2.4.20

\[ V_x = -\lambda a^3 \frac{(1+A)}{\mu} \left( \frac{\partial G_o}{\partial z} \right)_A - \lambda a^5 \frac{\partial}{\partial x} \left( \frac{\partial^2 G_o}{\partial x \partial z} \right)_A - \lambda a^3 \lambda x_A \left( \frac{\partial^2 G_o}{\partial x \partial z} \right)_A \]

\[ V_y = -\lambda a^3 \frac{A}{\mu} \frac{\partial}{\partial y} \left( \frac{\partial^2 G_o}{\partial x \partial z} \right)_A - \lambda a^3 \lambda y_A \left( \frac{\partial^2 G_o}{\partial x \partial z} \right)_A \]

\[ V_z = -\lambda a^3 \frac{(1+A)}{\mu} \left( \frac{\partial G_o}{\partial z} \right)_A - \lambda a^5 \frac{\partial}{\partial z} \left( \frac{\partial^2 G_o}{\partial z \partial x} \right)_A - \lambda a^3 \lambda z_A \left( \frac{\partial^2 G_o}{\partial z \partial x} \right)_A \]

2.4.25

To a higher degree of approximation he uses
\[ A = -10\mu (1 + \alpha c) \] 2.4.26

in which case

\[ \frac{\nu x + x_p}{2\mu} = -\frac{\lambda z a^5}{r^5} + \alpha \lambda c z \left(1 - \frac{a^5}{r^5}\right) + \lambda' p z \left(\frac{r^2}{a^2} - \frac{a^5}{r^5}\right) \]
\[ + \frac{5\lambda a z}{a^2} \left[ \frac{a^5}{r^5} \left(\frac{a^2}{r^2} - 1\right) - \frac{4\beta}{25} \left(1 - \frac{a^5}{r^5}\right) \right] \]

\[ \frac{\nu y + y_p}{2\mu} = \frac{5\lambda x y z}{a^2} \left[ \frac{a^5}{r^5} \left(\frac{a^2}{r^2} - 1\right) - \frac{4\beta}{25} \left(1 - \frac{a^5}{r^5}\right) \right] \]

\[ \frac{\nu z + z_p}{2\mu} = -\frac{\lambda x a^5}{r^5} + \alpha \lambda c x \left(1 - \frac{a^5}{r^5}\right) + \lambda' p x \left(\frac{r^2}{a^2} - \frac{a^5}{r^5}\right) \]
\[ + \frac{5\lambda a x z}{a^2} \left[ \frac{a^5}{r^5} \left(\frac{a^2}{r^2} - 1\right) - \frac{4\beta}{25} \left(1 - \frac{a^5}{r^5}\right) \right] \] 2.4.27

where \( \lambda' (1 - \beta) = \lambda \left(1 + \alpha c\right) \) and the pressure is

\[ P = -10\mu \frac{\lambda x y z}{a^2} \left[ \left(1 - \frac{4\beta}{25}\right) \frac{a^5}{r^5} - \frac{24\beta}{25} \right] \] 2.4.28

The two terms \( \alpha \) and \( \beta \) depend on the concentration

\[ \alpha = \frac{1 + \frac{1}{3} k b^2 + \frac{1}{3} k b^2}{(1 + \frac{1}{3} k b^2) - \frac{1}{3} c k^2 b^2} \approx 1 \] 2.4.29

\[ \frac{b^3}{\alpha} \beta = \frac{k^2 b^2 \left(1 + \frac{1}{3} k b^2 + \frac{1}{21} k^2 b^2\right)}{35(1 + \frac{1}{3} k b^2 + \frac{1}{21} k^2 b^2 + \frac{1}{231} k^3 b^2)} \] 2.4.30

where \( b \) is the interparticle distance between centres, and \( k \) the cut-off radius. Notice that the terms increase with distance from the sphere. If these expressions are used up to \( r = b \), the new terms are of the same order of magnitude as the main terms. Midway between particles Kynch believes the effects of at least three particles should be considered whereas here, only one has been considered.
To calculate the effective viscosity Kynch excludes a thin layer of thickness, \( d \), on the boundary of radius, \( r_o \), sufficiently large for \( G_0 \) to be practically zero along its narrow rim. Neglecting the term in \( \nabla^2 G_0 \) which is small of order, \( c \), the only term that remains is in \( v_z \) and is

\[
\dot{v}_z = n\lambda \alpha^2 \left(1 - A \right) \int G_0 dS 
\]

Assuming that \( G_0 \) in the bulk of the fluid does not depend on the thin surface layer, the necessary expression for it is a term in \( \frac{e}{k} \) due to the charge at \( p \) and a similar term due to the image. When \( kd \ll 1 \) the integration yields

\[
v_z = n\lambda \alpha^2 \left(1 - A \right) \frac{4\pi \sinh k\rho}{k} 
\]

Assuming that \( A = -10\mu \) there is a velocity gradient

\[
\left( \frac{\partial v_z}{\partial \rho} \right)_o = 5\lambda \alpha 
\]

and the total velocity gradient near the boundary is

\[
\frac{\partial u}{\partial x} + \frac{\partial v_z}{\partial x} = 2\lambda \left(1 + 2.5c\right) 
\]

as originally obtained by Einstein.

Greenstein and Heppel\(^\text{(42)}\) obtained an expression for viscosity of a dilute suspension of uniform spheres in Poiseuille flow in a parabolic field in a cylindrical tube, by summing the individual pressure drops for a uniform
suspension. The sphere moves with arbitrary constant translational velocity, \( U \), in the positive \( z \) direction parallel to the cylinder axis, and with constant angular velocity
\[
\omega = i\omega_1 + j\omega_2 + k\omega_3
\]
relative to the cylinder wall, while the fluid flows in a laminar flow with a superficial velocity, \( \frac{1}{2}U_o \), in the positive \( z \) direction. The sphere is of radius, \( a \), cylinder radius, \( R_o \), and the centre of the sphere is at a distance, \( b \), from the cylinder axis in the \( \hat{z} \) direction.

It is assumed that creeping flow exists; thus

\[
\begin{align*}
\mu_o \nabla^2 u &= \nabla p & \text{(2.4.13)} \\
\nabla \cdot u &= 0 & \text{(2.4.14)}
\end{align*}
\]

where \( \mu_o \) is the viscosity of the homogeneous fluid. Boundary conditions are:

At fluid-solid interfaces no relative motion exists.

At large distances from the disturbing influence of the sphere, \( z = \pm \infty \), the velocity distribution becomes parabolic.

The boundary value problem is solved by Greenstein and Happel by the method of reflections, where the general boundary conditions are that \( U^{(i+1)} = -U^{(i)} \). Since the equations of motion and boundary conditions are linear, the frictional force, \( F \), and torque, \( T \), (about the sphere centre) exerted on the sphere by the fluid and the additional pressure drop \( \Delta P_s \) (above that due to the parabolic field \( \Delta P_o \)) experienced by fluid passing through the cylinder are obtained by summing respective contributions of each of the individual fields:
Thus force and torque experienced by the sphere, and pressure drop experienced by the fluid as a result of the presence of the sphere are

\[ F \approx F^{(0)} + F^{(2)} \]
\[ T \approx T^{(0)} + T^{(2)} \]
\[ \Delta P_s \approx \Delta P_1 + \Delta P_2 \]

where the quantities with bracketed superscripts converge for all ratios of \( \frac{a}{b} \). For neutrally buoyant spheres \( F = 0, T = 0 \) and setting \( F = 0 \) in eq. 2.4.38

\[ u \approx u_0 \left( 1 - \frac{b^2}{R_o^2} \right) - \frac{2}{3} u_0 \left( \frac{a}{R_o} \right)^2 \]

and setting \( T = 0 \) in eq. 2.4.39

\[ a \omega_1 = 0 \]
\[ a \omega_2 \approx u_0 \frac{ab}{R_o^2} \]
\[ a \omega_3 = 0 \]
For conditions when rate of change of kinetic energy is neglected, rate of change of potential energy is neglected, $F = 0, T = 0$ the viscosity of a suspension is given by

$$\frac{\mu}{\mu_0} = 1 + \frac{\Delta P^*}{\Delta P_0}$$  \hspace{1cm} (2.4.43)

where $\Delta P_0$ is the pressure drop due to homogeneous fluid motion whereas $\Delta P^*$ is the additional pressure drop experienced by the fluid during its passage through the cylinder as a result of suspended solids. Using results in eq. 2.4.41 and 2.4.42 in eq. 2.4.40

$$\Delta P_0 = \frac{\varepsilon_0 \mu_0 c u_0 b^2}{3 K_0^2} + \frac{\varepsilon_0 \mu_0 u_0}{K_0^2} \frac{a^4}{K_0^4}$$  \hspace{1cm} (2.4.44)

and this is for a single particle only, in free translation and rotation. It can be seen that this is a function of position in the tube owing to the dependence on $b^2$. The total excess pressure drop for all particles is found by integrating over the whole cylinder. If $A = \pi R_0^2$, the area of the cylinder, the volume under consideration is $\pi R_0^2 \Delta z$, $c$ is the volume concentration of particles, assuming the particles to be uniformly and continuously distributed, the volume occupied by particles is $c\pi R_0^2 \Delta z$, and an element of area is given by $dA = bdcdb$. Therefore

$$\Delta P_{ex, single} = \frac{1}{A} \int_A \Delta P_0 \, dA$$

$$= \frac{2\pi}{\pi R_0^2} \int_{b=0}^{b=R_0} \Delta P_0 \, b \, db$$

$$= \frac{40}{3} \frac{\mu_0 c u_0}{K_0^2} + \frac{\varepsilon_0 \mu_0 u_0 a^4}{K_0^4}$$  \hspace{1cm} (2.4.45)
The number of particles in the cylinder is given by

\[ \frac{\Pi R^2 \Delta z c}{4 \pi a^3} = 3 \frac{R^2 \Delta z c}{4a^3} \]

and assuming overlapping disturbance of flow due to particle-particle interaction is negligible

\[ \Delta P_{\text{avg, all}} = \Delta P^* \]

\[ \simeq \left( \frac{40 \mu_0 \alpha u_0}{\varepsilon_0^4} + 8 \mu_0 \alpha^5 u_0 \right) \left( \frac{3}{4} \frac{R^2 \Delta z c}{a^3} \right) \]

\[ \simeq \frac{10 \mu_0 \alpha u_0 \Delta z c}{\varepsilon_0^4} + 6 \mu_0 \alpha u_0 \alpha^2 \Delta z c \]

2.4.46

and for Poiseuille flow \( \Delta P = \frac{4 \mu_0 u_0 \Delta z}{\varepsilon_0^4} \). Substituting this and eq. 2.4.46 into eq. 2.4.43

\[ \frac{L}{\mu_0} = 1 + \frac{5}{2} c + \frac{3}{2} \left( \frac{a}{\varepsilon_0} \right)^2 c \]

2.4.47

This result is obtained by assuming a parabolic field,

\[ u^{(c)} = \frac{1}{2} \left[ u_0 \left( 1 - \frac{r^2}{\varepsilon_0^2} \right) - u \right] \]

2.4.48

reflected from the sphere. Instead approximate the initial parabolic field in the vicinity of the sphere by a linear shear field,

\[ u^{(s)} = \frac{1}{2} \left[ -2 \frac{u_0}{\varepsilon_0^2} \right] + u_0 \left( 1 + \frac{r^2}{\varepsilon_0^2} \right) - u \]

2.4.49

and reflect this from the sphere. Thus

\[ F = F^{(c)} + F^{(s)} \]

\[ \simeq - \frac{1}{2} \frac{C \alpha \mu_0 a}{\varepsilon_0^2} \left[ u - u_0 \left( 1 - \frac{r^2}{\varepsilon_0^2} \right) \right] \]

2.4.50
\[ \Delta P_3 \approx \Delta P_1 + \Delta P_2 \]

\[ \alpha = \frac{12 \mu_c a u_c}{K_0^2} \left[ \left(1 - \frac{b_1^2}{K_0^2}\right) - \frac{2}{3} \left( \frac{a}{K_0} \right)^2 + \frac{2}{3} \left( \frac{a}{K_0} \right)^2 \left( \frac{b_1}{K_0} \right)^2 \right] \]

\[ - \frac{12 \mu_c a u_c}{K_0^2} \left[ \left(1 - b_2^2\right) - \frac{2}{3} \left( \frac{a}{K_0} \right)^2 \right] \left( -16 \mu_c a \right)^3 \frac{b_2}{K_0} \omega_2 \]

Eq. 2.4.50 and 2.4.51 are analogous to eq. 2.4.38 and 2.4.40. The expression for the torque remains unchanged. It follows that

\[ \Delta P_3 \approx \frac{3 \mu_c a u_c}{2 K_0^2} \frac{b_1^2 a^2}{K_0^2} \]

whence

\[ \frac{\mu}{\mu_0} \approx 1 + \frac{\delta c}{2} \]

the form first obtained by Einstein. Greenstein and Happel showed that their formula for suspension viscosity was more accurate than Einstein's but theirs reduced to his for \( a \to 0 \).

To obtain a more accurate result for viscosity it is required to take particle-particle interaction into account and use the method of reflections to further than the second reflection, and this is of the form

\[ \frac{\mu}{\mu_0} = 1 + c \left[ \frac{5}{2} + k_1 \frac{a}{K_0} + k_2 \left( \frac{a}{K_0} \right)^2 + k_3 \left( \frac{a}{K_0} \right)^3 + \ldots \right] \]

Fedors(37) investigated the viscosity of neutrally buoyant suspensions of glass and polymer spheres quoting a semi-empirical equation of the form

\[ \mu = \mu_0 \left( 1 + 2.5c + 10.05c^2 + 0.00273 \exp(16.6c) \right) \]

which does not reduce to the Einstein expression, eq. 2.4.1,
in the limit of vanishing particle concentration.

Fedors indicated that eq. 2.4.55 may be represented by a one parameter equation of the form

\[ \mu = \mu_0 \left[ 1 + \frac{2.5c}{1 - \left( \frac{c}{c_{\text{max}}} \right)} \right]^2 \]

where \( c_{\text{max}} \) is the maximum volume concentration to which the particles can pack, which for a random packing of spheres of uniform size is quoted as 63%. He further indicates that if aggregates of particles form, values of \( c_{\text{max}} \) varying from 36.5% to 43%, then the expression for viscosity is given by

\[ \mu = \mu_0 \left[ 1 + \frac{2.5c}{c_{\text{max}} - c} \right]^2 \]
2.5 Concentration Effects

Steinour (88, 89, 90) studied the effects of concentration on sedimentation and observed that in a suspension particles settle at a velocity less than that given by Stokes' law but showed that if individual particles fall in accordance with Stokes' law, are of uniform size and density and well distributed, the settling rate can be represented by the Stokes velocity, $U_s$, multiplied by a term which is a function of velocity only. He suggests that the spheres would all settle at a common constant rate if they were in a stable uniform arrangement and if wall and bottom effects were negligible. In an actual mixture the distribution of spheres is not uniform but he assumes that under best conditions a fixed arrangement and constant velocity are closely maintained and assumes that the fluid space maintains constant shape within which steady laminar flow exists. Furthermore, he assumes identical arrangements of spheres in different suspensions. In this case the problem becomes one of comparing laminar flows in composite flow spaces having the same shapes depending on the velocity gradients at corresponding points. Consequently, a suitably defined velocity gradient and a characteristic length are required. The velocity is proportional to the velocity gradient and the length, which he takes as the average spacing between spheres, which is proportional to sphere radius. He gives the average relative velocity as

$$ u = k_1 r \left( \frac{dv}{dn} \right)_{av} \phi_1 (c) $$

2.5.1
where \( \frac{\text{d}v}{\text{d}n} \) \(_{av} \) is the surface average derivative of velocity at the surface with respect to the normal to the surface, \( \phi_1(\varepsilon) \) is a size and shape factor, a function of voidage, \( \varepsilon \), only. The tangential forces give the rate of shear or surface average velocity derivative

\[
\frac{\mathcal{R}_T}{4\pi r^2} = h \mu \left( \frac{\text{d}v}{\text{d}n} \right)_{av} \tag{2.5.2}
\]

where \( h \) is a dimensionless factor to correct for the fact that not all the forces act in the line of motion of the particle considered. Because of the constant flow pattern the resultant of the tangential components of the viscous forces maintains a constant ratio to the resultant of the normal components at any given concentration. Thus at infinite dilution the resultant tangential force is twice the normal. As the concentration changes, however, both this and \( h \) may change because of the change in the shape of the flow space, giving rise to a more complicated expression for the total viscous resistance

\[
\frac{\mathcal{R}}{4\pi r^2} = k_2 \mu \left( \frac{\text{d}v}{\text{d}n} \right)_{av} \phi_2(\varepsilon) \tag{2.5.3}
\]

where \( k_2 \) is a dimensionless proportionality constant expressing the ratio of \( \frac{\mathcal{R}}{4\pi r^2} \) to \( \mu \left( \frac{\text{d}v}{\text{d}n} \right)_{av} \) at infinite dilution and \( \phi_2(\varepsilon) \) is a shape factor which is a function of \( \varepsilon \) only and approaches unity as \( \varepsilon \) approaches infinite dilution.

Eq. 2.5.3 shows that changes in size of flow space caused by changes in \( \varepsilon \) can effect \( \left( \frac{\text{d}v}{\text{d}n} \right)_{av} \) directly only through a possible effect on \( R \). Eliminating \( \left( \frac{\text{d}v}{\text{d}n} \right)_{av} \) from eq. 2.5.1
and 2.5.3

$$R = \frac{4\pi \mu r \Omega k_2 \phi_2(\varepsilon)}{k_1 \phi_1(\varepsilon)}$$ 2.5.4

Replacing the constants $k_1, k_2$ by a single term evaluated from Stokes' law and the two functions of $\varepsilon$ by $\phi(\varepsilon)$

$$R = \frac{4\pi \mu r \Omega}{\phi(\varepsilon)}$$ 2.5.5

The fluid friction, $R$, equals the motive force given by

$$R = \frac{4\pi r^3 (\rho_s - \rho_m)}{3} g$$ 2.5.6

where $\rho_s, \rho_m$ are the densities of solid and mixture respectively. These are related to the liquid density, $\rho$, by

$$\rho_s = (1-\varepsilon) \rho_s - \varepsilon \rho$$

$$\rho_m = (\rho_s - \rho) \varepsilon$$ 2.5.7

Substituting for $(\rho_s - \rho_m)$ in eq. 2.5.6 and then for $R$ in eq. 2.5.5

$$u = \frac{2g(\rho_s - \rho) r^2 \phi(\varepsilon)}{\Theta \mu}$$ 2.5.8

$U$ is defined as the average relative velocity between spheres and fluid, whereas the measured velocity, $U_m$, is of particles with respect to a fixed horizontal plane and these are related

$$(1-\varepsilon) U_m = \varepsilon (u - U_m)$$

$$U_m = \varepsilon U$$ 2.5.9
Equating loss in potential energy attending the fall of a sphere to the work done against viscous resistance

\[ \frac{4}{3} \pi r^3 (\rho_s - \rho) U_m = \frac{4}{3} \pi r^3 (\rho_s - \rho_m) U \]  
\[ \frac{U_m}{U} = \frac{\rho_s - \rho_m}{\rho_s - \rho} \]

Thus

\[ U_m = \frac{2g(\rho_s - \rho)r^2 \varepsilon^2 \phi(\varepsilon)}{\Theta \mu} \]

or in terms of Stokes' velocity, \( U_s \)

\[ U_m = U_s \varepsilon^2 \phi(\varepsilon) \]

Using the hydraulic radius, \( \frac{\varepsilon r}{3(1-\varepsilon)} \), for the relative spacing between spheres

\[ \phi(\varepsilon) = \frac{\varepsilon}{(1-\varepsilon)} \theta(\varepsilon) \]

where \( \theta(\varepsilon) \) represents the effects of shape not accounted for by the use of the hydraulic radius. Eq. 2.5.12 and 2.5.13 become

\[ U_m = \frac{2g(\rho_s - \rho)r^2 \varepsilon^3}{\Theta \mu} \frac{\theta(\varepsilon)}{(1-\varepsilon)} \]

\[ U_m = U_s \frac{\varepsilon^3}{1-\varepsilon} \theta(\varepsilon) \]

Steinour determined \( \theta(\varepsilon) \) empirically from a plot of his data in the form \([U_m (1-\varepsilon)]^{3/2} \) vs. \( \varepsilon \) for \( 0 \leq \varepsilon \leq 1 \) arriving at

\[ U_m = U_s 10^{-1.32(1-\varepsilon)} \]
Thus \[ \Theta(c) = \frac{1-c}{c} \ln^{1.32(1-c)} \]

Hawksley (47) discusses the work of Burgers concerning the concentration effect on settling suspensions. According to Burgers the sedimentation of any given particle in the suspension is subject to two influences, one arising from the motion of the other particles and one from the presence of the other particles. For example, if all the particles except one have the same density as the fluid and do not settle on their own, they will nevertheless tend to be carried down by the velocity imparted to them by a single sedimenting particle. When all the particles have the same density there is a mutual downward drag. This tendency is offset, however, by the upward return flow that occurs in a closed vessel. The net effect may be an accelerating or retarding one depending on the spatial arrangement of neighbouring particles. The second effect, due to the presence of other particles, is a retardation. Imagining all the particles but one again to have the same density as the fluid, the field of flow of the single sedimenting particle will be distorted because of the other particles which, owing to their rigidity, cannot take part fully in the deformation of the fluid. Consequently the sedimenting particle experiences a retardation, which appears to it to arise from the increased viscosity of the fluid. The effect is of the same nature as the increase in apparent bulk viscosity, but is not of the same magnitude.

In terminal velocity calculations of a single sphere in a suspension it has been assumed that the resultant
Gravitational effect is determined by the buoyancy of the suspension and not that of the fluid. However, any additional buoyancy is a consequence of the motion of the particles. This latter was adopted by Burgers. Although the rate of settling of a given particle is dependent on the distances and arrangements of all the other particles, Burgers gave no consideration to whether or not the process establishes an equilibrium arrangement. Hawksley considers that equilibrium must be achieved, otherwise the rate of settling of individual particles would vary with local particle distribution and the characteristic feature of suspensions, that they settle as a whole, would not be observed.

Burgers assumed that the return flow due to each sphere could be represented by a diffuse force field acting on the fluid in equilibrium with the gravitational force as a point force on the fluid, the effective gravitational force being the difference between particle weight and weight of liquid displaced. The final result gives the relative velocity, $U_r$,

$$u_r = \frac{U_c}{U_s} = \frac{1}{1 + (k_1 k_2) c}$$

for $c \ll 1$ 2.5.19

where $U_c$ is the mean settling rate of a suspension of equal spheres at volume concentration, $c$, and $k_1$ and $k_2$ represent the effect of the presence and motion, respectively, of the other spheres on the settling rate of the given sphere. For a random distribution Burgers finds $k_1 = \frac{15}{8}$, $k_2 = 5$.

Thus for dilute suspensions...
Hawksley considers more concentrated suspensions, assuming an equilibrium particle arrangement. He assumes that the average rate of settling can be found by considering the motion in a homogeneous suspension. Presence of the particles can be regarded as increasing the effective local viscosity of the fluid while their motion produces a uniform upward fluid velocity and an increased buoyancy. The motion of a representative particle can then be treated as the Stokes motion of a particle in a fluid of increased viscosity and density on which a uniform upward velocity has been impressed.

The presence of particles, he suggests, produces the effect given by Einstein's viscosity expression, eq. 2.4.1, the assumption being that each particle produces its own independent contribution to the local retardation and that these can be combined additively limits its application to an upper concentration limit of 2%. For a settling suspension in the case of equilibrium settling there is no relative motion of individual particles and the local resistance to shearing can be obtained by integrating Einstein's result. For more concentrated suspensions the mutual particle interaction must be included and this has been done by Vand giving an expression for the local effective viscosity

$$\mu = \mu_0 \exp \left( \frac{2.5c}{1 - 0.01c} \right)$$
This allows the effective viscosity on a representative sphere in a suspension to be derived as if there were no other spheres present excluding contributions from their motion and as if the viscosity were everywhere constant at \( \mu_L \). This value is not the apparent bulk viscosity which would exceed this value, \( \mu_L \).

The motion of the particles produces a number of effects. Two spheres settling in an infinite fluid acquire velocities in excess of the Stokes velocity, and a compact cloud of particles has been examined by Smoluchowski and by Burgers showing that the motion of each particle tends to drag down all other particles such that the cloud possesses a velocity in excess of the Stokes velocity of the individual particles. When the cloud fills the whole vessel the retarding effect of the return flow outweighs the downward drag and settling is reduced below the Stokes velocity. Cunningham made an estimate for the reduced velocity by assuming that the resulting extra drag was caused by each sphere carrying down liquid of diameter comparable with the distance between particle centres. Other workers assumed a surrounding shell of liquid round each particle. The representative sphere may be considered to be settling in a pure fluid subject to a constant upward velocity, the presence of other spheres being accounted for by the viscosity effect already described. The magnitude of the return flow velocity is determined by the continuity requirement

\[
(1-c)V + cu_c = 0
\]

2.5.22
where \( U_c \) is the mean velocity of the particles and \( V \) is the return flow velocity. The settling rate, \( U_c' \), is equal to the Stokes velocity, \( U_s' \), plus the return flow velocity, \( V \)

\[
U_c = U_s + V \tag{2.5.23}
\]

\[
U = \frac{U_c}{1-c} \tag{2.5.24}
\]

and the viscous drag on the representative sphere becomes

\[
D = \frac{3\pi\mu d U_c \exp\left(\frac{2.5c}{1-c}\right)}{(1-c)} \tag{2.5.25}
\]

A consequence of the return flow effect as treated above is that the addition of another sphere into a suspension will require the displacement of an equal volume of suspension. The effective gravitational force is, therefore, determined by the density of the suspension \( \rho_s \) \( (1-c)\rho_f \) and the effective gravitational force on the representative sphere is

\[
F = (1-c)(\rho_s \rho_f) g \frac{\pi d^3}{6} \tag{2.5.26}
\]

The terminal velocity for equilibrium settling of a suspension of spheres is obtained by equating eq. 2.5.25 and 2.5.26

\[
U_c = \left(\frac{\rho_s \rho_f}{\rho_s + \rho_f}\right) g \frac{d^2}{4} (1-c)^2 \exp\left(-\frac{2.5c}{1-0.61c}\right) \tag{2.5.27}
\]

and the relative settling rate becomes
\[ u_r = \frac{u}{u_s} = (1 - c)^2 \exp \left( \frac{-2.5c}{1 - 0.01c} \right) \] 2.5.28

which at very low concentrations approximates to

\[ u_r \approx 1 - 4.5c \] 2.5.29

Comparing eq. 2.5.29 to 2.5.20 there is a faster settling rate for an equilibrium arrangement of spheres than for a fixed random arrangement. Hawksley determined from experimental results that eq. 2.5.28 applies to suspensions with volume concentrations in excess of 5%.

Kawaguti (56) derived equations for the viscous flow of fluid past a sphere in a cylindrical column assuming the cylinder to be frictionless. He obtains an expression for the drag coefficient on a particle as

\[ c_D = \frac{24}{\text{Re}} \frac{1}{1 - 1.64931l c^{-1} + 0.79630 l c^{-3}} \] 2.5.30

where \( r_c \) is the ratio of cylinder radius to particle radius. He applies this to a sedimenting system considering the particle distribution to be random such that the effects of particles upstream and downstream of the particle under consideration are cancelled on average and assuming uniform flow. Particles near the particle considered cannot be neglected owing to an increase of effective mean velocity near the particle and thus an increase in drag. The drag on a single sphere is

\[ D = \frac{3 \pi \mu r U}{1 - 1.64931 l c^{-1} + 0.79630 l c^{-3}} \] 2.5.31
giving a terminal velocity

\[ u = \left(1 - 1.64931 r_o^{-1} + 0.79680 r_o^{-3}\right) \frac{2}{3} \left(\rho - \rho_0\right) \frac{r_o^2 \rho}{\mu} \]  \hspace{2cm} 2.5.32

If the mass concentration of particles expressed as the mass of the particles divided by the total mass is \( m \), the number of particles per unit mass, \( n \), is

\[ n = \frac{m \rho}{\frac{4}{3} \pi r^3 \left[ \rho - (\rho - \rho_0) m \right]} \]  \hspace{2cm} 2.5.33

and assuming simple cubic and body centred cubic lattices, relations between the number and interparticle distance are found as shown in Fig. 2.5.1

![Diagram of spherical arrangements](image)

**Simple cubic, \( b = n^{-\frac{1}{3}} \)  \hspace{2cm} Body-centred cubic, \( b = (\frac{r_o}{2})^{-\frac{1}{3}} \)**

Fig. 2.5.1 Number concentration of spheres in cubic arrays.

and assume that the effective radius of a frictionless cylinder, \( r_o r \), is given by \( 2(r_o r)^2 \pi = 2b^2 \) so that

\[ r_o = M^3 \sqrt{\frac{\rho - (\rho - \rho_0) m}{m \rho}} \]  \hspace{2cm} 2.5.34

where \( M \) is 0.9095 for the simple cubic array and 1.1458
for the body centred cubic array. Neglecting the term $(\rho_s - \rho)^m$ compared with $\rho_s$ since it is small in sedimenting systems

$$\tau = M^2 \sqrt{\frac{\rho_s}{\rho_m}} = Mc^{\frac{1}{3}}$$  \hspace{1cm} 2.5.35

where $c$ is the volume concentration. Thus the settling velocity is

$$u = \left[1 - 1.649 \sqrt{\frac{c}{M^2}} + 0.756 (\frac{s}{M^2}) \right] \frac{2}{3} \left( \rho_s - \rho \right) \frac{r^2 g}{\mu}$$  \hspace{1cm} 2.5.36

and the relative velocity compared to Stokes' velocity becomes the term in square brackets. The above expression gives velocities which are 20-30% too low.

McNown and Lin's\(^{(70)}\) analysis of the problem is concerned only with the vertical velocity since they consider the mean horizontal velocity to be zero. If $s$ is the interparticle distance, $r$ is the distance from a reference particle to any other and $z$ is the distance measured in the vertical direction, the vertical velocity induced on the reference particle by the other particles is determined by an infinite summation. The summation is begun with single particles below and above $(r = s, z = \pm s)$ and the four nearest particles in the horizontal plane through the reference particle $(r = s, z = 0)$. The summation is then continued as in Fig. 2.5.2.

For particles large distances away the summation is performed as an integral. The number of particles in an
Fig. 2.5.2 Interparticle distances for different particle configurations.

<table>
<thead>
<tr>
<th>No. of particles</th>
<th>2</th>
<th>4</th>
<th>4</th>
<th>8</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>r/s</td>
<td>1</td>
<td>1</td>
<td>√2</td>
<td>√2</td>
<td>√2</td>
</tr>
<tr>
<td>z/s</td>
<td>±1</td>
<td>0</td>
<td>0</td>
<td>±1</td>
<td>±1</td>
</tr>
</tbody>
</table>

Elementary volume $\Delta x \Delta y \Delta z$, where $x, y, z$ are the Cartesian axes, can be expressed as $\frac{\Delta x \Delta y \Delta z}{6 \theta}$ and the whole region is divided into two parts, $P$ the volume immediately surrounding the reference particle and $Q$ the remaining space. Thus the velocity induced by the other particles is

$$\sum_{P \cap Q} v' = \sum_P v' + \frac{1}{6 \theta} \iiint v' dxdydz \quad 2.5.37$$

where $v'$ is the velocity induced at the locus of the reference particle by the motion of the other particles. Because continuous variation is assumed throughout $Q$, $v'$ can be replaced by $v$, the velocity at any point induced by the motion of a single sphere. The continuity condition, no net flow across any horizontal plane, $A$, is

$$\iint_A v' dxdy = 0 \quad 2.5.38$$

The triple integral of $v$ throughout the entire space is, then, zero so that the integral throughout $P$ must be equal and opposite to that throughout $Q$. The change in fall velocity caused by the motion of other particles is more readily expressed as
\[ \Delta V = V - V_s = \sum_{r} \int_{r}^{\infty} V dV d\theta d\phi \]

2.5.39

Calculating \( \Delta V \) depends on the definition of \( V \) at any point in space. Using the stream function, \( \psi \), for a sphere in an infinite medium,

\[ \psi = \frac{3}{4} \frac{\mu d}{\rho} \left( 1 - \cos \theta \right) \left[ 1 - e^{r} \left( \frac{\psi}{2} \right) \frac{\psi}{2} \right] - \frac{1}{32} \frac{1}{\rho} \frac{d^3 \sin^2 \theta}{\phi} \]

2.5.40

where \( d \) is the particle diameter and \( \theta \) is the inclination from the \( z \) axis, and

\[ V = V_r \cos \theta - V_\theta \sin \theta = \frac{1}{r} \left( \frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \phi} \right) \]

2.5.41

eq. 2.5.39 is rearranged into the form

\[ \frac{\Delta V}{V} = \frac{3d}{8} \left[ \sum_{r} \frac{r}{s} \left( 1 - \frac{2r \theta}{r} - \frac{2r \phi}{2 \mu} \right) \exp \left( \frac{-V_p r}{2 \mu} \right) \right] 
- \frac{3r \mu^2}{\sqrt{2 \mu^2}} \left( \frac{23}{5} \exp \left( \frac{-V_p r}{2 \mu} \right) - 1 \right) 
- \frac{\mu}{2 \rho} \exp \left( \frac{-2V_p r}{2 \mu} \right) - 1 \] 

2.5.42

Fidleris and Whitmore (38) found that, for a falling sphere in a suspension of non-settling spheres, at small Reynolds numbers the falling speed of the sphere is unaffected by the size of the suspended spheres at constant volume concentration. At high Reynolds numbers, though, the falling speed diminishes as its size approaches that of the suspended particles.

The work of Oliver (72) considers an array of particles falling with velocity, \( U_p \), with a return flow of liquid, \( \bar{U} \),
averaged over the whole cross-section of the vessel, and must be \((1-c)\) times the solid settling rate. Particles move, therefore, with a velocity, \(U_p - \bar{U}\) relative to liquid as a whole, as shown in Fig. 2.5.3.

![Flow model based on average return flow velocity.](image)

Fig. 2.5.3 Flow model based on average return flow velocity.

The proportion of liquid always in the immediate vicinity of a sphere is carried downwards so the liquid which rises does so with a velocity, \(U\), greater than \(\bar{U}\). The velocity of liquid varies from a value \(U_p\) downwards at the sphere surfaces to \(U\), upwards, at points most distant from spheres. This is shown in Fig. 2.5.4.

![Flow model based on maximum return flow velocity.](image)

Fig. 2.5.4 Flow model based on maximum return flow velocity.

Using this model Oliver imagined a sphere moving in a
hypothetical cylinder as shown in Fig. 2.5.5. The flow round the sphere is identical to that obtained if the sphere is stationary and the liquid flows with a velocity, $U_0$. At large distances from the particle the axial flowrate is

$$Q = \pi R^2 U_0$$

2.5.43

and the stream function for Stokes flow past a sphere is

$$\psi = \frac{1}{2} U_0 r^2 \left(1 - \frac{3}{2} \frac{a}{r} + \frac{1}{2} \frac{a^3}{r^3}\right) \sin^2 \theta$$

2.5.44

Thus

$$Q' = 2\pi \int_{R}^{R} \psi dr$$

$$= \pi U_0 R^2 \left(1 - \frac{3}{2} \frac{a}{R} + \frac{1}{2} \frac{a^3}{R^3}\right)$$

Fig. 2.5.5 Hypothetical cylinder surrounding sphere.

Thus the volumetric flowrate, $Q-Q'$, of liquid passing outside the cylinder at $XX'$ does so because of the presence of the sphere. Assuming some constraint limits the flow.
to the cross-section XX' there will be an additional drift of the contents of the cylinder including the sphere, owing to the distribution of extra flowrate over the cross-section. This is given by

\[
\frac{Q - Q'}{\pi R^2} = u_o \left( \frac{3}{2} \frac{a}{R} - \frac{1}{2} \frac{a^3}{R^3} \right) \tag{2.5.45}
\]

which approximates to

\[
u_A = \frac{3}{2} \frac{a}{R} u_o, \quad \frac{a^3}{R^3} \ll \frac{a}{R} \tag{2.5.46}
\]

Oliver assumes that other spheres in a swarm cause a flow restriction as above and that the effective 'radius of action', \( R \), of a typical sphere is directly proportional to the mean separation, \( d \), and the ratio \( a \) is proportional to the one third power of concentration, \( c^\frac{1}{3} \). Thus

\[
\nu_A = \kappa c^\frac{1}{3} u_o \tag{2.5.47}
\]

For the case of spheres with a Stokes velocity \( U_s = U_o \)

\[
u_p = \nu_s - \nu_A = \nu_s \left( 1 - \frac{3}{2} \frac{a}{R} \right) = \nu_s \left( 1 - \kappa c^\frac{1}{3} \right) \tag{2.5.48}
\]

He assumes an expression for the increase in viscosity

\[
\frac{\mu}{\mu_o} = \frac{1}{1 - \kappa c} \tag{2.5.49}
\]

which will cause a reduction in particle settling velocity to \( U_k \)
Oliver finds from experiment that $K = 0.75$, and $k = 2.15$.

Thorpe (93) proposed an expression for uniformly dispersed spheres where he considered high velocity gradients set up by the proximity of particles to create high viscous forces compared to the inertial forces such that the suspension behaves as though it were viscous even though individual particles are outside the Stokes range for concentrations higher than 0.5%. Below this value inertial forces are important. For concentrations between 0.5% and 7% the relative change in fall velocity he described for particles up to 650 μm by

$$\frac{u_u}{u} = 1 + \left(\frac{d}{s}\right) + \left(\frac{d}{s}\right)^2 + \left(\frac{d}{s}\right)^3$$

2.5.51

where $d$ is the particle diameter and $s$ the interparticle spacing, $U$ is the fall velocity of a particle at any concentration and $U_s$ the Stokes velocity.

Barfod (3) continued work utilizing the results obtained by Kaye and Boardman, Johne, and Koglin, although unlike previous workers he found a maximum velocity of settling at about 0.1% volume concentration. He suggests, from his data, that at concentrations below 0.015% the settling rate is less than that given by Stokes' law.

Using the Stokes' law relationship $U = kd^2$ and for a solid in suspension whose sizes are normally distributed

$$U = k\left(d^2 + d^2\right)$$

2.5.52
where \( \sigma \) is the standard deviation of the diameter and \( d \) is the mean value of diameter. However, in his model to describe the relationship between concentration and settling velocity Barfod assumes the particles to be equal-sized spheres distributed at random in the fluid which sediment as if they were ordered in pairs with distance between centres equal to the mean value of the smallest distance between the particle centres in the suspension.

In a volume, \( V \), of suspension which is not too small compared to particle size the probability, \( P_r \), of finding \( r \) particles assuming a random dispersion is given by the Poisson distribution

\[
P_r = \frac{m^r e^{-m}}{r!}
\]

where \( m = cV \) and \( c \) is the mean number of particles per unit volume. The probability of finding more particles than the centre particle inside spherical shells with centre in a particle and volume, \( V \), and radius, \( R \), is

\[
P_{r+} = 1 - e^{-m} = 1 - \exp\left(-\frac{c4\pi R^3}{3}\right)
\]

The distribution function of the smallest distance between particles in random suspension can, therefore, be written

\[
\frac{dP_{r+}}{dR} = Z = c4\pi R^2 \exp\left(-\frac{c4\pi R^3}{3}\right)
\]

and the mean value is calculated

\[
\bar{R} = \frac{\int_0^\infty z dz dR}{\int_0^\infty z dR} = \frac{7}{4}
\]
where \( \int_0^\infty 2dR = 1 \)

\[
\bar{R} = \frac{3}{(c \frac{4}{3} \pi)^1/3} \int_{0}^{\infty} x^3 e^{-x^3} dx
\]

where \( x = R(c \frac{4}{3} \pi)^{1/3} \)

\[
\therefore \bar{R} = \frac{3}{(c \frac{4}{3} \pi)^{1/3}} \frac{1}{3} \Gamma \left( \frac{4}{3} \right) = 0.55296 c^{-\frac{1}{3}}
\]

2.5.58

The distribution is symmetrical which can be seen when comparing \( \bar{R} \) with the radius, \( R_{\text{max}} \), for the maximum value of \( z \).

\[
R_{\text{max}} = 0.54192 c^{\frac{1}{4}}
\]

2.5.59

Fluid will have an upward movement because of the settling particles and the volume of particles moving down equals the volume of fluid moving up, thus

\[
u_p c = u_c (1-c)
\]

2.5.60

where \( u_p \) and \( u_l \) are particle and liquid velocities respectively. The velocity of particles relative to the fluid is

\[
u = u_p + u_c
\]

\[
u_p = u(1-c)
\]

2.5.61

\[
u_p = u \left( 1 - \frac{\pi}{6 b^3} \right)
\]

2.5.62

where \( b \) is the ratio of particle diameter to the size of the cubic lattice in which it is found.
2.6 Statistical Hydrodynamics of Two Phase Dispersions

Statistical hydrodynamics in two phase dispersions were considered by Buyevich (16,17,18,19) based on the statistics of random flights reviewed by Chandrasekhar (24).

Consider a volume, V, of a monodisperse system containing \( n_v \) particles and assume that the macroscopic characteristics of the system are unchanged. This allows a coordinate system to be used in which the mean velocities of particles are zero. Assume, at present, that the random motion of a single particle is independent of the behaviour of other particles. The objective is to calculate \( W(n_A) \), the probability that \( n_A \) particles will be within a certain volume, \( A \), within \( V \). That is \( A \in V, A \ll V \).

Introduce the cell numbers \( N_v \), \( N_A \) equal to the number of particles which can be placed into the corresponding volumes in a state of close packing.

\[
N_v = \frac{V_{\text{cp}}}{\frac{4}{3} \pi a^3} \quad n_v \leq N_v
\]
\[
N_A = \frac{A_{\text{cp}}}{\frac{4}{3} \pi a^3} \quad n_A \leq N_A
\]  \[2.6.1\]

where \( c_{\text{cp}} \) is the concentration for close packing, \( a \) is the particle radius. The mean concentration of particles in an arbitrary state can be described by means of the fraction of filled cells, \( m \), or the mean volume concentration, \( c_m \),

\[
m = \frac{n_v}{N_v}
\]
\[
c_m = \frac{4 \pi a^3 n_A}{3 N_v} = mc_m
\]  \[2.6.2\]
Characterize the situation in the volume, \( A \), at time, \( t \), with the local concentration

\[
\sigma_A = \frac{n_A}{N_A} = \sigma + \delta n_A
\]

2.6.3

The individual cells and particles are indistinguishable. To calculate the quantity \( W(n_A) \), fill the lattice of \( N_v \) cells with \( n_v \) particles inserted in succession. The probability of the first particle entering into some cell of \( A \) is equal to \( \frac{N_A}{N_v} \). The probability of the second particle going into some empty cell of \( A \) equals \( (\frac{N_A-1}{N_v}) \) and so on.

Therefore, the probability that the first \( n_A \) particles fall into \( N_A \) cells of the volume, \( A \), and all the other particles into the remaining \( N_v-N_A \) cells is

\[
W(n_A) = \frac{N_A!}{(N_v-n_A)!} \cdot \frac{(N_v-n_v)!}{N_v!} \cdot \frac{N_B!}{(N_v-n_B)!}
\]

2.6.4

where \( B = V-A \). The probability that only \( n_A \) particles are in volume, \( A \), is obtained by multiplying by \( \left( \frac{n_v}{n_A} \right) \)

\[
W(n_A) = \left( \frac{N_v}{n_v} \right)^{-1} \left( \frac{N_B}{n_B} \right) \left( \frac{N_A}{n_A} \right)
\]

2.6.5

\[
\sum_{n_A=0}^{N_A} W(n_A) = \left( \frac{N_v}{n_v} \right)^{-1} \sum_{n_A=0}^{N_A} \left( \frac{N_B}{n_B} \right) \left( \frac{N_A}{n_A} \right) = 1
\]

2.6.6

In the problem of dilute systems determination of the upper limit of eq. 2.6.6 does not arise since close packing is very unlikely. Using Stirling’s formula for factorials
\[
\frac{W(n_A)}{n_B} \propto \left( \frac{N_A}{n_A} \right)^{n A (1-m)} (N_A - n_A)^{N_A - n_A} \]
\[
W(n_A) \propto \left( \frac{N_A}{n_A} \right)^{n A (1-m)} (N_A - n_A)^{N_A - n_A} \]

Taking \( n_A \rightarrow \infty \) and \( N_A \rightarrow \infty \) by \( m \) constant from eq. 2.6.7 the distribution of the number of particles in a large volume, \( A \), is

\[
W(n_A) \propto \frac{1}{[2\pi N_A m(1-m)]^{1/2}} \exp \left( - \frac{(n_A - m N_A)^2}{2N_A m(1-m)} \right) \]

The first two moments of the distributions of eq. 2.6.6, 2.6.8, are

\[
\left( \frac{n_A - m N_A}{n_A} \right) = \delta n_A = 0 \quad , \quad \bar{n}_A = m N_A
\]
\[
\left( \frac{n_A - m N_A}{n_A} \right)^2 = \left( \delta n_A \right)^2 = m \left( 1 - m \right) N_A
\]
\[
= \left( 1 - \frac{c_m}{c_{cf}} \right) \bar{n}_A \quad 2.6.9
\]

As \( m \rightarrow 0 \) and \( m \rightarrow 1 \), that is infinitely dilute and close-packed states, this quantity vanishes. From eq. 2.6.9 the average concentration follows

\[
\left( \delta c_A \right)^2 = \frac{\left( \frac{4}{3} \pi a^3 \right)^2 \left( \delta n_A \right)^2}{A^2}
\]
\[
= \frac{c_m^2}{\bar{n}_A} \left( 1 - \frac{c_m}{c_{cf}} \right) \quad , \quad \bar{n}_A > 1 \quad 2.6.10
\]

If \( n = n_A(0) \) be the number of particles in volume, \( A \), observed at time, \( t = 0 \), the probability can be found that at time, \( t \), the volume, \( A \), will contain \( n_A(t) = n+k \) where \(-n \leq k \leq N_A - n \). Introduce the probability, \( P(t) \), of the emergence of a particle from \( A \) during the time, \( t \). Any location of a particle within \( A \) is equally probable and
particles are independent. \( P(t) \) increases from 0 at \( t = 0 \) to 1 as \( t \to \infty \). Estimate the emergence probability for \( i \) particles in time, \( t \), under the condition that \( n \) particles were present in A at \( t = 0 \).

\[
S_n^{(i)}(t) = \binom{n}{i} P^i(t) \left[ 1 - P(t) \right]^{n-i}, \quad i \leq n
\]

2.6.11

The mean flux of particles through A is zero at statistical equilibrium between A and B = V - A. Thus the probabilities of passing particles into or out of A are equal, and the probability of \( j \) particles entering A is

\[
E_j(t) = S_j^{(n)}(t) = \sum_{n-j} \binom{n}{j} \binom{n}{j} \left[ m^{(1-m)} \right]^{(n-j)} \binom{m}{j} \left[ 1 - P(t) \right]^{n-j}
\]

\[
= \binom{n}{j} \left[ m P(t) \right]^{j} \left[ 1 - m P(t) \right]^{(n-j)}, \quad j \leq N_A
\]

2.6.12

Buyevich extended the analysis to polydisperse systems and here \( j \) refers to the size, \( j \), of particles so that for each size eq. 2.6.2 becomes

\[
c^{(i)}_m = \frac{n^{(i)}_v}{N_v} \frac{4}{3} \pi a_j^3
\]

\[
c_m = \sum c^{(i)}_m = \frac{1}{V} \sum n^{(i)}_v \frac{4}{3} \pi a_j^3
\]

2.6.13

and eq. 2.6.8 becomes

\[
W^{(i)}(n^{(i)}_A) \propto \frac{1}{\left[ 2 \pi N^{(i)}_A m (1-m) \right]^{\frac{3}{2}}} \exp \left( -\frac{(n^{(i)}_A - m N^{(i)}_A)^2}{2 N^{(i)}_A m (1-m)} \right)
\]

2.6.14

where
Eq. 2.6.9 can similarly be modified for polydisperse systems. In application of statistical hydromechanics to disperse systems Buyevich considers a system of uniform particles, radius, \( a \), density, \( \rho' \), in a fluid of density, \( \rho \), and viscosity, \( \mu \). Each particle has a velocity, \( \mathbf{w} \), the local concentration is \( c \) and the velocity and pressure fields in the fluid are \((\mathbf{v}, p)\). Averaged over the fluid volume in the vicinity of each particle, \( V - \frac{4}{3}\pi a^3 \),

\[
\begin{align*}
\mathbf{v}(t, \mathbf{r}) &= \frac{1}{V - \frac{4}{3}\pi a^3} \int_V \mathbf{v}_o(t, \mathbf{r})(1 - f(\mathbf{r} - \mathbf{r}')) d\mathbf{r}' \\
p(t, \mathbf{r}) &= \frac{1}{V - \frac{4}{3}\pi a^3} \int_V p_o(t, \mathbf{r})(1 - f(\mathbf{r} - \mathbf{r}')) d\mathbf{r}'
\end{align*}
\]

2.6.16

Here \( f(\mathbf{r}) \) is unity inside a particle and zero outside it, \( \mathbf{r} \) is the radius vector of the particle centre, \((\mathbf{v}_o(\mathbf{r}), p_o(\mathbf{r}))\) the true fluid field, and \( \bar{c} \) the average concentration is given by

\[
\bar{c} = \frac{1}{V} \int_V f(\mathbf{r} - \mathbf{r}') d\mathbf{r}'
\]

2.6.17

The distribution function, \( f(\mathbf{w}; t, \mathbf{r}) \) exists for the particle velocity normalized to the mean number concentration of particles, \( n(t, \mathbf{r}) \). Similar functions exist for \( c, v \) and \( p \). These may be written as the sum of an average value and a 'pseudo-turbulent' fluctuation.
\[ w = \bar{w} + w' \]
\[ v = \bar{v} + v' \]
\[ c = \bar{c} + c' \]
\[ p = \bar{p} + p' \]

and the distribution function is governed by the Kolmogoroff-Chapman equation

\[
f(w; t, r) = \int W(\Delta r, \Delta w, \Delta t | r_0, w_0, t_0) f(w_0, t_0, r_0) \, dw_0 \, dr_0
\]

2.6.19

where

\[
\Delta r = r - r_0 \\
\Delta w = w - w_0 \\
\Delta t = t - t_0
\]

and \( W(\Delta r, \Delta w, \Delta t | r_0, w_0, t_0) \) is the probability of transition for a particle passing from the volume element \((r_0, r_0 + dr_0; w_0, w_0 + dw_0)\) of the phase space where it was at \( t_0 \) to the volume element \((r, r + dr; w, w + dw)\) during the time interval, \( \Delta t \). This is valid provided that the functions \( f \) and \( W \) exist and are continuous, \( f \to 0 \) fast enough as \( |w| \to \infty \).

If eq. 2.6.19 exists then

\[
\frac{\partial f}{\partial t} + \frac{1}{\Delta t} \left[ \frac{\partial}{\partial r} f(\Delta r) + \frac{\partial}{\partial w} f(\Delta w) \right] + \frac{1}{2\Delta t} \left[ \frac{\partial^2}{\partial r \partial w} f(\Delta r, \Delta w) \right] = 0
\]

2.6.20
where

\[ \langle \Delta r \rangle = \iint \Delta r \, \mathcal{W}(\Delta r, \Delta w, \Delta t \mid r, w, t_0) \, dr_0 \, dw \]
\[ = w \, \Delta t + O((\Delta t)^2) \quad 2.6.21 \]

\[ \langle \Delta w \rangle = w \, \Delta t + O((\Delta t)^2) \quad 2.6.22 \]

\[ \langle \Delta w \Delta w \rangle = 2 \Delta t + O((\Delta t)^2) \quad 2.6.23 \]

where \( H \) is the force per unit mass acting on a particle and eq. 2.6.21 - 2.6.23 are taken as the limit \( \Delta t \to 0 \), \( \Lambda \) is some unknown tensor. From eq. 2.6.20 - 2.6.23 the Fokker-Planck equation results

\[ \frac{\partial f}{\partial t} + w \frac{\partial f}{\partial r} + \frac{\partial}{\partial w} \left( \mathbf{H} f \right) + \left( \frac{\partial}{\partial w^i} \frac{\partial}{\partial w^j} \right) : \left( \mathbf{A} f \right) = 0 \quad 2.6.24 \]

The force, \( \mathbf{H} \), can be represented as an average and fluctuating variable

\[ \mathbf{H} = \mathbf{\bar{H}} - B \mathbf{w'} \quad 2.6.25 \]

and the Fokker-Planck equation becomes

\[ \frac{\partial f}{\partial t} + w \frac{\partial f}{\partial r} + \frac{\partial}{\partial w} \left[ \left( \mathbf{\bar{H}} - B(w) \right) - \mathbf{B} \mathbf{w'} \right] f - \left( \frac{\partial f}{\partial w'} \right) : \left( \frac{\partial}{\partial w} \mathbf{w'} \right)
+ \left( \frac{\partial}{\partial w^i} \frac{\partial}{\partial w^j} \right) : \left( \mathbf{A} f \right) = 0 \quad 2.6.26 \]

Multiplying eq. 2.6.26 by \( mf \) and \( mw'f \) and integrating over \( -\infty < \mid w' \mid < \infty \)
\[ \frac{D}{Dt}(\rho_e \tilde{c}) + \rho_e \tilde{c} \frac{\partial \mathbf{w}}{\partial \mathbf{r}} = 0 \] 2.6.27

\[ \rho_e \tilde{c} \frac{D \mathbf{w}}{Dt} = - \frac{\partial \mathbf{T}'}{\partial \mathbf{r}} + \rho_e \tilde{c} \mathbf{g} + \bar{\mathbf{f}} \] 2.6.28

where \( \mathbf{T}' \) is the pseudo-turbulent stress tensor, \( \mathbf{g} \) the gravity vector and \( \bar{\mathbf{f}} \) is the interaction force between two phases.

\[ \mathbf{T}' = \rho_e \tilde{c} \mathbf{w}' \wedge \mathbf{w}' \] 2.6.29

For the fluid phase Buyevich presents

\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{c} - (1-c) \frac{\partial \mathbf{v}}{\partial \mathbf{r}} = 0 \] 2.6.30

\[ \rho(1-c) \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = - \frac{\partial \mathbf{p}}{\partial \mathbf{r}} + \mu \frac{\partial}{\partial \mathbf{r}} \left( \frac{\partial \mathbf{v}_i}{\partial r_j} + \frac{\partial \mathbf{v}_j}{\partial r_i} \right) + \rho(1-c) \mathbf{g} - \mathbf{f} \] 2.6.31

These are the continuity and Navier-Stokes equations respectively. For the random fluctuations Buyevich uses Fourier-Stieltjes integrals

\[ c' = \iint \exp i(\omega t + k \cdot \mathbf{r}) \, d\mathbf{z} \]
\[ p' = \iint \exp i(\omega t + k \cdot \mathbf{r}) \, d\mathbf{z} \]
\[ v' = \iint \exp i(\omega t + k \cdot \mathbf{r}) \, d\mathbf{z} \]
\[ w' = \iint \exp i(\omega t + k \cdot \mathbf{r}) \, d\mathbf{z} \] 2.6.32
where integration is carried out over all frequencies, $\omega$ and all wave number space, $k$. He solves the equations by a method of approximations. The solution obtained is valid when the time and space scales of the mean flow exceed considerably those for random pseudo-turbulent motion of particles in a fluid.
2.7 Models Involving Formation of Clusters of Spheres

In order to explain the enhancement of average particle settling velocity with increasing volume concentration of particles, as originally shown by Kaye and Boardman\(^{(57)}\), Johne\(^{(54,55)}\), and Koglin\(^{(58)}\), several workers have proposed that during sedimentation clusters of spheres may form whose velocity is greater than that of the individual particles comprising the cluster. Such a model was originally proposed by Johne\(^{(54,55)}\), and developed by Koglin\(^{(59,60,61)}\), Brauer and Thiele\(^{(9)}\), and Koglin and Al-Taweel\(^{(62)}\).

The Stokes settling velocity of a single sphere of diameter, \(x\), in a fluid of density, \(\rho\), and viscosity, \(\mu\), is given by

\[
U_s = \frac{g}{18} \frac{(\rho_s - \rho) x^2}{\rho} \tag{2.7.1}
\]

where \(\rho_s\) is the particle density. If now a cluster of discrete particles is considered to be a homogeneous sphere with diameter, \(x_k\), density, \(\rho_k\), and viscosity, \(\mu_k\), where internal circulation may take place, its settling velocity would be given by

\[
U_k = K_{HR} \frac{g}{18} \frac{(\rho_k - \rho) x^2}{\mu_k} \tag{2.7.2}
\]

where \(K_{HR}\) is the Hadamard-Rybczinski correction factor accounting for the increased settling velocity above that of a rigid sphere due to internal circulation, \(\mu_k\) is the viscosity of the drop and
The relative settling velocity of the cluster then becomes

\[ \frac{u_k}{u_s} = K_{HR} (\frac{\rho_e - \rho}{\rho_g - \rho}) (\frac{x_k}{x})^2 \]  \hspace{1cm} 2.7.4

and since

\[ (\rho_k - \rho) = \frac{k_x}{x_k^2} (\rho_g - \rho) \]  \hspace{1cm} 2.7.5

\[ \frac{u_k}{u_s} = K_{HR} \frac{k_x}{x_k} \]  \hspace{1cm} 2.7.6

If it is assumed that \( \mu = \mu_k \) then \( K_{HR} = 1.2 \), but if the viscosity is considered to increase with increasing concentration the correction factor lies between 1 and 1.2, and increases with decreasing concentration. Koglin and Al-Taweel(62) quote the expression for the suspension viscosity, \( \mu_k \), which is valid up to a volume concentration of 50%.

\[ \mu_k = \mu \left( 1 + 2.5c + 7.05c^2 + 37.37c^3 \right) \]  \hspace{1cm} 2.7.7

They carried out experiments dropping clusters of spheres in a suspension of volume concentration 0.5%. The clusters showed an inner circulation of the particles comprising the cluster. They indicate that the cluster attains a constant void fraction for clusters containing twelve or more particles. Curves for the settling velocity relative to the Stokes velocity of a single sphere typical of those comprising the cluster are shown plotted against the number of particles in the cluster as in Fig. 2.7.1
Other work of an experimental nature was carried out on clusters of spheres by Jayaweera, Mason and Slack\(^{(53)}\). Until their work predictions existed for systems of two spheres which they summarized as follows:

1. Pairs of spheres always fall faster under gravity than do single spheres falling alone.

2. This enhancement in the rate of fall is more marked when the spheres are close together.

3. Pairs of equal spheres falling together maintain a constant separation and orientation.

4. Members of a pair fall vertically only if their line of centres is either vertical or horizontal. Otherwise their velocity has a component in a downward sense along the line joining their centres.

5. If two spheres of unequal size have the same individual settling velocity, the smaller will always move faster than the larger.

These results are, of course, well known and have
already been described in section 2.3. Experiments by Jayaweera, Mason and Slack are summarized below.

For equal-sized pairs of spheres falling side by side, the rate of fall is greater than of either sphere individually and the enhancement is greater when the spheres are close together. This applies over the whole range of Reynolds numbers.

For Re<0.03 the spheres show no tendency to separate or rotate but for Re>0.05 each sphere rotates inwards, the rotation being indicated in Fig. 2.7.2, about a horizontal axis.

![Fig. 2.7.2 Rotation of two equal-sized spheres](image)

For equal-sized spheres falling vertically one behind the other, when Re>1 the rear sphere becomes accelerated in the wake of the front sphere. When Re>4 this acceleration is already noticeable at ten sphere diameters apart. At large distances the velocity of approach varies inversely as the separation. The relative velocity on apparent impact
is about half the terminal velocity of an individual sphere. Upon impact the rear sphere slides round the front one until the horizontal position is reached after which the spheres separate and begin to rotate.

A pair of equal spheres with their line of centres inclined to the horizontal, each with Re>1 appear to move along the line of centres as well as falling vertically.

For two spheres of unequal size but the same terminal velocity the smaller sphere falls faster than the larger.

For three equal-sized spheres released in a horizontal straight line Jayaweera, Mason and Slack indicate three cases.

For 0.06<Re<0.16, spheres in contact or equally spaced up to six diameters apart, the centre sphere moves slightly ahead, one of the laggards then moves between the other two, and the third, now trailing, passes between the other two. The interchanging of position continues but the spheres keep close together and do not separate. The sphere temporarily in the lead stops rotating. If the spheres are not initially equally spaced, one sphere is left behind. If the spheres are numbered and separated as

(1) ← a ← (2) ← b ← (3)

the sphere left behind depends on the ratio of \( \frac{b}{a} \) as shown in Fig. 2.7.3

<table>
<thead>
<tr>
<th>( \frac{b}{a} )</th>
<th>&lt;1.17</th>
<th>1.20-1.33</th>
<th>1.33-1.40</th>
<th>1.5</th>
<th>1.6-2.0</th>
<th>&gt;2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>sphere left behind</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Fig. 2.7.3 Sphere left behind in cluster of three spheres.
For $b<2$ one sphere is left behind after one or two interchanges, but if $b>2$, sphere three is left behind from the start.

For $0.16<Re<3$, if between three and six contacting spheres in a straight line are released they separate and form a regular polygon, all spheres lying the same horizontal plane.

For $Re>3$ the polygon is not regular and progressively breaks down until with $Re>7$ the spheres separate.

When three to six spheres, at $0.06<Re<7$, starting in a cluster fall together:

1. Their speed of fall is greater than that of a single sphere.
2. This enhancement is greater the more compact the cluster.
3. If the spheres are staggered by a few diameters they eventually drew together and form a regular polygon in the same horizontal plane.
4. The polygon expands slowly and at a decreasing rate during fall.
5. Three spheres arranged in a horizontal isosceles triangle show an oscillation of the apex sphere in a vertical spiral. The other two execute horizontal oscillations on the same line of centres. The oscillations eventually die out as the spheres form an equilateral triangle. This also happens with four, five and six sided polygons.
6. The final configuration is reached more slowly with larger numbers of spheres.
7. During the early life of the polygon each sphere
rotates inwards.

8. When separation of the spheres exceeds a certain amount (about six diameters at Re=1, three diameters at Re≈7) rotation ceases but separation continues. Regular polygons are not formed if the initial separation exceeds this distance.

9. If the spheres in the cluster are arranged asymmetrically the densest portion travels fastest causing a tilt in the cluster followed by motion in the direction of the tilt.

At Re<0.06 spheres tend to follow their initial configuration but fall faster than isolated spheres. They show no tendency to form regular polygons and if released in such a pattern are susceptible to small perturbations.

At Re>7 spheres of a cluster separate quickly with a sudden onset of rotation, but this soon ceases. There is no tendency to form regular polygons.

A compact cluster of more than seven spheres shows no tendency to form a regular polygon but tends to break up into two or more groups. A cluster arranged as a regular heptagon is unstable.

The practical work of Jayaweera, Mason and Slack was followed up by a theoretical study by Hocking\(^{(50)}\) as an attempt to see how many of their results could be explained by the use of the creeping motion equations, in which case the equations are only applicable at Re≪1 and a≪1 where a is the particle radius and s, the separation of spheres. Terms in the solution of \(O\left(\frac{s^3}{a^3}\right)\) and greater are neglected.

The force on a sphere, I, moving with velocity, \(v_i\), in the
presence of another moving with velocity, \( \mathbf{v}_2 \), and position vector, \( \mathbf{r}_2 \), relative to the first is

\[
\mathbf{F} = -6\pi \mu a \left[ \mathbf{v}_1 - \frac{3a}{2r_1^3} (\mathbf{v}_2 \cdot \mathbf{r}) \mathbf{r} - \frac{3a}{4r_1^2} \left( \mathbf{v}_2 - \frac{(\mathbf{v}_2 \cdot \mathbf{r})}{r_1^2} \right) \mathbf{r} \right]
\]

2.7.8

If the positions and velocities of \( i \) spheres are denoted by \( \mathbf{r}_i \), \( \mathbf{v}_i \) (\( i = 1,2,\ldots,n \)), the nondimensional equations expressing the balance of forces on each sphere are

\[
\mathbf{v}_i - \sum_{j \neq i} \frac{3}{4r_{ij}^3} \mathbf{v}_j - \sum_{j \neq i} \frac{3}{4r_{ij}^3} (\mathbf{v}_j \cdot \mathbf{r}_{ij}) \mathbf{r}_{ij} = \mathbf{z}
\]

2.7.9

where \( \mathbf{z} \) is a unit vector in the downward vertical, \( \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \) and terms \( O\left(\frac{1}{r_{ij}^3}\right) \) are neglected. The solution of these equations neglecting terms \( O\left(\frac{1}{r_{ij}^3}\right) \) is

\[
\mathbf{v}_i = \mathbf{z} + \sum_{j \neq i} \frac{3}{4r_{ij}^3} \mathbf{z} + \sum_{j \neq i} \frac{3}{4r_{ij}^3} (\mathbf{z} \cdot \mathbf{r}_{ij}) \mathbf{r}_{ij}
\]

2.7.10

If \( n = 2 \) the velocities of both spheres are the same and there is no change in configuration. The equations determining the paths of the spheres, and hence change in configuration are \( \frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i \) where \( \mathbf{v}_i \) is given by eq. 2.7.10, or by changing the time scale by a factor of \( \frac{4}{3} \)

\[
\frac{d\mathbf{r}_i}{dt} = \frac{4}{3} \mathbf{z} + \sum_{j \neq i} \frac{1}{r_{ij}^2} \mathbf{z} + \sum_{j \neq i} \frac{z \cdot \mathbf{r}_{ij}}{r_{ij}^3} \mathbf{r}_{ij}
\]

2.7.11

The relative motions of the spheres are unaffected by the term, \( \frac{4}{3} \mathbf{z} \). An alteration of scale in the configuration of the spheres is equivalent to a change in the time scale,
since the remaining terms in eq. 2.7.11 are homogeneous functions of the positions. Without loss of generality any convenient length in the initial configuration can be chosen as unit. With n = 3, the relative motion is given by the two vector equations

\[
\frac{d\mathbf{r}_{12}}{dt} = \left(\frac{1}{r_{12}} \mathbf{z} + \frac{z_2 r_{13}}{r_{13}^3} \mathbf{r}_{13}\right) - \left(\frac{1}{r_{23}} \mathbf{z} + \frac{z_3 r_{12}}{r_{23}^3} \mathbf{r}_{12}\right) \quad 2.7.12
\]

\[
\frac{d\mathbf{r}_{23}}{dt} = \left(\frac{1}{r_{21}} \mathbf{z} + \frac{z_1 r_{21}}{r_{21}^3} \mathbf{r}_{21}\right) - \left(\frac{1}{r_{31}} \mathbf{z} + \frac{z_3 r_{21}}{r_{31}^3} \mathbf{r}_{31}\right) \quad 2.7.13
\]

A simple result follows immediately from these equations. If A is the vector area of the triangle formed by the three spheres defined by \(2A = \mathbf{r}_{12} \times \mathbf{r}_{23}\), the rate of change of A is

\[
\frac{dA}{dt} = \frac{\mathbf{z} \times \left(\frac{r_{21}^2 + r_{23}^2 + r_{31}^2}{r_{12}}\right)}{2} \quad 2.7.14
\]

Hence \(\mathbf{z} \cdot \frac{dA}{dt} = 0\), that is the horizontal projection of the triangle is of constant area.

Three spheres initially placed in a horizontal line will always lie in the same vertical plane. The coordinates of two spheres relative to the third may be written \((x_1, z_1)\), \((x_2, z_2)\) and eq. 2.7.12 and 2.7.13 reduce to

\[
\begin{align*}
\frac{dz_1}{dt} &= \frac{2(z_1 - z_2)}{r_2^3} + \frac{(x_1 - x_2)^2}{r_2^3} - \frac{2z_2^2 + x_2^2}{r_2^3} \\
\frac{dz_2}{dt} &= \frac{2(z_1 - z_2)^2}{r_3^3} + \frac{(x_1 - x_2)^2}{r_3^3} - \frac{2z_2^2 + x_2^2}{r_3^3} \\
\frac{dx_1}{dt} &= \frac{(z_1 - z_2)(x_1 - x_2)}{r_2^3} - \frac{z_2 x_2}{r_2^3} \\
\frac{dx_2}{dt} &= \frac{(z_1 - z_2)(x_1 - x_2)}{r_3^3} - \frac{z_2 x_2}{r_3^3}
\end{align*}
\quad 2.7.15
The initial conditions are $z_1 = z_2 = 0$, $x_1 = 1$, $x_2 = -c$, so that the motion is given relative to the initially central sphere and all cases are covered by $1 \leq c < \infty$.

Eq. 2.7.15 were solved numerically by Hocking and typical results are shown in Fig. 2.7.4.

![Fig. 2.7.4](image)

*Fig. 2.7.4 Positions of three spheres initially in horizontal line, relative to the central sphere (a) $c = 3$, (b) $c = 1.5$, (c) $c = 1.2$, after Hocking.*

Four spheres initially placed symmetrically in a horizontal line will fall in a vertical plane and retain their symmetry. With the same axes as above, the coordinates of the spheres are $(x_1, z), (x_2, 0), (-x_1, z), (-x_2, z)$.
and eq. 2.7.11 gives

\[
\begin{align*}
\frac{dx_1}{dt} &= \frac{2}{z} \left[ \frac{x_1 - x_1}{((x_2 - x_1)^2 + z^2)^\frac{3}{2}} - \frac{x_2 - x_1}{((x_2 - x_2)^2 + z^2)^\frac{3}{2}} \right] \\
\frac{dx_2}{dt} &= -\frac{2}{z} \left[ \frac{x_2 - x_1}{((x_2 - x_1)^2 + z^2)^\frac{3}{2}} + \frac{x_1 - x_1}{((x_1 - x_1)^2 + z^2)^\frac{3}{2}} \right] \\
\frac{dz}{dt} &= \frac{1}{2x_1} - \frac{1}{2x_2}
\end{align*}
\]

2.7.16

The observed attainment of the steady configuration for three, four, five and six spheres cannot be explained by Hocking’s theory, neither does it describe the slow separation as clusters fall, Hocking assumes due to a small inertial effect, neglected in the low Reynolds number solution.

Observations were made by Woodward (98) using monodispersed polystyrene latex spheres of 2.12 \( \mu \)m diameter in water. He was under the impression that a uniform dispersion of spheres at volume concentration, \( c \), would settle with each particle moving at its Stokes velocity. However, if he introduced a small quantity of particles at the top of a sedimenting vessel with an initial volume concentration, \( c_0 \), then the rate of fall of the particles collectively increased to velocities much higher than the Stokes velocity. He observed that particles initially placed on the top of the liquid column drew into a tear-shaped drop and began to settle and a trail of particles was lost from the top of the drop. Fig. 2.7.5 shows the velocity of fall of the bottom boundary of particles at various initial concentrations plotted against time. Fig. 2.7.6 shows velocity of fall
plotted against time, for where the solid lines are for the same concentration, \( c_0 = 0.001 \) for different tube diameters. The dashed curve shows the effect of changing the initial quantity of particles added. Woodward suggests an upper limit on the downward accelerating force

\[
F = (\rho_0 - \rho) c_0 A z(t) g
\]

2.7.17

where \( \rho_0 \) and \( \rho \) are the densities of the particles and water respectively, \( A \) is the cross sectional area of the tube, \( z(t) \) is the coordinate describing the time displacement of the bottom particle boundary from the fluid surface.

Since only a fraction, \( f(c_0, t) \ll 1 \), of the particles are effective in causing the accelerating force for the collective motion, for the bottom boundary

\[
\frac{d}{dt} \left( m \frac{dz}{dt} \right) = f mg + F_f
\]

2.7.18
where \( m \approx (\rho_s - \rho) c_0 A z(0) \) and \( F_F = -k \frac{dz}{dt} \), the friction force retarding the particles; this is a modified form of Stokes' law.

\[ D = 15 \]
\[ D = 10 \]
\[ D = 7 \]
\[ D = 5 \]
\[ D = 3 \]

**Fig. 2.7.6** Bottom boundary velocity vs. time for 5ml of 2.12\( \mu \)m particles for various tube sizes.

Noting that the moving collection took up its final form quickly he assumed \( f \) to be constant, \( f = f(c_0, t) \) and that the rate of loss of particles, \( \frac{dm}{dt} \), decreased quickly so that

\[
\frac{k dz}{dt} \gg \frac{dm}{dt} \cdot \frac{dz}{dt}
\]

With these approximations

\[
\frac{d^2 z}{dt^2} + \frac{k}{m} \frac{dz}{dt} = g
\]

2.7.19

giving a solution

\[
z = C_1 (1 - \exp \left( -\frac{kt}{m} \right)) + \frac{f_{mg} t}{k} \]

2.7.20

and the zero-order result for the velocity of fall

\[
U = \frac{f_{mg} + C_1 \frac{k}{m} \exp \left( -\frac{km}{t} \right)}{\frac{k}{m}}
\]

2.7.21

where \( C_1 \) is an arbitrary constant.
2.8 Models Involving Flow Past Arrays of Spheres

Owing to the difficulty of solution of the creeping motion and continuity equations for a number of spheres immersed in a fluid imposed by the complicated boundary conditions, several workers have considered flow past arrays of particles.

Hasimoto (45) considered the steady motion of an incompressible viscous fluid past a periodic array of small obstacles with their centres at

\[ \mathbf{r}_n = n_1 \mathbf{a}^{(1)} + n_2 \mathbf{a}^{(2)} + n_3 \mathbf{a}^{(3)} \quad (n_1, n_2, n_3 = 0, \pm 1, \pm 2) \]

2.8.1

where \( \mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)} \) are the basic vectors determining the unit cell of the array. The continuity and creeping motion equations to be solved are respectively,

\[ \nabla \mathbf{v} = \mathbf{0} \]

2.8.2

\[ \mu \nabla^2 \mathbf{v} = \nabla \mathbf{p} + \sum_n F \delta(\mathbf{r} - \mathbf{r}_n) \]

2.8.3

where \( F \) is the force acting on one of the obstacles, \((x_1, x_2, x_3)\) defines, in Cartesian coordinates, the position vector, \( \mathbf{r} \), and \( \delta(\mathbf{r} - \mathbf{r}_n) \) denotes Dirac's delta function defined by the conditions

\[ \int_\mathbb{R} \delta(\mathbf{r} - \mathbf{r}_n) d\mathbf{r} = 1 \text{ if } \mathbf{r} = \mathbf{r}_n \]

\[ 0 \text{ if } \mathbf{r} \notin \mathbf{r}_n \]

2.8.4

Taking into account the periodicity of the flow field, \( \mathbf{v} \)
and \(-\nabla p\) are expanded in Fourier series

\[
\chi = \sum_k \gamma_k \exp(-2\pi i (k \cdot r)) \tag{2.8.5}
\]

\[
-\nabla p = \sum_k P_k \exp(-2\pi i (k \cdot r)) \tag{2.8.6}
\]

\[
k = n_1 b_1^{(y)} + n_2 b_2^{(z)} + n_3 b_3^{(z)} \tag{2.8.7}
\]

\(k\) represents the reciprocal lattice which satisfies

\[
k \cdot a^{(j)} = n_j \quad (j = 1, 2, 3) \tag{2.8.8}
\]

Using eq. 2.8.7 and 2.8.8, the reciprocal lattice vectors, \(b^{(j)}\), are found to be

\[
b^{(y)} = \frac{a \times a}{c_0}, \quad b^{(z)} = \frac{a \times a}{c_0}, \quad \Omega = a^{(y)}(a^{(y)} \times a^{(z)}) \tag{2.8.9}
\]

Multiplying eq. 2.8.2 and 2.8.3 by \(\exp 2\pi i (k \cdot r)\) and integrating over a unit cell in physical space

\[
-4\pi^2 k^2 \psi_k = -P_k + \frac{\Gamma}{c_0} \quad (k^2 = k \cdot k) \tag{2.8.10}
\]

\[
k \cdot \psi_k = 0 \tag{2.8.11}
\]

\[
P_k \times k = 0 \tag{2.8.12}
\]

Beginning by considering the terms for which \(k = 0\)

\[
P_0 = \frac{\Gamma}{c_0} \tag{2.8.13}
\]
which means that the force acting on an obstacle is balanced by the mean pressure gradient of the fluid.

Taking the scalar product of eq. 2.8.10 with \( \kappa \), for \( \kappa \neq 0 \)

\[
\kappa \cdot \mathbf{P}_k = \frac{1}{\zeta_0} (\kappa \cdot \mathbf{E}) = \kappa \cdot \mathbf{P}_o.
\]  \hspace{1cm} 2.8.14

or

\[
\mathbf{P}_k = \left( \frac{\kappa \cdot \mathbf{E}}{\zeta_0 \kappa^2} \right) \mathbf{k}.
\]  \hspace{1cm} 2.8.15

Substitution of eq. 2.8.15 into eq. 2.8.10 gives

\[
\mathbf{y}_k = \frac{1}{4\pi \mu \zeta_0} \left[ \left( \frac{\kappa \cdot \mathbf{E}}{\kappa^2} \right) \mathbf{k} - \frac{1}{\kappa^2} \right] (\kappa \neq 0).
\]  \hspace{1cm} 2.8.16

Eq. 2.8.5, 2.8.6, 2.8.13 - 2.8.16 are the periodic fundamental solutions of the creeping motion equations for the flow past a periodic array of obstacles. Their Cartesian components are given by

\[
\mathbf{v}_j = \mathbf{v}_{oj} - \frac{1}{4\pi \mu \zeta_0} \left( \sum_{i=1}^{3} F_i S_i - \sum_{i=1}^{3} F_i \frac{\partial S_i}{\partial x_i} \right)
\]  \hspace{1cm} 2.8.17

\[
- (\nabla F)_j = \frac{F_j}{\zeta_0} - \frac{1}{4\pi} \sum_{i=1}^{3} F_i \frac{\partial S_i}{\partial x_i}
\]  \hspace{1cm} 2.8.18

where

\[
S_1 = \frac{1}{\pi \zeta_0 \kappa \phi} \left( \exp - \frac{2\pi \iota (k \cdot r)}{\kappa^2} \right)
\]  \hspace{1cm} 2.8.19

\[
S_2 = - \frac{1}{4\pi^2 \zeta_0 \kappa \phi} \left( \exp - \frac{2\pi \iota (k \cdot r)}{\kappa^2} \right)
\]  \hspace{1cm} 2.8.20

and

\[
\Delta S_2 = S_1
\]  \hspace{1cm} 2.8.21

\[
\Delta S_1 = - 4\pi \left( \sum_{n} \delta (r - r_n) - \frac{1}{\zeta_0} \right)
\]  \hspace{1cm} 2.8.22
Hasimoto solves the problem for a periodic array of spheres of radius, \(a\), which is small compared to the distance between sphere centres, satisfying the boundary condition

\[
y = 0 \quad \text{at} \quad r = \sqrt{r^2} = a
\]

for three cases of cubic lattice, the simple cubic, body-centred cubic, and face-centred cubic, yielding an expression for the force on a sphere in the array

\[
F = \frac{6\pi \rho a U}{A}
\]

\[
A = 1 - ca + \frac{4\pi a^3}{3\zeta} - \left(\frac{16\pi^2}{45\zeta^2} + 6\zeta b^2\right)a^6 + O(a^8)
\]

where \(U\) is the mean velocity of fluid in any of the Cartesian directions. Values of \(b\) depend on the type of cubic array.

Childress\(^{(25)}\) considered the slow motion of a viscous fluid past a random array of nonoverlapping spheres where the expected force acting on a given sphere is a function of the statistical parameters defining the array. The method is asymptotic and assumes that the distribution is uniform and dilute, that is to say a uniform, small particle concentration. He solves the usual creeping motion equations with the boundary conditions

\[
y = -U \quad \text{on} \quad s_i \quad (i = 1, 2, \ldots, N) \quad \text{2.8.26}
\]

\[
y = 0, \quad p = 0 \quad \text{at} \quad \infty \quad \text{2.8.27}
\]
where a spherical volume of fluid, $V$, contains a number of spheres with surface, $s_i$, centred at a distance, $r_i$, from the fluid sphere centre. Childress writes the solution of the equations in the form

$$y = \sum_{i=1}^{N} y_i^{i} \tag{2.8.28}$$

where $y_i^{i}$ is a solution of the equations satisfying eq. 2.8.27 and $y_i^{i} = -\delta_{ij} U$ on $s_j$, $j = 1, 2, \ldots, N$. Summing as in eq. 2.8.28 gives the Stokes flow past $N$ spheres, the velocity at infinity being $U$. If the array is infinite, eq. 2.8.28 diverges and a bounded solution only exists for $U = 0$ and gives the same results obtained by Hasimoto, as already described. However, if $p$ in the creeping motion equation is defined as a perturbation a new body force is introduced, distributed uniformly over the volume, $V$, removing the divergent portion of eq. 2.8.28. The expected force on any given sphere is obtained by averaging over the ensemble of possible positions of all other spheres. $F_o$ denotes the Stokes drag on a single sphere, and the resulting expression for the expected force is

$$F = \left[1 + \frac{3}{2} \left(\frac{3}{2}\right)^{\frac{1}{2}} + \frac{125}{64} c \log c \right] F_o + c (D_{11} + D_{12}) F_o + \mathcal{O}(c^3 \log c) \tag{2.8.29}$$

The tensors, $D_{11}$ and $D_{12}$ are given by

$$D_{11} = \frac{3}{4 \pi c^3} \int_{|r_1| > 2a} \left( f_2 \Delta_{11} - E_{12} \right) d\Omega \tag{2.8.30}$$

$$D_{12} = \frac{3}{4 \pi c^3} \int_{|r_1| > 2a} \left( f_2 A_{12}^{12} \vec{E}_{12} + \vec{E}_{12}^3 + \frac{a^2}{3} \nabla^2 \vec{E}_{12} \right) d\Omega + k \vec{E} \tag{2.8.31}$$
\[ k = 5 + \frac{135}{64} \left( 2 \gamma_e + \log 2 + \frac{17}{5} \log 3 - \frac{760}{15} \right) \]
\[ \approx 6.08 \] 2.8.32

where \( \gamma_e \) is Euler's constant.

In eq. 2.8.30, \( f_2(r_1, r_2) \) is the two sphere distribution function for the array normalized so that \( f_2 \rightarrow 1 \) for large separation with the estimate

\[ f_2 = 1 + \mathcal{O} \left( \left| f_1 - f_2 \right|^{(2+\alpha)} \right) \quad \alpha > 0 \]

and satisfying the symmetry conditions \( f_2(r_1, r_2) = f_2(r_2, r_1) \).

The tensors \( A_j, j = 1, 2 \) are defined by the condition that

\[ \delta_{ij} F_0 + A_j : F_0 \quad \text{is the force acting on the sphere with centre } r_1 \text{ computed from a two sphere Stokes flow when the sphere at } r_j \text{ is instantaneously in motion with velocity, } -\mathbf{U}, \text{ and the other fixed}. \]

Finally \( B_{ij} = B(r_1 - r_2) \) is defined by

\[ B(r) = \frac{3 \alpha}{4} \left( r^{-1} I + r^{-3} \mathbf{r} \cdot \mathbf{r} \right) \]

and is \( \mathcal{O} \alpha \) times a fundamental solution of the Stokes equations.

Lundgren\(^{(67)}\) considered the uncertainty of using a viscosity or effective viscosity term when considering flow past spheres. For a suspension of spheres

\[ f(\mathbf{y}) = \lambda \nabla^2(\mathbf{y}) + B g \] 2.8.33

is used in the equation of motion

\[ \nabla(\mathbf{f} - \rho_3 \mathbf{r}) = \frac{\mathbf{f}}{1-c} \nabla^2(\mathbf{y}) - \frac{1}{1-c} f(\mathbf{y}) \] 2.8.34
where the velocity and pressure are averaged over the ensemble of particles, and an effective viscosity is defined

$$\tilde{\mu} = \frac{\mu - A}{1 - c} \quad 2.8.35$$

For small values of $c$, the expression for effective viscosity reduces to Einstein's ($33, 34$) expression, where

$$b = -\frac{c(1-c)}{\rho - \rho} \quad 2.8.36$$

$$\frac{\tilde{\mu}}{\mu} = \frac{1}{1 - 2.6c} \quad 2.8.37$$

The validity of the expression for effective viscosity is shown by Lundgren as a graphical comparison with experimental data. The theoretical curve is shown in Fig. 2.8.1

![Fig. 2.8.1 Effective viscosity vs. volume concentration, after Lundgren.](image)
Saffman considered slow flow round rigid spherical particles for three different cases; (1) regular periodic arrays of particles; (2) random free arrays; (3) random fixed arrays. For case (1) either the force or velocity is given, and the problem is to compute the other. In case (2), the force on each particle is given and the problem is to compute the statistics of the velocities. For case (3) the velocities are given and the average force is required. Particles are replaced by multiple distributions of forces at their centres, \( \mathbf{F}_i \). Thus the creeping motion equations become

\[
-\mu \nabla^2 \mathbf{v}_i + \mathbf{F}_i = \sum_{\alpha} \left[ F_{i\alpha} \delta(\mathbf{r}-\mathbf{r}_\alpha) + \mathbf{F}_{ij} \delta_{ij}(\mathbf{r}-\mathbf{r}_\alpha) + \mathbf{F}_{ijk} \delta_{ijk}(\mathbf{r}-\mathbf{r}_\alpha) + \ldots \right]
\]

2.8.38

The \( F_i \)'s in eq. 2.8.38 are calculated from the forces and moments. Thus, for instance, the drag, \( D_i \), and torque, \( T_i \), on particle \( \alpha \) are given by

\[
D_i^\alpha = -F_i^\alpha, \quad T_i^\alpha = -\varepsilon_{ijk} F_{jk}^\alpha.
\]

which for a single particle moving without rotation in a fluid otherwise at rest gives

\[
F_i = \frac{1}{2\eta} \left( \mathbf{F}_i + \frac{F_i r_i r_j}{r^3} \right) + \frac{\alpha^2}{24\eta^2 \rho^3} \left( \frac{F_i - 3F_i r_i r_j}{r^5} \right) - \frac{\alpha^2}{\rho} \mathbf{F}_i \delta(\mathbf{r})
\]

2.8.39

The corresponding velocity and pressure fields are
\[ P = \frac{1}{4\pi} \frac{F_{ik}}{r^3} + \frac{\alpha^2 F_i \delta_{ij}(r)}{\epsilon_0} \]  

2.8.40

The dipole component associated with \( F_{ijk} \) produces perturbations, \( O(\alpha^3 n) \), proportional to \( c \). Thus with an error, \( O(c) \), the finite size of the particles may be neglected. Thus eq. 2.8.38 may be simplified to

\[ -\mu \nabla^2 v_i + \rho v_i = \sum_{\alpha} F_i^\alpha(s_i - \epsilon^\alpha) \]  

2.8.41

Using the Fourier transform \( \hat{v}_i(k) \) of \( v_i(r) \) where

\[ \hat{v}_i(k) = \frac{1}{\tilde{2}\pi^3} \int v_i(r) \exp(-ik \cdot r) \, dr \]  

2.8.42

eq. 2.8.41 becomes

\[ \mu k^2 \hat{v}_i + ik \hat{k}_i \hat{P} = \frac{1}{\tilde{2}\pi^3} \sum_{\alpha} F_i^\alpha \exp(-ik \cdot r^\alpha) \]  

2.8.43

and the continuity equation

\[ k \cdot \hat{v}_i = 0 \]  

2.8.44

The pressure is now eliminated by operation on eq. 2.8.43 with

\[ P_j(k) = S_{ij} - \frac{k_i k_j}{k^2} \]  

2.8.45

noting that \( k^{-2} \) is a generalized function whose multiplication is non-trivial. Noting that
\[ \frac{1}{8\pi^3} \sum_x \exp(i k_i \cdot r^a) \approx n \delta(k) \quad \text{as } k \to 0 \quad 2.8.46 \]

and putting \( p = \hat{p} + p' \), \( \hat{p} = \hat{p} + p' \), where

\[ i k_i \hat{p} = n \bar{F}_s \delta(k) \quad 2.8.47 \]

\( F_i \) denotes the average force per particle. Then

\[ \hat{p} = in \bar{F}_i \frac{\partial}{\partial k_i} (\delta(k)) \quad 2.8.48 \]

which gives

\[ \bar{p} = n \bar{F}_i r_i \]

Eq. 2.8.43 is now written in modified form and can be multiplied by \( p_{ij}(k) \)

\[ \mu k^2 \psi_j = p_{ij}(k) \left[ \frac{1}{8\pi^3} \sum_a \bar{F}^a_i \exp(-i k_i \cdot r^a) - n \bar{F}_s \delta(k) \right] \quad 2.8.49 \]

This equation is then solved for each of the three cases.

For flow through a regular array

\[ F_i^x = F_i \quad 2.8.50 \]

\[ u_i = \frac{F_i}{\bar{6} \pi a} \quad = -\beta a \frac{F_i}{\bar{6} \pi a} \quad 2.8.51 \]

\[ \Delta u = \frac{\beta a}{\bar{6} \pi b} = \left( \frac{3 \beta c}{4 \pi^4} \right)^{\frac{1}{3}} \quad 2.8.52 \]

where \( b \) is the side of a simple cubic lattice and \( \beta \) is given by

\[ \beta = \int \left[ 1 - \sum_{(\vec{x} - \vec{x}^\prime)} \right] \frac{dx}{|x|^4} \quad 2.8.53 \]
where $k = \frac{2\pi}{b}$. According to the point particle approximation, the interaction effect on a particle is the difference between the velocity induced at the particle by forces concentrated at the other particles and the velocity induced at the particle by the same forces smeared out over space. The difference is measured by the geometrical factor, $\beta$.

For the random free array, if particle positions are completely uncorrelated, $\Delta U = 0$, because the randomness is equivalent to smearing out the force. Saffman verifies this conclusion.

For the third case, flow through a random fixed array, the forces, $F_i^\alpha$, are random but

$$\langle F_i^\alpha \exp(-i k \cdot r^i) \rangle = \bar{F}_i \langle \exp(-i k \cdot r^i) \rangle$$

and from eq. 2.8.49

$$\dot{\bar{V}}_i^\beta = \frac{k^2}{\mu} P_j^\beta \left[ \frac{\bar{F}_j}{3} \right] \exp(-i k \cdot r^j)$$

and because of eq. 2.8.54, $\langle \dot{\bar{V}}_i^\beta \rangle = 0$, and $\langle F_i^\beta \rangle = \bar{F}_i$.

This finally leads to the expression

$$U_i = \frac{\bar{F}_i}{\omega \mu a} \left( 1 - 3 \left( \frac{c_s}{2} \right)^\frac{1}{4} \right)$$

which is in agreement with the results obtained by Childress(25).
2.9 Other Mathematical Models for Sedimentation

Classic work on the sedimentation of a dispersion was carried out by Kynch (63) making the single main assumption that at any point in a dispersion the velocity of fall of a particle depends only on the local particle concentration, and the settling process is determined entirely from a continuity equation, without knowing details of the forces on particles. The theory predicts the existence of an upper surface to the dispersion in the liquid and that the motion of this surface and a knowledge of the initial distribution of particles is sufficient to determine the variation of velocity of fall with density for that particular dispersion.

Kynch considers a dispersion where the velocity, \( v \), of any particle is a function of local concentration, \( c \). The particle flux, \( Q \), is the product of these two. The concentration is assumed to be constant across any horizontal layer. As the concentration increases from \( c \) to \( c_{\text{max}} \) the velocity decreases from a finite value to zero. If \( z \) is the height of any level above the bottom of the column of dispersion in the time, \( dt \), the accumulation of particles between two heights, \( z \) and \( z+dz \) is the difference between the fluxes between the upper and lower layers, \( Q(z+dz) \) and \( Q(z) \) respectively.

\[
\frac{\partial}{\partial t} (cdz) \, dt = Q(z+dz) \, dt - Q(z) \, dt \\
\frac{\partial c}{\partial t} = \frac{\partial Q}{\partial z} \\
\text{and writing } V(c) = -\frac{dQ}{dc}
\]
On a graph where position, $z$, is plotted against time, $t$, curves are drawn through points with the same concentration. The coordinates $(z,t)$ and $(z + dz, t + dt)$ of two adjacent points on such a curve are related by

$$
\frac{\partial c}{\partial z} + V(c) \frac{\partial c}{\partial t} = 0
$$

2.9.3

Continuing eq. 2.9.3 and 2.9.4, the slope of such a curve is given by

$$
\frac{dz}{dt} = V(c)
$$

2.9.5

As $c$, and therefore $V$, is a constant along the curve, it must be a straight line. Therefore, on a $z$ against $t$ diagram, the concentration is constant along straight lines whose slope, $V$, depends only on concentration, as shown in Fig. 2.9.1

Fig. 2.9.1 Fall of surface of dispersion showing lines of density propagation, after Kynch.
The line KP of constant concentration has a slope $\frac{dz}{dt} = V$ determined by $c$ at the point, $K$. If the top of the dispersion is at $z = h$ the concentration increases from $c = c_a$ at $z = h$ to $c = c_b$ at $z = 0$. The line, OB, is the line of concentration, $c_b$. The equation of any line, KP, which crosses the $z$ axis at $z_0$

$$z = z_0 + V(c) t \quad 2.9.6$$

where $c_a < c < c_b$. Since $z_0$ is a known function of $c$, eq. 2.9.6 gives the concentration at any point, $z$, in the dispersion at time, $t$, where $(z,t)$ lies in the region $AOB$. At any point, $P$, since the speed of fall of the surface is that of the particles, then along $AB$

$$\frac{dz}{dt} = -V(c) \quad 2.9.7$$

The line, KP, represents the rise through the dispersion with a velocity, $V$, of a level, across which particles at concentration, $c$, fall with velocity, $v(c)$, downwards. In time, $t$, the number of particles which have crossed this level is $c(V+v)t$ per unit area. The level reaches the surface at the point, $P$, when this number equals the total number of particles, $n$, originally above the level, $K$. Thus

$$n(z_0) = \int_{z_0}^{h} c dz \quad 2.9.8$$

$$n(z_0) = c(V+v)t \quad 2.9.9$$
The lines, OC, correspond to the concentration, \( c_{\text{max}} \), at the bottom of the vessel and have slopes, \( v_{\text{max}} = v(c_{\text{max}}) \), and the equations of lines between OB and OC are

\[ z = V(c)t \]  \hspace{1cm} (2.9.10)

where \( c_b < c < c_{\text{max}} \). The fall of curve, BC, of the surface, the number of particles crossed by each level of constant density is now the total number of particles, \( N \), where

\[ N = \int_{c_b}^{c_{\text{max}}} c \, dc \]  \hspace{1cm} (2.9.11)

\[ N = c(V + v)t \]  \hspace{1cm} (2.9.12)

and since \( V \) and \( v \) are functions of \( z \) alone

\[ N = t f\left(\frac{z}{t}\right) \]  \hspace{1cm} (2.9.13)

The equations can be used to find values of \( c \) and \( v \) at points on ABC. At any point, P, the value of \( v \) is given according to eq. 2.9.7. If \( P \) lies close to A, assume that eq. 2.9.6, 2.9.7 and 2.9.9 are valid and eliminate \( V \) and \( v \):

\[ \left( z - \frac{tdz}{dt} \right)_P = z_o + \frac{n(z_o)}{c} \]  \hspace{1cm} (2.9.14)

which beyond B becomes

\[ \left( z - \frac{tdz}{dt} \right) = \frac{N}{c} \]  \hspace{1cm} (2.9.15)

Fasoli\(^{(34)} \) performed a theoretical analysis similar
to Kynch's but considered a distribution of particle sizes giving a distribution of velocities.

Smith(83,84), and Phillips and Smith(80) considered the differential sedimentation of two species and developed this into a model for sedimentation of a distribution of particle sizes. Each particle is assumed to be settling in a spherical fluid envelope which has a zero shear-stress condition at its boundary. These envelopes are assumed to be able to distort to fill void spaces which would be present for a rigid system of spheres. For a mixture of two solids each has a volume concentration and density, \((c_1, \rho_1)\) and \((c_2, \rho_2)\) respectively. The fluid density is \(\rho\). If \(S_1\) and \(S_2\) are the fluid-solid drag forces for each species and \(\frac{dp}{dz}\) a general pressure gradient, a force balance over each species for steady motion in the absence of particle surface forces gives

\[
\begin{align*}
&c_1(\rho_1 - \frac{dp}{dz}) + S_1 = 0 \\
&c_2(\rho_2 - \frac{dp}{dz}) + S_2 = 0
\end{align*}
\]

If each solid particle is associated with an envelope of fluid the effective concentrations are \(K_1\) and \(K_2\), so that a volume balance gives

\[
\frac{c_1}{K_1} + \frac{c_2}{K_2} = 1
\]

and force balances over the fluid envelopes give
The effective concentrations are explicitly obtained from eq. 2.9.16 and 2.9.18 as

\[
\begin{align*}
\kappa_1 &= \frac{\frac{d\rho}{dz} - \rho}{\rho_1 - \rho} \\
\kappa_2 &= \frac{\frac{d\rho}{dz} - \rho}{\rho_2 - \rho}
\end{align*}
\]

2.9.19

and an overall weight and pressure gradient balance

\[
\frac{d\rho}{dz} = c_1 \rho_1 + c_2 \rho_2 + (1-c_1-c_2) \rho
\]

2.9.20

leads to

\[
\begin{align*}
\kappa_1 &= \frac{c_1 \rho_1 + c_2 \rho_2 - (c_1 + c_2) \rho}{\rho_1 - \rho} \\
\kappa_2 &= \frac{c_1 \rho_1 + c_2 \rho_2 - (c_1 + c_2) \rho}{\rho_2 - \rho}
\end{align*}
\]

2.9.21

If the particle velocities are given by \(U_1, U_2\), and the fluid velocities in the envelopes as \(U'_1\) and \(U'_2\), for particles of diameters, \(d_1\) and \(d_2\), and a fluid viscosity, \(\mu\),

\[
\begin{align*}
\frac{S_1 \kappa_1}{c_1} &= - \frac{18 \mu}{d_1^2} \frac{\kappa_1 (U_1 - U'_1)(1-K_1)}{H_1} \\
\frac{S_2 \kappa_2}{c_2} &= - \frac{18 \mu}{d_2^2} \frac{\kappa_2 (U_2 - U'_2)(1-K_1)}{H_2}
\end{align*}
\]

2.9.22

where \(H\) is a function of \(K\) modifying the Stokes drag per unit volume, defined by
From a solution of the Navier-Stokes equation Lamb showed that for very slow motion

\[ u' = Ju \]  
2.9.23

where \( J \) is a function of \( K \). Applying this to the moving envelopes

\[ u_1' + J_1 (u_1 - u_1') (1 - \kappa_1) = u_2' + J_2 (u_2 - u_2') (1 - \kappa_2) \]  
2.9.24

Eq. 2.9.22 and 2.9.24 can both be rearranged and combined to give

\[ u_1 - \frac{s_1 h_1 d_1^2}{18 \mu c_1} \left( J_1 - \frac{1}{1 - \kappa_1} \right) = u_2 - \frac{s_2 h_2 d_2^2}{18 \mu c_2} \left( J_2 - \frac{1}{1 - \kappa_2} \right) \]  
2.9.25

\[ \frac{c_1}{K_1} \left( u_1 + \frac{s_1 h_1 d_1^2}{18 \mu c_1} \right) + \frac{c_2}{K_2} \left( u_2 + \frac{s_2 h_2 d_2^2}{18 \mu c_2} \right) = 0 \]  
2.9.26

These last two equations can be solved for the velocities of each species. Experimental measurements show that the agreement of the theory improves with decreasing ratio of large to small particle concentrations, and also increasing size ratio of large to small particles. These equations expand quite simply by replacing the subscript, 2, by \( i \) and summing over the distribution of sizes.

Phillips and Smith (79) also obtained a general model of multispecies systems suitable for solution by the method
of superposition of flow fields, considering an envelope containing a number of spheres which interact locally within that volume, but which is small compared to the size of the system. The drag on a particle chosen in the envelope due to the other \( n \) particles in the envelope is then

\[
F = - \sum_{j=0}^{n} 4\pi \nabla \left( \frac{3a\mu r\cdot u}{2} \right)
\]

This is done for each particle and the value compared with the weight of each and the difference removed by the application of a secondary field within the envelope. The operation is repeated until the drag on the object converges. The scheme is indicated in Fig. 2.9.2

---

Fig. 2.9.2 Primary (solid) and correcting (dotted) velocity fields in a multiparticle system. Velocity fields, \( v_{ij} \), emanating from particle, \( j \), of order, \( i \).

Thus the settling velocity of the particle considered is given by the sum of the field velocities taken at the particle centre.
where $U_o$ is the primary field velocity, $(\Delta U_o)_i$ is the drag correcting velocity and is summed over $m$ corrections, $(\delta U_o)_{ij}$ is the velocity contribution from the field of each of the $n$ interacting particles and summed over each particle at each correction.

Happel and Brenner\textsuperscript{(43)} carried out a theoretical investigation on the behaviour of sedimenting multiparticle systems for spherical particles in viscous flow assuming that a particle moves against a parabolic flow profile of fluid following solution of the creeping motion equations resulting in the viscous drag and pressure drop equations given in eq. 2.9.29 and 2.9.30

\[
D = -\frac{\rho u}{\mu} \left[ U - U_{CF} \left( 1 - \frac{b^2}{R^2} \right) \right] \quad 2.9.29
\]

\[
\Delta P = \frac{12 \rho a}{R^2} \left( 1 - \frac{b^2}{R^2} \right) \left[ (U_{CF} - U_o) - U_{CF} \frac{b^2}{R^2} \right] \quad 2.9.30
\]

where $U$ is the velocity of the particle relative to the cylinder and $U_{OF}$ is the fluid velocity along the cylinder axis.

The relationships above can be applied to the study of multiparticle systems. The simplest assemblage is the case when the particles suspended in the cylinder do not move relative to each other and are randomly distributed throughout the cylinder cross-section for an infinite distance axial. They are assumed not to interact with each other. For the annular volume, $dV = L(2\pi r dr)$, as in Fig. 2.9.4
consider $N$ spheres per unit volume, thus $N \pi r^2 dr$ spheres in the annular volume, the drag will be given by

$$D = \pi \mu a \left[ (u_{cf} - u) - \frac{U_{cf} r^2}{2} \right]$$  \hspace{1cm} (2.9.31)$$

The resulting drag from all the particles in the volume, $dV$, is

$$dD = N \pi r^2 \rho \pi a \left[ (u_{cf} - u) - \frac{U_{cf} r^2}{2} \right]$$  \hspace{1cm} (2.9.32)$$
which on integration gives the total drag

\[ D = 6\pi \mu a N R^2 L \left( \frac{U_{cf}}{2} - U \right) \]  \hspace{1cm} 2.9.33

\( \frac{U_{cf}}{2} \) is simply the average fluid velocity in the cylinder and eq. 2.9.33 is merely the application of Stokes' law to each of the spheres present and it can be shown that

\[ U = U_{MF} - U_s \]  \hspace{1cm} 2.9.34

where \( U_{MF} = \frac{U_{cf}}{2} \), the mean fluid velocity and \( U_s \) is the Stokes velocity of a single sphere. Similar considerations apply to the pressure drop

\[ d(\Delta P) = N L 2\pi r dr \left( \frac{12 \mu a}{R^2} \right) \left[ \left( U_{cf} - U \right) - \frac{U_{cf} R^2}{R^2} \right] \]  \hspace{1cm} 2.9.35

\[ \Delta P = N L \pi \mu a \left( \frac{U_{cf} - 3U}{2} \right) \]  \hspace{1cm} 2.9.36

The second case considered by Happel and Brenner is the one in which spheres are free to move relative to one another in an axial direction. The friction force in the direction of flow experienced by a sphere translating in the same direction with a constant velocity, \( u \), when the fluid flows with a mean velocity, \( \frac{U_{cf}}{2} \), is given by eq. 2.9.37.

The gravitational force experienced by a single particle is

\[ \frac{4}{3} \pi a^3 (\rho - \rho_f) g \]  \hspace{1cm} 2.9.37

which, if Stokes' law is valid, is equivalent to

\[ F = 6\pi \mu a U_0 \]  \hspace{1cm} 2.9.37
When no net force acts on the particle it moves at a constant velocity, \( u \), which is obtained by equating the drag to the gravitational forces, thus,

\[
    u = u_{\infty} \left(1 - \frac{r^2}{R^2}\right) - u_s
\]

which gives the equilibrium velocity of a particle situated at a distance, \( r \), from the cylinder axis. When all particles are free to move, a parabolic flow profile for the particles exists as shown in Fig. 2.9.5

![Particle velocity pattern](image)

**Fig. 2.9.5** Particle velocity pattern.

Eq. 2.9.38 implies that there is a radius, \( r = s \), \( R > s > 0 \), for which particle velocity \( u = 0 \), the stagnation radius, which corresponds to the point where gravitational forces equal frictional forces. Putting \( u = 0 \) at \( r = s \) in eq. 2.9.38

\[
    \frac{s}{R} = \left(1 - \frac{u_s}{u_{\infty}}\right)^{\frac{1}{2}}
\]

In the outer annular area, \( R > r > s \), from eq. 2.9.33 it is
found that \( u < 0 \), an indication that particles in this volume have a downward velocity and in the inner cylindrical space, \( s > r > 0, u > 0 \) corresponding to an upward motion of the spheres. The pressure drop in such an assemblage is for each sphere

\[
\Delta p = \frac{12\mu a}{R^2} \left(1 - \frac{r^2}{R^2}\right) u_s
\]

2.9.40

and the pressure drop in the differential element, \( dV \), is given by

\[
d(Ap) = \frac{24\pi \mu a l}{R^2} u_s \left(1 - \frac{r^2}{R^2}\right) nr dr
\]

2.9.41

A solution is possible without regarding the dependence of \( n \) upon \( r \). If \( \Psi \) is the net number of particles transported per unit time in a positive axial direction,

\[
\Psi = \int_0^R n u 2\pi r dr
\]

2.9.42

\[
\Psi = 2\pi \int_0^R \left[ u_{cf} \left(1 - \frac{r^2}{R^2}\right) - u_s \right] nr dr
\]

2.9.43

and finally

\[
\int_0^R (1 - \frac{r^2}{R^2}) nr dr = \frac{\Psi}{2\pi u_{cf}} + \frac{u_s}{2\pi u_{cf}} \int_0^R \eta 2\pi r dr
\]

2.9.44

and if \( N_m \) is taken as the mean number of particles per unit volume averaged over the assemblage,

\[
N_m \pi R^2 = \int_0^R n 2\pi r dr
\]

2.9.45

Thus the pressure drop is expressible as

\[
\Delta p = \frac{24\pi \mu a l}{R^2} u_s \left[ \frac{\Psi}{2\pi u_{cf}} + \frac{u_s N_m R^2}{2u_{cf}} \right]
\]

2.9.46
Bungey and Brenner\textsuperscript{(15)} derived an expression for the additional pressure drop due to a relatively small sphere moving near to the wall of a circular tube through which there is Poiseuille flow. The hydrodynamic force, torque and additional pressure drop are linear functions of the translational and rotational velocities of the particle, \(u\) and \(\omega\) respectively and the mean velocity in Poiseuille flow of the fluid, \(V_m\)

\[
\begin{pmatrix}
F \\
T \\
\Delta P_A
\end{pmatrix} = -\mu
\begin{pmatrix}
\kappa^t & \kappa^r & \kappa^s \\
L^t & L^r & L^s \\
\rho^t & \rho^r & \rho^s
\end{pmatrix}
\begin{pmatrix}
u \\
\omega \\
-V_m
\end{pmatrix}
\] 2.9.47

The grand resistance matrix is symmetric. Thus \(\kappa^t = L^t\), \(\kappa^s = \rho^t\), \(L^s = \rho^r\). The resulting equations for the particle velocity relative to Stokes velocity and additional pressure drop relative to Stokes drag are given by

\[
\frac{u}{u_s} = \left[ \kappa^t - \frac{L^t \kappa^r}{L^r} \right]^{-1} \circ \left( \frac{a}{R} \right)
\] 2.9.48

\[
\frac{\Delta P_A}{\rho \pi \mu a u_s} = \frac{4h}{R} \left( \frac{\kappa^s L^r - \kappa^r L^s}{\kappa^s L^r - \kappa^r L^s} \right) \circ \left( \frac{a}{R} \right) 2.9.49
\]

where \(R\) is the tube radius and \(h\) is the distance of the sphere centre to the tube wall.

Buyevich and Safrai\textsuperscript{(20)} calculated the drag forces in a distribution of particles where \(f(a)\) is the distribution of particle radii, \(a\). The forces in the direction of flow on the particles in a unit volume are given by

\[
F = -c \nabla P + \kappa V_{\text{mean}}
\] 2.9.50
where $c$ is the volume concentration, $V_{\text{mean}}$ and $p$ are the mean fluid velocity and pressure respectively; $\alpha$ is a root of the transcendental equation

$$\alpha = \mu \frac{c}{2 - c}; \quad \gamma^2 = \frac{c}{2(1-c)} \left( \frac{1}{b_3} \left[ \left( 1 - \frac{2c}{3} \right) (b_1 b_2) - \frac{b_3}{3} \gamma^2 \right] \right)$$

where the positive root is taken. Here $\mu$ is the fluid viscosity and $b_m$ ($m = 1, 2, 3$) are the moments of $f(a)$

$$b_m = \int_{a_{\text{min}}}^{a_{\text{max}}} f(a) a^m da$$

An individual particle of radius, $a$, experiences a drag force

$$F = \frac{4}{3} \pi a^3 \nu_f + 6 \pi \mu \kappa u$$

where

$$\kappa = \frac{1}{1-c} \left[ (1 - \frac{2c}{3}) (1 + \alpha \gamma) + \frac{1}{2} (\alpha \gamma)^2 \right]$$

Eq. 2.9.51 is solved as

$$\gamma = \frac{c}{2} \left( 1 - \frac{2c}{3} \right) \left[ \frac{b_2}{b_3} + \left( \frac{b_3}{b_3} \right)^2 + \frac{4}{3} \left( \frac{2 - 3c}{c(1 - \frac{2c}{3})} \right) \frac{b_1}{b_3} \right]^{\frac{1}{2}}$$

Thus the values of $F$ and $f$ can be determined. Buyevich and Safrai then obtain solutions to the equations for the case of a monodispersed system where $b_m = a^m$ and a normal size distribution where

$$f(a) = \frac{4}{\sqrt{\pi} \kappa} a^2 \exp\left( - \frac{a^2}{\kappa} \right) \quad 0 < a < \infty$$
Buyevich and Markov (21, 22) dealt with a suspension as the superposition of two interpenetrating continua. They obtained equations of conservation of mass, momentum and moment of momentum in terms of the mean stresses acting at the surface of an individual suspended sphere

\[ b_1 = \frac{2\rho}{\sqrt{\pi}}, \quad b_2 = \frac{3\rho}{2}, \quad b_3 = \frac{4\rho}{\sqrt{\pi}} \]

\[ \bar{F} = \int_{S_j} \left( -I_3 + 2\mu \left( \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \right) \right) \cdot \mathbf{n} \, d\mathbf{r} \quad 2.9.56 \]

\[ \bar{M} = \int_{S_j} (\mathbf{r} - \mathbf{r}') \times \left( -I_3 + 2\mu \left( \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \right) \right) \cdot \mathbf{n} \, d\mathbf{r} \quad 2.9.57 \]

where \( \bar{F} \) and \( \bar{M} \) are the average force and average moment of interaction respectively, on the surface, \( S_j \), of the jth particle.
2.10 Interpretation of the Literature with Respect to this Work

The work of other authors quoted in the preceding sections 2.1 - 2.9, describes experimental and theoretical contributions which at times might seem to present conflicting ideas and interpretations. One fact is common to all the contributions, however, and this is that they are all concerned with the motion of small rigid spherical particles in an incompressible Newtonian liquid under the influence of gravity.

The purpose of this section is to relate the work done by these other authors to the subject of this thesis and to highlight certain of the observations or interpretations made to indicate the reasoning behind the author's actions and interpretations.

Possibly the earliest, and a most important contribution was Stokes' solution of the equations of motion for a single sphere. This simultaneously provided a mathematical understanding of the motion of a single sphere under the conditions stated above, and indicated that by the same method, the solution when extended to cover a large number of particles would be extremely difficult, if not impossible. The difficulty arises since the boundary conditions to be satisfied are of zero fluid velocity at an infinite distance from the particle and a no-slip condition at the particle-fluid interface. Thus Stokes' solution is only an approximation, albeit a very accurate one. With an increasing number of spheres the no-slip condition must be applied to an increasing number of boundaries, and what
is more difficult, the application of the zero velocity condition at infinite distance from each sphere. Firstly, it is difficult to decide at what distance from a particle this occurs, and indeed with a large number of settling particles in a vessel with finite walls it becomes even more difficult to say whether such a condition arises at all owing to the fluid return flow from continuity considerations.

This issue was avoided by Kynch (63) by assuming that the velocity of fall of a particle at any point in a dispersion is a function only of the local particle concentration and that this allows a determination of the settling process entirely from the continuity equation. The underlying suggestion here is that particle velocity may change with concentration. Here it must be admitted that Kynch was concerned with the hindered settling region, where from continuity considerations one might reasonably expect velocity to decrease with increasing concentration. However, a suggestion of localized concentration gradients requires velocity gradients also. Unfortunately in avoiding the equations of motion, the problem changes from a microscopic to a macroscopic one with bulk movements of particles rather than the motions of individual particles within that bulk.

This statement was rationalized by Batchelor (6). By choosing a reference volume intermediate between the microscopic and macroscopic scales it may be possible to study the motions of a number of particles applying the principle of local relative equilibrium, such that the stress at the
bounding walls of the volume are equally opposed by those at neighbouring volume bounding walls.

Here, perhaps, theoretical considerations should give way, temporarily, to results obtained from experimentation, and these were initially performed on sedimenting systems of spheres by Kaye and Boardman (57) followed by Johne and Koglin (54, 55, 58, 59). As already observed, a single particle falls vertically with its Stokes velocity. In a multi-particle sedimenting system these workers obtained results which show no substantial change from Stokes' law up to concentrations of 0.05% by volume. One might suggest that half the interparticle distance corresponding to a volume concentration of 0.05% is the practical value for the 'zero fluid velocity at infinity' condition required by the Navier-Stokes equation. It is shown by these authors that as the volume concentration of particles is increased the average particle velocity also increases. Beyond about 1% volume concentration the average particle velocity begins to decrease and the hindered settling region is entered. The inference here is that particles do not fall at a constant velocity, since each worker reports an average particle velocity. Two schools of thought then develop as a result of the enhanced velocity observation with concentration increasing to 1%. Kaye and Boardman quote the increasing particle proximity with increasing particle concentration as 'a region of viscous interaction'. Johne and Koglin, on the other hand, quote a region of 'dynamic clustering' giving rise to the enhanced velocity, and when a particular particle is isolated it falls with its Stokes
velocity, which on average yields an enhanced average velocity. Kaye and Boardman quote an unstable region above 1% 'where clusters form and return flow is irregular and localized'. These interpretations are well documented and arguments for and against are presented in section 2.1. At this stage the most that may be said is that particles have a mutual effect on one another, they do not fall with constant velocity and in general are not confined to vertical translation only as in the Stokes law case.

Johne's (54,55) interpretation is of a fluid volume containing a number of particles, the number of which at any particular set of observations will be Poisson distributed. Batchelor (4), in a theoretical study, allows a particle velocity to differ from the Stokes velocity due to hydrodynamic interaction depending on the particle concentration. He determines an average particle velocity composed of the sums of the Stokes velocity, the effect of the presence of the other particles on a test particle, and the reflection from all the other particles of the test particle's velocity back onto itself. The solution demands a knowledge of the probability density function of the location of only one sphere relative to the test sphere, and it may be this fact which accounts for the non-agreement of the concentration dependence of velocity with the experiments quoted by Kaye and Boerdman, Johne, and Koglin.

Once again, recognizing the difficulty of analytical solution of the sedimentation of many spheres, two forms of investigation are open to workers in this field. One is to apply simple mathematical models from a microscopic point
of view and the other is to adopt a macroscopic approach similar to that of Kynch's, described above, and interpret sedimentation as a modification of bulk properties.

The problem of the motion of two spheres was first solved by Stimson and Jeffery (91). The general problem of two spheres was solved analytically by Goldman, Cox, and Brenner (41) who showed that for two neighbouring spheres the velocity of each will be greater than the Stokes velocity of either, that the magnitude of the velocity is a function of the relative sizes of the spheres, their distance apart and their orientation, and that for orientations other than a vertical or horizontal line between the sphere centres, each will possess a horizontal component of velocity. The spheres may also rotate at conditions other than the vertical orientation. The velocities increase with decreasing distance between particles, and this fact readily explains the increase in average velocity of particles in a suspension with increasing concentration. The implications of the two sphere solution are not wholly transferable to the suspension situation for two reasons. The rigorous hydrodynamic solution assumes a motionless fluid, which is not the case in a sedimenting system, for continuity reasons, and assumes particles to be point forces, which becomes inapplicable with increasing particle proximity. At such distances a lubrication theory approach similar to that of Cox (26) is necessary.

Clearly then, a two sphere approach indicates the more complicated nature of the motion of a particle neighbouring another particle, but is not sufficient to describe the
sedimentation phenomenon. Advances in this topic were made by the development from two spheres to several spheres, experimentally by Jayeweera, Mason and Slack (53), and theoretically by Hocking (50), who extended the theoretical solution to the relative motion of three spheres. Again, the experiments show enhanced velocities, rotation, and horizontal velocity components, but cannot account for the return flow of fluid in a sedimenting system.

The cluster formation as an explanation of the velocity enhancement at increasing concentrations is much favoured by such workers as Johne (54, 55), Koglin (58, 59, 60, 61) and others, but is here used merely to indicate that all the spheres neighbouring any particular sphere will tend to increase its velocity in the vertical direction and provide it with horizontal, translational and rotational components. With more than two spheres, spheres are free to change their relative positions, and hence velocities.

The logical development from several spheres is the solution of the equations of motion for arrays of particles. Workers such as Childress (25) obtain expressions which effectively modify the viscosity similar to Einstein's (32, 33) expression. Hasimoto (46) obtains a modification to Stokes' law involving a power series in terms of interparticle distance. Saffman (82) considered the cases of a regular periodic array, a random free array, and a random fixed array, yielding a non-linear relationship between velocity and concentration.

In contrast to the microscopic approach, the macroscopic approach has largely been concerned with expressing
the average particle velocity as a function of viscosity or concentration, or what amounts to the same thing, obtaining a suspension viscosity in terms of the particle concentration and pure liquid viscosity. In an analysis due to Kynch (64) the analogy of electrostatics is invoked. A particle is affected by those surrounding it. As the concentration increases a particle is shielded by those close to it from those far from it and the effect on the velocity of the particle in question is reduced. Kynch indicates that the effects of particles on one another are, as a consequence, not additive. This solves the problem, at least qualitatively, of what is meant by influencing neighbouring particles.

Barfod (3) assumed a distribution of particle velocities and adopted a model based on the Poisson distribution of the distances between two spheres to obtain an expression for particle velocity in terms of concentration.

Buyevich (16,17,18,19) was able to derive the equations of motion in statistical mechanical terms for a two phase suspension and expressed velocities, pressure and volume concentration in terms of a constant portion and a random or 'pseudo-turbulent' fluctuation, and these fluctuating terms are expressible in terms of Fourier-Stieltjes integrals.

Thus, it has been shown that one may either consider the bulk behaviour of a suspension by considering the modification of its properties, or study the motion of a particle of that bulk, such that during its sedimentation it will translate vertically and horizontally and rotate
and that these motions need not be constant with position or time, but rather their variation brought about by small changes in local concentration due to the random return flow of fluid. Experiments may be performed to show qualitatively and determine quantitatively these motions.

Experiments will also show the effect of increasing particle concentration on the average particle velocity. At very low concentrations only very small deviations from Stokes' law should be expected since the effect of a particle on its surrounding fluid is attenuated to a negligibly small value at distances smaller than half the average interparticle distance. At higher concentrations the motion of a particle will affect and be affected by the motion of neighbouring particles, yielding an enhanced average velocity. As the concentration rises still further the effect of the return flow of fluid becomes more significant and causes the average particle velocity to decrease with increasing concentration until eventually the hindered settling region is entered.

A theoretical analysis should allow for fluctuations in particle and fluid velocity as a function of the particle concentration or interparticle distance. The problem should be scaled such that a force balance can be performed over a representative volume of suspension and the relevant boundary conditions satisfied.
3. Experimental Methods

3.1 Introduction

At low volume concentrations of solids in a settling suspension viscous interaction is a significant effect whereas at higher volume concentrations the return flow of fluids begins to be significant. This shows itself as a rise in average settling velocity of the particles due to viscous interaction followed by a fall in average velocity of the particles due to the increased return flow of fluid, leading to the region of hindered settling. This region is usually quoted as starting at approximately 2% volume concentration of solids. This work, being largely concerned with the nature of the enhancement of average particle velocity due to viscous interaction, considered only experiments at low volume concentration of particles from a nominal 0% to about 1.2%.

The nature of the equations of fluid mechanics and their boundary conditions are such that solution is complicated even for simple systems, consequently experimentation was carried out with a simple reliable rig keeping all physical parameters except volume concentration of solids constant. This allowed comparison of the data at different concentrations to be carried out without fear of attributing results to phenomena other than viscous interaction due to gravity settling.

The experimental vessel was a right circular glass cylinder. Glass was used to enable observation of sedimentation at any point in the vessel. A right circular
cylinder was chosen as the vessel shape since a circular cross-section gives the simplest conditions of fluid flow, most amenable to solution of the equations of fluid mechanics, not offering discontinuities, as would, say, the corners of a rectangle in a rectangular section vessel. Also a vertical walled vessel eliminates convection of material near the bounding walls of the vessel which would detract from the simple sedimentation observations required.

A cubical perspex box was fitted to the glass cylinder and filled with the same fluid as that used in sedimentation to eliminate spherical aberration of the image on photographic film due to the shape of the vessel.

Experimentation was carried out at low Reynolds numbers. Stokes' solution of the equations of motion is only strictly applicable as the Reynolds number of a spherical particle approaches zero, but is correct with only a 10% error up to a Reynolds number 0.2. Consequently experiments were performed such that the Reynolds number of a single sphere of the mean size of the distribution of particles fell below the value 0.2. For the two dimensional particle measurement, described in section 3.4.1, the Reynolds number value for a marker particle was 0.00328. For the three dimensional particle measurement, described in section 3.4.2, its value was 0.00951.

Thus, relating the particle velocities to the Stokes settling velocity the modification from Stokes' law due to particle interaction can be seen, comparison is allowed with the sedimentation studies carried out by Kaye and Boardman(57), Johne(54,55), and Koglin(58,59,60), and
analysis of the equations of motion may be performed where their linear form is valid.

The particles used were small compared to the vessel diameter to eliminate wall effect in all regions of the vessel except very close to the vessel wall, but large enough to render the effects of Brownian motion negligible. The particles were spherical in shape since this shape offers the simplest fluid mechanical characteristics. They formed an approximately normal size distribution, with a narrow spread and mean and median sizes differing by only 20\mu m. Details of the particle selection and size measurement are given in section 3.2.1.

To maintain a low Reynolds number the fluid used had to be fairly viscous, and to keep the fluid mechanics simple had to be a Newtonian liquid. Observation of particle motion was carried out using a matched refractive index technique. This involves the use of a few opaque particles having the same properties as the bulk of the particles, but to allow observation of these opaque particles the remaining particles and fluid must be transparent, and this is achieved by matching the refractive index of the fluid to that of the remaining particles. Thus a mixture of two liquids was made such that the refractive index of the resulting mixture of liquids was the same as that of soda glass. The opaque particles were coloured black. Their manufacture is described in section 3.2.3.

The next problem encountered was that of obtaining a homogeneous mixture of particles and fluid, and not to mix the particles and fluid too violently giving residual fluid
motion after mixing had ceased. In addition the mixing device had to be removed from the fluid before sedimentation began, otherwise the obstruction would cause circulation effects in much the same way as non-vertical vessel walls would have done. An impeller, for instance, would give rotation of the fluid, and a non-homogeneous mixture of the particles, there being a concentration gradient across the vessel. Mixing was achieved quite successfully using a reciprocating perforated perspex disc fitting closely inside the cylindrical vessel driven by a piston mounted vertically above the vessel. When mixing stopped the perforated disc always 'parked' in the raised position, thus not interfering with particle motion in the fluid below. Its operation was fairly slow but traversed the whole of the vessel thus achieving mixing of all the particles without imparting rapid motion to the fluid giving rise to high Reynolds numbers, or much circulation of the fluid after the perforated disc had stopped. The holes in the disc were four in number and drilled symmetrically in the disc and accounted for about 25% of the area of the disc, this being found to be the best arrangement to achieve homogeneous mixing and no channelling of fluid during mixing.

A baffle plate at the top of the vessel ensured that operation of the perforated disc did not entrain air from above the liquid in the vessel, and did not deposit glass particles on the walls of the vessel above the liquid level. This ensured a constant total volume of the sedimenting mixture and a constant volume concentration of particles.

A sketch of the mixing disc and baffle plate shows
their arrangement in the experimental rig. This is shown in Fig. 3.1.

The methods used for observing particle and fluid motions were time lapse ciné photography and laser Doppler anemometry respectively. These techniques avoided the necessity of placing foreign matter, such as pitot tubes or neutral buoyancy marker particles, into the fluid, thereby upsetting the sedimentation characteristics. The ciné photography technique was used since this allowed observation at any point in the fluid thus allowing a whole range of particle velocities throughout a single trajectory to be observed. By this method large quantities of data were obtained in order that statistically correct results could be ensured. This involved the use of accurate timing equipment to operate the camera at regular time intervals putting observations on a constant time base. The camera was 155 cm. from the experimental vessel to allow a depth of focus of the camera to cover the diameter of the experimental vessel. Owing to the small size of the particles and the distance of the camera from the vessel, the most easily definable photographic image of the marker particles was as dark objects on a bright field. This was achieved by rear illumination from a white screen.

Measurement of fluid velocities with the laser Doppler anemometer produced accurate values of a mean fluid velocity and its standard deviation. The results obtained were for vertical motion only, and the equipment had no means of detecting the difference between upward and downward motion. However, downward fluid motions were measured, and these
Fig. 3.1 Sectional sketch of rig showing baffle plate and mixing disc mounted on shafts, and perspex box.
were not of the return flow of fluid but that near to particle-fluid interfaces, as may be shown from the continuity equation

\[ Uc + V(t-c) = 0 \]

where \( U \) is the particle velocity, \( V \) is the fluid velocity, and \( c \) is the volume concentration of solids. The order of magnitude of particle velocity does not change throughout these experiments but particle concentration spans more than two complete orders of magnitude. Consequently, the fluid return flow velocity will cover at least two complete orders of magnitude. The anemometer's oscilloscope trace shows velocity counts in each of 48 channels which only span a single order of magnitude, and the instrument was set up for these experiments to measure velocities in the range, 0.0001 to 0.001 cm/s, roughly the middle of the range of values expected for return flow velocities according to continuity considerations. The accuracy of the average velocity is justified as follows. The apparatus measures the time taken for minute particles of foreign matter, naturally present in gases and liquids, to pass from one fringe to the next of the interference pattern set up by two intersecting laser beams. The distance between fringes is known and, therefore, the velocity can be calculated. A correlation function of velocities is built up of such events falling into one of 48 velocity ranges to form a 48 point correlation. In this particular case the instrument settings were arranged to give a sampling time of 4x10^{-2}s with a photomultiplier count of 7x10^3.
photon/s giving a count of $2.8 \times 10^2$ counts for each of 10 samples on which to base the average velocity.

Practical details of the selection of particles and fluid, construction of the experimental rig, photographic and anemometer details are given in the remainder of this chapter.

3.2 Details of Particles and Liquid

3.2.1 Particle Selection and Size Measurement

The particles used were between 460 and 650 $\mu$m, being the sieved and retained particles of glass ballotini from original samples between 200 and 1200 $\mu$m. The non-spherical particles were removed from these by rolling small samples down an inclined plane at approximately $20^\circ$. The spherical particles, those reaching the bottom of the inclined plane, were retained. These particles were then resieved retaining those between 460 and 650 $\mu$m.

Ordinary optical techniques and sedimentation methods for particle size measurement could not be used since the particles were too large and too great in number. Consequently samples were sized by placing them on a microscope slide on the stage of a Leitz Wetzlar binocular microscope. To be observable the transparent glass particles were made opaque by illuminating from beneath focused onto the stage with a wide-angle short focal length stage condenser with a polarizing filter. A short focal length ($\times 1$) narrow field objective lens was used. Above the stage and objective lens
was a television camera, an Imanco Vidicon head, as the input to an Image Analysing Computers Ltd., Quantimet 720 particle size analyser. Touching particles had to be separated mechanically since the Quantimet cannot distinguish contiguous particles from a single particle. The analyser was set up to measure the longest horizontal chord of each particle and count this into one of seven size ranges. Sufficient number of particles were sized to satisfy B.S. 3704. The particle size data is presented in Table 3.1 and the distribution shown in Fig. 3.2.

<table>
<thead>
<tr>
<th>Size range (µm)</th>
<th>Mid point of range (µm)</th>
<th>Number in range (%)</th>
<th>Cumulative number (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>450.29-481.33</td>
<td>465.81</td>
<td>2.38</td>
<td>2.38</td>
</tr>
<tr>
<td>481.33-512.38</td>
<td>496.86</td>
<td>5.15</td>
<td>7.53</td>
</tr>
<tr>
<td>512.38-543.44</td>
<td>527.91</td>
<td>11.19</td>
<td>18.72</td>
</tr>
<tr>
<td>543.44-574.49</td>
<td>553.97</td>
<td>36.47</td>
<td>52.19</td>
</tr>
<tr>
<td>574.49-605.55</td>
<td>590.02</td>
<td>25.40</td>
<td>80.59</td>
</tr>
<tr>
<td>605.55-636.60</td>
<td>621.08</td>
<td>5.15</td>
<td>85.74</td>
</tr>
<tr>
<td>636.60-667.65</td>
<td>652.13</td>
<td>4.15</td>
<td>89.89</td>
</tr>
<tr>
<td>667.65-698.71</td>
<td>683.18</td>
<td>10.06</td>
<td>99.95</td>
</tr>
</tbody>
</table>

Table 3.1 Particle size data from Quantimet particle size analyser.
Fig. 3.2 Cumulative undersize distribution of glass particles. Data obtained from Quantimet particle size analyser.

The mean particle size, $\bar{a}_n$, is calculated as

$$\bar{a}_n = \frac{\Sigma n a_n}{\Sigma n}$$

$$= 562 \mu m \pm 1.85\%$$

3.2.2 Selection of Sedimenting Medium

For the matched refractive index technique a mixture of two plasticizers was made in proportions which gave a refractive index the same as that of soda glass.
The proportions used were:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reomol DiDP</td>
<td>74%</td>
</tr>
<tr>
<td>Reofos 95</td>
<td>26%</td>
</tr>
</tbody>
</table>

The rheological properties of this mixture of liquids were obtained using a Weissenberg R18 rheogoniometer. Torque and cone speed of the liquid were obtained and plotted as stress against strain rate in Fig. 3.3. The straight line through the origin indicates a Newtonian liquid and the viscosity, obtained from the slope of the graph is 1.92 P, this figure being obtained at 23°C, the temperature at which sedimentation experiments were carried out.

<table>
<thead>
<tr>
<th>Stress (\text{dyne/cm}^2)</th>
<th>Strain rate ((s^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.6063</td>
<td>11.2025</td>
</tr>
<tr>
<td>35.4771</td>
<td>17.7572</td>
</tr>
<tr>
<td>54.9483</td>
<td>23.1851</td>
</tr>
<tr>
<td>87.1252</td>
<td>44.5314</td>
</tr>
<tr>
<td>136.4632</td>
<td>70.8097</td>
</tr>
<tr>
<td>216.1831</td>
<td>112.0254</td>
</tr>
<tr>
<td>341.5707</td>
<td>177.5722</td>
</tr>
</tbody>
</table>

Table 3.2 Rheological data for reofos/reomol.

3.2.3 Manufacture of Marker Particles

Having matched the refractive index of the sedimenting fluid to the refractive index of the bulk of the soda glass particles, dark coloured marker particles were required.
Fig. 3.3 Rheological data for reofos/reomol mixture.
These were made by selecting particles of the desired size by observation under a microscope. The particles selected were rapidly vacuum-coated with a gold-palladium layer approximately 0.04 μm thick. This gave the spheres a dark grey appearance which rendered them easily observable in the reofos/reomol mixture. The coated particles were selected further by allowing them to settle individually in the liquid, retaining those which settled with the desired velocity or range of velocities, and discarding the remainder. Details of the marker particles retained are described in section 3.4.

3.3 Details of Experimental Rig

The experimental vessel was a 6 in. diameter, 2 ft. long Q.V.F. glass cylinder mounted on a brass plate fitted with a drain tap. A cubical perspex box was fitted to the glass cylinder which, when filled with the reofos/reomol mixture, eliminated lens effect due to the circular cross-section vessel. The perspex box was also fitted with a drain tap. The perspex box was fitted with three cross-wires, the upper and lower wires each 8.5 cm. distant from the centre wire. These were used to calculate the parallax correction for particle coordinates.

Glass particles and liquid were placed in the glass cylinder and liquid in the perspex box. Mixing was achieved using a reciprocating perforated perspex disc. This was a circular disc fitting closely inside the glass cylinder and drilled with four 1 in. holes. The perspex disc was mounted
on the ends of two rods, one being connected to the piston of a 3 ft. stroke, 1\frac{1}{2} in. bore double-acting air cylinder, the other being fitted with two collars operating an on/off switch switching a solenoid valve admitting air at 10 psig. to the top or bottom of the cylinder. Reciprocation led to swirling of the liquid and entrainment of bubbles and this was eliminated by placing a perspex disc above the 2 ft. cylinder and a 6 in. long cylinder above this also filled with the liquid. Three holes were drilled in the disc to accommodate the piston rod and control rod and to allow quantities of glass beads to be added. Photographs of the mixing cylinder and experimental vessel are shown in Figs. 3.4 and 3.5 respectively.

The experimental rig was erected in a constant temperature laboratory maintained at 23°C.

3.4 Photography for Particle Measurements

3.4.1 Two Dimensional Particle Measurements

Two series of experiments were carried out involving photography of the phenomenon of sedimentation from a front view only in order to ascertain the projected vertical and horizontal motions of particles.

In the first series of experiments marker beads were made as described in section 3.2.3 and those retained from the sedimentation selection method possessed a Stokes velocity corresponding to that of the median sized particles from the particle size distribution used in these experiments, namely 0.1167 cm/s. With this selection of marker particles there is no likelihood of attributing phenomena
Fig. 3.4 Pneumatic mixing cylinder for experimental vessel.
Fig. 3.5 Experimental vessel showing camera and time lapse equipment.
to different sizes of particles, only to particle-particle and particle-fluid interactions.

In the second series of experiments a wider range of marker particles was used. Again marker particles were selected as described in section 3.2.3 but those retained were particles which settled with a velocity within ±5% of the Stokes velocity of the median sized particle. The purpose of this wide selection of markers was to observe the settling characteristics with a spread of Stokes velocities and to compare these results with the sedimentation using very closely sized marker particles.

The ciné camera was set up in front of one face of the perspex box of the experimental vessel with the centre of the field of view corresponding to the central cross-wire on the experimental vessel. The field of view included the upper and lower cross-wires to the top and bottom respectively and the edges of the glass cylinder to the sides.

Photography was carried out in the darkened laboratory with rear illumination from a white screen by two 75 W photographic floodlights. The ciné camera was a Paillard Bolex H-16 fitted with a Kern Vario Switar 1:1.9 f16 100 mm. POE BOLEX H16RX zoom lens. The aperture was f16 at 25 ASA and Kodak Plus-X film was used.

The operation of the camera was controlled using Paillard-Wild time lapse equipment. This consisted of a synchro-stepping motor instantly starting and stopping driving the camera. This was operated by a controller and timer using the feeding current as the base for the timer resulting in flicker-free exposure of frames. The variable timer
was set to 1 s. The exposure time was 0.2 s and the wind-on time, 0.3 s giving a total time between frames of 1.5 s.

The vessel was charged with liquid and a known volume concentration of glass beads and these mixed by turning on the solenoid switch commencing reciprocation. After a few strokes the solenoid was switched off and the reciprocating disc returned to the upper position. The mixed glass particles then began to settle under the influence of gravity. A few seconds were allowed to elapse before the camera was switched on to allow any residual swirling to cease. Approximately 90 frames were exposed. This ensured that film was taken covering the field of view only while a constant concentration zone existed. The constant concentration zone exists after mixing has stopped while the particles are still homogeneously mixed. During this period at a given horizontal differential element, the number of particles entering at the top equals the number of particles leaving through the bottom. After a certain time the number of particles entering such a differential element is not as high as the number leaving the differential element. At this time and thereafter the concentration is, therefore, no longer constant and a concentration gradient exists. Measurements were not made during this period since the effective local concentration is no longer either constant or at the value of the initial concentration and results would apply to some lower value than that at which the experiment was said to have been carried out.
Fifteen such runs were made at one concentration. After these runs the concentration was increased by addition of a known weight of glass particles to the vessel and filming recommenced. This procedure was continued until the concentration range was completed. Run numbers and concentrations were marked on the perspex box with a felt-tip pen. A photograph of the rig is shown in Fig. 3.5 and schematic diagram shown in Fig. 3.6.

Fig. 3.6 Schematic plan view of experimental set-up.
Processed films of the sedimentation experiments were analysed using a P.C.D. Digital Data Reader. The film was loaded into a Vanguard M-16C projection head capable of forward and reverse motion of either a single frame at a time or with variable speed up to 16 frame/s. Projector illumination was a 300 W lamp giving rear illumination onto a translucent screen measuring 30x50 cm. of the P.C.D. Digital Data Reader type ZAE 1B. The reader was equipped with two cursors moveable in the x and y Cartesian directions inputted to an electronic drive unit containing an analog-digital converter, serializer and output providing scale lengths of ±9999 on a visual display and also to a Data Dynamics teletype type ASR33 and paper tape punch. The drive unit had zeroing and scaling facilities for both x and y scales. The position of the projector head was such that a particle falling vertically downwards appeared on the projector screen to be travelling horizontally from right to left. Thus the x axis had a scale length -3400 to +3400 representing a vertical distance, 17 cm. and the y axis a scale length -3200 to +3200 representing a distance, 16 cm. This is shown in Fig. 3.7. The position of a marker particle was recorded by centring the two cursors over the particle and recording their coordinates on punched paper tape. The film was advanced a single frame, the cursors repositioned over the particle's new position and the coordinates recorded on punched paper tape. The analysing procedure differed for the first and second series of data, and the different techniques are based on the following argument.
Consider the trajectory of a marker particle. During its settlement in the vessel it is influenced to a greater and lesser extent as it passes towards and away from, respectively, a neighbouring sphere and this effect may also be amplified or attenuated should the marker particle pass close to and become influenced by a number of spheres. In addition all the other spheres are in motion and so the other spheres exert what might be called a 'dynamic influence' on the marker sphere. If the degree of the influence is random, one might reasonably expect the marker particle to exhibit a distribution of velocities. Each of the particles in the sedimenting system will exhibit a distribution of velocities, that of the median sized marker particles being one such distribution from the set of distributions, one for each size. A composite distribution may be expected of the sum of the distributions of velocities of each size. Thus the distribution of velocities of the median sized particle will only cover a small part of the complete velocity distribution of all the particles in the system.

Thus for the first series of experiments with monosized marker particles about 1000 frames of film were exposed at each concentration, and in the analysis on the Vanguard projector sets of 30 consecutive frames were used to determine a range of velocities. This procedure was repeated until about 1000 values of the median sized particle's velocity were obtained. The whole was then repeated for each concentration, obtaining projected vertical and horizontal velocity distributions, and mean values at each concentration.
Considering the motion of two particles whose sizes differ by, say 5\( \mu \)m, the upper limit of the distribution of velocities of the smaller particle may overlap the lower limit of the velocity distribution of the larger particle. For this reason analysis of the second series of experiments consisted of analysing about 70 frames of a marker particle's trajectory, once again repeating the procedure until about 1000 values of velocity were obtained. The whole was then repeated for each concentration, obtaining projected vertical and horizontal velocity distributions and mean values for each concentration.

In the case of series 1 and series 2 experiments, a single paper tape was obtained for each concentration of a particle's coordinates throughout its trajectories, and these tapes were analysed using computer programs, details of which are given in Appendix 1.

\[
\begin{array}{c}
-3400, -3200 \\
\vdots \\
0,0 \\
\vdots \\
3400, 3200
\end{array}
\]

\[
\begin{array}{c}
-3400, -3200 \\
0,0 \\
3400, 3200
\end{array}
\]

Fig. 3.7 Schematic view of projector screen showing cross-wires and coordinates.
The recorded position of a particle on film was subject to a parallax error whose effect could be corrected, in both the x and y directions. This correction is only approximately correct since it assumes the position of the particle to be on the vessel centreline. The correction applies to both the vertical and horizontal data since in each case the scale length is the same. No better estimate of a particle's position may be made since a photograph only gives a projected view thereby not giving the horizontal distance from the vessel wall to the particle. This may be accounted for in the case when both horizontal coordinates and the vertical coordinate are determined, as will be indicated in section 3.4.2. Thus in Fig. 3.8 with an observer at B, a sphere apparently at E is at C. The real coordinate, $x$, is given in terms of the apparent coordinate, $x_2$, where the only two parameters required are $a_1$, the horizontal distance between the observer and the perspex box fitted to the experimental vessel, and $a_2$, the distance from the perspex box to the experimental vessel centreline.

By similar triangles, ABC and CDE

$$\frac{AB}{BC} = \frac{DE}{DC}$$

$$\frac{x_1 + x_2}{a_1 + a_2} = \frac{x_1}{a_1}$$

$$x_1 = \frac{a_2}{a_1 + a_2} (x_1 + x_2)$$

$$x_1 \left(1 - \frac{a_2}{a_1 + a_2}\right) = \frac{a_2}{a_1 + a_2} x_2$$

$$x_1 = \frac{a_2}{a_1} x_2$$
Therefore real coordinate, to first approximation

\[ x = x_2 \left( 1 + \frac{a_1}{a_2} \right) \]

---

**Fig. 3.8** Schematic side view of experimental rig showing geometry for parallax correction.

**3.4.2 Three Dimensional Particle Measurements**

A third series of experiments was conducted to observe the sedimentation of particles in the three Cartesian directions. Experiments were carried out in the same vessel as for the first and second series of experiments. A mirror placed at 45° to the left face of the perspex box surrounding the sedimentation vessel allowed simultaneous photographing of the front and side views of the sedimentation phenomenon, but this necessitated halving the observed
volume of the vessel in order to accommodate the front and side views simultaneously. Thus the magnification of the marker particles on each camera frame was also halved. In order to make marker particles visible, the vacuum coating procedure was repeated several times with larger glass particles giving markers within ±5% of the Stokes velocity of a 770μm particle. Thus, at once, three-dimensional observation was facilitated and a series of experiments was carried out using marker particles not belonging to the particle size distribution of the bulk of the sedimenting particles.

For the three-dimensional observation the camera was set up slightly differently to series 1 and 2 of experiments.

Four vertical cross-wires were added to the perspex box, one to each of its four vertical faces such that the plane between opposite pairs of cross-wires passed through the centreline of the vessel.

With the camera positioned as for the first series of experiments a plane mirror was placed at 45° to the left face and with one vertical edge touching that face of the perspex box such that a view of the left side of the vessel was visible; the mirror was known to be at 45° to the left face of the perspex box when the no parallax condition that the two vertical cross-wires on the sides of the perspex box were coincident, was observed.

A second white screen was erected parallel to the right face of the perspex box giving rear illumination of the left face with another two 75 W photographic floodlights.
The camera was moved to the left to an arbitrary position such that the field of view contained to the right, that portion of the front view of the glass cylinder to the left of the centreline, and to the left, that portion of the side view of the glass cylinder to the right of the centreline, thus giving a view of one quadrant of the vessel.

Two more vertical cross-wires were added to the perspex box such that the no parallax condition obtained on the camera centreline.

Details of the three-dimensional experimental rig are shown in the schematic view in Fig. 3.9 and photographic view in Fig. 3.10.

Fig. 3.9 Schematic plan view of three-dimensional experimental set-up.
Fig. 3.10 Experimental vessel showing 45° mirror.
Filming procedures were the same as for series 2 of experiments except that only approximately 500 frames were exposed at each concentration; however, analysis of the films was slightly different since two views of the sedimenting particles were recorded simultaneously on each frame.

The camera was set up such that a particle falling vertically downwards appeared to move in a similar direction on the projector screen such that the y axis had a scale length -3400 to +3400 representing a vertical distance, 17 cm. The x axis had a scale length -5550 to +3450 representing a horizontal distance, 30 cm. This is shown in Fig. 3.11.

![Diagram](image)

Fig. 3.11 Schematic view of projector screen showing cross-wires, coordinates and actual distances.
The vertical parallax correction for the three-dimensional measurements was the same as for the two-dimensional measurements but the horizontal correction was different and a third correction was required for the third dimension. Corrections are correct only to a first order of approximation. Details of the horizontal parallax corrections, $y_0$, $z$, are shown below and schematically in Fig. 3.12.

With the origin at $B$, an observer at $O$ sees an object at $E$ with real coordinates $(z_1+z_2)$, $y_0$ as an image at $z_1$, $(y_0+y_1)$. Triangles $OBC$ and $CDE$ are similar. Therefore

\[
\frac{y_4}{y_0} = \frac{z_1}{z_2}
\]

\[
z_2 = \frac{z_1 y_0}{y_4}
\]

and $z_1$ may be corrected by the addition of $z_2$. However, $y_0$ is not known accurately, it is subject to a parallax error, and the observer measures the distance $y_0 + y_1$ for $y_0$. To a first approximation

\[
y_0 \approx y_0 + y_1
\]

and the distance, $y$, is obtained from measurement as

\[
y_0 = y_0' = y_{\text{max}} - y_2 = y_{\text{max}} - y_2
\]

\[
z_2 = z_1 \left( \frac{y_{\text{max}} - y_2}{y_4} \right)
\]

\[
z = z_1 \left[ 1 + \left( \frac{y_{\text{max}} - y_2}{y_4} \right) \right]
\]
Fig. 3.12 Schematic plan view of experimental rig showing geometry for parallax corrections.

Similarly, for the y coordinate, triangle $E'F'G'$ is a mirror.
image of triangle EFG and each of these is similar to triangle OF'J. Therefore

\[
\frac{y_2 + y_4}{z_0 + z_1 + z_2} = \frac{y_4}{y_4}
\]

\[
y_1 = \frac{y_2(z_0 + z_1 + z_2)}{y_3 + y_4}
\]

and the true value, \( y_o \), may be found as

\[
y_o = y_1 + y_2 = y_2\left[1 + \frac{z_0 + z_1 + z_2}{y_3 + y_4}\right]
\]

It can be shown that the calculated values of the coordinates are only approximately correct. Choosing a pair of horizontal coordinates from the experimental data such that

\[
z, (y_o + y_4) = (7.73149, 7.77925)
\]

Successive corrections give the values quoted in Table 3.3

<table>
<thead>
<tr>
<th>( z )</th>
<th>( y_o + y_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.73149</td>
<td>7.77925</td>
</tr>
<tr>
<td>8.2662</td>
<td>8.83167</td>
</tr>
<tr>
<td>8.80923</td>
<td>8.91993</td>
</tr>
<tr>
<td>9.93537</td>
<td>9.59437</td>
</tr>
</tbody>
</table>

Table 3.3 Example of successive corrections to horizontal coordinates.

It is common to perform calculations on vectors such
that a right-hand Cartesian coordinate system is used. If the front view showing two coordinate directions is chosen to be within a right-hand Cartesian coordinate system, then the lateral inversion of the image of the side view in the $45^\circ$ mirror results in an image which is compatible with this right-hand system when projected into two dimensions with the origin of the projected horizontal dimension at the origin of the front view. This is shown diagrammatically in Fig. 3.13.

![Diagram](image)

Fig. 3.13 Diagrammatic view of right-hand Cartesian coordinate system showing lateral inversion correction.

3.5 Fluid Motion Measurements

A fourth set of experiments was carried out to investigate the flow of fluid close to the particle-fluid interface during sedimentation. All experiments
were carried out under the same experimental conditions as for particle measurements as described in section 3.4.1. However, for fluid measurements a laser Doppler anemometer was used.

Details of the principles of laser Doppler anemometry are given in Appendix 2. The laser and photomultiplier tube were used in the reference beam mode. The principle was to detect the Doppler shift in two convergent laser beams caused by microscopic dust particles suspended in the fluid in the sedimenting system. These are sufficiently small in size to be measured in interference fringes (in this case \(4.8255 \mu m\) apart). No interference was observed due to sedimenting particles since they were too large (\(\approx 500 \mu m\)) to be detectable. The apparatus consisted of a Spectra-Physics model 120, 15 mW He-Ne laser with polarized beam of 632.8 nm wavelength, fitted with a Precision Devices & Systems (U.K.) Ltd. RF307 transmitter beamsplitter and polarization unit, which provided the capability of splitting the beam into two beams of equal intensity. The laser and beamsplitter were placed in front of the experimental vessel with the two beams arranged one vertically above the other crossing at a point in the vessel on its diameter half way between the vessel's centreline and perimeter. The RF313 receiver photon detection unit comprised an iris and polarization system mounted on the front of a photomultiplier, selected for its low correlation and dark count (\(\approx 50 s^{-1}\)), a dynode and filter system together with magnetic and electrostatic shielding. At the rear of this end aligned coaxially was a \(x100\) gain amplifier and
discriminator which amplified and shaped photon signals into 50 ns wide pulses. In the reference beam mode the photon detector was aligned with the laser beam in which was a neutral density filter which cut down the beam intensity to about 10% of that of the other beam. Analog pulses from the photon detector were stored in digital form in a Malvern K7023 digital correlator, allowing high precision since instability and noise is removed from the system. The system itself introduces no noise and with sufficiently long integration times the system can extract a signal from background noise much greater than itself. A bank of scalers used as store permitted real time analysis over a range 50 ns to 1 s. Output was in graphical form on an oscilloscope and digital output of any of 48 count channels. A diagram of the reference beam set-up is shown in Fig. 3.14.

Fig. 3.14 Diagram of reference beam experimental set-up for laser Doppler anemometry.
The apparatus was set up to take 10 samples counting $4 \times 10^3$ $\mu$s/channel with a photomultiplier count rate of 7000 photon/s. With reference to Fig. 3.14 fringe separation is calculated as:

\[
\text{Fringe separation} = \frac{t \lambda}{d \mu}
\]

\[
= \frac{120 \times 10^{-4}}{16.5 \times 10^{-2}} \times \frac{0.6328 \mu m}{1.51}
\]

\[
= 3.289 \mu m
\]

Sampling time = time/channel x number of samples

\[
= 4 \times 10^{-3} \times 10
\]

\[
= 4 \times 10^{-2} \text{s}
\]

A photographic view of the apparatus arrangement is shown in the photograph, Fig. 3.15.

The experimental procedure was to operate the reciprocating disc to mix the fluid in the vessel containing a known weight of particles. The mixing was stopped with the reciprocating disc in the upper position and residual swirling allowed to stop. Throughout experimentation the laser, photomultiplier and digital correlator were left on, but the counting operation could be switched on or off or the digital counter's memory cleared to allow switching on and counting to commence after residual swirling had stopped and only gravity sedimentation was taking place.

Operation of the digital counter gave a trace on the oscilloscope screen of the type shown in Fig. 3.16 and the
Fig. 3.15 Experimental vessel showing laser Doppler anemometer.
values of $g_1$, $g_2$, and $g_3$ were made. This was done by switching the digital output control to the channels corresponding to $g_1$, $g_2$ and $g_3$ respectively. This procedure was carried out twenty times for one particle concentration and then the concentration increased by addition of glass beads to the experimental vessel. The procedure was repeated for all concentration values until the concentration range was completed.

![Diagram of typical oscilloscope trace from laser Doppler anemometer.](image)

Thus the average velocity and turbulence intensity for each oscilloscope trace are given by

$$u_{av} = \frac{\text{fringe separation}}{\text{sampling time} \times \text{channel number}}$$

$$= \frac{4.8256 \times 10^{-4}}{4 \times 10^{-2} \times Gz}$$

$$\eta = \frac{u_{av}}{U} \sqrt{\frac{1}{2} \left( \frac{g_2-g_1}{g_2-g_3} \right) + \frac{1}{2n^2}}$$
Justification for these expressions is given in Appendix 2. A computer program was used to handle the data, details of which are given in Appendix 3.
4. Experimental Results

4.1 Introduction

Analysis of the ciné film of particle motion during sedimentation with the Vanguard projector and P.C.D. Digital Data Reader yielded punched paper tapes of the coordinates of a particle's position on each frame of film. Sets of one pair of coordinates were obtained in the cases of series 1 and 2 of experiments, the two-dimensional observations, and sets of two pairs of coordinates in the case of series 3 of experiments, the three-dimensional observations. In all three series of experiments one paper tape for each value of solid volume concentration, comprising a set of pairs of coordinates of particle position, was analysed on a digital computer. Considering two consecutive pairs of particle coordinates, \((y_1, z_1)\) and \((y_2, z_2)\), where \(Oy\) is the horizontal axis and \(Oz\) is the vertical axis, the particle's velocity components in the projected horizontal and vertical directions are respectively

\[
\begin{align*}
  u_y &= \frac{y_2 - y_1}{400t} \\
  u_z &= \frac{z_2 - z_1}{400t}
\end{align*}
\]

where \(t\) is the time between camera frames, in these experiments 1.5 s, and 400 is the scale factor used in series 1 and 2 of experiments to convert the scale length on the projector to centimetres. Sets of velocity values were
obtained for each concentration and from these projected vertical and horizontal velocity distributions were calculated. In addition, average values, standard deviations and 95% confidence limits were calculated for the projected vertical and horizontal velocity components.

The distributions at each concentration are presented in graphical form. The vertical velocity distributions are shown relative to the Stokes velocity of a particle of the median size of the size distribution in series 1 and 2 and relative to the Stokes velocity of the mean sized marker particle in series 3 of experiments. Negative velocities indicate downward motion under the influence of gravity and positive velocities upward motion due to return flow of fluid.

Distributions of horizontal velocities are shown with positive and negative velocities. These correspond to motions to the right and left, respectively, so that velocities are defined in a right hand Cartesian coordinate system.

Graphs of average particle velocity plotted against particle volume concentration are presented for the projected vertical and horizontal motions. For a set of velocities as defined in eq. 4.1 the average values were calculated as

\[
\begin{align*}
\bar{u}_y &= \frac{1}{n} \sum_{j=2}^{n} \left( \frac{y_j - y_{j-1}}{400t} \right) \\
\bar{u}_z &= \frac{1}{n} \sum_{j=2}^{n} \left( \frac{z_j - z_{j-1}}{400t} \right)
\end{align*}
\]

4.2
The average velocities were divided by the Stokes velocity appropriate to the series of experiments, defined above. In the case of series 3 of experiments, two sets of average horizontal velocities are plotted, one for the front view and one for the side view of the experimental vessel. All average velocity graphs show the 95% confidence limits calculated at each concentration.

Data from the fourth series of experiments, the laser Doppler experiments on fluid velocity, were transformed from oscilloscope and digital output into punched paper tape. Average velocity, its standard deviation and 'turbulence intensity' at each concentration were obtained by processing this data on a digital computer. A graph of average fluid velocity plotted against volume concentration of solids is shown with its 95% confidence limits.

4.2 Series 1 of Experiments

The two-dimensional photographic observation of particle motion during sedimentation. The marker particles were 562μm in diameter possessing a Stokes velocity of 0.1167 cm/s, being the median size of the distribution of sedimenting particles.

The graphs of Fig. 4.2.1 and 4.2.2 show the average particle velocity in the projected vertical and horizontal directions relative to the Stokes velocity of a 562μm particle plotted against volume concentration of solids, showing the 95% confidence limits. The graphs of Fig. 4.2.3—4.2.19 show the frequency distributions of projected vertical and horizontal velocity of 562μm marker particles.
Fig. 4.2.1 Mean relative vertical velocity vs. concentration, with 95% confidence limits.

Fig. 4.2.2 Mean relative horizontal velocity vs. concentration, with 95% confidence limits.
Fig 4.2.3  Volume Concentration = 0.00368 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Figure 4.2.4: Volume Concentration = 0.0184 %

- Relative Vertical Velocity
- Horizontal Velocity (cm/s)
Fig 4.2.5 Volume Concentration = 0.0368 %

- Relative Vertical Velocity vs. Frequency
- Horizontal Velocity (cm/s) vs. Frequency
Fig 4.2.6  Volume Concentration = 0.0716%
Fig 4.2.7  Volume Concentration = 0.1472 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig 4.2.8  Volume Concentration = 0.1842 %

Frequency (%)

Relative Vertical Velocity

-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3

Horizontal Velocity (cm/s)

Frequency (%)

Horizontal Velocity (cm/s)
Fig. 4.2.9  Volume Concentration = 0.2576 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig 4.2.10 Volume Concentration = 0.3 mg/l.
Fig. 4.2.11  Volume Concentration = 0.4416 %

Volume Concentration

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Frequency (%)
Fig. 4.2.12 Volume Concentration = 0.5520 %

Frequency (%)

-9 -7 -5 -3 -1 0 1 2 3
Relative Vertical Velocity

Frequency (%)

0.3 0.2 0.1 0 -0.1 -0.2 -0.3 0
Horizontal Velocity (cm/s)
Fig 4.2.13  Volume Concentration = 0.0246

<table>
<thead>
<tr>
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<th>Horizontal Velocity (cm/s)</th>
</tr>
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<tbody>
<tr>
<td>-9</td>
<td>-0.3</td>
</tr>
<tr>
<td>-8</td>
<td>-0.2</td>
</tr>
<tr>
<td>-7</td>
<td>-0.1</td>
</tr>
<tr>
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<td>-4</td>
<td>0.2</td>
</tr>
<tr>
<td>-3</td>
<td>0.3</td>
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</tbody>
</table>

Frequency (%)
Fig. 4.2.15  Volume Concentration = 0.8096 %

Volume Concentration

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Frequency (%)
Fig. 4.2.16  Volume Concentration = 0.9200 %
Fig. 4.2.1. Volume Concentration = 0.9936 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig. 4.2.18  Volume Concentration = 1.1040 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig 4.2.15  Volume Concentration = 1.1776 %

- Relative Vertical Velocity
- Horizontal Velocity (cm/s)
4.3 Series 2 of Experiments

The two-dimensional photographic observation of particle motion during sedimentation. The marker particles were within ±5% of the Stokes velocity of a 562μm particle.

The graphs of Fig. 4.3.1 and 4.3.2 show the average particle velocity in the projected vertical and horizontal directions relative to the Stokes velocity of a 562μm particle plotted against volume concentration of solids with the 95% confidence limits. The graphs of Fig. 4.3.3 - 4.3.17 show the frequency distributions of projected vertical and horizontal velocity of the marker particles.
Fig. 4.3.1 Mean relative vertical velocity vs. concentration, with 95% confidence limits.

Fig. 4.3.2 Mean relative horizontal velocity vs. concentration, with 95% confidence limits.
Fig. 4.3.3  Volume Concentration = 0.0365 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig. 4.3.4  Volume Concentration = 0.0736 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig. 4.5.5  Volume Concentration = 0.1472 %

Relative Vertical Velocity

Frequency (f)

Horizontal Velocity (cm/s)
Fig 4.3.6  Volume Concentration = 0.1842 %

-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3
Relative Vertical Velocity

0.3 0.2 0.1 0.0 0.1 0.2 0.3
Horizontal Velocity (cm/s)

Frequency (%)
Fig. 4.37 Volume Concentration = 0.2576 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig 4.3.8  Volume Concentration = 0.3680 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Volume Concentration = 0.4415 %
Fig. 4.3.10  Volume Concentration = 0.5520 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig 4.3.11. Volume Concentration = 0.6256 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig. 4.3.12  Volume Concentration = 0.7360 \%
Fig. 4.3.13  Volume Concentration = 0.8096

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig. 4.3.14  Volume Concentration = 0.9300 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig. 4.3.15  Volume Concentration = 0.9936 %

![Graph showing Relative Vertical Velocity and Frequency](image)

![Graph showing Horizontal Velocity and Frequency](image)
Fig. 4.3.17  Volume Concentration = 1.1776 %

- Relative Vertical Velocity
- Horizontal Velocity (cm/s)
4.4 Series 3 of Experiments

The three-dimensional photographic observation of particle motion during sedimentation. The marker particles were approximately 770µm in diameter, within ±5% of the Stokes velocity of a 770µm particle.

The graphs of Fig. 4.4.1 and 4.4.2 show the average velocity in the projected vertical and horizontal directions relative to the Stokes velocity of a 770µm particle plotted against volume concentration of solids with the 95% confidence limits. The graphs of Fig. 4.4.3 - 4.4.20 show the frequency distributions of vertical and horizontal velocity of the marker particles.
Fig. 4.4.1 Mean relative vertical velocity vs. concentration, with 95% confidence limits.
Fig. 4.4.2 Mean relative horizontal velocities vs. concentration.

--- Front view with 95% confidence limits.

---- Side view only showing 95% confidence limits where values do not fall within 95% confidence limits of front view.
Fig 4.4.3  Volume Concentration = 0.00368 %

Front View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Side View

Horizontal Velocity (cm/s)
Volume Concentration = 0.01472 %

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Front View

Side View
Fig 4.4.5 Volume Concentration = 0.0184%
Fig. 4.4.6
Volume Concentration = 0.0068

---

Volume Concentration

---

Relative Vertical Velocity

---

Horizontal Velocity (cm/s)

---

Horizontal Velocity (cm/s)

---

Front View

---

Side View
Figure 4.4.7  Volume Concentration = 0.0736 %

Front View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Side View

Horizontal Velocity (cm/s)
Fig. 44.2  Volume Concentration = 0.1472 %
Fig 4A.9. Volume Concentration = 0.1842 %

Front View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Side View

Horizontal Velocity (cm/s)
Fig 4.4.10  Volume Concentration = 0.2576 g

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Front View

Side View

Horizontal Velocity (cm/s)
Fig 4.4.11  Volume Concentration = 0.3680 %

Front View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Horizontal Velocity (cm/s)
Fig 4.4.12  Volume Concentration = 0.4416 %

Front View

Side View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Horizontal Velocity (cm/s)
Fig 4.4.13  Volume Concentration = 0.9520 \( g \)

Front View

Side View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Horizontal Velocity (cm/s)
Fig 4.4.14 Volume Concentration = 0.6256 %
Fig 4.4.15  Volume Concentration = 0.7360 %

Front View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Side View

Horizontal Velocity (cm/s)
Fig 4.4.16 Volume Concentration = 0.8096 %

Front View

Side View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Horizontal Velocity (cm/s)
Fig 4.4.1] Volume Concentration = 0.9200%
Volume Concentration = 0.9956 %

Front View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Frequency (f)

0 [0.3]

0.2

0.1

0

-0.1

-0.2

-0.3

Horizontal Velocity (cm/s)

Frequency (f)

0 [0.3]

0.2

0.1

0

-0.1

-0.2

-0.3

Frequency (f)

0 [0.3]

0.2

0.1

0

-0.1

-0.2

-0.3

Side View
Fig 4.4.15 Volume Concentration = 1.1040 %

Front View

Side View

Relative Vertical Velocity

Horizontal Velocity (cm/s)
Fig 4A.20  Volume Concentration = 1.1776 %

Front View

Side View

Relative Vertical Velocity

Horizontal Velocity (cm/s)

Horizontal Velocity (cm/s)
4.5 Series 4 of Experiments

The observation of the vertical component of fluid motion during sedimentation by a laser Doppler anemometry technique, measuring close to particle-fluid interfaces.

Fig. 4.5.1 Mean vertical fluid velocities vs. concentration, with 95% confidence limits.
5. Discussion

5.1 Qualitative Remarks on the Sedimentation Curves

5.1.1 Series 1 of Experiments

The graphs of Fig. 4.2.1 and 4.2.2 show the average particle velocity in the projected vertical and horizontal directions respectively plotted against volume concentration of particles. The graphs refer to marker particles which were 562μm in diameter and represented the median size of the bulk of the sedimenting particles. The vertical axes are velocities plotted relative to the Stokes velocity of a 562μm particle settling in the fluid used in the experiments. The graphs of Fig. 4.2.3 - 4.2.19 show the frequency distributions of the projected vertical and horizontal velocities of the same marker particles. In this work a right-handed Cartesian coordinate system is used and so on the left hand graphs a negative relative velocity refers to downward motion and a positive relative velocity refers to upward motion. The right hand graphs are of absolute horizontal velocity. Negative velocities represent motion to the left and positive velocities represent motion to the right.

In common with the results of other workers in the field (54,55,57,58) Fig. 4.2.1 shows a velocity enhancement with increasing average particle concentration up to about 1% volume concentration followed by a decrease in average particle velocity. Although this work only considers sedimentation up to 1.1776% concentration, the
further decline in average settling velocity with increasing concentration is now well documented. As concentration increases still further the hindered settling region is entered. That is to say, the average particle velocity falls below the Stokes velocity of a single particle and will eventually fall to zero when a sedimenting system no longer exists but particles are touching and constitute a packed bed.

Returning to the description of Fig. 4.2.1, as would be expected, as the concentration approaches a nominal 0% the condition of a single particle is approximated and the velocity tends to the Stokes velocity. As particle concentration increases the velocity rises to $1.29U_s$ at 0.0368% concentration. The average velocity then begins to fall reaching a minimum of $1.09U_s$ at 0.1840%. This is repeated with maxima of $1.2U_s$ at 0.2576%, $1.36U_s$ at 0.6256% and the largest, $1.37U_s$ at 0.9936%, and minima of $1.12U_s$ at 0.3680% and $1.23U_s$ at 0.8096%. These data do not allow a smooth curve to be drawn of average particle velocity against concentration. This is consistent with the results of Kaye and Boardman, but it should be borne in mind that their average velocities were each obtained from 20 values and may be subject to statistical inaccuracy. Despite the fact that approximately 1,000 measurements were made for each average velocity in these experiments a smooth curve cannot be drawn between the 95% confidence limits at each concentration. Johne and Koglin were able to plot a smooth curve of best fit between the 95% confidence limits of their data, and their average velocities, too, were
obtained from only 20 values. Although far more data have been obtained in these experiments than by previous workers, it is still not certain whether the discontinuous appearance of the average velocity graphs represents accurately the physical phenomena or whether it is caused by statistical inaccuracy due to insufficient data. This will be discussed in section 5.1.4 with respect to all the average velocity graphs of Fig. 4.2.1, 4.2.2, 4.3.1, 4.3.2, 4.4.1, 4.4.2 and 4.5.1.

The greatest velocity enhancement is 37% at 0.9636% volume concentration, compared with maximum enhancements of 60% due to Kaye and Boardman and 140% due to Johne and Koglin. Koglin explains the differences in maximum velocity enhancements in the work of Johne and himself compared to that of Kaye and Boardman due to considerable wall effect causing the letters' velocities to be diminished. It will be shown in section 5.3 that the difference may be explained still further by measurement and averaging procedures adopted by these workers. However, the results reported by these workers are still higher than those reported in this work. This may be due to the fact that 1,000 data points were used to evaluate averages compared to the 20 used by other workers. The quantity of data for averaging is a topic dealt with in section 5.1.4.

The horizontal average velocity graph of Fig. 4.2.2 shows the same trends as the graph of Fig. 4.2.1 for vertical velocity although the maxima and minima are not always at corresponding concentrations. This, again, suggests that 1,000 data points may not be enough to
obtain a statistically accurate average value, although this fact requires verification and is suggested as a topic for further study in section 6. The maximum horizontal average velocity occurs at 0.9936% concentration as does the maximum vertical average velocity.

The velocity distributions indicate that there is interaction among particles giving rise to modifications of velocity from Stokes' law by the fact that there is not a single constant vertical velocity and that there is a horizontal velocity at all. Inferences can be drawn from these graphs to describe the nature of sedimentation and these will be dealt with in detail in section 5.2.

The distributions of vertical velocity are skewed such that the modal value appears approximately at the Stokes velocity, \(-U_s\). As concentration increases the curves generally become wider and the modal frequency becomes lower. In general, as concentration increases the distributions become more discontinuous. This is particularly noticeable with the distributions of horizontal velocity, although anomalies do occur, for instance a fairly discontinuous vertical distribution occurs at 0.0368% and a very smooth one at 0.6256%. The horizontal distributions are roughly symmetrical about the zero velocity value, as would be expected, since there is no horizontal driving force and, therefore, no net horizontal transport. Above 0.8096% the distributions become more discontinuous and some of the horizontal ones appear to be bimodal. This may be caused by an erratic return flow causing a preferential retardation of particles at certain
velocities or at certain positions in the vessel, or it may be that at increasing concentrations with larger numbers of particles more particle interactions occur and 1,000 data points are no longer sufficient to give an accurate average value.

Maximum downward velocities rarely exceed \(-2U_s\) at concentrations below 0.3680\% and rarely exceed \(-3U_s\) above this. Very few upward velocities are noticed. Similarly with horizontal velocities, absolute values rarely exceed 0.1 cm/s below 0.3680\% but at higher concentrations they may reach or sometimes exceed 0.15 cm/s.

5.1.2 Series 2 and 3 of Experiments

Series 2 of experiments represents a series of experiments which are analogous to those of series 1. Fig. 4.3.1 and 4.3.2 are analogous to Fig. 4.2.1 and 4.2.2 of series 1 in representing the projected vertical and horizontal average velocities of particles plotted against volume concentration, and Fig. 4.3.3 - 4.3.17 are analogous to Fig. 4.2.3 - 4.2.19, representing the frequency distributions at each concentration of projected vertical and horizontal particle velocities. Series 2 differs from series 1, however, in that rather than using marker particles of the mean diameter of the distribution of the bulk of particles, namely 562\(\mu\)m, the markers of series 2 are within \(\pm 5\%\) of the Stokes velocity of a 562\(\mu\)m particle. Using a distribution of marker sizes allows observation of particles in the suspension other than of the mean size.
The velocity enhancement of a sphere smaller than the mean size is greater than that of a sphere greater than the mean size. This statement may be verified by studying the interaction between pairs of spheres of different sizes, and this is done in section 5.2, Fig. 5.2.2. Since the particles belong to an approximately normal size distribution with symmetry about the mean, marker particles smaller than the mean size will encounter more particles larger than themselves and marker particles larger than the mean size will encounter more particles smaller than themselves. This will give rise to the observation of velocity enhancements larger and smaller, respectively, than those obtained by marker particles of the mean size. With a large number of observations of marker particles one would reasonably expect to observe as many 'large' markers as 'small' markers, the combined effect of which would be to give data corresponding to the average velocities observed in series 1. However, in series 2 of experiments, although 1,000 measurements were made at each concentration, they represent observations not of a single size of marker, but a range of sizes, that is to say fewer observations at each particle size. If the numbers of 'large' and 'small' markers are unequal, average velocities will be obtained which do not correspond to those obtained from the 562 \( \mu \)m marker particles. This is the significant difference between series 1 and series 2 of experiments, since, in series 2, vertical average velocities range from a minimum of 0.78\( U_s \) at 0.0363% concentration to a maximum of 1.66\( U_s \) at 0.7360% concentration, as shown in Fig. 4.3.1.
Fig. 4.3.2 compares with Fig. 4.2.2 in the same way, for the horizontal average velocities. Although the spread of marker particle sizes is noted as the prime cause of the differences between series 1 and 2 of results, the observation of fewer particles of each size also leads to a maximum velocity enhancement of 66% which compares favourably with the 60% enhancement obtained by Kaye and Boardman, though this argument is associated more with the statistical accuracy of results, which is to be discussed in section 5.1.4.

It was observed from the graphs of Fig. 4.2.3 - 4.2.19 that there is a distribution of vertical and horizontal velocities for a particle of a single size. Therefore, one might reasonably expect data for a number of particle sizes to give rise to a superposition of distributions for each of the sizes. This is, in fact, the case. The frequency distributions of Fig. 4.3.3 - 4.3.17 are in general wider than those of corresponding concentrations in Fig. 4.2.3 - 4.2.19 of series 1. In the graphs of Fig. 4.3.3 - 4.3.17 more upward velocities are observed, and these are due predominantly to particles less than 562 \( \mu \text{m} \). Similarly, velocities exceeding -2\( \mu \text{m/s} \) are also observed, and are due predominantly to particles greater than 562\( \mu \text{m} \). On occasions velocities of -3.5\( \mu \text{m/s} \) are observed, for instance at 0.4416% and 0.7360%. The distributions are smoother than those of series 1, and this again is due to the superposition of frequency distributions.

As with series 2 of experiments, the markers of series 3 covered a distribution of sizes, but here within
±5% of the Stokes velocity of a 770μm particle, and were thus larger than the largest of the bulk of sedimenting particles. In addition to using markers outside the range of the bulk of sedimenting particles, these experiments were conducted making measurements in three dimensions.

The graph of vertical average velocity of Fig. 4.4.1 shows good agreement with a similar experiment carried out by Kaye and Boardman. For particles larger than the bulk of sedimenting particles the graph of average velocity is much flatter and smoother showing a general decline with increasing concentration. There is a slight peak at 1.104%. In general very little velocity enhancement is noticed under such conditions, and at higher concentrations retardations are observed. Here experiments allow the observation of the effects of smaller particles only on the marker particles. At low concentrations slight enhancements of marker velocity are observed as would be expected since small particles effect large particles only to a small extent. The depression of velocity at increased concentration displays the retarding effect on particles when surrounded by a large number of particles. This effect will be described in section 5.2. Although at higher concentrations the marker particles have velocities less than their Stokes velocities they are still about 45% higher than the Stokes velocity of a 562μm particle and will thus affect those of the bulk of the particles which they neighbour quite considerably.

The superposition of velocity distributions for each size in the range of marker particle sizes led, in series 2
of experiments, to wider smoother frequency distributions. Here, however, the frequency distributions of Fig. 4.4.3 - 4.4.20 are quite narrow and bear more resemblance to those of series 1 of experiments, and this can again be described as due to the large difference in the marker particle sizes and those of the bulk of the particles. The influence of smaller particles on the large markers is only slight resulting in these narrow distributions.

The pairs of horizontal frequency distributions of Fig. 4.4.3 - 4.4.20 are in most cases similar, showing corresponding modal values and ranges. Those of 0.1472% concentration are the most noticeable example of a pair which are not similar. This effect is more pronounced when the graphs of average horizontal velocity of Fig. 4.4.2 are compared. These would be expected to follow each other fairly closely. Of the 18 points, 14 have overlapping 95% confidence limits and five of the mean velocities appear within the 95% confidence limits of the other. The poor comparison is attributable to two factors. Had larger quantities of data been taken, the results might have been better. In addition, the method of analysis required the simultaneous observation of a pair of photographs on a screen to determine the position of the particle from both front and side views and the observation of subsequent particle position by the advancement of pairs of frames of ciné film. Thus one cannot be certain of always observing the same particle throughout its trajectory from both views. An alternative is that since the side view shows consistently higher average velocities, there is net
particle transport to the side in the experimental vessel. This can be discounted because of the symmetrical nature of the horizontal frequency distributions and the similarity of front end side view horizontal frequency distributions at each concentration.

5.1.3 Series 4 of Experiments

Fig. 4.5.1 shows a plot of the average fluid velocity obtained using the laser Doppler anemometer, plotted against volume concentration of solids during sedimentation. Comparing this graph with the average relative particle velocity against concentration of Fig. 4.2.1, the two can be seen to be in fairly close agreement up to about 0.7360% concentration. Beyond this value, taking into account the 95% confidence limits of Fig. 4.5.1, a fluid velocity curve may be drawn similar in shape to the average particle velocity curve of Fig. 4.2.1. However, reference to table 5.1.1 shows that the fluid velocities obtained experimentally using the laser Doppler anemometer do not agree with those obtained by calculating an average return flow velocity, $\bar{U}_f$, from continuity considerations, using the average particle velocity, $\bar{U}_p$, and the particle concentration.

The results of table 5.1.1 are plotted in Fig. 5.1.1. The line with the 95% confidence limits is the experimental data already given in Fig. 4.5.1 and the line rising steeply is average fluid return flow velocity calculated from the average particle velocity through continuity considerations.
<table>
<thead>
<tr>
<th>Conc. %</th>
<th>Experimental values</th>
<th>From continuity and vertical average velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0184</td>
<td>$6.180 \times 10^{-4}$</td>
<td>$3.6127 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.0368</td>
<td>$6.719 \times 10^{-4}$</td>
<td>$7.7950 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.0736</td>
<td>$6.164 \times 10^{-4}$</td>
<td>$1.5115 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.1472</td>
<td>$5.346 \times 10^{-4}$</td>
<td>$2.8859 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.1840</td>
<td>$5.524 \times 10^{-4}$</td>
<td>$3.3224 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.2576</td>
<td>$5.830 \times 10^{-4}$</td>
<td>$5.0990 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.3680</td>
<td>$6.286 \times 10^{-4}$</td>
<td>$6.8310 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.4416</td>
<td>$6.613 \times 10^{-4}$</td>
<td>$9.0588 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.5520</td>
<td>$6.106 \times 10^{-4}$</td>
<td>$1.1328 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.6256</td>
<td>$5.706 \times 10^{-4}$</td>
<td>$1.4099 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.7360</td>
<td>$6.971 \times 10^{-4}$</td>
<td>$1.6571 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.8096</td>
<td>$7.830 \times 10^{-4}$</td>
<td>$1.6608 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.9200</td>
<td>$6.176 \times 10^{-4}$</td>
<td>$1.9154 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.9936</td>
<td>$8.203 \times 10^{-4}$</td>
<td>$2.2742 \times 10^{-3}$</td>
</tr>
<tr>
<td>1.1040</td>
<td>$7.574 \times 10^{-4}$</td>
<td>$2.3095 \times 10^{-3}$</td>
</tr>
<tr>
<td>1.1776</td>
<td>$9.768 \times 10^{-4}$</td>
<td>$2.3960 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 5.1.1 Average fluid velocity data at each concentration as obtained experimentally from laser Doppler anemometer and by calculation from continuity equation and vertical average particle velocity.

Clearly, then, the data obtained from the laser Doppler anemometer do not refer to the return flow of fluid during sedimentation. As already indicated in section 3.5, the equipment was set up to cover only one order of magnitude of velocities. The order of magnitude of the experimental particle velocities is $10^{-1}$ cm/s and the orders of magnitude of the fluid velocities calculated from continuity considerations span the range $10^{-5} - 10^{-3}$ cm/s. The laser Doppler anemometer results, being of order of magnitude $10^{-4}$ cm/s, cannot be a measure of particle velocities, nor of fluid return flow velocities, except possibly at around 0.3680% concentration.
Fig. 5.1.1 Mean vertical fluid velocity vs. concentration with 95% confidence limits and return flow fluid velocity from continuity considerations.
For a no-slip condition at a fluid-particle interface, the fluid at the interface possesses the same velocity as the particle. In the case of a single particle, the fluid velocity attenuates from this value at the interface to zero some distance away from the particle. With a large number of particles the situation is complicated by the interactions among particles, the fluid return flow, and the no-slip interface of each particle with the fluid. With increasing distance from a particle of the suspension the fluid velocity will attenuate from the particle velocity demanded by the no-slip condition at the fluid-particle interface to zero and then increase in magnitude but in the opposite direction to the particle's motion, to a return flow velocity demanded by continuity considerations. This local return flow velocity will depend on the local particle concentration and the magnitudes and directions of individual particles in that local volume. In general, at a small distance from a fluid-particle interface the fluid possesses a velocity equal in direction but smaller in magnitude than the velocity of the particle, and not, in general, equal to the fluid return flow velocity. This is the velocity measured by the laser Doppler anemometer. The magnitude of this velocity would be expected to change with the changing average particle velocity at different concentrations and this is borne out by the shapes of the graphs of fluid and particle velocity of Fig. 4.5.1 and 4.2.1 respectively.
5.1.4 The Statistical Accuracy of Results

The question now arising is that of the statistical accuracy of the results presented as the average velocity graphs of Fig. 4.2.1, 4.2.2, 4.3.1, 4.3.2, 4.4.1, 4.4.2 and 4.5.1. As already indicated, a smooth curve cannot be drawn through the average velocity values or between the corresponding 95% confidence limits. It was suggested in section 5.1.1 that unless the phenomena could account for this behaviour of the average velocity with respect to concentration, then the alternative explanation is that of insufficient data to describe the phenomena accurately.

Another problem which also arose in section 5.1.1 was that although these results agree qualitatively with those of Kaye and Boardman (57) and Johne and Koglin (54,55,58,59) they do not agree quantitatively. It will be shown in this section that these two problems can be answered simultaneously.

As a measure of the statistical accuracy of the results, the data of series 1 presented in Fig. 4.2.1 and 4.2.2 have been used. Each of the points on these graphs was obtained by averaging about 1,000 measurements of particle velocity, and these measurements were made by filming the trajectory of about 30 particles over about 35 frames each. In Fig. 5.1.2 and 5.1.3 average velocities have been plotted from the data of series 1 but using fewer data points. That is to say, average vertical and horizontal velocities have each been calculated from every third and every fifteenth value of velocity from the set of about
Fig. 5.1.2 Coarsely measured vertical mean velocities, Series 1.

Fig. 5.1.3 Coarsely measured horizontal mean velocities, Series 1.
1,000. The 95% confidence limits shown on these graphs are obtained from the whole of the data, and are the same as those appearing in Fig. 4.2.1 and 4.2.2.

At low concentrations these new averages calculated from fewer data approximate closely to the averages of the whole data but the approximation is not so good at higher concentrations. Recalling the velocity frequency distributions of Fig. 4.2.3 - 4.2.19, it will be remembered that, in general, the distributions were narrower at lower concentrations. Though it is perhaps intuitively obvious, this demonstrates that fewer data points are required from a narrow distribution than from a wide distribution to obtain an average value which is a good estimate of the average velocity occurring during the experiment. It may be concluded that at low concentrations 1,000 data points are sufficient to obtain an accurate estimate of the average particle velocity, though as concentration increases the quantity of data points needs to be more than 1,000. This is confirmed by the close approximation of the data of series 1 and series 4 of experiments up to about 0.7360% concentration shown by the similar shapes of Fig. 4.2.1 and 4.5.1 up to this concentration.

In Fig. 5.1.2 maximum enhancements of about 60% occur and these may explain why Kaye and Boardman, and Johne and Koglin obtained results which showed greater velocity enhancements than those reported in this work, since both these sets of workers calculated average velocities from 20 data points, considerably fewer than were used for averaging procedures in this work.
Implications for sedimentation experiments on a laboratory scale and in industry follow from these considerations. It has been shown that at concentrations less than about 0.1%, 1,000 measurements are sufficient to obtain an accurate measure of the average particle velocity in a suspension. Indeed, a third or a fifteenth of this number will give reasonably accurate values though particle interaction takes place resulting in deviations from Stokes' law even at these concentrations. At higher concentrations because of the increased number of particles there will be a wider range of interparticle distances and relative orientations yielding a wider distribution of particle velocities. In this region averages from one third and one fifteenth of the data compare less well with the averages from the whole data. Thus, it would be true to say that averages from small quantities of data in the concentration range between 0.1% and 1.1776% concentration are not necessarily sufficient to give a good estimate of average velocity. This can be extended further. Even averaging over 1,000 data points leads to a certain degree of statistical inaccuracy, though obviously to a much less extent. Results will obviously be more accurate, the larger the quantity of data over which averaging is performed. This, however, is more difficult to quantify from these data, since the effect of a particle's history is also involved.

Only in an infinitely long sedimenting column will a particle exhibit all the velocities of its velocity distribution due to particle interaction. In a finite vessel
a series of measurements can only cover part of the velocity distribution, since in a short distance a particle initially observed to be travelling at the 'fast' end of the velocity distribution may accelerate and decelerate to other velocities also at the 'fast' end of the distribution. A particle observed at the 'slow' end of the distribution will be more likely to accelerate and decelerate to other velocities also at the 'slow' end of the velocity distribution. This was also demonstrated by Koglin and is discussed in section 2 with particular reference to Fig. 2.1.7. From the graphs of Fig. 5.1.2 and 5.1.3 it can be seen that, at some concentrations, the average of every fifteenth value approximates more closely to the overall average than does the average of every third value. Since the original data were obtained as about 30 sets of 35 values for each concentration, the average of every fifteenth value represents roughly two values from each set, whereas the average of every third value represents roughly ten values from each set. Thus, the latter shows the particle's 'history' which sometimes presents an unrealistically high or low average velocity.

The comments made with regard to series 1 of experiments apply equally to series 2 of experiments, but not to the same extent with series 3 of experiments. Despite the fewer measurements, the velocity distributions for series 3 are much narrower, and the difference between the front and side views is due to the difficulty of observation of the latter.
5.2 Description of the Sedimentation Phenomenon

Having analysed the data obtained from experimentation a description of sedimentation may now be given. A single particle translating in an otherwise stationary fluid does so under the force of gravity with only a vertical component. Solution of the Navier-Stokes equation requires the boundary conditions:

\[
\begin{align*}
\frac{\partial \psi}{\partial r} &= -U_s \quad \text{at} \quad r = a \\
\psi &= 0 \quad \text{at} \quad r = \infty
\end{align*}
\]

5.2.1

Thus the velocity field in the fluid attenuates from the Stokes velocity, \( U_s \), at the particle's surface, where \( a \) is the particle radius to zero at some distance away from the particle, which for the sake of convenience is referred to as infinity. The stream function, \( \psi \), is defined as

\[
\psi = U_s \sin^2 \theta \left[ \frac{3}{4} \frac{a^2}{r} - \frac{1}{2} \frac{e^2}{r^2} - \frac{1}{4} \frac{a^3}{r^3} \right]
\]

5.2.2

and from this the velocity and pressure fields, \((v, p)\), in a fluid surrounding a particle become

\[
\begin{align*}
vr &= -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \\
&= u \cos \theta \left[ 1 - \frac{3}{2} \frac{a}{r} + \frac{1}{2} \frac{a^3}{r^3} \right] \\
\theta &= \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \\
&= -u \sin \theta \left[ 1 - \frac{3}{4} \frac{a}{r^2} - \frac{1}{4} \frac{a^3}{r^3} \right] \\
\phi &= 0
\end{align*}
\]

5.2.3 5.2.4 5.2.5
where $v_r, v_\theta, v_\phi$ are the components of velocity in spherical polar coordinates. Thus, when a single particle translates in an otherwise stationary fluid, the velocity imparted to the fluid by the particle's motion may be calculated at any distance, $r$, from the centre of the particle from eq. 5.2.3 and 5.2.4, and a practical value for the boundary condition at infinity of eq. 5.2.1 may be found as the value of $r$ which gives an asymptotic approach of $v$ to zero. Let this value be $r_\infty$.

Extending this to two particles it can be seen that if two particles are separated by a distance greater than $2r_\infty$ they will each behave as single particles in a stationary fluid. If the two particles are closer than the distance, $2r_\infty$, they no longer behave as single isolated particles because the velocity field in the fluid now depends on the motion of both particles and this gives rise to a more general boundary condition on the surface of each sphere that they may each possess a vertical translation, a horizontal translation in the same direction as the line between their centres, and a rotation about an axis through each particle centre perpendicular to the horizontal direction. Particular cases arise when the line between particle centres is horizontal, when there is no horizontal velocity component, and when the line between particle centres is vertical when there is only vertical translation. The second boundary condition of eq. 5.2.1 must still be applied but the practical definition of the distance $r = \infty$ is now
more difficult since the velocity field now includes two particles. The influence of one particle upon the other is the particle interaction. For isotropic spheres of radius, \( a \), and distance between centres, \( l \), the vertical velocity, \( U_v \), and horizontal velocity, \( U_h \), of each are given by

\[
U_v = U_0 + \frac{3}{4} \frac{a}{l} U_0 \left(1 + \sin^2 \theta \right) \tag{5.2.7}
\]

\[
U_h = \frac{3}{4} \frac{a}{l} U_0 \sin \theta \cos \theta \tag{5.2.8}
\]

From eq. 5.2.7 and 5.2.8 it can be seen that the interaction varies with the interparticle distance and the angle between particle centres and the horizontal. In particular it increases with increasing particle proximity. The closer two particles are, the faster they will translate. Increasing particle proximity is equivalent to increasing the concentration. Defining concentration, \( c \), as

\[
c = \frac{4}{3} \pi a^3 \frac{3}{4 \pi r^3} = \left( \frac{a}{r} \right)^3 \tag{5.2.9}
\]

the vertical velocity relative to the Stokes velocity of a particle of radius, \( a \), can be determined at different values of interparticle distance and plotted as in Fig. 5.2.1 in terms of concentration. The upper curve shows increasing velocity with increasing particle proximity or concentration. The velocity was obtained from
Fig. 5.2.1 Two sphere model of gravity settling with and without return flow.
eq. 5.2.7 averaging over all values of $\theta$.

As a model for sedimentation the two sphere model is very elementary. Return flow of fluid caused by particle settling can be crudely modelled by applying the continuity criterion. For various values of particle concentration, $c$, values of $r$, half the interparticle distance, can be found for a given particle size, according to eq. 5.2.9. Values of interparticle distance can then be used in eq. 5.2.7 and 5.2.8 to give values of the vertical and horizontal velocity components. These can then be averaged for all angles, $\theta$. Denoting vertical particle velocities by $U_p$ and fluid velocities by $U_f$, continuity demands that

$$U_p c = U_f (1-c) \quad 5.2.10$$

and values of the fluid return flow velocity, $U_f$, can be calculated for corresponding values of vertical particle velocities, $U_p$, at concentrations, $c$. Subtraction of the fluid velocity from the particle velocity yields a velocity, which when plotted against concentration, appears as the lower curve of Fig. 5.2.1, showing particle velocity rising to a maximum and then declining due to the increased influence of fluid return flow. The model, however, is a poor analogy to the sedimentation phenomenon since the particle velocity used in eq. 5.2.10 does not represent conditions in a settling suspension, but the average velocity of a particle neighbouring another particle in an otherwise stationary fluid.

In a real suspension the gravity driving force causes
a general downward motion of particles, and in a closed vessel the net downward motion of particles is balanced by a net upward flow of fluid. This is expressed mathematically by the continuity equation and eq. 5.2.10 is now generalized such that the total flowrates of particles and fluid are equal in magnitude and opposite in direction. Thus denoting average velocities of particles and fluid by $U_p$ and $U_f$ respectively

$$\bar{U}_p c = \bar{U}_f (1-c)$$ 5.2.11

During the sedimentation process particles are no longer constrained by the first boundary condition of eq. 5.2.1 but are free to undergo vertical and horizontal translations and to rotate, and do so because the velocity and pressure field in the fluid is modified by the motions of all the particles in it. If the suspension is imagined to be composed of monodispersed spheres of radius, $a$, at random positions in the fluid at any time, then there exists a distribution of interparticle distances. The velocities of the particles are influenced by the position and motion of all neighbouring particles. Thus there will be a distribution of velocities, both in magnitude and direction. At a subsequent time, the particles will have changed their relative positions, and then in turn their velocities. Throughout sedimentation any particle of the suspension undergoes dynamic interaction with its neighbours which manifests itself as a distribution of velocities. From the definition of particle concentration of eq. 5.2.9 it
can be seen that a distribution of interparticle distances at different times and positions in the sedimenting vessel leads to concentration variations with time and position. Consequently, in the sedimenting vessel there are dynamic concentration fluctuations. Observation of the continuity equation demands that localized regions of suspension will display fluid return flow fluctuations. This is so since the continuity equation may be integrated over any arbitrary volume. Thus, an observer would expect to see distributions of fluid and particle translations and rotations in both magnitude and direction in different regions of the sedimentation vessel at various times.

It has been shown that for two spheres, particle interaction in an otherwise stationary fluid yields a velocity enhancement which is inversely proportional to interparticle distance. Returning again to the monodispersed suspension, as the initial concentration of particles is increased, the average interparticle distance decreases and one expects by analogy that the average particle velocity should increase. However, as the particle velocity increases so also must the fluid velocity, for continuity reasons, and as a result the particles must experience increased drag.

The dynamic nature of the interactions owes itself to some extent to the random positioning of particles at any time. However, a further contribution to the dynamic nature of interactions is a distribution of sedimenting particle sizes. This can be demonstrated with two spheres. The interaction of two equal sized spheres at Reynolds numbers less than about 0.05 results in both possessing
velocities of the same magnitude and direction. The interaction of two unequally sized spheres results in the smaller having its velocity enhanced above its Stokes velocity by a greater proportion than the enhancement on the larger, as shown in Fig. 5.2.2. This requires that the two do not maintain their relative positions. At Reynolds numbers between 0.05 and 1 pairs of spheres will also possess a rotational component of velocity which means that under these conditions also spheres will not maintain their relative positions. In a suspension this can happen among all the spheres and this, too, leads to a range of interparticle distances and consequent velocities.

The distributions of particle velocity due to dynamic interaction are shown by the graphs of Fig. 4.2.3 - 4.2.19 where particles of the median size of the distribution are influenced by their neighbours whose sizes constitute a distribution. Since there is a vertical driving force due to gravity the vertical distributions show a net average velocity, but since there is no horizontal driving force no net average horizontal velocity occurs from the random motion. No data have been collected for rotational motion of the spheres, but this, too, would be expected to show a zero average as for horizontal translations, though this might be a useful topic for further work.

Increasing velocity enhancement due to increasing particle concentration is shown by graphs of Fig. 4.2.1 and 4.2.2 for average vertical and average absolute horizontal velocities. At very low concentrations particle interaction is only slight since particles rarely approach
Fig. 5.2.2 Interaction of two spheres of unequal size compared to that of a pair of equal size.
each other closer than the distance, $2n_0$. A 1% enhancement in the vertical velocity of a sphere due to a neighbouring sphere corresponds to a particle concentration of 0.00024%, though in the practical case of a real suspension 0.005% is a more realistic figure, from Fig. 4.2.1. As concentration increases the average particle velocity for vertical and horizontal motion increases due to increased particle proximity as, too, does the fluid return flow velocity, the former predominating up to about 1% concentration, and the latter thereafter, as shown by the decrease in average particle velocity.

The suggestion that a distribution of particle sizes leads to a wider distribution of particle velocities is shown by the graphs of Fig. 4.3.3 - 4.3.17 where a range of marker particle sizes was used.

Solution of the Navier-Stokes equation for a number of spheres has hitherto been impossible owing to the difficulty of satisfying the boundary conditions on all the spheres simultaneously. In addition it has been difficult to define a representative volume of the suspension. Eq. 5.2.9 defined concentration in terms of a single particle of radius, $a$, surrounded by fluid forming a spherical volume of radius, $r$. The suspension can be considered as a number of these unit cells of fluid each containing a particle. This definition of concentration is only an approximation, however, since fluid also exists in the interstices among the fluid spheres of radius, $r$, and concentration is overestimated by an amount derived from this interstitial volume. In spite of this drawback particle concentration
has been defined and a solution attempted for the motion of a typical particle of the suspension.

The force vector, \( \mathbf{dF} \), acting on a surface, \( dS \), is given by

\[
\mathbf{dF} = \mathbf{n} \cdot \mathbf{T} \, dS
\]  

where \( \mathbf{n} \) is the unit normal vector on the elemental area, \( dS \), and \( \mathbf{T} \) is the stress tensor expressed as

\[
\mathbf{T} = -\mathbf{P} + \rho (\nabla \mathbf{v} \cdot \nabla \mathbf{v})
\]

The total force on a sphere of the suspension is found by integrating over the particle surface bounded by a

\[
\mathbf{F} = \int \int_{S} (\mathbf{n} \cdot \mathbf{T}) \, a^2 \sin \theta \, d\phi \, d\theta
\]  

In spherical polar coordinates \( \mathbf{n} = \hat{\mathbf{r}} \) and the stress tensor is written as

\[
\mathbf{T} = \hat{\mathbf{r}} \hat{\mathbf{r}} T_{rr} + \hat{\mathbf{\theta}} \hat{\mathbf{\theta}} T_{\theta\theta} + \hat{\mathbf{\phi}} \hat{\mathbf{\phi}} T_{\phi\phi} \\
+ \hat{\mathbf{r}} \hat{\mathbf{\theta}} T_{r\theta} + \hat{\mathbf{r}} \hat{\mathbf{\phi}} T_{r\phi} + \hat{\mathbf{\theta}} \hat{\mathbf{\phi}} T_{\theta\phi} \\
+ \hat{\mathbf{\theta}} \hat{\mathbf{\theta}} T_{\theta\theta} + \hat{\mathbf{\phi}} \hat{\mathbf{\phi}} T_{\phi\phi}
\]

Therefore

\[
\mathbf{n} \cdot \mathbf{T} = \hat{\mathbf{r}} T_{rr} + \hat{\mathbf{\theta}} T_{\theta\theta} + \hat{\mathbf{\phi}} T_{\phi\phi}
\]

It has been shown in the results of Fig. 4.2.3 - 4.2.19, 4.3.3 - 4.3.17 and 4.4.3 - 4.4.20 that there is no net horizontal driving force during the sedimentation process, only
the vertical gravity driving force, and the mean force on the sphere becomes

\[ \mathbf{F} = \mathbf{i}_z \cdot \mathbf{F} \]

\[ = 2 \pi \int \int (T_{rr} \mathbf{i}_r + T_{r\theta} \mathbf{i}_\theta + T_{r\phi} \mathbf{i}_\phi) \sin \theta \, d\theta \, d\phi \]

5.2.15

Now \( \mathbf{i}_z \cdot \mathbf{i}_r = \cos \theta, \mathbf{i}_z \cdot \mathbf{i}_\theta = -\sin \theta \) and \( \mathbf{i}_z \cdot \mathbf{i}_\phi = 0 \) and eq. 5.2.15 becomes

\[ \mathbf{F} = \int \int (T_{rr} \cos \theta - T_{r\theta} \sin \theta) \sin \theta \, d\theta \, d\phi \]

5.2.16

In spherical polar coordinates, the required components of the stress tensor, \( T_{rr} \) and \( T_{r\theta} \), are

\[ T_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r} \]

\[ T_{r\theta} = \mu \left( \frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right) \]

and these are obtained by the substitution of the velocity and pressure components of eq. 5.2.3 - 5.2.6, but the velocity, \( U \), in these equations is not that for a single particle which resolves itself as the Stokes velocity but is a local particle velocity dependent on the distribution of interparticle distances pertaining to the particle being considered at that particular time. Therefore, the velocity, \( U \), in eq. 5.2.3 - 5.2.6 is replaced by \( U_i(r) \). Substitution of the stress tensor components into eq. 5.2.16, averaging \( U_i(r) \) over the distribution of interparticle distances to give \( \bar{U}(r) \) and integrating yields
The form of $U_i(r)$ is defined as an infinite series up to the $N$th power in terms of the reciprocal of interparticle distance. The average particle velocity is found by summing over $m$ observations of particle configurations, i.e.,

$$
\bar{U}(r) = \frac{U_o}{m} \sum_{i=1}^{m} \left( \sum_{n=0}^{N} k_n \left( \frac{a}{r} \right)^n \right)
$$

The expression must reduce to Stokes' law for a single particle. In this case the velocity is constant. There is only one particle configuration, $i = 1$, and since there are no other particles present, $n = 0$. Thus eq. 5.2.18 reduces to $\bar{U}(r) = k_o$ and obviously $k_o = 1$. The values of the other constants, $k_n$, and the practical limit of $n = N$ may be found by curve fitting techniques, or more satisfactorily by analytical methods, and this latter is proposed as a topic for further work.

5.3 Inferences for Velocity Measurement in Sedimentation Equipment

5.3.1 Measurement of Velocities

The motion of a particle in a suspension due to gravity is such that the interactions with other particles cause it to accelerate and decelerate. Thus, over a series of measurements the velocity will vary and must be averaged to give some idea of a typical particle's velocity during the experiment. The velocity measurements may be made in one
of three ways: by measuring the times elapsed in falling a number of equal distances, by measuring the distances travelled in a series of constant time intervals, or by keeping neither a constant time base nor a constant distance of travel. Clearly the last of these three methods is more difficult to perform since two variables have to be measured for each determination of velocity. The first two methods are the commonest in use in sedimentation experiments. Thus, to find the average of \( n \) velocities, \( U_i \), the observer may measure the times, \( t_i \), taken to fall \( n \) equal distances, \( \Delta x \), or measure the distances, \( x_i \), fallen in \( n \) equal time intervals, \( \Delta t \).

A velocity may be defined in one of two ways

\[
U_i = \frac{x_i}{\Delta t} \quad 5.3.1
\]

\[
U_i = \frac{\Delta x}{t_i} \quad 5.3.2
\]

5.3.2 Theory of Average Velocity Calculations

The experiments carried out by the author involved the measurement of velocities as the distances, \( x_i \), travelled in successive equal time intervals, \( \Delta t \), obtaining velocities as in eq. 5.3.1. There are several different sorts of average velocities, and the quantity calculated from these experiments referred to as the average velocity is an arithmetic mean velocity. This quantity appears throughout the work in the graphs of Fig. 4.2.1, 4.2.2, 4.3.1, 4.3.2, 4.4.1, 4.4.2. The arithmetic mean velocity, \( \bar{U} \), is defined
\[ \bar{U}_A = \frac{1}{n} \sum_{i=1}^{n} u_i \quad 5.3.3 \]

and using the definition of eq. 5.3.1
\[ \bar{U}_A = \frac{1}{n\Delta t} \sum_{i=1}^{n} U_i \quad 5.3.4 \]

However, a harmonic mean velocity, \( \bar{U}_H \), may also be defined
\[ \bar{U}_H = \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{U_i} \right]^{-1} \quad 5.3.5 \]

and using the definition of eq. 5.3.1
\[ \bar{U}_H = \frac{n}{\Delta t} \sum_{i=1}^{n} \frac{1}{U_i} \quad 5.3.6 \]

It may be shown that the harmonic mean of a distribution of quantities is less than or equal to the arithmetic mean of the distribution. Suppose two variables are \( U_1 \) and \( U_2 \). Their arithmetic and harmonic means may be defined by eq. 5.3.3 and 5.3.5 respectively. Define also the quantities \( m \) and \( d \) such that
\[ m = \frac{U_1 + U_2}{2} \]
\[ d = \frac{U_1 - U_2}{2} \]
\[ \therefore U_1 = m + d \]
\[ u_2 = m - d \]

Now
\[ \frac{1}{U_1} + \frac{1}{U_2} = \frac{1}{m+d} + \frac{1}{m-d} = \frac{2m}{m^2 - d^2} \]
\[ \frac{1}{2} \left( \frac{1}{U_1} + \frac{1}{U_2} \right) = \frac{m}{m^2 - d^2} \quad 5.3.7 \]
Now the left hand side of eq. 5.3.7 is the reciprocal of the harmonic mean velocity, \( \bar{U}_H \), \( m \) is equivalent to the arithmetic mean velocity, \( \bar{U}_A \) and eq. 5.3.7 may be written

\[
\frac{1}{\bar{U}_e} = \frac{\bar{U}_A}{\bar{U}_A^2 - d^2}
\]

\[
\therefore \quad \bar{U}_H = \bar{U}_A - \frac{d^2}{\bar{U}_A}
\]

5.3.8

Now \( d^2 > 0 \) except when \( d = 0 \), when \( U_1 = U_2 = m \), and it becomes apparent that

\[
\bar{U}_H \leq \bar{U}_A
\]

5.3.9

This proof can be generalized to \( n \) variables by the method of induction. The result of eq. 5.3.9 may be demonstrated by Fig. 5.3.1 which shows a plot of the arithmetic mean velocity against concentration, as in Fig. 4.2.1, formerly defined as the average velocity. The velocity is plotted relative to that of a 562\( \mu \)m sphere, the 95\% confidence limits are shown, and refers to the vertical velocity data of series 1 of experiments. The lower plot is of the same data, relative to the Stokes velocity of a 562\( \mu \)m sphere, but calculated as harmonic mean velocities. Fig. 5.3.2 and 5.3.3 show similar plots for series 2 and 3, respectively, of experiments. Table 5.3.1 shows the arithmetic and harmonic mean velocity data presented in Fig. 5.3.1, 5.3.2 and 5.3.3 together with the percentage difference between corresponding arithmetic and harmonic mean velocities, where the arithmetic mean is taken as 100\%.
Fig. 5.3.1 Series 1, arithmetic and harmonic mean velocities.

Fig. 5.3.2 Series 2, arithmetic and harmonic mean velocities.
Fig. 5.3.3 Series 3, arithmetic and harmonic* mean velocities.

*harmonic mean velocity curve shown as lower curve without 95% confidence limits
<table>
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<th>Conc. %</th>
<th>Series 1</th>
<th>Series 2</th>
<th>Series 3</th>
</tr>
</thead>
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<td>1.0958</td>
<td>0.9605</td>
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</tr>
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<td>1.1999</td>
<td>1.0172</td>
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<td>1.1954</td>
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<td>1.2146</td>
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| Av.   | 22.26 | 18.10 | 16.70 |

Table 5.3.1 Arithmetic and harmonic mean velocities at different concentrations for Series 1, 2 and 3 of experiments.
From the data in table 5.3.1 it can be seen that in series 1 of experiments the harmonic mean velocity is between 12.35% and 34.41% lower than the arithmetic mean velocity, or an average 22.26% lower. The harmonic mean is, on average, 18.10% lower than the arithmetic mean velocity in series 2 of experiments, and for series 3 of experiments the figure is 16.70%.

5.3.3 Interpretation of Other Workers' Results

Whereas, from experiments performed by the author, velocities were measured as the distances, \( x_i \), travelled in a series of equal times, \( \Delta t \), and calculated according to eq. 5.3.1, most of the work of other authors quoted in the literature has involved the measurement of velocities as the times, \( t_i \), taken to fall a series of equal distances, \( \Delta x \), and velocities calculated according to eq. 5.3.2. Notably are the experiments of Kaye and Boardman(57) where particles were timed between two fiducial marks 5 cm. apart, and Johne(54,55) and Koglin(57,58) where particles were timed between three scintillation counters giving distances of fall of 10 cm.

In Kaye and Boardman's paper they quote average velocities measured by summing the total distance travelled and dividing by the total time elapsed, where the averages were each obtained from 20 values. Thus, denoting their average velocity by \( \bar{U}_{KB} \)

\[
\bar{U}_{KB} = \frac{\Delta x_1 + \Delta x_2 + \ldots + \Delta x_{20}}{t_1 + t_2 + \ldots + t_{20}}
\]
where \( n = 20, \Delta x = 5 \text{ cm} \). Although in eq. 5.3.10 time, \( t_i \), is the variable, the calculation of average velocity, \( \bar{U}_{KB} \), is analogous to the calculation of a harmonic mean velocity, \( \bar{U}_H \), according to eq. 5.3.6 since it involves the reciprocal of a sum, \( (t_i)^{-1} \).

According to Koglin\(^{58}\), Johne and Koglin calculated average velocities as the average of path-time pairs, and implies that averages were taken of 20 measurements. Denoting their average velocity by \( \bar{U}_{JK} \)

\[
\bar{U}_{JK} = \frac{1}{n} \left( \frac{\Delta x_1}{t_1} + \frac{\Delta x_2}{t_2} + \ldots + \frac{\Delta x_{20}}{t_{20}} \right) 
\]

\[
= \frac{\Delta x}{n} \sum_{i=1}^{20} \frac{1}{t_i} \quad 5.3.11
\]

where \( n = 20, \Delta x = 10 \text{ cm} \). Again time is the variable, but the average involves the calculation of the sum of reciprocals, \( \Sigma \left( t_i^{-1} \right) \), which is equivalent to calculating an arithmetic mean, \( \bar{U}_A \), according to eq. 5.3.4. Thus

\[
\bar{U}_{KB} \equiv \bar{U}_H \quad 5.3.12
\]

\[
\bar{U}_{JK} \equiv \bar{U}_A \quad 5.3.13
\]

In view of the relationships of eq. 5.3.12 and 5.3.13 it is not surprising that Keye and Boardman obtained velocity enhancements which were consistently lower than those of Johne and Koglin. At 1% concentration Keye and Boardman
obtained a 60% velocity enhancement whereas Johne and Koglin obtained a 140% enhancement. The velocities concerned are 1.6U_s and 2.4U_s respectively. In section 5.3.2 it was shown from experimental data in table 5.3.1 that the harmonic mean velocity can be expected to be 22.26% lower, on average, than the arithmetic mean velocity. Increasing Kaye and Boardman's maximum velocity by this amount yields a value of 1.956U_s. A further enhancement of about 30% is required to give the value of 2.4U_s obtained by Johne and Koglin, and in Koglin's (58) paper, this value is the percentage increment he calculated to account for the wall effect which he suggests was present in Kaye and Boardman's measurements.

5.3.4 Average Velocities with Time as the Measured Variable

Calculation of arithmetic mean velocities according to eq. 5.3.3 with velocity defined as in eq. 5.3.1 is not dependent on the interval of time chosen for Δt since it involves the sum, \( \Sigma x_i \). However, with velocity measured as times, \( t_i \), to fall fixed distances, Δx, as in eq. 5.3.2, the value of Δx chosen influences the average velocity since its calculation involves the sum \( \Sigma (\frac{\Delta x}{t_i}) \). Using eq. 5.3.2 and 5.3.3 results in the expression for average velocity with time, \( t_i \), as the variable, as

\[
\bar{U}_{A_t} = \frac{\Delta x}{n} \sum_{i=1}^{n} \frac{1}{t_i} \quad 5.3.14
\]
Suppose a larger distance, $\Delta X$, were chosen over which to measure velocity such that

$$\Delta X = n \Delta x$$

5.3.15

Since previously a particle was measured to travel a distance, $\Delta x$, in time, $t_i$, it will travel a distance, $\Delta X$, in time, $\sum_{i=1}^{n} t_i$,

$$T = \sum_{i=1}^{n} t_i$$

5.3.16

and the average velocity measured in this way, $\bar{U}_{AT}$, is given as

$$\bar{U}_{AT} = \frac{\Delta X}{T}$$

5.3.17

The velocity, $\bar{U}_{AT}$, approximates more closely to a harmonic mean velocity, $\bar{U}_H$, as in eq. 5.3.6, than to an arithmetic mean, $\bar{U}_A$, in eq. 5.3.3. Obviously, in practice it is unlikely that an average velocity would be quoted on the basis of a single velocity, $\bar{U}_{AT}$. However, consider tests made where $\Delta x$ is, say, 1 cm., and $\Delta X$ is 10 cm., and tests are carried out over settling distances of 30 cm. The calculation of $\bar{U}_{AT}$ involves the summation of 30 values of time, $t_i$, whereas to obtain a value of $\bar{U}_{AT}$ the harmonic mean of ten values of $t_i$ is found three times, each of the three times covering 10 cm., and the arithmetic mean of these
three values is calculated. Thus, in practice $\bar{U}_{AT}$ is found as the arithmetic mean of several values, each of which may be regarded as the harmonic mean of settling velocities measured over smaller distances. This value would be expected to be less than $\bar{U}_{AT}$ in the same manner as eq. 5.3.9:

$$\bar{U}_{AT} \leq \bar{U}_{AL} \quad \text{for } t < T$$ 5.3.18

Consideration of even larger distances, $X$, over which velocity measurements are made such that $X = m\Delta X$ and $T' = \sum_{j=1}^{m} T_j$ gives expressions similar to eq. 5.3.15 and 5.3.16 so that

$$\bar{U}_{AT'} = \frac{X}{T'} = \frac{m\Delta X}{\sum_{j=1}^{m} T_j}$$

and by comparison with eq. 5.3.18, $\bar{U}_{AT'} \leq \bar{U}_{AT}$ for $T < T'$ and it can be seen that, in general, the greater the difference in the distance of measurement, the greater is the difference in the average velocities obtained by this method of calculation.

As an example a set of results at 0.1840% concentration were selected from series 2 of experiments. Thirty-one consecutive vertical particle coordinates are presented in table 5.3.2, giving 30 distances travelled each in 1.5 s. These give 30 velocities measured at constant time intervals, and for this demonstration must be converted to velocities measuring times taken to fall a series of constant distances. The times, $t_i$, are calculated to fall 1 cm., each based on
the actual velocity. The times, \( T \), are calculated as the sum of sets of 10 of the times, \( t_i \). This is the time to fall 10 cm. and three velocities are found based on 10 cm. falling distance in times, \( T \). The arithmetic mean of these is found and this evaluates \( \bar{U}_{AT} \). For this example

\[
\bar{U}_{AL} = 0.1320 \text{ cm/s}
\]
\[
\bar{U}_{AT} = 0.1285 \text{ cm/s}
\]

and the inequality of eq. 5.3.18 is demonstrated, though the example is somewhat contrived since the experimental data obtained by the author are for distances travelled in constant time intervals, which have been converted to times taken to travel a series of constant distances, in this case.

Until now the arithmetic mean velocity, \( \bar{U}_A \), of eq. 5.3.3 has been used in conjunction with velocities defined in terms of a variable distance or variable time, as in eq. 5.3.1 and 5.3.2 respectively. However, if experiments are carried out with time as the variable, as in eq. 5.3.2, then use of this equation in conjunction with the definition of the harmonic mean velocity, \( \bar{U}_H \), of eq. 5.3.5 allows direct comparison of such experimental results with those of the author. Substituting eq. 5.3.2 into eq. 5.3.5 yields the harmonic mean velocity with time as the variable, \( \bar{U}_{HT} \)

\[
\bar{U}_{HT} = [\frac{1}{n} \sum_{i=1}^{n} \frac{L_i}{\Delta x}]^{-1}
\]

\[
= \frac{n \Delta x}{\sum_{i=1}^{n} L_i}
\]
\[5.3.19\]
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<th>( \frac{1}{T} )</th>
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</table>

\( \bar{u}_{At} = 0.1324 \quad \bar{u}_{AT} = 0.1285 \)

Table 5.3.2 Velocity data for 30 consecutive measurements at 0.1840% concentration converted to velocities as a function of time showing calculations of \( \bar{u}_{At} \) and \( \bar{u}_{AT} \).

Eq. 5.3.4 and 5.3.19 are comparable since they each express velocity as the total distance travelled divided by the total time taken. Moreover, eq. 5.3.19 corresponds to Kaye and Boardman's average velocity, \( \bar{u}_{KB} \), eq. 5.3.10, and it is their experimental results which compare with the author's.
In addition, eq. 5.3.9 still holds since, there, velocity was defined irrespective of distance or time measurement, and Johne and Koglin's average velocity, $\bar{U}_{JK}$, eq. 5.3.11, can be obtained by substitution of eq. 5.3.2 into the definition of the arithmetic mean velocity of eq. 5.3.3, and again it can be seen that their average velocities will be greater than those of Kaye and Boardman.

Great care must be exercised when interpreting the results of sedimentation experiments. Average velocities should be compared only when they are calculated from the same definition of average velocity and velocities measured in terms of the same variable, either time or distance. In addition, a special case exists where comparison is allowed of data measured with time as the variable and averaged as a harmonic mean and data measured with distance as the variable and averaged as an arithmetic mean, since both express average velocity as the total distance travelled divided by the total time taken.

The relative merits of various methods of particle size analysis are well documented (99,100). Further consideration of these methods may be given as a result of the work reported here. Methods which utilize large distances or times will give better estimates of average velocity since a particle will cover a greater proportion of its velocity distribution. Most analytical methods rely on Stokes' law, but above about 0.05% concentration substantial velocity enhancements are observed, and according to these experiments rise to about a 40% velocity enhancement above the
Stokes' law value at about 1% particle concentration. It should be remembered, though, that suspensions of powders contain asymmetrical particles which possess greater surface areas than spheres, and as a result experience a greater drag yielding a diminished velocity enhancement to spheres at corresponding concentrations.
6. Conclusions and Suggestions for Further Work

6.1 Conclusions

From the work done on spherical particles sedimenting under the force of gravity in a Newtonian liquid at low Reynolds numbers, the following conclusions may be drawn.

Particles in a suspension are free to translate and rotate in all directions and random relative positions give rise to a distribution of velocities, yielding new relative positions and consequent velocities at a later time. Thus distributions of a particle's velocity components are observed.

The gravity driving force gives the vertical distribution of particle velocities a modal value corresponding roughly to the Stokes velocity. Since there is no horizontal driving force the modal value of the horizontal velocity distribution is approximately zero, and the distribution is roughly symmetrical about this value.

Velocity distributions are, in general, wider with increasing particle concentration due to the increased number of particle configurations and consequent velocities. A spread of marker particle sizes yields wider velocity distributions owing to the superposition of velocity distributions for each size. The velocity distributions of particles larger than the bulk of suspended particles are very narrow since large particles are affected only to a small extent by small particles.

The no-slip condition at fluid particle interfaces is
demonstrated by the correspondence of the average velocities of particles and fluid near the interface with concentration.

The interaction of particles with increasing concentration leads to enhancements in average vertical velocity above the Stokes' law value due to the increased number of particle configurations and velocities. For continuity reasons the return flow of fluid also increases until at about 1% concentration a maximum enhancement of about 40% is observed followed by a decline at higher concentrations due to the increasing effect of fluid return flow. The absolute average of horizontal velocities shows similar behaviour with concentration. These results agree qualitatively with experiments performed by Johne and Koglin and quantitatively with those of Kaye and Boardman if allowance is made for statistical inaccuracy due to the paucity of their data.

Differences in the results of Kaye and Boardman, and Johne and Koglin are explained in terms of different velocity averaging procedures. This is extended to allow comparison of Kaye and Boardman's data with the author's.

Differences in definitions of velocity and averaging procedures are demonstrated quantitatively and inferences made for the interpretation of average velocity data from experiments with regard to particle size analysis using sedimentation methods: arithmetic means of velocities measured with distance as the variable and harmonic means of velocities calculated with time as the variable may be compared since each defines the average velocity as the total distance travelled divided by the total time taken.
A force balance on a typical particle of the suspension yields an expression for the departure from Stokes' law in terms of a power series in inverse powers of inter-particle distance.

6.2 Suggestions for Further Work

The rotation of spheres in pairs or clusters has been studied experimentally by other workers by photography of spheres painted different colours on different halves of their surfaces. This might usefully be extended to spheres in a sedimenting system to see if results can be obtained similar to those for horizontal motion, of a symmetrical velocity distribution with zero mean, and a mean of absolute values which increases with increasing particle concentration up to 1% concentration.

Experiments might be performed with marker particles smaller than the bulk of sedimenting particles. By analogy with experiments performed with large marker particles and analytical calculations for two spheres, such experiments might be expected to show very large velocity enhancements.

Since spherical particles form an idealized case, experiments might be considered using irregularly shaped glass particles. Comparing a sphere and an irregular particle of equal volume, the surface area of the irregular particle is greater, so, therefore, is the drag, and velocity enhancements in such a system might well be expected to be diminished.

Experiments with spherical particles might be performed
further to see how many values of velocity are required at each concentration to give a good estimate of average velocity. This would also help to resolve the validity of the shape of the average velocity against concentration graph. The fact that a smooth curve cannot be drawn through the author's data has not been thoroughly vindicated.

The laser Doppler technique could be adapted to the measurement of interstitial fluid velocities of return flow.

Finally, solution might be attempted of the expression for the drag on a sphere in a suspension expressed as a power series in inverse powers of interparticle distance.
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Appendix 1

Computer Programs for Solid Data

Three FORTRAN programs were written for use on the University's ICL 1904a digital computer to handle the data obtained for the solid glass spheres during sedimentation.

Al.1 PROGRAM GO20, PTWRIT

Analysis of cine film on the Vanguard projector and P.C.D. Digital Data Reader provided data on punched paper tape. Ensuring that this paper tape data possessed even parity, batches of paper tape, one for each volume concentration, were read. An example, showing the format, of the paper tape input is shown in Fig. Al.1.

```
+3654 +3107 -1884 +3091
+3660 +3015 -1897 +2997
+3657 +2936 -1916 +2885
+3651 +2858 -1928 +2790
+3629 +2753 -1921 +2698
+3664 +2664 -1927 +2602
+3665 +2577 -1927 +2509
+3633 +2490 -1924 +2405
+3646 +2398 -1924 +2108
+3652 +2303 -1923 +2210
+3643 +2205 -1940 +2117
```

Fig. Al.1 Example of paper tape input showing format.

The program was written to scan the data to find any errors in the paper tapes, indicate where the errors lay and delete the line containing the error. The read paper tape was written to magnetic tape. This procedure was continued until all the paper tapes were read. The program then wrote the data to the line printer.
AI.2.1 PROGRAM G019, READER

This was the organizational program to handle the data written to magnetic tape by program G020. Its purpose was to call the magnetic tape containing the data, call certain subroutines to perform calculations on the data, write the processed data to the line printer and to a second magnetic tape for use by a graph plotting routine, PROGRAM G016, GRAPHS described in section AI.3. The subroutines VELDIS, HARVEL & STDVEL called by READER are described below.

AI.2.2 Subroutine VELDIS

This subroutine read a set of data for a particular concentration from magnetic tape and made parallax corrections to it. Then vertical and horizontal velocities were calculated from the data. The maximum and minimum vertical and horizontal velocities were calculated and each velocity put in one of fifty subranges of the range of velocities. The frequency of each subrange was then calculated, in order to give a frequency distribution of velocities at each concentration. Average values of vertical and horizontal velocities were calculated. Frequency distribution data and average velocity data were written to magnetic tape.

AI.2.3 Subroutine HARVEL

This subroutine read the velocity data calculated in subroutine VELDIS and calculated average velocities in vertical and horizontal directions as the harmonic mean velocity. These data were written to the line printer.
Al.2.4 Subroutine STDVEL

This subroutine calculated the standard deviations of vertical and horizontal velocities calculated in subroutine VELDIS and wrote them to the line printer.

Al.3.1 PROGRAM GOl6, GRAPHs

This was the organizational program to handle processed data written to magnetic tape by program GOl9. Its purpose was to call the magnetic tape, call the library functions UTPOP and UTPCL which open and close, respectively, a computer operated graph plotting facility and call subroutines DISTRB and LOGRAF which are graph plotting routines.

Al.3.2 Subroutine DISTRB

This subroutine read the velocity distribution data from magnetic tape. It then called a library routine, DEFBUF, to write the volume concentration value onto the graph plotter and library routines, UTP4A and UTP4B, to draw axes and plot data as graphs of vertical and horizontal velocity distributions at each concentration.

Al.3.3 Subroutine LOGRAF

This subroutine read the average velocity data from magnetic tape. It then called library function UTP1OA to plot a pair of logarithmic/linear axes and then library function UTP1OB to plot the vertical average velocity data against concentration as calculated in subroutines VELDIS and HARVEL of program GOl9, READER. It then repeated the procedure for the horizontal velocity data.
Appendix 2

**Principles of Laser-Doppler Anemometry**

A2.1 Principles of Doppler Shift

The frequency of radiation scattered by an object moving relative to a radiating source is changed by an amount which depends on the velocity and scattering geometry. Thus for a wave vector $\vec{k}_0$ incident on a scatterer moving with velocity $\vec{v}$, the scattered wave vector $\vec{k}_s$ gives rise to a Doppler shift $\Delta u$, as in Fig. A2.1.

$$\Delta u = (\vec{k}_s - \vec{k}_o) \cdot \vec{v} \quad \text{A2.1.1}$$

![Doppler shift from single incident beam.](image)

For light of 0.5μm wavelength the frequency shift is 4MHz in full backscatter. Interferometers exist for such high resolution but more practical fluid mechanics problems require an order of 100MHz, and this is overcome with two crossed incident beams, $\vec{k}_{o1}$ and $\vec{k}_{o2}$, measuring with a 'square law' detector the difference of the two Doppler-shifted frequencies in the scattering direction, as in Fig. A2.2.
\[ \Delta v_1 = (k_{o1} - k_{o2}) \cdot v \]
\[ \Delta v_2 = (k_{o1} - k_{o2}) \cdot v \]
\[ \therefore \Delta v_1 - \Delta v_2 = (k_{o2} - k_{o1}) \cdot v \]

Fig. A2.2 Doppler shift from two incident beams.

\[ \Delta v_1 - \Delta v_2 = k_{o} \cdot v \]

Now \( |k_{o1}| = |k_{o2}| = \frac{2\pi}{\lambda} \)

so that \( |k_{o}| = |(k_{o2} - k_{o1})| = \frac{4\pi}{\lambda} \sin \frac{\theta}{2} \)
\[ \therefore \Delta v_1 - \Delta v_2 = \frac{4\pi u \sin \frac{\theta}{2}}{\lambda} \quad A2.2.2 \]

where \( u \) is the component of \( v \) in the direction of \( k_{o} \) or \( \Delta f = \frac{2u \sin (\theta/2)}{\lambda} \) which is independent of the scattering direction. The scattering direction, \( k_{s} \), is taken to be \( 0^\circ \) or \( 180^\circ \) with respect to the axis of geometry of the incident beams.

A2.2 Optical Arrangement

Light from a laser is split into two equal parts by a beamsplitter and made to cross in a liquid at a point at which velocity measurement is required. If two or more particles are present in the liquid in the scattering
region at the same time, radiated fields are in general out of phase and the detected signal will have a lower depth of modulation due to partial cancellation. In the limit of large particle concentration the Doppler difference signal vanishes for incoherent detection. Consequently the reference beam arrangement is more suitable for these conditions, indicated in Fig. A2.3.

![Diagram](neutral density filter, laser, photomultiplier tube)

Fig. A2.3 Reference beam arrangement.

The detector aperture is sufficiently small that rays from any point of the scattering volume arrive in approximately the same relative phase, and cancellation, referred to above, does not take place. The criterion is that the angle subtended at the scattering volume by the detector aperture should be smaller than \( \lambda/w \) where \( w \) is the largest dimension of the scattering volume. In small angle forward or backward scattering, the reference light can conveniently be parasitic scattering from wall imperfections or dust particles.

### A2.3 Practical Considerations

Measurement depends on a knowledge of \( \theta \), which is not constant over the whole measuring region because the beam has a hyperbolic contour and Gaussian intensity profile. Only at the beam waist where the diameter is a minimum is
the wavefront planar. At a distance, \( z \), from the beam waist, radius \( r_0 \) (defined as the radial distance to the \( e^{-2} \) intensity point) this is given by

\[
[r(z)]^2 = r_0^2 \left[ 1 + \left( \frac{\lambda z}{\pi r_0^2} \right)^2 \right]
\]

as in diagram A2.4.

![Diagram of a Gaussian laser beam](image)

Fig. A2.4 Contour of Gaussian laser beam.

\[
R(z) = z \left[ 1 + \left( \frac{\pi r_0^2}{\lambda z} \right)^2 \right]
\]

and the divergence of the beam is defined as the angle \( \phi \) between the axis and the hyperbola asymptotes. Thus

\[
\phi = \frac{\lambda}{\pi r_0}
\]

For commercially available lasers \( \phi \) is typically \( \sim 0.5 \text{mr} \) and the waist is inside the laser cavity.
A2.4 Laser Doppler Anemometry

For a laser beam with a Gaussian intensity profile assume particles traverse the scattering volume in the plane of maximum fringe contrast, as in Fig. A2.5. For laminar flow with a velocity component $u$ perpendicular to the fringe system an instantaneous value of the probability, $I(t)$, of absorption of a photon by the detector is given by

$$I(t) = \sum_i a_{i0} \exp\left(-\frac{2u^2(t-t_{io})^2}{r^2}\right) \cos^2 \frac{\Pi u t}{\delta} \quad \text{A2.4.1}$$

where $a_{i0}$ = peak scattered light intensity for $i$th particle
Using the Weiner-Khinchine theorem, which assumes that the flow is statistically stationary (i.e. flow parameters are independent of the time origin), the autocorrelation function is derived. If it is assumed that the $t_{io}$ are distributed in random Poisson manner it can be shown that the photon correlation function given by

$$g(\tau) = \langle n(t, \tau) n(t+\tau, T) \rangle$$

\[= (\eta T)^2 \left\{ N\langle n_i \rangle \sqrt{\frac{\pi}{4}} \exp\left(-\frac{u^2 \tau^2}{r^2}\right) \left[ \frac{1}{2} + \cos^2 \left( \frac{\pi u \tau}{s} \right) \right] + \frac{1}{2} \exp\left(-\frac{\pi r^2}{s^2}\right) \right\}
\]

\[+ 2 \exp\left(-\frac{\pi r^2}{4s^2}\right) \cos\left( \frac{\pi u \tau}{s} \right) \right\} + N(N-1)\langle n_i^2 \rangle \frac{\pi r^2}{s^2} \left[ 1 + 2 \exp\left(-\frac{\pi r^2}{2s^2}\right) + \exp\left(-\frac{\pi r^2}{s^2}\right) \right]\]

\[+ \eta TN\langle n_i \rangle \sqrt{\frac{\pi}{2}} \frac{r}{2u} \left[ 1 + \exp\left(-\frac{\pi r^2}{2s^2}\right) \right] \delta(\tau) \quad \text{(A2.4.2)}\]

where $\tau$ = correlation delay

$n$ = number of photons absorbed in sample time $T$ (interval in which a sample acquired) centred on time $t$

$N$ = average number of particles present in scattering volume

$n_i$ = number of photons/s scattered by ith particle

$\eta$ = quantum efficiency

Provided $r > 2s$ (say) the time dependent part of the correlation function is very closely approximated by eq. A2.4.3.
In practice perfect fringe contrast cannot be achieved over the whole of the scattering volume and an accurate description of the observed autocorrelation function should include the fringe visibility factor, \( m \), yielding eq. A2.4.4

\[
G(\tau) = \frac{\alpha_o}{u} \exp\left(-\frac{u^2\tau^2}{r^2}\right) \left[ 1 + \frac{m^2}{2} \cos\left(\frac{2\pi u}{5} \tau\right) \right]
\]

This equation can be plotted and appears as Fig. A2.6. The output of a spectrum analyser operating on the signal of eq. A2.4.1 can now be calculated by applying the Weiner-Khinchine theorem (that the autocorrelation function and the power spectrum are a Fourier pair) to eq. A2.4.4.

\[
P(w) = \frac{2}{\pi} \int_0^\infty G(\tau) \cos(\omega \tau) d\tau
\]

\[
G(\tau) = \int_0^\infty P(\omega) \cos(\omega \tau) d\omega
\]

Fig. A2.6 Autocorrelation function.
P(w) is symmetrical about w = o. Denote the power spectrum corresponding to a laminar autocorrelation function \( G_\varepsilon(\tau) \) by \( P_\varepsilon(w) \), from A2.4.4, A2.4.5.

\[
P_\varepsilon(w) = \frac{2a_0}{\pi u} \int_0^\infty e^{-\frac{u^2 \tau^2}{r^2}} \left[ 1 + \frac{m^2}{2} \cos\left(\frac{2\pi u \tau}{s}\right) \right] \cos w \tau d\tau
\]

\[
\frac{\pi u}{2a_0} P_\varepsilon(w) = \int_0^\infty e^{-\frac{u^2 \tau^2}{r^2}} \cos w \tau d\tau + \frac{m^2}{2} \int_0^\infty e^{-\frac{u^2 \tau^2}{r^2}} \cos\left(\frac{2\pi u \tau}{s}\right) \cos w \tau d\tau \quad \text{A2.4.6}
\]

yielding

\[
P_\varepsilon(w) = \frac{a_0}{\sqrt{\pi}} \frac{r^2}{u^2} \exp\left(-\frac{r^2 w^2}{4u^2}\right) \left[ 1 + \frac{m^2}{2} \cosh\left(\frac{2\pi r}{u} w \right) \exp\left(\frac{-s^2}{5}\right) \right] \quad \text{A2.4.7}
\]

Now \( P_\varepsilon(w) \) is only approximately Gaussian. The finite width of the spectrum is referred to as 'ambiguity broadening' but is completely determined by optical parameters of the system.

Suppose the considered flow is not a single constant velocity but a velocity which fluctuates with time and position, as is the case of fluid passing through the sensing volume of the crossing laser beams during sedimentation in the fluid. Each particle in the scattering volume makes a contribution to the autocorrelation function of the form described in eq. A2.4.4. For a completely random distribution in an incompressible fluid, the rate at which particles pass through the scattering volume whose
dimensions are assumed to be small compared to the length scale of the fluctuation at a given time, will be proportional to the instantaneous velocity, \( u \). Hence, if \( p(u) \) is the probability density function describing the distribution of \( u \), the number of particles with velocities in the range \( u \) to \( u + \delta u \) which contribute to the autocorrelation function \( G_{\text{fluc}}(\tau) \) will be equal to \( k u p(u) \delta u \), where \( k \) is a constant depending on the dimensions of the scattering volume, density of the scatterers and the duration of the experiment. Hence:

\[
G_{\text{fluc}}(\tau) = a_1 \int_{-\infty}^{\infty} p(u) \exp\left(-\frac{u^2 \tau^2}{\tau^2}\right) \left[1 + \frac{m^2}{2} \cos\left(\frac{2\pi u \tau}{\tau^2}\right)\right] du \quad A2.4.8
\]

where \( a_1 = k a_0^2 \). As before:

\[
P_{\text{fluc}}(\omega) = \frac{2a_0}{\pi} \int_{0}^{\infty} \cos(\omega \tau) d\tau \int_{-\infty}^{\infty} p(u) \exp\left(-\frac{u^2 \tau^2}{\tau^2}\right) \left[1 + \frac{m^2}{2} \cos\left(\frac{2\pi u \tau}{\tau^2}\right)\right] du \quad A2.4.9
\]

Changing the order of integration, the physical function \( p(u) \) always possesses the necessary properties of convergence and continuity to allow this. Thus

\[
P_{\text{fluc}}(\omega) = \frac{2a_0}{\pi} \int_{-\infty}^{\infty} p(u) du \int_{0}^{\infty} \exp\left(-\frac{u^2 \tau^2}{\tau^2}\right) \left[1 + \frac{m^2}{2} \cos\left(\frac{2\pi u \tau}{\tau^2}\right)\right] \cos(\omega \tau) d\tau \quad A2.4.10
\]

\[
A2.4.11
\]

\[
P_{\text{fluc}}(\omega) = k \int_{-\infty}^{\infty} u p(u) \bar{P}(\omega) du
\]
No simple procedure for extracting $p(u)$ directly from the autocorrelation function A2.4.8 or the power spectrum A2.4.11 is available. Suppose $p(u)$ has a Gaussian form

$$p(u) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(u-\bar{u})^2}{2\sigma^2}\right)$$

where $\bar{u}$ = mean velocity

$\sigma$ = standard deviation

$$G_{fuc}(\tau) = \frac{a_1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(u-\bar{u})^2}{2\sigma^2}\right) \exp\left(-\frac{u^2 \tau^2}{r^2}\right) \left[1 + \frac{m^2}{2} \cos\left(\frac{2\pi u \tau}{s}\right)\right] du$$

A good approximation to the autocorrelation function can be obtained in explicit form for values of $\tau$ such that $\tau^2 \ll \frac{r^2}{2\sigma^2}$

$$G_{fuc}(\tau) \simeq a_1 \exp\left(-\frac{\bar{u}^2 \tau^2}{r^2}\right) \left[1 + \frac{m^2}{2} \exp\left(-\frac{2\pi \sigma^2 \tau^2}{s^2}\right) \cos\left(\frac{2\pi \bar{u} \tau}{s}\right)\right]$$

In terms of the turbulence intensity $\eta = \frac{\sigma}{\bar{u}}$

$$G_{fuc}(\tau) \simeq a_1 \exp\left(-\frac{\bar{u}^2 \tau^2}{r^2}\right) \left[1 + \frac{m^2}{2} \exp\left(-\frac{2\pi \bar{u}^2 \eta \tau^2}{s^2}\right) \cos\left(\frac{2\pi \bar{u} \tau}{s}\right)\right]$$

A2.4.12
Appendix 3

Computer Program for Fluid Data

A BASIC program was written for use on the department's Digital Equipment Corporation PDP 11/20 digital computer to handle data obtained from the laser-Doppler anemometry experiments on measurement of fluid velocity during sedimentation.

Data was input in punched paper tape form reading four variables corresponding to the values $G_2$, $g_1$, $g_2$, $g_3$ as defined in section 3.5. From these four data items the program calculated the average velocity and standard deviation. The program repeated this calculation 19 times for data at the same volume concentration and afterwards determined the overall average velocity, standard deviation and standard error on the mean.

The program was restarted for each new concentration.
Addendum

The velocity quoted on p.179 for marker particles of series 1 was obtained by measuring the speed of fall over 10 cm., and was subject to a stopwatch accuracy of ±0.2 s, resulting in a velocity of 0.1167 cm/s ± 0.25%.

The accuracy of measurement with the Vanguard projector and the constancy of operation of the time lapse camera equipment were obtained by filming the trajectory of a marker particle. A velocity of 0.1167 cm/s corresponds to 70.02 scale units travelled between camera frames, and the distances between frames obtained experimentally were within ±4 units of this value, corresponding to a velocity of 0.1167 cm/s ± 5.7%. This error and that due to the stopwatch during marker particle selection accumulate to give a velocity of 0.1167 cm/s ± 6%.

Fluid and particle densities, \( \rho \) and \( \rho_s \) were measured by the specific gravity bottle method. Weighing was accurate to 0.0001 gm., yielding densities: \( \rho_s = 2.41 \text{ gm/cm}^3 \pm 0.06\% \), \( \rho = 1.0006 \text{ gm/cm}^3 \pm 0.06\% \). The viscosity measurements of p.176 yield a viscosity which is accurate to ±5%.

Calculation of particle velocity by Stokes' law taking into account the errors involved in density and viscosity measurement, and average particle size measurement of p.175 yields a marker particle velocity of 0.1167 cm/s ± 8.84%.

The errors quoted here for series 1 of experiments apply equally to series 2 and 3 of experiments, where markers were chosen within ±5% of the Stokes velocity of particles with diameter 562\( \mu \)m for series 2 and 770\( \mu \)m for series 3 giving a possible spread of velocities of ±14.2%. 