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Trapped modes and acoustic resonances

By

Yuting Duan

Doctoral thesis

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Trapped modes and acoustic resonances

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Abstract

Keywords, trapped modes, complex resonances, wave scattering, mean flow, waveguides, Rayleigh-Bloch modes.

The scattering of waves by a finite thin plate in a two-dimensional wave guide and an array of finite thin plates, in the presence of subsonic mean flow, are formulated using a mode matching technique. The influence of mean flow on trapped modes in the vicinity of a finite thin plate in a two-dimensional wave guide is then investigated by putting the amplitude of the forcing term to zero in the scattering problem. The conditions for complex resonances are found, and numerical results are computed. The influence of mean flow on Rayleigh-Bloch modes is investigated by using a similar methodology.

The condition for embedded trapped modes to exist is introduced next, and then numerical results for embedded trapped modes without mean flow are presented. Complex resonances without mean flow are then found by fixing the geometry of the waveguide. The influence of mean flow on complex resonances and embedded trapped modes is investigated subsequently. In addition, the investigation of scattering coefficients is discussed when the frequency of an incident wave is near the real part of the frequency of complex resonances or embedded trapped modes.

Embedded trapped modes near an indentation in a strip wave guide, which may correspond to a two-dimensional acoustic wave guide or a channel of uniform water depth in water waves, are also found. Modes are sought which are either symmetric or anti-symmetric about the centreline of the guide and the centre of the indentation. In each case, a simple approximate solution is found numerically. Full solutions are then found by using a Galerkin approach in which the singularity near the indentation edge is modelled by choosing proper special functions.

The final part of the thesis is devoted to spinning modes (Rayleigh-Bloch modes) in a cylindrical waveguide in the presence of radial fins. A mode
matching technique is used to obtain the potential, and the coefficients in
the expansion are found numerically by using an efficient Galerkin procedure.
In addition, an existence proof for modes symmetric about the centre of the
guide and the centre of the section with radial fins is given by applying a
variational approach. The connection between Rayleigh-Bloch modes and
trapped modes is discussed thereafter, and numerical results for a number of
geometric configurations are presented.
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Chapter 1

Introduction

Trapped modes or trapped waves are the mathematical description of free oscillations, which are widely observed in the physical world. A typical and familiar example is the pendulum that will continue to oscillate for a long time once it is set in motion. A trapped mode is a free oscillation within an unbounded medium such that the medium motion is mostly confined to the vicinity of a structure, and decays to zero at large distance from the structure. The medium might be a fluid, or another elastic material. Physically, trapped modes are important because they are associated with resonances in forced initial value problems. In acoustics and water waves, a forced motion at a frequency near a trapped mode frequency can set up large fluid oscillations that may cause considerable damage to a structure, though the oscillations are limited by the action of non-linearity and the viscosity of the fluid.

Trapped modes or trapped waves are known to exist in a wide range of physical environments. For convenience the term trapped modes will be used, and we will concentrate on trapped modes in acoustics and water waves in this work.
1.1 Governing equation and boundary conditions

In acoustics a trapped mode solution is a solution of the Helmholtz equation under specific boundary and radiation conditions. A wave phenomenon in an unbounded medium may be described by the potential $\phi$ which satisfies the Helmholtz equation, specific boundary conditions and edge conditions under some assumptions. The Helmholtz equation reads as

$$(\nabla^2 + l^2)\phi = 0,$$  \hspace{1cm} (1.1)

and at boundaries $\phi$ satisfies

$$\frac{\partial \phi}{\partial n} = c_1,$$  \hspace{1cm} or $\phi = c_2,$  \hspace{1cm} (1.2)

where $l$ is the wave number, and $l = \omega/c$; $c$ and $\omega$ are the sound speed in the unbounded media and the circular frequency respectively; and $n$ is the unit normal directed into the fluid at boundaries; $c_1$ and $c_2$ are constants or functions depending on fluid-solid boundaries. If $\frac{\partial \phi}{\partial n} = c_1$ is given, then it is called as Neumann boundary condition, otherwise it is called Dirichlet boundary condition. Usually $l$ is also called the frequency since it has a simple relation with actual frequency. In this investigation, we also call $l$ the frequency for convenience.

In water waves the velocity potential satisfies the Laplace's equation in the fluid. In some cases, the governing equation may be reduced to the Helmholtz Equation, which will be explained in the following section. Therefore, the solution procedure in water wave applications is similar to that in acoustics in these cases.

In certain situations in which the region of interest either involves the behavior of potential at infinity or contains geometrical singularities, it is possible to derive several mathematically acceptable solutions, only one of
which is consistent with the phenomenon being investigated. Hence in these situations, it is necessary to apply certain additional physical constraints to ensure the uniqueness of the solution. These physical constraints include a radiation condition and an edge condition. The radiation condition governs the behaviour of the far field, whilst the edge condition governs the behavior of the near field, in the vicinity of a geometrical singularity which depends on the specific geometric structure and boundary conditions. For trapped modes to exist, the energy needs to be finite and the potential should decay to zero at infinity. Therefore the waves or modes are trapped in the vicinity of the structure, i.e., there are no waves propagating away from the structure.

In the following sections we will give a survey of trapped modes in water waves and acoustics.

1.2 Trapped modes in water waves

The most well-known trapped mode in the theory of water waves is the simple exponential solution derived by Stokes in 1846 based on the classical linear theory[97]. His solution described a wave over a uniformly sloping beach along which a wave can travel unchanged in the direction of the shore line, and decay exponentially to zero in the seawards direction. Stokes' solution is often referred to as the Stokes edge wave, and it is characterized by being confined to or trapped by the boundary, in this case the beach, despite the fact that the fluid region is unbounded. The Stokes' solution is simply an eigenfunction corresponding to a discrete eigenvalue embedded in the continuous spectrum of the Laplace operator, and it satisfies certain boundary conditions in the unbounded domain. It is the boundary conditions (surface conditions, beach conditions) and the unbounded domain which complicate the solutions.

After the first example of an edge wave mode over a sloping beach was
found in 1846 by Stokes, it took more than one hundred years before other
edge wave modes for this simple geometry were found. In the early 1950s,
Ursell[101, 102] announced some new edge wave modes and gave analytical
results as well as experimental data confirming the existence of these new
dge wave modes. A number of papers have been published since then in
which edge wave modes may be found for other geometries. An extensive
review of other geometries which permit the existence of edge waves was
given by Bonnet-ben Dhia and Joly[3]. Evans & Kuznetsov[15] give a state­
of-the-art review of trapped mode problems in water waves.

Briefly, the classical linear theory of water waves is based on the following
assumptions:

1. The fluid is considered to be heavy, incompressible, inviscid, and free
   from surface tension at the free surface;

2. The fluid motion is assumed to be irrotational and so may be described
   by a velocity potential \( \Phi(x, y, z, t) \);

3. The wave steepness is assumed to be small so nonlinear terms in the
   equations of motion and boundary conditions are neglected.

Cartesian coordinates \((x, y, z)\) are chosen such way that \(z\) is directed
vertically upwards, \(x\) and \(y\) lie in the unperturbed free surface. It follows
that the velocity potential satisfies Laplace's equation

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0,
\]

in the domain occupied by fluid. The linearised free surface condition reduces
to

\[
\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0, \quad \text{on} \quad z = 0,
\]

where \(g\) is the acceleration due to gravity. On rigid boundaries, we have

\[
\frac{\partial \Phi}{\partial n} = \mathbf{n} \cdot \mathbf{u},
\]
where $n$ is the unit normal directed into the fluid and $u$ the velocity of the body. The equations (1.3) - (1.5) form the basis for a particular problem in water wave theory. In some physical situations, when the dependence upon $z$ or $y$ can be eliminated, the remaining two-dimensional problems may have trapped mode solutions. In these cases equation (1.3) takes the form of the modified Helmholtz equation by eliminating the dependence on $z$ or $y$, see (1.7), or the Helmhiltz equation as in (1.11). Though the form of boundary conditions (see (1.9)) on any rigid bodies does not vary with the structure, it is the different structure which forms various trapped mode problems in water wave theory. Evans & Kuznetsov[15] classified these problems as

1. trapped modes above rigid surfaces with horizontal generators, and
2. trapped modes in water-wave channels.

We will briefly review these two types of trapped modes in water waves.

1.2.1 Trapped modes above rigid surfaces with horizontal generators

Let the fluid contain an obstruction which has a constant cross-section in the $y$ direction. The solution of (1.3) - (1.5) corresponding to the waves of frequency $\sigma$ and wave number $l$ traveling along the obstruction may then be sought in the form

$$\phi(x, y, z, t) = \phi(x, z) \cos(ly - \sigma t) \quad (1.6)$$

where $K = \sigma^2 / g$, with $g$ the gravity constant. It follows by substituting (1.6) into (1.3) - (1.5), that the function $\phi(x, z)$ satisfies the following boundary value problem

$$(\nabla^2 - l^2)\phi = 0 \text{ in the fluid,} \quad (1.7)$$

$$\phi_z - K\phi = 0 \text{ on the free surface,} \quad (1.8)$$
\[ \partial \phi / \partial n = 0 \text{ on solid boundaries.} \quad (1.9) \]

The solution of (1.7) - (1.9) is an eigenvalue problem. Jones[35] showed that, under some boundary conditions and a radiation condition, eigenvalues exist for this problem. In general, the boundary may be a sloping beach, a periodic coastline, or a submerged obstacle.

Trapped modes above submerged obstacles have been investigated by a number of authors. A submerged obstacle could be a thin plate, a cylinder, or simply a bottom protrusion. A submerged cylinder is the first structure other than sloping beach, for which trapped modes were found. The first paper demonstrating the existence of trapped modes other than edge waves was published by Ursell[101]. He demonstrated the possibility of trapped modes traveling along the top of a totally submerged horizontal circular cylinder of infinite extent and sufficient small radius, in infinitely deep water. Jones[35] removed the restriction that the radius of cylinder should be sufficient small, which had been assumed by Ursell[101], by considering cylinders of arbitrary section with symmetry about a vertical axis. Ursell[104] also showed that trapped modes exist above a horizontal cylinder of arbitrary cross section in water of infinite depth by applying potential theory.

Another geometry for which trapped modes exist is a protrusion from the sea bed, around which the water is of uniform depth apart from the protrusion. This geometry models an ocean of finite depth with an under water mountain range. The existence of trapped modes in this case was first demonstrated by Jones[35] in the case of a protrusion which is symmetric about the vertical axis. Maz'ya[62] and Kuznetsov[42] gives uniqueness theorems for a number of different bottom and obstacle geometries in water waves. A rigorous proof of the existence of trapped modes antisymmetric with respect to the centre of a submerged horizontal flat plate was given by Fernyhough[28].

A number of authors have considered the trapped modes over a sub-
merged body numerically. McIver & Evans[74] computed the zeros of the determinant of an infinite matrix which corresponds to trapped mode frequencies directly by using the formulation of Ursell[101]. Linton & Evans[59] obtained trapped modes above a submerged thin plate by matching eigenfunction expansion across the common boundaries. Parsons & Martin[86] confirmed these results by showing how to compute the trapped waves over a symmetric, curved, thin plate that is convex-upwards in the water of infinite depth, using hyper-singular integral equations. By considering different limiting cases they obtained agreements with both the submerged flat plate results of Linton & Evans[59] and the submerged cylinder results of McIver & Evans[74].

In addition to trapped modes over a single submerged body, a number of authors have focused on the investigation of trapped modes above more than one submerged obstacle. Examples of both two- and three-dimensional obstacles that can support trapped modes are demonstrated by McIver & McIver[73], and Evans & Porter[13]. Linton & Kuznetsov[49] have given the numerical evidence of trapped modes between pairs of surface-piercing angled barriers. In her paper McIver[66] further showed that trapped modes exist over a pair of bodies submerged in deep water in which an inverse procedure is used whereby a flow that represents a local oscillation is constructed and closed streamlines that surround the singularities of the flow are interpreted as the boundaries of the bodies. More recently Kuznetsov, McIver & Linton[41] have given the evidence of trapped waves by four surface-piercing barriers. Porter[88] has also shown that zero transmissions are possible for certain families of submerged bodies that include an ellipse. This suggests that waves could be trapped in these cases. In his further work[89], Porter gave a strong numerical evidence for the existence of trapped modes for pairs of such ellipses. Trapped modes may also exist over a submerged three dimensional bodies. McIver & Porter[63] have investigated the trapping of surface
waves by totally submerged three-dimensional toruses and strong numerical evidence for the existence of trapped modes was found.

1.2.2 Trapped modes in water-wave channels

In a channel of uniform water depth $h$ and parallel vertical walls which are described by $y = \pm d, -h \leq z \leq 0, -\infty < x < +\infty$, the depth dependence may be removed by writing the potential as

$$\Phi(x, y, z, t) = \phi(x, y) \cosh l(z + h) \cos \sigma t,$$  \hspace{1cm} (1.10)

where $l$ is the real positive root of $\sigma^2 = gl \tanh lh$. Substituting (1.10) into (1.3) and (1.5), we have

$$\nabla^2 \phi + l^2 \phi = 0 \text{ in the fluid},$$  \hspace{1cm} (1.11)

$$\dfrac{\partial \phi}{\partial n} = 0 \text{ on } S,$$  \hspace{1cm} (1.12)

$$\dfrac{\partial \phi}{\partial y} = 0 \text{ on } y = \pm d.$$  \hspace{1cm} (1.13)

where $S$ denotes the boundaries of obstacles which are considered. The resulting two-dimensional equation (1.11) is the Helmholtz equation.

A number of authors have investigated the trapped mode problem in this configuration. Evans & Linton[25] found trapped modes near a rectangular cylinder that extends through out the entire water depth and is positioned at the centre of the channel. They showed numerically that there exists at least one trapped mode and that other modes appear as the size of rectangular cylinder increases in the direction parallel to the walls. Following this Evans[23] proved that for a vertical plate positioned on the mid-plane and parallel to the channel walls, a trapped mode exists provided that the plate is sufficiently wide. A numerical scheme for determining trapped modes near a vertical cylinder of arbitrary cross-section was developed by Linton & Evans[57] through the application of Green theorem to the potential by using an
appropriate Green function, and the numerical solution of a homogeneous Fredholm integral equation of the second kind. Evans, Linton & Ursell[21] also proved that trapped modes exist when the plate is off-centre line and parallel to the channel walls. In this geometric configuration the operator can not be decomposed into symmetric or anti-symmetric part. A general existence proof for trapped modes near a cylinder of fairly general cross-section was given by Evans, Levitin & Vassiliev[19].

In addition to trapped modes near an rectangular cylinder positioned in the centre of a channel, Evans & Linton[53] found trapped modes near an indentation in a channel, which extends through out the entire water depth. Duan & McIver[11] also found embedded trapped modes near an indentation in a channel of uniform depth.

1.3 Trapped modes in acoustics

In acoustics trapped modes are usually referred to as acoustic resonances. The trapped modes occur at frequencies that correspond to eigenvalues which are embedded in the continuous spectrum of the relevant operator if no restrictions are placed on the symmetry of the solutions. However, if the structure is symmetric about the centre line of the guide and the motion is split into symmetric and anti-symmetric parts then the operator may be decomposed so that the essential spectrum of the anti-symmetric part has a non-zero lower limit. Consequently, the trapped modes correspond to eigenvalues that are below this value. We call the square root of this value the cut-off since it represents the value of non-dimensional wave number below which waves that are anti-symmetric about the guide centreline can not propagate along the guide.

One of the earliest experiments on acoustic resonances was made by Parker[82]. He found that (a) resonances associated with regular shedding
of wakes from parallel plates in a fluid stream are caused by purely acoustic effects which are unrelated to the structural vibration of plates, and (b) the resonance frequencies strongly depends on the geometry of shedding structure. On the basis of his experiment, Parker[83] computed resonant frequencies by means of a numerical relaxation technique for different shedding geometry, and found good correlation with his experiments. These resonance frequencies are defined as different modes, and often referred to as Parker modes.

In the early stages of the investigation of acoustic resonances, most attention was devoted to the study of unsteady flow phenomena in turbo-machinery, particularly in the field of aeronautical applications, see Tyler & Sofrin[100], Mani & Horvay[61], Kaji & Okazaki[36, 37], Koch[38, 39], and Rawlins[92], due to the strict noise regulations for commercial aircraft, and cases of acoustic fatigue.

The term trapped modes in acoustics first appeared in Evans[23], in which he proved the existence of trapped modes in the case of a sufficiently long vertical plate positioned parallel to and midway between the walls of the wave guide. In a following paper, Evans, Linton, & Ursell[21] considered the case of a sufficiently long plate off the centreline and parallel to the walls of wave guide, in which it is not possible to separate the problem into solutions symmetric and anti-symmetric with respect to the centre plane, and demonstrated that in this case a trapped mode also exists. In a following paper Evans & Linton[18] recovered all of the Parker modes by using a mode matching technique and residue calculus approach. Linton & Evans[57] also found trapped modes in the case of a cylinder of fairly general cross-section on the centreline of the guide. In a later paper Evans, Levitin & Vassiliev[19] proved that in a two dimensional wave guide, there exists at least one mode of oscillation anti-symmetric about the centreline of the wave guide.

In addition to proving the existence of trapped modes in two-dimensional
wave guides, Ursell[105] showed that trapped modes exist in a circular cylindrical wave guide in the presence of sufficiently small sphere. In his proof, he introduced a cut-off frequency below which waves can not propagate down the guide and away from the sphere, to restrict the modes of a particular angular variation. Motivated by the trapped modes near an off-centre plate in a two-dimensional wave guide in which symmetry is not a necessary requirement[21], Linton & McIver[48] proved that trapped modes exist in cylindrical wave guides in the presence of other obstacles, in which the restriction on the size of sphere was removed. The first class of the obstacles around which trapped modes exist is that the obstacle is thin and has a normal that is perpendicular everywhere to the generators of the cylinder. The second class of obstacles for trapped modes to exist is an axi-symmetric obstacle within a circular cylindrical wave guide.

Apart from trapped modes which have frequencies that are below the first cut-off frequency, embedded trapped modes have also been found. Embedded trapped modes are trapped modes above the first cut-off frequency. Evans & Porter[14] first provided numerical evidence for the existences of an isolated trapped mode in the presence of a circular obstacle on the center line of a two-dimensional wave guide, above the cut-off for anti-symmetric wave propagation in the guide. McIver, Linton & Zhang[55] further showed numerically that rather being isolated, this mode is one of a continuous branch which exists for ellipses of varying aspect ratio. The branch of modes begins with a trapped mode for a flat plate parallel to the wall of the guide and ends with a standing wave for a flat plate perpendicular to the walls of the guide. In addition they also found that the mode found by Evans & Porter[14] is a point on a branch of modes which exists for hypercircles, that is, obstacles with shape $|x/a|^\nu + |y/a|^\nu = 1$, $-a \leq x \leq a$, where $x$ is measured along the guide, $y$ across the guide, and $a, \nu$ vary along the branch. In a following paper, McIver et al [55] further investigated the existence of branches of em-
bedded trapped modes in the vicinity of symmetric obstacles that are placed on the centre line of a two-dimensional acoustic wave guide. In their paper they chose the obstacle to be a rectangular block of length $2a$, and width $2b$. Modes are sought which are anti-symmetric about the centreline of the guide and the block, and which have the frequencies that are above the first antisymmetric cut-off frequency and below the second antisymmetric cut-off frequency. The evidence that a two dimensional wave guide with an obstacle on its centre line can support embedded trapped modes indicates, that embedded trapped modes may also be found in the vicinity of an indentation in a two-dimensional acoustic waveguide. In this thesis we investigated this problem and part of the results have been presented in Duan & McIver[11].

Periodic structures are important in water waves and acoustics since another type of trapped mode, Rayleigh-Bloch modes, may be found. The potential satisfies a certain periodic condition in the presence of a periodic structure, which is often referred to as Rayleigh-Bloch condition, and the corresponding trapped modes are called Rayleigh-Bloch modes. Evans & Linton[20] found Rayleigh-Bloch surface waves along a periodic coastline which was modelled by an array of thin plates. Later Evans & Fernyhough found Rayleigh-Bloch modes along a periodic coastline which was modelled as an array of rectangular blocks[16]. McIver, Linton & McIver[69] also found Rayleigh-Bloch modes along an array of circular cylinders. Porter & Evans[90] investigated Rayleigh-Bloch surface waves along a periodic grating and their connection with the trapped modes in waveguides. Motivated by the absence of existence results, Linton & McIver[45] proved that Rayleigh-Bloch surface waves exist for a wide class of periodic structures. In another paper, Linton & McIver[46] provided an existence proof of trapped modes in the wave guides with periodic structures. They considered the structures of period 2 spanning a two dimensional wave guide of width $2N$. In this case, scattering problems in which Neuman conditions are imposed on the bound-
ary of the structure and either Neuman or Dirichlet conditions are applied on the guide walls, are decomposed into \( N + 1 \) independent problems. Based on this approach, they proved the existence of at least \( N \) trapped modes for the Neumann guide, and \( N - 1 \) such modes for the Dirichlet guide (The number of trapped modes in the Dirichlet guide increases to \( N \) if a certain geometrical condition on the obstacle is satisfied).

Trapped modes have been shown to exist in the vicinity of a number of obstacles in a two-dimensional wave guide, in which the solutions have real frequencies. In some situations, the frequencies are complex with a small imaginary part. In this case they are referred to as complex resonances. One example of a complex resonance was given by Aslanyan et al [1]. They investigated a problem of acoustic resonance in which an arbitrarily shaped obstacle is placed in a two dimensional, infinitely long acoustic waveguide, in which the obstacle is shifted from the centreline of the guide. By assuming the obstacle has smooth boundary (a disc was used in their calculation), they showed that only complex solutions may be found. They were particularly interested in those resonances which are perturbations of real eigenvalues. They also studied how an eigenvalue becomes a complex resonance, as the obstacle is shifted from the symmetric position in the guide to an off-set position. Numerical results show that the imaginary part of the complex resonance frequency increases with the offset.

Another example of complex resonances may be found is a two-dimensional waveguide with subsonic flow, in which a finite thin plate is placed midway from the guide walls. Few literatures can be found on complex resonances in the presence of mean flow although the impact of mean flow on acoustic wave scattering have been widely investigated by, for example, Kaji & Okazaki[37], Tyler & Sofrin[100], Mani & Horvay[61]. Koch[38] found complex resonances in a two-dimensional waveguide, in the presence of subsonic flow. He followed the idea of finding acoustic resonances in a duct without mean flow used by
Nayfeh & Huddleston[80]. In reality, the acoustic resonances can only be excited by the presence of mean flow. Therefore the first part of this thesis will be devoted to investigate the complex resonances caused by the subsonic uniform flow by using an approach which is different from that of Koch[38].

1.4 Mathematical solution techniques

There are several techniques for the solution of analytic acoustic problems, each of which has its own advantages and disadvantages. We describe two of them below, the Wiener-Hopf technique and the mode matching approach, since they are widely used in waveguide boundary problems in water waves and acoustics.

The Wiener-Hopf technique is a powerful method for the solution of partial differential equations. Generally, in waveguide problems, the potential satisfies Helmholtz equation and certain boundary conditions. The Fourier transform are then employed in one of two ways. In the first, the integral equation formulation, the boundary value problem is formulated in terms of an integral equation that is subsequently transformed into an equation in the complex $\alpha$-plane, $\alpha$ being the Fourier transform variable. In an alternative procedure, Jones' method of formulation, one formulates the problem directly in the transform domain. This involves the representation of the fields, followed by the application of boundary and continuity conditions in the transform domain. Both procedures lead to a single equation in the complex $\alpha$-plane for the two unknowns which have to be introduced in the process of formulating the problem. These two unknowns are then solved for, in a simultaneous manner, by applying a method called Wiener-Hopf technique, which involves the application of analytic continuation procedures in the complex $\alpha$-plane. The application of this method in waveguide problems was described in detail by Mittra & Lee[77]. A general discussion of
this method in the solution of partial differential equations can be found in Noble[81]. This method has been widely used in acoustic wave scattering problems, for example, by Meister[75], Koch[38], Mani & Horvay[61], and Rawlins[92]. A more recent paper by Hassan & Rawlins[32], using Wiener-Hopf technique, investigated the radiation of sound from a semi-infinite rigid duct, in the presence of mean flow.

Another powerful method widely used in waveguide problem is the mode matching technique which is one of the most frequently employed methods for formulating boundary-value problems. Generally, this method is useful when the geometry of structures can be identified as a combination of two or more regions, each belonging to a separable coordinate system. The first step of the mode matching procedure is to expand the unknown fields in the individual regions in terms of respective normal modes. The problem reduces to that of determining the set of modal coefficients (or modal amplitudes) associated with the field expansion in each region. The modal representation is followed by the application of continuity conditions for the field quantities at the interfaces of different regions. This procedure, in conjunction with the orthogonality property of the normal modes, eventually leads to an infinite set of linear simultaneous equations for the unknown modal coefficients. In general, it is not possible to extract an exact solution of this infinite system of equations, and one is forced to resort to approximate techniques, such as truncations. However, for a certain class of problems, there does exist an exact solution from the associated infinite system of equations. A powerful way to solve for the resultant infinite set of equations for certain problems is the application of residue calculus theory. The application of this technique in waveguide boundary problems was detailed by Mittra & Lee[77], and first used by Whitehead[111] for solving for the scattering by an infinite array of semi-infinite thin plates. Tyler and Sofrin[100] showed in their fundamental and influential study of axial flow compressor noise, that modes play an
important role in the sound generation and transmission in ducted rotors. The mode matching technique has been extensively used in a series papers in acoustics and water waves by, for example, Evans[23], Linton[54], McIver et al [65]. Mode matching technique will be used in this investigation for its apparent simplicity in dealing with certain waveguide problems in acoustics and water waves.

1.5 Work in this thesis

The work in this thesis is arranged in the following way.

In chapter 2, the scattering of waves by a finite thin plate in a two-dimensional wave guide, in the presence of subsonic mean flow, is formulated firstly by using a mode matching technique, followed by the formulation of scattering of waves by an array of finite thin plates. This chapter forms the preliminaries to Chapter 3 and 4.

In chapter 3, we investigate the influence of mean flow on trapped modes in the vicinity of a finite thin plate in a two-dimensional wave guide. We will use the formulation in Chapter 2, following the idea of Nayfeh & Huddleston[80], put the amplitude of the forcing term to zero in the correspondent scattering problem. The conditions for complex resonances will be found and numerical results will be presented and compared with those of Koch[39], for frequencies whose real part is below the lowest antisymmetric cut-off. The influence of mean flow on Rayleigh-Bloch modes will be investigated by using a similar methodology.

In chapter 4, we first introduce the condition for which embedded trapped modes exist and present numerical results for trapped modes without mean flow. The complex resonances without mean flow are then found by fixing the geometry. We then study the influence of mean flow on complex resonances and embedded trapped modes. The effect of mean flow on scattering coef-
ficients is then investigated when the frequency of the incident wave is near the real part of the frequency of complex resonances or embedded trapped modes.

Chapter 5 is devoted to the investigation of embedded trapped modes near an indentation in a strip wave guide, which may correspond to a two-dimensional acoustic wave guide or a channel of uniform water depth in water waves. Modes are sought which are either symmetric or anti-symmetric about the centreline of the guide and the centre of indentation. In each case, a simple approximate solution is found numerically. Full solutions are then found by using Galerkin approach in which the singularity near the indentation edge is modelled by choosing proper special functions.

The final chapter investigates spinning modes (Rayleigh-Bloch modes) in a cylindrical waveguide in the presence of radial fins. A mode matching technique is used to obtain the potential, and the coefficients in the expansion are found numerically by using an efficient Galerkin procedure. In addition, an existence proof for modes symmetric about the centre of the guide and the centre of the section with radial fins is given by applying a variational approach. The main results of this chapter have been submitted to *Wave Motion*, and it has been accepted[12].
Chapter 2

Impact of mean flow on acoustic wave scattering by thin plates

2.1 Introduction

In this chapter, we will use the mode matching technique and residue calculus theory[77] to formulate the boundary value problem for the scattering of waves by a finite thin plate in a two-dimensional waveguide (channel) and by an array of finite thin plates (cascades), in the presence of subsonic flow. The presence of mean flow adds complexity to the formulation although the procedure of using mode matching in scattering problems is a routine manipulation. The numerical calculation of the reflection and transmission coefficients will not be given in this chapter as they are available, for example, in Koch[39]. However, the results of the formulation are used in the subsequent chapters for the investigation of complex resonances in the presence of subsonic flow.
2.2 Scattering of waves by a thin plate in a two-dimensional waveguide in the presence of subsonic flow

2.2.1 Introduction

A thin plate with finite length $2L$ is placed on the centreline of a two-dimensional waveguide of width $2d$, as shown in Figure 2.1. It is assumed that the disturbances due to the incident sound wave are small perturbations in a uniform, inviscid flow at the subsonic Mach number $M$, so that the linear theory may be applied. Henceforth the velocity of the fluid may be expressed as

$$V_t = u_\infty (1 + \nabla \phi), \quad p_t = p_\infty (1 + p), \quad (2.1)$$

where $V_t$ is the velocity and $p_t$ pressure; and $u_\infty$ and $p_\infty$ denote the main stream velocity and pressure respectively; $i$ is the unit vector in $x$-direction. Thus from Koch[39] the perturbation velocity potential $\phi$ satisfies the wave equation for a flowing medium

$$\nabla^2 \Phi - M^2 \frac{\partial^2 \Phi}{\partial x^2} - \frac{2M}{c_\infty} \frac{\partial^2 \Phi}{\partial x \partial t} - \frac{1}{c_\infty^2} \frac{\partial^2 \Phi}{\partial t^2} = 0, \quad (2.2)$$

where $c_\infty$ denotes the free-stream sound speed. The perturbation pressure $p$ is given by

$$p = -\frac{\gamma u_\infty}{c_\infty^2} \left( \frac{\partial}{\partial t} + u_\infty \frac{\partial}{\partial X} \right) \Phi, \quad (2.3)$$

where $\gamma$ is the specific heat ratio $c_p/c_v$. Following Koch[39], we use the transformation

$$x = X/(d\sqrt{1-M^2}), \quad y = Y/d, \quad (2.4)$$

which is equivalent to the Prandtl-Glauert transformation in steady subsonic aerodynamics. The dimensionless length of the plate is given by

$$a = \frac{L}{d\sqrt{1-M^2}}, \quad (2.5)$$
Figure 2.1: A finite plate placed on the centreline of a strip waveguide where $0 < M < 1$ is the Mach number, defined by

$$M = \frac{u_\infty}{c_\infty}. \quad (2.6)$$

Figure 2.2 shows the definition sketch after using the Prandtl-Glauert transformation (2.4).

Thus the reduced potential $\phi(x, y)$ defined by

$$\Phi(X, Y, t) = d \phi(x, y)e^{-i\left(\frac{\omega t + \omega M X}{\omega_\infty(1-i\nu^2)}\right)}, \quad (2.7)$$

satisfies the Helmholtz equation

$$(\nabla^2 + t^2)\phi = 0, \quad (2.8)$$
Figure 2.2: A finite plate placed on the centreline of a strip waveguide after using the Prandtl-Glauert transformation

where \( l = \frac{\omega}{c_\infty (1 - M^2)^{1/2}} d \), is non-dimensional due to the Prandtl-Glauert transformation (2.4). The perturbation pressure is related to \( \phi \) by

\[
p = -\frac{\gamma M^2}{(1 - M^2)^{1/2}} \left( -\frac{i \phi}{M} + \frac{\partial \phi}{\partial x} \right) e^{-i(\omega t + i M x)}. \tag{2.9}
\]

The perturbation pressure is zero when the Mach number is zero. However, the perturbation pressure tends to infinity when the Mach number tends to one.
Hard wall boundary conditions are applied, namely

\[ \frac{\partial \phi(x, y)}{\partial y} \bigg|_{y=\pm 1} = 0, \quad -\infty < x < \infty, \tag{2.10} \]

and

\[ \frac{\partial \phi(x, y)}{\partial y} \bigg|_{y=0} = 0, \quad -a < x < a. \tag{2.11} \]

The whole waveguide consists of three regions. They are defined as I: \((-\infty < x < -a, \ 0 \leq y \leq 1)\), II: \((-a \leq x \leq a, \ 0 \leq y \leq 1)\), III: \((a < x < +\infty, \ 0 \leq y \leq 1)\) because of the symmetry of the geometry. Pressure continuity across the trailing edge requires that

\[ \lim_{y \to 0^+} \left( -\frac{i\lambda y}{M} + \frac{\partial \phi}{\partial x} \right) = \lim_{y \to 0^-} \left( -\frac{i\lambda y}{M} + \frac{\partial \phi}{\partial x} \right). \tag{2.12} \]

The integration of the above equation gives

\[ \phi(x, 0^+) - \phi(x, 0^-) = D e^{i\lambda x/M}, \tag{2.13} \]

where \(D\) is a constant to be determined by the Kutta condition at the trailing edge of the plate. Equation (2.13) means that the potential has a jump across the vortex sheet down stream of the plate when the Mach number is not zero. The perturbation pressure is identically zero from (2.9) if Mach number \(M = 0\). Thus there is no potential jump without mean flow.

2.2.2 Formulation

A plane wave is assumed to be incident from the up stream of the waveguide, then part of it is reflected by the plate and part of it is transmitted down the waveguide. The mode matching approach will be used to formulate the acoustic scattering problem.

In region I, the potential satisfies equation (2.8) and boundary condition (2.10). We assume that the total potential consists of three parts:

\[ \phi_I(x, y) = \phi_{\text{inc}} + \phi_{\text{ref}} + \phi_{\text{per}}, \tag{2.14} \]
where $\phi_{\text{inc}}, \phi_{\text{ref}}, \phi_{\text{per}}$ are the incident, reflected and perturbed potentials. By separating variables, we may solve the perturbed potential $\phi_{\text{per}}$ as

$$\phi_{\text{per}}(x, y) = \sum_{n=1}^{\infty} \frac{C_n e^{j_n(x+a)} h_n^I(y)}{j_n}, \quad -\infty < x \leq -a, \quad (2.15)$$

where $h_n^I(y)$ is the eigenfunction defined by

$$h_n^I(y) = 2^{1/2} \sin(\mu_n y), \quad \mu_n = (n - 1/2)\pi, \quad j_n = \sqrt{\mu_n^2 - l^2}, \quad n \geq 1, \quad (2.16)$$

where $h_n^I(y)$ form a complete orthonormal sets on $(0, 1)$, i.e.,

$$\int_0^1 h_n^I(y) h_m^I(y) = \delta_{m,n}. \quad (2.17)$$

The value of $j_n$ may be real or imaginary, depending on the value of $l$ and $n$. If $j_n$ is imaginary, it represents a wave-like term, otherwise it is a decaying term. If we assume $\pi/2 < l < \pi$, then $j_1$ is imaginary, and $j_n$ is real for all $n \geq 2$. In this case there is only one wave-like term. If a wave of form $e^{-j_1(x+a)} \sin \frac{n\pi}{2}$ is incident from upstream, and part of it is reflected and part of it is transmitted by the plate, we may write the incident and reflected potential in region I as

$$\phi_{\text{inc}}(x, y) = e^{-j_1(x+a)} h_1^I(y), \quad \phi_{\text{ref}} = e^{j_1(x+a)} h_1^I(y). \quad (2.18)$$

Substituting $\phi_{\text{inc}}, \phi_{\text{ref}}, \phi_{\text{per}}$ into (2.14), we have

$$\phi_I(x, y) = \left(e^{-j_1(x+a)} + e^{j_1(x+a)}\right) h_1^I(y) + \sum_{n=1}^{\infty} \frac{C_n e^{j_n(x+a)} h_n^I(y)}{j_n}, \quad -\infty < x \leq -a. \quad (2.19)$$

And the reflection coefficient is given by

$$R = 1 + \frac{C_1}{j_1}. \quad (2.20)$$

In Region II, the potential may be written as

$$\phi_{II}(x, y) = \sum_{n=0}^{\infty} \frac{1}{k_n} \left(A_n \cosh(k_n x) \sinh(k_n a) + A_n \frac{\sinh(k_n x)}{\cosh(k_n a)} \right) h_n^{II}(y), \quad -a \leq x \leq a, \quad (2.21)$$
where \( k_0 = -i \ell \), \( A^+_n, A^-_n \) are the coefficients of the symmetric and anti-symmetric parts respectively, and

\[
h^+_n(y) = \varepsilon_n^{1/2} \cos \lambda_n y, \quad \lambda_n = n\pi, \quad k_n = \sqrt{\lambda^2_n - \ell^2},
\]

(2.22)

where \( \varepsilon_n = 1, \ n = 0; \ \varepsilon_n = 1/2, n \geq 1 \), and \( h^+_n(y) \) form a complete orthonormal set on \((0,1)\), i.e.,

\[
\int_0^1 h^+_n(y)h^+_m(y) = \delta_{m,n}.
\]

(2.23)

In region III, the total potential may be divided two parts. One part is produced by the vortex sheet at the trailing edge of the plate, whilst the other part is anti-symmetric about the mid plane of the guide. We write

\[
\phi_{III} = \phi_1(x, y) + \phi_2(x, y),
\]

(2.24)

where \( \phi_1 \) is the anti-symmetric part, and \( \phi_2(x, y) \) is produced by the vortex sheet. Thus we may write the anti-symmetric part of the potential as

\[
\phi_1(x, y) = \sum_{n=1}^{\infty} \frac{B_n}{-j_n} e^{-j_n(x-a)} h^+_n(y), \quad x \geq a,
\]

(2.25)

where \( h^+_n(y) \) is the same as \( h^+_n(y) \). The potential produced by the vortex sheet is given by

\[
\phi_2(x, y) = \frac{D}{2} \frac{\cosh(k(1-y))}{\cosh(k)} e^{\frac{(x-a)}{\ell^2}},
\]

(2.26)

for \( x \geq a, \ 0 < y \leq 1 \), and

\[
\phi_2(x, y) = -\frac{D}{2} \frac{\cosh(k(1+y))}{\cosh(k)} e^{\frac{(x-a)}{\ell^2}},
\]

(2.27)

for \( x \geq a, \ -1 \leq y < 0 \). We shall focus on \((x \geq a, \ 0 < y \leq 1)\) in region III due to the symmetric geometry. Substitute (2.25) and (2.26) into (2.24) to give

\[
\phi_{III}(x, y) = \frac{D}{2} \frac{\cosh(k(1-y))}{\cosh(k)} e^{\frac{(x-a)}{\ell^2}} + \sum_{n=1}^{\infty} \frac{B_n}{-j_n} e^{-j_n(x-a)} h^+_n(y),
\]

(2.28)
where $x \geq a$, $0 \leq y \leq 1$ and $k = \frac{1}{M} \sqrt{1 - M^2}$.

Continuity of potentials and their first horizontal derivatives across $x = -a$ yields

$$2h_1^I(y) + \sum_{n=1}^{\infty} C_n h_n^I(y) = \sum_{n=0}^{\infty} \frac{A_n^a \coth(k_n a) - A_n^a \tanh(k_n a)}{k_n} h_n^{II}(y), \quad (2.29)$$

and

$$\sum_{n=1}^{\infty} C_n h_n^I(y) = \sum_{n=0}^{\infty} (-A_n^a + A_n^a) h_n^{II}(y). \quad (2.30)$$

Similarly, continuity of potentials and their first horizontal derivatives across $x = a$ gives

$$\sum_{n=0}^{\infty} \frac{A_n^a \coth(k_n a) + A_n^a \tanh(k_n a)}{k_n} h_n^{II}(y) = \frac{D}{2} \frac{\cosh(k(1-y))}{\cosh(k)} + \sum_{n=1}^{\infty} \frac{B_n}{j_n} h_n^{III}(y), \quad (2.31)$$

and

$$\sum_{n=0}^{\infty} (A_n^a + A_n^a) h_n^{II}(y) = \frac{D i l}{2 M} \frac{\cosh(k(1-y))}{\cosh(k)} + \sum_{n=1}^{\infty} B_n h_n^{III}(y). \quad (2.32)$$

Note that the unknowns $A_n^a, A_n^a, B_n, C_n$ are the coefficients in the expansions of horizontal velocity and so we can determine their behaviour as $n \to \infty$. We expect a square root singularity of velocity in the vicinity of both ends of the plate without mean flow, i.e.,

$$\frac{\partial \phi}{\partial r} = O(r^{-1/2}), \quad \text{as} \quad r = \sqrt{(x-a)^2 + y^2} \to 0, \quad x > a, \quad (2.33)$$

and

$$\frac{\partial \phi}{\partial r} = O(r^{-1/2}), \quad \text{as} \quad r = \sqrt{(x+a)^2 + y^2} \to 0, \quad x < -a. \quad (2.34)$$

Using the formula

$$\sum_{n=1}^{\infty} n^p e^{-nz} \sim \Gamma(p+1)z^{-(p+1)}, \quad \text{as} \quad z \to 0^+, \quad (2.35)$$
with \( z > 0 \) and \(-1 < p < 1\), see Mittra & Lee[77], we have

\[
\sum_{n=1}^{\infty} n^p e^{-n(x-a)} \sim \Gamma(p+1)(x-a)^{-(p+1)}, \text{ as } x \to a^+, \tag{2.36}
\]

and

\[
\sum_{n=1}^{\infty} n^p e^{n(x+a)} \sim \Gamma(p+1)(-x-a)^{-(p+1)}, \text{ as } x \to -a^-, \tag{2.37}
\]

where \( \Gamma(\omega) \) is gamma function defined by

\[
\Gamma(\omega) = \int_{0}^{\infty} t^{\omega-1} e^{-t} dt, \tag{2.38}
\]

with \( \text{Real}(\omega) > 0 \). By doing this, we can show

\[
\sum_{n=1}^{\infty} n^p e^{-n(x-a)} = O(n^{-1-p}), \text{ as } x \to a^+, \tag{2.39}
\]

and

\[
\sum_{n=1}^{\infty} n^p e^{n(x+a)} = O(n^{-1-p}), \text{ as } x \to -a^- . \tag{2.40}
\]

As a consequence, we have

\[
A_n^a, A_n^a, B_n, C_n = O(n^{-1/2}) \text{ as } n \to \infty. \tag{2.41}
\]

In the presence of mean flow, the velocity at the trailing edge is bounded. Therefore the Kutta condition has to be applied at the trailing edge, see for example, Koch[39], and Hassan & Rawlins[32]. Consequently, the proper behavior of the velocity near the trailing edge may be expressed as

\[
B_n = O(n^{-3/2}), \text{ as } n \to \infty, \tag{2.42}
\]

to ensure the velocity to be bounded near the trailing edge.

Following Evans & Linton[25], we can convert (2.29) - (2.32) into an infinite system of linear algebraic equations. Multiply (2.29) and (2.30) by \( h_m^H(y) \) and integrate over \([0, 1]\) to give

\[
2c_{m1} + \sum_{n=1}^{\infty} \frac{C_n}{j_n} \frac{c_{mn}}{j_m} = \frac{A_m^a \coth(k_m a) - A_m^a \tanh(k_m a)}{k_m}, \quad m \geq 0, \tag{2.43}
\]

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and
\[ \sum_{n=1}^{\infty} C_n c_{mn} = -A_m^a + A_m^a, \quad m \geq 0. \]  
(2.44)

Similarly, multiply (2.31) and (2.32) by \( h_n^I(y) \) and integrate over \([0,1]\) to yield
\[ \frac{A_m^a \coth(k_m a) + A_m^a \tanh(k_m a)}{k_m} = \frac{D \varepsilon_m^{1/2} k \tanh(k)}{2} + \sum_{n=1}^{\infty} \frac{B_n}{-j_n} c_{mn}, \quad m \geq 0, \]  
(2.45)

and
\[ A_m^a + A_m^a = \frac{D \varepsilon_m^{1/2} k \tanh(k)}{2M} - \sum_{n=1}^{\infty} B_n c_{mn}, \quad m \geq 0, \]  
(2.46)

where
\[ c_{mn} = \int_0^1 h_n^I(y) h_m^I(y) dy = (2 \varepsilon_m)^{1/2} \frac{\mu_n}{j_n - k_m^2}. \]  
(2.47)

By combining (2.43) and (2.44), we may solve for \( A_m^a \) and \( A_m^a \), to obtain
\[ A_m^a = \frac{2 k_m c_{m1} + \sum_{n=1}^{\infty} C_n c_{mn} (k_m/j_n + \coth(k_m a))}{\coth(k_m a) - \tanh(k_m a)}, \]  
(2.48)

and
\[ A_m^a = \frac{2 k_m c_{m1} + \sum_{n=1}^{\infty} C_n c_{mn} (k_m/j_n + \tanh(k_m a))}{\coth(k_m a) - \tanh(k_m a)}. \]  
(2.49)

Substituting \( A_m^a \) and \( A_m^a \) into (2.45), and (2.46) respectively, we have
\[ \sum_{n=1}^{\infty} \frac{C_n \mu_n}{j_n} \left( \frac{e^{2k_m a}}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) + \sum_{n=1}^{\infty} \frac{B_n \mu_n}{j_n} \left( \frac{1}{j_n - k_m} - \frac{1}{j_n + k_m} \right) \]  
\[ = \frac{Dk \tanh(k) k_m}{\sqrt{2(k_m^2 + (l/M)^2)}} - 2 \mu_1 \cosh(2k_m a) \left( \frac{1}{j_1 - k_m} - \frac{1}{j_1 + k_m} \right), \]  
(2.50)

and
\[ \sum_{n=1}^{\infty} \frac{C_n \mu_n}{j_n} \left( \frac{e^{2k_m a}}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) - \sum_{n=1}^{\infty} \frac{B_n \mu_n}{j_n} \left( \frac{1}{j_n - k_m} + \frac{1}{j_n + k_m} \right) \]  
\[ = \frac{Dk \tanh(k)}{\sqrt{2M(k_m^2 + (l/M)^2)}} - 2 \mu_1 \sinh(2k_m a) \left( \frac{1}{j_1 - k_m} - \frac{1}{j_1 + k_m} \right), \]  
(2.51)

where \( m \geq 0 \). The unknowns \( B_n \) and \( C_n \) may be solved from (2.50) and (2.51) since there are two systems of equations for two series of unknowns.
Unfortunately, it is difficult to find the unknowns $B_n$ and $C_n$ directly from (2.50) and (2.51) since it is hard to write $B_n$ in terms of $C_n$, or vice versa. Instead of solving for $B_n$ and $C_n$ directly, we try to first solve for $C_n + B_n$ and $C_n - B_n$, then $B_n$ and $C_n$ may be found. In order to do this, we add (2.50) to (2.51) to give

$$\frac{\sum_{n=1}^{\infty} C_n \mu_n}{j_n} \left( \frac{1}{j_n - k_m} \right) - \frac{\sum_{n=1}^{\infty} B_n \mu_n}{j_n} \left( \frac{e^{-2k_m a}}{j_n + k_m} \right) = \frac{D k \tanh(k)}{2\sqrt{2}} \frac{e^{-2k_m a}}{k_m - il/M} - \mu_1 \left( \frac{1}{j_1 - k_m} - \frac{1}{j_1 + k_m} \right),$$

and subtract (2.51) from (2.50) to give

$$\frac{\sum_{n=1}^{\infty} C_n \mu_n}{j_n} \left( \frac{-e^{-2k_m a}}{j_n + k_m} + \frac{1}{j_n - k_m} \right) + \frac{\sum_{n=1}^{\infty} B_n \mu_n}{j_n} \left( \frac{1}{j_n - k_m} - \frac{1}{j_n + k_m} \right) = \frac{D k \tanh(k)}{2\sqrt{2}} \frac{1}{k_m + il/M} - \mu_1 e^{-2k_m a} \left( \frac{1}{j_1 - k_m} - \frac{1}{j_1 + k_m} \right).$$

The equation (2.52) and (2.53) may easily be combined to give a system of equations which contain only the unknowns $C_n - B_n$ or $C_n + B_n$. By subtracting (2.53) from (2.52), we have

$$\frac{\sum_{n=1}^{\infty} (C_n - B_n) \mu_n}{j_n} \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) = \frac{D k \tanh(k)}{2\sqrt{2}} \left( \frac{1}{-il/M - k_m} + \frac{e^{-2k_m a}}{-il/M + k_m} \right) - \mu_1 \left( \frac{1}{j_1 - k_m} + \frac{e^{-2k_m a}}{j_1 + k_m} \right) + \mu_1 \left( \frac{1}{j_1 + k_m} + \frac{e^{-2k_m a}}{j_1 - k_m} \right),$$

and by adding (2.52) to (2.53), we have

$$\frac{\sum_{n=1}^{\infty} (C_n + B_n) \mu_n}{j_n} \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) = \frac{D k \tanh(k)}{2\sqrt{2}} \left( \frac{1}{il/M + k_m} - \frac{e^{-2k_m a}}{il/M + k_m} \right) - \mu_1 \left( \frac{1}{j_1 - k_m} - \frac{e^{-2k_m a}}{j_1 + k_m} \right) + \mu_1 \left( \frac{1}{j_1 + k_m} - \frac{e^{-2k_m a}}{j_1 - k_m} \right).$$
The equations (2.54) and (2.55) contain $C_n - B_n$ or $C_n + B_n$ separately. By examining (2.54) and (2.55), we find that the second term in the right hand side of (2.54) and (2.55) may be absorbed into the sum at their left hand side when $n = 1$. We define

$$E_n = \frac{(C_n - B_n)\mu_n}{j_n}, \quad F_n = \frac{(C_n + B_n)\mu_n}{j_n}$$

(2.56)

for $n \geq 2$, and

$$E_1 = \mu_1 \left( \frac{(C_1 - B_1)}{j_1} + 1 \right), \quad F_1 = \mu_1 \left( \frac{(C_1 + B_1)}{j_1} + 1 \right).$$

(2.57)

Therefore (2.54) and (2.55) become

$$\sum_{n=1}^{\infty} E_n \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) = \mu_1 \left( \frac{1}{j_1 + k_m} + \frac{e^{-2k_m a}}{j_1 - k_m} \right)$$

$$+ \frac{Dk \tanh(k)}{2\sqrt{2}} \left( \frac{1}{-iL/M - k_m} + \frac{e^{-2k_m a}}{-iL/M + k_m} \right),$$

(2.58)

and

$$\sum_{n=1}^{\infty} F_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) = \mu_1 \left( \frac{1}{j_1 + k_m} - \frac{e^{-2k_m a}}{j_1 - k_m} \right)$$

$$+ \frac{Dk \tanh(k)}{2\sqrt{2}} \left( \frac{1}{iL/M + k_m} + \frac{e^{-2k_m a}}{-iL/M + k_m} \right).$$

(2.59)

It is easy to obtain $B_n$ and $C_n$ from (2.56) and (2.57) if we can solve for $E_n$ and $F_n$. As a result, the remaining problem is how we can solve for $E_n$ and $F_n$.

Notice that (2.58) and (2.59) still contain unknown constant $D$ which is to be determined by the Kutta condition at the trailing edge of the plate. At this stage we decompose $E_n$ and $F_n$ by writing

$$E_n = E_n' + \frac{Dk \tanh k}{2\sqrt{2}} E_n'', \quad F_n = F_n' + \frac{Dk \tanh k}{2\sqrt{2}} F_n''$$

(2.60)

where $n = 1, 2, \ldots$. Thus we have

$$\sum_{n=1}^{\infty} E_n' \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) + \mu_1 \left( \frac{1}{-j_1 - k_m} + \frac{e^{-2k_m a}}{-j_1 + k_m} \right) = 0,$$

(2.61)
\[ \sum_{n=1}^{\infty} E''_n \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) - \left( \frac{1}{j_0 - k_m} + \frac{e^{-2k_m a}}{j_0 + k_m} \right) = 0, \]  
(2.62)

\[ \sum_{n=1}^{\infty} F'_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) = 0, \]  
(2.63)

and

\[ \sum_{n=1}^{\infty} F''_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) + \mu_1 \left( \frac{1}{-j_1 - k_m} - \frac{e^{-2k_m a}}{-j_1 + k_m} \right) = 0. \]  
(2.64)

where \( j_0 = -il/M \) and \( m = 0, 1, 2, \ldots \).

From (2.41) and (2.42) we have

\[ E_n = \frac{(C_n - B_n)\mu_n}{j_n} \propto O(n^{-1/2}), \quad F_n = \frac{(C_n + B_n)\mu_n}{j_n} \propto O(n^{-1/2}). \]  
(2.65)

If we can solve for \( E'_n, E''_n, F'_n, \) and \( F''_n \), then \( E_n, \) and \( F_n \), we have

\[ B_n = \frac{j_n(F_n - E_n)}{2\mu_n}, \quad C_n = \frac{j_n(E_n + F_n)}{2\mu_n}, \quad n \geq 2, \]  
(2.66)

and

\[ R = 1 + \frac{C_1}{j_1} = \frac{E_1 + F_1}{2\mu_1}, \quad T = \frac{B_1}{-j_1} = \frac{F_1 - E_1}{2\mu_1}, \]  
(2.67)

where \( R \) is the reflection coefficient and \( T \) the transmission coefficient.

The solution of (2.61), (2.62), (2.63) and (2.64) by using residue calculus theory may be found, for example, in Mittra & Lee[77], or Evans, Linton & Ursell[21]. By the formulation above, we may find the impact of mean flow on trapped modes in a channel. The details of the solution procedure will be given in section 3.2.

### 2.3 Scattering by an array of parallel plates in the presence of mean flow

#### 2.3.1 Introduction

The geometry of this problem is shown in Figure 2.3. The coordinate system is chosen such a way that the axis \( x \) is parallel to the plates, \( y \) is perpendicular...
to the plates and the origin is at the centre of one plate. The coordinate system is also transformed in the same way as in the previous section by (2.4), as shown in Figure 2.4. The potential $\phi$ still satisfies the Helmholtz equation (2.8) in the fluid. On the boundaries, we have

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = m, -a < x < a, \ m = 0, \pm 1, \pm 2, \ldots \quad (2.68)$$

A plane wave is incident from up stream at an angle $\theta$ to the x-axis. The potential of the incident wave may be described by

$$\phi_I = e^{i\alpha(x+a)}e^{i\beta y}, \quad (2.69)$$

where $a = L/(d\sqrt{1-M^2})$ and

$$\alpha = l \cos \theta; \ \beta = l \sin \theta; \ 0 \leq \theta \leq \pi/2. \quad (2.70)$$

Since the incident wave is periodic in the y direction and the array of plates extends over the whole y-axis, we seek a scattered wave field which has the term $e^{idy}$ in common with the incident wave. This, together with the periodicity of the geometry implies that the total potential must satisfy

$$\phi(x, y + m) = e^{im\beta}\phi(x, y). \quad (2.71)$$

The problem is set up the same way as in the previous section. We divide the whole domain into three regions, region I: $(-\infty < x < -a, -\infty < y < \infty)$, region II: $(-a < x < a, -\infty < y < \infty)$, and region III: $(a < x < \infty, -\infty < y < \infty)$, as shown in Figure 2.4. The potentials in each region can be found in the same way as in the previous section.

### 2.3.2 Formulation

In region II, the potential is expressed as

$$\phi(x, y) = \sum_{n=0}^{\infty} \frac{1}{k_n} \left( A_n^s \frac{\cosh(k_n x)}{\sinh(k_n a)} + A_n^a \frac{\sinh(k_n x)}{\cosh(k_n a)} \right) \cos(\lambda_n y), \quad (2.72)$$
Figure 2.3: Wave scattering by an array of equally spaced, identical thin plates in the presence of mean flow.

where $-a < x < a$, $0 < y < 1$, and

$$\lambda_n = n\pi, \quad k_n = (\lambda_n^2 - l^2)^{1/2}, \quad n = 0, 1, 2, \ldots$$

We restrict our attention to the case when only one wave mode is possible between the plates, i.e. $0 < l < \pi$, thus $k_0 = -il$ is imaginary, and $k_n$ are real for $n \geq 1$.

In region I, according to Linton & Evans[54], the potential may be written
Figure 2.4: Wave scattering by an array of equally spaced, identical thin plates in the presence of mean flow.

as

\[ \phi(x,y) = (e^{i\alpha(x+a)} + e^{-i\alpha(x+a)}) e^{iy} + \sum_{n=\infty}^{\infty} \frac{C_n}{j_n} e^{i\beta_n y + j_n(x+a)}, \]

(2.74)

where \( x < -a, \ 0 \leq y \leq 1, \) and

\[ \beta_n = \beta + 2n\pi, \ j_n = (\beta_n^2 - l^2)^{1/2}, \text{ and } n = 0, 1, 2, \ldots \]

(2.75)

From (2.70) we know that \( 0 \leq \beta \leq l \) since \( 0 \leq \theta \leq \pi/2. \) We have chosen \( 0 < l < \pi, \) therefore from (2.75), \( j_0 = (\beta^2 - l^2)^{1/2} = -i\alpha \) is pure imaginary, and \( j_n \) are real for all \( n \neq 0. \) It follows that there is only one reflected wave.
As $x \to -\infty$,

$$\phi \sim e^{ia(x+a)}e^{i\beta y} + Re^{i\beta y-ia(x+a)} \quad (2.76)$$

where $R$ is the reflection coefficient. Thus comparing (2.74) and (2.76), we have

$$R = 1 + \frac{C_0}{j_0} \quad (2.77)$$

In region III, the potential can be written as

$$\phi(x, y) = \phi_1(x, y) + \phi_2(x, y), x > a, \ 0 \leq y \leq 1, \quad (2.78)$$

where $\phi_1(x, y)$ is the potential excited by the vortex sheet in the trailing edge of each plate, and $\phi_2(x, y)$ is the perturbed potential that can be written as

$$\phi_2(x, y) = \sum_{n=-\infty}^{\infty} \frac{B_n}{j_n} e^{i\beta_n y-j_n(x-a)}, x > a, 0 \leq y \leq 1. \quad (2.79)$$

There is only one wave term in region III Since $j_0$ is purely imaginary and $j_n, n = \pm 1, \pm 2, \ldots$ are real and positive. Therefore the transmission coefficient can be expressed as

$$T = \frac{B_0}{j_0}. \quad (2.80)$$

To this stage, we need to find out the proper form of $\phi_1(x, y)$. Pressure continuity across the trailing edge requires that

$$\lim_{y \to 1^+} \left( \frac{-il\phi}{M} + \frac{\partial\phi}{\partial x} \right) = \lim_{y \to 1^-} \left( \frac{-il\phi}{M} + \frac{\partial\phi}{\partial x} \right). \quad (2.81)$$

The integration of above equation gives

$$\phi(x, 1^+) - \phi(x, 1^-) = D e^{i\beta(x - a)/M}. \quad (2.82)$$

The vertical derivative of potential across trailing edge should be continuous, i.e.

$$\frac{\partial}{\partial y} \phi(x, 1^+) = \frac{\partial}{\partial y} \phi(x, 1^-). \quad (2.83)$$

The Rayleigh-Bloch condition (2.71) requires

$$\phi(x, 1^+) = e^{i\beta} \phi(x, 0^+), \quad (2.84)$$
and so

\[ \frac{\partial}{\partial y} \phi(x,1^+) = e^{i\beta} \frac{\partial}{\partial y} \phi(x,0^+). \]  

(2.85)

Combining (2.82) - (2.85) yields the periodic conditions

\[ e^{i\beta} \phi(x,0^+) - \phi(x,1^-) = De^{i\beta(x-a)/M} \]  

(2.86)

and

\[ e^{i\beta} \frac{\partial}{\partial y} \phi(x,0^+) = \frac{\partial}{\partial y} \phi(x,1^-). \]  

(2.87)

We write

\[ \phi_1(x,y) = g(y)e^{i\beta(x-a)/M}, \]  

(2.88)

and substitute (2.88) into (2.8) to give

\[ \frac{d^2 g}{dy^2} - k^2 g = 0 \]  

(2.89)

where \( k = \frac{1}{M}(1 - M^2)^{1/2}. \) The boundary conditions (2.86), (2.87) may be written as

\[ e^{i\beta} g(0) - g(1) = D, \]  

(2.90)

and

\[ g'(1) = e^{i\beta} g'(0). \]  

(2.91)

The solution of (2.89) which satisfies (2.90) and (2.91) is given by

\[ g(y) = Ae^{ky} + Be^{-ky} \]  

(2.92)

where \( A = \frac{D}{2(e^{ky} - e^{-ky})}, \) and \( B = \frac{D}{2(e^{ky} - e^{-ky})}. \) Finally we may write the total potential in region III as

\[ \phi(x,y) = (Ae^{ky} + Be^{-ky})e^{i\beta(x-a)/M} + \sum_{n=-\infty}^{\infty} \frac{B_n}{j_n} e^{i\beta_n y - j_n(x-a)}, \quad x > a. \]  

(2.93)

As in the previous section, the continuity of potential and its horizontal derivative across the common boundaries yields systems of equations for the unknown coefficients which, after some manipulation, are given by

\[ 2d_m + \sum_{n=-\infty}^{\infty} C_n j_{mn} = \frac{A_m^* \coth(k_m a) - A_m^* \tanh(k_m a)}{k_m}, \]  

(2.94)

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\[
\sum_{n=-\infty}^{\infty} C_n d_{mn} = -A_m^a + A_m^s, \quad (2.95)
\]
\[
\frac{A_m^s \coth(k_m a) + A_m^a \tanh(k_m a)}{k_m} = \frac{\xi_m}{\lambda_m^2 + k^2} + \sum_{n=-\infty}^{\infty} \frac{B_n}{-j_n} d_{mn}, \quad (2.96)
\]
and
\[
A_m^s + A_m^a = \frac{il}{M(\lambda_m^2 + k^2)} + \sum_{n=-\infty}^{\infty} B_n d_{mn}, \quad (2.97)
\]
where
\[
d_{mn} = \int_0^1 \cos(\lambda_m y) e^{i\lambda y} dy = \frac{i\beta_n (1 - e^{i\beta}(-1)^m)}{j_n^2 - k_m^2},
\]
\[
\int_0^1 \cos(\lambda_m y) (A e^{k y} + B e^{-k y}) dy = \frac{\xi_m}{\lambda_m^2 + k^2},
\]
and
\[
\xi_m = \frac{Dk}{2} \frac{e^{-k} - e^k (1 - (-1)^m e^{i\beta})}{(e^{i\beta} - 1)(e^{i\beta} - e^{-k})}.
\]
Elimination of \(A_m^s, A_m^a\) from (2.94) - (2.97) yields
\[
\sum_{n=-\infty}^{\infty} \left( \frac{C_n - B_n}{j_n} \beta_n \left( \frac{1}{j_n - k_m} + \frac{e^{-2km a}}{j_n + k_m} \right) \right) \left( -1/j_0 - k_m \right) + \beta \left( \frac{1}{j_0 + k_m} + \frac{e^{-2km a}}{j_0 - k_m} \right), \quad (2.98)
\]
and
\[
\sum_{n=-\infty}^{\infty} \left( \frac{C_n + B_n}{j_n} \beta_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2km a}}{j_n + k_m} \right) \right) \left( 1/j_0 - k_m \right) + \beta \left( \frac{1}{j_0 + k_m} - \frac{e^{-2km a}}{j_0 - k_m} \right), \quad (2.99)
\]
where
\[
\delta = \frac{Dk}{2} \frac{i(e^k - e^{-k})}{(e^{i\beta} - 1)(e^{i\beta} - e^{-k})}. \quad (2.100)
\]
The second term on the right hand sides of (2.98) and (2.99) may be absorbed in their left hand sides by writing

\[ U_n = \frac{(C_n - B_n)\beta_n}{j_n}, \quad V_n = \frac{(C_n + B_n)\beta_n}{j_n}, \quad n = \pm 1, \pm 2, \ldots, \tag{2.101} \]

and

\[ U_0 = \beta \left( \frac{(C_0 - B_0)}{j_0} + 1 \right), \quad V_0 = \beta \left( \frac{(C_0 + B_0)}{j_0} + 1 \right). \tag{2.102} \]

Thus

\[ C_n = \frac{j_n}{2\beta_n} (U_n + V_n), \quad B_n = \frac{j_n}{2\beta_n} (V_n - U_n). \tag{2.103} \]

It follows that

\[ \sum_{n=-\infty}^{\infty} U_n \left( \frac{1}{j_n - k_n} + \frac{e^{-2k_n a}}{j_n + k_n} \right) = \delta \left( \frac{1}{-il/M - k_m} + \frac{e^{-2k_m a}}{-il/M + k_m} \right) + \beta \left( \frac{1}{j_0 + k_m} + \frac{e^{-2k_m a}}{j_0 - k_m} \right), \tag{2.104} \]

and

\[ \sum_{n=-\infty}^{\infty} V_n \left( \frac{1}{j_n - k_n} + \frac{e^{-2k_n a}}{j_n + k_n} \right) = \delta \left( \frac{1}{il/M + k_m} + \frac{e^{-2k_m a}}{-il/M + k_m} \right) + \beta \left( \frac{1}{j_0 + k_m} - \frac{e^{-2k_m a}}{j_0 - k_m} \right). \tag{2.105} \]

We further decompose \( U_n, V_n \) as

\[ U_n = U_n' + \delta U_n'', \quad V_n = V_n' + \delta V_n'' , \tag{2.106} \]

and look for solutions for \( U_n', U_n'', V_n', V_n'' \) which satisfy

\[ \sum_{n=-\infty}^{\infty} U_n' \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) + \beta \left( \frac{1}{-j_0 - k_m} + \frac{e^{-2k_m a}}{-j_0 + k_m} \right) = 0, \tag{2.107} \]

\[ \sum_{n=-\infty}^{\infty} U_n'' \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) - \left( \frac{1}{-j_0 - k_m} + \frac{e^{-2k_m a}}{-j_0 + k_m} \right) = 0, \tag{2.108} \]

\[ \sum_{n=-\infty}^{\infty} V_n' \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) + \beta \left( \frac{1}{-j_0 - k_m} - \frac{e^{-2k_m a}}{-j_0 + k_m} \right) = 0, \tag{2.109} \]
and

\[ \sum_{n=-\infty}^{\infty} V_n''\left( \frac{1}{j_n - k_m} - e^{-2k_m a} \right) + \left( \frac{1}{j'_0 - k_m} - e^{-2k_m a} \right) = 0, \quad (2.110) \]

where \( j'_0 = -il/M. \) The above equations may be solved by the residue calculus theory. The constant \( D \) which is in \( \delta \), is also unknown and will be fixed by applying the Kutta condition at the trailing edge. By the formulation above, we may investigate the impact of mean flow on the Rayleigh-Bloch modes, which will be discussed in detail in Chapter 3.3

### 2.4 Conclusion

We have given the formulation for the wave scattering problem for a plate in a two-dimensional wave guide and an array of finite parallel plates in the presence of mean flow by applying a mode matching technique. In each case, sets of infinite systems of equations for the unknown coefficients have been obtained, which can be solved for by residue calculus theory.
Chapter 3

Impact of mean flow on trapped modes

3.1 Introduction

In this chapter, two trapped mode problems in the presence of mean flow will be discussed based on the formulation in the previous chapter. First, the impact of subsonic flow on the trapped modes near a finite thin plate in a two-dimensional guide is investigated by using residue calculus theory. Numerical results will be given and compared with those of Koch[38], who used the Wiener-Hopf technique. Secondly, the influence of subsonic flow on Rayleigh-Bloch modes along an array of finite thin plates is investigated in a similar way.
3.2 Impact of mean flow on the trapped modes near a finite thin plate in a two-dimensional wave guide

A thin plate of length $2L$ is on the centreline of a two-dimensional wave guide of width $2d$ with inviscid, subsonic mean flow, as shown in Figure 2.1. Trapped modes may be found if there is no subsonic flow in the guide, for example, see Evans & Linton, [25], and Evans[23]. For a trapped mode problem in a two-dimensional waveguide with a finite thin plate in its centreline, one looks for a non-trivial solution which satisfies the Helmholtz equation (2.8), boundary conditions (2.10), and (2.11) and the radiation condition

$$\phi \to 0, \text{ as } x \to \pm \infty. \quad (3.1)$$

In the presence of subsonic mean flow, we still look for a non-trivial solution which satisfies (2.8), (2.10), (2.11) and (3.1). In this case, the frequency will become complex, see Koch[38]. We shall adopt residue calculus theory to solve this problem in this section.

3.2.1 Formulation

The difference between this problem and the wave scattering in the presence of subsonic flow formulated in the previous chapter lies in that there are no incident waves in this problem. Therefore the formulation for this problem may be easily derived from that in the wave scattering problem discussed in Chapter 2.2. This can be done by putting the amplitude of the incident wave to be zero in (2.54) and (2.55), and restricting the real part of the frequency
$l$ in the range of $0 < l < \pi/2$, i.e. below the first cut-off. Thus it follows that

$$\sum_{n=1}^{\infty} \frac{(C_n - B_n)\mu_n}{J_n} \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) = \frac{Dk \tanh(k)}{2\sqrt{2}} \left( \frac{1}{-il/M - k_m} + \frac{e^{-2k_m a}}{-il/M + k_m} \right), \quad m \geq 0, \quad (3.2)$$

and

$$\sum_{n=1}^{\infty} \frac{(C_n + B_n)\mu_n}{J_n} \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) = \frac{Dk \tanh(k)}{2\sqrt{2}} \left( \frac{1}{il/M + k_m} + \frac{e^{-2k_m a}}{-il/M + k_m} \right), \quad m \geq 0. \quad (3.3)$$

If we write

$$U_n = \frac{(C_n - B_n)\mu_n}{J_n}, \quad V_n = \frac{(C_n + B_n)\mu_n}{J_n}, \quad n = 1, 2, \ldots, \quad (3.4)$$

then it gives

$$C_n = \frac{j_n}{2\mu_n} (U_n + V_n), \quad B_n = \frac{j_n}{2\mu_n} (V_n - U_n), \quad n = 1, 2, \ldots. \quad (3.5)$$

The right-hand side of (3.2) and (3.3) may be absorbed into the sum in its left-hand side. By doing this, we have

$$U_0 = -\frac{Dk \tanh k}{2\sqrt{2}}, \quad V_0 = \frac{Dk \tanh k}{2\sqrt{2}}, \quad j_0 = -il/M. \quad (3.6)$$

Therefore we have

$$U_0 = -V_0, \quad (3.7)$$

$$\sum_{n=0}^{\infty} U_n \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) = 0, \quad m \geq 0, \quad (3.8)$$

and

$$\sum_{n=0}^{\infty} V_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) = 0, \quad m \geq 0. \quad (3.9)$$

The equations (3.8) and (3.9) may be solved by residue calculus, for example, see Mittra & Lee[77], and Evans, Linton & Ursell[21]. In the following section, we will follow their procedure to find out the approximate and full solutions.
3.2.2 Approximate solution

In the presence of mean flow, we assume that the frequency $l$ become complex, with a small imaginary part. The real part of the frequency is still restricted to below the first cut-off, i.e. $0 < \text{real}(l) < \pi/2$. Consequently, $k_m$ and $j_m$ become complex. When $m \geq 1$ the second term in the brackets of (3.8) and (3.9) decays exponentially. Therefore we may set all terms for which $m \geq 1$ to be zero to obtain approximate solutions. By means of this, we have

$$
\sum_{n=0}^{\infty} U_n \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_0a}}{j_n + k_0} \delta_{m0} \right) = 0, \ m \geq 0, \quad (3.10)
$$

and

$$
\sum_{n=0}^{\infty} V_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_0a}}{j_n + k_0} \delta_{m0} \right) = 0, \ m \geq 0. \quad (3.11)
$$

These two equations may be solved by residue calculus theory. We define the following integrals,

$$
I_{m1} = \frac{1}{2\pi i} \oint_{p_{1N}} f_1(z) \left( \frac{1}{z - k_m} + \frac{e^{-2k_0a}}{z + k_0} \delta_{m0} \right) \, dz, \ m \geq 0, \quad (3.12)
$$

and

$$
I_{m2} = \frac{1}{2\pi i} \oint_{p_{2N}} f_2(z) \left( \frac{1}{z - k_m} - \frac{e^{-2k_0a}}{z + k_0} \delta_{m0} \right) \, dz, \ m \geq 0, \quad (3.13)
$$

where the contours $p_{1N}, p_{2N}$ are circles with infinite radius. The functions $f_1(z), f_2(z)$ satisfy the following conditions:

1. $f_1(z), f_2(z)$ have simple poles at $z = j_n, n = 0, 1, 2, ...$

2. $f_1(z), f_2(z)$ satisfy the equations

$$
f_1(k_m) + e^{-2k_0a} f_1(-k_0) \delta_{m0} = 0, m = 0, 1, 2, ..., \quad (3.14)
$$

and

$$
f_2(k_m) - e^{-2k_0a} f_2(-k_0) \delta_{m0} = 0, m = 0, 1, 2, ... \quad (3.15)
$$
Therefore we have

\[ f_1(k_m) = 0, \quad f_2(k_m) = 0, \quad m \geq 1, \quad (3.16) \]
\[ f_1(k_0) + e^{-2k_0a}f_1(-k_0) = 0, \quad (3.17) \]

and

\[ f_2(k_0) - e^{-2k_0a}f_2(-k_0) = 0. \quad (3.18) \]

Thus \( f_1(z), f_2(z) \) have simple zeros at \( z = k_m \) from (3.16).

3. \( f_1(z), f_2(z) = O(z^{-1/2}) \) as \( |z| \to \infty \) on the contours \( p_{1N}, p_{2N} \) as \( N \to \infty \). Here \( p_{1N} \) and \( p_{2N} \) are a sequence of circles, the radii of which increase without bound as \( N \to \infty \), while avoiding the zeros of the integrand.

The last of these conditions ensures that these integrals defined by (3.12) and (3.13) vanish. By applying the conditions above, the evaluation of the integrals \( I_{m1}, I_{m2} \) in terms of their respective residue series gives

\[ \sum_{n=0}^{\infty} \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_0a}\delta_{m0}}{j_n + k_0} \right) \text{Res}(f_1 : j_n) = 0, \quad m \geq 0 \quad (3.19) \]

and

\[ \sum_{n=0}^{\infty} \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_0a}\delta_{m0}}{j_n + k_0} \right) \text{Res}(f_2 : j_n) = 0, \quad m \geq 0 \quad (3.20) \]

where \( \text{Res}(f_1 : j_n) \) and \( \text{Res}(f_2 : j_n) \) denote respectively, the residue of \( f_1(z) \) or \( f_2(z) \) at \( z = j_n \). Comparing (3.10) and (3.11) with (3.19) and (3.20), we have

\[ U_n = \text{Res}(f_1 : j_n), \quad V_n = \text{Res}(f_2 : j_n), \quad n = 0, 1, 2, \ldots \quad (3.21) \]

The remaining task is to construct the functions \( f_1(z) \) and \( f_2(z) \). Following Evans[23], we may write \( f_1(z), f_2(z) \) as

\[ f_1(z) = \frac{\zeta_1(z - k'_0)}{z - j_0} \prod_{m=1}^{\infty} \frac{1 - z/j_m}{1 - z/j'_m}, \quad (3.22) \]
and

\[ f_2(z) = \frac{\zeta_2(z - k''_0)}{z - j_0} \prod_{m=1}^{\infty} \frac{1 - z/k_m}{1 - z/j_m} \]  

(3.23)

where \( k'_0, k''_0 \) are determined by condition 2 for which \( m = 0 \), and \( \zeta_1, \zeta_2 \) are unknown coefficients. Evans [23] has shown that

\[ \prod_{m=1}^{\infty} \frac{1 - z/k_m}{1 - z/j_m} = O(z^{-1/2}), \]  

(3.24)

as \( z \to \infty \) through any sequence of values which avoids the points \( z = n\pi/2, n = 1, 2, \ldots \), for real \( l \). Similarly, it can be shown that (3.24) still holds for complex \( l \). Subsequently, \( f_1(z), f_2(z) = O(z^{-1/2}) \) when \( z \to \infty \) as required.

By applying condition 2 we may solve for \( k'_0, k''_0 \). Substitute \( f_1(k_0) \) into (3.17) and \( f_2(k_0) \) into (3.18) for \( m = 0 \) to obtain

\[ f_1(k_0) + e^{-2k_0a} f_1(-k_0) = 0, \]  

(3.25)

and

\[ f_2(k_0) - e^{-2k_0a} f_1(-k_0) = 0. \]  

(3.26)

The solution of these two equations gives

\[ k'_0 = \frac{k_0(1 - A)}{1 + A}, \quad k''_0 = \frac{k_0(1 + A)}{1 - A} \]  

(3.27)

where

\[ A = e^{-2k_0a} \prod_{m=1}^{\infty} \frac{1 + k_0/k_m}{1 + k_0/k_m} \frac{(1 - k_0/j_m)(1 - k_0/j_m)}{(1 - k_0/k_m)(1 + k_0/j_m)}. \]

It is easy to write the solution of \( U_n, V_n \) as

\[ U_n = \frac{\zeta_1 j_n(j_n - k'_0)(k_n - j_n)}{k_n(j_n - j_0)} \prod_{m=1, m \neq n}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m}, \]  

(3.28)

and

\[ V_n = \frac{\zeta_2 j_n(j_n - k''_0)(k_n - j_n)}{k_n(j_n - j_0)} \prod_{m=1, m \neq n}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m}, \]  

(3.29)
for \( n = 1, 2, 3, \ldots \), and

\[
U_0 = \zeta_1(j_0 - k'_0) \prod_{m=1}^{\infty} \frac{1 - j_0/k_m}{1 - j_0/j_m}, \quad V_0 = \zeta_2(j_0 - k''_0) \prod_{m=1}^{\infty} \frac{1 - j_0/k_m}{1 - j_0/j_m}.
\]  

(3.30)

From (3.7), we have

\[
\zeta_1(j_0 - k'_0) + \zeta_2(j_0 - k''_0) = 0.
\]

(3.31)

We know from condition 3 that \( U_n, V_n = O(n^{-1/2}) \). By means of a Taylor series expansion, we may write \( U_n, V_n \) as

\[
U_n = \zeta_1/n^{1/2} + O(n^{-3/2}), \quad V_n = \zeta_2/n^{1/2} + O(n^{-3/2}).
\]

(3.32)

The size of error terms in (3.32) follows from the general theory of Wigley[112] and the boundary conditions on the plate. Substituting \( U_n, V_n \) into (3.4) gives

\[
B_n = \frac{j_n}{2\mu_n} (\zeta_2 - \zeta_1)n^{-1/2} + O(n^{-3/2}).
\]

(3.33)

Here \( B_n = O(n^{-1/2}) \) if the Kutta condition at the trailing edge is not applied. However by applying Kutta condition, we need \( B_n = O(n^{-3/2}) \). Therefore by choosing \( \zeta_1 = \zeta_2 \) we have \( B_n = O(n^{-3/2}) \) as required. From (3.31) we have

\[
(j_0 - k'_0) + (j_0 - k''_0) = 0.
\]

(3.34)

This is the final equation which the frequency should satisfy. Generally, the solutions to this equation are complex. Substituting \( k'_0 \) and \( k''_0 \) from (3.27) into (3.34) gives, after some manipulation,

\[
e^{-2k_0a} \prod_{m=1}^{\infty} \frac{(1 + k_0/k_m)(1 - k_0/j_m)}{(1 - k_0/k_m)(1 + k_0/j_m)} = \pm \sqrt{\frac{1 + M}{1 - M}}.
\]

(3.35)

In this equation, the Mach number \( M \) only appears in the right-hand side. If we let Mach number \( M = 0 \), i.e. there is no mean flow in the waveguide, (3.35) becomes

\[
e^{-2k_0a} \prod_{m=1}^{\infty} \frac{(1 + k_0/k_m)(1 - k_0/j_m)}{(1 - k_0/k_m)(1 + k_0/j_m)} = \pm 1.
\]

(3.36)
This is exactly the same equation obtained for the existence of trapped modes without mean flow, where '+' corresponds to antisymmetric modes and '-' to symmetric modes. In the case of no mean flow, the modulus of \( \prod_{m=1}^{\infty} \frac{1+k_0/k_m(1-k_0/j_m)}{(1-k_0/k_m)(1+k_0/j_m)} \) is one since its denominator is the complex conjugate of the numerator if \( l \) is real. In this situation, equation (3.36) reduces to two equations that anti-symmetric or symmetric modes satisfy, as in Evans[23].

In the presence of mean flow, by assuming that the solution of frequency \( l \) has a small imaginary part, i.e. \( l = l_1 + il_2 \), we have

\[
k_0 = l_2 - il_1, \quad j_0 = \frac{l_2 - il_1}{M}.
\]  

(3.37)

We may write

\[
\prod_{m=1}^{\infty} \frac{1+k_0/k_m(1-k_0/j_m)}{(1-k_0/k_m)(1+k_0/j_m)} = r e^{2i\delta}
\]

(3.38)

where \( r \) is the modulus and \( 2\delta \) is the argument. Substituting (3.37) and (3.38) into (3.35) yields

\[
r e^{-2l_2a} e^{i(2l_1a + 2\delta)} = \pm \sqrt{\frac{1-M}{1+M}}
\]

(3.39)

Where '+' corresponds to complex resonances due to the influence of mean flow on antisymmetric modes and '-' complex resonances due to the impact of mean flow on symmetric resonances.

For complex resonances due to the influence of mean flow on symmetric trapped modes, we have

\[
re^{-2l_2a} = \sqrt{\frac{1+M}{1-M}}, \quad l_1a + \delta = (n + 1/2)\pi.
\]

(3.40)

for some integer \( n \).

For complex resonances due to the influence of mean flow on antisymmetric trapped modes, we have

\[
re^{-2l_2a} = \sqrt{\frac{1+M}{1-M}}, \quad l_1a + \delta = n\pi.
\]

(3.41)

for some integer \( n \).
3.2.3 Full solution

For the full solution of (3.8) and (3.9), following Evans[23] and Mittra & Lee[77], we may define the following integrals

\[ I_{m1} = \frac{1}{2\pi i} \oint_{p1N} f_1(z) \left( \frac{1}{z - k_m} + \frac{e^{-2km\alpha}}{z + k_0} \right) \, dz, \]

and

\[ I_{m2} = \frac{1}{2\pi i} \oint_{p2N} f_2(z) \left( \frac{1}{z - k_m} - \frac{e^{-2km\alpha}}{z + k_0} \right) \, dz \]

where the contours \( p_{1N}, p_{2N} \) are circles with infinite radius and \( f_1(z), f_2(z) \) satisfy the following conditions

1. \( f_1(z), f_2(z) \) have simple poles at \( z = j_n, n = 0, 1, 2, ... \).

2. \( f_1(z), f_2(z) \) satisfies the equations

\[ f_1(k_m) + e^{-2km\alpha} f_1(-k_m) = 0, m = 0, 1, 2, ..., \quad (3.42) \]

and

\[ f_2(k_m) - e^{-2km\alpha} f_2(-k_m) = 0, m = 0, 1, 2, ... \quad (3.43) \]

3. \( f_1(z), f_2(z) = O(z^{-1/2}) \) as \( |z| \to \infty \) on the contours \( p_{1N}, p_{2N} \) as \( N \to \infty \). Here \( p_{1N} \) and \( p_{2N} \) are a sequence of circles, the radii of which increase without bound as \( N \to \infty \), while avoiding the zeros of the integrand.

The last of these conditions ensures that these integrals vanish. By applying these conditions, the evaluation of integrals in terms of their respective residue series gives

\[ \sum_{n=0}^{\infty} \left( \frac{1}{j_n - k_m} + \frac{e^{-2km\alpha}}{j_n + k_0} \right) \text{Res}(f_1 : j_n) = 0, \quad (3.44) \]

and

\[ \sum_{n=0}^{\infty} \left( \frac{1}{j_n - k_m} - \frac{e^{-2km\alpha}}{j_n + k_0} \right) \text{Res}(f_2 : j_n) = 0. \quad (3.45) \]
Comparing (3.8) and (3.9) with (3.44) and (3.45), we have

\[ U_n = \text{Res}(f_1 : j_n), \quad V_n = \text{Res}(f_2 : j_n), \quad n = 0, 1, 2, \ldots \]  

(3.46)

The remaining problem is how we construct functions \( f_1(z) \), \( f_2(z) \). From the approximate solution we know that the frequency \( l \) is complex if there exists subsonic flow in the waveguide. From the definition of \( k_m = \sqrt{m^2 \pi^2 - l^2} \) we know that \( k_m \) is complex with dominant real part. Its modulus is given approximately by \( |k_m| \propto m \pi \). Therefore the term containing \( e^{-2kma} \) in (3.42) and (3.43) in condition 2 decays to zero exponentially. Consequently, the contribution of this term may be ignored if \( m \) is large enough for a given value of \( a \) and accuracy requirement. By assuming that the terms of \( m > R \) are ignored for a given \( a \) and an accuracy requirement in (3.42) and (3.43), we have

\[ f_1(k_m) = 0, \quad \text{and} \quad f_2(k_m) = 0, \quad m > R. \]  

(3.47)

This means that \( f_1(z) \), \( f_2(z) \) have zeros at \( z = k_m \) for \( m = R + 1, R + 2, \ldots \). As a result, we may write \( f_1(z) \), \( f_2(z) \) as

\[ f_1(z) = \zeta_1 \prod_{m=0}^{R} \frac{z - k'_m}{z - j_m} \prod_{m=R+1}^{\infty} \frac{1 - z/k_m}{1 - z/j_m}, \]  

(3.48)

and

\[ f_2(z) = \zeta_2 \prod_{m=0}^{R} \frac{z - k''_m}{z - j_m} \prod_{m=R+1}^{\infty} \frac{1 - z/k_m}{1 - z/j_m}, \]  

(3.49)

where \( k'_m, k''_m \) are determined by condition 2 whilst \( \zeta_1, \zeta_2 \) are unknown coefficients to be determined by the Kutta condition at the trailing edge. The form of \( f_1(z) \) and \( f_2(z) \) for the full solution here are different from those of Evans[23], but similar to those by Mittra & Lee[77]. By the evaluation of the integrals in terms of their respective residue series, we have the solution of \( U_n, V_n \) as

\[ U_n = \zeta_1 (j_n - k'_n) \prod_{m=0, m \neq n}^{R} \frac{j_n - j_m}{j_n - j_m} \prod_{m=R+1}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m}, \]  

\[ V_n = \zeta_2 (j_n - k''_n) \prod_{m=0, m \neq n}^{R} \frac{j_n - j_m}{j_n - j_m} \prod_{m=R+1}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m}. \]  

(3.50)
for \( n = 0, 1, 2, ..., R, \) and

\[
U_n = \frac{\zeta_1 j_n(k_n - j_n)}{k_n} \prod_{m=0}^{R} \frac{j_n - k_m'}{j_n - j_m} \prod_{m=R+1, m \neq n}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m},
\]

\[
V_n = \frac{\zeta_2 j_n(k_n - j_n)}{k_n} \prod_{m=0}^{R} \frac{j_n - k_m''}{j_n - j_m} \prod_{m=R+1, m \neq n}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m}
\]

for \( n = R+1, R+2, ..., \infty. \) Substitution of \( U_0, V_0 \) from (3.50) into (3.7) gives

\[
\zeta_1(j_0 - k_0') \prod_{m=1}^{R} \frac{j_0 - k_m'}{j_0 - j_m} + \zeta_2(j_0 - k_0'') \prod_{m=1}^{R} \frac{j_0 - k_m''}{j_0 - j_m} = 0,
\]

(3.51)

Substitution of \( U_n, V_n \) for \( n \geq 1 \) into (3.5) generates

\[
B_n = \frac{j_n}{2\mu_n} (\zeta_2(j_n - k_n')T_n'' - \zeta_1(j_n - k_n')T_n'),
\]

(3.52)

and

\[
C_n = \frac{j_n}{2\mu_n} (\zeta_2(j_n - k_n'')T_n'' + \zeta_1(j_n - k_n')T_n')
\]

(3.53)

where

\[
T_n' = \prod_{m=0, m \neq n}^{R} \frac{j_n - k_m'}{j_n - j_m} \prod_{m=R+1}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m},
\]

\[
T_n'' = \prod_{m=0, m \neq n}^{R} \frac{j_n - k_m''}{j_n - j_m} \prod_{m=R+1}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m}.
\]

Again substitute \( U_n, V_n \) for \( n \geq R+1 \) into (3.5) to give

\[
B_n = \frac{j_n}{2\mu_n} (\zeta_2 T_n'' - \zeta_1 T_n'),
\]

(3.54)

and

\[
C_n = \frac{j_n}{2\mu_n} (\zeta_2 T_n'' + \zeta_1 T_n')
\]

(3.55)

where

\[
T_n' = \frac{j_n(k_n - j_n)}{k_n} \prod_{m=0}^{R} \frac{j_n - k_m'}{j_n - j_m} \prod_{m=R+1, m \neq n}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m},
\]

\[
T_n'' = \frac{j_n(k_n - j_n)}{k_n} \prod_{m=0}^{R} \frac{j_n - k_m''}{j_n - j_m} \prod_{m=R+1, m \neq n}^{\infty} \frac{1 - j_n/k_m}{1 - j_n/j_m}.
\]
Similarly, we may write
\[ B_n = \frac{j_n}{2\mu_n}(\zeta_2 - \zeta_1)n^{-1/2} + O(n^{-3/2}) \]  \hspace{1cm} (3.56)
where \( B_n = O(n^{-1/2}) \). However by applying the Kutta condition, we need \( B_n = O(n^{-3/2}) \), thus by choosing \( \zeta_1 = \zeta_2 \) we have \( B_n = O(n^{-3/2}) \) as required. From (3.51) we have
\[ \prod_{m=0}^{R} (j_0 - k_m') + \prod_{m=0}^{R} (j_0 - k_m'') = 0. \]  \hspace{1cm} (3.57)
It is obvious that (3.57) is the same as (3.34) for the approximate solution if we choose \( R = 0 \). The remaining task of the full solution is how we solve for the shifted zeros \( k_m' \) from (3.42), and \( k_m'' \) from (3.43), which are complex. In the case of looking for real solutions, Mittra & Lee[77] provided several techniques. In the trapped mode problem, Evans[23] treated the imaginary \( k_0 \) and real \( k_m, m \geq 1 \) separately by constructing a slightly different integrand. Both of their techniques are not suitable for the solution of \( k_m' \) and \( k_m'' \) since they are complex. In the following section, we will solve for the problems by using a different technique.

### 3.2.4 Technique of full solution

The condition 2 in the formulation of the full solution states that zeros of the constructed functions are shifted from \( k_m \) to \( k_m' \) by equation (3.42) or from \( k_m \) to \( k_m'' \) by (3.43). Generally, we rewrite (3.42) in the condition 2 as
\[ f(k_m) + \rho_m f(-k_m) = 0, \]  \hspace{1cm} (3.58)
where \( \rho_m \) is a factor that decays as \( m \) increases, \( m = 0, 1, 2, \ldots \), and
\[ f(z) = \zeta_1 \prod_{r=0}^{R} \frac{z - k_r'}{z - j_r} \prod_{r=R+1}^{\infty} \frac{1 - z/k_r}{1 - z/j_r}. \]  \hspace{1cm} (3.59)
Thus it remains to determine how we solve for the unknowns $k_r$, where $r = 0, 1, 2, \ldots, R$. From the form of function $f(z)$ we know that (3.58) is a system of non-linear equations for $k_r, r = 0, 1, 2, \ldots, R$.

In solving for the scattering problem in electromagnetic wave guides, Mittra and Lee[77] provide three different ways of finding the solutions of the form of equation (3.58) by giving a good initial guess according to their corresponding physical conditions. In their cases all the unknowns are real, and initial guesses are easily obtained according to the relevant physical conditions. However in this case, the unknowns are complex and it is hard to find a good initial guess. Therefore Mittra and Lee’s[77] techniques are not appropriate though the function for approximate solution and full solution is of the same form.

In solving a trapped mode problem in acoustics, Evans, Linton and Ursell[21] consider $k_0$ and $k_m$, $m = 1, 2, 3, \ldots$ separately since $k_0$ is pure imaginary and $k_m$, $m = 1, 2, \ldots$, are all real. Thus they construct the function $f(z) = g(z)h(z)$, where $g(z), h(z)$ are written as

$$g(z) = \prod_{m=1}^{\infty} \frac{1 - z/k_m}{1 - z/j_m}, \text{ and } h(z) = 1 + \sum_{m=1}^{\infty} \frac{A_m}{z - k_m}. \quad (3.60)$$

Thus $A_m$ can be obtained by solving a system of linear equations. Again the constraint for this approach is that $k_m$ are all real for $m \geq 1$. This approach is particularly useful for real $l$ since the system of linear equations to be solved is real, unique and highly converged. In the case of complex $l$, the system of equations to be solved is still unique and highly converged. However, the constructed functions for approximate and full solutions are slightly different.

We will combine the advantages of these two techniques together to form a slightly different approach. Substitution of (3.59) into (3.58) gives

$$\prod_{r=0}^{R} (k_m - k_r') + A_m \prod_{r=0}^{R} (k_m + k_r') = 0, \quad m = 0, 1, 2, \ldots, R, \quad (3.61)$$
where

\[ A_m = \rho_m \prod_{r=0}^{R} \frac{k_m - j_r}{k_m + j_r} \prod_{r=R+1}^{\infty} \frac{(1 + k_m/k_r)(1 - k_m/j_r)}{(1 - k_m/k_r)(1 + k_m/j_r)}. \]  

(3.62)

It is still difficult to solve (3.61) directly since it is a system of nonlinear

equations. However (3.61) may be converted into a system of linear equations

by the following substitutions. We may write

\[ \prod_{r=0}^{R} (k_m - k'_r) = \sum_{r=0}^{R} (-1)^{r+1} k_m^{R-r} y_r + k_m^{R+1}, \quad m = 0, 1, 2, ..., R, \]  

(3.63)

and

\[ \prod_{r=0}^{R} (k_m + k'_r) = \sum_{r=0}^{R} k_m^{R-r} y_r + k_m^{R+1}, \quad m = 0, 1, 2, ..., R \]  

(3.64)

where

\[ y_0 = \sum_{r=0}^{R} k'_r, \quad y_1 = k'_0 k'_1 + k'_1 k'_2 + k'_2 k'_3, ..., \quad y_R = \prod_{r=0}^{R} k'_r. \]  

(3.65)

Thus equation (3.61) may be written as

\[ \sum_{r=0}^{R} k_m^{R-r} ((-1)^{r+1} + A_m) y_r = -(1 + A_m) k_m^{R+1}, \quad m = 0, 1, 2, ..., R. \]  

(3.66)

This is a system of inhomogeneous linear equations for \( y_r, \ r = 0, 1, 2, ..., R, \)

and its solution is straightforward. Instead of solving for \( k'_r \) directly, we

solve for their combinations \( y_r \). After solving for \( y_r \) from (3.66), the results

are substituted back into (3.59) where \( \prod_{r=0}^{R} (z - k'_r) \) is expressed in terms of

\( y_r, \ r = 0, 1, ... R, \) to get the final solution. Obviously, the approximate and

full solutions are sought by using the same equation, and there is no need for

initial guess which is essential in Mittra and Lee's techniques[77].

### 3.2.5 Numerical results

First we examine the convergence of the full solution approach discussed

above. Obviously, the key factor that affects the convergence of the full

solution and the effect of approximate solution is the geometric parameter \( a \)
Table 3.1: Convergence of full solutions for different $a$

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<th>$N=2$</th>
<th>$N=4$</th>
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which is $L/d$ if $M = 0$ in the relevant formulation. Therefore we discuss the convergence of the full solution when there is no mean flow in the waveguide, i.e. $M = 0$. In the equation (3.57), if we choose $R = 0$, it corresponds to the approximate solution. In this case we keep only wave-like term in (3.8) and (3.9), namely the truncation term $N = 1$. The trapped mode is computed for $N = 1, 2, 4$ for values of $a$ ranging between 0 and 1, and the results are shown in Table 3.1, in which $a = L/(d\sqrt{1 - M^2})$, $M = 0$.

Clearly, the value of $a$ affects the convergent speed of the solution. We may not find the approximate solution if $a = 0.2$. However the relative error is less than 1% if $a \geq 0.3$, and four digit accuracy may be achieved if $a \geq 0.9$. The convergence is very quick even if $a$ is small. For instance, three digit accuracy can be achieved if $N$ is increased from 2 to 4 when $a$ is as small as 0.1. In a real calculation, we may take the solutions for $N = 2$ as full solutions for $a < 1.0$, and there is essentially no difference between the
Koch[38] computed the resonant acoustic frequencies of flat plate cascades by using a Wiener-Hopf solution. Here we compare our numerical results with those of Koch[38]. In the case of flat plate cascades, the geometry of which resonances are sought may be expressed as a channel of width $d$, and a finite flat plate of length $L$, then the boundary conditions for which resonances exist may be expressed as

$$\frac{\partial \phi}{\partial Y} = 0, \quad Y = \pm d/2, -\infty < X < \infty; \quad Y = 0, -L/2 < X < L/2,$$  

(3.67)
Figure 3.2: Imaginary part of the resonant frequency $f*$ for modes symmetric about plane $x = 0$ as a function of $L/d$ and $M$ in a flat plate cascade, where dots represent the results from Koch[38]

where $d$ is the spacing between the two adjacent plates and $L$ is the length of the plates. After being nondimensionalized as in (2.4), the boundary conditions become

$$\frac{\partial \phi}{\partial y} = 0, \quad y = \pm 1/2, -\infty < x < \infty; \quad y = 0, -a/2 < x < a/2,$$

(3.68)

In what follows the eigenfunction in both the inner and outer regions are

$$h_n^I(y) = 2^{1/2} \sin((2n - 1)\pi y), \quad n = 1, 2, \ldots,$$

(3.69)

and

$$h_n^{II}(y) = e_n^{1/2} \cos(2n\pi y), \quad n = 0, 1, 2, \ldots,$$

(3.70)
Figure 3.3: Real part of the resonant frequency as a function of $L/d$ and $M$ in a channel with a finite thin plate.

Therefore the first cut-off below which trapped modes may be found is $\pi$. This is different from the channel problem where the first cut-off is $\pi/2$. Figure 3.1 and 3.2 show the impact of mean flow on symmetric modes, which is called $\beta$-mode$(0,0)$ by Parker[83]. Figure 3.1 shows the real parts of acoustic resonant frequencies $f^*$, as a function of geometric parameter $L/d$ and Mach number $M$. In order to compare with the results of Koch[39], the non-dimensional frequency $f^* = l \cdot \sqrt{1 - M^2}/2\pi c$ is plotted. In this figure, the real parts of $M = 0.1, 0.2$ are not plotted because they are almost overlapped with that of $M = 0$. Figure 3.2 shows the imaginary parts of acoustic resonant frequencies $f^*$, as a function of geometric parameter $L/d$. 
Figure 3.4: Imaginary part of the resonant frequency as a function of $L/d$ and $M$ in a channel with a finite flat plate and Mach number $M$.

It is obvious that our numerical results by a mode matching approach and the residue calculus theory agree with those obtained by the Wenier-Hopf technique. From the formulation in Chapter 2 and the solution by residue calculus theory in this section, we know that the mode matching approach combined with residue calculus theory is quite straightforward. In particular, the approximate solution gives a good approximation even if $L/d$ is quite small, i.e. $0.3 < L/d < 0.9$, and the approximate and full solutions are indistinguishable when $L/d > 0.9$.

From 3.1 and 3.2 we know that the trapped modes become complex res-
onances in the presence of mean flow. The imaginary part of the resonant frequencies is quite small compared with their real part, and the real part of the resonances decreases as Mach number increases. On the other hand, the magnitude of imaginary part increases as Mach number increases.

In a similar way, we may compute antisymmetric modes, or $\alpha$-mode as Parker called them paper[83], and the relevant acoustic resonance in the presence of mean flow.

For the channel problem, the frequency range is $0 < l < \pi/2$. Figure (3.3) and (3.4) show the numerical results for the trapped modes and acoustic resonances in a channel. In these figures, the dimensionless frequencies $\iota^* = l\sqrt{1 - M^2}$ are plotted against $L/d$. From these figures we may draw similar conclusions to those in the cascade problem.

3.3 The impact of mean flow on Rayleigh-Bloch modes

3.3.1 Introduction

In section 2.2 we have presented the formulation for the wave scattering by an array of identical thin plates. Assuming that there are no incident waves, the Rayleigh-Bloch mode problem corresponds to looking for the non-trivial solutions which satisfy

$$\phi(x, y) \rightarrow 0, \text{ as } x \rightarrow \pm\infty,$$

with the same boundary conditions (2.68) and the periodic condition (2.71). In the case of no mean flow, the solutions are called Rayleigh-Bloch modes because they satisfy the Rayleigh-Bloch condition, i.e. the periodicity condition (2.71). However, in the presence of subsonic uniform flow, there are no Rayleigh-Bloch modes. But acoustic resonances may also be found which
are similar to those in a two-dimensional waveguide. For the Rayleigh-Bloch problem, condition (3.71) requires that $\beta_0 = \sqrt{\beta^2 - T^2}$ should be real and positive since there are no waves propagating away from the array of plates. Thus it follows that

$$0 < l < \beta.$$  \hfill (3.72)

Notice that since the sum in (2.74) is over all $n$, it is only necessary to seek $\beta$ in the range $0 < \beta < 2\pi$. Furthermore, since we require $\beta_n > 0$ for all $n$, or $|\beta_n| > l$, for all $n$, it follows that we must have

$$l < \beta < 2\pi - l,$$  \hfill (3.73)

see, for example, Linton & Evans[54].

### 3.3.2 Formulation

In the Rayleigh-Bloch problem, let the amplitude of the incident waves be zero, i.e. there is no forcing term in equations (2.98) and (2.99). We end up with

$$\sum_{n=-\infty}^{\infty} \frac{(C_n - B_n)\beta_n}{J_n} \left( \frac{1}{J_n - k_m} + \frac{e^{-2k_m a}}{J_n + k_m} \right)$$

$$= \delta \left( \frac{1}{-il/M - k_m} + \frac{e^{-2k_m a}}{-il/M + k_m} \right), \quad m \geq 0,$$  \hfill (3.74)

and

$$\sum_{n=-\infty}^{\infty} \frac{(C_n + B_n)\beta_n}{J_n} \left( \frac{1}{J_n - k_m} + \frac{e^{-2k_m a}}{J_n + k_m} \right)$$

$$= \delta \left( \frac{1}{il/M - k_m} + \frac{e^{-2k_m a}}{il/M + k_m} \right), \quad m \geq 0,$$  \hfill (3.75)

We may write

$$U_n = \frac{(C_n - B_n)\beta_n}{J_n}, \quad V_n = \frac{(C_n + B_n)\beta_n}{J_n}$$  \hfill (3.76)
where \( n = 0, \pm 1, \pm 2, \ldots \). Thus it follows that

\[
C_n = \frac{j_n}{2\beta_n} (U_n + V_n), \quad \text{and} \quad B_n = \frac{j_n}{2\beta_n} (V_n - U_n).
\]

(3.77)

The right-hand sides of (3.74) and (3.75) may be absorbed into the sum of their left-hand sides by writing

\[
U_0' = -\delta, \quad V_0' = \delta, \quad \text{and} \quad j_0' = -\frac{i l}{M}.
\]

(3.78)

From this we have

\[
U_0' = -V_0',
\]

(3.79)

\[
\sum_{n=\infty}^{-\infty} U_n \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) = 0, \quad m \geq 0,
\]

(3.80)

and

\[
\sum_{n=\infty}^{-\infty} V_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) = 0, \quad m \geq 0,
\]

(3.81)

where \( \sum_{n=\infty}^{-\infty} \) means the sums also include \( U_0' \) in (3.80) or \( V_0' \) in (3.81). The solution of (3.80) and (3.81) by using residue calculus theory will be divided into two steps as in the previous sections, the approximate and full solutions. First we look for the approximate solutions and then the full solutions. The solution procedure is similar to that in the previous section, therefore only main steps will be described below.

### 3.3.3 Approximate solution

When \( m \geq 1 \), the second term in the brackets in equation (3.80) and (3.81) decays exponentially to zero. We set terms for which \( m \geq 1 \) all to be zero, and obtain

\[
\sum_{n=\infty}^{-\infty} U_n \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \delta_{m0} \right) = 0, \quad m \geq 0,
\]

(3.82)

and

\[
\sum_{n=\infty}^{-\infty} V_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \delta_{m0} \right) = 0, \quad m \geq 0.
\]

(3.83)
These two equations can be solved by residue calculus approach. We define
the following integrals.

\[ I_{m1} = \frac{1}{2\pi i} \oint_{p_{1N}} f_1(z) \left( \frac{1}{z - k_m} + \frac{e^{-2k_0\alpha}}{z + k_0} \delta_{m0} \right) dz, \]
\[ I_{m2} = \frac{1}{2\pi i} \oint_{p_{2N}} f_2(z) \left( \frac{1}{z - k_m} - \frac{e^{-2k_0\alpha}}{z + k_0} \delta_{m0} \right) dz \]

where the contour \( p_{1N} \) and \( p_{2N} \) are circles with infinite radii and \( f_1(z), f_2(z) \)
satisfy the following conditions

1. \( f_1(z), f_2(z) \) have simple poles at \( z = j_n, n = 0, \pm 1, \pm 2, \ldots \), and \( z = j_0 \).
2. \( f_1(z), f_2(z) \) satisfies the equation
   \[ f_1(k_m) + e^{-2k_0\alpha} f_1(-k_0) \delta_{m0} = 0, m = 0, 1, 2, \ldots \] (3.84)
   and
   \[ f_2(k_m) - e^{-2k_0\alpha} f_2(-k_0) \delta_{m0} = 0, m = 0, 1, 2, \ldots \] (3.85)
   therefore
   \[ f_1(k_m) = 0, f_2(k_m) = 0, m \geq 1. \] (3.86)

   This indicates that \( z = k_m, m \geq 1 \) are zeros of \( f_1(z), f_2(z) \), and
   \[ f_1(k_0) + e^{-2k_0\alpha} f_1(-k_0) = 0, \] (3.87)
   \[ f_2(k_0) - e^{-2k_0\alpha} f_2(-k_0) = 0. \] (3.88)

3. \( f_1(z), f_2(z) = O(z^{-1/2}) \) as \( z \to \infty \) on the contours \( p_{1N}, p_{2N} \) as \( N \to \infty. \)

The last of these conditions ensures that these integrals vanish. Therefore
by applying the above conditions the evaluation of integrals in terms of their
respective residue series gives

\[ \sum_{n=-\infty}^{\infty} \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_0\alpha} \delta_{m0}}{j_n + k_0} \right) \text{Res}(f_1 : j_n) = 0, \] (3.89)
and
\[ \sum_{n=-\infty}^{\infty} \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_0\delta_m\alpha_0}}{j_n + k_0} \right) \text{Res}(f_2 : j_n) = 0. \]  
(3.90)

Comparing (3.82) and (3.83) with (3.89) and (3.90), we have

\[ U_n = \text{Res}(f_1 : j_n), \quad V_n = \text{Res}(f_2 : j_n), \quad n = 0, \pm 1, \pm 2, \ldots, \]  
(3.91)

\[ U'_n = \text{Res}(f_1 : j'_n), \quad V'_n = \text{Res}(f_2 : j'_n). \]  
(3.92)

Following Linton and Evans[54], we may construct the functions \( f_1(z), f_2(z) \) as

\[ f_1(z) = \zeta_1 e^{-z\ln 2/\pi} (1 - z/\Gamma_1) \prod_{m=1}^{\infty} \frac{1 - z/k_m}{(1 - z/j_0)(1 - z/j'_0)(1 - z/j_m)(1 - z/j'_m)}, \]

and

\[ f_2(z) = \zeta_2 e^{-z\ln 2/\pi} (1 - z/\Gamma_2) \prod_{m=1}^{\infty} \frac{1 - z/k_m}{(1 - z/j_0)(1 - z/j'_0)(1 - z/j_m)(1 - z/j'_m)}, \]

where \( \Gamma_1, \Gamma_2 \) are to be determined by the condition 2 with \( m = 0 \) whilst \( \zeta_1, \zeta_2 \) are unknown coefficients, and the factor \( e^{-z\ln 2/\pi} \) is added to assure that \( f_1(z), f_2(z) \) have the right behaviour for the evaluation of residue series, i.e. \( f_1(z), f_2(z) = O(z^{-1/2}) \) as \( z \to \infty \) on \( p_{1N}, p_{2N} \) as \( N \to \infty \). It is easy to obtain

\[ \Gamma_1 = \frac{k_0(1 - A)}{1 + A}, \quad \Gamma_2 = \frac{k_0(1 + A)}{1 - A}, \]  
(3.93)

where

\[ A = e^{-2k_0\alpha + 2k_0\ln 2} \left( 1 - k_0/j_0 \right) \left( 1 - k_0/j'_0 \right) \prod_{m=1}^{\infty} \frac{1 + k_0/k_m}{(1 + k_0/j_0)(1 + k_0/j'_0)(1 + k_0/k_m)(1 + k_0/j'_m)(1 + k_0/j_m)(1 + k_0/j'_m)}. \]  
(3.94)

From the evaluation of residue series, we have the solution of \( U_n, V_n \) as

\[ U_n = \frac{\zeta_1 (1 - j_n/\Gamma_1)}{1 - j_n/j'_0} T_n, \quad V_n = \frac{\zeta_2 (1 - j_n/\Gamma_2)}{1 - j_n/j'_0} T_n \]  
(3.95)

where

\[ T_n = \frac{e^{-j_n\ln 2/\pi} j_n (k_n - j_n)}{k_n (1 - j_n/j'_{-n})(1 - j_n/j_0)} \prod_{m=1, m \neq n}^{\infty} \frac{1 - j_n/k_m}{(1 - j_n/j_m)(1 - j_n/j'_m)}, \]
From (3.79) we have

\[ \zeta_1 (1 - j_0' / \Gamma_1) + \zeta_2 (1 - j_0' / \Gamma_2) = 0. \]  \hspace{1cm} (3.96)

Substitution of \( U_n, V_n \) into (3.77) gives

\[ B_n = \frac{j_n}{2 \mu_n} (\zeta_2 (1 - j_n / \Gamma_2) - \zeta_1 (1 - j_n / \Gamma_1) T_n). \]  \hspace{1cm} (3.97)

Substitution of (3.96) into (3.97) yields

\[ B_n = \frac{j_n}{2 \mu_n} \left( \zeta_2 ((1 - j_n / \Gamma_2) + \frac{(1 - j_n / \Gamma_1)(1 - j_0' / \Gamma_2)}{1 - j_0' / \Gamma_1}) T_n. \]  \hspace{1cm} (3.98)

By applying the Kutta condition at the trailing edge, we need \( B_n = O(n^{-3/2}) \).

From (3.32), we may write

\[ B_n = \frac{j_n}{2 \mu_n} (\zeta_2 - \zeta_1) n^{-1/2} + O(n^{-3/2}). \]  \hspace{1cm} (3.99)

It is obvious that \( B_n = O(n^{-1/2}) \) from this relation. Therefore by choosing \( \zeta_1 = \zeta_2 \) we have \( B_n = O(n^{-3/2}) \) as required. Substitute \( \zeta_1 = \zeta_2 \) into (3.96) to give

\[ (j_0' - \Gamma_1) + (j_0' - \Gamma_2) = 0. \]  \hspace{1cm} (3.100)

This is the condition for which Rayleigh-Bloch resonances exist. Substitute \( \Gamma_1, \Gamma_2 \) into (3.100) to generate

\[ \frac{j_0 - k_0 e^{-2k_0 \alpha + 2k_0 \ln 2}}{j_0 + k_0} \frac{\infty}{\prod (1 - k_0 / k_m)(1 - k_0 / j_m)(1 - k_0 / j_{-m})} = \pm \sqrt{\frac{1 + M}{1 - M}}. \]  \hspace{1cm} (3.101)

This is final equation for which complex resonance frequency should satisfy.

We need to check if it agrees with the case of no mean flow.
If $M = 0$ equation (3.101) becomes
\[
\frac{j_0 - k_0}{j_0 + k_0} e^{-2k_0a + \frac{2k_0}{\pi}} \prod_{m=1}^{\infty} \frac{(1 + k_0/k_m)(1 - k_0/j_m)(1 - k_0/j_{-m})}{(1 - k_0/k_m)(1 + k_0/j_m)(1 + k_0/j_{-m})} = \pm 1.
\] (3.102)

Thus we may write
\[
\frac{j_0 - k_0}{j_0 + k_0} = r_1 e^{2i\delta_1}, \quad e^{-2k_0a + \frac{2k_0}{\pi}} = r_2 e^{2i\delta_2},
\]
and
\[
\prod_{m=1}^{\infty} \frac{(1 + k_0/k_m)(1 - k_0/j_m)(1 - k_0/j_{-m})}{(1 - k_0/k_m)(1 + k_0/j_m)(1 + k_0/j_{-m})} = r_3 e^{2i\delta_3}.
\]

Henceforth, for symmetric complex resonance we have
\[
r_1 r_2 r_3 = \sqrt{\frac{1 + M}{1 - M}}, \quad \delta_1 + \delta_2 + \delta_3 = (n + 1/2)\pi
\] (3.103)

for some integer $n$, and for symmetric complex resonance we have
\[
r_1 r_2 r_3 = \sqrt{\frac{1 + M}{1 - M}}, \quad \delta_1 + \delta_2 + \delta_3 = n\pi
\] (3.104)

for some integer $n$. These are exactly the approximate solutions for which Rayleigh-Bloch modes to satisfy if $M = 0$, therefore Rayleigh-bloch modes have been recovered.

### 3.3.4 Full solution

For the full solution (3.80) and (3.81), we may define the following integrals
\[
I_{m1} = \frac{1}{2\pi i} \oint_{p_{1N}} f_1(z) \left( \frac{1}{z - k_m} + \frac{e^{-2k_m a}}{z + k_0} \right) dz,
\]
\[
I_{m2} = \frac{1}{2\pi i} \oint_{p_{2N}} f_2(z) \left( \frac{1}{z - k_m} - \frac{e^{-2k_m a}}{z + k_0} \right) dz,
\]
where contour $p_{1N}, p_{2N}$ are circles with infinite radii and $f_1(z), f_2(z)$ satisfy the following conditions

1. $f_1(z), f_2(z)$ have simple poles at $z = j_n, n = 0, \pm 1, \pm 2, \ldots$, and $z = j'$. 
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2. \( f_1(z), f_2(z) \) satisfies the equation

\[
f_1(k_m) + e^{-2\pi m^2} f_1(-k_m) = 0, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{3.105}
\]

\[
f_2(k_m) - e^{-2\pi m^2} f_2(-k_m) = 0, \quad m = 0, \pm 1, \pm 2, \ldots. \tag{3.106}
\]

3. \( f_1(z), f_2(z) = O(z^{-1/2}) \) as \( z \to \infty \) on the contours \( p_{1N}, p_{2N} \) as \( N \to \infty \).

The last of these conditions ensures that these integrals vanish, therefore by applying the conditions above the evaluation of integrals in terms of their respective residue series gives

\[
\sum_{n=-\infty}^{\infty} \left( \frac{1}{j_n - k_m} + \frac{e^{-2\pi m^2}}{j_n + k_0} \right) \text{Res}(f_1 : j_n) = 0, \tag{3.107}
\]

and

\[
\sum_{n=-\infty}^{\infty} \left( \frac{1}{j_n - k_m} - \frac{e^{-2\pi m^2}}{j_n + k_0} \right) \text{Res}(f_2 : j_n) = 0. \tag{3.108}
\]

Comparing (3.80) and (3.81) with (3.107) and (3.108), we have

\[
U_n = \text{Res}(f_1 : j_n), \quad V_n = \text{Res}(f_2, j_n), \quad n = 0, \pm 1, \pm 2, \ldots, \tag{3.109}
\]

and

\[
U'_0 = \text{Res}(f_1 : j'_0), \quad V'_0 = \text{Res}(f_2, j'_0). \tag{3.110}
\]

The way of constructing functions \( f_1(z), f_2(z) \) is similar to the full solution in the waveguide problem. We may write \( f_1(z), f_2(z) \) as

\[
f_1(z) = \frac{\zeta_1 e^{-z \ln 2/\pi}}{z - j'_0} \prod_{m=0}^{R} \frac{z - k'_m}{(z - j_m)(z - j_m')}, \prod_{m=R+1}^{\infty} \frac{1 - z/k_m}{(1 - z/j_m)(1 - z/j_m')},
\]

and

\[
f_2(z) = \frac{\zeta_2 e^{-z \ln 2/\pi}}{z - j'_0} \prod_{m=0}^{R} \frac{z - k'_m}{(z - j_m)(z - j_m')}, \prod_{m=R+1}^{\infty} \frac{1 - z/k_m}{(1 - z/j_m)(1 - z/j_m')}
\]
where \( k'_m, k''_m \) are determined by condition 2 whilst \( \zeta_1, \zeta_2 \) are unknown coefficients, and \( f_1(z), f_2(z) = O(z^{-1/2}) \) as \( z \to \infty \). Henceforth we may write the solution of \( U_0, V_0 \) as

\[
U'_0 = \zeta_1 e^{-i\theta_0 \ln 2/\pi} \prod_{m=0}^{R} \frac{j'_0 - k'_m}{(\theta_0 - j_m)(\theta_0 - j_{m-1})} \prod_{m=R+1}^{\infty} \frac{1 - j'_0/k'_m}{(1 - \theta_0/j_m)(1 - \theta_0/j_{m-1})},
\]

and

\[
V'_0 = \zeta_2 e^{-i\theta_0 \ln 2/\pi} \prod_{m=0}^{R} \frac{j''_0 - k''_m}{(\theta_0 - j_m)(\theta_0 - j_{m-1})} \prod_{m=R+1}^{\infty} \frac{1 - j''_0/k''_m}{(1 - \theta_0/j_m)(1 - \theta_0/j_{m-1})}.
\]

From (3.79) we have

\[
\zeta_1 \prod_{m=0}^{R} \frac{j'_0 - k'_m}{(\theta_0 - j_m)(\theta_0 - j_{m-1})} + \zeta_2 \prod_{m=0}^{R} \frac{j''_0 - k''_m}{(\theta_0 - j_m)(\theta_0 - j_{m-1})} = 0. \tag{3.111}
\]

From (3.32), we may write

\[
B_n = \frac{j_n}{2\mu_n} (\zeta_2 - \zeta_1) n^{-1/2} + O(n^{-3/2}) \tag{3.112}
\]

where \( B_n = O(n^{-1/2}) \). However by applying the Kutta condition at the trailing edge of the plates, we need \( B_n = O(n^{-3/2}) \), thus by choosing \( \zeta_1 = \zeta_2 \) we have \( B_n = O(n^{-3/2}) \) as required. As a result, (3.111) becomes

\[
\prod_{m=0}^{R} (\theta_0 - k'_m) + \prod_{m=0}^{R} (\theta_0 - k''_m) = 0. \tag{3.113}
\]

The Rayleigh-Bloch modes are related to the parameter \( \beta \). From (3.72) and (3.73) we fix the value of \( \beta \), then \( 0 < l < \beta \). Linton & Evans[54] showed that the case \( \beta = \pi \) is in fact equivalent to the problem of a thin plate on the centreline of a waveguide considered in the previous section.

### 3.3.5 Numerical Results

Figure 3.5 shows Rayleigh-Bloch modes computed from the above formulation by taking \( M = 0 \), for \( \beta = 1, 2, \pi \) plotted against the geometric parameter.
Figure 3.5: Rayleigh-Bloch modes for $\beta = 1, 2, \pi$ as a function of $L/d$

$L/d$. These numerical results show that we have recovered the Rayleigh-Bloch modes by the formulation of taking account of mean flow.

Figure 3.6 shows the Rayleigh-Bloch complex resonances in the presence of mean flow. In this figure the real and imaginary parts of $l^* = l \sqrt{1 - M^2}$ are plotted against geometric parameter $L/d$ for different Mach number, $M = 0, 0.3, 0.5, 0.7$ respectively. The influence of mean flow on the Rayleigh-Bloch modes is very similar to that it has on the trapped modes discussed in the previous section. The real part decreases as the uniform flow speed increases, while the magnitude of imaginary part increases as the Mach number increases.

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Figure 3.6: Rayleigh-Bloch complex resonances for $\beta = 1$ as a function of $L/d$

3.4 Conclusion

The influence of mean flow on the trapped modes in channels, and the Rayleigh-Bloch modes along a cascade have been discussed by using a mode matching technique and the residue calculus theory. We have investigated these problems by letting the amplitude of incident wave to zero in the well-defined wave scattering problems in a two-dimensional waveguide or by an array of finite thin plates.

In the two-dimensional waveguide problem, i.e. the channel problem, we
first formulate the solution of the trapped mode problem in the presence of subsonic mean flow, and recover all trapped mode solutions in which Mach number $M = 0$. By assuming that the plate is long enough, the approximate solutions are obtained straightforwardly by applying the residue calculus theory. A convenient full solution technique, that can compute the shifted zeros numerically, has been developed. Numerical results show that we have successfully recovered the results obtained by Koch[38] using Wiener-Hopf technique. In the presence of mean flow, trapped modes become complex resonances with a small imaginary part, and the real part of resonant frequency decreases as Mach number $M$ increases, whilst the magnitude of the imaginary part increases as $M$ increases.

In the array of finite flat plates, we also recover the results of Rayleigh-Bloch modes by simply letting the Mach number $M = 0$. The influence of mean flow on Rayleigh-Bloch modes is very similar to that of mean flow on the trapped modes in channels.
Chapter 4

Wave Scattering near embedded trapped mode frequencies and complex resonances

4.1 Introduction

Recently, McIver et al. [64, 65] have found embedded trapped modes in two-dimensional wave guides. Embedded trapped modes are modes which are above the first cut-off for wave propagation along the guide. Therefore in this chapter, we will explore the wave scattering behavior near the frequencies of embedded trapped modes and complex resonances. In section 4.2, we first consider the embedded trapped modes near a finite thin plate in a two dimensional guide. In section 4.3 a discussion will be made on the complex resonances without mean flow. Section 4.4 will be devoted to a brief discussion of the influence of mean flow on embedded trapped modes. Section 4.5 will present the wave scattering near embedded trapped mode frequencies.
and complex resonances. A brief summary will be given in the final section.

4.2 Embedded trapped modes

McIver, Linton, & Zhang[64], and McIver, Linton, McIver, & Zhang[65] found embedded trapped modes in a two-dimensional wave guide in which a finite thin plate is placed on the centreline of the guide. Here we briefly introduce their results as a preliminary to this chapter.

The geometry of this problem is shown in Figure 2.1. We restrict \( l \) to the range \( \pi < l < 3\pi/2 \), which is above the first cut-off \( l = \pi/2 \), and below the second cut-off, \( l = 3\pi/2 \). The embedded trapped modes are the solutions of the following infinite system of equations

\[
\sum_{n=2}^{\infty} U_n \left( \frac{1}{j_n - k_m} \pm \frac{e^{-2\pi m a}}{j_n + k_m} \right) = 0, \quad m \geq 0
\]  

(4.1)

where '+' corresponds to modes symmetric about the plane \( x = 0 \) and '-' antisymmetric about the plane \( x = 0 \). Notice that the subscript starts from 2 in the sum of (4.1) since the coefficient of the first term in the potential expansion in the outer region is forced to be zero for an embedded trapped mode. The solution of (4.1) by residue calculus theory can be found, for example, from McIver et al [64]. Figure 4.1 shows the numerical results from the solution of (4.1) by using residue calculus theory.

In this figure, \( l \) is plotted against the non-dimensional geometric parameter \( L/d \). It should be observed that the embedded trapped modes in this geometric configuration take only discrete values, which means that the embedded trapped modes only exist for some specific geometries and frequencies. It is different from the trapped mode, which may be found for any given value of \( L/d \). Therefore the remaining question is, can we find the solution of (4.1) for fixed value of \( L/d \) when \( \pi/2 < l < 3\pi/2 \). This will be considered in the next section.
4.3 Complex resonances

4.3.1 Introduction

We know that embedded trapped modes are discrete, which means they exist for a specific pair of geometry and frequency values, namely $a, l$. In this case, we fix the geometry $a$, and look for the non-trivial solutions of the following system of equations

$$\sum_{n=1}^{\infty} U_n \left( \frac{1}{j_n - k_m} \pm \frac{e^{-2k_m a}}{j_n + k_m} \right) = 0, \quad m \geq 0.$$  \hspace{1cm} (4.2)
The difference between (4.2) and (4.1) is that \( n \) starts from 1 in equation (4.2). The non-trivial solutions of (4.2) occur when the determinant of the system of equations is zero. Obviously, the determinant of (4.2) is complex since the coefficients of \( U_n \) are complex. For a complex expression to be zero, both the real and imaginary part of the complex expression should be zero, therefore we have two equations to satisfy, which needs two unknown parameters. Generally, only when the frequency is complex, with unknown real and imaginary parts, can this requirement be satisfied. As a result, we call this complex frequency at a given geometry the complex resonance. It is not easy to find the complex frequencies by solving (4.2) directly, and the residue calculus approach will be used.

### 4.3.2 Complex resonances without mean flow

Assuming \( I \) consists two parts, real and imaginary, i.e. \( I = l_1 + il_2 \), we have

\[
k_0 = -il = -l_2 - il_1, \quad (4.3)
\]

\[
k_1 = -i\sqrt{l^2 - \pi^2} = -i\sqrt{l_1^2 - l_2^2 + 2il_1l_2}, \quad (4.4)
\]

and

\[
k_m = \sqrt{m^2\pi^2 - l_1^2 + l_2^2 - 2il_1l_2}, \quad m \geq 2. \quad (4.5)
\]

The solution of this problem will be sought by an approximate and full method. The approximate solution of (4.2) may be found by setting terms with \( e^{-2km} \), \( m > 2 \) to be zero. The solution procedure is routine, and only the main results are presented as below.

The approximate solutions of the complex resonances are given by

\[
(k_0 - k_1') \pm A_0(k_0 + k_1') = 0, \quad (4.6)
\]

where

\[
k_1' = (1 \pm A_1)/(1 \mp A_1)k_1,
\]
In (4.6), the '+' represents symmetric complex resonances, and '-' represents anti-symmetric resonances.

The full solutions of the complex resonances are given by

\[ \prod_{r=1}^{R} (k_0 - k_r^\prime) \pm A_0 \prod_{r=1}^{R} (k_0 + k_r^\prime) = 0, \tag{4.7} \]

where \( k_r^\prime, \ r = 1, 2, ..., R \) are the solution of the following system of equations

\[ \prod_{r=1}^{R} (k_m - k_r^\prime) \pm A_m \prod_{r=1}^{R} (k_m + k_r^\prime) = 0, \ m = 1, 2, ..., R, \tag{4.8} \]

and

\[ A_m = e^{-2k_m a} \prod_{r=1}^{R} \frac{k_m - j_r}{k_m + j_r} \prod_{n=R+1}^{\infty} \frac{(1 + k_m/k_n)(1 - k_m/j_n)}{(1 - k_m/k_n)(1 + k_m/j_n)} \tag{4.9} \]

for \( m = 0, 1, 2, ..., R \). In (4.7) and (4.8), the '+' represents symmetric complex resonances, and '-' represents anti-symmetric resonances.

### 4.3.3 Influence of mean flow on embedded trapped modes

The influence of mean flow on embedded trapped modes is similar to that on the trapped modes since we look for the non-trivial solutions of the same systems of equations (3.2) and (3.3) for \( \pi < \text{Real}(l) < 3\pi/2 \). For convenience, we rewrite (3.8) and (3.9) as below,

\[ \sum_{n=0}^{\infty} U_n \left( \frac{1}{j_n - k_m} + \frac{e^{-2k_m a}}{j_n + k_m} \right) = 0, \ m \geq 0, \tag{4.10} \]

and

\[ \sum_{n=0}^{\infty} V_n \left( \frac{1}{j_n - k_m} - \frac{e^{-2k_m a}}{j_n + k_m} \right) = 0, \ m \geq 0. \tag{4.11} \]
The approximate and full solution procedure of (4.10) and (4.11) is similar to that discussed in Section 3.2. Notice that there are two wave-like terms in inner region, therefore two terms should be kept for the approximate solution. Only the full solution result is given below since the approximate solution may be obtained by taking the truncation number \( R = 1 \).

![Figure 4.2: Comparison of real part of approximate and full solutions](image)

The complex resonances due to mean flow satisfy

\[
\prod_{r=0}^{R} (j_0 - k_r') + \prod_{r=0}^{R} (j_0 - k''_r) = 0
\]

(4.12)

where \( R = 0, 1, 2, \ldots \), is the truncation number determined by the accuracy requirement, and \( R = 1 \) represents the approximate solution; \( k'_r, k''_r \) are
shifted zeros determined by

\[
\prod_{r=1}^{R} (k_m - k'_r) + A_m \prod_{r=1}^{R} (k_m + k'_r) = 0, \ m = 1, 2, \ldots, R, \quad (4.13)
\]

and

\[
\prod_{r=1}^{R} (k_m - k''_r) - A_m \prod_{r=1}^{R} (k_m + k''_r) = 0, \ m = 1, 2, \ldots, R, \quad (4.14)
\]

where \( A_m, m = 0, 1, 2, \ldots, R \) are computed by (4.9).

---

**Figure 4.3:** Comparison of imaginary part of approximate and full solutions
4.3.4 Numerical results

Figure 4.2 shows the comparison between the real parts of approximate and full solutions for $0 < a \leq 1$, in which the solid lines represent the approximate solutions and the dots represent the full solutions. The numerical calculation suggest that we may find both symmetric and antisymmetric complex resonances when $a \geq 0.55$. Below this value only symmetric resonances may be found for a given value of $a$. It seems there is a minimum value of $a$ ($a \approx 0.10$) above which complex resonances may be found. A remarkable result is that the approximate solution is very accurate when $a \geq 0.4$, of at least three
Figure 4.5: Variation of imaginary part of complex resonances with $L/d$

digit accuracy, and this accuracy increase to 4 digits when $a \geq 0.5$. Figure 4.3 shows the imaginary parts of the approximate and full solutions, in which the lines correspond to the approximate solution while the dots represent the full solution. Compared with the real parts, the magnitude of the imaginary part is smaller.

Figure 4.4 and 4.5 shows the real and imaginary parts of the complex resonances for $1 \leq a \leq 5$, in which solid lines represent symmetric resonances, and dashed lines antisymmetric resonances, while filled circles represent symmetric embedded modes and filled diamonds antisymmetric embedded modes. Complex resonances are computed by fixing $a = L/d$, and
searching the solutions of (4.6) for real \( l \) that is in the range of \( \pi - 3\pi/2 \).

First we examine the real part of the complex resonances. From Figure 4.4 we notice that there are a number of complex resonances that the magnitude of real parts falls in the range of \( \pi - 3\pi/2 \), in which the embedded trapped modes exist for some specific value of \( a = L/d \), and the number of complex resonances increases with the increase of non-dimensional plate length \( a \), and the real part of the complex resonances starts from the second cut-off frequency \( 3\pi/2 \) and decreases to the first cut-off \( \pi \). All the lines of complex resonance (magnitude of the real part of complex resonances) intersect with the embedded trapped modes at which the imaginary part is zero. This can be seen clearly from (4.5) where only imaginary parts of the symmetric complex resonances are plotted against the geometric parameter \( a = L/d \).

Figure 4.4 and 4.5 suggest that we may find embedded trapped modes by computing complex resonances. By fixing geometry, we may compute complex resonances for which the magnitude of the real part of complex resonances is in the range of above first cut-off and below second cut-off. Embedded modes may be found when the magnitude of imaginary part is zero by variation of geometric parameter \( a \).

4.4 Wave scattering near complex resonances and embedded trapped mode frequencies

4.4.1 Introduction

There are no waves propagating down the wave guide if \( 0 < l < \pi/2 \). In this range trapped modes may be found for a given value of \( a \). However, waves can propagate down the wave guide if \( \pi/2 < l < 3\pi/2 \). In this case the scattering problem can be solved, and embedded trapped modes may be found if \( \pi < l < 3\pi/2 \). The remaining question is what happens if the
frequency of an incident wave is close to or near the frequency of an embedded trapped mode or a complex resonance. In order to answer this question, we need to solve the scattering problem in the presence of mean flow.

### 4.4.2 Solution of the scattering problem

In order to obtain reflection and transmission coefficients $R$ and $T$, we have to solve equations (2.61), (2.62), (2.63) and (2.64). The solution procedure using residue calculus theory is the same as before, and only the results are given below.

The terms containing $e^{-2km\alpha}$ in these equations decay exponentially for $m \geq 1$ when $\pi/2 < l < \pi$, and $e^{-2km\alpha}$ decay exponentially for $m \geq 2$ when $\pi < l < 3\pi/2$, which means there is only one wave-like term when $\pi/2 < l < \pi$, and two wave-like terms when $\pi < l < 3\pi/2$, in the inner region. For convenience, two terms are kept for the approximate solutions for $\pi/2 < l < 3\pi/2$. The full solution procedure is similar to that described in Section 3.2. Only the full solutions will be given since the approximate solutions may be easily obtained by taking the truncation number $N = 2$.

The reflection and transmission coefficients are given by

$$R = 1 + \frac{C_1}{\tilde{j}_1} = \frac{E_1 + F_1}{2\mu_1},$$

and

$$T = -\frac{B_1}{\tilde{j}_1} = -\frac{F_1 - E_1}{2\mu_1},$$

where $E_1$ and $F_1$ are given by

$$E_1 = E'_1 + \frac{Dk\tanh\tilde{k}'}{2\sqrt{2}}E''_n, \quad F_1 = F'_1 + \frac{Dk\tanh\tilde{k}'}{2\sqrt{2}}F''_n$$

where $\frac{Dk\tanh\tilde{k}'}{2\sqrt{2}} = -\frac{\xi_1 - \xi_2}{\xi_1 - \xi_2}$, and $E'_1, E''_1, F'_1, F''_1$ are given by

$$E'_1 = \frac{\xi_1(j_1 - k'_0)(j_1 - k'_1)}{2j_1} \prod_{m=0}^{R} \frac{j_1 - k'_m}{j_1 - j_m} \prod_{m=R+1}^{\infty} \frac{1 - j_1/k_m}{1 - j_1/j_m}.$$
\[
E''_1 = \xi_2 \left( j_1 - k''_0 \right) \left( j_1 - k''_0 \right) \frac{R}{j_1 - j_0} \prod_{m=2}^{R} \frac{j_1 - j_m}{j_1 - j_m} \prod_{m=R+1}^{\infty} \frac{1 - j_1/k_m}{1 - j_1/j_m},
\]
(4.19)
\[
F'_1 = \frac{\xi_3 (j_1 - k''_0) (j_1 - k''_0)}{2j_1} \prod_{m=2}^{R} \frac{j_1 - j_m}{j_1 - j_m} \prod_{m=R+1}^{\infty} \frac{1 - j_1/k_m}{1 - j_1/j_m},
\]
(4.20)
and
\[
F''_1 = \frac{\xi_4 (j_1 - k'''_0) (j_1 - k'''_0)}{j_1 - j_0} \prod_{m=2}^{R} \frac{j_1 - j_m}{j_1 - j_m} \prod_{m=R+1}^{\infty} \frac{1 - j_1/k_m}{1 - j_1/j_m},
\]
(4.21)
where the coefficients \( \xi_1, \xi_2, \xi_3, \xi_4 \) are given by
\[
\xi_1 = -\mu_1 \frac{\prod_{m=1}^{R} j_1 + j_m}{j_1 + k'_0} \prod_{m=R+1}^{\infty} \frac{1 + j_1/j_m}{1 + j_1/k_m},
\]
(4.22)
\[
\xi_2 = -\frac{1}{j_0 - k'_0} \prod_{m=1}^{R} j_0 - j_m \prod_{m=R+1}^{\infty} \frac{1 - j_0/j_m}{1 - j_0/k_m},
\]
(4.23)
\[
\xi_3 = -\mu_1 \frac{\prod_{m=1}^{R} j_1 + j_m}{j_1 + k''_0} \prod_{m=R+1}^{\infty} \frac{1 + j_1/j_m}{1 + j_1/k_m},
\]
(4.24)
and
\[
\xi_4 = \frac{1}{j_0 - k'''_0} \prod_{m=1}^{R} j_0 - j_m \prod_{m=R+1}^{\infty} \frac{1 - j_0/j_m}{1 - j_0/k_m}.
\]
(4.25)
The parameters \( k'_r, k''_r, k'''_r \) and \( k''''_r \) are shifted zeros, which satisfy
\[
(k_m - k'_0) \prod_{r=1}^{R} (k_m - k'_r) + A_1 (k_m + k'_0) \prod_{r=1}^{R} (k_m + k'_r) = 0,
\]
(4.26)
\[
(k_m - k''_0) \prod_{r=1}^{R} (k_m - k''_r) + A_2 (k_m + k''_0) \prod_{r=1}^{R} (k_m + k''_r) = 0,
\]
(4.27)
\[
(k_m - k'''_0) \prod_{r=1}^{R} (k_m - k'''_r) - A_1 (k_m + k'''_0) \prod_{r=1}^{R} (k_m + k'''_r) = 0,
\]
(4.28)
and
\[
(k_m - k''''_0) \prod_{r=1}^{R} (k_m - k''''_r) - A_2 (k_m + k''''_0) \prod_{r=1}^{R} (k_m + k''''_r) = 0,
\]
(4.29)
where \( m = 0, 1, \ldots, R \), and \( A_1, A_2 \) are given by
\[
A_1 = e^{-2k_m a} \frac{k_m + j_1}{k_m - j_1} \prod_{r=1}^{R} \frac{k_m - j_r}{k_m + j_r} \prod_{r=R+1}^{\infty} \frac{(k_r + k_m)(j_r - k_m)}{(k_r - k_m)(j_r + k_m)},
\]
(4.30)
Table 4.1: Magnitude of reflection coefficients vary with $N$ and $a$ for $l = \frac{3}{4}\pi$

<table>
<thead>
<tr>
<th>$a\backslash N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0118588104</td>
<td>0.0215588533</td>
<td>0.0221124354</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1730106196</td>
<td>0.1742016811</td>
<td>0.1742069675</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3069368296</td>
<td>0.3070398803</td>
<td>0.3070399130</td>
</tr>
<tr>
<td>0.7</td>
<td>0.2493455218</td>
<td>0.2493577560</td>
<td>0.2493577567</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1345620612</td>
<td>0.1345617180</td>
<td>0.1345617180</td>
</tr>
</tbody>
</table>

and

$$A_2 = e^{-2k_m a} \frac{k_m - j_0 \prod_{r=1}^{R} k_m + j_{r}}{k_m + j_{0}} \prod_{r=R+1}^{\infty} \frac{(k_r + k_m)(j_r - k_m)}{(k_r - k_m)(j_r + k_m)}$$

(4.31)

where $m = 0, 1, ..., R$. The approximate solutions may be obtained by taking $R = 1$, i.e. the truncation number $N = 2$.

4.4.3 Numerical results

First we discuss the accuracy of the numerical solution. Obviously, the accuracy of the approximate solution depends largely on the dimensionless plate length $a$. The accuracy of approximate solutions improves as $a$ increases. On the other hand, the full solution may take account of the contribution of more non wave-like terms that decay to zero exponentially. Another factor affecting the accuracy is $l$. There is only one wave-like term if $\pi/2 < l < \pi$. And there are two wave-like terms if $\pi < l < 3\pi/2$.

Table (4.1) shows the magnitude of reflection coefficient for different non-dimensional plate length $a$ and the different number of terms kept in equations (2.61) - (2.64) for a given $l = 3\pi/4$. In this case there is only one wave propagating along the wave guide. The first column in table 4.1 for which $N = 2$ represents the reflection coefficient given by the approximate solution. It is obvious from Table 4.1 that the accuracy of the approximate solution
Table 4.2: Magnitude of reflection coefficients vary with $N$ and $a$ for $l = \frac{5\pi}{4}$

<table>
<thead>
<tr>
<th>$a\backslash N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0489839357</td>
<td>0.0112358649</td>
<td>0.0115685909</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1733757029</td>
<td>0.1768951944</td>
<td>0.1769079927</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5035630926</td>
<td>0.5021108619</td>
<td>0.5021106391</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3437457506</td>
<td>0.3436618068</td>
<td>0.3436618059</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5390305624</td>
<td>0.5390256995</td>
<td>0.5390256995</td>
</tr>
</tbody>
</table>

$(N = 2)$ is not sufficient for $a = 0.1$, with a relative error of 45%. However the relative accuracy will be 2.5% if we increase the number of terms from $N = 2$ to $N = 4$, and four digit accuracy can be achieved for $N = 8$.

The accuracy of the approximate solution increases rapidly with $a$. The relative error is less than 2.0% when $a = 0.3$, and 0.034% when $a = 0.5$ and four digital accuracy is obtained when $a = 0.7$. Therefore the approximate solutions have sufficient accuracy for $a \geq 0.3$.

Table 4.2 shows the magnitude of reflection coefficient for different dimensionless plate lengths $a$ and different numbers of terms kept in equation (2.61) - (2.64) for $l = \frac{5\pi}{4}$. In this case there are two waves propagating down the wave guide. The column in the table for which $N = 2$ represents the reflection coefficient of approximate solutions. Similar conclusions about the accuracy of approximate solutions may be made from this table.

Figures 4.6 and 4.7 show the variation of the reflection and transmission coefficients with $l$ for given $L/d = 1$, with different Mach number $M$. From these figures we see that both the reflection and transmission coefficients decrease as the Mach number increases. This may be explained by the vortex effect at the trailing edge, in which part of the incident energy goes down into the vortex sheet, and this part of the energy increases as Mach number increases. The numerical results show that $|R|^2 + |T|^2 = 1$ when $M = 0$, 

83
which agrees with energy conservation, and $|R|^2 + |T|^2$ decreases as Mach number $M$ increases.

We notice from Figure 4.6 that the reflection coefficient curves have peaks after $l > \pi$, and that these peaks move to the left when Mach number $M$ increases since $l$ is a function of Mach number $M$, i.e.

$$l = l_0/\sqrt{1 - M^2}, \text{ and } l_0 = \omega d/c,$$  \hspace{1cm} (4.32)

where $\omega$ denotes the circular frequency, $d$ the half width of the wave guide, $c$ the free stream sound speed. The peaks are very sharp when $M = 0$, and
they become less sharper as $M$ increases.

The peaks of the reflection curves correspond to the troughs of the transmission coefficient curves in Figure 4.7. It is of interest to examine the significance of the peaks and troughs of these curves. We know that there exist embedded trapped modes when $\pi < l < 3\pi/2$ for some particular value of $a$, and for any given value of $a$ we may find complex resonances. We wish to determine the connections between the scattering behavior and the complex resonances or embedded trapped modes.

First let us look at the behavior of the reflection coefficient when the

Figure 4.7: Magnitude of transmission coefficients variation with frequency for given $L/d = 1$
Table 4.3: Frequencies at which reflection coefficient has peak values for $L/d = 1$

<table>
<thead>
<tr>
<th>$M=0$</th>
<th>$M=0.1$</th>
<th>$M=0.3$</th>
<th>$M=0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.428</td>
<td>3.411</td>
<td>3.259</td>
<td>2.938</td>
</tr>
<tr>
<td>4.074</td>
<td>4.054</td>
<td>3.851</td>
<td>3.414</td>
</tr>
<tr>
<td>4.712</td>
<td>4.688</td>
<td>4.457</td>
<td>2.938</td>
</tr>
</tbody>
</table>

Mach number $M = 0$. From Figure 4.4 we know that there are three complex resonances when $L/d = 1$. Their real parts are $\text{Real}(l) \approx 3.428, 4.074, 4.712$ respectively. From Figure 4.6 we know that the reflection coefficient takes peak values when the frequency of the incident waves is equal to the real part of the complex resonances.

Table 4.3 shows the frequencies of the complex resonances for different values of $M$, at which the reflection coefficient curves have peak values, as shown in figure 4.6. Further numerical results show that the number of peaks of the reflection coefficient or valleys on the transmission coefficient is equal to the number of complex resonances that may be found for a given value of $a$. Hence the number of peaks of the reflection coefficient increases as $a$ increases.

What happens if the frequency of an incident wave is equal to a trapped mode frequency? We know there exist an antisymmetric embedded trapped mode for $a = L/d \approx 3.0290$, and its frequency $l \approx 3.468$. The reflection coefficient $|R|$ is plotted against the frequency of an incident wave for $a = 3.0290$, the geometry that can sustain the antisymmetric embedded modes, in Figure 4.8. For the convenience of comparison, the real part of complex resonance frequencies (dots) and the trapped mode frequency (diamond) are also plotted on the same graph. From this figure we know that the number of peaks is equal to the number of complex resonances that may be found
for $a = 3.0290$, see Figure 4.4. However, when the frequency of the incident wave approaches the frequency of embedded trapped mode, the reflection coefficient tends to be zero. Numerical results show that the modulus of the reflection coefficient is around $10^{-6}$ in the vicinity of embedded trapped mode frequency. This means that the incident wave is completely transmitted if the frequency of the incident wave is equal to that of the embedded trapped mode.

### 4.5 Conclusion

In this chapter, we have investigated the complex resonances which are above the first cut-off frequency and below the second cut-off. The real part of the complex resonances are found to be in the range of $\pi/2 < l < 3\pi/2$ by fixing the geometric parameter $a$, and the imaginary part of the frequency is a small value compared with the real part. When the imaginary part is zero, the real part corresponds to the embedded trapped mode found by McIver, Linton, McIver, & Zhang[65]. The effect of mean flow on the embedded trapped modes is similar to that on the trapped modes. The wave scattering by a finite thin plate in a two dimensional wave guide has been investigated. When the frequency of the incident wave is equal to the magnitude of the real part of complex resonances, we obtain peak values of reflection coefficients, and the incident wave is totally transmitted when the frequency of the incident wave is equal to the frequency of embedded trapped modes.
Figure 4.8: Variation of reflection coefficients with frequency \( l \) for given \( a = 3.0290 \). There is no reflection if the frequency of the incident wave is equal to the trapped mode frequency.
Chapter 5

Trapped modes near an indentation in parallel wave guide

5.1 Introduction

Evans & Porter[14] first provided numerical evidence for the existence of an isolated trapped mode in the presence of a circular obstacle on the centre line of a two-dimensional wave guide, above the cut-off for anti-symmetric wave propagation in the guide. McIver, Linton, McIver, Zhang & Porter[65] further showed numerically that rather being isolated, this mode lies on a continuous branch which exists for ellipses of varying aspect ratio. The branch of modes begins with a trapped mode for a flat plate parallel to the wall of the guide and ends with a standing wave for a flat plate perpendicular to the walls of the guide. In addition they also found that the mode found by Evans & Porter[14] is a point on a branch of modes which exists for hypercircles, that is, obstacles with shape $|x/a|^\nu + |y/a|^\nu = 1$, $-a \leq x \leq a$, where $x$ is measured along the guide, $y$ across the guide, and $a, \nu$ vary along the branch.
In a following paper, McIver, Linton, & Zhang[64] further investigated the existence of branches of embedded trapped modes in the vicinity of symmetric obstacles that are placed on the centre line of a two-dimensional acoustic wave guide. In their paper they chose the obstacle to be a rectangular block of length $2a$, and width $2b$. Modes are sought which are anti-symmetric about the centreline of the guide and the block, and which have frequencies that are above the first antisymmetric cut-off frequency and below the second antisymmetric cut-off frequency. The evidence that a two dimensional wave guide with an obstacle on its centre line can support embedded trapped modes indicates that embedded trapped modes may also be found in the vicinity of an indentation in a two-dimensional acoustic waveguide. This motivates further investigation of embedded trapped modes near an indentation in two dimensional wave guides.

Section 2 of this chapter is devoted to the discussion of embedded trapped modes which are symmetric about the centreline of the wave guide. In section 5.3, embedded trapped modes which are anti-symmetric about the centreline of the wave guide are investigated. In each case, a crude approximation and a full solution with a Galerkin approach are presented.

A two-dimensional wave guide with a rectangular indentation on the wall may be an acoustic wave guide or a channel of uniform water depth in which the indentation starts from the bottom and extends through the depth. In both cases, a velocity potential may be defined which satisfies the Helmholtz equation in a fluid and proper boundary conditions (In the water wave problem, the depth dependence must be factored out.).

An infinite waveguide of width $2d$ has an indentation of length $2a$ and width $2b$, $b > d$. A Cartesian coordinate system is chosen in such a way that the origin of the system is at the centre of the indentation, $x$ is along the the centreline of the wave guide, and $y$ is perpendicular to the centreline of the wave guide. A quarter of the geometric configuration is shown in Figure 5.1,
Figure 5.1: Definition sketch of a two-dimensional wave guide with a rectangular indentation
which is divided into two regions, region I, $0 < x < a$, $0 < y < b$, and region II, $a < x < \infty$, $0 < y < b$. We will look for trapped modes at frequencies that are above the first cut-off and below the second cut-off, symmetric or anti-symmetric about the centreline of the waveguide.

The acoustic wave potential $\phi$ satisfies the Helmholtz equation
\[
(\nabla + k^2)\phi = 0
\]
(5.1)
in the fluid, where $k = \omega/c$ and $\omega$ is the angular frequency, and $c$ the speed of sound. There is assumed to be no mean flow. The boundary conditions are given by
\[
\frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = b, \quad 0 \leq x \leq a,
\]
(5.2)
\[
\frac{\partial \phi}{\partial x} = 0 \quad \text{on} \quad x = a, \quad d \leq y \leq b,
\]
(5.3)
and
\[
\frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = d, \quad a \leq x < \infty.
\]
(5.4)
The far field condition is given by
\[
\phi \to 0, \quad x \to \pm \infty, \quad 0 \leq y \leq d.
\]
(5.5)
The conditions on $x = 0$ or $y = 0$ depend on whether the modes are symmetric or antisymmetric in $x$ and $y$. Suppose that the motion is symmetric about the centreline and $kd < \pi$ but $\pi < kb < 2\pi$. If the indentation is long then the trapped mode must represent waves incident from region I which are totally reflected at $x = a$. This is physically plausible because the range of $kd$ and $kb$ means that there are two possible types of progressive waves in region I, but only one in region II. Thus a wave $e^{ikx}$ incident from region I produces a transmitted wave $T_1 e^{ikx}$ and a wave $e^{iax} \cos \pi y/b$, $a^2 + \pi^2/b^2 = k^2$ produces a transmitted wave $T_2 e^{ikx}$, and so a suitable combination of these waves produce no transmission.

We seek non-trivial solutions of (5.1)-(5.5) for certain discrete values of $kd$ corresponding to trapped modes.
5.2 Embedded trapped modes symmetric about the centreline of the waveguide

The acoustic potential $\phi$ can be either symmetric or anti-symmetric about the plane $x = 0$ in region I. First assuming that the potential in the indentation is symmetric about the centreline of the indentation in both $x$ and $y$ directions, the potential in region I may be written as

$$\phi_I = \sum_{n=0}^{\infty} U_n I \cosh(k_n x) h_n I(y), \quad k_n = (\lambda_n^2 - k^2)^{1/2},$$  \hspace{1cm} (5.6)

where $U_n I$ are unknown coefficients and $h_n I(y)$ are a complete orthonormal set of functions, given by

$$h_n I(y) = b^{-1/2} \varepsilon_n \cos \lambda_n y, \quad \lambda_n = \frac{n\pi}{b}, \quad n \geq 0,$$  \hspace{1cm} (5.7)

$$\varepsilon_0 = 1; \quad \varepsilon_n = 2^{1/2}, \quad n \geq 1,$$  \hspace{1cm} (5.8)

and

$$\int_0^b h_m I(y) h_n I(y) dy = \delta_{mn}, \quad m, n = 0, 1, 2, \ldots.$$  \hspace{1cm} (5.9)

From (5.6), $k_0 = -ik$ is imaginary, and $k_1 = \left(\left(\frac{\pi}{b}\right)^2 - k^2\right)^{1/2}$ is also imaginary for $\frac{\pi}{b} < k < \frac{\pi}{d}$. We write

$$k_1 = ik' = i \left( k^2 - \lambda_1^2 \right)^{1/2}.$$  \hspace{1cm} (5.10)

It is interesting to note that $b/d = 1, \quad kd = \pi$ corresponds to the following standing waves

$$\phi = \cos \left( \frac{y}{b} \pi \right), \quad \frac{b}{d} = 1, \quad kd = \pi,$$  \hspace{1cm} (5.11)

which satisfy the boundary conditions (5.2) - (5.4). The potential in region II may be expressed as

$$\phi_{II} = \sum_{n=1}^{\infty} U_n II e^{-j_n(x-a)} j_n h_n II(y), \quad j_n = (\mu_n^2 - k^2)^{1/2},$$  \hspace{1cm} (5.12)
where $U_n^{II}$ are unknown coefficients and $h_n^{II}(y)$ are a complete orthonormal set of functions, given by

$$h_n^{II}(y) = d^{-1/2} \varepsilon_n \cos \mu_n y, \quad \mu_n = \frac{n\pi}{d}, \quad n \geq 0,$$

$$\varepsilon_0 = 1; \quad \varepsilon_n = 2^{1/2}, \quad n \geq 1,$$

and

$$\int_0^b h_m^{II}(y)h_n^{II}(y)dy = \delta_{mn}, \quad m, n = 0, 1, 2, \ldots$$

Normally, the sum in (5.12) should start from $n = 0$. However, the term with $n = 0$ represents a progressing wave because $j_0 = -ik$ is imaginary, which should be forced to disappear for an embedded trapped mode problem. As a result, the sum of (5.12) starts from $n = 1$.

Note that we need two wave-like terms in region I and no wave like terms in region II. This means that $k_0, k_1$ should be imaginary, and $k_n$ real for $n \geq 2$ while $j_n$ real for $n \geq 1$. It follows that the non-dimensional indentation depth $b/d$ must be in the range of $1 < b/d < 2$ if we restrict $\pi/b < k < \pi/d$.

If $b/d$ is greater than 2, for example, $b/d$ is in the range of $2 < b/d < 3$, then we have $k_2 = \sqrt{4\pi^2/b^2 - k^2}$, which is also imaginary when $\pi/b < k < \pi/d$. This means that there are three waves in the inner region. For simplicity, we only look for the case for which there are only two waves in the inner region.

5.2.1 Formulation

Continuity of the potential and its derivatives across $x = a$ gives

$$\sum_{n=0}^{\infty} \frac{U_n^{II}}{k_n} h_n^{II}(y) = \sum_{n=1}^{\infty} \frac{1}{-j_n} h_n^{II}(y), \quad 0 \leq y \leq d,$$

$$\sum_{n=0}^{\infty} U_n^{I} \tanh(k_n a) h_n^{I}(y) = \left\{ \begin{array}{ll}
\sum_{n=1}^{\infty} U_n^{II} h_n^{II}(y), & 0 \leq y \leq d, \\
0, & d \leq y \leq b.
\end{array} \right.$$
Multiplication of both sides of (5.16) by \( h_m^I(y) \), \( m = 0, 1, 2, \ldots \) and integration over \([0, d]\) gives

\[
\sum_{n=0}^{\infty} \frac{U_n^I}{k_n} c_{nm} = \begin{cases} 
-\frac{u_{nm}^I}{k_n}, & m \geq 1, \\
0, & m = 0,
\end{cases}
\]  
(5.18)

where

\[ c_{nm} = \int_0^d h_n^L(y)h_m^I(y)dy = \frac{(-1)^m \varepsilon_m \varepsilon_n \lambda_n \sin(\lambda_n d)}{(bd)^{1/2}(\lambda_n^2 - \mu_m^2)}. \]  
(5.19)

Multiplication of both sides of (5.17) by \( h_m^I(y) \), \( m = 0, 1, 2, \ldots \) and integration over \([0, d]\) gives

\[ \tanh(k_m a) U_m^I = \sum_{n=1}^{\infty} U_n^H c_{mn}, \quad m \geq 0. \]  
(5.20)

Note that (5.19) is not valid if \( \lambda_n = \mu_m \). For any given value of \( b/d \) as a rational number, there are always \( \lambda_n = \mu_m \) which leads to the invalidity of (5.19). It is not difficult to show that

\[ \int_0^d h_n^L(y)h_m^I(y)dy = \left( \frac{d}{b} \right)^{1/2}, \quad \lambda_n = \mu_m. \]  
(5.21)

From (5.19), \( c_{01} = 0 \), then equation (5.20) reduces to

\[ \tanh(k_m a) U_m^I = 0. \]  
(5.22)

Either \( U_0^I = 0 \) or \( \tanh(k_m a) = 0 \) satisfies the above equation. However, only a trivial solution of equations (5.18), (5.20) exists if \( U_0^I = 0 \). Hence the non-trivial solution of (5.18), (5.20) requires \( U_0^I \neq 0 \). The only choice for the non-trivial solution to exist is

\[ \tan(ka) = 0, \]  
(5.23)

since \( \tanh(k_m a) = i \tan(ka) \). From this we have

\[ ka = n\pi, \quad n = 1, 2, 3, \ldots \]  
(5.24)

This is the first condition which has to be satisfied for embedded trapped modes to exist. It will be called the side condition for convenience.
5.2.2 Approximate solution

Following McIver, Linton & Zhang\cite{64}, a crude approximate solution may be made by truncating the series at two terms in region I and one term in region II. From (5.18) and (5.20) we have

\[
\begin{align*}
\frac{U_0^I}{k_0} c_{00} + \frac{U_1^I}{k_1} c_{10} &= 0, \\
\frac{U_0^I}{k_0} c_{01} + \frac{U_1^I}{k_1} c_{11} + \frac{U_1^{II}}{j_1} &= 0, \\
U_0^I \tanh(k_0 a) - U_1^{II} c_{01} &= 0,
\end{align*}
\]

and

\[
U_1^I \tanh(k_1 a) - U_1^{II} c_{11} = 0.
\]

Equation (5.27) is satisfied by the side condition (5.24) and so (5.25) - (5.28) reduces to the matrix equation

\[
AX = 0
\]

where

\[
A = \begin{pmatrix}
c_{00} & c_{01} & 0 \\
l_0 & c_{11} & \frac{1}{j_1} \\
0 & \tanh(k_1 a) & -c_{11}
\end{pmatrix}, \quad X = \begin{pmatrix}
U_0^I \\
U_1^I \\
U_1^{II}
\end{pmatrix}.
\]

The condition for a non-trivial solution of (5.29) to exist demands that

\[
\det |A| = 0
\]

Hence it follows that

\[
- \frac{c_{00} c_{11}}{k_0 k_1} - \frac{c_{00} \tanh(k_1 a)}{k_0 j_1} + \frac{c_{01} c_{10} c_{11}}{k_0 k_1} = 0.
\]

After some manipulations, we have

\[
\tan(k'a) = \frac{4(b/d) \sin^2(\pi d/b) j_1 d}{\pi^2 (1 - (b/d)^2)^2 k'd'}.
\]
This is another condition that embedded trapped modes must satisfy. Equation (5.33) is an equation for the non-dimensional parameters $kd$, $ka$, and $b/d$. The value of $kd$ may be found under different side conditions, i.e. $ka = n\pi$, $n = 1, 2, ...$ for a given $1 < b/d < 2$ if $d\pi/b < kd < \pi$. Therefore, trapped modes exist only for some specific geometry which satisfies both (5.33) and (5.24) for a given $1 < b/d < 2$. Equation (5.33) might have more than one solution when $n > 1$. Since $k'a$ is real and positive when $1 < b/d < 2$ and $\pi/b < k < \pi/d$, real solutions may be found from equation (5.33). Since the function $\tan(k'a)$ is a periodic function with periodicity $\pi$, then for a given $ka = n\pi, n > 1$, there may exist more than one solution.

Actually, from equation (5.33), $\tan(k'a)$ is positive since the right hand side of (5.33) is positive. It follows that

$$m\pi < k'a < \left(m + \frac{1}{2}\right)\pi, \ m = 0, 1, 2, ....$$

(5.34)

Combining this with the side condition ($ka = n\pi$), we have

$$\frac{\pi}{b/d (n^2 - m^2)^{1/2}} < kd < \frac{\pi}{b/d (n^2 - (m + 1/2)^2)^{1/2}}.$$ 

(5.35)

and

$$\frac{b}{d} (n^2 - m^2)^{1/2} < \frac{a}{d} < \frac{b}{d} (n^2 - (m + 1/2)^2)^{1/2}$$

(5.36)

where \{\(m < n; \ n = 1, 2, ..., \ m = 0, 1, 2, ..., n - 1\) since $k'a < ka$. From this relation, we know that, as the length of the indentation increases (the increase of $n$), the lowest mode frequency tends to be $\frac{\pi}{b/d}$ by taking the limit of $n \to \infty$ in (5.35).

For a given value of $1 < b/d < 2$ and $ka = n\pi$ with $n \geq 1$, the maximum number of possible embedded trapped modes is $n$. We denote any one of the modes as $(n, m)$ in which $n$ comes from $ka = n\pi$, and $m = 0, 1, 2, ..., n - 1$. Thus $(n, 0)$ represents the lowest mode and $(n, n - 1)$ the highest mode for a given $1 < b/d < 2$. 

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5.2.3 Full solution

From (5.18), if \( m = 0 \), we have \( \tanh(k_0 a)U^I_0 = 0 \) for \( c_0 = 0 \). A non-trivial solution requires \( U^I_0 \neq 0 \). Therefore the first condition for the full solution to exist is

\[
\tan(ka) = 0, \quad ka = n\pi, \quad n = 1, 2, 3, ... , \tag{5.37}
\]

which is the same as the approximate solution. It follows that (5.20) becomes

\[
\tanh(k_m a)U^I_m = \sum_{n=1}^{\infty} U^I_n c_{mn}, \quad m \geq 1 \tag{5.38}
\]

for \( m \geq 1 \), where \( m \) starts from 1 because both sides of (5.20) are identically zero when \( m = 0 \). The unknowns to be solved for are firstly \( \{U^I_n, n \geq 1\} \), and then \( U^I_0 \) will be determined. Actually, from (5.18), if \( m = 0 \), we have

\[
\sum_{n=0}^{\infty} \frac{U^I_n}{k_n} c_{n0} = 0. \tag{5.39}
\]

This equation may be used to calculate \( U^I_0 \) once \( \{U^I_n, n \geq 1\} \) have been found. Substitution of (5.18) into (5.20) gives

\[
U^I_m + \sum_{n=1}^{\infty} A_{mn} U^I_n = 0, \quad m \geq 1, \tag{5.40}
\]

where

\[
A_{mn} = \coth(k_m a) \sum_{r=1}^{\infty} \frac{j_r}{k_n} c_{nr} c_{mr}, \quad m, r \geq 1. \tag{5.41}
\]

From the definitions in (5.6), (5.7), (5.12) and (5.13), \( k_m a \) and \( j_r/k_n \) may be expressed as

\[
k_m a = ka \left( \frac{m\pi}{kb} \right)^2 - 1 \right)^{1/2}, \quad \frac{j_r}{k_n} = \frac{b}{d} \left( \frac{(r\pi)^2 - (kd)^2}{(n\pi)^2 - (kb)^2} \right)^{1/2}. \tag{5.42}
\]

Equation (5.40) is an infinite system of homogeneous equations which is dependent on non-dimensional parameters \( ka, kd \), and \( b/d \), since \( kb = kd \cdot b/d \). Equation (5.40) is an infinite system of homogeneous equations. The non-trivial solutions of (5.40) correspond to its determinant being zero. For a
given value of \( b/d \), the non-dimensional parameters \( ka \) and \( kd \) may be determined from the side condition (5.24) and equation (5.40). However, the convergence is slow since \( A_{mn} \propto 1/n\pi \), and it is not easy to find the zeros of the determinant of a large matrix.

Alternatively, substitution of (5.20) into (5.18) yields

\[
U''_m + \sum_{n=1}^{\infty} B_{mn} U''_n = 0
\]  

(5.43)

where

\[
B_{mn} = j_m \sum_{r=1}^{\infty} \coth(k_r a) k_r^{-1} c_r^2.
\]  

(5.44)

Let \( X_n = U'_n k_{n}^{-1} d^{-1} \), then we have

\[
X_m + \frac{\coth(k_m a)}{dk_m} \sum_{n=1}^{\infty} A_{mn}(k_n d) X_n = 0.
\]  

(5.45)

This can be written as

\[
X_m + f_m \sum_{n=1}^{\infty} g_{mn} X_n = 0.
\]  

(5.46)

Thus let \( X_n = -X_1 f_1^{-1} Y_n \), then rearrangement of (5.46) gives

\[
\sum_{n=1}^{\infty} K_{mn} Y_n = \delta_{m1}, \quad m = 1, 2, ...
\]  

(5.47)

where

\[
f_r = \frac{\coth(k_r a)}{k_r d},
\]  

(5.48)

\[
g_{mn} = A_{mn}(k_n d) = \sum_{r=1}^{\infty} (j_r d) c_{nr} c_{mr},
\]  

(5.49)

\[
K_{mn} = \sum_{r=2}^{\infty} f_r^{-1} \delta_{rm} \delta_{rn} + g_{mn},
\]  

(5.50)

and

\[
Y_1 = -f_1 = \frac{1}{k'd \tan(k'a)}.
\]  

(5.51)

Hence the homogeneous system (5.40) is converted into an inhomogeneous system (5.47). If we can show that \( K_{mn} \) is positive definite, in combination
with the side condition (5.37), equation (5.47) can be solved to obtain $Y_1$ with $\pi/b < kd < \pi/d$, and (5.51) may be then used to solve for the trapped modes. Evans & Linton[25] demonstrated that $K_{mn}$ is positive definite, so the solution of (5.47) is guaranteed. However, the convergence is still a problem if we solve (5.47) directly since the velocity singularity at the corner of the indentation has not been taken into account. In the following section, a Galerkin method will be adopted to solve this problem, in which the singularity at the corner of the indentation is considered.

**Galerkin method**

Here we follow the method described by Evans & Fernyhough[16]. The common boundary of the indentation and wave guide perpendicular to axis $x$ is denoted by $\{L : x = a, 0 \leq y \leq d\}$. From (5.17), we write the velocity on the boundary as

$$U(y) = \sum_{n=0}^{\infty} U_n^I \tanh(k_n a) h_n^I(y) = \begin{cases} \sum_{n=1}^{\infty} U_n^{II} h_n^{II}(y) & 0 \leq y \leq d \\ 0 & d \leq y \leq b \end{cases}.$$  

(5.52)

It follows that

$$U_n^I = \coth(k_n a) \int_L U(y) h_n^I(y) dy, \quad n = 1, 2, 3, \ldots,$$  

(5.53)

and

$$U_n^{II} = \int_L U(y) h_n^{II}(y) dy, \quad n = 1, 2, 3, \ldots,$$  

(5.54)

as $\tanh(k_0 a) = 0$. Substitution of $U_n^I, U_n^{II}$ into (5.16) gives

$$\int_L U(y') \left\{ \sum_{n=1}^{\infty} \frac{\coth(k_n a)}{k_n} h_n^I(y) h_n^I(y') + \sum_{n=1}^{\infty} j_n^{-1} h_n^{II}(y) h_n^{II}(y') \right\} dy' = 0.$$  

(5.55)

This is a homogeneous integral equation for $U(y)$. The oscillatory first term in the first sum is shifted to the right hand side to give

$$\int_L U(y') \left\{ \sum_{n=2}^{\infty} \frac{\coth(k_n a)}{k_n} h_n^I(y) h_n^I(y') + \sum_{n=1}^{\infty} \frac{h_n^{II}(y) h_n^{II}(y')}{j_n d} \right\} dy'$$

$$= \frac{\cot(k' a)}{k' d} h_1^I(y) \int_L U(y') h_1^I(y') dy'.$$  

(5.56)
If we define
\[
U(y) = \frac{\cot(k'a)}{k'd} U_1 u(y),
\]
then we have
\[
\int_L u(y') K(y, y') dy' = h'_1(y), \quad y \in L
\]
(5.57)
\[
\text{where } U_1 = \int_L U(y) h'_1(y) dy, \text{ and}
\]
\[
K(y, y') = \sum_{n=2}^{\infty} \frac{\coth(kn'a) h^I_n(y) h^I_n(y')}{kn'd} + \sum_{n=1}^{\infty} \frac{h^II_n(y) h^II_n(y')}{jn'd}. \quad (5.58)
\]
Equation (5.57) is multiplied by \( h'_1(y) \) and integrated over \( L \) to give
\[
\int_L u(y) h'_1(y) dy = k'd \tan(k'a). \quad (5.59)
\]
Consequently, the problem has been reduced to first solving (5.58) for \( u(y) \), for a given set of geometric parameters, and then looking for a trapped mode frequency which can be sustained by the given geometry by solving (5.60). We shall adopt a Galerkin approach for the solution of equations (5.58), and (5.60). We first write in the operator form
\[
\mathcal{R} u = h'_1, \quad (5.60)
\]
with
\[
(u, h'_1) = \int_L u(y) h'_1(y) dy = A \equiv k'd \tan(k'a). \quad (5.61)
\]
Rather than solve (5.58) directly, the Galerkin method seeks an approximation \( u \approx U \) such that
\[
(U, \mathcal{R}U) = (U, h'_1). \quad (5.62)
\]
Hence the approximation of \( A \) is
\[
\overline{A} = (U, h'_1). \quad (5.63)
\]
We choose
\[
u(y) = \sum_{n=1}^{N} a_n v_n(y) \quad (5.64)
\]
for some $u_n(y)$ and unknown $a_n$, substitute $u_n(y)$ into (5.61), multiply by $u_m(y)$ and integrate over $L$ to give

$$\sum_{n=1}^{N} K_{mn} a_n = F_{m1}, \ m = 1, 1, 2, ... \quad (5.66)$$

where

$$K_{mn} = (\Re u_n, u_m), \quad F_{m1} = (h_{1}^{I}, u_m), \quad (5.67)$$

thus it follows that

$$\overline{\Lambda} = \sum_{n=1}^{N} a_n F_{n1}. \quad (5.68)$$

If (5.59) is used in (5.66) we have

$$K_{mn} = \sum_{r=2}^{\infty} d^{-1} k_r^{-1} \coth(k_r a) F_{mr} F_{nr} + \sum_{r=1}^{\infty} d^{-1} j_r^{-1} G_{mr} G_{nr}, \quad (5.69)$$

where

$$F_{mn} = (h_{n}^{I}, u_m) = \int_{0}^{d} h_{n}^{I}(y) u_m(y) dy, \quad (5.70)$$

and

$$G_{mn} = (h_{n}^{I}, u_m) = \int_{0}^{d} h_{n}^{II}(y) u_m(y) dy. \quad (5.71)$$

The equation which is used to calculate the trapped mode frequencies may be written as

$$\tan(k'a) = \overline{\Lambda}/(k'd). \quad (5.72)$$

**Basis functions for the Galerkin method**

At this stage, we need to choose a suitable set functions of $u_n(y)$. The choice of $u_n(y)$ is directed by the requirements of the correct physical behavior and the simplicity of the final forms, see Porter & Evans[91]. Since $u(y)$ is proportional to the acoustic velocity in the wave guide, we might expect that at $y = d$, $u(y)(d - y)^{-1/3}$ is bounded (which can be derived by a simple conformal mapping argument).
In order to preserve simple forms for $F_{mn}, G_{mn}$ and hence $K_{mn}$, we choose

$$u_n(y) = \frac{2n! \Gamma(1/6)b^{2/6}}{(-1)^n \sqrt{2\pi} \Gamma(2n + 1/3)(d^2 - y^2)^{1/3}} C_n^{1/6} \left( \frac{y}{d} \right),$$  \hspace{1cm} (5.73)

where

$$C_n^\nu (\cos \theta) = \sum_{r=0}^{n} \frac{\Gamma(\nu + r) \Gamma(\nu + n - r)}{r!(n - r)! |\Gamma(\nu)|^2} \cos(n - 2r)\theta$$  \hspace{1cm} (5.74)

are the ultra-spherical Gegenbauer polynomials and $\Gamma(\nu)$ is the Gamma function. The $C_n^\nu (y)$ have the following properties

$$C_n^\nu (-y) = (-1)^n C_n^\nu (y),$$  \hspace{1cm} (5.75)

$$\int_0^1 (1 - t^2)^{-1/3} C_{2n}^{1/6} (t) \cos yt = (-1)^n P_{2n}(y),$$  \hspace{1cm} (5.76)

and

$$\int_0^1 (1 - t^2)^{-1/3} C_{2n+1}^{1/6} (t) \sin yt = (-1)^n P_{2n+1}(y)$$  \hspace{1cm} (5.77)

where

$$P_n(y) = \frac{\pi \Gamma(n + 1/3) J_{n+1/6}(y)}{n! \Gamma(1/6)(2y)^{1/6}},$$  \hspace{1cm} (5.78)

and $J_\nu(y)$ is a Bessel function. After some algebra, it can be shown that

$$F_{m1} = \int_0^d h_1^I u_m(y)dy = \frac{J_{2m+1/6}(n\pi/b)}{(2\pi d/b)^{1/6}},$$  \hspace{1cm} (5.79)

$$F_{mn} = \int_0^d h_\nu^I u_m(y)dy = \frac{J_{2m+1/6}(n\pi/d)}{(2n\pi d/b)^{1/6}},$$  \hspace{1cm} (5.80)

$$G_{mn} = \int_0^d h_\nu^I u_m(y)dy = \left( \frac{b}{d} \right)^{1/2} \frac{J_{2m+1/6}(n\pi)}{(2n\pi)^{1/6}},$$  \hspace{1cm} (5.81)

and

$$K_{mn} = \sum_{r=2}^{\infty} \frac{J_{2m+1/6}(r\pi d/b) J_{2n+1/6}(r\pi d/b)}{k_r d \tanh(k_r a)(2r\pi d/b)^{1/3}} + \sum_{r=1}^{\infty} \frac{J_{2m+1/6}(r\pi) J_{2n+1/6}(r\pi)}{d/b_r d(2r\pi)^{1/3}}.$$

$$\hspace{1cm} (5.82)$$
5.2.4 Potential anti-symmetric about the plane $x = 0$

When the velocity potential in region I is anti-symmetric about the plane $x = 0$, the $\cosh(k_n x)/\cosh(k_n a)$ in the potential expansion in region I is replaced by $\sinh(k_n x)/\sinh(k_n a)$. By a process similar to the previous section we may obtain the side condition, approximate and full solutions. The side condition is given by

$$\cot(ka) = 0, \quad ka = \left(n - \frac{1}{2}\right)\pi, \quad n = 1, 2, ..., \quad (5.83)$$

and the approximate solution may be written as

$$\tan(k'a) = -\frac{\pi^2(1 - (b/d)^2)^2}{4(b/d)^2} \frac{k'd}{\sin^2(\pi d/b) f_1d}. \quad (5.84)$$

The procedure of finding the full solution is the same as that discussed above. The difference lies in the $K_{mn}$ in (5.82), in which $\coth$ is replaced by $\tanh$. The final equation that is used to solve for trapped modes for the full solution is

$$\tan(k'a) = -k'd/A. \quad (5.85)$$

Again we can find the range and limiting value of trapped mode frequencies. Since the right hand side of (5.84) is negative, $\tan(k'a)$ should be negative, otherwise there are no real solutions. It follows that

$$\left(n - \frac{1}{2}\right)\pi < k'a < m\pi, \quad m = 1, 2, .... \quad (5.86)$$

Combining with the side condition $ka = (n - 1/2)\pi$, we have

$$\frac{\pi}{b/d} \frac{n - 1/2}{((n - 1/2)^2 - (m - 1/2)^2)^{1/2}} < k_d < \frac{\pi}{b/d} \frac{n - 1/2}{((n - 1/2)^2 - (m)^2)^{1/2}}, \quad (5.87)$$

and

$$\frac{b}{d}((n - 1/2)^2 - (m - 1/2)^2)^{1/2} < \frac{a}{d} < \frac{b}{d}((n - 1/2)^2 - m^2)^{1/2} \quad (5.88)$$

where $\{m < n; \quad m = 1, 2, ..., n - 1; \quad n = 1, 2, ...\}$. From this relation, we know that, as the length of the indentation increases (which correspond to
increasing \( m \), the lowest mode tends to \( \frac{\pi}{b/d} \). Trapped modes tends to the lower limits when the indentation length increases. However, there will be no trapped modes exist if \( ka = \pi/2 \) since the condition (5.87) does not hold for \( n = 1 \) and \( m = 0 \). The shortest length of indentation to support modes symmetric in \( y \) and antisymmetric in \( x \) is \( ka = \frac{3}{2}\pi \). One should notice that, in contrast to the number of possible modes symmetric in \( x \), the maximum number of modes anti-symmetric in \( x \) is \( n - 1 \), for given \( ka = (n - 1/2)\pi \).

5.2.5 Numerical results and discussion

We computed the embedded trapped modes symmetric about the centreline of the waveguide, and symmetric or anti-symmetric in \( x \) direction numerically by the approximate and full solution described above. In each case, the approximate solutions are used as the initial guess for the full solutions.

**Modes symmetric about the centreline of the waveguide and the plane \( x = 0 \)**

Figures (5.2) and (5.3) show the numerical results of approximate and full solutions for \( ka = 4\pi \), i.e, \( n = 4 \), in which \((4, 0)\) represents the lowest mode and \((4, 3)\) the highest mode. In Figure (5.2), \( kd \) is plotted against \( b/d \), in which lines represent the approximate solution, whilst the dots represent the full solutions. From this figure we see that the solution tends to a standing wave solution when \( b/d \to 1 \), and for a given length of indentation \((a/d)\), the number of trapped modes increase as \( b/d \) increase. The maximum number of trapped modes for \( ka = n\pi \) is \( n \). These results agree with our discussion in the previous section. Another feature of this figure is that all trapped modes start from \( \pi \), which is the second cut-off, and decrease as \( b/d \) increases. In Figure (5.3), \( a/d \) is plotted against \( b/d \), where \( a/d \) increases as \( b/d \) increases. An interesting feature of this graph is that all \( a/d \) start from almost the same value which is 4. This can be derived from the approximate solution.
Figure 5.2: Approximate and full solutions for modes symmetric about the centreline of the waveguide and the plane $x = 0$, $ka = 4\pi$, variation of $kd$ with $b/d$
Figure 5.3: Approximate and full solutions for modes symmetric about the centreline of the waveguide and the plane $x = 0$, $ka = 4\pi$, variation of $a/d$ with $b/d$. 
Actually, from (5.35) and (5.36) we may find that the lower bound of $a/d$ is $n$, i.e. $a/d > n$.

We can also see from Figures (5.2) and (5.3) that the approximate solutions are very close to the full solutions. The application of Gegenbauer polynomials in the Galerkin method make the infinite system converge very quickly. In real calculations, truncating the number of $m$ and $n$ in (5.82) from 4 to 8 yields the accuracy of four digits.

There will be more than two wave-like terms in region I if $b/d > 2$ for $\pi/b < k < \pi/d$. All oscillatory terms in the first sum in (5.55) should be moved to the right hand side. The proper mathematical treatment will result in a system of linear equations for which the condition for a non-trivial solution may be found is that the determinant of the matrix is zero. However, it is hard to find a simple approximate solution in this case.

**Modes symmetric about the centreline of the guide and anti-symmetric about the plane $x=0$**

Figures (5.4) and (5.5) show the numerical results of the approximate and full solutions of modes symmetric about the centreline of the waveguide and anti-symmetric about the plane $x=0$ for $ka = (4 - 1/2)\pi$, i.e. $n = 4$, in which $(4,0)$ represents the lowest mode and $(4,3)$ the highest mode. In Figure (5.4), $kd$ is plotted against $b/d$ where the lines represent the approximate solution, and the dots represent the full solutions. The maximum number of trapped modes for $ka = n\pi$ is $n-1$. Another feature of this figure is that all trapped modes start from $\pi$, which is the second cut-off, and decrease as $b/d$ increases. In Figure (5.5), $a/d$ is plotted against $b/d$, where $a/d$ increases as $b/d$ increases. An interesting feature for this graph is that all $a/d$ start from almost the same value which is 3.5. This can be derived from the approximate solution. Actually, from (5.87) and (5.88) we may find that the lower bound of $a/d$ is $n - 1/2$, i.e. $a/d > (n - 1/2)$. These results also show
Figure 5.4: Approximate and full solution for modes symmetric about the centreline of the waveguide and anti-symmetric about plane $x = 0$, $ka = 4\pi$, variation of $kd$ with $b/d$
Figure 5.5: Approximate and full solutions for modes symmetric about the centreline of the waveguide and anti-symmetric about the plane \( x = 0, \) \( k\alpha = 4\pi, \) variation of \( a/d \) with \( b/d.\)
a good agreement of the approximate and full solutions. From Figure (5.4) we also know that there exist a critical value of $b/d$, above which modes anti-symmetric about the plane $x = 0$ may be found. The range of this value can be obtained from (5.87) by letting $n = 4$ and $m = 1$, which is $7\sqrt{3}/12 < b/d < 7\sqrt{5}/15$. Actually we can find the approximate value of $b/d$ above which an extra mode emerges through (5.87) by substituting the correspondent values of $n$ and $m$ for $1 < b/d < 2$.

5.3 Embedded trapped modes anti-symmetric about the centreline of the waveguide

5.3.1 Introduction

If we impose $\phi = 0$ on $y = 0$ then the potentials in both region I and II are anti-symmetric about the centreline of the waveguide. In this case the eigenfunctions in region I and II become $H_n^I(y)$, $H_n^{II}(y)$, and they may be written as

$$H_n^I(y) = (2/b)^{1/2} \sin(f_n y), \quad f_n = \frac{(n - 1/2)\pi}{b}, \quad n \geq 1, \quad (5.89)$$

and

$$H_n^{II}(y) = (2/d)^{1/2} \sin(g_n y), \quad g_n = \frac{(n - 1/2)\pi}{d}, \quad n \geq 1 \quad (5.90)$$

where $H_n^I(y)$, $H_n^{II}(y)$ form the complete orthonormal set of functions.

The potential in region I may also be symmetric or anti-symmetric about the plane $x = 0$. Here we will firstly deal with the potential which is anti-symmetric about the plane $x = 0$ since the other case may be easily derived thereafter. The potentials in region I and II may be written separately as

$$\phi_I = \sum_{n=1}^{\infty} U_n \frac{\sinh(k_n x)}{k_n \cosh(k_n a)} H_n^I(y), \quad k_n = (\beta_n^2 - k^2)^{1/2}, \quad (5.91)$$
and

\[ \phi_{II} = \sum_{n=2}^{\infty} U_{n}^{II} e^{-j_n(x-\sigma)} H_{n}^{II}(y), \quad j_n = (\gamma_n^2 - k^2)^{1/2} \]  

(5.92)

where \( U_{n}^{I} \), \( U_{n}^{II} \) are unknown coefficients as before, in which \( U_{n}^{II} \) starts from \( n = 2 \) since we look for the embedded modes, which means that \( j_1 \) is imaginary and \( j_2 \) is real. This requires

\[ \frac{\pi}{2} < kd < \frac{3\pi}{2}. \]  

(5.93)

Evans & Linton[25] have investigated trapped modes near an indentation in an open channel in water waves. In their discussion, there exists only one wave-like term in the inner region. This corresponds to the case in which \( k_1 \) is purely imaginary, and \( k_2 \), and \( j_1 \) are real. Therefore in their investigation, \( \frac{\pi}{2d} < k < \frac{\pi}{2d} \), \( 1 < \frac{b}{d} < 3 \). But we look for \( \frac{\pi}{2d} < k < \frac{3\pi}{2d} \), which is the embedded trapped mode problem in which \( j_1 \) is purely imaginary whilst \( j_2 \) real. However there must be at least two-wave like terms in the inner region for embedded trapped modes to exist. The reason for this argument is the same as that in the previous section. It follows that \( k_1 \), \( k_2 \) must be purely imaginary. For simplicity, we keep only two wave-like terms in the indentation, then \( k_1, k_2 \) are imaginary and \( k_3 \) real. The requirement of the third term to be real in region \( I \) yields

\[ \left( \frac{(3 - 1/2)\pi}{b} \right)^2 - k^2 > 0, \]  

(5.94)

which gives \( kb < \frac{5\pi}{2} \). Thus it follows that \( b/d < 5/3 \) since \( \frac{\pi}{2d} < k < \frac{3\pi}{2d} \) ensure that the third term is real.

The second term could be real or imaginary for the given range of \( \frac{\pi}{2d} < k < \frac{3\pi}{2d} \) and \( 1 < b/d < 5/3 \). Actually, \( k_2d = \left( \left( \frac{3\pi}{2bd} \right)^2 - (kd)^2 \right)^{1/2} \) is real if \( \frac{\pi}{2} < kd < \frac{3\pi}{2bd} \), and imaginary if \( \frac{3\pi}{2bd} < kd < \frac{3\pi}{2} \). Thus the frequency space is divided into two parts. There is only one wave-like term if

\[ \frac{\pi}{2} < kd < \frac{3\pi}{2bd}, \quad 1 < b/d < 5/3, \]  

(5.95)
in which case there are no embedded trapped mode solutions. Embedded trapped modes can only be found if

\[
\frac{3\pi}{2b/d} < kd < \frac{3\pi}{2}
\]

(5.96)

for given value of \( b/d \) in the range of \( 1 < b/d < 5/3 \). Therefore we only look for the embedded trapped mode solutions for \( \frac{3\pi}{2b/d} < kd < \frac{3\pi}{2} \), and \( 1 < b/d < \frac{5}{3} \). For imaginary \( k_1 \) and \( k_2 \), they may be written as \( k_1 = -ik', k_2 = -ik'' \) where \( k' = (k^2 - \beta^2)^{1/2} \), \( k'' = (k^2 - \beta'^2)^{1/2} \) for \( 1 < b/d < \frac{5}{3} \) and \( \frac{3\pi}{2b/d} < kd < \frac{3\pi}{2} \).

### 5.3.2 Formulation

Continuity of the potential and its derivatives across \( x = a \) gives

\[
\sum_{n=1}^{\infty} \frac{U^I_n}{k_n} \tanh(k_na)H_n^I(y) = \sum_{n=2}^{\infty} U^I_n \frac{1}{j_n} H_n^I(y), \quad 0 \leq y \leq d,
\]

(5.97)

and

\[
\sum_{n=1}^{\infty} U^I_n H_n^I(y) = \begin{cases} 
\sum_{m=2}^{\infty} U^I_m H_m^I(y), & 0 \leq y \leq d \\
0, & d \leq y \leq b
\end{cases}
\]

(5.98)

Multiplication of both sides of (5.97) by \( H_m^I(y) \), \( m = 1, 2, ... \) and integration over \([0, \, d]\) gives

\[
\sum_{n=1}^{\infty} \frac{U^I_n}{k_n} \tanh(k_na) d_{nm} = \begin{cases} 
\frac{U^I_m}{j_m}, & m \geq 2 \\
0, & m = 1
\end{cases}
\]

(5.99)

where

\[
d_{nm} = \int_0^d H_n^I(y)H_m^I(y)dy = \frac{2(-1)^m \beta_n \cos(\beta_n d)}{(bd)^{1/2} (\beta^2 - \beta_n^2)}.
\]

(5.100)

Multiplication of both sides of (5.98) by \( h_m^I(y) \), \( m = 1, 2, ... \) and integration over \([0, \, b]\) gives

\[
U^I_m = \sum_{n=2}^{\infty} U^I_n d_{mn}, \quad m \geq 1.
\]

(5.101)
From (5.99), if \( m = 1 \), we have

\[
\sum_{n=1}^{\infty} \frac{U_n^I}{k_n} \tanh(k_n a) d_n = 0. \tag{5.102}
\]

Substitution of (5.99) into (5.101) for \( m > 1 \) we have

\[
U_m^I + \sum_{n=1}^{\infty} A_{mn} U_n^I = 0, \quad m > 1
\]

where

\[
A_{mn} = k_n^{-1} \tanh(k_n a) \sum_{r=2}^{\infty} j_r d_{mr} d_{nr}, \quad m > 1. \tag{5.104}
\]

For a given \( b/d \), there are two remaining unknowns, \( kd \) and \( a/d \). A trapped mode corresponds to non-trivial solutions for \( kd \) and \( a/d \) which satisfy both (5.102) and (5.103). In the following sections, both the approximate and full solution will be explored.

### 5.3.3 Approximate Solution

For a given value of \( 1 < b/d < 5/3 \) and \( \frac{\pi}{2} < kd < \frac{3\pi}{2} \), a trapped mode problem corresponds to having two progressing terms in region I and no progressive term in region II. However, there is only one wave-like term if \( \frac{\pi}{2} < kd < \frac{3\pi}{2b/d} \) and \( 1 < b/d < 5/3 \). Thus we will explore the only situation for which there are two wave-like terms in region I and no progressing terms in region II, i.e. \( \frac{3\pi}{2b/d} < kd < \frac{3\pi}{2} \) and \( 1 < b/d < 5/3 \).

Truncating the series at two terms in region I and one term in region II gives

\[
k_1^{-1} \tanh(k_1 a) d_{11} U_1^I + k_2^{-1} \tanh(k_2 a) d_{21} U_2^I = 0, \tag{5.105}
\]

\[
k_1^{-1} \tanh(k_1 a) d_{12} U_1^I + k_2^{-1} \tanh(k_2 a) d_{22} U_2^I + j_2^{-1} U_2^{II} = 0, \tag{5.106}
\]

\[
U_1^I - d_{12} U_2^I = 0, \tag{5.107}
\]

and

\[
U_2^I - d_{22} U_2^{II} = 0. \tag{5.108}
\]
The above equations may be written as the following matrix system of equations

\[ AX = 0, \quad (5.109) \]

where

\[
A = \begin{pmatrix}
\frac{\tanh(k_1a)d_{11}}{k_1} & \frac{\tanh(k_2a)d_{21}}{k_2} & 0 \\
\frac{\tanh(k_1a)d_{12}}{k_1} & \frac{\tanh(k_2a)d_{22}}{k_2} & j_2^{-1} \\
1 & 0 & -d_{12} \\
0 & 1 & -d_{22}
\end{pmatrix}, \quad X = \begin{pmatrix}
U_1^I \\
U_2^I \\
U_2^{II}
\end{pmatrix}.
\quad (5.110)

As the last two rows of the matrix are independent of \( a/d \) and \( kd \), it is convenient to express the condition that \( \text{rank}(A) \leq 2 \) as the two equations

\[
\frac{\tanh(k_1a)d_{11}}{k_1} d_{12} + \frac{\tanh(k_2a)d_{21}}{k_2} d_{22} = 0, \quad (5.111)
\]

and

\[
1 + \frac{\tanh(k_1a)d_{12}}{k_1} + j_2^{-1} \frac{\tanh(k_2a)d_{22}}{k_2} = 0. \quad (5.112)
\]

The expansion of (5.111) and (5.112) gives

\[
\frac{\tanh(k_1a)d_{11}d_{12}}{k_1} + \frac{\tanh(k_2a)d_{21}d_{22}}{k_2} = 0, \quad (5.113)
\]

and

\[
j_2^{-1} + \frac{\tanh(k_1a)d_{12}^2}{k_1} + \frac{\tanh(k_2a)d_{22}^2}{k_2} = 0. \quad (5.114)
\]

It will be helpful if (5.113) and (5.114) can be combined into one transcendental equation. The re-arrangement (5.113) and (5.114) generates

\[
\frac{\tan(k'a)}{k'a} + \frac{d_{11}d_{12} \tan(k''a)}{d_{21}d_{22} k''a} = 0, \quad (5.115)
\]

and

\[
j_2d = s \frac{k'd}{\tan(k'a)} = \frac{d_{21}}{d_{12}(d_{11}d_{22} - d_{12}d_{21})} \frac{k'd}{\tan(k'a)}. \quad (5.116)
\]
From the definition of \( k' \) and \( k'' \), we have

\[
(k'a)^2 = (ka)^2 - \frac{\pi^2 \alpha^2}{4b^2}, \quad (k''a)^2 = (ka)^2 - \frac{9\pi^2 \alpha^2}{4b^2},
\]

(5.117)

then it follows

\[
(k''a)^2 = (k'a)^2 - \frac{2\pi^2 \alpha^2}{b^2} = (k'a)^2 \left(1 - \frac{2\pi^2}{(k'd)^2(b/d)^2}\right).
\]

(5.118)

Again from the definition of \( j_2 \), we have

\[
(j_2d)^2 = \frac{9\pi^2}{4} - (kd)^2 = \frac{9\pi^2}{4} - \left((k'd)^2 + \frac{\pi^2 d^2}{4b^2}\right).
\]

(5.119)

and so

\[
(k'd)^2 = \frac{9\pi^2}{4} - \frac{\pi^2 d^2}{4b^2} - (j_2d)^2.
\]

(5.120)

Substitution of (5.116) into (5.120) gives

\[
(k'd)^2 = \frac{\pi^2}{4} \left(9 - \frac{(d/b)^2}{\tan(k'a)}\right).
\]

(5.121)

Let \( c = b/d \), substitute (5.121) into (5.118), then into (5.115) to give the following transcendental equation

\[
\frac{\tan(k'a)}{k'a} + q \frac{\tan(k''a)}{k''a} = 0
\]

(5.122)

where

\[
q = \frac{(9 - c^2) \cos^2(\pi/2c)}{(1 - 9c^2) \cos^2(3\pi/2c)},
\]

(5.123)

\[
k''a = k'a \left(1 - \frac{8(1 + s^2 \cot^2(k'a))}{9c^2 - 1}\right)^{1/2},
\]

(5.124)

and

\[
s = \frac{-9(1 - 9c^2)^2(c^2 - 1)^2\pi^2}{1024c^3 \cos^2(\pi/2c)}
\]

(5.125)

where \( s \) depends only on \( b/d \). By looking at the value of \( s \) we find that \( s \) is always negative for any given \( 1 < b/d < 5/3 \). The potential in region II is formed by a sum of decaying exponentials since there is no progressing terms
in region II, thus $j_2d$ is positive. Therefore, from (5.116), $\tan(k'a)$ should be negative, i.e.

$$\left(n - \frac{1}{2}\right)\pi < k'a < n\pi, n = 1, 2, 3, \ldots$$  \hspace{1cm} (5.126)

It is obvious that $q$ is negative from (5.123) since $1 < c < 5/3$. This indicates that $\tan(k''a)$ should be negative in order that the solution of (5.122) may be found. This requires

$$\left(m - \frac{1}{2}\right)\pi < k''a < m\pi, \quad m = 1, 2, 3, \ldots, n - 1$$ \hspace{1cm} (5.127)

where $m < n$ because $k''a < k'a$. Obviously, for a fixed $1 < b/d < 5/3$, there will be a series of $k'a$ that satisfy equation (5.122). After solving for $k'a$, we can obtain $k'd$ and $k''a$ from (5.121) and (5.118). Finally $kd$ and $a/d$ are given by

$$kd = \left((k'd)^2 + \frac{\pi^2d^2}{4b^2}\right)^{1/2},$$ \hspace{1cm} (5.128)

$$\frac{a}{d} = \frac{k'a}{kd \left(1 - \left(\frac{\pi}{2kd b/d}\right)^2\right)^{1/2}},$$ \hspace{1cm} (5.129)

and

$$k'a = \frac{k'a}{\left(1 - \left(\frac{\pi}{3kd b/d}\right)^2\right)^{1/2}}.$$ \hspace{1cm} (5.130)

In the real numerical calculation, we still need the effective range of $k'a$ for a given $n$. From (5.124), $k''a$ should be real and positive, thus we have

$$\cot(k'a) \leq \frac{3(c^2 - 1)^{1/2}}{2\sqrt{2s}}$$ \hspace{1cm} (5.131)

for a given $b/d = c$. Therefore, the roots of (5.122) can only be found in the range of $\left(n\pi, n\pi + \frac{3(c^2 - 1)^{1/2}}{2\sqrt{2s}}\right)$ since $\frac{3(c^2 - 1)^{1/2}}{2\sqrt{2s}}$ is very small therefore $\tan^{-1} \frac{3(c^2 - 1)^{1/2}}{2\sqrt{2s}} \approx \frac{3(c^2 - 1)^{1/2}}{2\sqrt{2s}}$. Substitution of $k'a$ and $k''a$ from (5.117) into (5.126) and (5.127) gives

$$\frac{d\pi}{2b} \left(\frac{9(n - 1/2)^2 - (m - 1/2)^2}{(n - 1/2)^2 - (m - 1/2)^2}\right)^{1/2} < kd < \frac{d\pi}{2b} \left(\frac{9n^2 - m^2}{n^2 - m^2}\right)^{1/2},$$ \hspace{1cm} (5.132)
\[ \frac{\sqrt{2} b}{2} \left( (n - 1/2)^2 - (m - 1/2)^2 \right)^{1/2} < \frac{a}{d} < \frac{\sqrt{2} b}{2} \left( (n^2 - m^2)^{1/2} \right). \]  

5.3.4 Full solution

The procedure of the full solution is similar to that in the section 5.2.3. However it is more complicated since there are two wave-like terms to be moved to the right hand side of the integral equation. The common boundary is denoted by \( \{ L : x = a, \ 0 \leq y \leq d \} \). From (5.98), we write

\[ U(y) = \sum_{n=1}^{\infty} U_n^{I} H_n^{I}(y) = \left\{ \begin{array}{ll}
\sum_{n=2}^{\infty} U_n^{II} H_n^{II}(y), & 0 \leq y \leq d, \\
0, & d \leq y \leq b \end{array} \right. \]  

(5.134)

It follows that

\[ U_n^{I} = \int_L U(y) H_n^{I}(y) dy, \ n = 1, 2, 3, ..., \]  

(5.135)

and

\[ U_n^{II} = \int_L U(y) H_n^{II}(y) dy, \ n = 2, 3, 4, .... \]  

(5.136)

Substitution of \( U_n^{I}, U_n^{II} \) into (5.97) gives

\[ \int_L U(y') \left\{ \sum_{n=1}^{\infty} \frac{\tanh(k_n a)}{k_n} H_n^{I}(y) H_n^{I}(y') + \sum_{n=2}^{\infty} j_n^{-1} H_n^{II}(y) H_n^{II}(y') \right\} dy' = 0. \]  

(5.137)

This is a homogeneous integral equation for \( U(y) \). We need to change it into a system of positive definite, inhomogeneous equations. In order to do this, the oscillatory first two terms (assuming \( \frac{3\pi}{2b/d} < kd < \frac{8\pi}{2} \)) in the first sum in (5.137) are shifted to the right hand side to give

\[ \int_L U(y') \left\{ \sum_{n=3}^{\infty} \frac{\tanh(k_n a)}{k_n d} H_n^{I}(y) H_n^{I}(y') + \sum_{n=2}^{\infty} \frac{H_n^{II}(y) H_n^{II}(y')}{j_n d} \right\} dy' = -\frac{\tan(k'a)}{k'd} H_1^{I}(y) U_1 - \frac{\tan(k''a)}{k''d} H_2^{I}(y) U_2. \]  

(5.138)

If we define

\[ U(y) = -\left( \frac{1}{\cot(k'a)k'd} U_1 v(y) + \frac{1}{\cot(k''a)k''d} U_2 w(y) \right), \]  

(5.139)
then we have

\[ U_1 = \int_L U(y)H_1^I(y)dy, \quad U_2 = \int_L U(y)H_2^I(y)dy, \quad (5.140) \]

and

\[
\begin{align*}
\frac{\tan(k'a)}{k'd} \int_L v(y')K(y,y')dy' + \frac{\tan(k''a)}{k''d} \int_L w(y')K(y,y')dy' \\
= \frac{\tan(k'a)}{k'd} H_1^I(y)U_1 + \frac{\tan(k''a)}{k''d} H_2^I(y)U_2.
\end{align*}
\]  

(5.141)

By choosing

\[ \int_L v(y')K(y,y')dy' = H_1^I(y), \quad y \in L, \quad (5.142) \]

and

\[ \int_L w(y')K(y,y')dy' = H_2^I(y), \quad y \in L \quad (5.143) \]

where

\[ K(y,y') = \sum_{n=2}^{\infty} \frac{\tanh(k_na)H_n^I(y)H_n^I(y')}{k_n d} + \sum_{n=2}^{\infty} \frac{H_n^{II}(y)H_n^{II}(y')}{j_n d}, \quad (5.144) \]

and multiplying both sides of (5.139) by \( H_1(y), \ H_2(y) \) separately, and integrating over \([0, d]\), we obtain

\[
\left(1 + \frac{\tan(k'a)}{k'd} \int_L v(y)H_1^I(y)\right) U_1 + \left(\frac{\tan(k''a)}{k''d} \int_L w(y)H_1^I(y)\right) U_2 = 0 \quad (5.145)
\]

and

\[
\left(\frac{\tan(k'a)}{k'd} \int_L v(y)H_2^I(y)\right) U_1 + \left(1 + \frac{\tan(k''a)}{k''d} \int_L w(y)H_2^I(y)\right) U_2 = 0. \quad (5.146)
\]

These are two homogeneous equations with two unknowns. The system has a non-trivial solution if and only if its determinant is equal to zero, i.e.

\[
\frac{\tan(k'a)}{k'd} \int_L w(y)H_1^I(y)dy - \frac{1+\tan(k''a)}{k''d} \int_L v(y)H_2^I(y)dy = 0. \quad (5.147)
\]
This is the first condition for the full solution to exist. We still have another condition to satisfy, namely condition (5.102). Substitution of (5.139) into (5.102) gives
\[
\frac{\tan(k'a)}{k'd}U_1 + \frac{\tan(k''a)}{k''d} \sum_{n=1}^{\infty} \frac{d_{n1} \tanh(k_n a)}{k_n} \int_L w(y) H_n^I(y) dy = 0. \tag{5.148}
\]
From (5.145) or (5.146), \(U_1\) may be expressed by \(U_2\), or vice versa. Here we express \(U_1\) by \(U_2\). From (5.145), we have
\[
U_1 = -\frac{\tan(k''a)}{k''d} U_2 \int_L w(y) H_1^I(y) dy \frac{\tan(k'a)}{k'a} \int_L v(y) H_1^I(y) dy.
\tag{5.149}
\]
Substitution of \(U_1\) into (5.148) gives
\[
\frac{\tan(k'a)}{k'a} \int_L w(y) H_1^I(y) dy \sum_{n=1}^{\infty} \frac{d_{n1} \tan(k_n a)}{k_n} \int_L v(y) H_1^I(y) dy \frac{\tan(k''a)}{k''d} \int_L w(y) H_1^I(y) dy = 0. \tag{5.150}
\]
This is another condition for the full solution to exist. Equations (5.147) and (5.150) have two independent parameters, \(kd\) and \(ka\) for a given \(b/d\), and they may be solved for simultaneously if we can find \(v(y)\) and \(w(y)\) for \(\frac{3\pi}{2b/d} < kd < \frac{3\pi}{2}\).

We will adopt the Galerkin method to solve this problem. In order to preserve simple forms for equations, we choose
\[
u_n(y) = \frac{b^{1/2}d^{-1/3}(2n + 1)!\Gamma(1/6)}{(-1)^n \sqrt{2\pi} \Gamma(2n + 1 + 1/3)(d^2 - y^2)^{1/3}} C_n^{1/6} \left(\frac{y}{d}\right), \tag{5.151}
\]
where
\[
C_n^*(\cos \theta) = \sum_{r=0}^{n} \frac{\Gamma(\nu + r)\Gamma(\nu + n - r)}{\Gamma(\nu + n)\Gamma(\nu)\Gamma(\nu + n)!! r! (n - r)! |\Gamma(\nu)|^2} \cos (n - 2r) \theta \tag{5.152}
\]
are the ultra-spherical Gegenbauer polynomials as before. Let
\[
v(y) \approx \sum_{r=1}^{N} V_n v_n(y), \quad w(y) \approx \sum_{r=1}^{N} W_n w_n(y) \tag{5.153}
\]
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Substitution of (5.153) into (5.142) and (5.143) yields

\[
\sum_{r=1}^{N} V_n \int_{L} v_n(y') K(y, y') dy' = H_1^{I}(y), \quad y \in L, \quad (5.154)
\]

and

\[
\sum_{r=1}^{N} W_n \int_{L} w_n(y') K(y, y') dy' = H_2^{I}(y), \quad y \in L \quad (5.155)
\]

which are multiplied by \(v_n(y), w_n(y)\) and integrated over \(L\) to give

\[
\sum_{n=1}^{N} K_{mn}^{v} V_n = F_{m1}^{v}, \quad m = 1, 2, ..., \quad (5.156)
\]

and

\[
\sum_{n=1}^{N} K_{mn}^{w} W_n = F_{m2}^{w}, \quad m = 1, 2, ... \quad (5.157)
\]

where

\[
F_{m1}^{v} = \int_{L} H_1^{I}(y) v_m(y) dy,
\]

\[
F_{m2}^{w} = \int_{L} H_2^{I}(y) w_m(y) dy,
\]

\[
K_{mn}^{v} = \sum_{r=3}^{\infty} \frac{\tanh(k_r \alpha) F_{mr}^{v} F_{nr}^{v}}{k_r d} + \sum_{r=2}^{\infty} \frac{G_{mr}^{v} G_{nr}^{v}}{j_r d},
\]

\[
K_{mn}^{w} = \sum_{r=3}^{\infty} \frac{\tanh(k_r \alpha) F_{mr}^{w} F_{nr}^{w}}{k_r d} + \sum_{r=2}^{\infty} \frac{G_{mr}^{w} G_{nr}^{w}}{j_r d},
\]

\[
F_{mr}^{v} = \int_{L} H_1^{I}(y) v_m(y) dy,
\]

\[
F_{mr}^{w} = \int_{L} H_2^{I}(y) w_m(y) dy,
\]

\[
G_{mr}^{v} = \int_{L} H_1^{II}(y) v_m(y) dy,
\]

and

\[
G_{mr}^{w} = \int_{L} H_2^{II}(y) w_m(y) dy.
\]

Let

\[
A_n = \int_{L} v(y) H_n^{I}(y) dy = \sum_{r=1}^{N} V_r \int_{L} v_r(y) H_n(y) dy = \sum_{r=1}^{N} V_r F_{rn}^{v}, \quad (5.158)
\]
and

\[ B_n = \int_L w(y) H_n^r(y) dy = \sum_{r=1}^{N} W_r \int_L w_r(y) H_n(y) dy = \sum_{r=1}^{N} W_r F_r^w. \]  

(5.159)

Thus substitution of \( A_n, B_n \) into (5.147) and (5.150) gives the final equations

\[ \frac{\tan(k''a)}{k''a} B_1 - \frac{\tan(k''a)}{k''a} A_1 = 0, \]

(5.160)

and

\[ \frac{\tan(k''a)}{k''a} B_1 - \frac{\tan(k''a)}{k''a} A_1 = 0, \]

(5.161)

through which the full solution of embedded trapped modes may be found.

In order to preserve the simple forms of \( F_{mn}^v, F_{mn}^w, G_{mn}^v, G_{mn}^w \) and hence \( K_{mn}^v, K_{mn}^w \), we choose \( v_n(y) = w_n(y) = u_n(y) \). After some algebra, it can be shown that

\[ F_{m1}^v = F_{m1}^w = \int_0^d H_n^r u_m(y) dy = \frac{J_{2m+1+1/6}(\frac{\pi}{2b})}{(\pi d/b)^{1/6}}, \]

(5.162)

\[ F_{mn}^v = F_{mn}^w = \int_0^d H_n^r u_m(y) dy = \frac{J_{2m+1+1/6}(\frac{2n-1}{2b})}{(2n-1)^{1/6}}, \]

(5.163)

\[ G_{mn}^v = G_{mn}^w = \int_0^d H_n^r u_m(y) dy = \frac{\left(\frac{b}{d}\right)^{1/2} J_{2m+1+1/6}((n-1/2)\pi)}{(2n-1)^{1/6}} \]

(5.164)

and

\[ K_{mn}^v = K_{mn}^w = \sum_{r=2}^{\infty} \frac{J_{2m+1+1/6}(\frac{(r-1/2)\pi d}{b}) J_{2m+1+1/6}(\frac{(r-1/2)\pi d}{b})}{k_r d \coth(k_r a)((2r-1)\pi d/b)^{1/3}} \]

\[ + \sum_{r=2}^{\infty} \frac{b}{d} J_{2m+1+1/6}((r-1/2)\pi) J_{2m+1+1/6}((r-1/2)\pi)}{(2r-1)^{1/3}}. \]

(5.165)

5.3.5 Embedded trapped modes symmetric about the plane \( x = 0 \)

For embedded modes symmetric about the plane \( x = 0 \), the \( \frac{\sinh(k_n a)}{\cosh(k_n a)} \) in potential expansion \( \phi_I \) is replaced by \( \frac{\cosh(k_n a)}{\sinh(k_n a)} \). The solution procedure is the
same as that described above. The approximate solutions are written as

\[ k'a \tan(k'a) + q'k''a \tan(k''a) = 0, \quad (5.166) \]

and

\[ j_2d = s'k'd \tan(k'a), \quad (5.167) \]

where

\[ q' = \frac{(9 - c^2) \cos^2(\pi/2c)}{(1 - 9c^2) \cos^2(3\pi/2c)} \quad s' = \frac{9(1 - 9c^2)^2(c^2 - 1)^2\pi^2}{1024c^3 \cos^2(\pi/2c)}, \quad (5.168) \]

\[ n\pi < k'a < (n + 1/2)\pi, \quad n = 0, 1, 2, ..., \quad (5.169) \]

and

\[ m\pi < k''a < (m + 1/2)\pi, \quad m = 0, 1, 2, ..., n - 1. \quad (5.170) \]

From these relations we may find

\[ \frac{d\pi}{2b} \left( \frac{9n^2 - m^2}{n^2 - m^2} \right)^2 < kd < \frac{d\pi}{2b} \left( \frac{9(n + 1/2)^2 - (m + 1/2)^2}{(n + 1/2)^2 - (m + 1/2)^2} \right)^2, \quad (5.171) \]

and

\[ \frac{\sqrt{2}}{2} b \left( \frac{n^2 - m^2}{n^2 - m^2} \right)^{1/2} < \frac{a}{d} < \frac{\sqrt{2}}{2} b \left( \frac{(n + 1/2)^2 - (m + 1/2)^2}{(n + 1/2)^2 - (m + 1/2)^2} \right)^{1/2}. \quad (5.172) \]

For the full solutions, the tanh in the formulation for the anti-symmetric case is replaced by coth in Section 5.3.4. Attention should be given to the change of sign that occurs when the argument of coth, as opposed to tanh, is imaginary. The final conditions for full solutions to exist can be expressed as

\[ \frac{\cot(k'a)}{k'd} B_1 - \frac{1 - \cot(k'a)}{k'd} A_1 = 0, \quad (5.173) \]

and

\[ \frac{\cot(k'a)}{k'd} B_1 \sum_{n=1}^{\infty} \frac{\coth(k_na)}{k_n d} d_1 A_n + \sum_{n=1}^{\infty} \frac{\coth(k_na)}{k_n d} d_1 B_n = 0 \quad (5.174) \]

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Figure 5.6: Variation of trapped mode frequencies with indentation depth $b/d$. Modes are antisymmetric in both $x$ and $y$, $n = 4$.

5.3.6 Numerical results

We computed the embedded trapped modes which are antisymmetric about the centreline of the waveguide, and anti-symmetric or symmetric in the $x$ numerically, by both the approximate and full solution described above. In each case, the results of the approximate solution are used as an initial guess for the full solution.
Figure 5.7: Variation of $a/d$ with indentation depth $b/d$, $n = 4$
Modes anti-symmetric in \(x\)

The numerical results of both the approximate and full solutions are plotted together in Figures 5.6 and 5.7. Figure 5.6 shows the numerical results of embedded modes for the approximate and full solutions anti-symmetric about the centreline of the waveguide and the planes \(x = 0\), in which lines represent the approximate solutions and dots represent the full solutions for \(n = 4\). From this figure we see that \(kd\) starts from the second cut-off, i.e. \(3\pi/2\), and decreases as \(b/d\) increases, the maximum number of modes is 3 for a given \(b/d\) provided that \(n = 4\), and there exists a critical value above which the anti-symmetric modes may be found. The behavior of the curves in Figure (5.6) may be analyzed roughly from the approximate solution. First we may estimate \(kd\) at the maximum value of \(b/d\), i.e. \(b/d \to 5/3\). From (5.132), we have

\[
\frac{3\pi}{10} \left( \frac{9(n - 1/2)^2 - (m - 1/2)^2}{(n - 1/2)^2 - (m - 1/2)^2} \right)^{1/2} < kd < \frac{3\pi}{10} \left( \frac{9n^2 - m^2}{n^2 - m^2} \right)^{1/2}, \quad (5.175)
\]

for \(n = 2, \ldots, m = 1, 2, \ldots, n - 1\), as \(\frac{b}{d} \to \frac{5}{3}\). When the length of indentation increases (increasing \(n\)), the lowest trapped mode frequency can be estimated. Actually, this value may be obtained by letting \(n \to \infty\) in (5.132), which gives

\[
kd = \frac{3\pi}{2b/d}, \quad (5.176)
\]

The lower bound of \(b/d\) that can sustain trapped modes for each pair of \((n, m)\) is given by

\[
\frac{b}{d} = \frac{1}{3} \left( \frac{9(n - 1/2)^2 - (m - 1/2)^2}{(n - 1/2)^2 - (m - 1/2)^2} \right)^{1/2}, \quad (5.177)
\]

from (5.132) by taking \(kd = 3\pi/2\).

Figure (5.7) plots \(a/d\) against \(b/d\), in which the lines represent the approximate solutions and the dots represent the full solution. From this figure we see that \(a/d\) starts from a fixed value when a new mode comes in for
a given \( n \). Actually this value can be estimated from equation (5.126) by letting \( kd = \frac{3\pi}{2} \). It gives

\[
\frac{a}{d} > \frac{2n - 1}{(9 - (d/b)^2)^{1/2}},
\]

(5.178)

The value of \( a/d \) given by (5.178) gives the lower bound of \( a/d \) above which an embedded trapped modes may be found. An estimate of maximum value of \( a/d \) can be obtained from (5.133) by letting \( \frac{b}{d} \rightarrow \frac{d}{b} \). It follows that

\[
\frac{a}{d} \geq \frac{5\sqrt{2}}{6} \left( \left( n - \frac{1}{2} \right)^2 - \left( m - \frac{1}{2} \right)^2 \right)^{1/2},
\]

(5.179)

where \( n = 2, 3, ..., m = 1, 2, ..., n - 1 \).

Modes symmetric in \( x \)

The numerical results of both the approximate and full solutions for embedded trapped modes which are antisymmetric about the centreline of the waveguide and symmetric about the plane \( x = 0 \) are plotted together in Figure 5.8 and 5.9. The lines represent the approximate solutions and the dots represent the full solutions for \( n = 4 \). From Figures 5.8 and 5.9 we see that \( kd \) starts from the second cut-off, i.e. \( 3\pi/2 \), and decreases as \( b/d \) increases, and the maximum number of modes is 4 for a given \( b/d \) provided that \( n = 4 \).

The behavior of the curves in Figure 5.8 may be analyzed roughly by the approximate solution. First we may estimate the maximum value of \( kd \) by letting \( b/d = 5/3 \) from (5.171). Namely, it gives

\[
\frac{3\pi}{10} \left( \frac{9n^2 - m^2}{n^2 - m^2} \right)^{1/2} < kd < \frac{3\pi}{10} \left( \frac{9(n + 1/2)^2 - (m + 1/2)^2}{(n + 1/2)^2 - (m + 1/2)^2} \right)^{1/2},
\]

(5.180)

where \( n = 1, 2, ..., m = 0, 1, ..., n - 1 \). When \( b/d \rightarrow 1 \), only one trapped mode can be found. Its value tends to \( \frac{3\pi}{2} \). This may also be obtained through (5.171) by letting \( m = 0 \). Actually, the following function

\[
\phi = \sin \frac{3\pi y}{2d},
\]

(5.181)
Figure 5.8: Variation of $kd$ with indentation depth $b/d$ that are anti-symmetric about the centreline of the waveguide and symmetric about the plane $x = 0$ for $n = 4$
Figure 5.9: Variation of $a/d$ with indentation depth $b/d$ for modes that are anti-symmetric about the centreline of the waveguide and symmetric about the plane $x = 0$ for $n = 4$
is a standing wave solution to equation (5.1) and satisfies the boundary conditions (5.2) - (5.4).

Figure (5.9) plots $a/d$ against $b/d$. From this we see that $a/d$ starts from a fixed value when a new mode comes in for a given $n$ and $m$. Actually, the lower bound of $a/d$ above which a new mode comes in, may be found through (5.171) and (5.172) by letting $kd = \frac{3\pi}{2}$. It follows that

$$\frac{a}{d} > \frac{\sqrt{2}}{3} \sqrt{9n^2 - m^2}. \quad (5.182)$$

The lower bound of maximum value of $a/d$ when $b/d = 5/3$ may also be obtained from (5.172) by substituting $\frac{k}{d} = \frac{5}{3}$. It follows

$$\frac{a}{d} = \frac{5\sqrt{2}}{6} (n^2 - m^2)^{1/2}, \quad n = 1, 2, \ldots, \quad m = 0, 1, \ldots, n - 1. \quad (5.183)$$

5.4 Conclusion

Embedded trapped modes that are above the first cut-off and below the second cut-off have been found near an indentations in two-dimensional wave guides which might be strip acoustic wave guides or water channels of uniform depth.

Modes are sought which are symmetric or anti-symmetric about the centreline of the waveguides. In each case both approximate and full solutions are derived. The crude approximation results are found to be close to the full solutions numerically. In the full solutions, a powerful Galerkin approach is used in which the one Third root singularity at the corners of indentation is modelled by the proper choice of special functions. This yields fast convergence in the solution of infinite linear system in which truncating 4 terms gives enough accuracy in each case.
Chapter 6

Rayleigh-Bloch acoustic modes in cylindrical waveguides

6.1 Introduction

Unforced fluid oscillations which have finite energy are known to occur in unbounded fluids and they are localised in regions in which there is some change in the properties or the geometry of the medium. In the context of water waves these oscillations are known as edge waves or trapped modes, whereas in acoustic waveguides they are known as acoustic resonances. The occurrence of acoustic resonances excited by vortex shedding from thin plates is of great practical importance and an excellent survey of engineering applications in which acoustic resonances play an important role has been given by Parker & Stoneman[84]. The practical consequences of the existence of these modes is that if the system is forced at a frequency near that of the mode, a large response of the fluid-structure system will occur. In particular Parker [85] made an experimental study of acoustic resonances in axial flow compressors and observe large vibrations of the blades. Clearly such large motions may result in damage to one or more components of the system and
a general degradation in performance. Further work on the occurrence and
effect of acoustic resonances in aeroengines has been done by Woodley &
Peake [113, 114] and Cooper & Peake [9].

Mathematically these modes are eigenfunctions of a linear operator and
the frequencies at which they occur are related to the eigenvalues of the oper­
ator. In general the modes which have been found and proven to exist have
frequencies which are below some cut-off frequency in the problem, which
means that they oscillate at frequencies at which no propagation of waves
to infinity is possible. Examples of such modes are edge waves which prop­
agate along periodic coastlines, which have been found by Evans & Linton
[20], and acoustic resonances in cascades of plates, which were found both
experimentally and numerically by Parker [82, 83].

Proofs of the existence of trapped modes in two and three dimensions are
given by Evans et al [19], Davies & Parnovski [10] and Groves [30], and all of
these proofs are based on variational principles. More recently the existence
of edge waves, or as they are sometimes referred to in the electromagnetic
literature Rayleigh-Bloch modes, has been established rigorously by Linton
& McIver [45]. Modes which have frequencies which are above a cut-off
frequency in a guide have also been found numerically by McIver et al [65]
and Evans & Porter [14]. These modes are said to be embedded, as their
frequencies correspond to eigenvalues which are embedded in the continuous
spectrum of the relevant operator. Unfortunately this means that the usual
variational arguments fail, and it is very difficult to prove rigorously that
such modes exist.

A typical example mentioned by Parker & Stoneman [84] is the case of
axial-flow compressors in turbo-machinery. To a first approximation, these
can be modelled as an infinite circular cylinder with thin radial fins of finite
length in the axial direction, distributed uniformly around the guide, though
in reality the geometry of these compressors may be extremely complex. This
geometry is a special case of the general class considered by Groves[31] in which an existence proof of trapped modes has been given, thus the existence of acoustic resonances is guaranteed.

The purpose of this work is to investigate the existence of trapped modes which correspond to rotational motion in a cylindrical acoustic waveguide which has circular cross-section. Such modes will be referred to as 'spinning modes' following the terminology of Parker[?], who found such modes experimentally in an axial flow compressor. Linton & McIver[47] computed 'standing' trapped modes for this geometry, but we will show that such modes are only one class of modes that exist, and additional spinning modes are possible.

We begin in Section 6.2 by formulating the problem, using matched eigenfunctions. In Section 6.3 we give an existence proof of Rayleigh-Bloch modes. Section 6.4 will be devoted to the discussion of the connection between Rayleigh-Bloch modes and the trapped modes found by Linton and McIver[47]. In Section 6.5 a Galerkin approach will be used to model the singularity at the edge of fins to find Rayleigh-Bloch modes numerically. Some results and discussion will be presented in Section 6.6, and a summary will be given in the final section.

6.2 Formulation

We consider a circular cylindrical waveguide of radius \( d \) and infinite length in which thin radial fins of finite length along the axial direction are uniformly distributed around the guide. A cylindrical polar coordinate system \((p, \theta, x)\) is chosen in such way that the origin of the system is in the centre of the section with radial fins, in which \( x \) is chosen to point along the guide, as shown in Figure 6.1, and the position of the fins is given by \( \{ \theta = m\beta, m = 0, \ldots, L - 1, 0 \leq \rho \leq d, -a \leq x \leq a \} \), where \( \beta = 2\pi/L \), and \( L \) is the number
Figure 6.1: A cylindrical waveguide containing a section of fins formed by uniformly distributed plates around the guide (by courtesy of Dr M. McIver).

of fins. We restrict our attention to any one of the sectors, \( \Omega = \{ \rho, \theta, x : 0 \leq \rho < d, 0 < \theta < \beta, x > 0 \} \), Trapped modes are functions \( \text{Re}[\phi e^{-i\omega t}] \), where \( \phi \) is a non-trivial solution of the boundary-value problem

\[
(\nabla + k^2)\phi = 0, \quad \text{in } \Omega \tag{6.1}
\]

\[
\frac{\partial \phi}{\partial \rho} = 0 \quad \text{on } \rho = d, \tag{6.2}
\]

\[
\frac{\partial \phi}{\partial \theta} = 0 \quad \text{on } \theta = 0 \text{ and } \theta = \beta, \quad 0 \leq x \leq a, \tag{6.3}
\]

and

\[
\phi \rightarrow 0 \text{ as } |x| \rightarrow \infty, \tag{6.4}
\]
The parameter $k = \omega/c$, where $\omega$ is the trapped mode frequency and $c$ is the speed of sound. An application of condition (6.4) to the eigenfunction expansion of $\phi$ in the far field ensures that the trapped mode has finite energy. The potential field may be symmetric or anti-symmetric about the plane $x = 0$ since the section with radial fins is symmetric about the plane $x = 0$. For symmetric or anti-symmetric potential in $x$ axis, one of the following boundary conditions should be imposed. Namely,

$$\frac{\partial \phi}{\partial x} = 0 \text{ on } x = 0,$$  \hspace{1cm} (6.5)

for a potential $\phi$ symmetric in $x$, and

$$\phi = 0 \text{ on } x = 0$$  \hspace{1cm} (6.6)

for a potential $\phi$ anti-symmetric in $x$.

Linton & McIver [46] restricted the type of modes which they considered to ones which satisfy $\phi = 0$ on $\theta = m\beta$, $|x| > a$, $m = 0, \ldots, L - 1$. However in order to obtain modes which represent a rotational motion around the axis ('spinning modes') different conditions must be applied. To determine these, we first consider the scattering of the wave $J_p(J_p',\rho/d)e^{i\rho\theta+i(|x^2-J_{p,1}^2|d^2)^{1/2}x}$, where $J_p$ is the Bessel function of the first kind of order $p$, $p$ is a given integer and $J_{p,1}$ is the first zero of $J_p'$. As the geometry is unchanged after rotation through an angle $\theta = \beta$, the only change in the scattered wave field is due to the phase change of $e^{ip\beta}$ in the incident wave.

We denote region I: \{0 < \theta < \beta, 0 < \rho < d, 0 < x < a\}, and region II: \{0 < \theta < \beta, 0 < \rho < d, a < x < \infty,\}. The difference of the potentials in different sectors in region I lies in their phases. Following to Linton & McIver[46], we may write

$$\phi(\rho, \theta + m\beta, x) = e^{im\beta} \phi(\rho, \theta, x), \; m = 0, 1, 2, \ldots, L - 1, 0 \leq \theta < \beta, 0 < x < a,$$  \hspace{1cm} (6.7)
where \( p \) is a given integer. There is no requirement for continuity of potential across the fins in the inner region (region I). This implies that we may confine our discussion to any one sector. However, in region II, the potential is required to be continuous when \( \theta \) varies across the extension line of the fins. This means that we should look for a solution of the form

\[
\phi(\rho, \theta, x) = e^{ip\theta} \varphi(\rho, \theta, x) \tag{6.8}
\]

where \( \varphi \) is a periodic function of periodicity \( 2\pi/L \).

Thus the potential \( \phi \) is symmetric about the plane \( x = 0 \) in region I, and may be written as

\[
\phi^I(\rho, \theta, x) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \frac{\cosh \alpha_{mn} x}{\alpha_{mn} \sinh \alpha_{mn} a} \psi^I_m(\rho) \Psi^I_m(\theta), \quad 0 < x < a \tag{6.9}
\]

and the potential in region II can be expressed as

\[
\phi^I(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} \frac{e^{-\beta_{mn} (x-a)}}{-\beta_{mn}} \psi^II_m(\rho) \Psi^II_m(\theta), \quad a < x < \infty, \tag{6.10}
\]

where

\[
\psi^I_m(\rho) = J_{mL/2}(j'_{mL/2,n} \rho/d), \quad \Psi^I_m(\theta) = \cos (mL\theta/2),
\]

and

\[
\psi^II_m(\rho) = J_{p+mL}(j'_{p+mL,n} \rho/d), \quad \Psi^II_m(\theta) = e^{i(p+mL)\theta},
\]

where \( j'_{mL/2,n} \) is the \( n \)th zero of the Bessel function \( J'_{mL/2} \), and \( j'_{p+mL,n} \) is the \( n \)th zero of Bessel function \( J'_{p+mL} \), and

\[
\alpha_{mn} d = \begin{cases} 
(j^2_{mL/2,n} - (kd)^2)^{1/2}, & kd \leq j'_{mL/2,n}, \\
-i((kd)^2 - j^2_{mL/2,n})^{1/2}, & kd > j'_{mL/2,n}, 
\end{cases} \tag{6.13}
\]

\[
\beta_{mn} d = \begin{cases} 
(j^2_{p+mL,n} - (kd)^2)^{1/2}, & kd \leq j'_{p+mL,n}, \\
-i((kd)^2 - j^2_{p+mL,n})^{1/2}, & kd > j'_{p+mL,n}.
\end{cases} \tag{6.14}
\]

For the potential antisymmetric about the plane \( x = 0 \), the \( \cosh(\alpha_{mn} x)/\sinh(\alpha_{mn} a) \) in (6.9) is replaced by \( \sinh(\alpha_{mn} x)/\cosh(\alpha_{mn} a) \).
Continuity of potential and its first derivatives across $x = a$ gives
\[
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \coth(\alpha_{mn} a) \psi_m^I(\rho) \psi_m^I(\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} \psi_m^{II}(\rho) \psi_m^I(\theta),
\] (6.15)

and
\[
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \psi_m^I(\rho) \psi_m^I(\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} \psi_m^{II}(\rho) \psi_m^I(\theta),
\] (6.16)

with $0 < \rho < d$ and $0 < \theta < \beta$ in both cases. These equations may be converted into an infinite system of equations by multiplying both of the two equations by $\rho \psi^I_\mu(\rho) \psi^I_\mu(\theta), \mu = 0, 1, 2, ..., \nu = 1, 2, ..., \text{or } \rho \psi^{II}_\mu(\rho) \psi^{II}_\mu(\theta), \mu = 0, 1, 2, ..., \nu = 1, 2, ...$, and integrating over the sector $0 < \rho < d, 0 < \theta < \beta$. We will first derive the infinite system of equations which contain only unknown coefficients $a_{mn}$ or $b_{mn}$, where $m = 0, 1, 2, ..., \infty, n = 1, 2, ..., \infty$, and then use integral equations with the Galerkin method to solve this problem.

Equations (6.15) and (6.16) are multiplied by $\rho \psi^I_\mu(\rho) \psi^I_\mu(\theta), \mu \in \mathcal{N}_0, \nu \in \mathcal{N}$ and $\rho \psi^{II}_\mu(\rho) \psi^{II}_\mu(\theta), \mu \in \mathcal{N}_0, \nu \in \mathcal{N}$ separately, and integrated over the sector $0 < \rho < d, 0 < \theta < \beta$ to give
\[
a_{\mu \nu} \frac{d^2 \beta}{2 \varepsilon_\mu \alpha_{\mu \nu}} \coth(\alpha_{\mu \nu} a) q_{\mu \nu} = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} G_{mn}^\nu R_{mn},
\] (6.17)
\[
a_{\mu \nu} \frac{d^2 \beta}{2 \varepsilon_\mu} q_{\mu \nu} = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} G_{mn}^\nu R_{mn},
\] (6.18)
\[
b_{\mu \nu} \frac{d^2 \beta Q_{\mu \nu}}{2 \beta_{\mu \nu}} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \coth(\alpha_{mn} a) H_{mn}^\nu S_{mn},
\] (6.19)
and
\[
b_{\mu \nu} \beta Q_{\mu \nu} d^2/2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} H_{mn}^\nu S_{mn},
\] (6.20)

where
\[
F_{mn}^\nu = \int_0^d \rho \psi^I_{mn}(\rho) \psi^I_\mu(\rho) \, d\rho,
\]
\[
G_{mn}^\nu = \int_0^d \rho \psi_{mn}^{II}(\rho) \psi^I_\mu(\rho) \, d\rho,
\]

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\[ H^{\mu \nu}_{m \nu} = \int_{0}^{d} \rho \psi^{I}_{mn}(\rho) \psi^{II}_{\mu}(\rho) \, d\rho, \]
\[ I^{\mu \nu}_{m \nu} = \int_{0}^{d} \rho \psi^{II}_{mn}(\rho) \psi^{I}_{\mu}(\rho) \, d\rho, \]
\[ R^{\mu \nu}_{m \nu} = \int_{0}^{\beta} \Psi^{II}_{m}(\theta) \Psi^{I}_{\mu}(\theta) \, d\theta = \frac{i(p + mL)((-1)^{m} e^{i2\pi m / L} - 1)}{(mL/2)^2 - (p + mL)^2}, \]
and
\[ S^{\mu \nu}_{m \nu} = \int_{0}^{\beta} \Psi^{I}_{m}(\theta) \Psi^{II}_{\mu}(\theta) \, d\theta = \frac{i(p + mL)(1 - (-1)^{m} e^{i2\pi m / L})}{(mL/2)^2 - (p + mL)^2}. \]

In the formulation above we have used the following results
\[ \int_{0}^{\beta} \Psi^{I}_{m}(\theta) \Psi^{I}_{\mu}(\theta) \, d\theta = \frac{\beta}{\epsilon_{\mu}} \delta_{m\mu}, \quad \epsilon_{0} = 1, \quad \epsilon_{\mu} = 2, \quad (6.21) \]
\[ \int_{0}^{\beta} \Psi^{II}_{m}(\theta) \Psi^{II}_{\mu}(\theta) \, d\theta = \beta \delta_{m\mu}, \quad (6.22) \]
and
\[ F^{\mu \nu}_{\mu \nu} = \frac{d^2}{2} q_{\mu \nu} \delta_{\mu \nu}, \quad I^{\mu \nu}_{\mu \nu} = \frac{d^2}{2} Q_{\mu \nu} \delta_{\mu \nu}, \quad (6.23) \]

where
\[ q_{\mu \nu} = \begin{cases} \left( 1 - \frac{(mL/2)^2}{J_{\mu}(J_{\mu} / 2, \nu)} \right) \left( J_{\mu}(J_{\mu} / 2, \nu) \right)^2, & (\mu L/2, \nu) \neq (0,1), \\ 1, & (\mu L/2, \nu) = (0,1), \end{cases} \quad (6.24) \]
and
\[ Q_{\mu \nu} = \begin{cases} \left( 1 - \frac{(p + mL)^2}{J_{p}(J_{p} / 2, \nu)} \right) \left( J_{p}(J_{p} / 2, \nu) \right)^2, & (p + mL, \nu) \neq (0,1), \\ 1, & (p + mL, \nu) = (0,1). \end{cases} \quad (6.25) \]

There are a number of ways to obtain a infinite system of equations from equations (6.17) - (6.20). The easiest way is to combine (6.17) and (6.18). Substitution of (6.20) into (6.17) gives
\[ a_{\mu \nu} \frac{\beta \coth(\alpha_{\mu} a)}{\epsilon_{\mu} \alpha_{\mu} d} q_{\mu \nu} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sum_{r=-\infty}^{\infty} \sum_{s=1}^{\infty} \frac{H^{n}_{mr} G^{e}_{rs} S_{rs} R_{rs}}{-\beta_{rs} Q_{rs}}, \quad (6.26) \]
where \( \mu = 0, 1, 2, \ldots, \nu = 1, 2, \ldots \), or alternatively, substitute (6.18) into (6.17) to give
\[ \sum_{m=-\infty}^{\infty} R_{m \nu} \sum_{n=1}^{\infty} \left( \frac{\coth(\alpha_{\mu} a)}{\alpha_{\mu} d} + \frac{1}{\beta_{mn} d} \right) \bar{G}^{mn}_{\mu \nu} b_{mn} = 0, \quad (6.27) \]
where $\mu = 0, 1, 2, \ldots$, $\nu = 1, 2, \ldots$. We may truncate the series in (6.26) or (6.27) to $M$ terms in the sum over $m$ and $N$ terms in the sum over $n$ to form a finite system of homogeneous equations with $(2M + 1)N$ variables. Rayleigh-Bloch mode frequencies can be found when the determinant of the coefficient of the matrix is zero for a given geometry and value of $p$ for which there is no progressing terms in the outer region. Thus the problem remains to show that we can find solutions from (6.26) or (6.27). In the following discussion we will first show that solutions exist analytically, and then find the frequencies numerically.

6.3 Proof of existence of Rayleigh-Bloch modes

We will use a standard variational argument to prove the existence of Rayleigh-Bloch modes. It is well known that the Rayleigh quotient defined by

$$Q[\phi] = \frac{\int_\Omega |\nabla \phi|^2 d\Omega}{\int_\Omega |\phi|^2 d\Omega}$$

(6.28)

has a minimum value that corresponds to the lowest point of the spectrum of equation (6.1) for all functions $\phi \in (H^1(\Omega) \cap S_1) \setminus \{0\}$, where $S_1$ consists of those functions which are antisymmetric about the plane $x = 0$, see Linton and McIver[46], and $H^1(\Omega)$ is the Sobolev space consisting of all those functions in $L^2(\Omega)$ which also have square-integrable first partial derivatives and satisfy the Rayleigh-Bloch condition in the outer region where $\Omega$ is defined by $\{0 < \rho < d, 0 < \theta < \beta, -\infty < x < \infty\}$. Therefore the Rayleigh quotient may be expressed as

$$Q[\phi] = \frac{\int_{-\infty}^{\infty} \int_0^d \int_0^\beta |\nabla \phi|^2 \rho d\theta d\rho dx}{\int_{-\infty}^{\infty} \int_0^d \int_0^\beta |\phi|^2 \rho d\theta d\rho dx}.$$  (6.29)

The potential $\phi$ may be either symmetric or anti-symmetric about the plane $x = 0$, while $|\nabla \phi|^2$ and $|\phi|^2$ are always symmetric about the plane $x = 0$. Thus

$$\int_{-\infty}^{\infty} \int_0^d \int_0^\beta |\nabla \phi|^2 \rho d\theta d\rho dx = 2 \int_{0}^{\infty} \int_0^d \int_0^\beta |\nabla \phi|^2 \rho d\theta d\rho dx,$$  (6.30)

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and
\[ \int_{-\infty}^{\infty} \int_0^d \int_0^\beta |\phi|^2 \rho d\theta dx = 2 \int_{-\infty}^{\infty} \int_0^d \int_0^\beta |\phi|^2 \rho d\theta dx. \] (6.31)

It follows that the Rayleigh quotient may be expressed as

\[ Q[\phi] = \frac{\int_{-\infty}^{\infty} \int_0^d \int_0^\beta |\nabla \phi|^2 \rho d\theta dx}{\int_{-\infty}^{\infty} \int_0^d \int_0^\beta |\phi|^2 \rho d\theta dx}, \] (6.32)

so we need only calculate the Rayleigh quotient in the half domain, namely, \(0 < x < \infty, 0 < \rho < d, 0 < \theta < \beta\). And this is divided into two sub domains \(D_1, D_2\). We denote \(\{D_1 : 0 < x < a, 0 < \theta < \beta, 0 < \rho < d\}\), and \(\{D_2 : a < x < \infty, 0 < \theta < \beta, 0 < \rho < d\}\). Following Linton & McIver[45], we may consider the following test function

\[ \phi = \begin{cases} J_p(\sigma_{p1}\rho)e^{ip\theta} + e^{1/2}(1 - |x|/a), & |x| < a, 0 < \theta < \beta, 0 < \rho < d, \\ J_p(\sigma_{p1}\rho)e^{ip\theta}e^{-\epsilon(|x|/a-1)}, & |x| > a, 0 < \theta < \beta, 0 < \rho < d, \end{cases} \] (6.33)

which satisfies the periodic condition (6.7) with \(\sigma_{p1}d = j_{p1}'\). For Rayleigh-Bloch modes to exist we need to prove

\[ Q[\phi] < \sigma_{p1}^2, \] (6.34)

for a small value of \(\epsilon > 0\) since \(\sigma_{p1}^2\) is the lowest point of the continuous spectrum.

At this stage, \(|\phi|^2\) and \(|\nabla \phi|^2\) are evaluated according to their definition in \(D_1\) and \(D_2\). Bearing in mind that we have

\[ \int_D |\phi|^2 dV = \int_{D_1} |\phi|^2 dV + \int_{D_2} |\phi|^2 dV, \] (6.35)

and

\[ \int_D |\nabla \phi|^2 dV = \int_{D_1} |\nabla \phi|^2 dV + \int_{D_2} |\nabla \phi|^2 dV. \] (6.36)

In \(D_1\) we have

\[ |\phi|^2 = J_p^2(\sigma_{p1}\rho) + 2\epsilon^{1/2}J_p(\sigma_{p1}\rho)\cos(p\theta)(1 - x/a) + O(\epsilon), \] (6.37)
|∇ϕ|^2 = A^2 \sigma_{p1}^2 + \frac{p^2}{(p^2)} J_p^2(\sigma_{p1}\rho) + O(\varepsilon), \quad (6.38)

and similarly in D_2 we obtain

|ϕ|^2 = J_p^2(\sigma_{p1}\rho)e^{-2\varepsilon(\varepsilon^2/\varepsilon^2 - 1)}, \quad (6.39)

and

|∇ϕ|^2 = \left( A^2 \sigma_{p1}^2 + \frac{p^2}{(p^2)} J_p^2(\sigma_{p1}\rho) \right) e^{-2\varepsilon(\varepsilon^2/\varepsilon^2 - 1)} + O(\varepsilon^2), \quad (6.40)

where A = \frac{1}{2}(J_{p-1}(\sigma_{p1}\rho) - J_{p+1}(\sigma_{p1}\rho)). It follows that

\[ \int_{D_1} |ϕ|^2 dV = a\beta I_1 + a\varepsilon^{1/2} I_2 \sin(p\beta)/p + O(\varepsilon), \quad (6.41) \]

\[ \int_{D_2} |ϕ|^2 dV = \frac{\beta}{2\varepsilon} I_1, \quad (6.42) \]

\[ \int_{D_1} |∇ϕ|^2 dV = a\sigma_{p1}^2 I_3 \beta + a\beta^2 I_3 \beta + O(\varepsilon), \quad (6.43) \]

and

\[ \int_{D_2} |∇ϕ|^2 dV = \frac{\beta}{2\varepsilon} (\sigma_{p1}^2 I_3 + \beta I_3) + O(\varepsilon), \quad (6.44) \]

where

\[ I_1 = \int_0^\rho \rho J_p^2(\sigma_{p1}\rho)d\rho, \quad I_2 = \int_0^\rho \rho J_p(\sigma_{p1}\rho)d\rho, \quad (6.45) \]

\[ I_3 = \int_0^\rho \rho \{J_{p-1}(\sigma_{p1}\rho) - J_{p+1}(\sigma_{p1}\rho)\}^2 d\rho, \]

\[ I_4 = \int_0^\rho \frac{1}{\rho/\rho^2} J_p^2(\sigma_{p1}\rho)d\rho. \]

Therefore substitution (6.41) - (6.44) into (6.35) and (6.36) gives

\[ \int_D |ϕ|^2 dV = (1 + \frac{1}{2a\varepsilon})a\beta I_1 + \frac{2a\varepsilon^{1/2} I_2}{p} \sin(p\beta) + O(\varepsilon), \quad (6.46) \]

and

\[ \int_D |∇ϕ|^2 dV = (1 + \frac{1}{2a\varepsilon})(a\beta I_3 \sigma_{p1}^2 + a\beta^2 I_3) + O(\varepsilon). \quad (6.47) \]

Substitution of (6.46) and (6.47) into (6.32), shows that the Rayleigh quotient may expressed as

\[ Q[ϕ] = \frac{(1 + \frac{1}{2a\varepsilon})(\sigma_{p1}^2 I_3 + p^2 I_4) + O(\varepsilon)}{a\beta I_1 + \frac{1}{2a}\varepsilon^{1/2} I_2 \sin(p\beta/2)/p + O(\varepsilon)}. \quad (6.48) \]
After some algebraic manipulation, we have

\[
Q[\phi] = \frac{1 + O(\varepsilon^2)}{1 + f(\varepsilon) + O(\varepsilon^2)} \frac{\sigma_{p1}^2 I_3 + p^2 I_4}{I_1}
\]

(6.49)

where

\[
f(\varepsilon) = \frac{4ae^{3/2}I_2 \sin(p\beta)}{(1 + 2ae)I_p \beta}.
\]

(6.50)

Since \( p\beta = 2\pi r/L < \pi \) for \( p < L/2 \) we have \( 0 < \sin(p\beta)/p\beta < 1 \) and \( I_1, I_2 > 0 \), therefore for any small value of \( \varepsilon > 0 \) we have \( f(\varepsilon) > 0 \). We may choose

\[
0 < \varepsilon < \left( \frac{p\beta I_1}{4aI_2 \sin(p\beta)} \right)^{2/3}
\]

(6.51)
to make \( 0 < f(\varepsilon) < 1 \). From (6.49), we have

\[
Q[\phi] = (1 - f(\varepsilon) + O(\varepsilon^2)) \frac{\sigma_{p1}^2 I_3 + p^2 I_4}{I_1}
\]

(6.52)

Next we need to show

\[
\frac{\sigma_{p1}^2 I_3 + p^2 I_4}{I_1} = \sigma_{p1}^2. \quad \text{The Bessel function of the first kind } J_p(z) \text{ satisfies}
\]

\[
z^2 \frac{d^2 J_p(z)}{dz^2} + z \frac{d J_p(z)}{dz} + (z^2 - p^2)J_p(z) = 0.
\]

(6.53)

Multiplying (6.53) by \( J_p(z)/z \) gives

\[
zJ_p(z) \frac{d^2 J_p(z)}{dz^2} + J_p(z) \frac{d J_p(z)}{dz} + \frac{z^2 - p^2}{z} J_p^2(z) = 0.
\]

(6.54)

This may be re-arranged as

\[
\frac{d}{dz} \left( zJ_p(z) \frac{d J_p(z)}{dz} \right) - z \left( \frac{d J_p(z)}{dz} \right)^2 + \frac{z^2 - p^2}{z} J_p^2(z) = 0.
\]

(6.55)

Integrating (6.55) over \((0, j_{p1}^p)\) gives

\[
\left[ zJ_p(z) \frac{d J_p(z)}{dz} \right]_{0}^{j_{p1}^p} - \int_{0}^{j_{p1}^p} z \left( \frac{d J_p(z)}{dz} \right)^2 dz + \int_{0}^{j_{p1}^p} \frac{z^2 - p^2}{z} J_p^2(z) dz = 0
\]

(6.56)

where \( j_{p1}^p \) is the first zero of \( J_p^p(z) \). The first term in this equation is identically zero. It follows that

\[
\int_{0}^{j_{p1}^p} z \left( \frac{d J_p(z)}{dz} \right)^2 dz + p^2 \int_{0}^{j_{p1}^p} J_p^2(z) dz = \int_{0}^{j_{p1}^p} z J_p^2(z) dz.
\]

(6.57)
Let \( z = \sigma_{p1}\rho \), thus we have

\[
I_1 = \frac{1}{\sigma_{p1}^2} \int_0^{T_p} z J_p^2(z) \, dz, \tag{6.58}
\]

\[
I_2 = \frac{1}{\sigma_{p1}^2} \int_0^{T_p} z \left( \frac{J_p(z)}{dz} \right)^2 \, dz, \tag{6.59}
\]

and

\[
I_4 = \int_0^{T_p} \frac{1}{z} J_p^2(z) \, dz. \tag{6.60}
\]

Consequently from (6.57), we have

\[
\frac{\sigma_{p1}^2 I_3 + p^2 I_4}{I_1} = \sigma_{p1}^2. \tag{6.61}
\]

Finally, we have shown that

\[
Q[\phi] = (1 - f(\varepsilon) + O(\varepsilon^2))\sigma_{p1}^2 < \sigma_{p1}^2 \tag{6.62}
\]

which is the requirement for Rayleigh-Bloch modes to exist. One should notice that \( p \) can't be zero for the proof given above since \( \sigma_{01} = 0 \).

### 6.4 Connection between Rayleigh-Bloch modes and standing modes

First we discuss the possible values of \( p \) for which solutions can be found. The principle to determine the values of \( p \) is to ensure that there is no progressing wave terms in region II. From condition (6.4), we need \( \beta_{mn}d > 0, m = 0, \pm 1, \pm 2, \ldots, n = 1, 2, \ldots \). Thus from (6.14) we must restrict

\[
kd < j'_{p+mL,1}, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{6.63}
\]

so that there will be no wave-like term in the outer region. This implies that

\[
p + mL \neq 0. \tag{6.64}
\]

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The integer $p$ cannot be zero since $j_{0,1} = 0$. Otherwise there will be wave-like terms in outer region. Thus possible value for $p$ should be

$$p = 1, 2, \ldots, L - 1, \quad L = 1, 2, \ldots.$$  \hspace{1cm} (6.65)

The cut-off frequency should be chosen as the smaller value of $\{j_{L-p,1}, j_{P,1}\}$ for we need all $\beta_{mn}d$ to be real and positive. Therefore the value of $p$ may only be chosen by

$$p = \begin{cases} 
1, 2, \ldots, L/2, & L \text{ even}, \\
1, 2, \ldots, (L-1)/2, & L \text{ odd}.
\end{cases}  \hspace{1cm} (6.66)$$

Bear in mind we have used the relation $j_{p-L,1} = j_{L-p,1}$ since $J_n(z) = (-1)^n J_n(z)$ for any integer $n$.

In the previous section we have found that our proof fails when $p = 0$, so in the following discussion, we will discuss the relations between Rayleigh-Bloch modes and trapped modes found by Linton and McIver[47].

Let $\phi_1(\rho, \theta, x)$ be a Rayleigh-Bloch spinning mode, i.e.

$$\phi_1(\rho, \theta + 2\pi/L, x) = e^{2ip\pi/L} \phi_1(\rho, \theta, x),$$  \hspace{1cm} (6.67)

where $\phi_1(\rho, \theta, x)$ is continuous for $|x| > a$ and $p$ an arbitrary integer. By symmetry

$$\phi_2(\rho, \theta, x) = \phi_1(\rho, -\theta, x),$$  \hspace{1cm} (6.68)
is also a spinning mode. It is easy to show that
\[ \phi_2(\rho, \theta, x) = e^{-2ip\pi/L} \phi_2(\rho, \theta - 2\pi/L, x), \] (6.69)
and
\[ \phi_2(\rho, \theta + 2\pi/L, x) = e^{-2ip\pi/L} \phi_1(\rho, \theta, x). \] (6.70)

Let
\[ \psi(\rho, \theta, x) = \phi_1(\rho, \theta, x) - \phi_2(\rho, \theta, x) = \phi_1(\rho, \theta, x) - \phi_1(\rho, -\theta, x), \] (6.71)
which is also a spinning mode solution because of the linearity. Therefore
\[ \psi(\rho, 0, x) = 0, |x| > a, \] (6.72)
which is Linton and McIver's [47J condition on the extension of a fin on \( \theta = 0 \). Again for \( x > a \) and any integer \( k \) we have
\[ \psi(\rho, 2k\pi/L, x) = \phi_1(\rho, 2k\pi/L, x) - \phi_2(\rho, 2k\pi/L, x) \]
\[ = e^{2ikp\pi/L} \phi(\rho, 0, x) \left( 1 - e^{-4ikp\pi/L} \right). \] (6.73)

Thus \( \psi(\rho, 2k\pi/L, x) = 0 \) if \( p = 0 \) or \( p = L/2 \) for \( L \) even which is also Linton and McIver's [47J condition on the extension of fins.

First we look at \( p = 0 \). From the potential expansion (6.10), we have
\[ \phi^{II}(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} \frac{e^{-\beta_{mn}(x-a)}}{-\beta_{mn}} J_m(L_j m_{L,n} \rho/d) e^{imL\theta}. \] (6.74)

Bearing in mind that there are no wave-like terms in the outer region, we let \( b_{01} = 0 \). By symmetry, if \( \phi(\rho, \theta, x) \) is a trapped mode solution so is \( \phi(\rho, -\theta, x) \)
\[ \phi^{II}(\rho, -\theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} \frac{e^{-\beta_{mn}(x-a)}}{-\beta_{mn}} J_m(L_j m_{L,n} \rho/d) e^{-imL\theta}. \] (6.75)

Substitution of \( m \) by \( -m \) yields
\[ \phi^{II}(\rho, -\theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{-m,n} \frac{e^{-\beta_{-m,n}(x-a)}}{-\beta_{m,n}} J_{-m}(L_j m_{L,n} \rho/d) e^{imL\theta}. \] (6.76)
Since $j'_{(m-1)L,n} = j'_{(m+1)L,n}$, $J_{-n} = (-1)^n J_n$, we have

$$\phi^H(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{m,n} e^{-\beta_{m,n}(x-a)} \frac{(-1)^m L J_{mL}(j'_{mL,n\rho/d}) e^{imL\theta}}{-\beta_{m,n}}.$$  

(6.77)

As $\beta_{m,n} = \beta_{m,n}$, we must have that, for all $m$

$$b_{m,n}(-1)^m = \mu b_{m,n} = \mu^2(-1)^m b_{m,n}.$$  

(6.78)

Thus

$$\mu = \pm 1, \quad b_{m,n}(-1)^m = \pm b_{m,n},$$  

(6.79)

and

$$b_{0n} = 0 \text{ if } b_{m,n}(-1)^m = -b_{m,n}.$$  

(6.80)

Therefore the potential in the outer region for $p = 0$ may be expressed as

$$\phi^H(\rho, \theta, x) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} b_{m,n} e^{-\beta_{m,n}(x-a)} J_{mL}(j'_{mL,n\rho/d}) e^{imL\theta}$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} e^{-\beta_{m,n}(x-a)} J_{mL}(j'_{mL,n\rho/d}) e^{-imL\theta}$$

$$= \sum_{n=1}^{\infty} b_{0n} e^{-\beta_{0n}(x-a)} J_0(j'_{0n\rho/d})$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} e^{-\beta_{m,n}(x-a)} J_{mL}(j'_{mL,n\rho/d}) \left( e^{imL\theta} \pm e^{-mL}\right).$$  

(6.81)

It follows that

$$\phi^H(\rho, \theta, x) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} 2b_{m,n} e^{-\beta_{m,n}(x-a)} J_{mL}(j'_{mL,n\rho/d}) \cos mL\theta, \quad \mu = 1,$$  

(6.82)

and

$$\phi^H(\rho, \theta, x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2ib_{m,n} e^{-\beta_{m,n}(x-a)} J_{mL}(j'_{mL,n\rho/d}) \sin mL\theta, \quad \mu = -1.$$  

(6.83)

It is obvious that (6.82) is not a trapped wave solution for waves travel down the guide and propagate away, whereas (6.83) satisfies $\phi = 0$ on $\theta = 146$
$2k\pi/L$ (extensions of fins) and the antisymmetry between fins, which are the conditions of mode solutions antisymmetric about the mid plane of a sector by Linton & McIver[47].

Now suppose that $L$ is even and $p = L/2$, therefore $\phi(\rho, \theta, x)$ becomes

$$
\phi^{II}(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} e^{-\beta_{mn}(x-a)} J_{(m+1/2)L} (J'_{(m+1/2)L,n}) \rho/d e^{i(m+1/2)L\theta}
$$

(6.84)

and

$$
\phi^{II}(\rho, -\theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} e^{-\beta_{mn}(x-a)} J_{(m+1/2)L} (J'_{(m+1/2)L,n}) \rho/d e^{-i(m+1/2)L\theta}
$$

(6.85)

Substitute $m$ by $-m - 1$ to give

$$
\phi^{II}(\rho, -\theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{m-1,n} e^{-\beta_{m-1,n}(x-a)} \frac{(-1)^{(m+1/2)L}}{J_{(m+1/2)L}(J'_{(m+1/2)L,n}) \rho/d} e^{i(m+1/2)L\theta}
$$

(6.86)

where we have used $j'_{(m+1/2)L,n} = j'_{(m+1/2)L,n}$ and $J_{(-m-1/2)L} = (-1)^{(m+1/2)L} J_{(m+1/2)L}$. As

$$
\beta_{-m-1,n} = (j^2_{-m-1/2}L_n - (kd)^2)^{1/2} = \beta_{mn},
$$

(6.87)

so for all $m$, we must have

$$
b_{-m-1,n}(-1)^{(m+1/2)L} = \mu b_{mn} = \mu^2(-1)^{(m+1/2)L} b_{-m-1,n}
$$

(6.88)

therefore $\mu = \pm 1$. Henceforth we may write

$$
\phi^{II}(\rho, \theta, x) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} b_{mn} e^{-\beta_{mn}(x-a)} J_{mL}(j'_{(m+1/2)L,n}) \rho/d \left( e^{i(m+1/2)L\theta} \pm e^{-(m+1/2)L\theta} \right).
$$

(6.89)

For $\mu = 1$, we have

$$
\phi^{II}(\rho, \theta, x) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} 2b_{mn} e^{-\beta_{mn}(x-a)} J_{mL}(j'_{(m+1/2)L,n}) \rho/d \cos(m + 1/2)L\theta,
$$

(6.90)
and for $\mu = -1$ we have

$$
\phi^{II}(\rho, \theta, x) = \sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \frac{2ib_{mn}e^{-\beta_{m,n}(x-a)}}{-\beta_{mn}} J_{mL}(j_{m+1/2}L_n \rho/d) \sin(m + 1/2)L\theta.
$$

(6.91)

The potential (6.90) satisfies $\partial \phi / \partial \theta = 0$ on $\theta = 2k\pi / L$, $k = 0, 1, 2, ..., L - 1$ and $\phi = 0$, on $\theta = (2k + 1)\pi / L$, which means waves can propagate down the full guide without change. In this case, there is no trapped mode solution. However, (6.91) satisfies $\phi = 0$, on $\theta = 2k\pi / L$, and $\partial \phi / \partial \theta = 0$, on $\theta = (2k + 1)\pi / L$, which means $\phi = 0$ on the extension of fins and symmetry between fins, the same condition of Linton and McIver[47].

Now we have recovered all the results given by Linton and McIver[47]. We may also pick up some interesting results for special $p$ and $L$. From (6.71), we have

$$
\frac{\partial \psi(\rho, \theta, x)}{\partial \theta} = \frac{\partial \phi_1(\rho, \theta, x)}{\partial \theta} - \frac{\partial \phi_2(\rho, \theta, x)}{\partial \theta}.
$$

(6.92)

From (6.67) and (6.68), we have

$$
\frac{\partial \phi_1(\rho, \theta + 2\pi / L, x)}{\partial \theta} = e^{2ip\pi / L} \frac{\partial \phi_1(\rho, \theta, x)}{\partial \theta},
$$

(6.93)

and

$$
\frac{\partial \phi_2(\rho, \theta + 2\pi / L, x)}{\partial \theta} = e^{-2ip\pi / L} \frac{\partial \phi_2(\rho, \theta, x)}{\partial \theta}.
$$

(6.94)

Therefore, it follows that

$$
\frac{\partial \phi_1(\rho, 2k\pi / L, x)}{\partial \theta} = e^{2ikp\pi / L} \frac{\partial \phi_1(\rho, 0, x)}{\partial \theta},
$$

(6.95)

and

$$
\frac{\partial \phi_2(\rho, 2k\pi / L, x)}{\partial \theta} = e^{-2ikp\pi / L} \frac{\partial \phi_2(\rho, 0, x)}{\partial \theta}.
$$

(6.96)

Substitution of (6.95) and (6.96) into (6.92) gives

$$
\frac{\partial \psi(\rho, 2k\pi / L, x)}{\partial \theta} = e^{2ikp\pi / L} \frac{\partial \phi_2(\rho, 0, x)}{\partial \theta} \left(1 + e^{-4ikp\pi / L}\right) = 0,
$$

(6.97)

only if

$$
4kp\pi / L = (2n + 1)\pi
$$

(6.98)
for some integer \( n \). The above condition can only be satisfied if \( p = L/4 \), i.e., \( L = 4p \), and \( k = 2n + 1, n = 0, 1, 2, \ldots \). Returning to (6.71), \( \psi(p, 2k\pi/L, x) = 0 \) for \( k \) even. This means that Neumann condition is applied on the extension of the fins when the number of the fins is even whereas Dirichlet condition is applied on the extension of fins when the number of the fins is odd. Table 6.1 shows the cut-off frequency of different \( L \) for which modes may be found. For example, \( L = 3 \) is the minimum number of fins for which Rayleigh-Bloch modes may be found.

### 6.5 Galerkin approach

We will adopt a Galerkin approach to solve this problem. Returning to the continuity equation (6.15) and (6.16), we write

\[
f(\rho, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} \psi_{mn}^I(\rho) \Psi_m^I(\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} b_{mn} \psi_{mn}^{II}(\rho) \Psi_m^{II}(\theta). \tag{6.99}
\]

Multiplying (6.99) by \( \rho \psi_{\mu\nu}(\rho) \Psi_\mu^I(\theta) \), and integrating over \{\rho, 0 \leq \rho \leq d; \theta, 0 \leq \theta \leq \beta \} gives

\[
a_{\mu\nu} \frac{d^2 \beta_{\mu\nu}}{2 \epsilon_{\mu}} = \int_0^d \int_0^\beta \rho f(\rho, \theta) \psi_{\mu\nu}(\rho) \Psi_\mu^I(\theta) d\rho d\theta. \tag{6.100}
\]

Again multiplying (6.99) by \( \rho \psi_{\mu\nu}^{II}(\rho) \overline{\Psi_\mu^{II}(\theta)} \), and integrating over \{\rho, 0 \leq \rho \leq d; \theta, 0 \leq \theta \leq \beta \} yields

\[
b_{\mu\nu} Q_{\mu\nu} \beta^2/2 = \int_0^d \int_0^\beta \rho f(\rho, \theta) \psi_{\mu\nu}^{II}(\rho) \overline{\Psi_\mu^{II}(\theta)} d\rho d\theta. \tag{6.101}
\]

For a given value of \( p \) and \( L \), suppose there are \( N \) wave-like terms in the region of fins, and shift the oscillatory terms in equation (6.15) to the right hand side. After re-arrangement, it gives

\[
\sum_{n=N+1}^{\infty} \frac{a_{0n} \coth(\alpha_{0n} a)}{\alpha_{0n}} \psi_{0n}(\rho) \Psi_0^I(\theta) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn} \coth(\alpha_{mn} a)}{\alpha_{mn}} \psi_{mn}^I(\rho) \Psi_m^I(\theta) \\
+ \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{b_{mn} \psi_{mn}^{II}(\rho) \Psi_m^{II}(\theta)}{\beta_{mn}} = -\sum_{n=1}^{N} \frac{a_{0n} \coth(\alpha_{0n} a)}{\alpha_{0n}} \psi_{0n}(\rho) \Psi_0^I(\theta). \tag{6.102}
\]
Substitution of (6.100) and (6.101) into the left hand side of (6.102) gives

\[
\int_0^d \int_0^\beta \rho' f(\rho', \theta') K(\rho, \theta, \rho', \theta') \, d\rho' d\theta' = \sum_{n=1}^{N} \frac{a_{0n} \cot(\alpha_{0n} a) \psi_{0,n}^I(\rho) \psi_0^I(\theta)}{2\beta^{-1} a_{0n} d},
\]

(6.103)
since \(\alpha_{0n} d = -i((kd)^2 - j_{0,n}^2)^{1/2}\), \(kd > j_{0,n}^2\) and \(\coth(\alpha_{0n} a) = -\cot(\alpha_{0n} a)/i\), in which \(K(\rho, \theta, \rho', \theta')\) is given by

\[
K(\rho, \theta, \rho', \theta') = \sum_{n=N+1}^{\infty} \frac{a_{0n} \cot(\alpha_{0n} a)}{d^2 a_{0n} d_{0n}} \psi_{0,n}^I(\rho) \psi_0^I(\theta) \psi_{0,n}^I(\rho') \psi_0^I(\theta')
+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn} \cot(\alpha_{mn} a)}{d^2 a_{mn} d_{mn}} \psi_{m,n}^I(\rho) \psi_m^I(\theta) \psi_{m,n}^I(\rho') \psi_m^I(\theta')
+ \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{d^2 a_{mn} d_{mn}} \psi_{m,n}^{II}(\rho) \psi_m^{II}(\theta) \psi_{m,n}^{II}(\rho') \psi_m^{II}(\theta').
\]

(6.104)

The simplest case is that there is only one progressing term in inner region, i.e.

\[
\int_0^d \int_0^\beta \rho' f(\rho', \theta') K(\rho, \theta, \rho', \theta') \, d\rho' d\theta' = \frac{a_{01} \cot(\alpha_{01} a) \psi_{0,1}^I(\rho) \psi_0^I(\theta)}{2\beta^{-1} a_{01} d}.
\]

(6.105)

By defining

\[
f(\rho, \theta) = \frac{a_{01} \cot(\alpha_{01} a)}{2\beta^{-1} a_{01} d} g(\rho, \theta),
\]

(6.106)

we have

\[
\int_0^d \int_0^\beta \rho' g(\rho', \theta') K(\rho, \theta, \rho', \theta') \, d\rho' d\theta' = \psi_{0,1}^I(\rho) \psi_0^I(\theta)
\]

(6.107)

and (6.106) is multiplied by \(\psi_{0,1}^I(\rho) \psi_0^I(\theta)\), and integrated over the sector \(0 < \rho < d, 0 < \theta < \beta\) to give

\[
\frac{1}{d^2} \int_0^\beta \int_0^\beta \rho g(\rho, \theta) \psi_{0,1}^I(\rho) \psi_0^I(\theta) \, d\rho d\theta = \alpha_{01} d \tan(\alpha_{01} a) q_{01}/\varepsilon_0.
\]

(6.108)

The problem has been reduced to first solving for \(g(\rho, \theta)\) for a given set of geometric parameters, wave number \(p\) and \(kd\) for which (6.108) is satisfied. The term \(d^2\) in (6.108) will be cancelled in the final form. We will adopt a
Galerkin approach to the solution of (6.107), (6.108) which we first write in the operator form

\[ K g(\rho, \theta) = \psi_{0,1}^f(\rho) \Psi_0^f \]  

(6.109)

with

\[ (g(\rho, \theta), \psi_{0,1}^f(\rho) \Psi_0^f(\theta)) = A \equiv \alpha_{01}^d \tan(\alpha_{01}^d q_{01}/\epsilon_0) \]  

(6.110)

where we define the complex inner product by

\[ (g, h) = \overline{(h, g)} = \int_{A_g} g(\rho, \theta) \overline{h(\rho, \theta)} d\rho d\theta \]  

(6.111)

with \( A_g \) being the cross area of the sector. Now from (6.104)

\[ K(\rho, \theta, \rho', \theta') = K(\rho', \theta', \rho, \theta), \]  

(6.112)

and it follows that

\[ (g(\rho, \theta), K h(\rho, \theta)) = (K g(\rho, \theta), h(\rho, \theta)). \]  

(6.113)

Further if \( g = h \)

\[
\begin{align*}
\int_{A_g} g(\rho, \theta) \int_{A_g} K(\rho, \theta, \rho', \theta') g(\rho', \theta') d\rho' d\theta' \\
= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\varepsilon_m \coth(\alpha_{mn} d)}{\alpha_{mn} d q_{mn}} \left| \int_{A_g} g(\rho, \theta) \psi_{mn}^f(\rho) \Psi_m^f(\theta) d\rho d\theta \right|^2 \\
+ \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\beta_{mn} d q_{mn}} \left| \int_{A_g} g(\rho, \theta) \psi_{mn}^H(\rho) \overline{\Psi_m^H(\theta)} d\rho d\theta \right|^2 \geq 0,
\end{align*}
\]

(6.114)

provided that the infinite series converge. Thus it follows

\[ A = (g(\rho, \theta), \psi_{0,1}^f(\rho) \Psi_0^f(\theta)) = (g(\rho, \theta), K g(\rho, \theta)) \geq 0. \]  

(6.115)

Despite the fact that \( K(\rho, \theta, \rho', \theta') g(\rho', \theta') \) is complex, we have shown that for \( g(\rho, \theta) \) satisfying (6.107), \( A = (g(\rho, \theta), \psi_{0,1}^f(\rho) \Psi_0^f(\theta)) \) is real and non-negative so that trapped Rayleigh-Bloch modes will exist if we may find the solutions of \( A = kd \tan ka \). It is obvious that \( ka \) should be in the range of \( ((n - 1/2)\pi, n\pi) \) since \( A \) is real and non-negative.
It remains to choose the right form of function \( g(\rho, \theta) \). This may be directed by the requirement of the correct physical behavior and the simplicity of the final function forms. We might expect that sufficiently close to the edge of the fins that

\[
g(\rho, \theta) = C(\rho)\theta^{-1/2}(\beta - \theta)^{-1/2}, \quad \text{at } \theta = 0, \text{ and } \theta = \beta, \quad (6.116)
\]

since \( g(\rho, \theta) \) is proportional to the velocity, in which \( \theta^{-1/2}(\beta - \theta)^{-1/2} \) may be changed into the weight function of Chebyshev polynomials of the first kind. This suggest that if we fix \( \rho \), \( f(\rho, \theta) \) may be expressed as a complete set of Chebyshev polynomials, i.e.

\[
g(\rho, \theta) = \sum_{r=0}^{\infty} \xi^r C_r(\rho)\theta^{-1/2}(\beta - \theta)^{-1/2}T_r \left( \frac{2\theta - \beta}{\beta} \right), \quad (6.117)
\]

where the coefficients of expansion are incorporated into \( C_r(\rho) \). In a cylindrical system, \( \phi(\rho, \theta, x) \) satisfies \( \partial \phi / \partial \rho = 0 \) on \( \rho = d \). This suggest that \( C_r(\rho) \) may be expressed as a complete set of Bessel functions, \( C_r(\rho) = \sum_{s=1}^{\infty} \xi^r A_{rs} J_r(\beta_s \rho / d) \), then it follows

\[
g(\rho, \theta) = \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \xi^r A_{rs} J_r(\beta_s \rho / d)\theta^{-1/2}(\beta - \theta)^{-1/2}T_r \left( \frac{2\theta - \beta}{\beta} \right). \quad (6.118)
\]

For approximation and simplicity, we may write

\[
g(\rho, \theta) = \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs} g_{rs}(\rho, \theta), \quad (6.119)
\]

where

\[
g_{rs}(\rho, \theta) = \frac{\xi^r}{\pi} J_r(\beta_s \rho / d)\theta^{-1/2}(\beta - \theta)^{-1/2}T_r \left( \frac{2\theta - \beta}{\beta} \right). \quad (6.120)
\]

The coefficients of the Chebyshev polynomials are chosen to make the final function form simple. Chebyshev polynomials have the following properties

\[
\int_{0}^{\beta} (\beta^2 - \theta^2)^{-1/2} T_{2n}(\theta / \beta) \cos(y \theta) d\theta = \frac{1}{2} (-1)^n \pi J_{2n}(\beta y), \quad (6.121)
\]
\[ \int_0^\beta (\beta^2 - \theta^2)^{-1/2} T_{2n+1}(\theta/\beta) \sin(y\theta) d\theta = \frac{1}{2} (-1)^n \pi J_{2n+1}(\beta y), \quad (6.122) \]

and

\[ \int_0^1 (1 - x^2)^{-1/2} T_n(x) T_m(x) dx = \delta_{nm} \zeta_n, \quad \zeta_n = \frac{\pi}{2}, \quad \zeta_n = \frac{\pi}{4}, \quad n \geq 1. \quad (6.123) \]

Substitution of (6.120) into (6.107), multiplying by \( \rho g_{\mu \nu}(\rho, \theta) \), and integrating over the sector gives

\[
\sum_{r=0}^R \sum_{s=1}^S A_{rs} \sum_{m=0}^\infty \frac{\epsilon_{0} \coth(q_{0n})}{\alpha_{0n} d q_{0n}} U_{\rho r} U_{\rho s} U_{0r}^* U_{0s}^* + \]

\[
\sum_{r=0}^R \sum_{s=1}^S A_{rs} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\epsilon_m \coth(q_{mn})}{\alpha_{mn} d q_{mn}} U_{\rho m} U_{\rho n} U_{\mu m} U_{\mu n}^* + \]

\[
\sum_{r=0}^R \sum_{s=1}^S A_{rs} \sum_{m=-\infty}^\infty \sum_{n=1}^\infty \frac{1}{\beta_{mn} d Q_{mn}} V_{\rho m} V_{\rho n} V_{\mu m} V_{\mu n}^* = U_{\rho \mu} U_{0 \mu} \quad (6.124) \]

where \(( \mu = 0, 1, \ldots, S, \nu = 1, 2, \ldots, R ), \) and

\[ U_{\rho \nu} = \int_0^1 \rho J_r(j_{r,s}^\rho) \psi_{\rho \mu}^I(\rho) d\rho = \int_0^1 \rho J_r(j_{r,s}^\rho) J_{mL/2}^r(j_{mL/2,s}^\rho) d\rho, \quad (6.125) \]

\[ V_{\rho \nu} = \int_0^1 \rho J_r(j_{r,s}^\rho) \psi_{\rho \mu}^I(\rho) d\rho = \int_0^1 \rho J_r(j_{r,s}^\rho) J_{p+mL}^r(j_{p+mL,s}^\rho) d\rho, \quad (6.126) \]

\[ u_{\rho \nu} = \int_0^\beta \psi^I(\pi) - 1 \theta^{-1/2}(\beta - \theta)^{-1/2} T_r \left( \frac{2\theta - \beta}{\beta} \right) \Psi_{\rho m}(\theta) \]

\[ = J_r(mL\beta/4) \cos(mL\beta/4 + r\pi/2) e^{ir\pi/2}, \quad (6.127) \]

\[ \overline{u}_{\rho \mu} = \int_0^\beta \psi^I(\pi) - 1 \theta^{-1/2}(\beta - \theta)^{-1/2} T_{\mu} \left( \frac{2\theta - \beta}{\beta} \right) \Psi_{\rho m}(\theta) \]

\[ = J_{\mu}(mL\beta/4) \cos(mL\beta/4 + \mu\pi/2) e^{-i\mu\pi/2}, \quad (6.128) \]

\[ v_{\rho \mu} = \int_0^\beta \psi^I(\pi) - 1 \theta^{-1/2}(\beta - \theta)^{-1/2} T_{\nu} \left( \frac{2\theta - \beta}{\beta} \right) \overline{\Psi}_{\rho m}(\theta) \]

\[ = J_{\nu}((p+mL)\beta/2) e^{-i((p+mL)\beta/2)}, \quad (6.129) \]
and

\[
\overline{u_{mq}} = \int_0^\beta (i^\mu (-1)^\mu (\pi)^{-1} - \theta^{-1/2}(\beta - \theta)^{-1/2} T_\mu \left(\frac{2\theta - \beta}{\beta}\right) \Psi_m(\theta) = J_\mu((p + mL)\beta/2)e^{i(p+mL)/2}. (6.130)
\]

Equation (6.124) is an inhomogeneous, real, and linear system of equations with RS variables. Attention should be given to \( u_{mr}, \overline{u_{mq}}, \) as

\[
u_{00} = 1, \quad \nu_{0r} = 0, \quad r > 0; \quad \text{and} \quad \overline{u_{00}} = 0, \quad \overline{u_{0r}} = 0, \quad \mu > 0. \quad (6.131)
\]

For \( m > 0, \) after some manipulation, we have

\[
u_{mr} \overline{u_{mq}} = P_{mr} J_r(m\pi/2) J_\mu(m\pi/2) \quad (6.132)
\]

where

\[
P_{mr} = i^\mu (-1)^\mu \cos \left(\frac{m + r}{2}\pi\right) \cos \left(\frac{m + \mu}{2}\pi\right)
\]

\[
= \frac{1}{4}\{(1)^m + (-1)^r\}\{(1)^m + (-1)^\mu\} \quad (6.133)
\]

and

\[
u_{mr} \overline{u_{mq}} = J_r((p + mL)\pi/L) J_\mu((p + mL)\pi/L) \quad (6.134)
\]

where we have used the following results,

\[
(i^{r+\mu} (-1)^\mu e^{-i(r+\mu)\pi/2} = \begin{cases} (-1)^r, & r + \mu \text{ even} \\ (-1)^{r+\mu+1}, & r + \mu \text{ odd} \end{cases} = 1. \quad (6.135)
\]

Therefore equation (6.124) becomes

\[
\begin{align*}
&\sum_{r=0}^R \sum_{s=1}^S A_{rs} \sum_{n=2}^\infty \frac{\epsilon_{00} \coth(\alpha_{00}a_n)}{\alpha_{0n} d_{0n}} U_{0r} U_{0m}^* \delta_{0r} \delta_{0m} + \\
&\sum_{r=0}^R \sum_{s=1}^S A_{rs} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\epsilon_{mn} \coth(\alpha_{mn}a)}{\alpha_{mn} d_{mn}} U_{mn} U_{mr} \overline{u_{mr}} + \\
&\sum_{r=0}^R \sum_{s=1}^S A_{rs} \sum_{m=-\infty}^\infty \sum_{n=1}^\infty \frac{1}{\beta_{mn} d_{Qmn}} V_{nr} V_{nr} \overline{u_{mr}} = U_{0r} \delta_{0r}, \quad (6.136)
\end{align*}
\]

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where $\mu = 0, 1, ..., S$, $\nu = 1, 2, ... R$, and $u_{0r}, \overline{u_{0r}}$ have been replaced by $\delta_{0r} \delta_{0\mu}$.

It follows that equation (6.108) maybe expressed as

$$
\sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs} U_{0s}^{l} \delta_{0r} = \alpha_{0l}' d \tan(\alpha_{01}' \alpha) q_{01}/\varepsilon_0
$$

(6.137)

since $\overline{u_{0r}} = \delta_{0r}$.

Returning to (6.103), there are $N$ oscillatory terms with $N > 1$. Similarly, we define

$$
f(\rho, \theta) = \sum_{n=1}^{N} \frac{\beta a_{om} \cot(\alpha_{on}')} {2 \alpha_{om}' d} g^{(n)}(\rho, \theta)
$$

(6.138)

where

$$
g^{(n)}(\rho, \theta) = \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(n)} g_{rs}(\rho, \theta).
$$

(6.139)

Substitute (6.138) into (6.103), and choose

$$
\int_0^d \int_0^\beta \rho' g^{(n)}(\rho', \theta') K(\rho, \theta, \rho', \theta') d\rho' d\theta' = \psi_0^I(\rho) \Psi_0^I(\theta),
$$

(6.140)

which is multiplied by $\overline{\rho g_{\mu\nu}(\rho, \theta)}$ and integrated over the sector. After some algebra, we have

$$
\sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(l)} \sum_{n=1}^{N} \alpha_{on} d q_{on} \frac{\varepsilon_0 \coth(\alpha_{on}) U_{0r}^{ms} U_{0s}^{nv} \delta_{0r} \delta_{0\mu} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\varepsilon_0 \coth(\alpha_{mn}) U_{mr}^{ns} U_{mr}^{nv} \delta_{0r} \delta_{0\mu}} {\alpha_{mn} d q_{mn}} + \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(l)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\beta_{mn} d Q_{mn}} V_{mr}^{ns} V_{mr}^{nv} \delta_{0r} \delta_{0\mu} = \psi_0^I(\theta),
$$

(6.141)

where $l = 1, ..., N$, $\mu = 0, 1, ..., S$; $\nu = 1, 2, ... R$. Multiply (6.138) by $\overline{\rho \psi_0^I(\rho)} \Psi_0^I(\theta)$ on both sides, and integrate over the sector to give

$$
a_{0l} q_{0l} - \sum_{n=1}^{N} \frac{a_{on} \cot(\alpha_{on}')} {\alpha_{on}' d} \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(n)} U_{0s}^{l} u_{0r} = 0, \ l = 1, ..., N.
$$

(6.142)

This is a set of homogeneous equation about $\{a_{0l}, l = 1, ..., N\}$. The condition for a non-trivial solution to exist is that the determinant is zero.
Now returning to (6.100), (6.101), and using (6.138) we may express $a_{mn}, b_{mn}$ as

$$a_{mn} = \frac{\varepsilon_m}{Q_{mn}} \sum_{t=1}^{N} a_{ot} \cot(\alpha_{ot} d) \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(t)} U_{mr}^{ns} u_{mr},$$

(6.143)

and

$$b_{mn} = \frac{1}{Q_{mn}} \sum_{t=1}^{N} a_{ot} \cot(\alpha_{ot} a) \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(t)} v_{mr}^{ns} v_{mr}.$$  

(6.144)

Thus the potential in both regions may be expressed as

$$\phi^I(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \sum_{t=1}^{N} a_{ot} \cot(\alpha_{ot} a) \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(t)} U_{mr}^{ns} u_{mr} \right) \cdot \frac{\cosh(\alpha_{mn} x)}{\alpha_{mn} d q_{mn} \sinh(\alpha_{mn} a)} \psi^I_m(\rho) \psi^I_m(\theta), \quad 0 < x < a,$$

(6.145)

and

$$\phi^H(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \sum_{t=1}^{N} a_{ot} \cot(\alpha_{ot} a) \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(t)} V_{mr}^{ns} v_{mr} \right) \cdot \frac{e^{-\beta_{mn} (x-a)}}{-\beta_{mn} d Q_{mn}} \psi^H_m(\rho) \psi^H_m(\theta), \quad a < x < \infty.$$

(6.146)

The real part of the potential which may be expressed as

$$\eta^I(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \sum_{t=1}^{N} a_{ot} \cot(\alpha_{ot} a) \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(t)} U_{mr}^{ns} U_{mr} \right) \cdot \frac{\cosh(\alpha_{mn} x)}{\alpha_{mn} d q_{mn} \sinh(\alpha_{mn} a)} J_{mL/2} \left( \frac{mL}{2} \cos \left( \frac{m \pi}{2} \cos \frac{r \pi}{2} \right) \right)$$

$$\left( \frac{(m + \pi) \pi}{2} \cos \frac{r \pi}{2} \right)$$

(6.147)

for $0 < \rho < \beta, \quad 0 < \theta < \beta, \quad 0 < x < a$, and

$$\eta^H(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \sum_{t=1}^{N} a_{ot} \cot(\alpha_{ot} a) \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs}^{(t)} V_{mr}^{ns} V_{mr} \right) \cdot \frac{e^{-\beta_{mn} (x-a)}}{-\beta_{mn} d Q_{mn}} J_{p+mL} \left( \frac{p + mL}{2} \cos(p + mL)(\theta - \beta/2)(\theta) \right)$$

(6.148)

for $0 < \rho < d, 0 < \theta < 2\pi, \quad a < x < \infty$. Since there is no forcing in the problem we may rescale the surface profiles arbitrarily. We choose

$$B = \frac{a_{ot} \cot(\alpha_{ot} a)}{\alpha_{ot}}$$

(6.149)
as a scaling factor. For simplicity we choose $B = -1$ so that $\eta'(\rho, \theta, x)$ may be written for only one wave-like term in inner region as

$$
\eta'(\rho, \theta, x) = - \sum_{m=0}^{\infty} \sum_{n=1}^{R} \sum_{r=0}^{S} A_{mr} U_{ns} \cosh(\alpha_{mn} x) J_r(mL\beta/4) J_{m/2L}(j_{mL/2,n,0}/d) \cos(r\pi/2) \cos(mL\beta/4 + r\pi/2) \cos(mL\theta/2)
$$

(6.150)

for $0 < x < a$, and

$$
\eta''(\rho, \theta, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{R} \sum_{r=0}^{S} A_{mr} V_{ns} e^{-\beta_{mn}(x-a)} J_r((p + mL)\beta/2)
J_{p+mL}(j_{p+mL,n,0}/d) \cos(p + mL)(\theta - \beta/2),
$$

(6.151)

for $a < x < \infty$. There will be two wave-like terms in inner region if $j_{0,2} < kd < j_{3,1}$ for $L = 8, p = 3$. From (6.142) we have

$$
\frac{a_{02} \cot(a_{02} a)}{a_{02}} = \frac{B}{S_2} (q_{01} a_{01} d \tan(a_{01} a) - S_1)
$$

(6.152)

where

$$
S_1 = \sum_{r=0}^{R} \sum_{s=1}^{S} A_{r0} U_{0s}^1 u_{0r}, \quad S_2 = \sum_{r=0}^{R} \sum_{s=1}^{S} A_{rs} U_{0s}^1 u_{0r}.
$$

(6.153)

### 6.6 Results and discussion

In order to compute Rayleigh-Bloch mode frequencies we must first determine the matrix size and truncation number in the Galerkin approach. The Galerkin approach is powerful if we could model the exact singularity at the edge. Usually $A$ converges to three significant figures when using $4 \times 4$ matrix for two-dimensional cases[18, 91]. Here we are dealing with three dimensional cases. Real calculations show that we need $9 \times 9$ matrix for $A$ to get three significant figures of accuracy. In the sum of (6.141), both $m$ and $n$ should be truncated. The main computing time consumed in these sums is the numerical evaluation of $U_{mr}^ns$, $V_{mr}^ns$ since the integrands are highly oscillatory when $m$ and $n$ become large. Real computing shows that the convergence of three
significant figures can be achieved by truncating at \( m = n = 64 \) in (6.136) and (6.141).

The minimum number of fins for which we may find modes other than those of Linton & McIver[47] is \( L = 3 \). According to Linton and McIver[47], only modes antisymmetric about the mid plane of a sector may be found, which is between cut-off \( j_{3/2,1} \) and \( j_{3,1} \) in this case, since \( L \) is odd. However, our investigation shows that, the Rayleigh-Bloch modes may also be found for which \( p = 1 \) and the frequency is between 0 and \( j_{1,1}^r \). The results are shown in figure 6.2. The wavenumber \( kd \) of modes antisymmetric about the mid plane of a sector starts from the cut-off \( kd = j_{3,1}^r \approx 4.2017 \) and tends to the cut-off \( kd = j_{3/2,1}^r \approx 2.4605 \) as \( a/d \to \infty \). The wavenumber \( kd \) of Rayleigh-Bloch modes starts from the cut-off \( j_{1,1}^r \approx 1.8412 \) and tend to be zero as \( a/d \to \infty \). In both cases more modes will come as \( a/d \) increases. An interesting observation is that there is a gap, \( j_{1,1}^r < kd < j_{3/2,1}^r \), in which no modes may be found. Actually, if the number of the fins is odd, this gap in which no modes may be found always exists. This gap may be expressed as

\[
j_{L-1/2,1}^r < kd < j_{L/2,1}^r, \quad L \text{ odd.} \tag*{6.154}
\]

Figure 6.3 shows the Rayleigh-Bloch modes for \( L = 4 \). The upper set of the figures corresponds to modes antisymmetric about the mid plane of the sectors. Their wavenumbers start from cut-off \( j_{4,1}^r \approx 5.3176 \) and tend to be cut-off \( j_{2,1}^r \approx 3.0542 \). The middle set of the figures shows the modes symmetric about the mid plane of sectors which is the same as Rayleigh-Bloch mode solution for \( p = 2 \). Their wavenumber \( kd \) starts from cut-off \( j_{2,1}^r \approx 3.0542 \) and tend to be zero as \( a/d \to \infty \). The lower set of figures represents the solution of Rayleigh-Bloch modes for \( p = 1 \), their values of \( kd \) starting from the cut-off \( j_{1,1}^r \approx 1.8412 \) and tending to zero as \( a/d \to \infty \). In all these cases, more modes come when \( a/d \) become increases. In comparison with \( L \) odd, there is no frequency gap for \( L \) even for which trapped modes
may not be found. Notice that the solutions tend to be the same as \( a/d \) become large and \( 0 < kd < j_{\ell,1}' \). This reveals the asymptotic behaviour of \( kd \) for large \( a/d \), and will be discussed later. Attention should be given to the lower set of figures \((p = 1)\) which is the same as the modes that are symmetric about the mid plane of the sectors for \( L = 2 \).

Returning to our potential expansion (6.9) and (6.10), \( \alpha_{01}d \) is always imaginary since \( j_{0,1}' = 0 \). Therefore we always have at least one wave in the inner region when \( kd < j_{\ell,1}' \). However \( \alpha_{02}d \) may be real or imaginary depending on the value of \( kd \). We know that \( j_{0,2}' \approx 3.8317 \), \( j_{2,1}' \approx 3.0542 \), \( j_{3,1}' \approx 4.2012 \), therefore when \( L \leq 5 \) there is only one wave in the inner region for any \( p \leq [L - 1/2] \) whilst there will be two waves in the inner region if \( j_{0,2}' < kd < j_{\ell,1}' \), \( p \geq 3 \) for \( L \geq 6 \). However, for convenience we use \( L = 8 \) as an example to show the different behaviour of the modes below and above first interior cut-off.

Figure 6.4 shows the Rayleigh-Bloch modes For \( L = 8 \), \( p = 1, 2 \). In these cases there is only one wave in inner region since \( j_{1,1}' < j_{2,1}' \) \& \( < j_{0,2}' \). Similarly, the solutions for both \( p = 1 \) and \( 2 \) tend to be the same as \( a/d \) become large and \( kd < j_{1,1}' \).

Figure 6.5 shows the results for \( L = 8 \) and \( p = 3 \). In this case the cut-off is \( j_{3,1}' \approx 4.2012 \), which is above the first interior cut-off \( j_{0,2}' \approx 3.8317 \). So there is only one wave in inner region for \( 0 < kd < j_{0,2}' \) and two waves when \( j_{0,2}' < kd < j_{3,1}' \). First we look at the figures for \( kd < j_{0,2}' \). The values of \( kd \) start from \( j_{0,2}' \approx 3.8317 \) and tend to zero as \( a/d \rightarrow \infty \). The fairly regular spacing between the modes may be observed. However this simple behaviour vanished when \( j_{0,2}' < kd < j_{3,1}' \). For \( a/d \) sufficiently large an extra set of modes exists and the two sets of curves appear to 'kiss' and then separate. Linton & McIver [47] explained this phenomenon in terms of so-called diabolical points, which were first observed by Berry \& Wilkinson[2] in bounded regions. Physically the behaviour first occurs when two waves
exist in the inner region, and so two different types of trapped oscillations are possible. The results for \( p = 4, L = 8 \), corresponding to modes symmetric about the mid plane of the sectors, is exactly the same as the Figure 8 in Linton and McIver[47], so it will not be shown here.

Returning to (6.26), the right hand side of (6.26) tends to zero as \( a/d \to \infty \). Thus we may find the asymptotic behaviour for large \( a/d \) from this behavior. From (6.26), if we have only one wave-like term in the inner region, then the approximate solution may be expressed as

\[
ka \approx \left( m - \frac{1}{2} \right) \pi, \ m = 1, 2, 3, ..., \quad (6.155)
\]

and if there are two wave-like terms in the inner region, then

\[
ka \approx \left( \left( m - \frac{1}{2} \right)^2 + \left( \frac{j''_0(a)}{d} \right)^2 \right)^{1/2}, \ m = 0, 1, 2, 3, .... \quad (6.156)
\]

For modes antisymmetric about the plane \( x = 0 \), the approximate solution for only one wave-like term in the inner region may be expressed as

\[
ka \approx m \pi, \ m = 1, 2, 3, ..., \quad (6.157)
\]

and for two wave-like terms in the inner region as

\[
ka \approx \left( m^2 + \left( \frac{j''_0(a)}{d} \right)^2 \right)^{1/2}, \ m = 0, 1, 2, 3, .... \quad (6.158)
\]

Figure 6.6 shows the comparison of numerical solutions and asymptotic approximations. It is obvious, that for large \( a/d \), the approximation may be used to estimate the trapped modes when \( kd < j''_{1,1} \).

6.7 Conclusion

In this chapter, we have discussed possible Rayleigh-Bloch modes in the presence of radial fins in a cylindrical waveguide. An analysis shows that
Rayleigh-Bloch modes do exist. We introduced an integer $p$ to represent the parameter in Rayleigh-Bloch mode representations and showed that modes exist for different values of $p$. After giving an existence proof, we discussed the connection between the Rayleigh-Bloch modes and trapped modes. A matched eigenfunction method was used to calculate modes numerically, and results were presented for a number of geometrical configurations.
Figure 6.2: Modes for $L = 3$. The upper set of figures corresponds to modes antisymmetric about the mid plane of the sectors ($p = 0$), whereas the lower set of figures corresponds to the Rayleigh-Bloch modes for $p = 1$. 
Figure 6.3: Modes for $L = 4$. The upper set of figures corresponds to the modes antisymmetric about the mid plane of the sectors ($p = 0$), whereas the middle set of the figures shows the modes symmetric about the mid plane of the sectors ($p = 2$) and the lower set for Rayleigh-Bloch modes $p = 1$.
Figure 6.4: Modes for $L = 8$. The upper set of the figures corresponds to $p = 2$ whilst the lower set of the figures represents $p = 1$. 
Figure 6.5: Variation of $kd$ with $a/d$ for $L = 8$ and $p = 3$
Figure 6.6: Solid lines represent modes symmetric about the plane $x = 0$ and dashed lines are modes anti-symmetric about the plane $x = 0$, in which thick lines represent approximation for large $a/d$. 
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