Quantum-bit devices inspired by classical stochastic analogies.

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Additional Information:

- A Doctoral Thesis. Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University.

Metadata Record: [https://dspace.lboro.ac.uk/2134/13022](https://dspace.lboro.ac.uk/2134/13022)

Publisher: © Zoe M. Washington

Please cite the published version.
This item was submitted to Loughborough’s Institutional Repository (https://dspace.lboro.ac.uk/) by the author and is made available under the following Creative Commons Licence conditions.

For the full text of this licence, please go to: http://creativecommons.org/licenses/by-nc-nd/2.5/
Quantum-bit devices inspired by classical stochastic analogies

Zoe M. Washington, MPhys.

A Doctoral Thesis submitted in partial fulfillment of the requirements for the award of

Doctor of Philosophy of

Loughborough University

July 2013

© Zoe M. Washington 2013
I would like to dedicate this thesis to my Nan, Mary Chapman, you have never left my thoughts to this day.
Abstract

As systems/structures get smaller we need to take into account noise and quantum effects and so, we need to develop some quantum devices. Quantum devices work using quantum principles like qubits that have already been developed, i.e., superconducting qubits that are going to be discussed in chapter 1. Initially, scientists wanted to use qubits to do quantum computations, this is not easy so scientists developed methods to do something different, e.g. quantum metamaterials [2]. Here in this thesis we describe two examples of quantum devices.

Our first device is the parametric quantum amplifier. Used when we need to amplify very weak signals. Amplifying a weak signal on the nanoscale is a very big challenge, this is due to classical and quantum noise, and so, we need to employ quantum physics to resolve this issue. The proposed two-qubit system amplifies weak signals at very small scales. We have shown that we can construct a multitude of novel devices on the nano-scale with the use of qubits.

Our second device uses harmonic mixing. It can be used where rectification is needed, for example, when we need to rectify some fluctuations and in principle some quantum fluctuations in order to pump either an excited or ground state of the two qubit device. In this thesis we propose how to do this. Firstly, we propose that if we apply harmonic mixing of two signals for two qubits, using the structure of the equation and basically the structure of quantum mechanics we can pump a desirable quantum state. We can pump either the upper or ground state by changing the signal.
Papers and conferences

The results presented in this thesis were published in two Physical Review A articles:


Our work on ”Two-qubit parametric amplifier: Large amplification of weak signals” was also presented at the following conferences:


Contents

1 Introduction 17

1.1 Noise ................................. 17
   1.1.1 Brownian motion .................. 19
   1.1.2 Chaos ............................... 20
1.2 Numerics ............................... 21
   1.2.1 Euler method ...................... 21
   1.2.2 Runge-Kutta method ............... 23
   1.2.3 Multiderivatives method .......... 25
1.3 Parametric amplifiers .................. 27
1.4 Harmonic mixing ........................ 30
1.5 Quantum computation and qubits ........... 35
1.6 Master equations ....................... 38
1.7 Superconducting qubits ................. 39

2 Two-qubit parametric amplifier: large amplification of weak signals 47

2.1 Parametric Amplifier .................... 47
   2.1.1 Outline of results ................. 48
2.2 Quantum amplification with qubits ........................................... 49
2.3 Simulation results ................................................................. 54
2.4 Noise influence ................................................................. 60
2.5 Away from the optimal regime: robust amplification of two-qubit amplifier ................................................................. 61
2.6 Conclusions to chapter 2 ...................................................... 64

3 Harmonic mixing in two coupled qubits: quantum synchronization via ac drives ................................................................. 66
   3.1 Introduction ................................................................. 66
   3.2 Model ................................................................. 67
   3.3 Simulation results ................................................................. 68
   3.4 Conclusions to chapter 3 ...................................................... 74

4 Conclusions ................................................................. 75
List of Figures

1.1 Stochastic simulation of an isomerisation reaction $\chi \rightleftharpoons A$. Molecules change from one to another as seen by the decrease of the blue curve and the increase of the green curve (the molecules change from the blue curve molecule to the green curve molecule predominantly). The plot shows multiple simulations that follow the same trend (see the multiple of fluctuating lines on top of one another). This example uses parameters and conditions as described in [3] . . . . . . . . . . . . . . . . 18

1.2 Illustration of the Euler method. The unknown curve or trajectory is in pink, and the green curve is its polygonal approximation using Euler’s method. It is the collection of line segments as a result of Euler’s Method. Each time Euler method is used another point is created and thus another line segment. Generally, the approximation gets less accurate the further you are away from the initial value. Better accuracy is achieved when the points in the approximation are closer together. . . . 23
1.3 Expression 1.26 plotted to show harmonic mixing of the two signals. A) Mixing of the two signals with $A_1 = 0.1, A_2 = 0.5, \omega = 10$ and $\alpha = -5$. It can be seen from the plot that if we integrate we will get a response of zero. B) $A_1 = 0.5, A_2 = 0.1, \omega = 10$ and $\alpha = -0.1$. It can be seen from the plot that if we integrate we will get a positive response. We have shown that a dc signal can be produced as we require by varying the parameters of the drive.

1.4 The Bloch sphere representation of the possible states of a qubit. The sphere provides a geometric representation of a pure qubit (quantum bit) state space as points on the surface of the unit sphere. Any point of the surface represents some pure qubit. The mixed qubit states can be represented by points inside of the unit sphere, with the maximally mixed state laying at the centre. The $|\psi\rangle$ line from the centre to the surface of the sphere corresponds to the pure state and has unit length. For mixed qubit state the length of line must be less than 1. The numbers $\theta$ and $\phi$ define a point on the sphere of Eq. 1.32.
1.5 A charge qubit is formed by a tiny superconducting island (known as the Cooper pair box) coupled by a Josephson junction reservoir. The state of the qubit is determined by the number of Cooper pairs which have tunnelled across the junction. A) Charge qubit schematic representation including the cooper pair box, marked by the green dashed rectangle, and a Josephson junction, marked by a square with an X. B) Realisation of a cooper pair box coupled to a single electron transistor (SET). Scanning electron micrograph of the nanofabricated sample used in the experiment. The superconducting box in the upper part is capacitively coupled to the electrometer (lower part) and the Josephson junctions are the bright dots [31][32].
1.6 Flux qubits (also known as persistent current qubits) are micrometer sized loops of superconducting loops interrupted by a number of Josephson junctions, shown by the squares with X inside. The junction parameters are engineered during fabrication so that a persistent current will flow continuously when an external flux is applied. The computational basis states of the qubit are defined by circulating currents which can flow either clockwise or anti-clockwise. These currents screen the applied flux limiting it to multiples of the flux quantum and give the qubit its name. When the applied flux through the loop is close to a half-integer number of flux quanta the two energy levels corresponding to the two directions of circulating current are brought together and the loop may be operated as a qubit. A) Flux qubit schematic representation, micrometer sized superconducting loops interrupted by a number of Josephson junctions, marked by squares with an X inside. B) Realisation of a flux qubit. Scanning electron micrograph showing the flux qubits with three Josephson junctions [35].
1.7 A) Phase qubit schematic-superconducting device based on the superconductor-insulator-superconductor (SIS) Josephson junction, shown by a square with an X inside and also shows a constant current source (two circles). B) Phase qubit realisation. The superconducting qubit is an aluminium circuit, a few hundred microns across but considered macroscopic from the point of view of quantum physics (which describes the atomistic scale world), and the low temperatures (30mk) brought out its quantum properties. Microwaves at a frequency of 4.743 GHz can be used to drive it only between its ground and first excited states the qubit’s 0 and 1 states. [36] . . . . . . 43

2.1 Schematic diagram of two flux qubits (persistent current qubits) coupled via a coupler loop. Josephson junctions are represented by crosses, and their thickness indicate their Josephson critical currents (drawn not to scale). The qubit states and the amplitude and sign of the coupling constant $g$ are controlled by the corresponding magnetic fluxes, $f^{(1,2)}_e$ and $f_{coupler}$ (in units of $\Phi_0$). Both the pumping drive and the weak signal $A \sin \omega t + \epsilon \sin \tilde{\omega} t$ are generated by a source in the bottom circuit. The left circuit is needed to pick up an output signal $Z_1(t)$. The top circuit controls the coupling $g$. Noise, shown by $\sqrt{2D} \xi_1(t)$, is coupled to each qubit. . . . . . . . . . . . . . 51
2.2 The spectrum $S_Z$ of the $Z_1$ matrix element (responsible for the occupation of the excited level of the first qubit) is shown for dephasing and relaxation given by $\Gamma_\phi/\Delta = \Gamma_r/\Delta = 10^{-3}$ (we use the same $\Gamma_\phi$ and $\Gamma_r$ for all results reported in this article). (a) When the weak signal amplitude is equal to zero ($\epsilon = 0$) and when the reduced-drive amplitude $A/\Delta = 15$, the ac monochromatic drive fed into the two-qubit amplifier is converted into a set of even harmonics $\omega_m = m\omega$, with $m = 2, 4, 6...$ Note the absence of response at $\omega_1 = \omega$ (no peak there). For comparison, $S_Z$ is shown in blue and indicated by the blue arrow for zero-drive amplitude $A = 0$ and weak signal $\epsilon/\Delta = 0.1$. Only one very small peak is hardly seen, indicated by the blue arrow, corresponding to $2\omega_{weak}$. (b) Spectrum $S_X$ of the off-diagonal matrix element $X_1$ (zero signal case). The same ac monochromatic drive fed into the amplifier is converted into a set of odd harmonics for $S_X$: $m = 1, 3, 5, ...$ Note that (a) and (b) show very unusual non-monotonic spectra. The observed non-monotonicity in the system response can be seen as a fingerprint of the qubits, characterizing their dynamical nonlinear response.
2.3 (a) The amplification of a weak signal $\epsilon/\Delta = 0.1$ by a strong drive $A/\Delta = 15$ (i.e., $\epsilon/A = 1/150$) can be seen in the spectrum of $S_Z$ shown by the black solid lines. The spectrum obtained is not a simple superposition of the two spectra $S_Z(A/\Delta = 15, \epsilon = 0)$ (also shown here by red vertical dotted peaks) and $S_Z(A = 0, \epsilon/\Delta = 0.1)$ [see also red and blue peaks in Fig. 2.2(a)]. Instead, a combination of harmonics appears for $k\omega + l\tilde{\omega}$, with integer $k$ and $l$. The height of these peaks is almost proportional to $\beta \epsilon$, for $0 < \epsilon/A < 0.005$, and it is strongly enhanced by a factor $\beta \sim 100$. The enhanced peak is marked by an arrow with symbol $I(\epsilon)$. Note that this peak is absent in Fig. 2.2(a). (b) Normalized output amplitude $I(\epsilon)/I(A)$ (ratio of the heights of the highest mixed peak $I(\epsilon)$ to the highest peak $I(A)$ of the spectrum at $\epsilon = 0$) as a function of the reduced weak signal amplitude $\epsilon/A$. This dependence shows an almost-linear increase of the peak height with $\epsilon$ for small $\epsilon/A$, followed by saturation and even decay at relatively large $\epsilon/A$.

2.4 Trajectories $Z_1(t)$ for different noise levels: $\sqrt{D}/\epsilon = 0$ for (a) and $\sqrt{D}/\epsilon = 0.066$ for (b). All other parameters are the same as in Fig. 2.3(a). As seen from (b), the applied white noise considerably affects the time dependence of $Z_1(t)$, making trajectories quite noisy with respect to the noiseless situation shown in (a).
2.5 Spectrum $S_z(\omega)$ for the same parameters used in Fig. 2.3 and different noise levels $\sqrt{D}/\epsilon = 0.066$ for (a) and 0.2 for (b). One can clearly see the mixed (pump-signal) peaks for several combination-frequencies $k\omega_{\text{pump}} + l\omega_{\text{weak}}$, even at high noise levels, when the trajectories $Z_1(t)$ shown in Fig. 2.4 are very noisy.

2.6 Amplification of a weak signal away from the optimal regime.

(a) Spectrum $S_z$ for off-resonance frequency $\omega_{\text{weak}} = 1.113\omega^{(3)}(g = 1)$ of a weak signal shown by red thin solid line. All other parameters of the simulations are the same as in Fig. 2.3. The thick black dotted line corresponds to the simulations with zero signal, thus, highlighting the combination-frequency amplified harmonics as red lines with no black dots on top. One can see that the amplification of off-resonance signal is weaker, but still remarkable with amplification factor $\beta$ about 30. (b) Spectrum of $S_z$ for weakly coupled qubits $g/\Delta = 0.1$ and weak signal amplitude $\epsilon/A = 24$, as shown by the red thin solid line; all other parameters are as in Fig. 2.3 and $A/\Delta = 12$. The same simulations but with zero signal is shown by a thick black dotted line to highlight the amplified mixed peaks. The estimated amplification factor $\beta$ is about 10.
3.1 Time-averaged Bloch tensor components $\langle X_1 \rangle = \langle \Pi_{0x} \rangle$ (a) and $\langle Z_1 \rangle = \langle \Pi_{0z} \rangle$ (b) for two coupled qubits driven by the two harmonic drive (3.1) with parameters: $A_1 = A_2 = 10$, $\phi = 0$, $\omega_1 = 2\sqrt{\Delta^2 + g^2}$. Other parameters are the simulation step $dt = 1.13 \times 10^{-5}$, the number of simulation steps $5 \times 10^9$, damping $\Gamma = 10^{-3}$, coupling constant $g = 1$, the tunnelling splitting energies $\Delta = 1$. The peaks of the time-averaged Bloch tensor element $\langle X_1 \rangle$ responsible for the qubit coherence shows peaks at $\omega_2/\omega_1 = 2/5, 4/5, 2, 4$, while the time averaged component $\langle Z_1 \rangle$ peaks at $\omega_2/\omega_1 = 3/5, 9/10, 3, 5$. Also, pumping the excited state for incommensurate frequencies are clearly seen: $|\langle Z_1 \rangle|$ increases for $\omega_2 \gtrsim 2\omega_1$. 70

3.2 Time-averaged Bloch tensor components $\langle X_1 \rangle = \langle \Pi_{0x} \rangle$ (a) for two coupled qubits driven by the two harmonic drive (3.1) with the same parameters as in Fig.3.1 and driving frequency $\omega_1 = \sqrt{\Delta^2 + g^2} - g$ (a) equal to an energy level transition frequency and for $\omega_1 = 2.113(\sqrt{\Delta^2 + g^2} - g)$ (b) which is away from the energy level transition. 71
3.3 Dependence of $\langle X_1 \rangle$ and $\langle Z_1 \rangle$ on a relative phases of two drives of the bi-harmonic signal at different frequency ratio $\omega_2/\omega_1 = 2$ (a), 3(b), 4(c). All other parameters are the same as in Fig. 1. For even frequency ratio, where $\langle X_1 \rangle$ has a peak (Fig.1), the strong dependence of $\langle X_1 \rangle(\varphi)$ and a weak dependence of $\langle Z_1 \rangle(\varphi)$ occurs, while, for odd ratio of $\omega_2/\omega_1$, dependence of $\langle Z_1 \rangle(\varphi)$ is clearly seen and $\langle X_1 \rangle(\varphi)$ is negligible. The periods of functions $\langle X_1 \rangle(\varphi)$ and $\langle Z_1 \rangle(\varphi)$ are controlled by the frequency ratio and is equal to $2\pi \omega_1/\omega_2$ (for $\omega_2 > \omega_1$).
Acknowledgements

To all my friends and family that have supported me through this ordeal (only kidding), this thesis, I thank you. You have all made my research much easier in a variety of ways. You all know who you are, even if it was just a little text telling me to just get on with it! It all helped.

I would like to thank Loughborough University for offering me the studentship to do this PhD, and for the excellent facilities that made my undergraduate, and then my postgraduate, an absolute pleasure. To all the other postgraduates (past and present), you have all helped in your own way and maybe, just maybe slightly hindered my work at times!

Finally, last but not least (by any stretch of the imagination), I would like to show my sincerest gratitude to my supervisor Professor Sergey Savel’ev; he has kept me motivated even though he has an amazing ability to make me feel very stupid but then clever in the next breath. You truly are a pleasure to be supervised by, without you this all would have been impossible. You’ve seen me laugh (mainly at myself), you’ve seen me cry and throughout it all you’ve never stopped supporting me and my work, thank you.
Chapter 1

Introduction

1.1 Noise

Theoretical science up to the end of the nineteenth century can be viewed as the study of solutions of differential equations and the modelling of natural phenomena by deterministic solutions of these differential equations. It was at the time commonly thought that if all initial data could only be collected, one would be able to predict the future with certainty.

We now know this is not so, in at least two ways. Firstly, the advent of quantum mechanics within a quarter of a century gave rise to a new physics and hence a new theoretical basis for all science, which had as an essential basis a purely statistical element. Secondly, more recently, the concept of chaos has arisen, in which even quite simple differential equation systems have the rather alarming property of giving rise to essentially unpredictable behaviour. To be sure, one can predict the future of such a system given it’s initial conditions is so rapidly magnified that no predictability is left.
In fact, the existence of chaos is really not surprising, since it agrees with more of our everyday experience than does pure predictability - but it is surprising perhaps that it has taken so long for the point to be made.

Figure 1.1: Stochastic simulation of an isomerisation reaction $\chi \rightleftharpoons A$. Molecules change from one to another as seen by the decrease of the blue curve and the increase of the green curve (the molecules change from the blue curve molecule to the green curve molecule predominantly). The plot shows multiple simulations that follow the same trend (see the multiple of fluctuating lines on top of one another). This example uses parameters and conditions as described in [3]

The experience of careful measurements in science normally gives us data like that of Fig. 1.1, representing the growth of the number of molecules of a substance $\chi$ formed by a chemical reaction of the form $\chi \rightleftharpoons A$. A quite well defined deterministic motion is evident, and this is reproducible, unlike the
fluctuations around this motion, which are not.

1.1.1 Brownian motion

For microscopic or nano-scale motors both thermal and quantum fluctuations become crucial; the influence from energy fluctuations within the system is of the same order of magnitude as the energy taken from any fuel driving motor, and hence detailed studies of how fluctuations affect performance of nano-motors are needed.

One of the most famous manifestations of fluctuations for small (micro or nano) objects was first discovered by Scottish botanist Robert Brown in 1827, whilst studying pollen floating on water under the microscope. While examining the particles immersed in the water, Brown observed many of them evidently in motion; their motion consisting of a change of place in the fluid [4]. This inherent, incessant motion of small particles suspended in a fluid is nowadays called Brownian motion in honour of Robert Brown. Similar observations had, in fact, been made earlier by other workers. Brown, however, was the first to give them a serious scientific study, and showed that the phenomenon was not one of biology, but one of physics [5]. It was not until later on that the jittery motion was explained.

In 1905 Albert Einstein brought the solution of the problem to the attention of the physicists, and presented it as a way to indirectly confirm the existence of atoms and molecules. Einstein predicted that Brownian motion of a particle in a fluid at a thermodynamic temperature $T$ is characterised by and proportional to $k_B T$, where $k_B$ is the Boltzmann’s constant [6].
Brownian motion is the seemingly random movement of particles suspended in a fluid or the mathematical model used to describe such random movements. Brownian motion is among the simplest of continuous time stochastic (or random) processes, and it is a limit of both simpler and more complicated processes. From the point of view of this thesis, Brownian motion demonstrates how thermal fluctuations can affect work of small motors or devices that should work in very noisy environments.

1.1.2 Chaos

The processes, that have an element of random behaviour, can also be observed in deterministic systems. For instance, it is known that deterministic differential equations can sometimes exhibit very unpredictable chaotic dynamics. Chaos theory studies the behaviour of dynamical systems that are highly sensitive to initial conditions, an effect which is popularly referred to as the butterfly effect. Small differences in initial conditions (such as those due to rounding errors in numerical computation) yield widely diverging outcomes for such dynamical systems, rendering long-term prediction impossible in general. This happens even though these systems are deterministic, meaning that their future behaviour is fully determined by their initial conditions, with no random elements involved. In other words, the deterministic nature of these systems does not make them predictable.
“Chaos: when the present determines the future, but the approximate present does not approximately determine the future.”
Edward Lorenz

Therefore, both stochastic and complex ordinary differential equations show unpredictable random dynamics. The question arises of if we can use such random dynamics for something useful in the nano-scale.

1.2 Numerics

Most results reported in this thesis were obtained by numerical simulations of stochastic/deterministic differential equations. Note, analytical solutions of both stochastic and complicated differential equations are very challenging, and often impossible. On the other hand, numerical calculations can be easily implemented on modern computers.

1.2.1 Euler method

For systems with a certain amount of noise, we used the Euler method.

In mathematics, the Euler method is a first-order numerical procedure for solving ordinary differential equations with a given initial value. The Euler method is a first-order method, which means that the local error (error per step) is proportional to the square of the step size, and the global error (error at a given time) is proportional to the step size. It also suffers from stability problems. For these reasons, the Euler method is not often used in practice for ordinary equations. However, the Euler method is usually very useful
for stochastic equations where the force strongly fluctuates and where higher order methods that usually require “acceleration” fail.

Consider the problem of calculating the shape of an unknown curve which starts at a given point and satisfies a given differential equation. Here, a differential equation can be thought of as a formula by which the slope of the tangent line to the curve can be computed at any point on the curve, once the position of that point has been calculated. The idea is that while the curve is initially unknown, its starting point, which is denoted by \(A_0\), is known (See fig 1.2). Then, from the differential equation, the slope to the curve at \(A_0\) can be computed, and so, the tangent line.

Take a small step along that tangent line up to a point \(A_1\). Along this small step, the slope does not change too much, so \(A_1\) will be close to the curve. If we pretend that \(A_1\) is still on the curve, the same reasoning as for the point \(A_0\) above can be used. After several steps, a polygonal curve \(A_0A_1A_2A_3\ldots\) is computed. In general, this curve does not diverge too far from the original unknown curve, and the error between the two curves can be made small if the step size is small enough and the interval of computation is finite.

The Euler method is the most basic explicit method for numerical integration of ordinary differential equations and is the simplest Runge-Kutta method.
1.2.2 Runge-Kutta method

If there is no noise in the system one can use a method of higher order. The simplest one is called the Runge-Kutta method.

In numerical analysis, the Runge-Kutta methods are an important family of implicit and explicit iterative methods for the approximation of solutions of ordinary differential equations. One member of the family of Runge-Kutta methods is often referred to as “classical Runge-Kutta method”. Let the
initial value problem be specified as follows.

\[ \dot{y} = f(t, y), y(t_0) = y_0 \]  

(1.1)

Here, \( y \) is an unknown function of time \( t \) which we would like to approximate; it is known that \( \dot{y} \), the rate at which \( y \) changes, is a function of \( t \) and \( y \) itself. At the initial time \( t_0 \) the corresponding \( y \)-value is \( y_0 \). The function \( f \) and the data \( t_0, y_0 \) are given.

Now pick a step size \( h > 0 \) and define

\[ y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \]  

(1.2)

\[ t_{n+1} = t_n + h \]  

(1.3)

for \( n = 0, 1, 2, 3, \ldots \) using

\[ k_1 = f(t_n, y_n), \]  

(1.4)

\[ k_2 = f(t_n + \frac{1}{2}h, y_n + \frac{h}{2}k_1), \]  

(1.5)

\[ k_3 = f(t_n + \frac{1}{2}h, y_n + \frac{h}{2}k_2), \]  

(1.6)

\[ k_4 = f(t_n + h, y_n + hk_3) \]  

(1.7)

Here \( y_{n+1} \) is the Runge-Kutta approximation of \( y(t_{n+1}) \), and the next value \( y_{n+1} \) is determined by the present value \( y_n \) plus the weighted average of four increments, where each increment is the product of the size of the interval, \( h \), and an estimated slope specified by function \( f \) on the right-hand
side of the differential equation. \( k_1 \) is the increment based on the slope at the beginning of the interval; \( k_2 \) and \( k_3 \) are the estimates of the increment based on the slope at the midpoint of the interval; and \( k_4 \) is the increment based on the slope at the end of the interval. In averaging the four increments, greater weight is given to the increments at the midpoint. The weights are chosen such that if \( f \) is independent of \( y \), so that the differential equation is equivalent to a simple integral.

Runge-Kutta method is the second order method with accuracy of up to \( h^2 \) in contrast to \( h \) for the Euler method. Unfortunately, for fast changing random force, this method cannot be easily generalised [7].

### 1.2.3 Multiderivatives method

For many practical problems, it is possible to derive formulae for the second, and higher, derivatives of \( y \), making use of the formula for \( y' \) given by a differential equation. This opens up many computational options, which can be used to enhance the performance of multi-stage and multi-value (multi-step) methods. If these higher derivatives are available, then the most popular option is to use them to evaluate a number of terms in Taylor’s theorem. Second order multi-derivative method gives accuracy to \( h^2 \) like the Runge-Kutta method. It requires the initial value and is based on Taylor series to give an approximate but accurate value of the solution.

If we have a function that is dependent on \( t \) and \( y \) we can follow the following procedure to find the solution to the ordinary differential equation:
\[ \frac{dy}{dt} = f(t, y) \] (1.8)

when the initial value \( y(t_0) = y_0 \) is given and is a known point on the solution curve.

If the existence of all higher order partial derivatives is assumed for \( y \) at \( t = t_0 \), then by Taylor series the value of \( y \) at any neighbouring point \( t + h \) can be written as

\[ y(t + h) = y(t) + hy'(t) + \frac{h^2}{2!}y''(t) + ... \] (1.9)

where \( ' \) represents the derivative with respect to \( t \). Since \( t_0, y_0 \) is known, \( y' \) at \( t_0 \) can be found by computing \( f(t_0, y_0) \). Similarly higher derivatives of \( y \) at \( t_0 \) can also be computed making use of the relation \( y' = f(t, y) \).

At the point \( t = t_i \)

\[ y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i ... \] (1.10)

But, for the required solution \( y(t) \), we know that

\[ y'_i \equiv \left( \frac{dy}{dt} \right)_{t_i} = f(t_i, y_i), \] (1.11)

and the value of the second derivative at \( t = t_i, y = y_i \) can be obtained from it:

\[ y''_i = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial y} \] (1.12)
Hence the value of $y$ at any neighbouring point $t_0 + h$ can be obtained by summing the above infinite series. However, in any practical computation, the summation has to be terminated after some finite number of terms. The same can be repeated to obtain $y$ at other points of $t$ in the interval $[t_0, t_n]$ in a marching process.

We will use the second order multi-derivative method to solve deterministic equations in this thesis.

### 1.3 Parametric amplifiers

Using the numerical methods, that have been described above, we can simulate an array of interesting dynamical effects and these effects can be implemented in useful devices. One such effect is called parametric amplification.

Parametric amplifiers were first used in 1913-1915 for radio telephony from Berlin to Vienna and Moscow, and were predicted to have a useful future [8]. The early parametric amplifiers varied inductances, but other methods have been developed since: e.g., the Varactor diodes, Klystron tubes, Josephson junctions and optical methods.

The simplest equation to describe parametric amplification or parametric resonance is:

$$\frac{d^2x}{dt^2} + \beta(t) \frac{dx}{dt} + \omega^2(t)x = 0$$

(1.13)

The equation is linear in $x(t)$. By assumption, the parameters $\omega^2$ and $\beta$ depend only on time and do not depend on the state of the oscillator. In general, $\beta(t)$ and/or $\omega^2(t)$ are assumed to vary periodically, with the same
A parametric amplifier is based on the idea of a parametric resonance occurring for a non linear oscillator with parameters oscillating in time. Such an amplifier is implemented as a mixer, where an input weak signal is mixed with a strong local oscillator signal producing the strong output. The mixer’s gain shows up in the output as amplifier gain. The input signal is mixed with a strong local oscillator signal, and the resultant strong output is used in the ensuing receiver stages.

The parametric oscillator equation can be extended by adding an external driving force $E(t)$.

$$\frac{d^2x}{dt^2} + \beta(t)\frac{dx}{dt} + \omega(t)x = E(t) \quad (1.14)$$

We assume that the damping $\beta$ is sufficiently strong that, in the absence of the driving force $E$, the amplitude of the parametric oscillations does not diverge. In this situation, the parametric pumping acts to lower the effective damping in the system. For illustration, let the damping constant be $\beta(t) = \omega_0b$ and assume that the external driving force is at the mean frequency $\omega_0$ i.e., $E(t) = E_0 \sin \omega_0 t$. The equation becomes:

$$\frac{d^2x}{dt^2} + b\omega_0 \frac{dx}{dt} + \omega_0^2 [1 + h_0 \sin 2\omega_0 t] x = E_0 \sin \omega_0 t \quad (1.15)$$

whose solution is roughly

$$x(t) = \frac{2E_0}{\omega_0^2 (2b - h_0)} \cos \omega_0 t \quad (1.16)$$
as $h_0$ approaches the threshold $2b$, the amplitude diverges $[9]$. When $h \geq 2b$, the system enters parametric resonance and the amplitude begins to grow exponentially even in the absence of a driving force $E(t)$.

Advantages of parametric amplifiers:

- It is highly sensitive
- Low noise level amplifier for ultra high frequency and microwave signal
- Unique capability to operate as a wireless powered amplifier that doesn’t require an internal power source.

A parametric amplifier is based on the idea of a parametric resonance occurring for a linear oscillator with parameters oscillating in time. Such an amplifier $[10]$ is implemented as a mixer, where an input weak signal is mixed with a strong local oscillator signal producing the strong output.

In recent years, several new mesoscopic systems have been proposed and implemented as parametric amplifiers. These systems include small molecules in intense laser fields $[11]$, polaritons in semiconductor microcavities $[12]$, current-voltage oscillations in SQUIDs $[13]$ and Josephson junctions $[14, 15, 16]$ . Another proposal uses active and tuneable metamaterials $[17]$ to amplify weak signals. Most of these proposals wish to develop a very compact parametric amplifier, which can even demonstrate quantum amplification $[18]$. This indicates that quantum qubit systems (and, in particular, superconducting qubit systems) could be very promising candidates for mesoscopic parametric amplifiers.

In this thesis we will show that a two coupled qubit system can be used as a quantum parametric amplifier on the nano-scale.
1.4 Harmonic mixing

Another interesting and useful class of classical devices are rectifiers or ratchets. For a long time now, scientists have been asking the question of whether it is possible, or not, to extract useful work out of unbiased random fluctuations, i.e., by means of a device where temperature gradients and all the applied forces average out to zero. There are various kinds of mechanical or electrical non-linear ratchets, or rectifiers, that, when under the affect of unbiased random perturbations, loop in either one direction or the other. These devices make the task doable as far as macroscopic fluctuations are concerned. They are currently in use for a new generation of green power stations. Among the diverse rectification schemes proposed and tested so far on nano-scales, some examples are: (1) rocked ratchets, where a (deterministic) symmetric periodic signal gets rectified due to the substrate asymmetry, even in the absence of noise; (2) harmonic mixing, where the substrate non-linearity “mixes” two symmetric zero-mean ac signals to yield a dc output; (c) signal recycling, where the frequency difference of the input ac signals is replaced by a time/phase delay to cause a spontaneous symmetry breaking; and (d) collective effects, where particle-particle interactions can replace the non-linearity of the substrate. Below we are going to focus on only one type of rectifier that is based on harmonic mixing.

The simplest realisation of a harmonic mixer is the following. Considering a Brownian particle in a bistable and symmetric potential in the presence of periodic forcing. The particle is subjected to viscous friction. Also, assuming that the inertia effects are negligible (over-damped dynamics), the driven
Langevin dynamics reads in scaled units:

$$\frac{d}{dt} x(t) = - \frac{d}{dx} V(x) + f(t)$$  \hspace{1cm} (1.17)

The harmonic mixing driving signal $f(t)$ has the form,

$$f(t) = B_1 \cos(\omega t + \beta_1) + B_2 \cos(2\omega t + \beta_2)$$ \hspace{1cm} (1.18)

The phase differences are denoted by $\beta_1$ and $\beta_2$, and it is predominantly used as a control parameter for stochastic resonance. As an example case we will show that harmonic mixing can be used to rectify a signal from ac to dc (i.e., to generate non-zero $\langle x \rangle$).

If our potential is given by

$$V(x) = \frac{\alpha}{2} x^2 + \frac{\beta}{4} x^4$$ \hspace{1cm} (1.19)

the dynamics are described by the following equation:

$$\frac{dx}{dt} = -\alpha x - \beta x^3 + f(t).$$ \hspace{1cm} (1.20)

For small enough $x$, we can solve Eq. 1.20 perturbatively $x = x^{(0)} + x^{(1)} + \ldots$. In the zero approximation we have

$$\frac{dx^{(0)}}{dt} = -\alpha x^{(0)} + f(t)$$ \hspace{1cm} (1.21)

who’s solution (for certain combinations $\beta_1, \beta_2, \beta$) can be written in the form:
\[ x^{(0)} = A_1 \cos(\omega t) + A_2 \cos(2\omega t + \alpha) \] 

(1.22)

And from this, one can see that we have no rectification in the zero approximation \( \langle x^{(0)} \rangle = 0 \). However, we will have rectification in the next order.

Indeed for \( x^{(1)} \), we have

\[ \frac{dx^{(1)}}{dt} = -\alpha x^{(1)} - \beta \left( x^{(0)} \right)^3 \] 

(1.23)

Averaging with respect to time, we derive

\[ \bar{x}^{(1)} \sim -\frac{\beta}{\alpha} \bar{(x^{(0)})^3} \] 

(1.24)

We are looking for terms that are non-zero after averaging. We can try to average \( (x^{(0)})^3 \) thus, if some terms are oscillating then we neglect these terms. We are only interested in terms that can produce a dc output. Simple algebra results in

\[ (x^{(0)})^3 = A_1^3 \cos(\omega t) + 3A_2^2 \cos^2(\omega t)A_2 \cos(2\omega t + \alpha) + 3A_1 \cos(\omega t)A_2^2 \cos^2(2\omega t + \alpha) + A_2^3 \cos^3(2\omega t + \alpha) \] 

(1.25)

We are going to focus on the second term in the equation because this is the only term that is affected by the asymmetric potential. It is known that the average of \( \cos^3 \) is zero and of all the other terms except the second term is zero. Thus, to calculate \( \bar{(x^{(0)})^3} \) we will have to consider the following combination:
Using the trigonometric identities:

we estimate the average of 1.26 as

\[
\frac{3}{2} A_1^2 A_2 \left( \frac{1}{2} \cos(-\alpha) \right)
\]  

(1.27)

This term will give us a constant (no time dependence) and so this generates a dc output:

\[
\bar{x} \propto -\frac{\beta}{2} \frac{3}{\alpha^2} A_1^2 A_2 \left( \frac{1}{2} \cos(-\alpha) \right)
\]  

(1.28)

Despite the fact that we started with an oscillating force we finally have a dc output. This can be seen as rectification, which is tuneable because \( \alpha \) can be used to control the polarity and strength of our rectifier. We can see from Fig 1.3 that in panel A) if we integrate the plot to get an average we would average to zero, in panel B) we get a positive average (i.e. a positive dc output). From this simple calculation it is clearly seen that if we have an initial force comprising of only oscillating components, we can still generate a dc current, by mixing them. This is the simplest possible example of the effect of the asymmetry of our potential. So, we have a dc current and it is controllable. Such an effect for two driven qubits, will be discussed in chapter 3.

Below we will show that both of these classical devices (parametric amplifier and harmonic mixer) can be realised in the system of two coupled
qubits.

Figure 1.3: Expression 1.26 plotted to show harmonic mixing of the two signals. A) Mixing of the two signals with $A_1 = 0.1, A_2 = 0.5, \omega = 10$ and $\alpha = -5$. It can be seen from the plot that if we integrate we will get a response of zero. B) $A_1 = 0.5, A_2 = 0.1, \omega = 10$ and $\alpha = -0.1$. It can be seen from the plot that if we integrate we will get a positive response. We have shown that a dc signal can be produced as we require by varying the parameters of the drive.
1.5 Quantum computation and qubits

Both Brownian/chaotic systems and quantum computation can be described by dynamical or stochastic equations, thus there is hope yet to find similar properties for quantum qubit systems.

Initially, qubit structures have been fabricated and studied with a goal to perform quantum computations. Code cracking quantum computers seem the reality of Hollywood films or sc-fi geek dreams but they are becoming more and more probable. The pursuit of quantum computation has been one of the major contributing factors to the gigantic developments achieved in quantum mesoscopic physics and quantum nano-devices over the last decade and a half. The efforts have already resulted in progress towards a new form of mesoscopic analogue and digital devices [19, 20, 21, 22, 23, 24] and even new types of materials known as quantum metamaterials [2, 25], allowing to control quantum coherent media.

In the case of a qubit, the state space is a 2 dimensional Hilbert space and the state vector is a 2 dimensional complex vector which we describe using Dirac’s familiar notation $|\psi\rangle$. Any state in the state space can be expressed as a linear combination of basis states. A basis is a set of linearly independent vectors that span the state space, an example of which is the computational basis;

$$
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

(1.29)

These vectors are both normalised (the square of the components sum to 1) and orthogonal ($<0|1>=<1|0>=0$), hence they are known as an
orthonormal basis. Therefore a qubit in an arbitrary state can be described by a linear combination (or superposition) of the orthonormal basis states $|0\rangle$ and $|1\rangle$;

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$ (1.30)

where $\alpha$ and $\beta$ are in general both complex numbers and satisfy the normalisation condition;

$$<\psi|\psi> = |\alpha|^2 + |\beta|^2 = 1 \quad (1.31)$$

as the state vector $|\psi\rangle$ is a unit vector. $\alpha$ and $\beta$ are sometimes known as probability amplitudes because $|\alpha|^2$ and $|\beta|^2$ give the probability of the qubit being found in the state $|0\rangle$ or $|1\rangle$ respectively.

The state space of a qubit can be represented as a unit sphere in phase space. This is known as the Bloch sphere representation and provides a very convenient geometrical picture of the possible states of a qubit. Using the normalisation condition, we can express the general state vector of a qubit as

$$|\psi\rangle = \cos \frac{1}{2} \theta |0\rangle + e^{i\phi} \sin \frac{1}{2} \theta |1\rangle$$ (1.32)

The two real parameters $\theta$ and $\phi$ completely describe any point on the Bloch sphere, as shown in Fig 1.4.
Figure 1.4: The Bloch sphere representation of the possible states of a qubit. The sphere provides a geometric representation of a pure qubit (quantum bit) state space as points on the surface of the unit sphere. Any point of the surface represents some pure qubit. The mixed qubit states can be represented by points inside of the unit sphere, with the maximally mixed state laying at the centre. The $|\psi\rangle$ line from the centre to the surface of the sphere corresponds to the pure state and has unit length. For mixed qubit state the length of line must be less than 1. The numbers $\theta$ and $\phi$ define a point on the sphere of Eq. 1.32

From the dynamical point of view and for the purpose of this thesis, the qubit can be described by two dynamical variables $\theta$ and $\phi$ (or $\alpha(t), \beta(t)$) which satisfy ordinary differential equations (Schrödinger-like for one qubit). These variables can be driven by an external field, thus allowing analogy with a driven nano-particle. If the dissipation is essential to the system, one should use the master equation, to describe the open quantum system,
instead of the Schrödinger equation.

1.6 Master equations

Since experiments on quantum mechanical systems get more and more complicated, it is no longer sufficient to describe these systems as closed. In practice any realistic system has an uncontrollable coupling to the environment which influences the time evolution of the system[26]. Therefore the theory of open quantum systems describes their behaviour by considering different assumptions about the coupling to the environment, since a complete description of the environment degrees of freedom is not feasible. The dynamical evolution of the open system is then described with an effective equation of motion- the master equation.

The master equation determines how the density matrix of the system alone evolves in time. This uses the Schrödinger picture where evolution of the state (represented by its density matrix) is considered. The master equation is:

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar} \left[ \hat{H}(t), \hat{\rho}(t) \right] + L_D[\hat{\rho}(t)]$$  \hspace{1cm} (1.33)

where $\hat{H}(t) = H_s + \Delta$ is the system’s Hamiltonian, along with a (possible) unitary contribution $\Delta$ from the bath, and $L_D$ is the Lindblad decohering term.

For the system of qubits we would use the qubit density matrix:
\[ \hat{\rho} = \frac{1}{4} \sum_{\alpha, \beta = 0, x, y, z} \Pi_{\alpha \beta} \sigma^1_\alpha \otimes \sigma^2_\beta \]  

(1.34)

This is a straightforward generalization of the standard representation of the single-qubit density matrix expression using the Bloch vector; the components \( \Pi_{\alpha \beta} \) thus constitute what can be called the Bloch tensor. In our case we can simply substitute our qubit density matrix into the Schrödinger picture equation 1.33. We will then have an ordinary differential equation that is the master equation for our qubits

\[ \frac{d\Pi_{\alpha \beta}(t)}{dt} = \Theta_{\alpha \beta \gamma \delta}(t) \Pi_{\gamma \delta} \]  

(1.35)

\( \Theta_{\alpha \beta} \) are now coefficients that can be time dependent and play the role of parametric drive of qubits, while \( \Pi_{\gamma \delta} \) are the dynamical variables of the quantum system.

### 1.7 Superconducting qubits

Experimental realisation of the theoretically predicted effects below can be realised by using superconducting qubits. The reasons this is possible is because of a few points. There are now well developed fabrications techniques, measuring methods and quite long decoherence times in these systems. Below is an overview of different types of superconducting qubits.

From the point of view of quantum computing, a qubit is any two-state quantum system which satisfies certain control and readout requirements and can therefore be used for the execution of quantum algorithms (see, e.g.,
Many efforts have been devoted to the study of theoretical quantum information processing and, more recently, significant progress has been achieved on experimental aspects of this field. For example, one of the most fascinating results obtained in experimental mesoscopic physics over the past decade has been the realization of several types of superconducting qubits, which demonstrated many of the qualities required for quantum information processing. Moreover, superconducting qubits in quantum electronics have a potentially much wider range of applications than just quantum information processing (see, e.g., Refs. [19, 20, 21, 2, 28, 29, 30]), including: single-photon generators, producing quantum (squeezed or Fock) states of the electromagnetic field, quantum transmission lines, quantum amplifiers, etc. Various types of superconducting qubits have been produced, including charge, flux, and phase qubits. These use different physical mechanisms to control their states and store information. For example, in a charge qubit, the state $|1\rangle$ has one extra Cooper pair, compared to the state $|0\rangle$, while for a flux qubit, the two logical states differ by a fraction of the magnetic flux quantum. Superconducting qubits are mesoscopic (e.g., their working quantum states may differ by dozens of millions of single-particle states) and scalable (i.e., it is possible to link these together). Moreover, their fabrication, control, and readout require techniques already well developed in solid state electronics.

A charge qubit is a qubit whose basis states are charge states (i.e. states which represent the presence or absence of excess Cooper pairs in the island). A charge qubit is formed by a tiny superconducting island (known as a Cooper-pair box) coupled by a Josephson junction to a superconducting
Figure 1.5: A charge qubit is formed by a tiny superconducting island (known as the Cooper pair box) coupled by a Josephson junction reservoir. The state of the qubit is determined by the number of Cooper pairs which have tunnelled across the junction. A) Charge qubit schematic representation including the cooper pair box, marked by the green dashed rectangle, and a Josephson junction, marked by a square with an X. B) Realisation of a cooper pair box coupled to a single electron transistor (SET). Scanning electron micrograph of the nano-fabricated sample used in the experiment. The superconducting box in the upper part is capacitively coupled to the electrometer (lower part) and the Josephson junctions are the bright dots [31][32].

The phase qubit is a superconducting device based on the superconductor-insulator-superconductor or (SIS) Josephson junction. Each of the superconductors that make up the Josephson junction is described by a macroscopic
Figure 1.6: Flux qubits (also known as persistent current qubits) are micrometer sized loops of superconducting loops interrupted by a number of Josephson junctions, shown by the squares with X inside. The junction parameters are engineered during fabrication so that a persistent current will flow continuously when an external flux is applied. The computational basis states of the qubit are defined by circulating currents which can flow either clockwise or anti-clockwise. These currents screen the applied flux limiting it to multiples of the flux quantum and give the qubit it’s name. When the applied flux through the loop is close to a half-integer number of flux quanta the two energy levels corresponding to the two directions of circulating current are brought together and the loop may be operated as a qubit. A) Flux qubit schematic representation, micrometer sized superconducting loops interrupted by a number of Josephson junctions, marked by squares with an X inside. B) Realisation of a flux qubit. Scanning electron micrograph showing the flux qubits with three Josephson junctions [35].

The difference in the complex phases of the two superconducting wavefunctions is the most important dynamic variable for the Josephson
Figure 1.7: A) Phase qubit schematic-superconducting device based on the superconductor-insulator-superconductor (SIS) Josephson junction, shown by a square with an X inside and also shows a constant current source (two circles). B) Phase qubit realisation. The superconducting qubit is an aluminium circuit, a few hundred microns across but considered macroscopic from the point of view of quantum physics (which describes the atomistic scale world), and the low temperatures (30mk) brought out its quantum properties. Microwaves at a frequency of 4.743 GHz can be used to drive it only between its ground and first excited states the qubit’s 0 and 1 states.\[36\] junction.

Several superconducting qubit designs of various degrees (see discussion above) of complexity and performance have been realized (reviewed in, e.g., Refs. [19, 20, 22, 28, 29, 30]). Below we focus on the best possible candidate (flux qubit) for implementation of the effects described in this thesis. The so-called persistent-current flux qubit [33, 34] combines relative simplicity with
decent decoherence times and scalability. It consists of a small superconducting loop (approximately 10 µm across) interrupted by three Josephson junctions. The flux quantization condition is

\[ \phi_1 + \phi_2 + \phi_3 + 2\pi \Phi/\Phi_0 = 2\pi n, \quad (1.36) \]

where \( \phi_j \) is the phase difference across the \( j \)th junction, \( \Phi \) is the total magnetic flux through the loop, \( \Phi_0 = h/2e \), and \( n \) is an integer. This allows to eliminate one of the phases (e.g., \( \phi_3 \)). Due to the small self-inductance of the loop, the difference between the flux \( \Phi \) and the external flux \( \Phi_x \), as well as the magnetic energy of the system, can be neglected. On the other hand, one must include the contribution to the energy from the charges on the Josephson junctions, \( Q_j^2/2C_j \), and the Josephson energy \( -E_j \cos \phi_j \), where \( C_j \) and \( E_j \) is the capacitance and the Josephson energy of the \( j \)th junction, respectively. Up to numerical factors, the junction charges are the momenta canonically conjugate to the Josephson phase differences (see, e.g., [37], §2.3).

Introducing variables \( \phi_{\pm} = (\phi_1 \pm \phi_2)/2 \), it is straightforward to show that the classical Hamilton function of the system (assuming that two junctions are identical, \( C_1 = C_2 = C, E_1 = E_2 = E \), and that \( C_3 = \alpha C, E_3 = \alpha E \)) is given by

\[ \mathcal{H} = \frac{\Pi_+^2}{2M_+} + \frac{\Pi_-^2}{2M_-} - E \left[ 2\cos \phi_+ \cos \phi_- + \alpha \cos \left( 2\phi_- + 2\pi \frac{\Phi_x}{\Phi_0} \right) \right], \quad (1.37) \]

where \( \Pi_{\pm} \) are the corresponding momenta and \( M_{\pm} \) are determined by the junction capacitances. The potential energy term in the interval is periodic
and, for the proper choice of $\alpha$ and with $\Phi_x \approx \Phi_0/2$, it contains a double well, the minima of which are almost degenerate and correspond to the current flowing (counter)clockwise around the loop. These states are chosen as physical qubit states. Transitions between them are enabled by quantum tunnelling through the barrier. The quantum-mechanical Hamiltonian of the persistent-current qubit is obtained from (1.37) by using the relation $\hat{N} = -i\partial/\partial\phi$ between the superconducting phase and the number of extra Cooper pairs (proportional to the charge on the Josephson junction). Its detailed analysis is given in [33, 34]. Truncating the Hilbert space of the system to the two lowest states, which can be done due to the strong anharmonicity of the potential in (1.37), its Hamiltonian can be reduced to the standard pseudospin form,

$$H = -\frac{1}{2} (\epsilon \sigma_z + \Delta \sigma_x).$$

Here $\epsilon$ is proportional to the external flux through the qubit, and $\Delta$ is determined by the tunnelling matrix element between the potential minima. Here we note that persistent-current flux qubits have decoherence times in excess of $10\mu$s at the operating frequency $\sim 1$ GHz, allow successful coherent coupling of several qubits, and show steady improvements. Therefore, these superconducting qubits are promising new elements for versatile quantum circuits.

Two coupled qubits can be described by the Hamiltonian:

$$H = -\frac{1}{2} \sum_{j=1,2} [\Delta_j \sigma_z^j + \epsilon_j(t) \sigma_x^j] + g \sigma_x^1 \sigma_x^2$$

where $\sigma_z^j$ and $\sigma_x^j$ are Pauli matrices corresponding to either the first ($j = 1$)
or the second \((j = 2)\) qubit; the eigenstates of \(\sigma_j^z\) are the basis states of the \(j\)th qubit at zero coupling.

Below we will use the Hamiltonian Eq. 1.39 to construct the master equations for two coupled qubits, these will just be a set of ordinary differential equations for dynamical variables related to energy level occupation and current in the qubit. So these dynamical variables can be directly measured and allow us to check the results predicted in this thesis. We will then show that these can show parametric amplification and harmonic mixing.
Chapter 2

Two-qubit parametric amplifier: large amplification of weak signals

2.1 Parametric Amplifier

While the desire of building code-cracking quantum computers [27] remains as elusive as ever since its inception, its pursuit has been one of the major contributing factors to the enormous progress achieved in quantum mesoscopic physics and quantum nanodevices over the last decade and a half. These efforts have already resulted in the development of a new branch of mesoscopic digital and analogue devices [19, 20, 21, 22, 23, 24] and even new types of materials known as quantum metamaterials [2, 25], allowing to control quantum coherent media. In this chapter we describe how two coupled qubits can be used as a parametric amplifier.
2.1.1 Outline of results

The main results of this chapter can be summarized as follows.

- Consider an ac drive \( A \sin(\omega_{pump} t) \) applied to a four-level quantum system (e.g., a couple of flux qubits (e.g.,[38, 39]) placed in an ac magnetic field, see Fig.2.1 with \( \omega_{pump} = \omega \) in resonance with a transition between a pair of its energy levels.

Even though the evolution of the density matrix is described by a linear equation, the spectrum of the density matrix elements (Fig. 2.2) shows a sequence of peaks corresponding to different harmonics of the ac drive (which is commonly thought to occur in nonlinear systems). More interestingly, the strongest oscillations are not at the main frequency \( \omega \), but at some of its harmonics, forming a hierarchy of resonances, which is unusual for nonlinear systems. The explanation of this phenomenon lies in the structure of the master equations’ set for the density matrix elements, in which the external signal enters multiplicatively rather than additively.

- Such a hierarchy of parametric resonances makes this two qubit system (e.g., two coupled flux qubits) an efficient parametric amplifier and frequency shifter, especially if the weak signal has its frequency \( \omega_{rm\_weak} = \tilde{\omega} \) close to another inter-level transition of the system. The latter can be achieved by tuning the qubit coupling (e.g., like in Ref. [39], see also Fig. 2.1 below). In this case, a weak signal \( \epsilon \cos \tilde{\omega} t \) generates a combination of harmonics \( m\omega + k\tilde{\omega} \). The amplitudes of these harmonics increase with \( \epsilon \), and can reach a height of the order of the main
harmonic of the drive, even for very small $\epsilon$. Thus, the weak signal can be amplified by a factor of up to several hundred.

- The amplification effects (Fig. 2.3) are not suppressed by a realistic amount of decoherence (dephasing and relaxation) in the system. Thus, currently available superconducting flux qubits with relatively short coherence times can be used as nanoscale, coherent amplifiers of weak signals in the frequency range of several hundred MHz.

- Noise (Fig. 2.4) of the order of the signal cannot suppress signal amplification (Fig. 2.5), which is also robust (but, of course, weaker) with respect to both: changing the frequency of the weak signal (Fig. 2.6a) and the parameters of the coupled-qubit system (Fig. 2.6b).

### 2.2 Quantum amplification with qubits

One of the challenging tasks for which superconducting qubits seem to be well suited is the amplification of a weak signal, a crucially important tool for both technological and scientific applications. For this goal, different types of linear or nonlinear resonance devices are commonly used (e.g., Ref. [10, 40]). The problem of signal amplification becomes very difficult at the nanoscale. Here we demonstrate that two coupled qubits can be employed as a parametric amplifier [10] based on the effect of the parametric resonance between a weak signal and quantum oscillations between the quantum levels of the system, driven by an external ac signal (the pump signal). While the actual realization details of the qubits is irrelevant for the mathematics of
the problem, we stress that the implementation of the proposed scheme is ideally suited for mesoscopic superconducting qubits because these are very controllable and versatile.

Two coupled qubits can be described by the Hamiltonian:

$$H = -\frac{1}{2} \sum_{j=1,2} \left[ \Delta_j \sigma_z^j + \epsilon_j(t) \sigma_x^j \right] + g \sigma_x^1 \sigma_x^2$$

where $\sigma_z^j$ and $\sigma_x^j$ are Pauli matrices corresponding to either the first ($j = 1$) or the second ($j = 2$) qubit; the eigenstates of $\sigma_z^j$ are the basis states of the $j$th qubit at zero coupling.

To be more specific, let us consider two coupled nominally-identical superconducting persistent-current flux qubits (e.g., Ref. [33]), where each of the latter consist of a superconducting loop interrupted by three Josephson junctions, along which persistent-currents can circulate controlled by applied magnetic fluxes. When two such loops are placed next to each other, so that they feel each other’s magnetic fields, this situation naturally produces the “antiferromagnetic” coupling represented in Eq. (2.1) by the $\sigma_x^1 - \sigma_x^2$ term, with $g > 0$ (see, e.g., Ref. [38]). More elaborate designs can produce a tunable coupling (see, e.g., Refs. [39, 41]), with the amplitude and sign of $g$ controlled externally by the magnetic flux, $\Phi_{\text{coupl}}$, through the coupler loop [39] (see Fig. 2.1).

The state of each qubit is controlled by the magnetic flux $\Phi_{e}^{(j)} = f_{e}^{(j)}(t) \Phi_0$ through it, where $\Phi_0 = h/2e$ is the flux quantum. In the vicinity of $f_{e}^{(1)} = f_{e}^{(2)} = 1/2$, the ground state of each qubit is a symmetric superposition of states $|L\rangle$ and $|R\rangle$ with, respectively, clock- and anticlockwise circulat-
Figure 2.1: Schematic diagram of two flux qubits (persistent current qubits) coupled via a coupler loop. Josephson junctions are represented by crosses, and their thickness indicate their Josephson critical currents (drawn not to scale). The qubit states and the amplitude and sign of the coupling constant $g$ are controlled by the corresponding magnetic fluxes, $f_{1,2}$ and $f_{\text{coupler}}$ (in units of $\Phi_0$). Both the pumping drive and the weak signal $A \sin \omega t + \epsilon \sin \tilde{\omega} t$ are generated by a source in the bottom circuit. The left circuit is needed to pick up an output signal $Z_1(t)$. The top circuit controls the coupling $g$. Noise, shown by $\sqrt{2D} \xi_i(t)$, is coupled to each qubit.

In the basis $\{ |L\rangle, |R\rangle \}^{(1)} \otimes \{ |L\rangle, |R\rangle \}^{(2)}$ the two-qubit system can be described by the four-level Hamiltonian (Eq 2.1) with $\epsilon_j = I_p \Phi_0 \delta f_{j}^{(j)}$; here $\delta f_{j}^{(j)}(t)$ contains both the pump and the input signals. The tunnelling amplitudes $\Delta_j$ are usually fixed by the fabrication process, but can be tuned if one of the qubit junctions is replaced by two junctions in parallel, in a dc SQUID configuration.
(see, e.g., [42, 43]). The interaction constant $g$, as mentioned above, can be made tuneable using a coupler loop (Fig. 2.2). Note that the density matrix spectrum is directly related to the immediately measurable current/voltage spectrum in the tank, which was exploited in [45] to detect Rabi oscillations in a flux qubit.

For simplicity, we consider two identical qubits; that is, we assume $\Delta_1 = \Delta_2 = \Delta$. A direct diagonalisation of the Hamiltonian (2.1) leads to the inter-level transition frequencies

$$
\omega^{(1)} = 2 \sqrt{\Delta^2 + g^2}, \quad \omega^{(2)} = \sqrt{\Delta^2 + g^2} - g,
$$

$$
\omega^{(3)} = \sqrt{\Delta^2 + g^2} + g, \quad \omega^{(4)} = 2g,
$$

which are tuneable by changing $g$. Therefore, $\omega^{(1)}, ..., \omega^{(4)}$ can be adjusted to a desirable frequency, that was used below.

Let us drive the qubits simultaneously by a control ac pump signal, with frequency $\omega_{\text{pump}} = \omega$ and amplitude $A$, as well as a weak input signal with frequency $\omega_{\text{weak}} = \tilde{\omega}$ and amplitude $\epsilon \ll A$, to be amplified:

$$
\epsilon_j(t) = A \sin(\omega t) + \epsilon \sin(\tilde{\omega} t) + \sqrt{2D} \xi_j(t)
$$

where we also take into account a noise term (with intensity $D$) which can be due to fluctuations in the signals or an environmental noise. As an example, we consider white noise with zero mean: $\langle \xi(t) \rangle = 0$ and $\langle \xi_j(t) \xi_l(t') \rangle = \delta_{jl}\delta(t-t')$, where $\delta$ refers to either the Dirac delta function or the Kronecker delta.

Using equations 1.33 and 1.34, the ordinary differential equations 1.35 for the qubit density matrix $\Pi_{\alpha\beta}$ can be written directly:
\[ \dot{\Pi}_{0x} = \Delta_2 \Pi_{0y} - \Gamma_{\phi} \Pi_{0x} \]
\[ \dot{\Pi}_{0y} = -\Delta_2 \Pi_{0x} + \epsilon_2(t) \Pi_{0z} - 2g \Pi_{xx} - \Gamma_{\phi} \Pi_{0y} \]
\[ \dot{\Pi}_{0z} = -\epsilon_2(t) \Pi_{0y} + 2g \Pi_{xy} - \Gamma_2 (\Pi_{0z} - Z_{T2}) \]
\[ \dot{\Pi}_{x0} = \Delta_1 \Pi_{y0} - \Gamma_{\phi} \Pi_{x0} \]
\[ \dot{\Pi}_{y0} = -\Delta_1 \Pi_{x0} + \epsilon_1(t) \Pi_{z0} - 2g \Pi_{xx} - \Gamma_{\phi} \Pi_{y0} \]
\[ \dot{\Pi}_{z0} = -\epsilon_1(t) \Pi_{y0} + 2g \Pi_{yx} - \Gamma_1 (\Pi_{z0} - Z_{T1}) \]
\[ \dot{\Pi}_{xx} = \Delta_2 \Pi_{xy} + \Delta_1 \Pi_{yx} - (\Gamma_{\phi} + \Gamma_{\phi}) \Pi_{xx} \]
\[ \dot{\Pi}_{xy} = -2g \Pi_{0z} - \Delta_2 \Pi_{xx} + \Delta_1 \Pi_{yy} + \epsilon_2(t) \Pi_{xz} - (\Gamma_{\phi} + \Gamma_{\phi}) \Pi_{xy} \]
\[ \dot{\Pi}_{yx} = -2g \Pi_{0z} - \Delta_1 \Pi_{xx} + \Delta_2 \Pi_{yz} + \epsilon_1(t) \Pi_{xz} - (\Gamma_{\phi} + \Gamma_{\phi}) \Pi_{yx} \]
\[ \dot{\Pi}_{xz} = 2g \Pi_{0y} - \epsilon_2(t) \Pi_{xy} + \Delta_1 \Pi_{yz} - (\Gamma_{\phi} + \Gamma_{2}) \Pi_{xz} \]
\[ \dot{\Pi}_{zx} = 2g \Pi_{0y} - \epsilon_1(t) \Pi_{yx} + \Delta_2 \Pi_{zy} - (\Gamma_{\phi} + \Gamma_{1}) \Pi_{zx} \]
\[ \dot{\Pi}_{xy} = -\Delta_1 \Pi_{xy} - \Delta_2 \Pi_{yx} + \epsilon_2(t) \Pi_{yz} + \epsilon_1(t) \Pi_{zy} - (\Gamma_{\phi} + \Gamma_{2}) \Pi_{yy} \]
\[ \dot{\Pi}_{yz} = -\Delta_1 \Pi_{xx} - \epsilon_2(t) \Pi_{yy} + \epsilon_1(t) \Pi_{zz} - (\Gamma_{\phi} + \Gamma_{2}) \Pi_{yz} \]
\[ \dot{\Pi}_{xy} = -\Delta_2 \Pi_{sx} - \epsilon_2(t) \Pi_{yy} + \epsilon_2(t) \Pi_{zz} - (\Gamma_{1} + \Gamma_{2}) \Pi_{zy} \]
\[ \dot{\Pi}_{zz} = -\epsilon_1(t) \Pi_{yz} - \epsilon_2(t) \Pi_{zy} - (\Gamma_{1} + \Gamma_{2})(\Pi_{zz} - Z_{T1}Z_{T2}) \]

Here we use the standard approximation for the dissipation operator $\hat{\Gamma}$ via the dephasing ($\Gamma_{\phi}$), and relaxation ($\Gamma_1, \Gamma_2$) rates, to characterize the intrinsic noise in the system. Where the symbols are explained in Eqs. (2.1)-(2.3).

In the absence of qubit-qubit coupling, $g = 0$, the first three equations in (2.4) describe the evolution of the Bloch vector components ($\Pi_{0x}, \Pi_{0y}, \Pi_{0z}$) of qubit
1, and the second three equations in (2.4) describe those \((\pi_x, \pi_y, \pi_z)\) of qubit 2. Also, for simplicity, hereafter we assume that the relaxation rates are the same for both identical qubits, i.e., \(\Gamma_1 = \Gamma_2 = \Gamma_r\) and \(\Gamma_1 = \Gamma_2 = \Gamma_r\), and the temperature is low enough, resulting in \(Z_{T2} = Z_{T1} = 1\), where \(Z_{Tj} = \text{tanh}(\Delta_j/2k_BT_j)\) is the equilibrium value of the \(z\)-component of the Bloch vector.

In the limit of zero coupling, \(g = 0\), there exists a solution of Eqs. (2.4) with no entanglement between the qubits. This solution can be written as a direct product of two independent density matrices expressed through their Bloch vectors:

\[
\hat{\rho}_j = \frac{1}{2}(1 + X_j \sigma_x + Y_j \sigma_y + Z_j \sigma_z).
\]

(2.5)

The components of the Bloch tensor \(\pi_{\alpha\beta}\) are all zero with the exception of

\[
(\pi_{ox}, \pi_{oy}, \pi_{oz}) = (X_1, Y_1, Z_1); \quad (\pi_{xo}, \pi_{yo}, \pi_{zo}) = (X_2, Y_2, Z_2),
\]

(2.6)

which are just the separate qubits’ Bloch vector components. If the interaction is non-zero, \(g \neq 0\), the entanglement between these qubits makes the components of the Bloch tensor non-zero, and such an entangled state can persist for some time even if the interaction is switched off later on.

### 2.3 Simulation results

Formally, the set of equations (2.4) might look complicated but these are just a set of fifteen coupled ordinary differential equations which can be easily integrated numerically [7]. Indeed, measuring time in units of \(1/\Delta\), we
numerically solved Eqs. (2.4) by the Euler method for two coupled qubits, driven by the field (2.3) with $\omega_{\text{pump}} = \omega = \omega^{(2)}$ and $\omega_{\text{weak}} = \tilde{\omega} = \omega^{(3)}$ [see Eq. (2.2)]. Our numerical integration produces a set of data for $\Pi_{a,b}(t)$ which can be analysed using a Fourier transform.

$$Z_1(\omega) \propto \int_0^\infty \exp^{i\omega t} Z(t) dt$$

Thus, we can numerically evaluate the spectrum of $Z_1 = \Pi_{oz}$, defined as

$$S_Z(\omega) = \langle |Z_1(\omega)| \rangle,$$  \hspace{1cm} (2.8)

and the similarly defined spectrum of $X_1 = \Pi_{0x}$.

If one considers a flux qubit as a particular realization of our two-qubit parametric amplifier, then $Z_1(t)$ and $X_1(t)$ (and, thus, their spectra $S_Z$ and $S_X$) can be directly obtained by using impedance measurements [44] of the circulating current in the first qubit of the proposed device. These spectra are shown in Fig. 2.2 for $g/\Delta = 1$. Below we will analyse the influence of a weak signal on $S_Z$ and $S_X$.

When the weak signal is switched off [Fig. 2.2(a)], the spectrum of $Z_1$ exhibits several peaks corresponding to the harmonics, $\omega_m = m\omega$, of the applied drive ($A/\Delta = 15$). In contrast to the standard nonlinear response, where the spectrum starts from $\omega_1$ and has harmonic peaks $\omega_m$, whose heights decreases with $m$, in our two-qubit system, $S_z$ starts with $\omega_2$ and contains only even harmonics. The peak heights show a surprising non-monotonic dependence on $m$, with the two highest peaks occurring at $m = 20$ and $m =$
Figure 2.2: The spectrum $S_Z$ of the $Z_1$ matrix element (responsible for the occupation of the excited level of the first qubit) is shown for dephasing and relaxation given by $\Gamma_\phi/\Delta = \Gamma_r/\Delta = 10^{-3}$ (we use the same $\Gamma_\phi$ and $\Gamma_r$ for all results reported in this article). (a) When the weak signal amplitude is equal to zero ($\epsilon = 0$) and when the reduced-drive amplitude $A/\Delta = 15$, the ac monochromatic drive fed into the two-qubit amplifier is converted into a set of even harmonics $\omega_m = m\omega$, with $m = 2, 4, 6...$ Note the absence of response at $\omega_1 = \omega$ (no peak there). For comparison, $S_Z$ is shown in blue and indicated by the blue arrow for zero-drive amplitude $A = 0$ and weak signal $\epsilon/\Delta = 0.1$. Only one very small peak is hardly seen, indicated by the blue arrow, corresponding to $2\omega_{\text{weak}}$. (b) Spectrum $S_X$ of the off-diagonal matrix element $X_1$ (zero signal case). The same ac monochromatic drive fed into the amplifier is converted into a set of odd harmonics for $S_X$: $m = 1, 3, 5, \ldots$. Note that (a) and (b) show very unusual non-monotonic spectra. The observed non-monotonicity in the system response can be seen as a fingerprint of the qubits, characterizing their dynamical nonlinear response.
22. A somewhat similar non-monotonic peak dependence on the harmonic number is seen for $S_X$, the spectrum of the off-diagonal matrix element $X_1 = \Pi_{ox}$. Even though this spectrum starts with the main harmonic $\omega_1$, it contains only odd peaks which heights still show a non-monotonic dependence on $m$ [Fig. 2.2(b)]. Let us clarify one point about notations, the super-indices in $\omega^{(1)}, \omega^{(2)}$, etc., refer to the level-splitting (intrinsic properties) of the qubits, while the lower indices in $\omega_1, \omega_2, \ldots$ label the harmonics of the response to the input ac monochromatic drive. We measure all frequencies in units of the energy-splitting of independent qubits $\omega^{(2)}(g = 0)$, since this provides a characteristic frequency in the system which is fixed even if one tunes the inter-level frequencies $\omega^{(i)}(g)$. All spectral peak heights are proportional to the drive amplitude $A$. Thus, a weak signal should produce a spectrum with strongly-suppressed peaks. This is consistent with our simulations for zero-drive amplitude $A = 0$ and weak-signal amplitude $\epsilon/\Delta = 0.1$. The resulting spectrum has only one peak, with an amplitude about 150 times lower than the highest peak of the $S_Z$ spectrum for $A/\Delta = 15, \epsilon = 0$ [Fig. 2.2(a), the weak peak is shown there as “weak” and it is almost invisible](i.e., no weak signal and strong drive $A/\Delta = 1$).

When mixing the strong drive with a weak signal, the resulting spectrum $S_Z$ at $A/\Delta = 15, \epsilon/\Delta = 0.1$ (i.e., $\epsilon/A = 1/150$) [Fig. 2.3(a)] strongly differs from the simple superposition of the two spectra described above, $S_Z(A/\Delta = 15, \epsilon = 0)$ and $S_Z(A = 0, \epsilon/\Delta = 0.1)$.

It is remarkable that the weak signal produces a considerable change in the spectrum: it generates peaks at combination frequencies, $k\omega + l\tilde{\omega}$, with
Figure 2.3: (a) The amplification of a weak signal $\epsilon/\Delta = 0.1$ by a strong drive $A/\Delta = 15$ (i.e., $\epsilon/A = 1/150$) can be seen in the spectrum of $S_Z$ shown by the black solid lines. The spectrum obtained is not a simple superposition of the two spectra $S_Z(A/\Delta = 15, \epsilon = 0)$ (also shown here by red vertical dotted peaks) and $S_Z(A = 0, \epsilon/\Delta = 0.1)$ [see also red and blue peaks in Fig. 2.2(a)]. Instead, a combination of harmonics appears for $k\omega + l\tilde{\omega}$, with integer $k$ and $l$. The height of these peaks is almost proportional to $\beta \epsilon$, for $0 < \epsilon/A < 0.005$, and it is strongly enhanced by a factor $\beta \sim 100$. The enhanced peak is marked by an arrow with symbol $I(\epsilon)$. Note that this peak is absent in Fig. 2.2(a). (b) Normalized output amplitude $I(\epsilon)/I(A)$ (ratio of the heights of the highest mixed peak $I(\epsilon)$ to the highest peak $I(A)$ of the spectrum at $\epsilon = 0$) as a function of the reduced weak signal amplitude $\epsilon/A$. This dependence shows an almost-linear increase of the peak height with $\epsilon$ for small $\epsilon/A$, followed by saturation and even decay at relatively large $\epsilon/A$. integer $k$ and $l$; their heights are determined by the weak signal but can be of the order of the highest peak in $S_Z(A/\Delta = 15, \epsilon = 0)$. Interestingly, the enhancement of combination-frequency harmonics occurs by borrowing some
energy from the pumping drive, which own harmonics (harmonics existing at $\epsilon = 0$) decay when the weak signal is applied. Indeed, if one compares the spectrum at $\epsilon = 0$ shown by red dotted lines, with the spectrum at $\epsilon/\Delta = 0.1$ shown by black solid lines in Fig. 2.3 (a), one can see that the heights of the harmonics at $\epsilon = 0$ are higher that the corresponding peaks at $\epsilon/\Delta = 0.1$, even though the total energy pumped in the system is larger when both the pumping drive and the weak signal are applied.

The heights of the combination-frequency peaks increase with the weak signal amplitude, $\epsilon$, followed by saturation at large values of $\epsilon$. Of course, the heights of the combination-frequency peaks tend to zero when $\epsilon \to 0$. It is clearly seen [Fig. 2.3(b)] that their height can be approximated by a linear function, $\beta \epsilon$, for the weak signal amplitude in the range $0 < \epsilon/A < 0.005$, with an amplification coefficient $\beta$ of about 100. This level of amplification is remarkable.

It is also useful to stress that many peaks associated with different combinations of two frequencies, $k\omega_{\text{pump}} + l\omega_{\text{weak}}$, appear in the spectrum of $S_Z$ for different integers $k$ and $l$. In other words, the spectrum has many combination-frequency harmonics with different intensities.

This allows to pick up the signal on a frequency which better fits an available experimental set-up. Thus, the proposed two-qubit amplifier is also a frequency shifter, allowing to shift the frequency of a weak signal to a desirable frequency range. Note that it is usually hard to predict which peak should be the highest one [see Fig. 2.3 with highest peak marked as $I(\epsilon)$]. However, by measuring the output signal $I_{\pm}$ of the first mixing harmonics, $\omega_{\text{weak}} \pm \omega_{\text{pump}}$, usually allows to pick up a strongly amplified signal. For
instance, the ratio of the highest peak $I_\epsilon$ to $I_\pm$ is about 1.7 in our simulations shown in Fig. 2.3(a). Therefore, choosing the harmonics $\omega_{\text{weak}} \pm \omega_{\text{pump}}$ would be a good guide to observe the predicted signal amplification.

### 2.4 Noise influence

Since a weak signal can be considerably amplified by a strong drive in our parametric amplifier, one can wonder if an uncontrollable noise could also be amplified making the weak signal indistinguishable. To check this we performed simulations at different noise levels choosing all other parameters as in Fig. 2.3, i.e., $A/\Delta = 15$, and $\epsilon/A = 1/150$. A noise with intensity of the order of 6.6% with respect to the weak signal (i.e., $\sqrt{D}/\epsilon \sim 0.066$) has already considerably affected the time dependence of the measured signal (see, Fig. 2.4), but it does not strongly influence the spectrum of $S_Z(\omega)$, as seen in Fig. 2.5(a).

The reason for this is that the noise contains all frequencies (or at least a broad frequency spectrum), thus, its energy pumped in the signal harmonic is relatively small.

Even stronger noise, $\sqrt{D}/\epsilon = 0.2$, is still not enough to suppress the peaks attributed to the weak signal. Moreover, a sort of stochastic resonance (increase of the peak heights with noise) is also seen on Fig. 2.5(b). Therefore, the proposed two-qubit parametric amplifier is robust with respect to noise and, moreover, the noise can even be used to further amplify the signal.
Figure 2.4: Trajectories $Z_1(t)$ for different noise levels: $\sqrt{D}/\epsilon = 0$ for (a) and $\sqrt{D}/\epsilon = 0.066$ for (b). All other parameters are the same as in Fig. 2.3(a). As seen from (b), the applied white noise considerably affects the time dependence of $Z_1(t)$, making trajectories quite noisy with respect to the noiseless situation shown in (a).

2.5 Away from the optimal regime: robust amplification of two-qubit amplifier

In order to realize the strongest amplification of a weak signal (optimal working regime), a tuneable coupling must be used. In other words, the coupling should be tuned to adjust a level splitting frequency $\omega^{(i)}$ to the frequency $\omega_{weak}$ of the applied weak signal. This adjustment cannot be always real-
Figure 2.5: Spectrum $S_z(\omega)$ for the same parameters used in Fig. 2.3 and different noise levels $\sqrt{D}/\epsilon = 0.066$ for (a) and 0.2 for (b). One can clearly see the mixed (pump-signal) peaks for several combination-frequencies $k\omega_{\text{pump}} + l\omega_{\text{weak}}$, even at high noise levels, when the trajectories $Z_1(t)$ shown in Fig. 2.4 are very noisy.

ized. Thus, the amplification of a weak signal of an arbitrary frequency (different from the level splitting) is worth studying. As we expected, the amplification of the weak signal on the frequency $\omega_{\text{weak}} = 1.113\omega^{(3)} \neq \omega^{(i)}$ is weaker (Fig. 2.5(a)) by a factor of about 3.5 (ratio of the highest combination-frequency peaks in Figs. 2.3 and 2.6) with respect to the optimal amplification $\omega_{\text{weak}} = \omega^{(3)}$, i.e., in the case when the signal frequency is equal to the level splitting.
Figure 2.6: Amplification of a weak signal away from the optimal regime. (a) Spectrum $S_z$ for off-resonance frequency $\omega_{weak} = 1.113\omega^{(3)}(g = 1)$ of a weak signal shown by red thin solid line. All other parameters of the simulations are the same as in Fig. 2.3. The thick black dotted line corresponds to the simulations with zero signal, thus, highlighting the combination-frequency amplified harmonics as red lines with no black dots on top. One can see that the amplification of off-resonance signal is weaker, but still remarkable with amplification factor $\beta$ about 30. (b) Spectrum of $S_z$ for weakly coupled qubits $g/\Delta = 0.1$ and weak signal amplitude $\epsilon/A = 24$, as shown by the red thin solid line; all other parameters are as in Fig. 2.3 and $A/\Delta = 12$. The same simulations but with zero signal is shown by a thick black dotted line to highlight the amplified mixed peaks. The estimated amplification factor $\beta$ is about 10.

Nevertheless, this amplification is still strong enough ($\beta$ is about 20–30) allowing to use the proposed amplifier for a weak signal with arbitrary frequency.
A similar situation occurs if the qubit coupling is not strong enough to reach the optimal regime (see Fig. 2.6(b)). Indeed, the amplification of a weak signal ($\epsilon/\Delta = 0.5$) by the strong drive ($A/\Delta = 12$) is weaker, but still essential ($\beta$ about 10) for the coupling $g = 0.1$.

2.6 Conclusions to chapter 2

The spectrum of two coupled qubits driven by an ac signal with the frequency in resonance with inter-level transitions has an unusual structure, with a hierarchy of harmonic peaks with heights non-monotonically dependent on the harmonic’s number. This peak-height hierarchy is a fingerprint of any two-qubit system and can be used to characterize both individual qubit parameters as well as the interqubit coupling.

Exploiting the analogy between a parametric amplifier and a system of two coupled qubits, we propose a method of amplification of a weak signal via its mixing with a strong pump signal applied to the two-qubit system. If both signals are relatively close to the inter-level transitions in the four-level quantum system (which can be achieved by tuning the qubit coupling), then the amplification coefficient can be of the order of 100.

When the weak signal frequency is different from the inter-level splitting then the amplification is still strong enough, allowing the proposed amplifier to work efficiently both in inter-level resonance and off inter-level resonance regimes. Weakening the qubit coupling also suppresses a weaker signal enhancement, thus, requiring strongly-coupled qubits for this remarkable parametric amplification.
We also show that noise, which is of the order of a weak signal, can strongly affect the time dependence of the output signal $Z_1(t)$, but it modifies the spectrum $S_Z$ much more weakly. Therefore, the proposed amplifier can work efficiently in noisy conditions.

This large amplification offers a different way of using multiqubit circuits as parametric amplifiers.
Chapter 3

Harmonic mixing in two coupled qubits: quantum synchronization via ac drives

3.1 Introduction

This work was motivated by the analogy between driven Brownian nanoparticles and a system of coupled qubits. It is well-known that a particle driven by two harmonic ac signals through the substrate can drift in any desirable direction if frequency of the two drives are commensurate. This effect known as harmonic mixing [48, 49, 50, 51, 52, 53, 54, 55, 56] have been already observed in many systems, including vortices in superconductors [57], nanoparticles driven through a pore [58], current driven Josephson junctions [59] etc. This suggests an idea that a couple two-qubit system also should exhibit a harmonic mixing behaviour, but in contrast to the usual classical
harmonic mixing for overdamped particles, quantum harmonic mixing should be via parametric coupling of two drives in the quantum master equation. This effect can be used to synchronize quantum oscillations in two qubits and can control the average probability for a qubit to stay in either ground and excited states by changing the relative phase or frequencies of bi-harmonic drive.

Further, our results can be applied to the case when one needs to control qubits, which do not have their own control circuitry for the reasons of limiting the decoherence brought in by such extra elements, or because of accessibility (e.g., in case of 2D or 3D qubit arrays [60, 61] with control circuitry placed at the boundary viz. surface of the device). Similar problems arise in the so-called indirect quantum tomography (see e.g., Ref [62]) or in quantum computations with access to a limited number of qubits (see e.g., Ref. [63]). In all these cases, the proposed method of harmonic mixing in qubits allows to control the state of the second (not directly accessible qubit) by varying the frequency and/or phase of the first (accessible) qubit.

3.2 Model

In order to describe a two-qubit system we will use a Hamiltonian Eq. 1.39 in a spin-representation for each qubit with so called $\sigma_x - \sigma_x$ coupling. For simplicity, we again consider two identical qubits; that is, we assume $\Delta_1 =$
\[ \Delta_2 = \Delta. \] Let us drive the qubits by a control bi-harmonic drive:

\[
\begin{align*}
\epsilon_1(t) &= A_1 \sin(\omega_1 t) \\
\epsilon_1(t) &= A_2 \sin(\omega_2 t + \varphi)
\end{align*}
\tag{3.1}
\]

In other words, each qubit is driven by its own signal and amplitudes, frequencies and relative phase of these two signal can be varied at will. The question arises if and under what conditions the second qubit can influence the coherence and occupation of the ground and excited states of the first one. Therefore, we are interested if the second qubit can be used to control the state of the first qubit via dynamical synchronization of their quantum oscillations.

As in Chapter 2 we will simulate ordinary differential equations 2.4 for the density matrix \( \pi_{\alpha\beta} \) with \( \Gamma_{\psi_1} = \Gamma_{\psi_2} = \Gamma_1 = \Gamma_2 = \Gamma \) and \( Z_1 = Z_2 = 1 \), but for two differently driven qubits Eq.3.1 and to study harmonic mixing.

### 3.3 Simulation results

We simulated the set (Eq 2.4) by using the standard Euler method which has been proved to converge well for low-noise drives [47, 64] and analysed the time-averaged diagonal element of density matrix

\[ \langle X_1 \rangle = \langle \Pi_{ox} \rangle = \lim_{T \to \infty} \int_0^T \Pi_{ox} dt / T, \]

responsible for the mean coherence in the first qubit, as well as the time-averaged density matrix element \( \langle Z_1 \rangle = \langle \Pi_{oz} \rangle = \lim_{T \to \infty} \int_0^T \Pi_{oz} dt / T, \)

responsible for the mean occupation of the ground and excited state in the first qubit. To verify validity of our numerical
results we have also used higher-order multi-derivative methods to prove the stability of our numerical procedure (compare the open circles for Euler methods and the filled circles for the second order method in Fig.3.1 and 3.2).

As we expected, there is no mean coherence $\langle X_1 \rangle \approx 0$ for most of frequency ratio $\omega_1/\omega_2$ apart of specific commensurate cases (e.g., $\omega_1/\omega_2 = 2/5, 4/5, 2, 4$, see Fig 3.1(a)). Such a situation reminds a usual classical harmonic mixing for nanoparticles (see, e.g.,[55]), however, the frequency ratios where peaks occur, are also tuneable by changing the absolute value of signal frequency in either the first or the second qubit. Indeed, by choosing the frequency $\omega_1$ to be equal to the inter-level spacing frequency $\omega_1 = \sqrt{\Delta^2 + g^2} - g$ (Fig 3.1(a)) or $\omega_1 = 2\sqrt{\Delta^2 + g^2}$ (Fig 3.2(a)) or even away from the inter-level resonances $\omega_1 = 2.113(\sqrt{\Delta^2 + g^2} - g)$ (Fig 3.2(b)) results in a qualitatively similar peak structure, but showing different sequence of the frequency ratios. Indeed, for $\omega_1 = 2.113(\sqrt{\Delta^2 + g^2} - g)$ (Fig 3.2(b), the several new frequency ratios corresponding to the enhancement of qubit coherence occur at $\omega_2/\omega_1 = 2/5, 4/5, 6/5, 8/5, 2, 12/5, 14/5, 4$. Moreover, some peaks can even invert its sign (compare peaks at $\omega_2/\omega_1 = 4/5$ in Fig. 3.1(a),3.2(a),3.3(b)) indicating that both the frequency ratio and the absolute value of frequency can be used to tune qubit harmonic mixing. Such a behaviour is quite unusual with respect to classical harmonic mixing (see e.g.,[50]) where the frequency ratio is defined by nonlinearity of the system. In contrast, the master equation set (2.4) is linear and harmonic mixing occurs via mixture of multiplicative drives as in parametric oscillator. As we have recently shown, this results is a quite unusual spectra of $\Pi_{\alpha,\beta}$, and in particular $X_1$ and $Z_1$ with
Figure 3.1: Time-averaged Bloch tensor components $\langle X_1 \rangle = \langle \Pi_{0x} \rangle$ (a) and $\langle Z_1 \rangle = \langle \Pi_{0z} \rangle$ (b) for two coupled qubits driven by the two harmonic drive (3.1) with parameters: $A_1 = A_2 = 10, \phi = 0, \omega_1 = 2\sqrt{\Delta^2 + g^2}$. Other parameters are the simulation step $dt = 1.13 \times 10^{-5}$, the number of simulation steps $5 \times 10^9$, damping $\Gamma = 10^{-3}$, coupling constant $g = 1$, the tunnelling splitting energies $\Delta = 1$. The peaks of the time-averaged Bloch tensor element $\langle X_1 \rangle$ responsible for the qubit coherence shows peaks at $\omega_2/\omega_1 = 2/5, 4/5, 2, 4$, while the time averaged component $\langle Z_1 \rangle$ peaks at $\omega_2/\omega_1 = 3/5, 9/10, 3, 5$. Also, pumping the excited state for incommensurate frequencies are clearly seen: $|\langle Z_1 \rangle|$ increases for $\omega_2 \gtrsim 2\omega_1$.

many harmonic peaks showing complex hierarchy. This can explain a non-trivial behaviour of harmonic mixing changes when varying $\omega_1$ or $\omega_2$. Note
also, that the quantum harmonic mixing occurs in both cases: when (i) $\omega_1$ is equal to inter-level spacing and (ii) away from this situation. Therefore, there is no need to tune parameters of the external drives to any characteristic internal frequency of the two qubit system to observe quantum harmonic mixing.

Figure 3.2: Time-averaged Bloch tensor components $\langle X_1 \rangle = \langle \Pi_{0x} \rangle$ (a) for two coupled qubits driven by the two harmonic drive (3.1) with the same parameters as in Fig.3.1 and driving frequency $\omega_1 = \sqrt{\Delta^2 + g^2} - g$ (a) equal to an energy level transition frequency and for $\omega_1 = 2.113(\sqrt{\Delta^2 + g^2} - g)$ (b) which is away from the energy level transition.
We have also observed a similar harmonic mixing in time-averaged matrix element $\langle Z_1 \rangle$ responsible for the occupation of the excited and ground states (Fig 3.1(b)). Interestingly, the peaks in $\langle Z_1 \rangle$ occurs at different ratio of bi-harmonic drive $\omega_2/\omega_1 = 3/5, 9/10, 3, 5$. However, such a behaviour is perfectly consistent with the harmonic spectra of $\langle Z_1 \rangle$ and $\langle X_1 \rangle$ studied in Chapter 2.

Indeed, the spectrum of $X_1$ contains only odd while the spectra of $Z_1$ consists of even harmonics in agreement with the fact that peaks of $\langle X_1 \rangle$ and $\langle Z_1 \rangle$ have a different parity. Moreover, apart from the peaks at the specific commensurate frequencies, the value of $|\langle Z_1 \rangle|$ gradually increases with frequency for $\omega_2 \gtrsim 2\omega_1$ indicating pumping in the excited state even for incommensurate frequencies. Such an incommensurate harmonic mixing is very unusual for classical nonlinear systems.

Following analogy with classical harmonic mixing [55], we expect the dependence of $\langle X_1 \rangle$ and $\langle Z_1 \rangle$ on the relative phase $\varphi$ of bi-harmonic drive at commensurate frequencies where peaks have been observed. Indeed, we obtained such a dependence $\langle X_1 \rangle(\varphi)$ and $\langle Z_1 \rangle(\varphi)$ shown in Fig. 3.3(a-c) for the same simulation parameters as in Fig 3.1 and for frequency ratio $\omega_2/\omega_1 = 2$ (a), 3 (b), 4 (c). The well resolved peak of $\langle X_1 \rangle$ at even frequency ratio $\omega_2/\omega_1 = 2$ and 4 exhibits a strong dependence on relative phase, while a very weak peak of $\langle Z_1 \rangle$ at these frequency ratios shows almost no dependence on $\varphi$.

Comparing figures 3(a) and 3(c), we also conclude that periodicity of the $\langle X_1 \rangle(\varphi)$ changes with increasing frequency ratio following the rule: $\langle X_1 \rangle(\varphi + 2\pi \omega_1/\omega_4) = \langle X_1 \rangle(\varphi)$. Therefore, the number of full oscillations increases
Figure 3.3: Dependence of $\langle X_1 \rangle$ and $\langle Z_1 \rangle$ on a relative phases of two drives of the bi-harmonic signal at different frequency ratio $\omega_2/\omega_1 = 2$ (a), 3(b), 4(c). All other parameters are the same as in Fig. 1. For even frequency ratio, where $\langle X_1 \rangle$ has a peak (Fig.1), the strong dependence of $\langle X_1 \rangle(\varphi)$ and a weak dependence of $\langle Z_1 \rangle(\varphi)$ occurs, while, for odd ratio of $\omega_2/\omega_1$, dependence of $\langle Z_1 \rangle(\varphi)$ is clearly seen and $\langle X_1 \rangle(\varphi)$ is negligible. The periods of functions $\langle X_1 \rangle(\varphi)$ and $\langle Z_1 \rangle(\varphi)$ are controlled by the frequency ratio and is equal to $2\pi\omega_1/\omega_2$ (for $\omega_2 > \omega_1$).

with frequency ratio $\omega_2/\omega_1$. This dependence of the $\varphi$-periods of $\langle X_1 \rangle$ and $\langle Z_1 \rangle$ oscillations on the winding number of the harmonic drives is analogues
to the similar dependence of classical harmonic mixing of a Brownian particle
driven by bi-harmonic drive on nonlinear substrate [55].

3.4 Conclusions to chapter 3

We have predicted quantum harmonic mixing in a two-qubit system driven
by a bi-harmonic drive. It manifests itself in a set of peaks of time-averaged
density matrix components responsible for both qubit coherence and occupa-
tion of ground and excited states of qubits. These peaks can be controlled not
only by the ratio of frequencies of two signals but also by tuning frequencies
itself and by relative phase of two signals. Such quantum harmonic mixing
can be used to manipulate one driven qubit by applying an additional ac sig-
nal to the other qubit coupled with the one we have to control. This effect is
obviously robust to a reasonably strong decoherence and energy dissipation
in the system.
Chapter 4

Conclusions

The main aim of the enormous efforts in quantum technology is for the development and fabrication of qubit structures that can perform quantum computations. This is extremely tricky. Quantum computations are hard to realise right now but physicists, scientists, one and all are trying to work on this problem. There have been some great steps forward in the quantum computational world recently, the $15m computer that uses “quantum physics” effects to boost its speed is to be installed at a NASA facility [1]. This machine uses quantum tunnelling and is up to 3600 times faster than a conventional computer. There are small steps happening all the time, and surely with Moore’s law losing it’s trend we’ll see some great leaps in the field of quantum computation.

One of the benefits of the vast amounts of research into quantum computation is the development of well controllable and precisely driven systems of coupled qubits which can be used for quantum nanodevices. Even though lots of time and money is going into the fabrication and development, we have lots
of other interesting research areas that are becoming more and more useful. Using qubit arrays, new quantum nanodevices and quantum metamaterials have been proposed and some of these have been implemented.

As systems/structures get smaller we need to take into account noise and quantum effects and so, we need to develop some new working principles for nanodevices. Quantum devices work using quantum principles like that have already been developed, but now some aspects of quantum physics should be reconsidered for macroscopic quantum systems (e.g., Superconducting qubits) and very small measurement devices like a SET. In this thesis we describe two operations of quantum apparatus utilising qubit structures for signal amplification and rectification.

Firstly, we describe the parametric quantum amplifier, which can be used when we need to amplify very weak signals. Our system can amplify weak signals at very small scales. This can be achieved if the two qubits are biased simultaneously by this weak signal and a strong pump signal, both of which having frequencies close to the inter-level transitions in the system. We show that the amplification is robust with respect to noise. We propose to use coupled qubits as a combined parametric amplifier and frequency shifter.

We also describe a harmonic quantum mixer, which can be used to pump quantum states. This device can pump either the excited or ground state by changing the signal. This can be useful if we would like to control a large system of many qubit (e.g. D-wave)structures by only using several accessible qubits. Such a dynamic synchronisation of several differently driven qubits has an analogy with harmonic mixing of Brownian particles forced by two signals through a substrate. Nevertheless, the quantum synchronization in
two qubits occurs due to multiplicative coupling of signals in the qubits rather than via a nonlinear harmonic mixing for a classical nano-particle.

These are just two examples in quite a big picture of possible quantum devices based on qubit arrays and structures. There is quite a long way to go in the world of quantum devices and quantum metamaterials.
Bibliography


82


