

Loughborough University
Institutional Repository

*Subglacial bedforms reveal
an exponential
size-frequency distribution*

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Citation: HILLIER, J.K. ... et al., 2013. Subglacial bedforms reveal an exponential size-frequency distribution. *Geomorphology*, 190 pp. 82 - 91.

Additional Information:

- This article was published in the journal, *Geomorphology* [© Elsevier B.V.] and the definitive version is available at: <http://dx.doi.org/10.1016/j.geomorph.2013.02.017>

Metadata Record: <https://dspace.lboro.ac.uk/2134/13039>

Version: Accepted for publication

Publisher: © Elsevier B.V.

Please cite the published version.

This item was submitted to Loughborough's Institutional Repository (<https://dspace.lboro.ac.uk/>) by the author and is made available under the following Creative Commons Licence conditions.



CC creative commons
COMMONS DEED

Attribution-NonCommercial-NoDerivs 2.5

You are free:

- to copy, distribute, display, and perform the work

Under the following conditions:

BY: **Attribution.** You must attribute the work in the manner specified by the author or licensor.

Noncommercial. You may not use this work for commercial purposes.

No Derivative Works. You may not alter, transform, or build upon this work.

- For any reuse or distribution, you must make clear to others the license terms of this work.
- Any of these conditions can be waived if you get permission from the copyright holder.

Your fair use and other rights are in no way affected by the above.

This is a human-readable summary of the [Legal Code \(the full license\)](#).

[Disclaimer](#) 

For the full text of this licence, please go to:
<http://creativecommons.org/licenses/by-nc-nd/2.5/>

Subglacial bedforms reveal an exponential size-frequency distribution

J. K. Hillier^{a,*}, M. J. Smith^b, C. D. Clark^c, C. R. Stokes^d, M. Spagnolo^e

^a*Department of Geography, Loughborough University, LE11 3TU, UK.*

^b*School of Geography, Geology and the Environment, Kingston University, KT1 2EE, UK.*

^c*Department of Geography, University of Sheffield, S10 2TN, UK.*

^d*Department of Geography, Durham University, DH1 3LE, UK.*

^e*School of Geosciences, University of Aberdeen, AB24 3UF, UK.*

Abstract

Subglacial bedforms preserved in deglaciated landscapes record characteristics of past ice-sediment flow regimes, providing insight into subglacial processes and ice sheet dynamics. Individual forms vary considerably, but they can often be grouped into coherent fields, typically called flow-sets, that reflect discrete episodes of ice flow. Within these, bedform size-frequency distributions (predominantly height, width and length) are currently described by several statistics (e.g., mean, median, and standard deviation) that, arguably, do not best capture the defining characteristics of these populations. This paper seeks to create a better description based upon semi-log plots, which reveal that the frequency distributions of bedform dimensions (drumlin, mega-scale glacial lineation, and ribbed moraine) plot as straight lines above the mode (ϕ). This indicates, by definition, an exponential distribution, for which a simple and easily calculated, yet statistically rigorous, description is designed. Three descriptive parameters are proposed: gradient (λ ; the exponent, characterising bedforms likely least affected by non-glacial factors), area-normalised y -intercept (β_0 ; quantifying spatial density), and the mode (ϕ). Below ϕ , small features are less prevalent due to i) measurement: data, sampling and mapping fidelity; ii) possible post-glacial degradation; or iii) genesis: not being created sub-glacially. This new description has the benefit of being insensitive to the impact of potentially unmapped or degraded smaller features and better captures properties relating to ice flow. Importantly, using λ , flow sets can now be more usefully compared with each other across all deglaciated regions and with the output of numerical ice sheet

*Corresponding author. Tel.: ++44 1509 223727; Fax: ++44 1509 223930

Email addresses: j.hillier@lboro.ac.uk (J. K. Hillier), mike@hsm.org.uk (M. J. Smith), C.Clark@Sheffield.ac.uk (C. D. Clark), C.R.Stokes@durham.ac.uk (C. R. Stokes), m.spagnolo@abdn.ac.uk (M. Spagnolo)

models. Applications may also exist for analogous fluvial and aeolian bedforms. Identifying the characteristic exponential and that it is typical of ‘emergent’ subglacial bedforms is a new and potentially powerful constraint on their genesis, perhaps indicating that ice-sediment interaction is fundamentally stochastic in nature.

Keywords: Subglacial; Bedform; Exponential; Stochastic; Flow-set; Fluvial.

1. Introduction

Subglacial bedforms are a group of landforms created at the interface between glaciers and the terrain underneath (e.g., Benn and Evans, 2010). Mainly comprised of glacial sediments (e.g., Stokes et al., 2011), they are often assigned to one of four categories based on their size and shape: (i) flutes (e.g., Boulton, 1976), (ii) drumlins (e.g., Menzies, 1979a), (iii) ribbed moraine (Hättestrand and Kleman, 1999) and (iv) mega-scale glacial lineations (MSGSL) (Clark, 1993). Taken together, these range between 10^1 and 10^5 m long (Clark, 2010). Ribbed moraine form transverse to ice flow direction, whilst flutes, drumlins and MSGSL form parallel to ice flow and are possibly a continuum of landforms (e.g., Aario, 1977; Rose, 1987) that are created by similar processes that operate under variable conditions. For example, it has been suggested that bedform length may be related to ice velocity (e.g., Clark, 1993; Hart, 1999; Stokes and Clark, 2002). Glacial bedforms are generally argued to be created directly by overriding ice flow (e.g., Benn et al., 2006; King et al., 2007; Clark, 2010; Ó Cofaigh et al., 2010), although an origin through sub-glacial floods has also been proposed (e.g., Shaw, 1983; Shaw et al., 2008). Due to their prevalence, drumlins have been most heavily studied, but even these remain enigmatic with their exact mode of formation still undetermined (e.g., Smalley and Unwin, 1968; Menzies, 1979b; Shaw, 1983; Boulton and Hindmarsh, 1987; Hindmarsh, 1998; Fowler, 2000; Clark, 2010).

The shapes of bedforms (e.g., height H , width W , length L , and orientation) preserve key information about the dynamics and mechanics of former ice sheets, an important guide as to how existing ice sheets will behave in the future. Observations are typically used descriptively to indicate properties such as ice extent or flow direction (e.g., Hollingsworth, 1931; Livingstone et al., 2008), for example to assess consistency with numerical ice sheet models (e.g., Evans et al., 2009), and only rarely to directly consider the mechanics of ice-sediment interaction and flow (Chorley, 1959; Morris and Morland, 1976; Smalley and Piotrowski, 1987; Smalley and Warburton, 1994).

25 Indeed, few theories of subglacial bedform genesis are yet to explicitly engage with empirical data
26 on their shape and size. One that has made predictions of bedform dimensions is the instability
27 theory (e.g., Hindmarsh, 1998; Fowler, 2000; Stokes et al., 2013), but it only considers them as
28 quantitative constraints in the broadest sense, as an order of magnitude scale ground-truth (Dunlop
29 et al., 2008; Chapwanya et al., 2011). A disconnect therefore exists between glacial geomorphology
30 and glaciological modelling (e.g., Bingham et al., 2010).

31 As a step towards forming a link between the subglacial bedform record and the nature and
32 mechanics of ice flow, this paper presents a descriptive development: a new statistical charac-
33 terisation of bedform populations. The need for an improved description is two-fold. Firstly,
34 population metrics should capture the signal of ice-sediment interaction, not artefacts of measure-
35 ment or preservation. Secondly, individual population metrics should ideally capture key aspects
36 of data allowing inter-comparison of data types, localities, and palaeo-environments. The origin
37 and nature of potential artefacts and the implications of this for current metrics are considered in
38 Section 2. It is demonstrated graphically in Section 3, using semi-log plots, that the size-frequency
39 distributions of key properties (e.g. H , W , L , and L/W) of subglacial landforms are exponentially
40 distributed above the mode. Following this, a simpler objective parameterisation of the data is
41 created in Section 4 which consists of individual metrics better suited to isolating characteristics
42 of bedform populations relating to ice flow. Then, by collating data sets for a variety of areas,
43 Section 5 demonstrates the general applicability of the proposed description to subglacial bed-
44 forms. Finally, in Section 6, the selection of the exponential-based parameterisation is discussed
45 and initial thoughts are offered on implications for the process of drumlin genesis.

46 **2. Quantifying subglacial bedforms**

47 Subglacial bedforms have been quantified in a variety of ways, both as individuals and popu-
48 lations (e.g., Gardiner, 1983; Smalley and Warburton, 1994). Individual forms vary considerably,
49 even within a locality (e.g., Hollingsworth, 1931), so they are likely to best reflect flow regimes
50 when grouped into spatially and temporally co-located flow-sets. Thus, quantifications for popu-
51 lations (e.g., Fig. 1a) are considered here, although error bars for parameters may be large enough
52 to warrant particular attention for small populations (i.e., $n \lesssim 50$). For clarity we use the terms
53 ‘metric’ or ‘parameter’ exclusively to refer to quantification statistics such as the mean or mode,

54 as distinct from measurements to which they are applied such as L or ‘aspect ratio’ (i.e., L/W).
55 Populations of observations from which the metrics are calculated are the final products of ap-
56 plication of three compounding processes. Artefacts are due to i) measurement, ii) post-glacial
57 preservation and iii) the process of glaciological interest i.e., bedform genesis itself. The artefacts
58 must be accounted for to reveal information about ice-sediment interaction. In light of this, each
59 metric has its strengths and weaknesses as a descriptor. Consequently, in attempting to faithfully
60 capture process-related characteristics of the bedform populations it is necessary to choose metrics
61 that will be minimally sensitive to systematic biases.

62 Measurement is the translation from the real, currently observable landscape to geometric quan-
63 tities describing bedforms (e.g., H and L). In terms of size-frequency populations, this presents
64 three specific issues concerning the efficacy of the measurements taken:

65 1. Effect of source data on mapping (e.g., Smith and Clark, 2005): Smith and Wise (2007)
66 outline the primary controls on the ‘detectability’ of landforms mapped from satellite imagery
67 or visualised digital elevation models (DEMs); namely solar elevation, solar azimuth and
68 sensor spatial resolution. These factors resolve to sampling issues: there exists a population
69 of phenomena from which our observational method necessarily involves the selection of a
70 subset. Solar azimuth can, for instance, systematically reduce all L values. Perhaps the
71 best understood sampling bias is sensor resolution; small landforms are not detectable in
72 coarse, low resolution data. Resolution therefore may contribute towards the low number
73 of small bedforms (e.g., Fig. 1a) by imposing a threshold below which sampling becomes
74 more difficult. Spagnolo et al. (2012), for instance, note this with respect to H in previous
75 databases (Francek, 1991; Wysota, 1994; Hättestrand et al., 2004), although inability to
76 observe in no way precludes the landforms not being there in the first place. Without knowing
77 the actual population or error associated with the sampling, true values for statistics derived
78 from the whole population cannot be ascertained with certainty.

79 2. Quantification method: even for a given mapped outline and digital terrain model (DTM),
80 a variety of algorithms exist to compute a bedform’s properties (H , W , L , and volume V)
81 (e.g., Spagnolo et al., 2010; Hillier and Smith, 2012). Values will vary, e.g. for H (Spagnolo
82 et al., 2012), depending upon the method selected. Identical geometries, however, will be
83 affected by the same proportion at all scales. Removing post-glacial clutter (e.g., trees) to

84 create a DTM will affect H for mapped forms (Hillier and Smith, 2012). This has not been
85 systematically studied, but it seems probable that bedforms with small heights will more
86 commonly be rendered unmappable.

- 87 3. Subjectivity of interpretation: Manual mapping of bedforms is subjective and reliant upon
88 the expertise and experience of the mapper. Whilst the process is not objectively repeatable,
89 procedures are employed to maintain consistency and minimise bias (e.g., Smith and Clark,
90 2005; Hughes et al., 2010). Interpretations may, perhaps inevitably, vary more towards both
91 perceived limits of the size range of a bedform, creating the largest uncertainties there. This
92 subjectivity may, in future, be alleviated by automated mapping (e.g., Hillier, 2008; Saha
93 et al., 2011; Kalbermatten et al., 2012; Rutzinger et al., 2012), but most benefits depend
94 upon agreement being reached on an exact formal definition of each bedform (e.g., Evans,
95 2012).

96 After measurement, post-glacial preservation rates also affect bedform populations. If the mea-
97 surement issues could all be accounted for, it would be possible to interpret frequency information
98 across the size spectrum in terms of physical processes. Even then, however, a low prevalence
99 for palaeo-landforms does not necessarily mean they are not abundant in active environments.
100 Relative abundances could still be an artefact of post-glacial degradation that varies with size,
101 e.g. diffusive hillslope-type erosion (e.g., Putkonen and Swanson, 2003). The preservation of small
102 features, flutes for instance, is thought to be low. Therefore, to best interpret mapped subglacial
103 bedforms in terms of subglacial processes, it is likely important to use measures least affected by
104 all the issues identified above. At the very least, doing this has no detrimental effects.

105 In terms of a size-frequency distribution, non-glacial distortions may be summarised as follows
106 (also Fig. 1b). Artefacts affecting all sizes by a single factor do not change the distribution's
107 shape, and are a minor issue. Most seriously, there is potentially significant undersampling of small
108 features due to several limitations in source data, post-glacial erosion, perhaps the quantification
109 method, and potentially the views of an interpreter when mapping landforms. This latter factor
110 also introduces uncertainty into the upper end of the size distribution, potentially increasing or
111 decreasing detections or introducing outliers by including genetically unrelated landforms. So,
112 'good' metrics will be insensitive to the potential absence of small landforms and either not be
113 unduly influenced by outliers at the upper end of the size range or provide means to identify and

114 exclude them. They should also not, if possible, depend on sample size or arbitrary choices. For
115 utility, it is also desirable to have as succinct yet complete a description of the distribution as
116 possible.

117 Currently both simple (H , W , and L) and derived morphometric measures such as ‘elongation’
118 (i.e., L/W) are collated for populations (e.g., Hoppe and Schytt, 1953; Boulton, 1976; Stokes and
119 Clark, 2002; Dunlop and Clark, 2006; Clark et al., 2009; Smith et al., 2009; Phillips et al., 2010).
120 Up to eight parameters or metrics (e.g., Clark et al., 2009) are used to describe each measure (e.g.,
121 Fig. 1a): minimum, maximum, mean, standard deviation, modal class, median, skewness and
122 kurtosis. Whilst undoubtedly useful for initial assessment, the number and nature of these metrics
123 is not necessarily ideal for describing populations. Problems include: (i) extreme values depend
124 upon the number of observations (unless estimated using appropriate statistical techniques e.g.,
125 van der Mark et al. 2008), observational completeness, and distribution shape, (ii) modal class is
126 dependent upon the selection of a bin width, and (iii) the use of all of four moments (i.e., mean,
127 standard deviation, skew and kurtosis) to describe the shape of the distribution; comparisons
128 between shapes are more straightforward for single characteristic shape parameters. Lastly, (iv)
129 the mean is affected in the first order by the steepness and length of the right-hand tail (e.g.,
130 Fig. 1a), the location of the ‘roll-over’ at the mode, ϕ , and any outliers. Thus, this ensemble
131 of metrics is somewhat unsatisfactory, primarily because smaller bedforms *may* be substantively
132 under-represented (Fig. 1b), perhaps leaving larger bedforms best reflecting glacial processes (Fig
133 1b). A simpler description may be possible, however, whose parameters likely better relate to ice–
134 sediment interaction and only requires the assumption that larger features are accurately observed.
135 This would be a weaker requirement than that of accurate quantification at all sizes implicit in
136 present analyses.

137 **3. Graphical investigation**

138 Appropriate parameterisation of a distribution requires knowledge of its form. Many univariate
139 statistical distributions contain exponential or power-law elements (e.g., Leemis and McQueston,
140 2008). Exponential functions or forms plot as straight lines on semi-log plots, as do power law
141 relationships on log–log ones. These plots are therefore useful in preliminary investigations of
142 the characteristics of observed data. This section illustrates the utility of these approximations

143 to subglacial bedform data through different plots of L for one data set relating to one type of
144 bedform.

145 Through the production of a semi-log histogram (Fig. 1b) and plotting a linear fit through the
146 data (see Section 4) above the modal ‘roll over’, it is possible to visually demonstrate that counts
147 of drumlin lengths above the mode conform to an exponential distribution. Though being non-
148 linear, Fig. 1c clearly demonstrates that no large part of the the distribution is power-law. Power-
149 law segments in distributions are typical of fractals such as topography (e.g., Mandelbrot, 1983;
150 Weissel et al., 1994; Cheng and Agterberg, 1996), natural phenomena (e.g., floods, earthquakes, and
151 wildfires) (e.g., Main et al., 1999; Malamud et al., 2005; Kidson et al., 2006; Malamud and Turcotte,
152 2006), and linked to the notion of self-organising criticality in systems (e.g., Bak, 1996; Tebbens
153 et al., 2001). Importantly, Haschenberger (1999) empirically relate the observed exponent of
154 exponential distributions for fluvial bedforms to estimates of basal shear stress in that environment.
155 Gradients of the fitted lines such as that in Fig. 1b may therefore not only capture an important
156 property related to flow but also encapsulate it in a single value, facilitating easy intercomparison
157 between data sets. Descriptively, e.g., in Fig. 1b, the exponential only applies to data above
158 the mode. There are no grounds for plotting it at smaller sizes other than extrapolation. In the
159 simplest possible model, continuing the trend may be seen as a continuation of the signature of
160 a subglacial process, but there is no evidence here to support this. As noted in Section 2, the
161 difference between data and extrapolation due to i) measurement: data, sampling and mapping
162 fidelity, ii) possibly post-glacial preservation or iii) the roll-over being a signature of the processes
163 of ice-sediment interaction resulting in smaller features not being created subglacially in the first
164 place i.e., their genesis. Insufficient work has been published to make definitive, comprehensive
165 comments upon which one dominates, but there are strong hints that commonly observed bedforms
166 lack numerous smaller versions. For instance, in extension of the results of Smith and Wise (2007),
167 Clark et al. (2009) suggest that a clear lower bound for W in UK drumlins is unlikely to be
168 an artefact of imagery resolution, attributing it to glacial processes (i.e., smaller forms are less
169 commonly created). For the smallest bedforms this is very probably true, and the exponential
170 should certainly not be extended to the y-axis. Consider drumlins; size observations from recently
171 deglaciated terrain (Johnson et al., 2010) conform with palaeo data, and very small drumlins are
172 not reported. However, measurement and preservation issues seem to affect significant fractions

173 of drumlins that are larger and yet below the mode (Smith et al., 2006). So, speculatively, the
174 existence of a well-defined modal peak is a signature of physical processes. However, its location
175 is not yet necessarily well determined, with the possibility that smaller features are not recorded.
176 As such, non-glacial factors may have a large effect on measures such as the mean, particularly for
177 mapping in areas where high-resolution DEMs are not available.

178 4. Objective parameterisation

179 Given that bedform size-frequency distributions appear well described by a right-hand expo-
180 nential decay above the mode and a roll-over to low numbers below it (Fig. 1), a description
181 using three parameters is proposed that is designed to best represent subglacial processes, facili-
182 tate comparison between regions and data sources, and whose computation is readily accessible to
183 geomorphologists. The selected metrics to approximate the distributions, (see Fig. 2c), are:

- 184 1. Gradient (λ): magnitude of the gradient of the fitted line (e.g., Fig. 1b), which is the exponent
185 of the decay (Eq. 2 in Appendix). This characterises the part of the distribution that is
186 least likely affected by non-glacial factors. Larger bedforms will have greater endurance in
187 the landscape and the observed frequency should be close to the expected frequency. No
188 disproportionate weight in the fit is placed on the largest features whose interpretation may
189 be uncertain (e.g., Fig. 2f), and features unrelated to the distribution can be identified and
190 excluded.
- 191 2. Mode (ϕ): estimates the point at which bedforms are no longer representatively sampled,
192 non-glacial factors become dominant, or ice-flow related behaviour somehow changes in its
193 nature or effect. If many smaller features are missed, it will be influenced (e.g. Smith and
194 Wise, 2007), but is much more robust than the mean or median.
- 195 3. Intercept (β_0): intercept of the exponential with the y-axis represents the spatial density of
196 the landforms (i.e., number per unit area) in a way that is insensitive to the efficiency with
197 which small ones are detected, unlike the mean (e.g., Smalley and Unwin, 1968; Miller, 1972;
198 Menzies, 1979b). Whilst the area, A , of a bedform field remains inexactly defined, the use
199 of this for subglacial bedforms is limited at present, but is a key parameter compared for
200 seamount distributions (e.g., Jordan et al., 1983; Scheirer and Macdonald, 1995; Hillier and
201 Watts, 2007) illustrating its potential.

202 It is anticipated that λ values, either individually or when plotted against each other for H , W
203 or L (e.g., x - y or ternary diagrams), will be a powerful means of characterising landforms. This
204 could, for example, contribute to the debate as to whether bedforms constitute a continuum (e.g.,
205 Rose and Letzer, 1977; Rose, 1987; Clark, 1993; Clark et al., 2009), with data points for localities
206 for each bedform type either forming separate domains or a merging in a progression from one to
207 the other. Using λ should make such analyses robust to the dataset or resolution used. Note also
208 that λ will not vary with the size of the data set. ϕ is a natural measure of unimodal bedform
209 distributions and is a useful metric whatever it is thought to represent. For instance, ϕ is a good
210 indicator of the size at which imperfect detection arises perhaps due to data type where this
211 dominates (e.g., Smith and Wise, 2007), and will reflect glacial processes where measurement is
212 not an issue.

213 The question then is how to estimate values for these metrics. Various methods to estimate
214 parameters of plotted data exist (e.g., Cornell and Speckman, 1967); fitting a line (e.g., by ordinary
215 least squares – OLS) to counts from a selected portion of a histogram considered to be linear may
216 be done for simplicity (e.g., Wessel, 1997), but is not optimal (e.g., Smith and Jordan, 1988;
217 Solow et al., 2003; Bauke, 2007). OLS fits of power-laws to log–log frequency plots, for instance,
218 are known commonly to introduce significant, systematic, unpredictable biases into estimates of
219 gradient (e.g., Newman, 2005; Clauset et al., 2009). The results also depend on i) bin width
220 and construction (e.g., Newman, 2005) and ii) range chosen. An insight into the limitations of
221 applying OLS to plots such as Fig. 2 may be gained by considering that it fits to the x - y plot
222 rather than the underlying data, and each point on the plot is assigned equal weight and accuracy
223 despite containing a different number of data, although larger counts tend to be less variable. An
224 objective, statistically valid method based upon the underlying data (i.e., not fitting a frequency
225 plot) that minimises arbitrary choices is proposed to estimate λ , ϕ and β_0 . The method of moments
226 (e.g., Freund and Walople, 1980, p. 325) is used to estimate the mode using a Gamma distribution,
227 then the gradient obtained through a maximum likelihood fit (e.g., Freund and Walople, 1980, p.
228 327) of an exponential distribution for data larger than ϕ . This may be performed without any
229 specialist statistical software, requiring only the calculation of the mean and standard deviation
230 (i in Appendix). Not only is this approach relatively straightforward, but with minor adaptation
231 it allows parameter estimation from the published literature using data presented in histograms

232 (iii in Appendix), although it is not able to recover information lost during binning. In fact, data
233 digitised from published histograms were deliberately used in Fig. 2 to specifically illustrate this
234 point.

235 The fitted lines (solid lines in Figs. 1 and 2) show the efficacy of the method, and whilst the
236 Gamma distribution shown by dashed lines provides a poorer fit to the population, it is able to
237 estimate ϕ particularly well. In Fig. 2a, ϕ is estimated as 424 m (3 s.f.), inside the range of the 393–
238 441 of the modal bin of Clark et al. (2009). The same is true for W and H with 177 m inside 173-183
239 and 3.7 m inside 3.5–4.0, respectively (Fig. 2b,c). Note that no selection of a bin width is necessary.
240 This approach of fitting a line to data larger than an objectively determined value for ϕ overcomes
241 *ad hoc* criteria previously used to determine the range of data fitted (e.g., n in bin > 5 ; Rappaport
242 et al., 1997). Furthermore the method outlined here avoids the systematic overestimation of
243 λ that occurs when it is estimated by fitting a Gamma distribution (ii in Appendix). Once
244 the exponential distribution is fitted, β_0 is calculated by simple geometry. A worked example
245 detailing the procedure is provided in a Microsoft Excel spreadsheet as Supplementary Material
246 accompanying this paper. In anticipation that readers may want to compare bedform populations,
247 a method of statistically evaluating whether λ is significantly different for those populations is
248 given in iv) in Appendix

249 5. Prevalence of the exponential tail

250 Fig. 2 demonstrates that a form of size-frequency distribution with roll-over and right-hand
251 exponential tail is typical of subglacial bedforms and derived measurements. Data for Fig. 2 were
252 selected to demonstrate this via a number of specific points. Fig. 2a,d depicts an exponential tail
253 for large samples ($n > 10,000$) of the same measure, L , of a particular bedform (i.e., drumlins) in
254 two discrete study areas. Evidence for the form is therefore not location dependent, and it may
255 occur wherever bedforms do. Fig. 2a,e,g,h shows the output of at least four independent mappers
256 demonstrating that the form is not a result of an individual’s style or preference. Independent
257 mapping of a sub-area of Fig. 2c is shown in Fig. 2e for the same measure of UK drumlins,
258 H . So, the occurrence of the form over large areas is not purely the result of aggregation, but
259 is directly related to and applicable to individual flow sets. Furthermore, Fig 2e illustrates the
260 form’s utility for even a relatively small sample ($n < 200$), although the error bars for descriptive

261 parameters are larger; $\lambda = 0.209 \pm 0.003 \text{ m}^{-1} 2\sigma$ and $\lambda = 0.238 \pm 0.024 2\sigma$ for Fig. 2c and Fig.
262 2e, respectively, using the estimation method in Section 4. To be explicit, Fig. 2 demonstrates
263 that the form applies to all glacial bedforms considered in this paper for which adequate data are
264 available for assessment with drumlins in Fig. 2a–f, ribbed moraine in Fig. 2g and MSGL in Fig.
265 2h. Tentatively, this description may also apply to flutes (e.g., H), but evidence is limited (i.e.,
266 $n \sim 50$) (e.g., Hoppe and Schytt, 1953; Boulton, 1976) leading to much scatter in semi-log plots.

267 Despite the weight of evidence presented in Fig. 2, it is important to note that the fit to the
268 right hand tail is neither perfect nor ubiquitous. Firstly, then, it is notable that the plots (Fig.
269 2) show some scatter for large bedforms, and exponentials fit imperfectly. Most data were, quite
270 deliberately, digitised from published histograms, but its presence in Fig. 2h demonstrates that
271 errors due to this re-use of data are not the main cause. Perhaps it originates from uncertainty
272 in categorising and thus selecting larger forms. Secondly, some data sets show distinct deviations
273 from a linear trend on semi-log plots. Height data from northern Sweden (Hättestrand et al.,
274 2004; data pers. comm.), where ‘crag-and-tail’ bedrock-influenced drumlins dominate, show a
275 distinct bend in their trend on a semi-log plot (Fig. 3a). Why? Few published bedform frequency
276 plots exist to assess this. Elongation ratio (i.e., L/W) data digitised and re-plotted from Fig. 2
277 of Phillips et al. (2010) also shows trends of two distinct gradients. These data, however, come
278 from neighbouring regions in which Phillips et al. (2010) consider the influence of bedrock in
279 creating landforms. Fig. 3b shows a steep trend (blue line) in Zone 1 ‘*dominated by an extensive*
280 *drumlin field*’ and a shallower one in Zone 2 (red line) where landforms are of ‘*ice moulded bedrock*’.
281 Speculatively, it seems possible that bedrock influence can create geometric extremes beyond those
282 of till-dominated landforms consistent with local formational conditions. Thus, the Swedish data
283 may be exhibiting the signature of bedrock influence. Note the contrasting studies in Fig. 2 (e.g.,
284 Clark et al., 2009; Spagnolo et al., 2012) pointedly seek to exclude bedrock influenced landforms
285 from their drumlin catalogues, as do most studies (cf. Stokes et al., 2011). More generally, distinct
286 trends will likely exist if a plot contains data aggregated from distinctly different flow regimes,
287 perhaps forms attributed to streaming ice (e.g., Fig. 2h) and ‘typical’ drumlins (e.g., Fig. 2d).

288 In summary, an exponential right-side tail is typical, and perhaps characteristic, of till-dominated
289 ‘emergent’ subglacial bedforms (Clark, 2010), and the spread of observations is sufficient to suggest
290 that this is a general characteristic. The consistency of form is remarkable considering the gradient

291 and perhaps mode likely change with local conditions.

292

293 **6. Discussion**

294 *6.1. Choice of parameterisation*

295 This paper concerns the description of subglacial bedforms and, in particular, how to most
296 usefully quantify populations within an area. The approach taken is that size-frequency distribu-
297 tions may contain information not best captured by currently used metrics. Primary difficulties
298 for current statistical metrics in reflecting ice-sediment interaction are that many are not natural
299 descriptors of heavily skewed distributions and are sensitive in the first order to imperfect detec-
300 tion rates for smaller features. Additional computational issues exist for some in that they are
301 dependent upon sample size (e.g., min., max., range) or bin selection for aggregation (e.g., mode).
302 Size-frequency data for measures of co-located subglacial bedforms display linearly on semi-log
303 plots (Figs. 1 to 3). This demonstrates that they are commonly distributed exponentially, at least
304 above their modal values. Noting this form creates the possibility to design a simpler description.

305 The simplest description would be an exponential probability density function. This has been
306 used to characterise domains of submarine volcanoes (e.g., Jordan et al., 1983; Scheirer and Mac-
307 donald, 1995; Hillier and Watts, 2007) and fluvial scour depths (Haschenberger, 1999), but fre-
308 quencies of these do not roll-over at small sizes. The single parameter, the exponent λ , could
309 not capture this. In fluvial geomorphology a variety of two-parameter distributions (e.g., Gamma,
310 Gaussian, Gumbel, Log-normal, and Weibull) have been evaluated for their potential to describe
311 bedform size-frequency populations (Leemis and McQueston, 2008; van der Mark et al., 2008).
312 The distributions approximate, with variable degrees of success, the *shape* of the size-frequency
313 distributions of the populations including a roll-over. Whilst entirely statistically valid, their utility
314 when applied to subglacial landforms may suffer as both their parameters are influenced by data
315 across the whole size range. Ideally, for the purposes of description, the characteristics of the right-
316 hand tail that likely best represent subglacial processes should not be influenced by potentially
317 unmapped small features.

318 The method proposed here minimises the influence by fitting an exponential distribution, λ , to
319 only data above the mode, ϕ : two shape parameters. Admittedly, λ and ϕ incompletely describe
320 observations below the mode, giving only its starting point, but this is where observations are least

321 securely related to glacial processes. λ represents a part of the distribution least likely affected by
322 factors unrelated to ice-sediment interaction, and ϕ ensures that data selection for its calculation is
323 objective. If features larger than the mode are unreliably detected it may not be entirely accurate,
324 but biases due to this will be no worse than for other parameterisations. Synthetic landscapes
325 (Hillier and Smith, 2012) may allow this to be quantified. β_0 is an additional scaling factor
326 normalised for area to make it a useful measure i.e., of landform spatial density.

327 *6.2. Utility of the parameterisation*

328 Typically, in parameterising data, there is a trade-off between computational simplicity and
329 ease (e.g., requirement for statistical software), and objectivity and rigour. This is optimised in
330 the method suggested here as no subjective choices (e.g., bin width) exist: it fits underlying data,
331 not a plot, and the whole calculation is possible without specialist software. It requires only the
332 calculation of means and standard deviations (see i in Appendix or the accompanying worked
333 example using Excel. These calculations may be biased by large, mis-identified outliers, but these
334 are rare and the exponential form provides a mechanism for assessing if observations are consistent
335 with the bulk of a population, leading to an iterative fitting solution if necessary. The method to
336 estimate λ , ϕ and β_0 demonstrably (Fig. 2) works on both raw data and those already derived
337 from published histograms.

338 The parameterisation proposed is entirely descriptive and non-genetic in that it is not neces-
339 sarily related to any formational process: the description will be valid whether or not future work
340 identifies it as a signature of any particular ice-sediment process. Its non-genetic nature is useful
341 in a characterisation as it avoids tying it to process-related debates. It, and particularly λ , not
342 only has the power to present a single, generally applicable measure of bedforms, but also apply it
343 to a wide range of published size catalogues, mapped from data of various types and ages, allowing
344 inter-comparison. For instance, flow sets can now be more usefully compared with each other
345 across all deglaciated regions and with the output of numerical ice sheet models (e.g., flow veloc-
346 ity or basal shear stress). A method of determining whether λ is significantly different between
347 flow sets is also given. With the same governing equations proposed to control the evolution of
348 bedforms created by ice, water or wind (e.g., Fowler, 2002), and a similarity between glacial (e.g.,
349 Fig. 2) and fluvial size-frequency distributions (e.g., van der Mark et al., 2008), applications for
350 the parameterisation may also exist for analogous fluvial and aeolian bedforms.

351 6.3. *Bedform genesis*

352 Perhaps the most exciting aspect of the work is the future potential to use the explanatory
353 power of the exponential characterisation in terms of understanding physical processes that are
354 operating. Some insights, however, are feasible now. The caveat is that caution is necessary as
355 multiple processes or histories can lead to the same statistical distributions (e.g., Tuckwell, 1995;
356 Beven, 2006; Newman, 2005).

357 Tentatively, it is possible to suggest that the similarity between distributions for different
358 bedforms indicates some commonalities between processes creating them and progressions in the
359 processes between the bedform types. In depth modelling of the underlying processes of bedform
360 genesis is beyond the scope of this work, but the few indicators available suggest that λ may
361 directly reflect aspects of physical processes. Specifically, Haschenberger (1999) empirically relate
362 λ for fluvial scour depths to basal shear stress in that environment. Furthermore another simple
363 form of size-frequency distribution, the power-law, has been interpreted and modelled in terms
364 of process (e.g. Newman, 2005), for instance ‘self-organised criticality’ (e.g. Bak, 1996; Tebbens
365 et al., 2001). ‘Self-organised critically’ involves a set of simple rules and randomness acting at
366 multiple locations that combine to produce characteristic size-frequency distributions. Subglacial
367 bedforms originating in the presence of random variations at multiple locations may have a similar
368 ability to produce characteristic distributions. Indeed, fluvial bedforms with similar heavy-tailed
369 size-frequency distributions (e.g., van der Mark et al., 2008; Singh et al., 2011) are considered
370 to originate in random fluctuations in turbulent flow (e.g., Fredøse, 1996; McElroy and Mohrig,
371 2009; Coleman and Nikora, 2011) with H and L described there as ‘stochastic variables’ (van der
372 Mark et al., 2008). Similarly, ice-sediment interaction may be fundamentally stochastic in nature
373 i.e., bedform growth may be a process involving the convolution of randomness with simple rules
374 about the rate of growth. This is consistent with geophysical studies that have revealed spatio-
375 temporally variable bed conditions (Vaughan et al., 2003; Smith, 2006; Murray et al., 2008) and
376 subglacial landforms (King et al., 2007; Smith and Murray, 2009) that evolve rapidly on sub-
377 decadal timescales (Smith et al., 2007; King et al., 2009) under Antarctic ice streams. It is unclear,
378 however, whether this variability at the bedform scale originates dominantly in the dynamics of
379 ice-sediment-water interactions (e.g., water incursions or basal stick-slip events) or those between
380 bedforms. This stochastic approach contrasts to a deterministic view whereby proto-bedforms of

381 known size and shape always evolve similarly with time to a predictable final morphology; perhaps,
382 each bedform's size may be individually limited by local physical conditions that vary in space such
383 that an exponential distribution is created. It is not immediately clear, however, how neighbouring
384 bedforms of dramatically different sizes, as commonly observed, originate in this theory, although
385 it is likely that bedforms are 'born' at different times (cf. Smith et al., 2007), even within a single
386 flow-set. So for this reason, and by a loose analogy with the processes creating exponential tails
387 for fluvial bedform populations, we suspect that conditions that give rise to subglacial bedforms
388 are fundamentally variable and stochastic.

389 Many possible processes can be conceived in which bedforms are created and destroyed under
390 ice using randomness and growth with various rate characteristics. A limited number, however,
391 will produce exponential size-frequency distributions. The observations are therefore a constraint
392 on models of bedform genesis. For instance, can stochastic variability be incorporated into till
393 instability theory of Hindmarsh (1998)? Considering the stability or otherwise of bedform pop-
394 ulations with respect to time may also prove valuable. Are bedforms in steady state dynamic
395 equilibrium? If so, λ values may relate to properties of ice flow, such as velocity. Alternatively, if
396 size-frequency distributions continue to evolve with time, ϕ and λ might combine to provide some
397 constraint upon both the rate and duration of bedform growth. Finally, note that for accurately
398 measured and well-preserved size-frequency distributions, a different two-parameter distribution
399 (see Section 6.1) may assist in providing further constraints by describing the roll-over as well as
400 the exponential tail. So, we suggest that future progress will come through understanding the
401 observed exponential in terms of the statistics and mechanics of ice flow.

402 7. Conclusions

403 This paper presents a simple yet robust descriptive parameterisation that can be used to sum-
404 marise and compare populations of subglacial bedforms, e.g. in flow-sets. Whilst a variety of
405 distributions have been used in other disciplines, an exponential characterisation is appropriate in
406 this area and offers potential explanatory power in terms of the processes in operation. Through
407 plotting observations of landform size, specifically for ribbed moraine, drumlins and MSGL, and
408 for populations of different sizes, the following main conclusions may be drawn:

- 409 • Till-dominated subglacial bedform size-frequency distributions characteristically have an ex-

ponential right-hand tail.

- Semi-log plots are a useful tool with which to initially assess this since exponentials plot as straight lines.
- The distributions may be rigorously, objectively and practically approximated by using the method of moments and the Gamma distribution to estimate the mode ϕ , and then using a maximum likelihood method to estimate the exponent λ (i.e. gradient of the semi-log plot) for measurements larger than the mode.
- For observations *below* the mode, a combination of possible sampling error and probable absence means that there is some uncertainty here depending upon the data type used for mapping.
- λ is likely to reflect glacial processes significantly better than previously used metrics.

This description uses three parameters, rather than the selection of up to eight currently used. This simplicity makes it a preferable approach to developing understanding in unresolved areas such as the subglacial bedform continuum or spatial patterns of palaeo-flow. Future insights may come through the comparison of the spatial distribution of observed λ with the output of numerical ice sheet models, or through creating statistical models to link the mechanics of physical processes to observable characteristics of bedform populations. Indeed, it is consistent with the observed exponentially-tailed distribution that the growth and development of subglacial bedforms may be fundamentally stochastic in nature and involve the convolution of randomness with some, as yet unknown, simple rules about the rate of growth.

Acknowledgements

The authors wish to thank Clas Hättestrand for data. We are grateful to Prof. A. James and an anonymous reviewer for their constructive comments, and Prof. T. Oguchi for his help as editor.

Appendix

i) Parameterisation Method

The method proposed below is not the only possible solution (e.g., Fraile and Garcia-Ortega, 2005), but is objective, statistically valid, and easily implemented. Firstly, determine the range

438 of the measured variable, x , that conforms to the exponential distribution: the linear part of the
439 semi-log plot (Section 3). The mode is a visually reasonable, objective, estimate of the lower end
440 of this range. Despite some previous practice to the contrary (e.g., Abers et al., 1988; Smith and
441 Cann, 1992; Rappaport et al., 1997), all data of larger x are included here. To calculate the mode,
442 based on the underlying data x_i where $i = 1 \dots n$ and n is the number of individual observations, a
443 Gamma distribution is used. A Gamma distribution is a two-parameter distribution (α, λ_g) which
444 tends to an exponential at large x , but which rolls over to zero at small x , with a probability
445 density function (pdf) (Tuckwell, 1995, , p. 62):

$$f(x) = \frac{\lambda_g^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda_g x}, x > 0; \lambda_g, \alpha > 0 \quad (1)$$

446 Fig. 2 shows that the Gamma distribution approximates ϕ well. Maximum likelihood estimators
447 (MLEs) of α and λ_g require numerical techniques, but may also be estimated by the method of
448 moments as $\hat{\alpha} = (\bar{x}/s_x)^2$ and $\hat{\lambda}_g = \bar{x}/(s_x)^2$ where \bar{x} is the sample mean, and s_x is the sample
449 standard deviation (Tuckwell, 1995, , p. 326). The mode of the Gamma distribution, ϕ , is then
450 $(\hat{\alpha} - 1)/\hat{\lambda}_g$. For length, L , of UK drumlins this is shown in red on Fig. 2a, as 424 m (3 s.f.) and
451 is inside the range of the 393–441 modal bin of Clark et al. (2009).

452 Now, determine the gradient and intercept based upon data of size greater than ϕ . Data are
453 fitted as a left-truncated exponential, which is equivalent to an exponential shifted by ϕ . Let
454 $k_i = x_i - \phi$, and then for $k > 0$ the MLE estimator of the gradient of the exponential, $\hat{\lambda}$, is $\hat{\lambda} = 1/\bar{k}$
455 where \bar{k} is the mean of the data (Tuckwell, 1995, , p. 329). This fully describes the pdf of the
456 exponential distribution, which is defined by the following equation and has an area of 1 unit under
457 its curve (Tuckwell, 1995, , p. 86 and 196).

$$f(x) = \lambda e^{-\lambda x} \quad (2)$$

459
460 Histograms and frequency plots are considered inferior to pdfs by many statisticians, but are
461 common in the wider literature. So, how are the results related to the more familiar histogram
462 (Figs. 1 and 2) deliberately used in this paper? The short answer is the line to be plotted on a
463 histogram is given by the following equation

$$y = \hat{\lambda}x + \ln(n_\phi \hat{\lambda} w_b) + \hat{\lambda}\phi \quad (3)$$

464 where $\hat{\lambda}$ and ϕ have been calculated as above, w_b is bin width, n_ϕ is the number of measurements
 465 greater than ϕ , and x and y are the variables relating to the axes of the semi-log plot.

466 To calculate the equation of the best-fit line for a frequency plot, firstly obtain the x -intercept
 467 x_0 through setting the exponential distribution equal to zero. Scaled up to an area of n_ϕ under
 468 its curve, the equation for the linear part of the histogram becomes $f(x) = n_\phi \lambda e^{-\lambda x}$. Taking
 469 logs and setting this to zero gives a horizontally shifted x -intercept of $\ln(n_\phi \hat{\lambda})/\hat{\lambda}$. This becomes
 470 $x_0 = [\ln(n_\phi \hat{\lambda} w_b)/\hat{\lambda}] + \phi$ when bin width w_b is used to multiply up for the conversion from count
 471 density (per unit x) to count within bins and the line is un-shifted and put back to its original
 472 location. With x_0 and λ estimated for the line, y_0 , the y -intercept (i.e. y at $x = 0$) is by simple
 473 geometry $\hat{\lambda}x_0$. The equation of the line is therefore $y = \hat{\lambda}x + \hat{\lambda}x_0$ or $y = \hat{\lambda}x + \ln(n_\phi \hat{\lambda} w_b) + \hat{\lambda}\phi$.

474 By plotting this equation it appears that the data are well-approximated (Fig. 2), and no
 475 arbitrary upper cut-off is required unless data clearly outlying from the distribution are known
 476 e.g. $L > 4$ km; an iterative technique may be used, perhaps excluding data by the probability that
 477 they could exist in the fitted distribution, calculated from the pdf of the exponential. The spatial
 478 density of landforms, β_0 is y_0/A , where A is the area of the study (km^2), although rigorous use of
 479 this will require work to define criteria by which to calculate A .

480 *ii) Using Gamma distribution*

481 Paola and Borgman (1991) estimated λ for fluvial bedforms by fitting a Gamma distribution.
 482 This assumed that fitted Gamma distributions become linear on semi-log plots after the mode;
 483 Fig 2b illustrates that this is not the case. The fitted distribution (dashed line) systematically
 484 increases with x and λ_g and so the gradient at large x is not reached on the plot; λ is systematically
 485 overestimated ($\lambda_g = 0.00589$ whilst $\lambda = 0.00313$ for calculations as in Section (i) of this Appendix).
 486 Furthermore, this approach was not preferred since Fig. 2 a–c show that the gamma distribution
 487 (dashed line) fits more poorly than the exponential ($x > \phi$) and the fit gets poorer as α increases
 488 from 1 (exponential distribution): $\alpha_L = 3.58$, $\alpha_W = 6.68$, $\alpha_H = 2.11$. Namely, the extent of
 489 over-estimation depends upon the overall shape of the distribution, which is not desirable.

490 *iii) Parameterising histogram data*

491 Using, n underlying data points, x_i , the mean, \bar{x} , and standard deviation, s_x , of the sample are
 492 calculated using standard formulae. For n data with counts, c_j , of bins at x_j the related equations
 493 used are:

$$\bar{x} = \frac{1}{n} \sum c_j x_j \quad (4)$$

494

495

$$s_x = \sqrt{\frac{1}{n-1} \sum c_j (x_j - \bar{x})^2} \quad (5)$$

496

497 iv) Comparing populations or sub-populations

498 Confidence intervals can be determined for $\hat{\lambda}$. Strictly, $\hat{\lambda}$ is distributed as $2n\lambda\bar{x} \sim \chi_{2n}^2$, assuming
 499 ϕ correctly delimits the linear portion of the size-frequency distribution. However, with large
 500 n , usually > 30 , (i.e. using the central limit theorem) the sampling distribution of $\hat{\lambda}$ becomes
 501 approximately normal (Tuckwell, 1995, , p. 255–9). So, for large n an asymptotic unbiased
 502 approximation to the variance of a MLE estimate of a parameter may be determined using the
 503 Cramer-Rao lower bound (Tuckwell, 1995, , p. 313–4), giving

504

$$\lambda \sim N\left(\hat{\lambda}, \frac{\hat{\lambda}^2}{n_\phi}\right). \quad (6)$$

505 Stated more fully, the exponent of the observed sample (the gradient of the linear part of the
 506 semi-log plot) is distributed according to the normal distribution with a mean of $\hat{\lambda}$ and variance
 507 of $\hat{\lambda}^2/n_\phi$. The standard error of the sampling distribution of $\hat{\lambda}$ for an exponential distribution is

508

$$s \simeq \sqrt{\frac{\hat{\lambda}^2}{n_\phi}} \quad (7)$$

509

510 which, using standard tabulations for the normal distribution (Tuckwell, 1995, , p. 520), gives
 511 a 95% confidence interval of

512

$$\pm 1.96 \sqrt{\frac{\hat{\lambda}^2}{n_\phi}}. \quad (8)$$

513

514 This then allows the difference between two sub-populations with estimated gradients to be
 515 assessed using a standard t-test. If independent random samples, of sizes $n_{1\phi}$ and $n_{2\phi}$ values above
 516 modes ϕ_1 and ϕ_2 , are drawn from distributions $N(\lambda_1, \sigma_1^2)$ and $N(\lambda_2, \sigma_2^2)$, with standard deviations
 517 unknown *a priori*, $H_0 : \lambda_1 = \lambda_2$ can be tested using the test statistic

$$518 \quad t_{n_{1\phi}+n_{2\phi}-2} = \frac{\hat{\lambda}_1 - \hat{\lambda}_2}{s_p \sqrt{\frac{1}{n_{1\phi}} + \frac{1}{n_{2\phi}}}} \quad (9)$$

519

520 where s_p is the pooled variance

521

$$s_p = \frac{(n_{1\phi} - 1)s_1^2 + (n_{2\phi} - 1)s_2^2}{n_{1\phi} + n_{2\phi} - 2} \quad (10)$$

522

523 within which s_1 and s_2 are estimated by Eq. 7 (Tuckwell, 1995, , p. 348). $t_{n_{1\phi}+n_{2\phi}-2}$ is the t
 524 statistic for $n_{1\phi} + n_{2\phi} - 2$ degrees of freedom, and can be compared to critical values obtained from
 525 standard tables or elsewhere. Note that this is a two-tailed test, so for 95% confidence the 0.025
 526 tabulated value is the critical one. A z statistic may also be useful because samples are relatively
 527 large.

528 References

- 529 Aario, R., 1977. Classification and terminology of moranic landforms in Finland. *Boreas* 6, 77–100.
- 530 Abers, G.A., Parsons, B., Weissel, J. K., 1988. Seamount Abundances and Distributions in the
 531 southeast Pacific. *Earth and Planetary Science Letters* 87, 137–151.
- 532 Bak, P., 1996. *How Nature Works: the Science of Self-Organized Criticality*. Springer, New York.
- 533 Bauke, H., 2007. Parameter estimation for power-law distributions. *Eur. Phys. J. B* 58, 167–173.
- 534 Benn, D.I., Evans, D.J.A., 2010. *Glaciers and Glaciation*, 3rd Edition. Hodder, Oxford, UK.
- 535 Benn, D.I., Evans, D.J.A., Shaw, J., Munro-Stasiuk, M., 2006. Subglacial megafloods: outrageous
 536 hypothesis or just outrageous? / reply. In: Knight, P.G. (Ed.), *Glacier Science and Environ-*
 537 *mental Change*. Wiley-Blackwell, p. 42-60.
- 538 Beven, K., 2006. A manifesto for the equifinality thesis. *J. Hydrology* 320, 18–36.
- 539 Bingham, R.G., King, E.C., Smith, A. M., Pritchard, H.D., 2010. Glacial geomorphology: Towards
 540 a convergence of glaciology and geomorphology. *Progress in Physical Geography* 34, 327–355.

541 Boulton, G., 1976. The origin of glacially fluted surfaces - observations and theory. *J. Glaciology*
542 17, 287–309.

543 Boulton, G., Hindmarsh, R. C.A., 1987. Sediment deformation beneath glaciers - rheology and
544 geological consequences. *Journal of Geophysical Research-Solid Earth and Planets* 92 (B9), 9059–
545 9082.

546 Chapwanya, M., Clark, C.D., Fowler, A.C., 2011. Numerical computations of a theoretical model
547 of ribbed moraine formation. *Earth Surface Processes and Landforms* 36, 1105–1112.

548 Cheng, Q.M., Agterberg, F.P., 1996. Multi-fractal modelling and spatial statistics. *Mathematical*
549 *Geology* 28, 1–16.

550 Chorley, R.J., 1959. The Shape of drumlins. *J. Glaciology* 3, 339–344.

551 Clark, C.D., 1993. Mega-scale glacial lineations and cross-cutting ice-flow landforms. *Earth Surface*
552 *Processes and Landforms* 18, 1–29.

553 Clark, C.D., 2010. Emergent drumlins and their clones: from till dilatancy to flow instabilities. *J.*
554 *Glaciology* 51, 1011–1025.

555 Clark, C.D., Hughes, A.L., Greenwood, S.L., Spagnolo, M., Ng, F.S., Apr. 2009. Size and shape
556 characteristics of drumlins, derived from a large sample, and associated scaling laws. *Quaternary*
557 *Science Reviews* 28, 677–692.

558 Clauset, A., Shalizi, C.R., Newman, M. E.J., 2009. Power-law distributions in empirical data.
559 *SIAM Review* 51, 661–703.

560 Coleman, S.E., Nikora, V.I., 2011. Fluvial dunes: initiation, characterization, flow structure. *Earth*
561 *Surface Processes and Landforms* 36, 39–57.

562 Cornell, R.G., Speckman, J.A., 1967. Estimation for a simple exponential model. *Biometrics* 23,
563 717–737.

564 Dunlop, P., Clark, C.D., 2006. The morphological characteristics of ribbed moraine. *Quaternary*
565 *Science Reviews* 25, 1668–1691.

566 Dunlop, P., Clark, C.D., Hindmarsh, R. C.A., 2008. Bed Ribbing Instability Explanation: Testing
567 a numerical model of ribbed moraine formation arising from coupled flow of ice and subglacial
568 sediment. *Journal of Geophysical Research* 113 (F3).

569 Evans, D. J.A., Livingstone, S., Vieli, A., Ó Cofaigh, C., 2009. The palaeoglaciology of the central
570 sector of the British and Irish Ice Sheet: reconciling glacial geomorphology and preliminary ice

571 sheet modelling. *Quat. Sci. Rev.* 28, 739–757.

572 Evans, I.S., 2012. Geomorphometry and landform mapping: What is a landform? *Geomorphology*
573 137, 94–106.

574 Fowler, A.C., 2000. An instability mechanism for drumlin formation. In: Maltman, A.J., Hub-
575 bard, B., Hambrey, M.J. (Eds.), *Deformation of Glacial Materials*, geological Edition. Geol. Soc.
576 Publishing House, London, pp. 307–319.

577 Fowler, A.C., 2002. Evolution equations for dunes and drumlins. *Revista de la Real Academia de*
578 *Ciencias Exactas, Físicas y Naturales, Serie A. Mat.* 96, 377–387.

579 Fraile, R., Garcia-Ortega, E., 2005. Fitting an exponential distribution. *J. Applied Meteorology*
580 44, 1620–1625.

581 Francek, M., 1991. A spatial perspective on the New York drumlin field. *Physical Geography* 12,
582 1–18.

583 Fredø se, J., 1996. The stability of a sandy river bed. In: Nakato, T., Ettema, R. (Eds.), *Issues*
584 *and Directions in Hydraulics*. Balkema, Rotterdam, pp. 99–114.

585 Freund, F.E., Walople, R.E., 1980. *Mathematical Statistics*, 3rd Edition. Prentice Hall, London.

586 Gardiner, V., 1983. The relevance of geomorphometry to studies of Quaternary morphogenesis. In:
587 Briggs, D.J., Waters, R.S. (Eds.), *Studies in Quaternary Geomorphology*. Geo Books, Norwich,
588 pp. 1–18.

589 Hart, J.K., 1999. Identifying fast ice flow from landform assemblages in the geological record: a
590 discussion. *Annals of Glaciology* 28, 59–67.

591 Haschenberger, J., 1999. A probability model of scour and fill depths in gravel-bed channels. *Water*
592 *Resources Research* 35, 2857–2869.

593 Hättestrand, C., Gotz, S., Naslund, J., Fabel, D., AP, S., 2004. Drumlin formation time: Evidence
594 from northern and central Sweden. *Geografiska Annaler Series A-Physical Geography* 86A, 155–
595 167.

596 Hättestrand, C., Kleman, J., 1999. Ribbed moraine formation. *Quat. Sci. Rev.* 18, 43–61.

597 Hillier, J.K., 2008. Seamount detection and isolation with a modified wavelet transform. *Basin*
598 *Research* 20, 555–573.

599 Hillier, J.K., Smith, M.J., 2012. Testing 3D landform quantification methods with synthetic drum-
600 lins in a real DEM. *Geomorphology* 153, 61–73, doi:10.1016/j.geomorph.2012.02.009.

- 601 Hillier, J.K., Watts, A.B., 2007. Global distribution of seamounts from ship-track bathymetry
602 data. *Geophysical Research Letters* 34, L113304, doi:10.1029/2007GL029874.
- 603 Hindmarsh, R. C.A., 1998. Drumlinization and drumlin-forming instabilities: viscous till mecha-
604 nisms. *J. Glaciology* 44, 293–314.
- 605 Hollingsworth, S.E., 1931. The glaciation of western Edenside and adjoining areas and the drumlins
606 of Edenside and the Solway basin. *Quart. J. Geol. Soc. London* 87, 281–359.
- 607 Hoppe, G., Schytt, V., 1953. Some observations on fluted moraine surfaces. *Geografiska Annaler*
608 2, 105–115.
- 609 Hughes, A., Clark, C.D., Jordan, C., 2010. Subglacial beforms of the last British ice sheet. *Journal*
610 *of Maps*, 543—563.
- 611 Johnson, M.D., Schomacker, A., Benediktsson, I.O., Geiger, A.J., Ferguson, A., Ingolfsson, O.,
612 Sep. 2010. Active drumlin field revealed at the margin of Mulajokull, Iceland: A surge-type
613 glacier. *Geology* 38, 943–946.
- 614 Jordan, T.H., Menard, H.W., Smith, D.K., 1983. Density and size distribution of seamounts in
615 the Eastern Pacific inferred from wide-beam sounding data. *Journal of Geophysical Research*
616 88, 10508–10518.
- 617 Kalbermatten, M., van der Ville, D., Turberg, P., Tuia, D., Joost, S., 2012. Multiscale analysis of
618 geomorphological and geological features in high resolution digital elevation models using the
619 wavelet transform. *Geomorphology* 138, 352–363.
- 620 Kidson, R., Richards, K.S., Carling, P.A., 2006. Fractal analysis for natural hazards. *Geol. Soc.*
621 *Special Publication* 261, 141–153.
- 622 King, E.C., Hindmarsh, R.C.A., Stokes, C.R., 2009. Formation of mega-scale glacial lineations
623 observed beneath a west Antarctic ice stream. *Nature Geoscience* 2, 585–596.
- 624 King, E.C., Woodward, J., Smith, A.M., 2007. Seismic and radar observations of subglacial bed
625 forms beneath the onset zone of Rutford Ice Stream Antarctica. *J. Glaciology* 53, 665–672.
- 626 Leemis, L.M., McQueston, J.T., 2008. Univariate distribution relationships. *The American Statis-*
627 *tician* 62, 45–53.
- 628 Livingstone, S., Cofaigh, C., Evans, D. J. A., 2008. Glacial geomorphology of the central sector of
629 the last British-Irish Ice Sheet. *Journal of Maps*, 358–377.
- 630 Main, I., Irving, D., Musson, R., Reading, A., 1999. Constraints on the frequency-magnitude rela-

631 tion and maximum magnitudes in the UK from observed seismicity and glacio-isostatic recovery
632 rates. *Geophys. J. Int.* 137, 535–550.

633 Malamud, B.D., Millington, J. D.A., Perry, G. L.W., 2005. Characterizing wildfire regimes in
634 the United States. *Proceedings of the National Academy of Sciences of the United States* 102,
635 4694–4699.

636 Malamud, B.D., Turcotte, D.L., 2006. The applicability of power-law frequency statistics to floods.
637 *J. Hydrology* 322, 168–180.

638 Mandelbrot, B.B., 1983. *The Fractal Geometry of Nature*. W. H. Freeman and Company, New
639 York.

640 McElroy, B., Mohrig, D., 2009. Nature of deformation of sandy bed forms. *J. Geophys. Res.* 144,
641 F00A04.

642 Menzies, J., 1979a. The mechanics of drumlin formation with particular reference to the change in
643 pore-water content of the till. *J. Glaciology* 22, 373–383.

644 Menzies, J., 1979b. A review of the literature on the formation and location of drumlins. *Earth*
645 *Science Reviews* 14, 315–59.

646 Miller, J.W., 1972. Variation in New York drumlins. *Annals of the Association of American Geog-*
647 *raphers* 62, 418–23.

648 Morris, E.M., Morland, L.W., 1976. A theoretical analysis of the formation of glacial flutes. *J.*
649 *Glaciology* 17, 311–323.

650 Murray, T., Corr, H., Forieri, A., Smith, A.M., 2008. Contrasts in hydrology between regions of
651 basal deformation and sliding beneath Rutherford ice stream, West Antarctica, mapped using
652 radar and seismic data. *Geophys. Res. Lett.* 35, 10.1029/2006GL028207.

653 Newman, M. E.J., 2005. Power laws, Pareto distributions and Zipf’s law. *Contemporary Physics*
654 46, 323–351.

655 Ó Cofaigh, C., Dowdeswell, J.A., King, E.C., Anderson, J.B., Clark, C.D., Evans, D. J.A., Evans,
656 J., Hindmarsh, R. C.A., Larter, R.D., Stokes, C.R., 2010. Comment on Shaw J., Pugin, A. and
657 Young, R. (2008): ”A meltwater origin for Antarctic shelf bedforms with special attention to
658 megalineations, *Geomorphology* 102, 364-375”. *Geomorphology* 117, 195–198.

659 Paola, C., Borgman, L., 1991. Reconstructing random topography from preserved stratification.
660 *Sedimentology* 38, 553–656.

- 661 Phillips, E., Everest, J.D., Diaz-Doce, D., Dec. 2010. Bedrock controls on subglacial landform
662 distribution and geomorphological processes: Evidence from the Late Devensian Irish Sea Ice
663 Stream. *Sedimentary Geology* 232, 98–118.
- 664 Putkonen, J., Swanson, T., 2003. Accuracy of cosmogenic ages for moraines. *Quaternary Research*
665 59, 255–261.
- 666 Rappaport, Y., Naar, D.F., Barton, C.C., Liu, Z.L., Hey, R.N., 1997. Morphology and Distribution
667 of Seamounts Surrounding Easter Island. *Journal of Geophysical Research* 102, 24713–24728.
- 668 Rose, J., 1987. Drumlins as part of a glacier bedform continuum. In: Menzies, J., Rose, J. (Eds.),
669 Drumlin Symposium. Balkema, Rotterdam, pp. 103–116.
- 670 Rose, J., Letzer, J. M., 1977. Superimposed drumlins. *J. Glaciology* 18, 471–480.
- 671 Rutzinger, M., Hofle, B., Kringer, K., 2012. Accuracy of automatically extracted geomorpholog-
672 ical breaklines from airborne LiDAR curvature images. *Geografiska Annaler Series A-Physical*
673 *Geography* 94A, 33–42.
- 674 Saha, K., Wells, N., Munro-Stasiuk, M., 2011. An object-orientated approach to landform mapping:
675 A case study of drumlin. *Computers and Geosciences* 37, 1324–1336.
- 676 Scheirer, D.S., Macdonald, K. C., 1995. Near-axis seamounts on the flanks of the East Pacific Rise,
677 8N to 17N. *Marine Geophysical Research* 100, 2239–2259.
- 678 Shaw, J., 1983. Drumlin formation related to inverted melt-water erosional marks. *J. Glaciology*
679 29, 461–479.
- 680 Shaw, J., Pugin, A., Young, R.R., 2008. A meltwater origin for Antarctic shelf bedforms with
681 special attention to megalineations. *Geomorphology* 102, 364–375.
- 682 Singh, A., Lanzoni, S., Wilcock, P. R., 2011. Multiscale statistical characterization of migrating
683 bed forms in gravel and sand rivers. *Water Resources Research* 47, W12526.
- 684 Smalley, I., Unwin, D., 1968. The formation and shape of drumlins and their distribution and
685 orientation in drumlin fields. *J. Glaciology* 7 (51), 377–390.
- 686 Smalley, I., Warburton, J., 1994. The shape of drumlins, their distribution in drumlin fields, and
687 the nature of the sub-ice shaping forces. *Sedimentary Geology* 91, 241–252.
- 688 Smalley, I., Piotrowski, J.A., 1987. Critical strength / stress ratios at the ice-bed interface in the
689 drumlin forming process: from "dilatancy" to "cross-over". In: Menzies, J., Rose, J. (Eds.),
690 Drumlin Symposium. Balkema, Rotterdam, pp. 81—86.

691 Smith, A.M., 2006. Microearthquakes and subglacial conditions. *Geophys. Res. Lett.* 33,
692 10.1029/2006GL028207.

693 Smith, A.M., Murray, T., 2009. Bedform topography and basal conditions beneath a fast-flowing
694 West Antarctic ice stream. *Quat. Sci. Rev.* 28, 584–596.

695 Smith, A.M., Murray, T., Nicholls, K.W., Makinson, K., Athalgerirdottir, G., Behar, A., Vaughan,
696 D. G., 2007. Rapid erosion and drumlin formation observed beneath a fast-flowing Antarctic ice
697 stream. *Geology* 35, 127–130.

698 Smith, D.K., Cann, J.R., 1992. The role of seamount volcanism in crustal construction at the
699 Mid-Atlantic Ridge. *Journal of Geophysical Research* 97, 1645–1658.

700 Smith, D.K., Jordan, T.H., 1988. Seamount Statistics in the Pacific Ocean. *Journal of Geophysical*
701 *Research* 93, 2899–2918.

702 Smith, M.J., Clark, C.D., 2005. Methods for the visualization of digital elevation models for
703 landform mapping. *Earth Surface Processes and Landforms* 30, 885–900.

704 Smith, M.J., Rose, J., Booth, S., 2006. Geomorphological mapping of glacial landforms from
705 remotely sensed data: an evaluation of the principal data sources and an assessment of their
706 quality. *Geomorphology* 76, 148–165.

707 Smith, M.J., Rose, J., Gousie, M.B., 2009. The Cookie Cutter: A method for obtaining a quanti-
708 tative 3D description of glacial bedforms. *Geomorphology* 108, 209–218.

709 Smith, M.J., Wise, S.M., 2007. Mapping glacial lineaments from satellite imagery: an assessment
710 of the problems and development of best procedure. *Int. J. Applied Earth Observation and*
711 *Geoinformation* 9, 65–78.

712 Solow, A.R., Costello, C.J., Ward, M., 2003. Testing the power law model for discrete size data.
713 *The American Naturalist* 162, 685–689.

714 Spagnolo, M., Clark, C.D., Hughes, A.L., 2012. Drumlin relief. *Geomorphology* 153-154, 179–191.

715 Spagnolo, M., Clark, C.D., Hughes, A.L., Dunlop, P., Stokes, C.R., 2010. The planar shape of
716 drumlins. *Sedimentary Geology* 232, 119–129.

717 Stokes, C.R., Clark, C.D., 2002. Are long subglacial bedforms indicative of fast ice flow? *Boreas*
718 31, 239–249.

719 Stokes, C.R., Clark, C.D., 2003. The Dubawnt palaeo-ice stream: evidence for dynamic ice sheet
720 behaviour on the Canadian Shield and insights regarding the controls on ice stream location and

721 vigour. *Boreas* 32, 263–279.

722 Stokes, C.R., Fowler, A.C., Clark, C.D., Hindmarsh, R. C.A., Spagnolo, M., 2013. The instabil-
723 ity theory of drumlin formation and its explanation of their varied composition and internal
724 structure. *Quat. Sci. Rev.* 62, 77–96.

725 Stokes, C.R., Spagnolo, M., Clark, C.D., 2011. The composition and internal structure of drumlins:
726 complexity, commonality, and implications of a unifying theory of their formation. *Earth Science*
727 *Reviews*, 10.1016/j.earscirev.2011.05.001.

728 Tebbens, S.F., Burroughs, S.M., Barton, C.C., Naar, D. F., 2001. Statistical self-similarity of
729 hotspot seamount volumes modeled as self-similar criticality. *Geophys. Res. Lett.* 28, 2711–2714.

730 Tuckwell, H.C., 1995. *Elementary Applications of Probability Theory*, 2nd Edition. Chapman and
731 Hall, London.

732 van der Mark, C.F., Blom, A., Hulscher, S. J. M.H., 2008. Quantification of variability in bedform
733 geometry. *J. Geophys. Res.* 113, F03020, 10.1016/j.earscirev.2011.05.001.

734 Vaughan, D.G., Smith, A.M., Nath, P.C., Le Meur, E., 2003. Acoustic impedance and basal shear
735 stress beneath four Antarctic ice streams. *Annals of Glaciology* 36, 225–232.

736 Weissel, J.K., Pratson, L.F., Malinverno, A., 1994. The length-scaling properties of topography.
737 *J. Geophys. Res.* 99, 13997–14012.

738 Wessel, P., 1997. Sizes and ages of seamounts using remote sensing: implications for intraplate
739 volcanism. *Science* 277, 802–805.

740 Wysota, W., 1994. Morphology, internal composition and origin of drumlins in the southeastern
741 part of Chelmno-Dobrzy Lakeland, north Poland. *Sedimentary Geology* 91, 345–364.

742

743 **Figure captions**

744 **Fig. 1:.** Frequency plots of the lengths, L , of UK drumlins. Black dots are data digitised
745 from Fig. 8 of Clark et al. (2009); bin width ~ 50 m. Larger drumlins ($L > \phi$) are, to a good
746 first approximation, fit (see text) by a straight line in b), an exponential distribution. They are
747 not power law, i.e. linear in c). Mode, ϕ , in b) estimated by fitting gamma distribution. Crosses
748 indicate zero counts, placed at a nominal value of 1 in b) and c).

749 **Fig. 2:.** Semi-log plots for subglacial bedform properties (H , W , L and L/W) and types
750 (drumlin, ribbed moraine, MSGL). Data (black dots) are exact (e, h) and digitised (a, b, c, d, f and
751 g). Bin widths vary, and crosses indicate zero counts, placed at a nominal value of 1 if $n > 10,000$.
752 Solid lines are the exponential distributions fitted to data above the mode ϕ . The exponent is the
753 plotted gradient, λ . The red bars indicate ϕ , estimated from fitted gamma distributions, shown
754 as dashed lines in a) to c) only. Hiller and Smith data are for ‘best’ isolation technique. Spagnolo
755 et al. (2012) discard superimposed (i.e. cross-cutting) (e.g., Rose and Letzer, 1977) or slightly
756 overlapping drumlins of Clark et al. (2009). MSGL are from Dubawant Lake ice stream flow-set
757 (Stokes and Clark, 2003).

758 **Fig. 3:.** Size-frequency data possibly exhibiting the influence of bedrock. a) Swedish drumlin
759 observations. H categorised discretely as 2, 5, 7, 10, 20, 30 ... 80 m, so the number of drumlins
760 per unit bin width (count density) is plotted. Lines fitted manually. b) Frequencies of drumlins
761 (black dots) and streamlined bedrock forms (open circles) for L/W from neighbouring regions in
762 Anglesey, UK. Lines fitted as in Section 4.

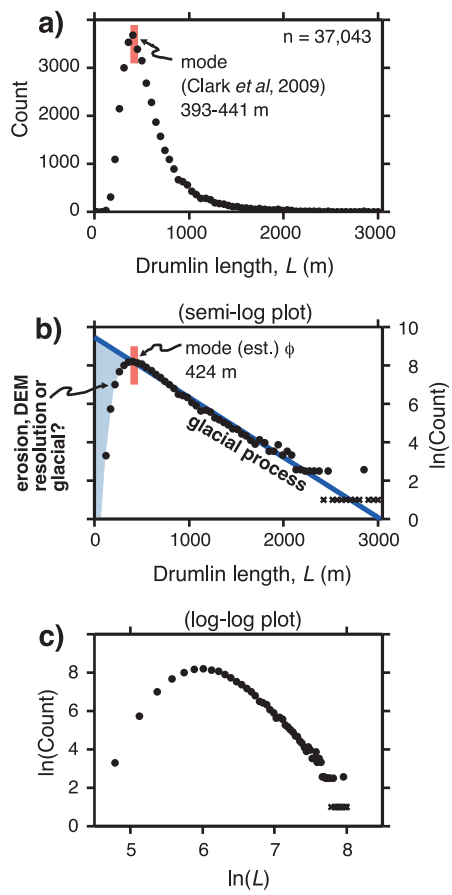


Figure 1:

763

764

765

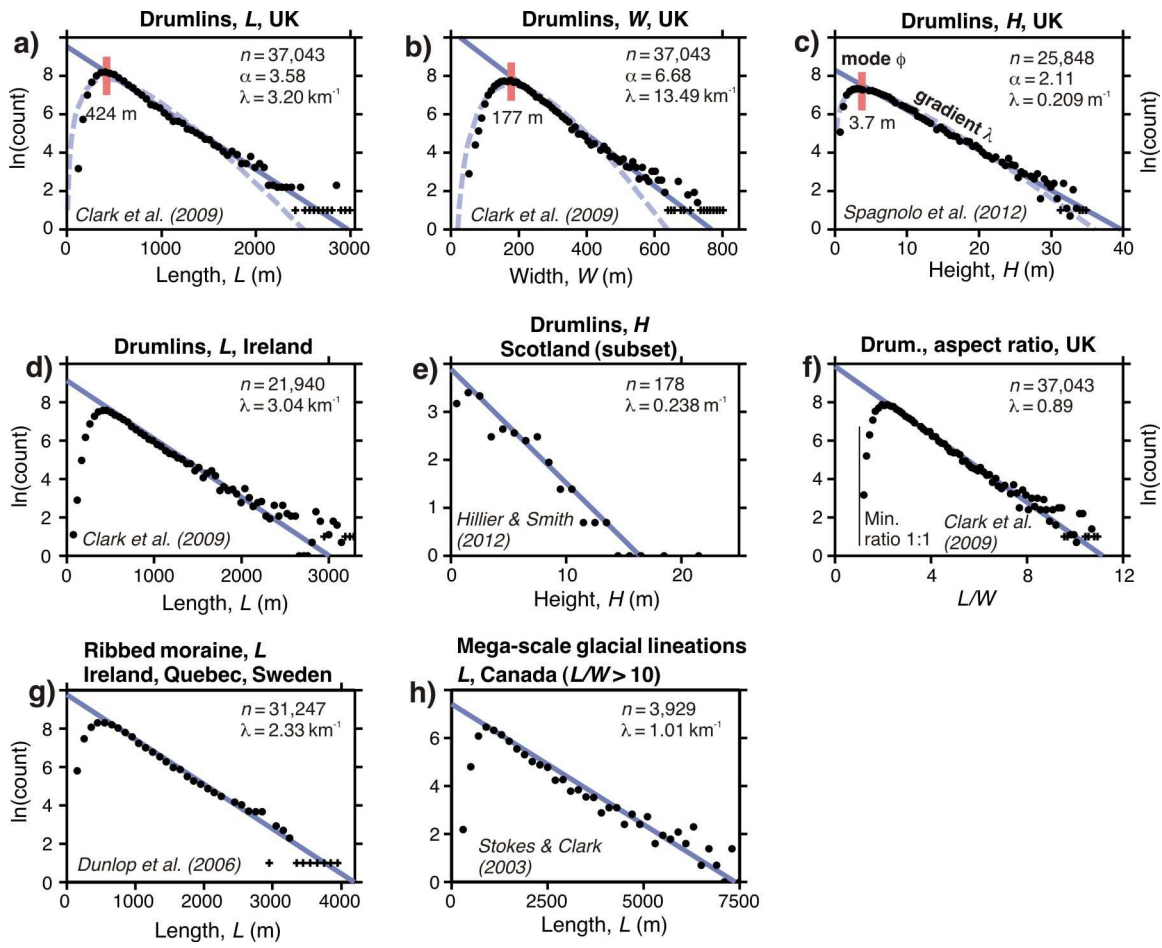


Figure 2:

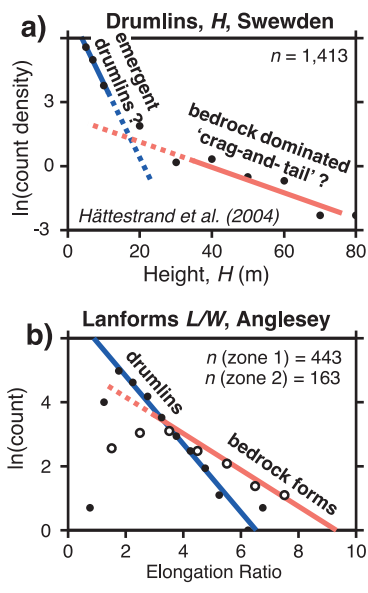


Figure 3: