The role of ‘S’ level mathematics

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<tr>
<th>VOL NO.</th>
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THE ROLE OF "S" LEVEL MATHEMATICS

by

PHILIP JOHN STEPHENS, B.Sc., Ph.D., A.F.I.M.A.

A Master's Dissertation submitted in partial fulfilment of the requirements for the award of the degree of M.Sc. in Mathematical Education of the Loughborough University of Technology, December 1975.

Supervisor: R. P. KNOTT, M.Sc., Ph.D.

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ABSTRACT

The dissertation sets out to examine the role of the General Certificate of Education Special papers in mathematics. An account is given of the development of these so-called "S" level papers from the time of the first external school examinations to the present day. A review of the mathematical philosophy and the educational theory underlying "S" level mathematics is engaged upon; in particular, there is discussion of mathematical giftedness, the aims of educating the mathematically gifted, Bloom's taxonomy of educational objectives, and the role of examinations.

A survey of current practices was made with particular reference to the reasons why pupils were entered for the examinations. It was found that a vast majority of teachers entered candidates for "S" level because they considered it beneficial to set the more able "A" level students harder questions. A big majority also entered candidates because "S" level was thought to provide suitable practice for the entrance examinations of Oxford and Cambridge. Correspondence with university Tutors for Admissions concerning this last point is chronicled.

Variations in "S" level entries between the different Boards are examined - some disturbing evidence is discovered. The mathematical content of the syllabuses and the difficulty of the examination questions are explored. A comparison is made between "S" level mathematics and the entrance examinations of the ancient universities.

The teaching of a particular "S" level syllabus in one school is reported, and a method of broadening the education of the mathematically gifted is studied.

The future of "S" level is discussed on two planes - what is likely to happen and what should happen. It is argued that "S" level is
obsolescent, that the needs of the gifted would not be served by the introduction of N and F levels and that the gifted should receive separate education after the age of fifteen.
ACKNOWLEDGEMENTS

I wish to thank Dr. Knott for his helpful suggestions during the preparation of this dissertation. I am grateful to the officials of the G.C.E. Examinations Boards, to the officers of the Mathematical Association, the Association of Teachers of Mathematics, the Schools Council and to the Tutors for Admissions of six Oxbridge colleges; all have been generous and most helpful in their responses to my correspondence. I must also thank 67 Heads of Mathematics in schools throughout the country who completed the questionnaire.

Words of thanks are also due to James Woodhouse, Head Master of Rugby School, and to Fred Norton, Senior Mathematics Master, both of whom have given me much encouragement and have made my attendance at Loughborough possible.

Finally, I would like to pay a special tribute to my wife, Jennet, without whose patience and support this dissertation would not have been completed.
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>2</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>4</td>
</tr>
<tr>
<td>Introduction</td>
<td>6</td>
</tr>
<tr>
<td>Chapter 1 The Development of &quot;S&quot; Level</td>
<td>10</td>
</tr>
<tr>
<td>Chapter 2 Mathematical Philosophy and Educational Theory</td>
<td>20</td>
</tr>
<tr>
<td>Chapter 3 Educational Practice</td>
<td>44</td>
</tr>
<tr>
<td>Chapter 4 The Mathematical Content of &quot;S&quot; Level Syllabuses</td>
<td>86</td>
</tr>
<tr>
<td>Chapter 5 Preparation of Candidates for &quot;S&quot; Level Mathematics</td>
<td>141</td>
</tr>
<tr>
<td>Chapter 6 The Future of &quot;S&quot; Level</td>
<td>155</td>
</tr>
<tr>
<td>Bibliography</td>
<td>168</td>
</tr>
<tr>
<td>Appendix 1 The Schools Questionnaire on the Role of &quot;S&quot; Level Mathematics</td>
<td>173</td>
</tr>
<tr>
<td>Appendix 2 Summary of the Schools Council's N and F Proposals</td>
<td>181</td>
</tr>
</tbody>
</table>
INTRODUCTION

In the academic year 1971-72 some 645 thousand boys and girls left school in England and Wales (Department of Education and Science, 1972). Of these, 275 thousand had attempted neither the Certificate of Secondary Education (C.S.E.) nor the General Certificate of Education (G.C.E.) examinations. 119 thousand school leavers entered for the G.C.E. "A" level examinations in 1972; almost an equal number of candidates — some of whom were to remain at school and others who had already left — also attempted the examination.

In the Summer 1972 "A" level examinations there were 64,395 entries in mathematical subjects. Generally an "A" level examination in a mathematical subject comprises two papers. There is, in addition, an optional Special paper which is taken by some of the more able candidates. In 1972 there were 3,632 entries for the Special papers in mathematical subjects. (The number of candidates involved is somewhat smaller since some candidates attempt the Special papers in two mathematical subjects; a reasonable estimate for 1972 would be about three thousand.) Expressed as a percentage, the number of entries in the Special papers in mathematical subjects was 5.6% of the total number of "A" level entries in those subjects. Over all subjects, the number of entries in the Special papers was 6.2% of the total number of "A" level entries.

The purpose of this dissertation is to examine the role of the Special ("S" level) papers in Mathematics. The figures quoted in the previous paragraphs perhaps explain why so little research has been attempted into "S" level generally and this subject in particular. The candidates at "S" level form 6% of the "A" level population who, in turn, form 18% of school leavers. The percentage of candidates offering "S" level is approximately 1% of the school population. The discussion will
thus revolve about a subset of 1% of the school population. While this may be a small proportion it is, nonetheless, an important proportion. For almost all will go on to a university education, many to Oxford or Cambridge, and a number will become problem solvers and innovators in society.

It is a necessary task in describing the role of "S" level to set the examination in its historical context. Thus the first chapter is used to describe the development of "S" level from the time of the first secondary school external examinations in 1858. It also contains a discussion of the current aims and objectives of the "S" level examinations.

This is followed by a consideration of the mathematical philosophy and educational theory involved in "S" level mathematics examinations. In particular, it seems appropriate to discuss the nature of mathematics, the aims of education, Bloom's taxonomy of educational objectives, the problem of giftedness, and the role of examinations.

Chapter 3 contains an account of current educational practice. The reasons why schools enter candidates for "S" level mathematics examinations are considered. Since there are no published results on this theme, a questionnaire was circulated to one hundred schools in England and Wales in an attempt to obtain first hand information; an analysis of the results is given. Later parts of the chapter are taken up with a discussion of correspondence with various professional bodies and universities. Statistics of the "S" level examinations are also considered; special attention is given to the difference in the practices of the various Boards and Projects.

In the sixth form, students can spend the same amount of time as for any other "A" level subject; in this case they are 'single-subject' students. 'Double-subject' students spend twice the time in their
studies and sit two examinations as a result. The 'double-subject' syllabuses are organised into two subjects either as Pure Mathematics and Applied Mathematics or as Mathematics and Further Mathematics. The Mathematics and Further Mathematics syllabuses both contain some pure and some applied mathematics. The advantage of the latter arrangement is that 'single-subject' students sit Mathematics as their examination.

The mathematical content of "S" level syllabuses is examined in Chapter 4. In particular, attention will be paid to the difficulty of the examinations both in the 'single-subject' and in the 'double-subject'.

This leads quite naturally to the question of how best to prepare candidates for "S" level examinations in mathematics. Can such syllabuses be 'taught' in the traditional sense? If so, which methods (if any) ought the teacher to use? If not, are there other approaches?

The final chapter contains a summary of the present position and a discussion of the future of "S" level.

The reader will find that from time to time the Universities of Oxford and Cambridge enter the discussion. These universities hold their own entrance examinations each year. The vast majority of their successful applicants sit these examinations, although a few are awarded places on the basis of their "A" and "S" level results alone. The universities also award Scholarships and Exhibitions for performances in their entrance examinations. These examinations are known collectively as Oxbridge Entrance Examinations. Occasionally the term Oxbridge Scholarship is used - there is no distinction.

This exordium is concluded by listing the abbreviations used in the dissertation together with their meanings.
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Full Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.E.B.</td>
<td>Associated Examining Board</td>
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<tr>
<td>J.M.B.</td>
<td>Joint Matriculation Board</td>
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<td>O</td>
<td>Oxford Delegacy of Local Examinations</td>
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<td>O &amp; C</td>
<td>Oxford and Cambridge Schools Examination Board</td>
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<td>S</td>
<td>Southern Universities' Joint Board for School Examinations</td>
</tr>
<tr>
<td>C</td>
<td>University of Cambridge Local Examinations Syndicate</td>
</tr>
<tr>
<td>L</td>
<td>University of London School Examinations Council</td>
</tr>
<tr>
<td>W</td>
<td>Welsh Joint Education Committee</td>
</tr>
<tr>
<td>S.M.P.</td>
<td>School Mathematics Project</td>
</tr>
<tr>
<td>M.E.I.</td>
<td>Mathematics in Education and Industry Project</td>
</tr>
<tr>
<td>C.S.</td>
<td>University of Cambridge Entrance Scholarships and Exhibitions Examinations</td>
</tr>
<tr>
<td>O.S.</td>
<td>University of Oxford Examination for Admission and Entrance Awards</td>
</tr>
<tr>
<td>G.C.E.</td>
<td>General Certificate of Education</td>
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<td>C.S.E.</td>
<td>Certificate of Secondary Education</td>
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</table>
CHAPTER 1
THE DEVELOPMENT OF "S" LEVEL

The first external examinations conducted in British secondary schools took place in 1858 with the introduction of the Senior and Junior Local Examinations of the Universities of Oxford and Cambridge. These emanated from an experiment undertaken at Exeter the previous year when T. D. Acland organised an examination for the Bath and West of England Society for the Encouragement of Agriculture, Arts, Manufactures and Commerce (Montgomery, 1965, p. 45). From the beginning the local examinations were intended to provide schools with guidance and standards neither of which had previously been available. As Acland himself put it (Acland, 1858, p. vii), the Oxford Local Examination was proposed "to bring the honours of the universities to bear upon what is called a middle-class education".

The local examinations were not used for matriculation purposes. Separate examinations existed. At Oxford these examinations were called 'Responsions' while at Cambridge they were called 'Previous' - although not all students were required to pass in order to qualify for entry. London University, too, made its own arrangements by setting entrance examinations.

The number of school examinations proliferated during the last decades of the 19th Century. New examining bodies such as the Oxford and Cambridge Schools Examination Board and the Northern Universities Joint Matriculation Board were founded. But the universities were not the only bodies conducting their own examinations. The Army, the Civil Service, and professional bodies such as the College of Preceptors (a teachers' organisation) also held examinations.

It can be seen that, even at an early stage of their history, a variety of examinations existed for a variety of reasons. First,
examinations were held to test the curriculum of a school and the general education of its pupils. Second, they were conducted as a hurdle to university entrance. Third, they were used by the professions to test suitability for entry into those professions.

As early as 1868, the Taunton Commission saw the need to introduce a central examination organisation. Unfortunately their recommendation was ignored by the government of the day. As a result it was virtually impossible for a school to satisfy the curriculum requirements of all of its pupils—simply because of the number and differing natures of the syllabuses.

In 1899 the Board of Education was founded and the local education authorities were formed three years later. (The Board was later to become the Ministry of Education, now the Department of Education and Science.) By 1909 the Board had set up a Consultative Committee to consider the whole subject of examinations in secondary schools. But it was not until 1917 that the Secondary Schools Examination Council came into existence. With it came the two new School Certificate examinations first held in 1918. The function of these examinations is described in Statistics of Education (Department of Education and Science, 1972, Vol. 2, p. xii):

"(i) The First School Certificate Examination was for pupils of about 16 years, and was meant to foster in the schools the provision of a good general education and to test the results of courses followed during the first 5 grammar school years. The examination, which was designed on the basis of the existing curriculum, was to assess the average competence of the pupils in an efficient secondary
school. The form, rather than the individual pupil, was the unit to be tested.

(ii) The Second (Higher) School Certificate Examination was designed for pupils at about 18 years, being considered a test of pupils' attainments after a further two years' study."

The First Examination was a test of general education. There were two grades of pass in each subject - Pass and Credit. To qualify for a Certificate a candidate had to pass in five subjects including at least one from each of three groups (i) English subjects, (ii) Foreign languages, (iii) Mathematics and Science). In addition, at least one of the passes had to be a credit.

The Second Examination, on the other hand, demanded a high degree of specialisation. Candidates offered subjects from exactly one of three groups (i) Classics and History, (ii) Modern Humanistic Studies, (iii) Science and Mathematics), together with subsidiary work in topics outside the group.

The First Examination was a type of school assessment. Each state-aided school had to enter a whole form, rather than its most able pupils, in order to obtain a grant from the Board of Education. This aspect of the First Examination was supported by the system of school inspection. On the other hand, employers and universities were paying more attention to individual performances in the examinations. Indeed, in some circumstances, it was possible to gain exemption from matriculation or from the preliminary examinations of professional bodies. The Higher School Certificate became increasingly more important to applicants for university places.
These factors contributed to the Norwood Committee's reasons for recommending in 1943 the introduction of an examination without subject grouping and with a certificate recording the candidate's performance in each subject. The recommendation, accepted in principle in 1947, was implemented in 1951. The General Certificate of Education examinations were first conducted that year at two levels - Ordinary and Advanced - and are still run by the University Examining Boards. The Ordinary level examination, which replaced the First Examination, was conducted on a pass/fail basis. The Advanced level examination took the place of Higher School Certificate and had a Distinction grading as well as a pass. The Ordinary and Advanced levels were not intended to be competitive. They were designed to be qualifying examinations and were accepted as such by the professions.

To describe the next phase of the story it is necessary to discuss the financial assistance given to university students. At the end of the 19th century a number of open and closed university scholarships existed, but there was little in the way of state aid to students. The situation improved gradually from the beginning of the present century. In 1911, there were 1,400 state-maintained awards (at an average value of £43) to universities or colleges (Ministry of Education, 1948). In 1920, State Scholarships were initiated and were awarded on the results of the Higher School Certificate examinations. At first there were only 200 such scholarships but this number increased to 360 by 1936 and to 750 in 1947 (Ministry of Education, 1948, p. 4). Other financial aid was through local authority grants or loans.

The Norwood Report contained proposals which aimed at unifying the system of providing financial assistance. Montgomery (op. cit., pp. 152-153) described the proposals thus:
"Two sorts of papers were therefore envisaged by the Norwood Committee for pupils at the age of 18+. The qualifying examination has already been described as giving evidence of ability to carry on with further education at university or professional level. A competitive examination, with more difficult questions and on no detailed syllabus, was to cater for the abler candidates, and to be of use in selecting them for university grants. The university examining boards were to set the papers as they did already, but boards composed of teachers and university and local authority representatives were to take school records and other relevant information into account when classifying successful candidates. The most promising of these were to receive awards directly from the State, while others might receive local grants, half the cost of which was to be defrayed by the State."

The competitive Scholarship papers thus served the purpose of helping the fair distribution of State and other scholarships. Whatever the intentions of the Ministry of Education and the Secondary Schools Examination Council in suggesting that Advanced level should be only a qualifying examination, events turned out differently. Advanced level marks were sent by some examining bodies to university selectors. By 1960 an unofficial grading system sprang up. Performance in Advanced level mattered a great deal to those students who hoped to continue their education at universities. The universities in turn put pressure on the Examining Boards to introduce an official grading system at Advanced level. Such a system (still in operation) was introduced in 1963. There were to be five grades of pass - A, B, C, D, E.
The reason why the qualifying examination became competitive was simply due to the surplus of students over university places. The situation was aggravated by the decision in 1962 to make grants available to all students entering universities. At the same time State Scholarships were discontinued.

With the abandonment of State Scholarships the main reason for the existence of Scholarship papers disappeared. Indeed the Scholarship papers were discontinued in 1963 when the new grading system at Advanced level was introduced. They were, however, replaced by Special papers. Distinctions or Merits were to be awarded to those candidates with an A, B, or C in their Advanced level papers who performed sufficiently well in their Special papers. (These supplementary grades are now called 1 and 2, respectively; a candidate who fails to be awarded a supplementary grade is unclassified with a U!) These papers were intended to be set on the Advanced level syllabuses but would contain more searching questions. It was expected that candidates would normally offer the Special paper in one subject. They were not allowed to enter the Special papers in more than two subjects; indeed some Boards restricted candidates to just one Special paper. About 15% of the Advanced level candidates were expected to enter the Special papers.

The reader must be wondering just why Special papers were provided. Two reasons may be found in the Third Report of the Secondary Schools Examination Council (Ministry of Education, 1960). First, it was felt desirable that the more able candidates should be allowed to show their intellectual grasp by answering more searching questions. Second, the Special paper would provide universities with more information about a candidate. These reasons have since been repeated in the Regulations and Syllabuses of various Examination Boards.
A third reason has not been so well publicised. It is that the Boards were under pressure from schools to maintain an examination for the more able student. For instance, the Oxford and Cambridge Schools Examination Board circulated a questionnaire to schools when the draft of the Third Report of the Secondary Schools Examination Council was being considered. H. F. King, Secretary to the Board, in a letter (26 June, 1975) wrote:

"One question was on the Report's proposal that there should be no extension of the A syllabus for S papers. Many replies were lengthy and analysis was difficult, but the majority of schools wanted the syllabuses to remain as before (i.e. with extensions), although a few wanted no extensions. Nearly all felt that extensions were essential in Maths, Physics and Chemistry - this corresponds to the present position with our syllabuses."

These remarks are supported by J. M. Sharp, Assistant to the Secretaries, University of Cambridge Local Examinations Syndicate who wrote (13 June, 1975):

"... the Syndicate, in common with other examining bodies, received representations from schools that papers were still needed to 'stretch the minds' of pupils of more than average merit. Special papers were, therefore, introduced although in Mathematics and Science these were less 'difficult' than the former scholarship papers."
The relationship between "S" papers and "A" papers in a subject was thus seen as not differing greatly from that between Scholarship and Advanced papers, but the purpose would be different. The "S" paper would provide a qualitative, rather than a competitive, test. The need for it - especially in Mathematics and Science - was emphasised by the schools.

In the first year (1963) of the new Special papers, the forecast that about 15% of the "A" level candidates would sit the Special papers was not too inaccurate. In mathematical subjects that year there were 6,250 entries for the "S" papers out of an "A" level entry of 48,392; this was 12.9% of the Advanced entry. During the first decade of its existence, the number of entries for the Special paper in mathematical subjects, in common with most other subjects, steadily decreased. The number of "S" entries expressed as a percentage of the number of "A" entries in mathematical subjects dropped even more quickly. For example, in 1967, the 5,207 "S" entries formed 8.8% of the "A" entry. By 1970 only 6.9% of the 64,381 "A" entry sat the Special paper in mathematics; in 1972 the proportion had dropped to 5.6%.

The publication of the Robbins Report in 1963 and subsequent government action may partially explain the decline in popularity. The Report drew attention to the lack of university places for suitably qualified applicants. (In 1963, the Association of University Teachers claimed that 5,000 young people who possessed university entrance qualifications would be turned away that year because of the lack of places (see K. Hartley (1963, p. 84)). The government took action. New universities were completed, nine Colleges of Advanced Technology were given university status, and thirty Polytechnics received their Charters. Each of these institutions could offer degree courses, and places in higher education establishments became more plentiful.
The situation prior to 1963 was so serious that "even one of the lesser-known London colleges could demand a State Scholarship as a condition of entry in mathematics" (Montgomery, op. cit., p. 146). In 1963 the author was accepted by University College, London, on the result of an interview in September having obtained two grade As and a grade C at "A" level, together with a Merit in the Special paper and passes in the General Paper and Use of English. These results were unacceptable to Southampton. While such rigorous demands were not made by all university mathematics departments, it was common practice at that time to ask for high grades at "A" level. Students are often now accepted with grade C and grade D passes in two "A" level subjects. Polytechnics will gladly accept students on to degree courses with only the minimum qualifications of two grade E passes at "A" level.

There are two other factors which may have contributed to the decline in popularity of "S" level. One is the 'swing away from Science', resulting in fewer applicants to science and mathematics courses than had been anticipated in the Robbins Report. The other is the tendency of less able children to stay longer at school and to attempt Advanced level courses. (This may have been because certain employers raised their standards of entry; for instance, banking demanded "A" level passes where previously Ordinary level sufficed.) While this tendency may not account for the decrease in the actual numbers trying "S" level, it may explain some of the increase in numbers at "A" level, and thus some of the decrease in the proportion of "S" entries in mathematical subjects.

In 1963, and for one or two years after, it is understandable that schools should have entered candidates for the Special papers. There was such a shortage of university places that a good performance in the Special paper could improve the chances of a candidate being accepted by
the university of his choice. But, in recent years, the supply of places — in mathematics at least — has exceeded the demand by suitably qualified applicants.

The question one perhaps should pose about "S" level mathematics is not 'why has the proportion of "S" entries decreased so rapidly?' but rather 'why is there still a substantial "S" entry each year?'. An attempt to answer this last question will be made in Chapter 3. Before doing so, however, it seems right to discuss some of the underlying principles and assumptions of both educational theory and mathematical philosophy.

It is clear that giftedness must be discussed as must the nature of mathematics and the role of examinations — "S" level Mathematics is an examination in mathematics for the most able "A" level candidates. Consideration must also be given to the aims of education and, in particular, to the aims of sixth form education. For these form the nucleus around which the sixth form curriculum is built. The next chapter is used to these ends; it is intended to give an account of the theory underlying "S" level mathematics and not of its practical relevance to the subject — the reader is asked to be patient!
2.1 The Need for Definition

If there is one idea that mathematical education inculcates in students then it is the need for agreed definitions. Nor is this concept the sole property of mathematics students. In every academic subject a need has been felt to define terms; it is simply the desire for effective communication. How many hours have been wasted discussing some advanced theological concept when the two participants in the debate have failed to agree on a basic premise such as the existence of God? There must be some truth in the maxim 'one cannot measure what one cannot define' (although I would not give it whole-hearted support).

In the field of education it is no simple matter to present a list of accepted definitions. For there may be dozens of different definitions of the same term. As an example, the term 'gifted' was reported by Abraham (1958) to have been defined in 113 ways. It is thus with extreme caution that the reader should accept the definitions presented later in the chapter. The reader may indeed disagree with my definitions but at least he will be able to judge my arguments based on these definitions.

Although it is my intention to provide an objective summary of the educational theory, it will be impossible to present a complete picture. The reader who believes that mathematical philosophy is much easier to define is, in my view, mistaken and deserves to be quickly disillusioned.

2.2 Mathematical Philosophy

There are three main schools of thought: the logistic school, the intuitionist school and the formalist school.
The logistic thesis is that mathematics is a branch of logic. The school supports the view that mathematical concepts can be expressed as logical concepts. All theorems of mathematics can be expressed and developed as theorems of logic. The distinction between mathematics and logic is merely one of practical convenience.

Whitehead and Russell in *Principia Mathematica* (1910, 1927) attempted to reduce the whole of mathematics to logic, but their attempt was not totally satisfactory. First they introduced an artificial 'axiom of reducibility' which avoided the contradictions of set theory (Russell's paradox for instance); but this appeared to be the sole reason for the presence of the axiom. A second more weighty criticism is that the systematic development of logic itself assumes mathematical concepts such as iteration.

The intuitionist school believed that mathematics should be built solely by finite constructive methods from the intuitively given sequence of natural numbers. The intuitionists, led by Brouwer in 1908, did not accept that a set can be thought of as a ready-made collection. Rather they held that a set must be considered as a law by means of which its elements can be constructed one step at a time. In this way the intuitionists also avoided the paradoxes of set theory.

Another consequence of intuitionist mathematics is that the law of the excluded middle can no longer be held to be universally true. While the intuitionists could accept the law for finite sets, they had to reject it for infinite sets. This rejection leads to what many mathematicians regard as a major drawback in intuitionist mathematics; for there is much classical mathematics which, for the present at least, has to be abandoned. However, this may not always be the case, since intuitionist mathematics could conceivably be used to reconstruct the whole of classical mathematics. What can be said in its favour is that its methods do not lead to contradictions.
The formalist thesis is that mathematics can be regarded as a system in which the terms are symbols and the statements are formulae involving these symbols. In order to justify this point of view it was necessary for the formalists to establish the consistency of the various branches of mathematics.

Hilbert and Bernays attempted to prove the consistency of mathematics in *Grundlagen der Mathematik* (1934-39). The attempt was doomed to failure from the beginning. For, in 1931, Gödel proved by a method acceptable to all three schools that it is impossible for a sufficiently rich formalised deductive system (such as Hilbert's for all classical mathematics) to prove the consistency of the system by the methods of that system.

It is clear that all three schools of thought have their shortcomings. But mathematicians like Russell, Brouwer and Hilbert were interested in placing mathematics on a sound footing; in particular, they wished to avoid the much publicised paradoxes of set theory. My view of mathematics cuts across the three schools of thought. My own concern is to explain what mathematics is and how new results are discovered.

I believe that the study of mathematics has a dual nature - intuition (in the mathematician) and abstraction (of the subject itself). Let me illustrate with an example which must be all too familiar to a thirteen year-old school child - the problem of solving a pair of simultaneous linear equations. Some modern syllabuses insist upon the use of matrix methods to solve the equations. I am interested in the old-fashioned method of eliminating the variables. Consider two equations of the form

\begin{align*}
ax + by &= c \\
Ax + By &= C
\end{align*}

(where \(a, b, c, A, B, C\) are known constants) which must be simultaneously
satisfied. Then, by the familiar rules of algebra, one is allowed to add equations or to multiply through by non-zero constants without altering the basic relations between the variables. For instance consider
\[
\begin{align*}
x + y &= 2 \\
x + 2y &= 1
\end{align*}
\]
We could add the two equations to obtain \(2x + 3y = 3\). But this relationship, although true, is not very helpful. Suppose instead we multiply the second equation by \(-1\) to obtain \(-x - 2y = -1\). If we now add this equation to our first equation, then we find that \(y - 2y = 2 - 1\), or \(y = -1\). We have found the value of one of the variables. It is a simple matter now to find the value of the other variable. The person who first solved this type of problem may have arrived at his solution to a particular problem by accident. But it was a flash of inspiration which led to the realisation that there is a general principle to be recovered from the particular example, that there exists a general method for solving any pair of simultaneous linear equations. All we have to do is to make the coefficients of \(x\) (or \(y\)) equal in magnitude and opposite in sign by multiplying through one equation by a suitable constant. This realisation is intuition; it must be verified.

The verification is a process of abstraction, a manipulation of equations according to the fundamental laws of algebra. I shall not bother to go into the details here, although there is one point which the abstraction highlights. It produces an exceptional case. Suppose that the variable \(x\) has been eliminated; then an equation of the form \(ky = 1\), where \(k\) and \(1\) are constants, remains. This equation yields a unique value of \(y\) except in the case \(k = 0\). How can one explain this exceptional case? Well, we know that \(ax + by = c\) represents a straight
and that two straight lines meet in a unique point except when they are parallel. Intuition tells us that the case \( k = 0 \) must correspond to the case of two parallel lines. Indeed this is just what happens. Another process of abstraction yields the result.

I have spent some time on this simple example since it illustrates two most important features of mathematics. First, it shows the two-stage process of intuition and abstraction in solving a mathematical problem. The other feature demonstrated by the example is the repetitive nature of the process - intuition and abstraction followed by intuition and abstraction.

The example also touches on a third aspect of mathematics which is closely linked to the intuition - abstraction dichotomy. I refer to concept and technique. Intuition sometimes throws up a concept, a general principle. A technique, on the other hand, is the way in which a concept is applied; this occurs during abstraction. The difference between concept and technique is akin to the difference between strategy and tactics. Intuition leads to the realisation of a concept, the formulation of a strategy. Abstraction is the manufacture of a technique, the production of tactics - the testing of the concept or strategy. Sometimes a concept is bad or even incorrect (just as a strategy can fail) in which case a technique leads to poor results or no technique can be found. One has to recall Gauss' famous remark "I have the result but I do not yet know how to get it".

The dual process I have described applies to all mathematics whether it is pure or applied. However, there is a difference in emphasis when one talks about pure or applied mathematics.

The applied mathematician has to set up a model of the physical system he is examining. This requires an intuitive understanding of what is important in that system, what it is that makes the system behave in
the way it does. He must then solve the mathematical equations he obtains from the model. These equations are usually familiar in type and the applied mathematician has merely to follow the rules in applying a known technique. The final stage in the process is to interpret the solutions of the equations in terms of the physical system. But there are difficulties; occasionally the applied mathematician sets up a model which leads to hitherto unsolved equations. He then has the choice of a compromise by altering his model (after all any model is a simplification of the physical system) or of turning pure mathematician and solving the equation.

The pure mathematician, on the other hand, is not concerned with any physical system. His is an abstract subject. It is a subject of axioms and definitions, of theorems and proofs. Yet to think that intuition plays no part in pure mathematics is completely to misunderstand the subject. It is precisely intuition that mathematics hinges on. From where do the axioms come? Intuition tells the pure mathematician, or anyone else for that matter, what a continuous curve looks like; it is quite simply one which can be drawn without removing pencil from paper. But this is unsuitable as a rigorous definition. Nor is a rigorous definition easy to obtain, as any first year mathematics undergraduate will testify. Such a definition was not forthcoming until the beginning of the nineteenth Century. The process of abstraction lasted 150 years in this case! However, once a suitable definition is obtained it becomes a simple matter to test whether a certain function is continuous (although I suspect that some first year undergraduates would contest this point).

Intuition plays another role in pure mathematics. Many pure mathematicians have an intuitive feeling for the abstraction process itself. Such surely is the case with Walter Feit and John Thompson, the American mathematicians, who used almost 300 closely packed pages of
mathematics to prove that every group of odd order is soluble. It is
unimportant if the reader does not understand the technicalities; the
point is that Feit and Thompson could see their way through a mass of
complicated algebra.

This then is my opinion of what mathematics is. In order to under­
stand more fully how new mathematical results are obtained it seems
appropriate to discuss the nature of mathematical giftedness.

2.3 Mathematical Giftedness

Although, as mentioned earlier, Abraham (1958) reported over one
hundred definitions of giftedness, it is fortunate that they fall into
five basic classes.

The first class consists of **ex post facto** definitions. For instance,
one such definition could be "the gifted are those who achieve outstand­
ing stature in their profession". These definitions, by their nature,
permit identification only after the subject has 'made the grade'. Thus
Shakespeare's plays still attract wide audiences **ergo** Shakespeare was
gifted. Or, Gauss contributed much to mathematics **ergo** Gauss was gifted.
Quite clearly this type of definition is totally unsuited to the needs
of parent and teacher when discussing a child's future career prospects.
Another disadvantage of this type of definition is that it does not
include the 'underachieving' gifted child - one who does not realise,
for whatever reasons, his full potential. Newton, Abel and Galois were
mathematicians who did not impress at school, while Shakespeare's talent
was recognised only as he approached middle age. Finally, the **ex post
facto** definitions make it impossible to estimate the number of gifted
children in a community.

A second class of definitions may be termed IQ definitions, where
gifted children are defined as those having an IQ above a certain level.
This type of definition is the most popular. However, the cut-off point can be chosen arbitrarily and immediately gives rise to a large number of interpretations of giftedness. Perhaps the most famous study of giftedness is Terman's long-term study of 1528 gifted children. Terman used an IQ of 140, measured by the Stanford-Binet Intelligence Scale, as his cut-off point. Various authors have used intelligence quotients varying between 115 and 180 as dividing lines. Indeed De Haan and Havighurst (1957, p. 1) divided the intellectually gifted into two groups for educational purposes - the top 1% they called 'first-order' gifted while the remainder of the upper 10% they termed 'second-order' gifted.

A third type of definition is social. The American Association for Gifted Children terms gifted children as those whose "performance, in a potentially valuable line of human activity, is consistently remarkable". However, what may appear remarkable to one person may seem ordinary to another. Ogilvie (1973, p. 6) researching into giftedness in British primary schools gave the following definition:

"The term 'gifted' is used to indicate any child who is outstanding in either a general or specific ability, in a relatively broad or narrow field of endeavour."

Ogilvie claims several advantages for this definition, but when all is said and done it merely replaces 'gifted' by 'outstanding' - and we are left to wonder what 'outstanding' means.

A fourth class of definition may be called the percentage definitions. For example, Conant considers the academically talented to be those students within the upper 15-20% of the secondary school population. In some ways this class of definition is an amalgam of the first three.
While these definitions (like the IQ and ex post facto definitions) are satisfactory in an operational sense, they do not give a clue as to the type of child who ought to be regarded as gifted. (A definition which gives such a clue is termed a logical definition.)

Recent authors have given more credence to a fifth class of definitions — those involving creativity. Getzels and Jackson (1962) is the standard, early American work while Liam Hudson’s fascinating books (1967, 1970) represent a British viewpoint. It is worth emphasising the significant points in recent work since much of it will be relevant in later discussion.

The creative process is familiar to mathematicians, scientists, composers, artists and playwrights alike. Poincaré (1914), Helmholtz (1895), Henry James (1908) and Nietzsche (1955) all report similar experiences. G. Wallas (1926) identified four stages in the creative process:

(i) preparation - where the problems are identified, the data understood and trial solutions are initiated;
(ii) incubation - a period when the mind may not even be consciously considering the problem;
(iii) inspiration - the moment when the solution is suddenly clear;
(iv) verification - the checking of the solution.

Jackson and Messick (1965) considered creativity from a different angle by examining the work itself, the creative product. They believed that the creative product is unusual, appropriate and both transforms and condenses existing work. The personal qualities considered necessary for creative production are originality, sensitivity, flexibility and a poetic nature.

There exist open-ended tests which, some psychologists claim, measure creativity. They differ from IQ tests in that there are many
good answers to an open-ended question (for example, 'how many uses can you think of for a brick?') instead of a single correct answer to an item in an IQ test.

Thus, if we are prepared to accept that open-ended tests do indeed measure creativity then creativity figures both as a logical and as an operational definition of giftedness.

What is really required is a logical definition which can be easily translated into an operational one. Lucito (1963, p. 184) gives a definition which depends on Guilford and Merrifield's adaptation (1960) of Bloom's taxonomy of educational objectives:

"The gifted are those students whose potential intellectual powers are at such a high ideational level in both productive and evaluative thinking that it can be reasonably assumed they could be the future problem solvers, innovators and evaluators of the culture if adequate educational experiences are provided."

Lucito quotes the four basic operations of cognition, memory, production (convergent and divergent), and evaluation in Guilford and Merrifield's model of the intellect. He argues that since the functions of cognition and memory are necessary for production and evaluation to occur, emphasis should be placed on these two latter functions. The open-ended tests are designed to measure divergent production while the more traditional IQ tests measure convergent production. However, it is much more difficult to measure 'evaluation'; indeed it is not easy to conceive how tests measuring 'evaluation' could be constructed. For we are dealing here with the highest powers of the human intellect. Indeed often the most remarkable advances in science and the arts have occurred
when an individual has swum against the tide of contemporary educated opinion. For example, Copernicus' theory that the earth moves around the sun, a commonplace observation in the twentieth century, was heresy in the fifteenth. No wonder Copernicus did not share this thought with the world until the year he died.

Thus Lucito's logical definition can be translated into an operational one only in part. Further, it is my belief that no complete logical definition of giftedness can ever be made totally operational. There will always be a Copernicus, a Newton, or a Gauss who will make a test involving critical thinking or value judgments unmarkable.

The remainder of this section is devoted to extending the general discussion of giftedness to the formulation of logical and operational definitions of mathematical giftedness.

Solving a mathematical problem consists of a combination of intuitive and abstract processes. The intuitive process involves finding a likely idea. The abstract process involves a rigorous examination of the idea to test whether it is a good one. But this situation is reminiscent of the chicken and the egg. For how can one know whether an idea is appropriate before it is tested? This is the crux of the problem. Certainly it is this feature of mathematics which distinguishes the great from the merely competent - the large number of appropriate ideas conceived by great mathematicians. It is for this reason that I believe mathematicians, particularly pure mathematicians, have an intuitive feel for the abstract process. In my view this is obtained through experience using the functions of perception, memory, reason and others related to number, space and form. These qualities are ones which intelligence tests are designed to measure; a good mathematician would be expected to score highly on this type of test.
But there is more to a great mathematician. He produces a large number of likely ideas. In the process he will probably consider, perhaps subconsciously, and dismiss a considerable number of unfruitful ones. Thus I believe that such a mathematician would score highly on open-ended tests, as well as on the conventional IQ tests.

This reasoning leads me to believe that both kinds of productive thinking should be present in very good mathematicians. But these abilities alone are insufficient. Some considerable critical judgment is required in deciding which ideas may or may not be appropriate. Thus, in my view, evaluative thinking is an integral part of an ability to do mathematical research.

The nature of mathematics and of mathematical research leads me to accept Lucito's definition of giftedness. But it is also my view that mathematical giftedness is different from, say, literary giftedness. Yet I would still accept Lucito's definition for literary giftedness. The distinction is a question of degree, not of content. For example, where the mathematician must be very able in convergent thinking (indicated by a high IQ), the novelist needs less ability in this direction. The novelist will, however, need greater sensitivity. Hence while I am prepared to accept a common logical definition of giftedness, I do believe it necessary to translate this into an operational definition dependent on the type of gift being considered.

What operational definition of mathematical talent can be given? One might accept the standards, reported by Johnson and Rising (1972, p. 357), laid down by an American school:

1. An IQ score of 120 or above.
2. A percentile rank of 90 or better on a standardised mathematics achievement test appropriate at his grade level.
3. An achievement record of A in previous mathematics courses.

4. A strong interest in learning mathematics.

5. A record of good work and study habits."

My own view is that the minimum IQ chosen here is too low for identifying the productive mathematician. It is doubtful whether there are many mathematical innovators (using the term in its loosest sense) with IQ scores much below 130. The second and third items are difficult to apply in the British educational system, while the last two items do not permit classification of the 'underachieving' gifted child. Nor does this definition include a measure of divergent thinking; this is not unusual since open-ended tests are still regarded by some as experimental.

My operational definition of mathematical giftedness restricts the IQ measure more than the American definition and it includes some measure of divergent thinking. The mathematically gifted child should, in my view,

(i) lie in the top 2% of an IQ scale;

(ii) lie in the top 5% of an open-ended test scale;

(iii) have a strong interest or a high ability in solving mathematical problems (measured by teachers' reports in current and previous years).

The first two items are unequivocal although the actual cut-off points are, of course, arbitrary. The relatively lower division line in the second condition indicates my belief in the stronger role played by abstraction in mathematics. The third condition is less precise and relies heavily on teachers' judgments. Further, it does not allow for the potentially gifted student who is not currently interested in mathematics (perhaps because of poor teaching). It is, therefore,
included with reluctance and has, in my view, a lesser standing than the first two criteria.

It is my belief that my definition would include holders of higher degrees in mathematics, 'good honours' graduates in mathematics, students with grade A in "A" level mathematics and those with grade 1 (but not necessarily the new grade A) at "O" level. I would also contend that the definition would cover those students awarded a supplementary grade on the Special papers in mathematics at "A" level. My definition might also include some who had none of these qualifications, but there would not be many in today's 'exam-conscious' society.

The reader now has my view of mathematical giftedness as well as, I hope, an accurate and comprehensive summary of giftedness in general. If he feels that much of the subject is conjecture and contention then I feel satisfied that the summary has been accurate.

2.4 Aims of Education of the Mathematically Gifted

There are two distinct groups of aims implicit in the title of this section. First, why is mathematics taught in school at all? Second, what should be the aims in educating the gifted as opposed to the non-gifted? The section is accordingly divided along these lines.

Mathematics, or certain branches of mathematics, has always featured as an important part of the school curriculum. There should therefore be strong reasons for its exalted place in the curriculum. Over sixty years ago, T. P. Nunn (1914, pp. 16-17) made the following 'brief statement of general principles':

"Mathematical truths always have two sides or aspects. With the one they face and have contact with the world of outer realities lying in time and space. With the other
they face and have relations with one another. Thus the fact that equiangular triangles have proportional sides enables me to determine by drawing or by calculation the height of an unscaleable mountain peak twenty miles away. This is the first or outer aspect of that particular mathematical truth. On the other hand I can deduce the truth itself with complete certainty from the assumed properties of congruent triangles. This is its second or inner aspect. . . .

. . . Our purpose in teaching mathematics in school should be to enable the pupil to realise, at least in an elementary way, this two-fold significance of mathematical progress. A person, to be really "educated", should have been taught the importance of mathematics as an instrument of material conquests and of social organisation, and should be able to appreciate the value and significance of an ordered system of mathematical ideas. There is no need to add that mathematical instruction should also aim at "disciplining his mind" or giving him "mental training". So far as the ideas intended by these phrases are sound they are comprehended in the wider purpose already stated. Nor should we add a clause to safeguard the interests of those who are to enter the mathematical professions. The treatment of the subject prescribed by our principle is precisely the one which best supplies their special needs."

Thus Nunn saw mathematics as both useful and intrinsically interesting, and took for granted that it should aim at 'disciplining the mind'. This last point is also made by another writer of that period, the American, J. W. A. Young (1924, pp. 17-19).
If more recent authors are consulted, then similar views can be found although with less emphasis on 'mental training'. James (1958, p. 1) says simply:

"the reasons generally advanced for the inclusion of mathematics in the school curriculum may be broadly classified under two headings: (i) its utilitarian value and (ii) its cultural value."

A year later, the Mathematical Association (1959, p. 2) put forward five reasons for including mathematics in the education of all primary school children.

"(1) Mathematics is the language of orderliness and ordered thinking.
(2) Mathematics is the tool and language of science.
(3) Mathematics is an inheritance of the race. We live in a community using the language of numbers, measurement and shape in everyday talk.
(4) Mathematics gives pleasure. Children find pleasure in mathematical thinking and achievement.
(5) Mathematics develops patterns of thinking which are fundamental patterns of all thinking."

These views are echoed by Johnson and Rising (1972, pp. 44-45). In all the quotations and references spanning sixty years there can be found a common thread. There are two basic reasons why mathematics is studied:
(i) it is useful - in science, engineering, geography, and in everyday life;
(ii) it is worth studying for its own sake and, more generally, for its cultural value.

In most sixth form mathematics courses a balance has to be struck between these two aims. Where mathematics is studied as a 'single' subject at "A" level, it is my belief that more attention should be paid to the first aim. For the 'double' subject I think the first aim should be less dominant (and I believe it is, although I have nothing apart from my own observations and experience on which to base this claim).

To complete the section it is necessary to consider the aims and objectives in the education of the gifted. The first point to settle is that many of the aims for educating the gifted will also be aims in educating the majority. For instance, the gifted child, like any other, should:

(i) be encouraged to develop his or her own personality;
(ii) be encouraged to become a good citizen - one who is a self-supporting member of the community and who recognises his or her own role in society;
(iii) be able to communicate effectively with others - to converse, to be both literate and numerate;
(iv) be aware of his or her own body, its powers and limitations - physical education in the widest sense.

However, there are some aims, common to both gifted and non-gifted, which differ in levels. Thus, while one hopes to develop both productive and evaluative thinking in all children, one expects that gifted children could cope with a more abstract presentation of the same material. With this understanding the aims for educating the gifted should also include:
(v) the development of productive and evaluative thinking;
(vi) the provision of a wide scope of information - this would increase the probability of productive and evaluative thinking;
(vii) the exposure of the gifted to higher levels of concepts - the gifted usually reach a level of conceptualisation at a younger age and may eventually be able to cope with levels the average student will never reach;
(viii) the encouragement of attitudes favourable to productive and evaluative thinking - independent work skills and habits, independence from peer group pressures on judgments, predisposition to critically evaluate accepted ways of doing things, ability to delay gratification of rewards, predisposition to examine many sides of an issue, the assumption of responsibility for the betterment of society;
(ix) the development of leadership skills and of techniques of changing leaders through orderly processes.

2.5 Bloom's Taxonomy of Educational Objectives

To be a successful teacher one must have some idea of how students learn, of how they acquire concepts. It is, of course, quite unnecessary to know the educational jargon, but it is essential to have some practical knowledge. The educational jargon centres around Bloom's Taxonomy of Educational Objectives (1956, 1964). Although the taxonomy is concerned with objectives, it is really a theory of concepts. It also demonstrates three aspects of learning - awareness, expertise and experience - familiar to most teachers. These three aspects are applied to three domains (cognitive, affective, and psycho-motor), the first being concerned with the acquisition of knowledge, the second with attitudes and the third (not yet fully developed by Bloom) with movement and motor skills.
"S" level mathematics is too advanced for the psycho-motor domain to be relevant. This domain will, therefore, be ignored in the discussion that follows. Bloom organised the various factors hierarchically to demonstrate that the objectives are cumulative; thus higher classes are built on skills involved in the lower classes.

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Table 2.1 Bloom's Taxonomy of Educational Objectives

In the cognitive domain, awareness consists of knowledge and comprehension - knowing facts and understanding their meaning. Quite clearly there is a difference; for example, the author is able to quote several Welsh phrases (and quite a few English words) without having any idea of their meanings. Application makes use of remembering and combining material to give generalisations for use in solving problems. Analysis means the breakdown of material into its constituents in order to find relationships between them. The difference between application and analysis can be thought of as the difference between the ability to apply knowledge to solve familiar and unfamiliar problems. Synthesis is the
ability to reorganise material. (It is, in my view, the lowest level at which a teacher can effectively operate.) The highest level in the cognitive domain is evaluation which requires value judgments about materials or ideas. Each level can be reached only with a thorough knowledge of the preceding levels.

The effective domain emphasises emotive qualities expressed in attitudes, interests, values and emotional biases. At the lowest level the student merely receives. In classroom terms one can think of the child who dutifully copies down notes (without day-dreaming while doing so). Responding might be illustrated by the child who asks a relevant question, although a child can also 'respond' silently. The student who values a subject is the one who sees the point of it, even if it has few or no practical uses. Very few school-children proceed beyond this level, but the ones who do are quite often the gifted. At the highest levels come organising ability and characterisation; the child who is prepared to rewrite his notes or who will construct his own notes on a topic or who will question (with good reasons!) the foundations of a subject serve as illustrations. What is meant here is a deep appreciation of the subject.

It would be a mistake to think that the columns in the table are not finely interwoven. The level of cognitive understanding reached may depend on a student's attitude, and conversely.

There is, moreover, one important influence only present implicitly in the table, the influence of psychology. Motivation can lead to a will to understand; a student's attitude can be changed from 'receiving' to 'responding' simply because he has the will to succeed. A student may succeed in solving a problem purely because he has a better memory than many of his class-mates. Of course, a better memory could be accounted for by a better learning technique which may in turn depend on better
'comprehension' of the material involved. Once again we return to the spider's web that is Bloom's taxonomy!

2.6 The Role of Examinations

Any educational system includes the need to evaluate a student's learning. There needs to be some means of assessing the student's progress, relative to other students and relative to his own potential. Examinations have evolved as one method of assessment which is readily understood by all sections of society. (Continuous assessment is another method but when one discusses assessment in Britain one is usually referring to examinations.) Examinations are an accepted part of every student's way of life; indeed his whole career may be affected by his performance in a single examination. This dissertation is a discussion of such an examination. Thus the need for examinations requires careful justification.

There are seven inter-connected purposes which examinations serve. First, they measure attainment: how much the student understands the aims and objectives of the course as well as a test of the acquisition and application of specific knowledge. The second purpose follows on directly from the first: examinations are diagnostic. Provided they are well-constructed they inform the teacher of the weaknesses of individual candidates and (a fact which is more often ignored) the weaknesses of the course presented by the teacher. A third justification of examinations is that they are used, rightly or wrongly, as predictors of future success; candidates successful at "A" level may be considered suitable for university education or professional training. Another reason in favour of the examination is that it provides motivation; practising school teachers realise the importance of this - learning is not easy and some students need a spur to make them work. (There are some who would claim
that the interest of the course itself could provide the necessary spur; 
I dismiss this argument in the case of older children, although there 
may be some truth in it for younger primary school children.) A fifth 
advantage of the examination is that it provides a vehicle for the student 
to develop powers to think quickly, to make correct decisions under 
pressure, to be able to organise ideas, to learn how to study, and to be 
able to concentrate for considerable periods of time. A sixth argument 
in favour of the examination is that it can provide and maintain standards; 
it is (theoretically) possible to arrange "A" level papers to be of the 
same standard from one year to the next and from one Board to another. 
Finally, examinations allow social mobility; the Chinese used examinations 
specifically to avoid nepotism.

Despite these seven persuasive arguments for examinations there are, 
however, some severe drawbacks. These drawbacks are mainly practical in 
the sense that the theoretical advantages of examinations may simply not 
exist in practice. The design of an examination may be so far from 
perfect that the examination no longer accurately measures attainment nor 
acts as a good predictor of future success.

The two main failings of examinations appear under the headings of 
reliability and validity. An examination is said to be reliable if it 
produces a consistent score from one occasion to the next. Certain types 
of examination, notably where essay questions are set, consistently cause 
problems. Fortunately, mathematics examinations do not generally come 
under this category.

An examination is valid if it succeeds in achieving what the 
examiners designed it to achieve. There are two main kinds of validity: 
content validity and predictive validity. The first tests whether the 
whole syllabus has been covered both qualitatively and quantitatively. 
Dale (1954) reported cases of "O" level mathematics examinations where
an improvement in marks between a sitting in June and a resit in September was so pronounced that hard work and good coaching alone could not have been responsible. This situation may have been due to sampling from a big syllabus, or the examinations may have been unreliable, or a combination of these two could have been responsible. Predictive validity is concerned with how well an examination provides a forecast of a candidate's future success; there has been some disturbing research on this at all levels from the old 11 plus examination to university degree examinations.

It is important to realise that an unreliable examination cannot possibly be valid. However, even a highly reliable examination may be invalid, for a highly reliable examination may simply test qualities which an examiner neither wished to test nor believed had been tested.

There are further limitations to examinations. For example, is it justifiable to compare marks in a Pure Mathematics examination with those in an Applied Mathematics examination, let alone those (say) in English and Physics? Some standardisation of marks is first required. But even then one must be careful in making a comparison for the two examinations are possibly testing quite different qualities.

Another limitation is not of the examination, but of the examinee. The examinee may perform poorly in an examination for a variety of intellectual, personal, and social reasons. This, in some ways, is the thorniest of the problems facing educators. Could these factors be allowed for and, if so, should they be? My belief is that poor examination technique and bad study strategies can and should be improved. But I regard certain examinations as successes if they are not only valid in content but also rank candidates on their ability to stand up to pressure. For example, a university degree is often used by industry as a predictor of future success; in industry an ability to meet deadlines is paramount.
Does and should "S" level mathematics also serve this purpose?

Finally, one may ask whether examinations, which were designed to test a syllabus, ever play a role in determining that syllabus. Do we include in a syllabus only that which is easily examinable?
CHAPTER 3
EDUCATIONAL PRACTICE

3.1 The Schools Questionnaire

A brief outline of the development of "S" level was given in Chapter 1. Three reasons for the existence of the examination were enunciated:

(i) more able candidates should be allowed to show their individual grasp by answering more searching questions;
(ii) universities would be provided with more information about a candidate;
(iii) the G.C.E. Examining Boards received representations from schools to maintain an examination for the more able student.

It was also reported that the entry for the Special papers in mathematical subjects fell steadily from 12.9% of the "A" level entry in 1963 to 5.6% of the "A" level entry in 1972. Several reasons for the decrease were conjectured. But there must remain a doubt whether the reasons for the inception of the examination are valid for its continued existence.

Further, in the last dozen years or so, great changes have occurred in secondary schools. First, many grammar and secondary modern schools have merged to become comprehensives. Has the formation of these conglomerates affected traditional values? A second great change has taken place in school mathematics in this period. 'Modern' Mathematics courses, notably the S.M.P., have become popular. The main feature of their syllabuses is that a wider section of mathematics is covered. Believers in traditional mathematics courses (and possibly some who are not convinced traditionalists) would argue that the breadth is achieved in modern syllabuses at the expense of depth. Has modern mathematics affected teachers' attitudes to the "S" level mathematics examination?
It would have alleviated my burden to have been able to refer to some learned text or erudite Journal to find an answer to these questions. Unfortunately there is, to my knowledge, no such work. Indeed, I know of no published research on any aspect of "S" level. The Department of Education and Science in a letter dated 17th June, 1975, stated that "there was nothing on the subject in this Department". It was, therefore, decided (with some reluctance on the author's part) to construct a modest questionnaire for circulation to schools.

The primary aim of the questionnaire was to determine why schools entered candidates for the Special papers in mathematical subjects at "A" level. It had several other functions. First, it would give some idea of the proportion of schools entering candidates for "S" level mathematics, and of the numbers of candidates from schools which did. Second, it might provide some interesting information on "S" level entries from different types of school. Third, it would provide an opportunity to receive valuable comments from teachers.

The first problem to arise was how to phrase the questions asking schools why they entered candidates for "S" level. If an 'open-ended' question, such as "what were the major factors in deciding whether or which candidates should be entered for the examination?" were asked, then difficulties were foreseen in collating the results! On the other hand, an 'objective' question could have been too restrictive. The reader may refer to Questions 5 and 6 of the questionnaire which is reproduced on the following three pages to evaluate the compromise reached.
QUESTIONNAIRE ON THE ROLE OF S-LEVEL MATHEMATICS

1. Name and Address of School .........................................................
   .................................................................................................
   .................................................................................................

2. Has the school entered candidates for S-level examinations in
   mathematics (either 'single-subject' or 'double-subject') during
   the past three years?
   YES / NO  (Please circle the appropriate reply.)

   If the answer to Question 2 is NO, then please turn to Question 7.

3. Please indicate the number of candidates entered for:
   (a) the 'single-subject' in     (i) 1973 ...........
      (ii) 1974 ...........
      (iii) 1975 ...........
   (b) the 'double-subject' in     (i) 1973 ...........
      (ii) 1974 ...........
      (iii) 1975 ...........

   (For the 'double-subject' please give the total number of S-level
   mathematics entries; for example, if a candidate was entered for
   S-level in both Pure Mathematics and Applied Mathematics then this
   should count as two entries.)

4. Which Board(s) or Project(s) has the school used for this
   purpose?

   O C O & C L AEB S W JMB SMP MEI

   (Please circle the appropriate reply or replies.)
5. Below is a list of reasons for entering candidates for the S-level examinations. If any of these played a major part in deciding whether or which pupils should be entered for the examination then please circle the appropriate letter(s).

(a) it is beneficial to subject the more able student to harder questions;
(b) it is beneficial to subject the more able student to a broader range of material;
(c) S-level provides an opportunity for the teacher to undertake more challenging work;
(d) S-level provides suitable practice for potential Oxbridge candidates;
(e) S-level provides an extra qualification for university applicants;
(f) some reason(s) other than (a) - (e).

6. Please elaborate if (f) is circled in the answer to Question 5

........................................................................................................................................
........................................................................................................................................
........................................................................................................................................
........................................................................................................................................

If the answer to Question 2 is YES, then please omit the next question.

7. Do you believe that S-level has a place in the education of able A-level candidates?
   YES / NO   (Please circle the appropriate reply.)

8. Do you wish to be informed of the results of this questionnaire?
   YES / NO   (Please circle the appropriate reply.)
9. If you have any comments about any aspect of S-level, then please use the space below.
It was decided to ask for the number of candidates entered in each of the past three years because a single year's figures could well have been unreliable. These figures could also be used to attach a weight to a school's replies to Question 5. (It seemed unreasonable to be forced to attach the same weight to a reply from a school entering one candidate in the last three years as to a school with a strong tradition of "S" level entries.)

Another factor thought to be significant was the G.C.E. Board or Project used by the school. In particular, some of the Boards set their Special papers entirely on the "A" level syllabus. Thus it would be unreasonable to expect schools using their examinations to claim that (b) of Question 5 formed a major factor in whether or which candidates should be entered. Furthermore, answers to other parts of the same question might equally well show some correlation with the Board or Project used.

A question (Question 7) was included for those schools which did not enter candidates for "S" level mathematics to test their attitudes to the examination. A negative reply was to be regarded as more interesting than an affirmative answer. It was anticipated that some schools which had insufficient experience of the examination might well circle 'YES'. A negative reply, on the other hand, would indicate a definite rejection of the examination.

A test of interest in "S" level (as well as a bribe to encourage a greater number of replies!) was introduced by asking recipients whether they wished to be informed of the results of the questionnaire.

Finally it was anticipated that some recipients would wish to make comments on "S" level. Indeed these would be most valuable; the teachers in the sample would have a cumulative wealth of experience. In any case it was important to obtain as much information as possible on a topic so
little researched. Thus an open-ended question inviting comments on any aspect of "S" level mathematics completed the questionnaire.

A covering letter addressed to Headmasters (out of politeness) was drafted explaining the purpose of the questionnaire and urging them to pass it on to their Head of Mathematics. The questionnaire and letter were then passed to a number of friends and colleagues for criticism and comment. As a result, the final covering letter was addressed directly to Heads of Mathematics (see Appendix I)! The questionnaire remained unchanged.

The next task was to decide how many and which schools should form the sample. The answer to the first question was simple - as many as the author could afford! There are approximately 6,000 secondary schools in England and Wales. But there are so many possible combinations of types of school (e.g. comprehensive, grammar, sixth form college, public, direct grant - any of which could be rural or urban) using so many different examining bodies (10 offering "S" level) that one would be dissatisfied with a sample containing fewer than about 400 schools. However, at a cost of 25 pence per questionnaire, the author would need a deeper pocket than he possesses to finance such research. It was eventually decided to send out 100 questionnaires.

The most difficult decision was how to select schools for the sample. If the schools were specially selected by the author, then bias would immediately be introduced into the sample. On the other hand, if schools were selected completely at random then the replies might be unrepresentative of those entering "S" level candidates. It was conjectured by the author that there would be a much higher proportion of "S" level entries from the independent sector than from the wholly maintained schools - a belief later to be vindicated by the results. Yet only 7% of the nation's school children are educated independently. Some compromise had to be found.
After much deliberation the 100 schools were divided into four unequal groups: 60 wholly maintained schools, 20 public schools, 10 direct grant schools, and 10 specially selected schools. Only the first three groups would count in the analysis. This was because the schools in these groups would be randomly chosen whereas the schools in the fourth group would not. The numbers in each group are arbitrary but were thought to be a reasonable compromise between the two conflicting views mentioned in the previous paragraph.

The 90 schools to be used in the analysis were selected using the Education Authorities Directory and Annual (1974). The standard method of random selection is to number all possible choices - suppose there are 6,214. Then a four-digit random number table is used to provide 90 selections. Any four-digit number larger than 6214 produced by the table is ignored for selection purposes.

This method was not employed in this case simply to avoid copying out the names of and numbering some 6,000 schools. It was possible to do this because there were approximately the same number of schools on each of the relevant pages of the Directory. The method used was to select a page number using a table of three-digit random numbers. Once the page had been located the number of schools on that page were counted and a two-digit random number table isolated the school to be included in the sample. Any schools clearly designated 'Middle' or 'Secondary Modern' were excluded from the sample. All others were accepted.

The ten specially selected schools were chosen for a number of reasons. All are well-known; most are household names. Many, if not all, of the schools have extremely fine records of success in placing their pupils at Oxford or Cambridge. Most were thought to enter candidates for "S" level. If this were the case then further information on "S" level mathematics might be obtained.
A full list of schools in all the groups can be found in Appendix 1. In the wholly maintained group of schools there were 18 grammar schools, 31 comprehensive schools, 4 high schools, 3 bilateral schools, 1 sixth form college and 3 which were not categorised. There were 15 boys schools, 10 girls schools and 35 mixed schools. Of the public schools 8 were boys schools, 10 took girls only and 2 were mixed. The 10 direct grant schools were equally divided into boys and girls schools.

3.2 Analysis of Replies

It was Andrew Lang who said "he uses statistics as a drunken man uses lamp-posts - for support rather than illumination". The point is not that statistics themselves which are dangerous and misleading but rather it is the person using them. It is the interpretation of statistics which causes disagreements, arguments and which perhaps led Benjamin Disraeli to his considered and well-known opinion of the subject.

It is thus with some hesitation that the author presents an analysis of the results. The actual statistics were easy to obtain (even if the task were laborious). It is their interpretation which is difficult. In educational research, in particular, there are so many factors - only some of which may be known - which can affect the results that interpretation becomes close to impossible at times. The reader is asked to accept any conclusions with the same caution the author used in drawing them.

The analysis that follows refers to the sample of 90 schools. The results from the other 10 schools will be considered separately at the end of this section. The order of the analysis follows that of the questionnaire and the numbers at the head of each paragraph refer to the number of the question.
Of the 90 schools in the sample, 62 submitted replies, representing a 69% response. In addition, it was learned that 1 school had closed; another returned the questionnaire stating that lack of time did not permit its completion.

<table>
<thead>
<tr>
<th>Name of group</th>
<th>Number of responses</th>
<th>Number in sample</th>
<th>Percentage response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wholly maintained</td>
<td>42</td>
<td>60</td>
<td>70%</td>
</tr>
<tr>
<td>Public</td>
<td>15</td>
<td>20</td>
<td>75%</td>
</tr>
<tr>
<td>Direct grant</td>
<td>5</td>
<td>10</td>
<td>50%</td>
</tr>
<tr>
<td>TOTAL</td>
<td>62</td>
<td>90</td>
<td>69%</td>
</tr>
</tbody>
</table>

Table 3.1 Distribution of responses within groups

Table 3.1 shows the response to have been roughly the same in each group. It would be wrong to conclude that the percentage response of 50% in the Direct grant group indicates a much lower level of response. A reply from one more school in this group would mean an increase of 10% in the percentage response rate.

Of the 62 responses, 26 schools had entered candidates for "S" level mathematics during the three years 1973, 1974, 1975. There were 14 wholly maintained schools (out of 42 responses), 8 public schools (out of 15 responses) and 4 direct grant schools (out of 5 responses). It is worthwhile examining these figures in greater detail. First consider the wholly maintained schools. Table 3.2 shows the number of different types of schools entering "S" level candidates in mathematics during the years 1973-75.
The table shows that the response rate amongst the various types of school was similar. Yet none of the High schools entered candidates for "S" level mathematics. But perhaps the more remarkable figures are those for the Grammar and Comprehensive schools. About 60% of the Grammar schools responding entered candidates, while less than one in four Comprehensive schools used the examination. Opponents of comprehensive education would no doubt use these statistics as ammunition in their battle against reorganisation. While it may be true that comprehensive education pays less attention to the gifted, other explanations are also possible. For instance, most of the schools which turned comprehensive prior to 1974 did so out of the choice of the local authority. This usually meant a Labour-controlled council in a predominantly working-class area where there may, in any case, have been little tradition of "S" level. However, the figures would suggest that further research is necessary; perhaps a controlled experiment in half
a dozen paired areas of the country (three which had turned comprehensive and three which had not) could shine more light on the situation.

If one examines the distribution of "S" level entries in wholly maintained schools by sex, then the results are not exceptional — provided it is remembered that most Comprehensive schools are mixed, and conversely. Table 3.3 contains the figures.

<table>
<thead>
<tr>
<th>Schools by sex of pupils</th>
<th>Number of schools entering &quot;S&quot; candidates</th>
<th>Number of responses</th>
<th>Number in sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>5</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>Girls</td>
<td>3</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>Mixed</td>
<td>6</td>
<td>22</td>
<td>35</td>
</tr>
<tr>
<td>TOTAL</td>
<td>14</td>
<td>42</td>
<td>60</td>
</tr>
</tbody>
</table>

Table 3.3 Distribution of "S" level entries in wholly maintained schools (by sex of pupils)

This picture is drastically altered if similar tables are drawn for Public schools and Direct grant schools. In Public schools, especially, there is a marked difference in the proportion of boys and girls schools entering "S" level candidates (see Table 3.4).
<table>
<thead>
<tr>
<th>Schools by sex of pupils</th>
<th>Number of schools entering &quot;S&quot; candidates</th>
<th>Number of responses</th>
<th>Number in sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>5</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>Girls</td>
<td>2</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>Mixed</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>TOTAL</td>
<td>8</td>
<td>15</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 3.4 Distribution of "S" level entries in Public schools (by sex of pupils)

<table>
<thead>
<tr>
<th>Schools by sex of pupils</th>
<th>Number of schools entering &quot;S&quot; candidates</th>
<th>Number of responses</th>
<th>Number in sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>3</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Girls</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>TOTAL</td>
<td>4</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 3.5 Distribution of "S" level entries in Direct grant schools (by sex of pupils)

(3) It should already be evident that there is a considerable difference between various types of school in the practice of entering candidates for "S" level mathematics. But it is only when the numbers of candidates entered by schools are considered that the true situation becomes clear. The 26 schools between them entered 374 candidates during the three years, at an average of about 5 per year. However, the
distribution of entries was non-uniform. Six schools entered only one candidate in the three years while 8 schools each made more than 20 entries during the same period. (One school put in 82 entries for the examinations.) At least 315 of the 374 entries were boys; only 15 are known to be girls - the remainder came from mixed schools.

<table>
<thead>
<tr>
<th>Name of group</th>
<th>Number of entries SINGLE 1973</th>
<th>Number of entries SINGLE 1974</th>
<th>Number of entries SINGLE 1975</th>
<th>Number of entries DOUBLE 1973</th>
<th>Number of entries DOUBLE 1974</th>
<th>Number of entries DOUBLE 1975</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wholly maintained</td>
<td>17</td>
<td>12</td>
<td>17</td>
<td>22</td>
<td>14</td>
<td>8</td>
<td>90</td>
</tr>
<tr>
<td>Public schools</td>
<td>29</td>
<td>30</td>
<td>17</td>
<td>54</td>
<td>33</td>
<td>48</td>
<td>211</td>
</tr>
<tr>
<td>Direct grant</td>
<td>6</td>
<td>5</td>
<td>1</td>
<td>22</td>
<td>17</td>
<td>22</td>
<td>73</td>
</tr>
<tr>
<td>TOTAL</td>
<td>52</td>
<td>47</td>
<td>35</td>
<td>98</td>
<td>64</td>
<td>78</td>
<td>374</td>
</tr>
</tbody>
</table>

Table 3.6 Distribution of numbers of "S" level entries within groups

The figures in Table 3.6 show that, during the three year period, the 14 wholly maintained schools entered 90 candidates, the 8 Public schools 211 candidates, and the 4 Direct grant schools 73 candidates. Put another way, this means that each year one would expect 2 entries from a wholly maintained school in the sample, 6 entries from a Direct grant school and 9 entries from a Public school.

These figures are relevant only if the sizes of the various schools are comparable. The average size of the 14 wholly maintained schools is 750 pupils, of the 8 Public schools 590, and of the 4 Direct grant schools 530 pupils. But there are complicating factors which affect a direct comparison. For instance, the wholly maintained schools generally take
pupils from 11 to 18 years of age, although some operate on a 14 to 18 basis and 1 is a sixth form college. On the other hand, the Public schools accept pupils at 13 years of age. This implies that a Public school will have more pupils in a particular age group than an 11 to 18 school of the same size. Another complicating factor is that Comprehensive schools will contain many non-G.C.E. pupils. Direct grant, Grammar and Public schools do not.

These complications in fact mean that the three groups of schools will be roughly comparable in size, with the Public schools being perhaps the largest if one considers the population from which the "S" level candidates are being drawn. If this is the case, then one would expect "S" level entry numbers in the three groups of schools to be in the ratio 2 : 8 : 6, that is, 1 : 4 : 3 (Wholly maintained : Public : Direct grant).

In retrospect, the author is not completely happy with the design of this particular question; rather, he feels that more information is needed - in particular, the size of the sixth form is a factor which might have made a comparison easier to accomplish. More instructions were needed to help those schools using the SMP examination answer this question. The SMP course, like several "A" level courses, is organised on a Mathematics/Further Mathematics basis. However, unlike the other similar "A" level courses, the SMP does not have a Special paper in Further Mathematics. As a result, SMP "S" level Mathematics is taken both by candidates studying mathematics as a 'single' subject and by those studying the subject as part of a 'double' subject course. Respondents who used SMP were not clear how to answer the question and entered the number of candidates under a heading "SMP Mathematics".
(4) The Boards or Projects used by the 26 schools are shown in Table 3.7. It should be noted that 3 of the schools used two Boards or Projects. Also shown are the numbers of candidates sitting the examinations.

While it does not seem worth breaking down the information further, two points are remarkable. Six of the 7 JMB schools came from the Wholly maintained group. The other point may help settle a popular misconception. Many teachers call the Oxford and Cambridge Schools Examination Board the 'Public schools board'. It is true that most schools using the Board's examinations are Public schools. But there are many Public schools using other Boards. In this instance 3 of the Public schools used O & C, 3 used L, 2 used SMP, 2 used C and 1 used JMB. (All 3 schools using two Boards were Public schools – the combinations were SMP/L, O & C/L, and O & C/SMP.)

<table>
<thead>
<tr>
<th>Board or Project</th>
<th>Number of schools</th>
<th>Number of candidates</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>89</td>
</tr>
<tr>
<td>O &amp; C</td>
<td>4</td>
<td>136</td>
</tr>
<tr>
<td>L</td>
<td>7</td>
<td>51</td>
</tr>
<tr>
<td>AEB</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S</td>
<td>1</td>
<td>29</td>
</tr>
<tr>
<td>W</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>JMB</td>
<td>7</td>
<td>38</td>
</tr>
<tr>
<td>SMP</td>
<td>4</td>
<td>41</td>
</tr>
<tr>
<td>MEI</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3.7 Boards or Projects used by the schools
The reader may be disturbed by the lack of information from schools using three of the Boards. The AEB caters for very small numbers of Special paper entries; for example, in 1973, there were 75. In comparison, the JMB received over 800 entries, while the SMP paper attracted over 500 entries, the Cambridge Syndicate's papers were also taken by more than 500 candidates. Three other Boards had more than 400 candidates sitting the Special papers. The Welsh Joint Education Committee, like the AEB (and indeed the Southern Board), does not have a large number of "S" entries; in 1973, there were 119. Thus it is unsurprising that Boards with so few candidates are omitted from a sample of this size. Indeed there is further evidence to support this view. The Welsh Joint Education Committee caters almost exclusively for Welsh schools, of which there were only 3 in the sample.

These arguments cannot hold for the Oxford Delegacy which, in 1973, received 420 "S" entries. The small size of the sample may be responsible for the lack of information. Alternatively, there may be some item(s) in the questionnaire which made it difficult for respondents to complete. A third factor which may have discouraged replies is that the Oxford Delegacy is in the process of revising the "A" level Mathematics syllabus and of remodelling the examination structure.

(5) It has already been explained in the previous section that the number of candidates a school entered for the Special papers could be used to weight the replies to this question. This would allow for schools with greater experience to count more in the analysis. However, the unweighted responses are also interesting. To give the reader some idea of the number of reasons a school gave for entering candidates for the Special papers in mathematics, he is referred to Table 3.8.
In what follows, the reasons in Question 5 are denoted by the appropriate letter to save space. The most popular reason was (a) which was circled by 22 of the 26 schools. Next came (d) with 17, (b) and (e) with 9, and (c) with 8; 5 schools had other reasons for entering candidates in the Special papers (see the analysis of Question 6).

To weight these figures according to the number of candidates a school entered during the three years, one simply counts the number of candidates each time a particular reason is circled. The weighted figures are shown in Table 3.9 under the unweighted responses.
respectively. The figures then appear as in Table 3.10. Although they are quoted to two decimal places, the author believes they can only be relied on to the first place. For example, a difference of 1 in the number of responses would make a difference of 0.04 in the unweighted ratio.

<table>
<thead>
<tr>
<th>Reason</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unweighted ratio</td>
<td>0.85</td>
<td>0.35</td>
<td>0.31</td>
<td>0.65</td>
<td>0.35</td>
<td>0.19</td>
</tr>
<tr>
<td>Weighted ratio</td>
<td>0.84</td>
<td>0.31</td>
<td>0.22</td>
<td>0.78</td>
<td>0.46</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Table 3.10 Ratios of unweighted responses to total number of responses and of weighted responses to total number of candidates

These figures may be interpreted to show that schools with more experience of entering "S" level candidates regard (c) as less important and (d) as more important than those schools with little experience. One point must be settled at once before the reader places too much faith in the figures for (e). These figures include replies from two Heads of Mathematics who amended the response before circling the reason. One prefixed the response by "in a very few cases", and added "rarely applies since nearly all places are conditional on "A" level results only". The other was more specific - "if they are future Cambridge candidates otherwise not important". These two were Heads of Mathematics at Public schools entering 82 and 26 "S" level candidates in the three year period. If their replies were omitted from the tally for (e) then the unweighted and weighted ratios would be 0.27 and 0.17, respectively.
A further point which the reader may wish to be reminded of is that some Heads of Mathematics might have wished to circle (b) but could not do so because the syllabuses of the Boards they used contained no extra material. The author regards this as an important point, but not one which should unduly affect the accuracy of the figures. After all, Question 5 asked for reasons why schools actually entered candidates not for reasons they would support. However, it is interesting to see the importance attached to the broader range of material by those schools using appropriate syllabuses. Eleven schools are in this category; the unweighted and weighted ratios for these 11 schools are 0.64 and 0.41, respectively.

If Table 3.10 is examined, bearing in mind the considerations of the previous two paragraphs, then it seems fair to draw the following conclusions about why candidates are entered for "S" level mathematics. First, there is a widespread belief that it is beneficial to subject the more able student to harder questions. Second, there is a considerable body of opinion (particularly amongst teachers with more experience of the examination) that the course provides suitable practice for potential Oxbridge candidates. Third, about one in three respondents believe that "S" level provides an extra qualification for university applicants. (There is some evidence to believe that it is an extra qualification in only a few cases; we shall return to this point later.) Fourth, about the same proportion believe that it is beneficial to subject the more able student to a broader range of material.

These conclusions hold also if the figures for the three groups of schools are given separately (see Tables 3.11, 3.12, 3.13). There are, of course, some differences in emphasis. Perhaps the most noteworthy is the difference in the importance attached to the belief that "S" level provides suitable practice for potential Oxbridge candidates. In wholly
maintained schools the unweighted and weighted ratios were 0.5 and 0.6, whereas in Public schools they were $\frac{7}{8}$ and 0.88, respectively. The author puts this down to the view that Public schools are more 'Oxbridge-conscious' than state schools.

<table>
<thead>
<tr>
<th>Reason</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unweighted ratio</td>
<td>0.79</td>
<td>0.28</td>
<td>0.36</td>
<td>0.50</td>
<td>0.36</td>
<td>0.28</td>
</tr>
<tr>
<td>Weighted ratio</td>
<td>0.86</td>
<td>0.23</td>
<td>0.44</td>
<td>0.60</td>
<td>0.32</td>
<td>0.32</td>
</tr>
</tbody>
</table>

Table 3.11 Unweighted and weighted ratios. Wholly maintained schools.

<table>
<thead>
<tr>
<th>Reason</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unweighted ratio</td>
<td>$\frac{7}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{2}{8}$</td>
<td>$\frac{7}{8}$</td>
<td>$\frac{3}{8}$</td>
</tr>
<tr>
<td>Weighted ratio</td>
<td>0.78</td>
<td>0.21</td>
<td>0.20</td>
<td>0.88</td>
<td>0.68</td>
</tr>
</tbody>
</table>

Table 3.12 Unweighted and weighted ratios. Public schools.

<table>
<thead>
<tr>
<th>Reason</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unweighted ratio</td>
<td>$\frac{4}{4}$</td>
<td>$\frac{2}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Weighted ratio</td>
<td>1.00</td>
<td>0.68</td>
<td>0.01</td>
<td>0.70</td>
<td>0.01</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Table 3.13 Unweighted and weighted ratios. Direct grant schools.
Only 5 of the 26 schools circled (f) in reply to Question 5. Four of these were in the Wholly maintained group while the fifth was a Direct grant school. The author believes it is worthwhile quoting all five replies.

(i) A boys Grammar school entering 5 candidates in the three year period (all for the 'single-subject' paper):

"Oxbridge insisted that one of the candidates mentioned on these forms should take "S" level: two other good mathematicians then asked to try "S" level also."

(ii) A sixth form college entering 16 'double-subject' candidates during the three years:

"Some of our students gain entry to Oxbridge on "A" level results rather than take the relevant Entrance Exams. This necessitates their taking an "S" level paper - and obtaining a good grade. Others take it for the satisfaction they derive in being able to cope with the more difficult type of question."

(iii) A mixed Grammar school entering 7 candidates (3 'single-subject' and 4 'double-subject'):

"There seems very little reason for entering any candidates unless outstanding and trying for Oxford or Cambridge. To be honest there seems little point in "S" level work."

(iv) A Roman Catholic boys Comprehensive which entered only 1 candidate:

"Candidate had already passed "A" level. In order to occupy his time while he completed another "A" level course, "S" level Maths was considered appropriate."

(v) A Direct grant school entering 22 candidates for the 'double-subject' paper of a Board which timetabled the Special paper to
be taken a few days before the corresponding "A" level papers:  
"A demanding examination is excellent preparation for  
the "A" levels following; pupils are more confident  
and less prone to 'nervous' errors having first taken  
the "S" levels."

The connection between the "S" level examinations and Oxbridge  
entrance is reaffirmed. But the last two quotations show that "S" level  
may be used for other purposes.

(7) Of the 36 schools not entering "S" level candidates, 25 circled YES,  
8 circled NO, and 3 claimed they had insufficient experience to reply.  
In addition, 3 schools of the 26 which entered candidates for the  
examination ignored the invitation to omit the question and circled NO.  
One could conclude that Headmasters and Heads of Mathematics do not  
always see eye to eye (but see (9)).

(8) 49 of the 62 respondents wished to be informed of the results of the  
questionnaire.

(9) One who asks an open-ended question has only himself to blame if a  
wide range of responses has to be analysed! Comments were received from  
26 schools. Four of the replies said simply that the school had never  
had pupils of such high calibre, while another 4 said that they hoped to  
enter candidates for the examination in 1976 or 1977. Of the 18  
remaining, all shades of opinion were represented. Six of these were  
from teachers who answered Question 7 in the negative (including the 3  
whose schools had entered candidates for "S" level). The other 12 were  
divided between those outlining some of the disadvantages or practical  
difficulties involved and those which spoke of the examination with  
unreserved praise.
An attempt will be made to summarise the points made; unless otherwise stated each point was made by only one respondent.

(i) "S" level tends to increase the already high level of specialisation in sixth forms (2 replies).

(ii) Most universities do not require (or take note of) the extra qualification (4 replies).

(iii) Some candidates feel that there is no point in doing extra work for no obvious gain.

(iv) The gifted could be given harder examples during the "A" level course and be encouraged to explore topics not on the syllabus if they are interested enough.

(v) Girls do not wish to attempt "S" level (2 replies).

(vi) There is too much manipulation in the present "S" level (a school using JMB).

(vii) Many schools cannot make provision for "S" level candidates through lack of suitable staff and/or of allocated teaching time (2 replies).

(viii) One SMP school used "S" level Mathematics to extend more able students because Further Mathematics 'is very time-consuming and difficult to time-table'.

(ix) By tackling harder questions of "S" level type, "A" level questions would become relatively simple; the deeper analysis and understanding necessary at "S" level can help student and teacher.

(x) "S" level provides a suitable course to stretch the very able (5 replies).

(xi) The Special papers in mathematics are good, requiring considerable understanding and originality (an O & C school which also uses SMP to a lesser extent).
(xii) A hope that "if N and F proposals go through then "S" level would either remain or be replaced by an exam equally demanding."

(xiii) Since "S" levels are not usually used as selective examinations for university purposes, they are not 'divisive' or socially unacceptable.

There are two replies which seem worthy of quotation in their entirety. The first from a Head of Mathematics at a boys Comprehensive includes several points not made by other respondents:

"At my previous school, which was a Grammar school until a few years ago, "S" level was offered. Its value is I believe limited to

(i) acting as a stimulus for able boys who intend to study Mathematics at University;
(ii) acting as a time-filler for prospective Oxbridge entrants;
(iii) providing prestige for the school (!).

I have not been led to believe that it is considered to any extent by prospective universities; indeed, when I was at school we were told that to attempt and fail "S" level was worse than never to have taken the exam."

The second quotation comes from the Head of Mathematics at a Public school with some tradition of "S" level work (35 entries in the three year period):

"It is a very great pity that Universities do not use this examination for assessing suitable entrants as was originally intended. Some Universities (Oxford and
Cambridge) are a thorough nuisance to schools by continuing to hold separate Entrance Examinations in the autumn. Others appear to ignore it altogether. Few offer any positive encouragement to students to make the extra effort involved in doing the Special papers."

To complete this section let us examine the 5 returns from the 10 specially selected schools. The 5 schools all entered candidates for "S" level; in fact, between them they entered 301 during the last three years!

One school, using JMB and entering 115 candidates during the three year period, circled (a) and (d) in response to Question 5, NO in answer to Question 7, did not wish to be informed of the results and offered no further comment!

A second school, entering 46 candidates for the O & C papers in the three year period, circled (a), (b), (c), (d) and (f) in response to Question 5 and added that (e) applied to Oxbridge candidates only. A need was felt to implant some of the concepts students would later need while they were still at school. The respondent wished to be informed of the results. He also made it clear that he would continue to teach material beyond "A" level even if "S" level were abolished. He added that the benefits both to pupil and to teacher of the extra challenge and stimulation are enormous. He continued:

"One can safely say to most pupils who will read maths, physics, engineering, etc., that they will be bound to need some of this (e.g. partial differentiation) later on at college, and it is very likely that it can be presented in a more palatable form at school than they are likely
to get when it is introduced at University. Indeed this motivation is the one which is the most acceptable to pupils when they realise they are being taught material beyond the syllabus."

A third school entered 73 candidates for the MEI examinations, circled 5(a), 5(b) and 5(d), wished to be informed of the results, but offered no further comment.

The Head of Mathematics from the fourth school did not, I suspect, see Question 9 on the last sheet of the questionnaire. For he attached a slip of paper to the front which stated:

"I have not been able to fill the questionnaire in the manner I suspect you expected; because

(1) "S" level is NOT available in Further Mathematics for SMP, so that my desire to have such entries is nowhere recorded;

(2) 5(b) cannot be completed by SMP schools because the syllabus for the "S" paper is NOT broader - a blank here is different from a blank which would indicate disagreement with this proportion."

The author believes that he has already dealt with the second point in the discussion of Question 5 in the main sample. The first point (and indeed the second) could have been made in answer to Question 9. The school, in fact, entered 50 candidates for the SMP examination. 5(a) and 5(d) were circled, as was YES in reply to Question 7. The respondent wrote three times that SMP Further Maths "S" level was unavailable. The author takes this to be an indication of the respondent's frustration!
The fifth school used "S" level for an entirely different purpose. The school entered 17 candidates for the MEI examinations. Only (f) was circled in response to Question 5; the respondent wrote:

"I'd like to circle all these but honesty impels me to circle (f)! On the whole we find these days that boys who obtain 'moderate' "A" levels on their first run are the ones who stay on and take "S". The really good leave after "A" and university entrance."

He circled YES in response to Question 8. His comments in answer to Question 9 reflect, I believe, the views of a number of respondents:

"I am sorry to see its decline over the last few years: unfortunately, universities seem to take very little cognisance of it and I am forced to recommend to even good boys that the first priority is a good "A". I think that "S" level offers challenging work to both pupil and teacher: I shall be sorry to see it decline further, but am steeled to the inevitable!"

3.3 Correspondence

Although the replies from the questionnaire were interesting and provided a wealth of material, they did not suppress the author's disquiet about the small size of the sample. It seemed worthwhile to contact a number of professional bodies to enquire whether any had views on the role of "S" level mathematics. In this way collective views on "S" level might be obtained. Unfortunately, neither the Mathematical Association nor the Association of Teachers of Mathematics had
formulated any considered views on the role of "S" level. (Replies were not received from the Institute of Mathematics and its Applications or from the National Association of Gifted Children.)

However, two interesting letters expressing personal views were received from officials of the Mathematical Association. Miss N. L. Squire, the Hon General Secretary, wrote that she entered some of her candidates for the Special papers in the University of London "A" level examination mainly as an added incentive to tackle some really demanding questions. She added:

"In some cases (though not in the majority of questions I find) the questions do lead them more into the type of approach they will need when they get into a University course."

She concluded with a promise to pass the author's letter on to the Teaching Committee of the Association.

The second letter came from the Chairman of the Teaching Committee, J. W. Hersee, who until recently was also Head of Mathematics at Clifton College. His reply is all the more interesting because he is the newly appointed Director of the School Mathematics Project:

"I don't think the Association has any particular view on "S" level; we would regard it as a matter for each school to consider in the light of its own situation. Since university entrance is usually based on "A" level only I know that many schools do not take the "S" papers, apart from the difficulty they might have in arranging teaching for them."
Speaking personally, I like "S" level for the more able pupils as it gives them a more challenging target and extends their knowledge; as some of them are potential Oxbridge candidates it takes them further and thus gives them a better start on Oxbridge scholarship work."

This view is one which many teachers supported in their replies to the questionnaire. The connection between "S" level and Oxbridge entrance is so firmly emphasised by so many teachers that it seemed the natural thing to write to Tutors for Admissions at Oxford and Cambridge colleges. The letter would have two purposes: first to ascertain the importance attached by the Tutors to "S" level results, and second, to enquire whether Tutors thought "S" level mathematics to provide suitable practice for potential Oxbridge candidates. Six colleges were selected - three from Oxford and three from Cambridge - using the standard method of random selection. The full text of the letter is given overleaf.

The three Oxford colleges contacted were New College, St. Peter's College, and St. John's College. New College and St. John's saw no distinction between a candidate with three As at "A" level but with a U at "S" level and a candidate with three As who was not even entered for "S" level. On the other hand, C. A. Caine, Fellow and Tutor in Mathematics at St. Peter's, wrote:

"As far as this college is concerned a candidate's results in A and S-levels are taken as an indication of his ability, and in certain circumstances, for example a candidate with three As at A-level but not a
The Tutor for Admissions,  
New College,  
Oxford.

Dear Tutor,

I am conducting some research into the role of "S" level mathematics. One of the fundamental questions appears to be what importance universities attach to a candidate's performance in the Special papers at Advanced level.

I wonder whether you, as Tutor for Admissions, pay much attention to "S" level grades. In particular, is a candidate who has obtained (say) three grade A's at "A" level but has achieved only a U in the Special paper less favourably regarded than a candidate with the same "A" level grades who was not even entered for "S" level? On the other hand, does a 1 at "S" level in a subject in which a candidate scored a grade B at "A" level count more favourably than a grade A at "A" level unsupported by an "S" level grade? I realise that it is very difficult to generalise and that each individual's case must be examined separately. But what I am really after is the importance (or lack of it) attached by the Tutor for Admissions to a candidate's "S" level results.

One reason many schools enter pupils for "S" level is that the examination is thought to provide suitable practice for potential Oxbridge candidates. Do you agree that the examination does provide suitable practice and would you support its use for this purpose?

I would be most grateful for your valuable comments.

Yours sincerely,
U in the S paper, we might well award a place on those results."

St. John's replied that a "1 at "S" level in a subject with a B at A level would certainly be thought preferable to an A at "A" level with no "S" level". The New College Tutor could not answer this second question "since in the past eight years no maths candidate whom I can recall has had Bl in Maths".

The three Cambridge colleges (Clare, Selwyn and Trinity) all thought that a 1 at "S" level and a B at "A" level as being better than an A at "A" level unsupported by an "S" level grade, although one of the colleges (Clare) said that this situation does not often arise in the case of Mathematics.

Four of the colleges, two from each University, pointed out that "S" level is only one criterion used in assessing candidates and that more attention was paid to the University Entrance Examinations. The following is the text of the letter from Dr. H. J. Easterling, the Tutor for Admissions at Trinity College. (The standards set by Trinity appear to be representative of Cambridge but seem to be very slightly higher than those required by Oxford.)

"We pay a great deal of attention to S level grades in Mathematics. At Trinity we make conditional offers to pre-A level candidates in Mathematics (a good many Cambridge Colleges refuse to do this) and our normal standard for a conditional offer is three grade A's together with grade 1 on an S Paper either in Mathematics or in Physics. We should be unlikely to stretch a point in favour of a candidate who has
failed to achieve the stipulated grade 1. When we are considering post-A level candidates, we are more interested in their performance in our own entrance examination than in their A level grades, but it is fair to say that, when we come to look at marginal candidates, a U in the S Paper would count against a man and that we should be more impressed by a grade B1 than by AU or A alone.

What I have said above applies to candidates for Mathematics. Much the same is true, though to a lesser extent, in the case of scientific subjects, though we should not normally require an S paper grade when making a conditional offer for any subject except Mathematics. I am sure you are right in suggesting that S level provides good practice for Oxbridge papers. I do not see any harm in using the examination in this way."

Four of the replies supported the suggestion that "S" level provides suitable practice for potential Oxbridge candidates; one respondent said that he could not comment on the proposition. The sixth did not comment. One of the Cambridge colleges (Clare) said they would like to see all candidates taking the "S" level papers.

The author feels that these responses indicate that "S" level is regarded with some importance by Oxbridge colleges and that there is, on the whole, support for the view that the examination provides suitable practice for Oxbridge candidates.

That the same importance is not attached to the examination by provincial universities is easy to demonstrate. These universities have as the great majority of their applicants pre-"A" level pupils and make
conditional offers on the basis of "A" level results alone (specifying grades in certain subjects perhaps). "S" level plays no part in the selection procedure, except possibly as additional information about post-"A" level applicants. This is not to say that the provincial universities regard the examination as not providing an extra qualification. Shortage of time prevented the author from properly verifying these remarks, but from his discussions with lecturers in a number of provincial universities it appears that most universities receive few (if any) applications from candidates with "S" level qualifications.

3.4 Statistics of the Examination

The reader has been presented with certain figures about "S" level entries taken both as a proportion of "A" level entries and as raw numbers in the Introduction and in Chapter 1. The discussion of the last two sections may have left him wondering how the "S" level entry is distributed between the various Examination Boards and what proportion of candidates are awarded supplementary grades (i.e. a 1 or 2 at "S" level). This section should answer these questions.

To see how the "S" level entry is distributed between the various Boards, the figures for 1973, the last year for which complete statistics are available, will be examined.

It should be noted that the table includes figures for Mathematics, Pure and Applied Mathematics, and Pure Mathematics with Statistics. The London Board held an "S" level examination in Pure Mathematics with Statistics but not in Mathematics (Pure and Applied), for which there were 3518 "A" level entries, nor in Mathematics. Despite these various anomalies, the figures would still seem to indicate either a disparity between the Boards of the quality of the "S" entry or a difference in standard of the examinations!
<table>
<thead>
<tr>
<th>Board</th>
<th>No. of A entries</th>
<th>No. of S entries</th>
<th>No. awarded Distinction (1)</th>
<th>No. awarded Merit (2)</th>
<th>% of S entry awarded supplementary grades</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>4558</td>
<td>323</td>
<td>34</td>
<td>42</td>
<td>23.5%</td>
</tr>
<tr>
<td>C</td>
<td>4776</td>
<td>248</td>
<td>59</td>
<td>69</td>
<td>51.6%</td>
</tr>
<tr>
<td>O&amp;C</td>
<td>2054</td>
<td>212</td>
<td>67</td>
<td>69</td>
<td>64.1%</td>
</tr>
<tr>
<td>L</td>
<td>3697</td>
<td>9</td>
<td>0</td>
<td>2</td>
<td>2/9</td>
</tr>
<tr>
<td>AEB</td>
<td>2323</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>S</td>
<td>*</td>
<td>-</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>W</td>
<td>1449</td>
<td>53</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>JMB</td>
<td>11203</td>
<td>799</td>
<td>94</td>
<td>167</td>
<td>32.7%</td>
</tr>
<tr>
<td>SMP</td>
<td>4600</td>
<td>504</td>
<td>75</td>
<td>127</td>
<td>40.1%</td>
</tr>
<tr>
<td>MEI</td>
<td>1169</td>
<td>86</td>
<td>25</td>
<td>17</td>
<td>48.8%</td>
</tr>
<tr>
<td>TOTAL+</td>
<td>34380</td>
<td>2181</td>
<td>354</td>
<td>493</td>
<td>38.8%</td>
</tr>
</tbody>
</table>

Table 3.14  Distribution of numbers of A and S entries amongst the 10 Examining Boards or Projects
Subject: Mathematics

The symbols used in the tables in this section have the following meaning:

The notation * denotes that figures were not issued by the Boards, while - denotes that no examination was set by the Board.

+ The figures in the total exclude those for W and S.

The figures for Further Mathematics do not show too much since only three Boards held an "S" level examination in this subject, and only two could boast of a substantial entry. The figures are given in Table 3.15.
The statistics for Pure Mathematics are less disturbing than those for Mathematics, showing much less variation about the mean (see Table 3.16). It should be emphasised that many schools use the University of London School Examination Council's examinations in Pure Mathematics for their 'single-subject' candidates. This can be proved by examining the statistics of Applied Mathematics for the same Board (see Table 3.17). The number of candidates involved must exceed 4000 at "A" level, and over 300 "S" level Pure Mathematics candidates do not also take the Applied Mathematics examination at the same level. While similar conclusions can be drawn for other Boards, the numbers concerned are much smaller.

<table>
<thead>
<tr>
<th>Board</th>
<th>No. of A entries</th>
<th>No. of S entries</th>
<th>No. awarded Distinction (1)</th>
<th>No. awarded Merit (2)</th>
<th>% of S entry awarded supplementary grades</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>859</td>
<td>89</td>
<td>21</td>
<td>26</td>
<td>52.8%</td>
</tr>
<tr>
<td>C</td>
<td>109</td>
<td>19</td>
<td>5</td>
<td>7</td>
<td>12/19</td>
</tr>
<tr>
<td>O&amp;C</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>L</td>
<td>326</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>AEB</td>
<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>S</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>W</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>JMB</td>
<td>2049</td>
<td>179</td>
<td>28</td>
<td>29</td>
<td>31.8%</td>
</tr>
<tr>
<td>SMP</td>
<td>491</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>MEI</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>TOTAL</td>
<td>3842</td>
<td>287</td>
<td>54</td>
<td>62</td>
<td>40.4%</td>
</tr>
</tbody>
</table>

Table 3.15  Distribution of A and S entries amongst the 10 Examining Boards or Projects
Subject: Further Mathematics
<table>
<thead>
<tr>
<th>Board</th>
<th>No. of A entries</th>
<th>No. of S entries</th>
<th>No. awarded Distinction (1)</th>
<th>No. awarded Merit (2)</th>
<th>% of S entry awarded supplementary grades</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>319</td>
<td>8</td>
<td>0</td>
<td>1</td>
<td>1/8</td>
</tr>
<tr>
<td>C</td>
<td>1106</td>
<td>232</td>
<td>47</td>
<td>72</td>
<td>51.3%</td>
</tr>
<tr>
<td>O&amp;C</td>
<td>374</td>
<td>118</td>
<td>32</td>
<td>30</td>
<td>52.5%</td>
</tr>
<tr>
<td>L</td>
<td>8716</td>
<td>446</td>
<td>72</td>
<td>128</td>
<td>44.8%</td>
</tr>
<tr>
<td>AEB</td>
<td>2085</td>
<td>54</td>
<td>8</td>
<td>11</td>
<td>35.2%</td>
</tr>
<tr>
<td>S</td>
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<td>*</td>
</tr>
<tr>
<td>JMB</td>
<td>655</td>
<td>16</td>
<td>4</td>
<td>3</td>
<td>7/16</td>
</tr>
<tr>
<td>SMP</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>MEI</td>
<td>307</td>
<td>102</td>
<td>22</td>
<td>29</td>
<td>50.0%</td>
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<tr>
<td>TOTAL</td>
<td>13562</td>
<td>976</td>
<td>185</td>
<td>274</td>
<td>47.0%</td>
</tr>
</tbody>
</table>

Table 3.16  Distribution of A and S entries amongst the 10 Examining Boards or Projects
Subject: Pure Mathematics

The one remarkable feature of the Applied Mathematics statistics is the enormous discrepancy between one of the examinations and the remainder. Once again one must conclude that there is either a difference in the quality of entry between the Boards or a difference in standard of the examinations (or both). Since the syllabuses of the various Boards do not exactly coincide for any particular examination, practice amongst schools on "S" level entry may be different. Thus it seems worthwhile to regard the total numbers of "S" entries in mathematical subjects for each Board; in this way an overall picture is obtained for each Board.
<table>
<thead>
<tr>
<th>Board</th>
<th>No. of A entries</th>
<th>No. of S entries</th>
<th>No. awarded Distinction (1)</th>
<th>No. awarded Merit (2)</th>
<th>% of S entry awarded supplementary grades</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>C</td>
<td>862</td>
<td>132</td>
<td>26</td>
<td>46</td>
<td>54.5%</td>
</tr>
<tr>
<td>O&amp;C</td>
<td>338</td>
<td>87</td>
<td>25</td>
<td>18</td>
<td>49.4%</td>
</tr>
<tr>
<td>L</td>
<td>3962</td>
<td>111</td>
<td>19</td>
<td>32</td>
<td>45.9%</td>
</tr>
<tr>
<td>AEB</td>
<td>718</td>
<td>21</td>
<td>6</td>
<td>6</td>
<td>12/21</td>
</tr>
<tr>
<td>S</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
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<td>W</td>
<td>284</td>
<td>5</td>
<td>*</td>
<td>*</td>
<td>*</td>
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<td>JMB</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>SMP</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>MEI</td>
<td>284</td>
<td>90</td>
<td>31</td>
<td>42</td>
<td>81.1%</td>
</tr>
<tr>
<td>TOTAL</td>
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<td>441</td>
<td>107</td>
<td>144</td>
<td>56.9%</td>
</tr>
</tbody>
</table>

Table 3.17  Distribution of A and S entries amongst the 10 Examining Boards or Projects

Subjects: Applied Mathematics
<table>
<thead>
<tr>
<th>Board</th>
<th>No. of A entries</th>
<th>No. of S entries</th>
<th>No. awarded Distinction (1)</th>
<th>No. awarded Merit (2)</th>
<th>% of S entry awarded supplementary grades</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>5736</td>
<td>420</td>
<td>55</td>
<td>69</td>
<td>29.5%</td>
</tr>
<tr>
<td>C</td>
<td>6853</td>
<td>631</td>
<td>137</td>
<td>194</td>
<td>52.5%</td>
</tr>
<tr>
<td>O&amp;C</td>
<td>2766</td>
<td>417</td>
<td>124</td>
<td>117</td>
<td>57.8%</td>
</tr>
<tr>
<td>L</td>
<td>16701</td>
<td>566</td>
<td>91</td>
<td>162</td>
<td>44.7%</td>
</tr>
<tr>
<td>AEB</td>
<td>5134</td>
<td>75</td>
<td>14</td>
<td>17</td>
<td>41.3%</td>
</tr>
<tr>
<td>S</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>W</td>
<td>2111</td>
<td>119</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>JMB</td>
<td>13907</td>
<td>994</td>
<td>126</td>
<td>199</td>
<td>32.7%</td>
</tr>
<tr>
<td>SMP</td>
<td>5091</td>
<td>504</td>
<td>75</td>
<td>127</td>
<td>40.1%</td>
</tr>
<tr>
<td>MEI</td>
<td>1760</td>
<td>278</td>
<td>78</td>
<td>88</td>
<td>59.7%</td>
</tr>
<tr>
<td>TOTAL</td>
<td>57948</td>
<td>3885</td>
<td>700</td>
<td>973</td>
<td>43.1%</td>
</tr>
</tbody>
</table>

Table 3.18 Distribution of A and S entries in mathematical subjects amongst the 10 Examining Boards or Projects

The overall picture is, in fact, quite disturbing at first sight. The figures displaying the percentage of the "S" entry being awarded a supplementary grade indicate a wide variation about the mean of 43%, ranging from 30% to 60%. What is not quite so apparent from the table is the wide range of the proportions of "S" entries to "A" entries from Board to Board. The 3885 "S" entries represent 6.7% of the 57948 "A" level entries. But they range from 15.8% for MEI and 15.1% for O & C down to 3.4% for L and a mere 1.5% for AEB. The low figure for AEB may partly be explained by the fact that many Further Education establishments use this Board. The London Board's figures may be lower than other
Boards for two reasons. First, in 1973, it permitted candidates to offer only one Special paper whereas most other Boards allowed their candidates to offer two. Second, the London Board did not set Special papers for a number of their syllabuses.

A further disturbing point is that the three Boards awarding the highest percentages of supplementary grades to "S" level candidates also had very high proportions of their "A" level entry opting to sit the examinations. Quite amazingly (for one might reasonably expect a negative correlation between these two rankings), Spearman's rank correlation coefficient $\rho = +0.55$ ($p = 0.05$, using a one-tailed test) for the eight Boards about which complete information is available. This supports the argument that the differences in the figures in the final column of Table 3.18 are not due to schools of one Board entering for the Special papers a greater proportion of inferior candidates than schools of another Board. There is either a difference in standard of the candidates of the Boards or a difference in the standard of the examinations. The latter possibility is very difficult to test analytically, although attempts have been made by J. F. Scott (1975) for the Schools Council to do this at "A" level (not "S" level).

The author does not have the means, the time, or the authority to conduct a similar investigation at "S" level. However, a subjective comparison between the various syllabuses can be attempted by examining their contents and by discussing examination questions on particular topics. This will be done in the next chapter. Indeed, since "S" level mathematics and Oxbridge entrance are so closely connected, the comparison will include questions from the Oxbridge Entrance Examinations.

Before proceeding, however, there is one further area of "S" level statistics which may be of interest. Some Boards publish figures on the numbers of boys and girls entering "S" level and "A" level
examinations. Table 3.19 contains these figures (where known).

<table>
<thead>
<tr>
<th>Board or Project</th>
<th>Boys</th>
<th></th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of A entries</td>
<td>No. of S entries</td>
<td>S entry as percentage of A entry</td>
</tr>
<tr>
<td>O</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>C</td>
<td>5060</td>
<td>492</td>
<td>9.7%</td>
</tr>
<tr>
<td>O&amp;C</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>L</td>
<td>12948</td>
<td>412</td>
<td>3.2%</td>
</tr>
<tr>
<td>AEB</td>
<td>4421</td>
<td>65</td>
<td>1.5%</td>
</tr>
<tr>
<td>S</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>W</td>
<td>1623</td>
<td>88</td>
<td>5.4%</td>
</tr>
<tr>
<td>JMB(^a)</td>
<td>10714</td>
<td>839</td>
<td>7.8%</td>
</tr>
<tr>
<td>SMP</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>MEI</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>TOTAL</td>
<td>34766</td>
<td>1896</td>
<td>5.5%</td>
</tr>
</tbody>
</table>

Table 3.19 Distribution of A and S entries in mathematical subjects between boys and girls

\(^a\) The JMB figures include 642 "A" level entrants to SMP examinations and 19 "A" level entrants to MEI examinations conducted by the Board.

The table shows that (where the figures are known) there is no appreciable difference between the "S" entry for boys regarded as a percentage of the "A" entry and that for girls. Further, the 10601 "A" level girl candidates represented 23.4% of the total "A" level entry while the 516 female "S" level candidates formed 21% of the total "S"
level entry. Thus these figures do not support the view that girls do not wish to go on to do "S" level. "A" level mathematics may be more popular with boys than girls but "S" level does not alter the 'popularity ratio' (if I am allowed to use the term).

This chapter has been used to discuss why teachers enter candidates for "S" level mathematics. It was found that most teachers used the examination to set the more able "A" level student harder questions. There was also a large body of teachers supporting the contention that "S" level provided suitable practice for Oxbridge candidates. Some entered candidates for the examination because they believed it provided an extra qualification for universities, particularly for Oxford and Cambridge, but most dismissed it as a qualification for provincial universities. Correspondence with Tutors for Admissions from Oxbridge colleges confirmed that "S" level mathematics is taken into account when assessing candidates and is regarded with some importance. Examination of statistics issued by the G.C.E. Boards showed there was a discrepancy between the Boards on the percentages of "S" entries awarded supplementary grades and on the "S" entries regarded as a percentage of "A" entries.
CHAPTER 4

THE MATHEMATICAL CONTENT OF "S" LEVEL SYLLABUSES

4.1 General Discussion

It is the author's intention in this chapter to present an analysis of the content of the various "S" level syllabuses. There are so many factors involved that a full analysis becomes impossible in a dissertation of this length. For example, one of the questions which will not be formally answered is whether "S" level is indeed harder than "A" level — although a discussion of "S" level questions taken from the examinations of various Boards may well convince the reader that this is the case! Nor will it be possible to consider all topics in "S" level mathematics. There is such a wealth of material that it is impossible to discuss it all. We restrict ourselves to what generally comes under the heading of Pure Mathematics. (Indeed the author would not feel completely happy about initiating discussion about the difficulty of "S" level or Oxbridge questions in mechanics.)

The chapter is organised in two sections apart from this general discussion. The 'single' subject and the 'double' subject are treated separately. There can be little doubt that an examination for a candidate who has studied mathematics as a single subject will necessarily be incompatible with an examination for a candidate who has spent twice the time on mathematical work. This is illustrated by the Report on the work in mathematical subjects, 1971, published by the Cambridge Syndicate (1972):

"A comment is perhaps desirable concerning choice of subject for the candidate who takes only one mathematical subject at A level. Most of these candidates choose the subject
Mathematics, but a minority (approximately 300 in 1971, half of them girls) choose Pure Mathematics. Analysis of the results of these candidates shows that those who took Pure Mathematics generally achieved worse grades in this subject than in others; on the other hand, the grades in the subject Mathematics of those candidates taking it as their only mathematical subject were strictly comparable with their other grades. It seems that those who take Pure Mathematics inevitably suffer in comparison with candidates who are spending probably twice as much time on mathematical work."

This quotation, although not specifically on "S" level, not only indicates the difference between the 'single' subject and the 'double' subject, but also exemplifies the choice available to schools. (For both subjects, Special papers were available.) The variety of material examined at "S" level is enormous. So that the reader may be made fully aware of the range, the author made a list of all topics examined at "S" level in the subject Pure Mathematics in the year 1973:

1. Equations and Identities.
2. Induction.
4. Complex Numbers.
5. Hyperbolic functions.
6. Inverse trigonometric functions.
7. Inequalities.
8. Series and expansions.
9. Recurrence relations.
10. Limits.
11. Differentiation (including max. and min.).
12. Curve-sketching.
13. Partial differentiation.
15. Integration (methods of).
17. Areas and volumes of rotation.
20. Iteration.
22. Gaussian elimination.
23. Matrices.
24. Determinants.
27. Eigenvectors and eigenvalues.
28. Coordinate geometry.
29. Geometry.
30. Projective geometry.

The list is not exhaustive. Quite clearly some topics were more popular than others; for example, all but one Board set a question on complex numbers, whereas there was only one question on number theory. It is equally obvious that, with an average of ten or eleven questions on each paper, every Board failed to examine many of these topics. Some of the topics which a Board did not examine would lie outside its syllabus. There are, however, many topics in every syllabus which remained unexamined. This does not mean that they were not included
in either the previous year's or the following year's papers. It is
this problem of sampling from large syllabuses that makes comparison
between the Boards so difficult.

Another factor which makes comparison difficult is the style of
the examinations. Although all the examinations in Pure Mathematics in
1973 were three hour papers, the rubrics were very different (see Table
4.1).

<table>
<thead>
<tr>
<th>Board or Project</th>
<th>Maximum number of questions to be attempted</th>
<th>Number of questions on paper</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>8</td>
<td>14</td>
</tr>
<tr>
<td>O&amp;C</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>L</td>
<td>8</td>
<td>15*</td>
</tr>
<tr>
<td>AEB</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>S</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>W</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>JMB</td>
<td>6</td>
<td>14+</td>
</tr>
<tr>
<td>SMP</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>MEI</td>
<td>7</td>
<td>13a</td>
</tr>
</tbody>
</table>

Table 4.1 Rubrics in Pure Mathematics "S" papers, 1973

* Candidates had a choice of only 10 questions dependent on which "A" level papers they took.

+ The paper contained 8 short questions each carrying half the marks of any of 6 long questions. Candidates could attempt the equivalent of 6 long questions.

a Candidates were expected to answer questions outside the "A" level syllabus as well as inside in order to obtain a Distinction.
Although, as has been mentioned, it is not the intention to verify that "S" level questions are harder than "A" level (this will be assumed), an attempt is made to discuss the difficulty of "S" level questions. Indeed this is the crux of any discussion about "S" level. What makes a question difficult? In the exposition reference will be made to Chapter 2 on educational theory.

In Chapter 3 it was found that many candidates were entered for the examination partly because their teachers thought the papers provided suitable practice for potential Oxbridge candidates. Oxbridge Tutors for Admissions agreed that they too thought the examination provided suitable practice. Moreover they took considerable notice of performances in "S" level. Thus it seems imperative to include some discussion of Oxbridge Entrance Examination questions.

The method of approach adopted will be to give an outline of the material common to most "S" level syllabuses, to mention some of the topics that distinguish certain courses, and to examine in depth some isolated themes by considering questions from recent past papers. The connection with Oxbridge work will be highlighted by considering questions from Entrance Examinations which are relevant to those "S" level questions being discussed. The topics discussed will fall under the general heading of pure mathematics - partly to save space and partly because the author does not feel competent to analyse the difficulties of "S" level mechanics questions.

4.2 The 'Single-subject' Syllabuses

There is no simple answer to the question "what is contained in the 'single-subject' syllabus?". There are nine "S" level syllabuses for which there is a substantial 'single-subject' entry. Five of these may be termed traditional while four are modern. (I do not propose to
define exactly what I mean by these terms, but it is essentially a
difference in syllabus content — geometry is traditional while vectors,
groups, and numerical methods are modern. A further distinguishing
feature is that modern syllabuses are broader while traditional
syllabuses tend to look more deeply at a topic.) The nine syllabuses
are thus divided into two camps. But the situation is worse than this.
For, within each syllabus, there are topics which distinguish it from
another in the same camp.

A second possible method of classifying syllabuses is to examine
separately those which are based on the "A" level syllabus alone and
those which contain additional material. Unfortunately this too leads
to differences in the content of syllabuses in the same class.

Despite these difficulties it is possible to give the reader some
idea of the content of the syllabuses, which contain a common core of
work. Thus all Boards use questions involving differentiation (including
maximum/minimum and curve sketching), methods of integration, and
equations, identities and inequalities. Most set questions on complex
numbers. Coordinate geometry is a favourite amongst the traditional
syllabuses. The modern syllabuses favour vectors, groups, differential
equations, numerical methods, number theory and partial differentiation,
although not all these topics are their sole province.

The Oxbridge Entrance Examinations provide no problem when discuss­
ing their syllabuses. Although there are no formal syllabuses for these
examinations, past papers indicate that if a syllabus were published it
would approximately be the union of all "S" level syllabuses, together
with a few topics of a more general mathematical nature. Thus most "S"
level questions have their counterparts in Oxbridge papers. This does
not mean that the questions at "S" level and those in Oxbridge examina­
tions are similar (merely that they cover the same topic). It has yet
to be decided how much alike the two examinations are. In particular, attention will be paid to the difficulty of "S" level in comparison with the Oxbridge Entrance Examinations.

Let us first consider differentiation as treated at "S" level. The questions can be divided into two classes – those which involve maxima and minima, and those which are more concerned with curve sketching. It is true, of course, that curve sketching often involves finding stationary points, but it is hoped that the reader will appreciate the difference in due course.

Curve sketching has always been a popular topic in the sixth form, and many past papers reflect this. As a first example, consider the following "S" level question:

"Sketch the curve

\[ y = \frac{(x - 1)(x - 4)}{(x - 2)(x - 3)} \]

and also, using this curve if you wish and explaining your argument carefully, the curve

\[ y^2 = \frac{(x - 1)(x - 4)}{(x - 2)(x - 3)} \]

[C, Syllabus A]

The first part of the question would cause most candidates with any experience of this type of question little difficulty – provided they remembered the undefined points 2 and 3 and correctly determined the behaviour of y for large values of x. The differentiation involved is straightforward (with much cancellation to relieve candidates) and there is only one stationary point at \( x = \frac{5}{2} \), which must be a minimum. Thus most candidates would have obtained:
Fig. 4.1  Sketch of $y = (x - 1)(x - 4)/(x - 2)(x - 3)$

The second part of the question requires two realisations on the part of the candidate. First that the function is undefined between 1 and 2 and between 3 and 4 (where the values of $y$ in his first sketch were negative), and second that the new graph is symmetrical about the $x$-axis. The solution is now trivial.
Fig. 4.2 Sketch of $y^2 = (x - 1)(x - 4)/(x - 2)(x - 3)$

A second example of curve-sketching can be taken from any of a number of the Welsh Joint Education Committee's Special papers in Mathematics. The following is typical:

"Sketch the graphs of the following functions, showing the important features:
(i) \( y = \frac{1}{x} \sin \pi x \),

(ii) \( y = \frac{x^2 + x - 6}{x^2 - 1} \),

(iii) \( r = a \sin 3 \theta \) where \((r, \theta)\) are the usual polar coordinates.

(Detailed calculations are unnecessary.)

At first sight this question appears harder than the first example quoted. The second part of this question bears a strong resemblance to the first part of the previous example, but there are a greater number of ways a candidate may go wrong on the other parts of the second example. However, the rubrics on the two papers were different (candidates could attempt only seven on the Welsh paper but eight in the Cambridge examination); this makes a straightforward comparison hazardous.

The first part of the question has many hidden obstacles for the unwary. No doubt most candidates would soon establish that the function takes zero values at all non-zero integers. (There would also no doubt be some who would claim \( y(0) = 0 \).) Many would show that there is exactly one turning value between two consecutive integers, but one has to ask how many would spot that \( y = \frac{1}{x} \sin \pi x \) is an even function, that it tends to 0 for large values of \( x \), and how many would correctly interpret its behaviour near 0.
Fig. 4.3 Sketch of $y = \frac{1}{x} \sin \pi x$

The second part of the question would cause few problems — except possibly the behaviour between 1 and infinity. A candidate would need to be careful in order to include the maximum (see Fig. 4.4).

The last part of the question has its problems in that candidates must realise that the function is undefined for values of $\theta$ in the ranges $(\frac{\pi}{3}, \frac{2\pi}{3})$, $(\pi, \frac{4\pi}{3})$ and $(\frac{5\pi}{3}, 2\pi)$. 
Fig. 4.4 Sketch of \( y = \frac{x^2 + x - 6}{x^2 - 1} \)

Fig. 4.5 Sketch of \( r = a \sin 3 \theta \)
Curve sketching is also a regular feature of Oxbridge Entrance Examinations, as the following two examples show.

"Show that \( x^{-1} \sin x \) has just one turning point between \( x = n \pi \) and \( x = (n + \frac{1}{2}) \pi \), where \( n \) is a positive integer. Show that these turning points are all maxima or minima, and find which is which. Sketch the graphs of \( y = x^{-1} \sin x \) and \( y = \frac{1}{x} \) between \( x = 0 \) and \( x = 3\pi \) to the same scale on the same diagram and mark the positions of the turning points of \( x^{-1} \sin x \).

(You may assume that \( x^{-1} \sin x \to 1 \) as \( x \to 0 \))."

[O.S.]

Quite clearly there is no need to spend any time discussing this after considering the last example. The second Oxbridge example is rather similar to two already considered.

"Sketch the function \( y = \frac{(x - a)(x - b)}{(x - c)} \)
for the cases \( a < c < b \) and \( c < a < b \). If \( c < a < b \) show that \( y \) cannot lie within a certain range of values, where the length of this range is \( 4 \sqrt{(a - c)(b - c)} \).

[C.S.]

Candidates attempting this question who had also succeeded in doing one or other of the two Special paper questions quoted would have little trouble with the first part.
Fig. 4.6 Sketch of $y = \frac{(x - a)(x - b)}{(x - c)}$ with $a < c < b$

Fig. 4.7 Sketch of $y = \frac{(x - a)(x - b)}{(x - c)}$ with $c < a < b$
The second part of the question is not difficult once the candidate has realised that what is required is the distance between the minimum and maximum values. But it does require considerable manipulative skill and a high degree of accuracy. An alternative method which might impress the examiners is to transform the equation into \((x - c)y = x^2 - (a + b)x + ab\), to apply the condition "\(B^2 \geq 4AC\)" for real roots of a quadratic to obtain \(y^2 + 2(a + b - 2c)y + (a - b)^2 \geq 0\) and then to deduce that \(y\) cannot lie in the range \((a + b - 2c) - 2\sqrt{(a - c)(b - c)}, (a + b - 2c) + 2\sqrt{(a - c)(b - c)}\) which is of the required length.

Manipulative skill also features in some "S" level questions on differentiation. Consider the following SMP question:

"Let \(f\) be the function which is defined for all real \(x\) by

\[f(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} .\]

By considering the derivative of \(f\), or otherwise, prove that the equation \(f(x) = 0\) has no real roots.

How many real roots has the equation

\[1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} = 0 ?\]

Can you suggest any general result concerning the numbers of real roots of polynomials similar to those above but of higher order?"

[SMP]

Most candidates would successfully find \(f'(x) = -(1 - x)(1 + x^2)\) and prove that \(f(x)\) had a single turning value (a minimum) at \(x = 1\). By examining \(f(1) \geq 0\) they could conclude that \(f(x) = 0\) has no real roots. The first part of the question would undoubtedly lead many
candidates to consider a function \( g(x) = f(x) - \frac{x^5}{5} \) in an attempt to solve the second part. But the expression for \( g'(x) \) is more difficult to deal with. In fact

\[
g'(x) = -1 + x - x^2 + x^3 - x^4.
\]

If candidates could express this in the form

\[
g'(x) = \begin{cases} 
-\frac{x^5 + 1}{x + 1} & \text{if } x \neq -1 \\
-5 & \text{if } x = -1
\end{cases}
\]

then they could easily conclude that \( g'(x) \) never vanishes and moreover that it is negative for all values of \( x \). Since \( g(x) \to +\infty \) as \( x \to -\infty \) and \( g(x) \to -\infty \) as \( x \to +\infty \) it follows that there is exactly one real root of \( g(x) = 0 \). But to see that \(-1 + x - x^2 + x^3 - x^4 \equiv -\frac{5 + 1}{x + 1} \) for \( x \neq -1 \) requires either considerable manipulative skill or a good memory; it also requires some ingenuity - for it is not clear that the new expression is any better than the old. Indeed this part of the question illustrates very well the dual nature of mathematics, intuition and abstraction, discussed in Chapter 2.

A candidate who has proceeded thus far would have no difficulty in generalising the results to complete the question.

The discussion on differentiation at "S" level is concluded by looking at some examples on maxima and minima which do not involve curve sketching.

"If \( y \) is defined in terms of \( x \) by the equation

\[
x^3 + y^3 = 3axy,
\]

where \( a \) is constant, prove that \( y \) has a turning value when \( x = 2^{1/3}a \). Determine whether this value is a maximum or a minimum."
This is another example which has many traps for the unsuspecting - including the examiner. The implicit differentiation is easy, but the candidate must remember to consider the case \( a = 0 \) separately. [If this is done then it is easy to show that \( y \) has no turning value when \( a = 0 \).] If \( a \neq 0 \) then \( \frac{dy}{dx} = 0 \) when \( ay = x^2 \) from which it follows that \( x = \frac{2}{3}a \). The candidate has the choice between examining the sign of \( \frac{dy}{dx} \) near \( x = \frac{2}{3}a \) and obtaining \( \frac{d^2y}{dx^2} \) in order to determine whether the turning value is a maximum or a minimum. The first is fraught with difficulty. The second can be tedious for the candidate who is not fully awake.

Since

\[
\frac{dy}{dx} = \frac{(ay - x^2)}{(y^2 - ax)}
\]

it follows that

\[
\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(\frac{dy}{dx} - 2x\right) - (ay - x^2)(2y\frac{dy}{dx} - a)}{(y^2 - ax)^2}
\]

The unsuspecting candidate would simplify this expression before substituting \( x = \frac{2}{3}a, y = \frac{2}{3}a \) to find \( \frac{d^2y}{dx^2} \). The good candidate, on the other hand, will see there is no point in evaluating the denominator (which is positive), will recall that, at \( x = \frac{2}{3}a, ay = x^2 \) and \( \frac{dy}{dx} = 0 \). From these remarks it immediately follows that

\[
\frac{d^2y}{dx^2} = \frac{-2^{4/3}a \left(2^{4/3}a^2 - 2^{1/3}a^2\right)}{(y^2 - ax)^2}
\]

\[
= \frac{-2^{5/3}a^3}{(y^2 - ax)^2}
\]

which is negative if \( a > 0 \) and positive if \( a < 0 \). Even this is a trap; for many candidates expect a mathematics answer to be one thing or the other and not 'it depends'.
A second example of this type of question is the following set by JMB:

"A tank is to be made consisting of a cylinder of radius \( r \) and height \( h \), closed at one end by a disc of radius \( r \) and at the other end by a hemisphere of radius \( r \) with its convex side outwards. The material is of negligible thickness. The capacity and surface area of the tank are denoted by \( V \) and \( S \), respectively. Show that if \( S \) is fixed the maximum value of \( V \) is obtained by taking \( h = r \) and that if \( V \) is fixed the minimum value of \( S \) is obtained by taking \( h = r \).

Show also that if both ends of the tank are hemispherical, with their convex faces outwards, the maximum value of \( V \) for fixed \( S \) is obtained by taking \( h = 0 \)."

[JMB Syllabus A]

Where most candidates go wrong in answering this sort of question is in writing down initial expressions for \( V \) and \( S \). A few also make the mistake of thinking that \( h \) is constant! But there is nothing intrinsically difficult about the question.

Once again this type of question appears in Oxbridge Entrance Examinations. For example, the following Oxford Scholarship question is easy once the expression for total profit has been written down.

"A baker finds that the number \( N \) of cakes he sells each day is related to the selling price \( Q \) pence of each cake by \( N = N_0 \ e^{-\alpha Q} \) where \( N_0 \) and \( \alpha \) are constants. Each cake costs \( C \) pence to produce. What is the selling price which maximises the total profit if \( \alpha C = 1 \)?"

[O.S.]
Another topic which is popular at "S" level is complex numbers. Most questions do not test computational skills although algebraic manipulation is necessarily a feature of some. The great majority test geometrical reasoning, as the next example shows:

"If the points $P_1$ and $P_2$ correspond in the Argand diagram to the complex numbers $z_1$ and $z_2$, prove that the area of the triangle $OP_1P_2$, taken with the positive sign if the anticlockwise description of the triangle gives the points $0, P_1, P_2$ in that order, and the negative sign otherwise, is $\frac{1}{4} i (z_1 \bar{z}_2 - \bar{z}_1 z_2)$, where $\bar{z}_1, \bar{z}_2$ are the conjugates of $z_1, z_2$. ($\bar{z}$ is called the conjugate of $z$ if $\bar{z} = x - iy$ when $z = x + iy$.) Hence, or otherwise, prove that if $P_3$ corresponds to $z_3$, the area of the triangle $P_1P_2P_3$, with the same sign convention as before, is

$$\frac{1}{4} i \left[(z_3 \bar{z}_2 - \bar{z}_2 z_3) + (z_3 \bar{z}_1 - \bar{z}_3 z_1) + (z_1 \bar{z}_2 - \bar{z}_1 z_2)\right].$$"

The first part of the question would be 'book-work' for many of those candidates who received separate tuition for the examination. But, as was evident from the response to the questionnaire, many candidates do not receive special attention; for these candidates the question would be more difficult. If one thinks of the triangle $OP_1P_2$ as being contained in a rectangle, then the manipulation involved is relatively simple and the result follows. An appeal to symmetry, which less able pupils always seem reluctant to make ("that's cheating" echoes around the classroom walls!), yields the sign change if $P_1$ and $P_2$ are interchanged.
The final part of the question is straightforward for the candidate who has proceeded along these lines. The essence of the solution is contained in Fig. 4.9.

**Fig. 4.8** The triangle $O_{P_1}P_2$ contained in the rectangle $OLMN$

**Fig. 4.9** Relationship between $\Delta P_1P_2P_3$ and the triangles $O_{P_1}P_2$, $O_{P_2}P_3$ and $O_{P_3}P_1$
The second example of complex numbers appeared in the same year as the first.

"Four complex numbers are defined by the relations

\[ z_1 = x_1 + iy_1, \quad \bar{z}_1 = x_1 - iy_1, \]
\[ z_2 = x_2 + iy_2, \quad \bar{z}_2 = x_2 - iy_2, \]

where \( x_1, y_1, x_2, y_2 \) are real. Prove that \( z_1 \bar{z}_2 + z_2 \bar{z}_1 \)
is real and that \( z_1 \bar{z}_2 - z_2 \bar{z}_1 \) is pure imaginary.

The triangle \( \triangle OP_1 P_2 \) in an Argand diagram has vertices at the origin \( O \) and at the points corresponding to \( z_1, z_2 \).

Prove that the numerical value of the area of the triangle is \( \frac{1}{4} \left| z_1 \bar{z}_2 - z_2 \bar{z}_1 \right| \).

\[ C, \text{ Syllabus B} \]

The coincidence is remarkable! At least, in this case, the questions are of a comparable standard!

To be more serious, these examples are yet another illustration of mathematics in its two forms of intuition and abstraction. Geometrical intuition is required on the part of the candidate in order to regard triangle \( \triangle OP_1 P_2 \) as a constituent part of the rectangle \( OLMN \), and to realise that the following abstraction will be straightforward.

Geometrical intuition is also a feature of Cambridge Scholarship papers. (Oxford pays less attention to complex numbers for 'single-subject' mathematicians.)
"(a) Show on a diagram how to interpret the product of complex numbers. What can be said of $z$ if $z^2 = z^2$?

(b) Prove that if $z_1$, $z_2$ and $z_3$ are complex numbers satisfying $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3|$, then they represent the vertices of an equilateral triangle in the complex plane."

[c.s.]

The first part of the question requires only geometrical intuition, once the book-work has been completed. If, however, a candidate were asked to prove his assertion then the following process of abstraction would be necessary:

$$z^2 = z^2 \Rightarrow x^2 - y^2 + 2jxy = x^2 - y^2 - 2jxy \Rightarrow 4xy = 0 \Rightarrow x = 0 \text{ or } y = 0.$$  

The second part is really a test of geometrical knowledge. Even so abstraction can play a role in its solution. The author solved the problem as follows. The second condition implies that the three points $P_1$, $P_2$, $P_3$ representing $z_1$, $z_2$, $z_3$ all lie on a circle centre the origin. The first condition can be manipulated to give $z_3 = -(z_1 + z_2)$. This suggested looking at the point $Q_3$ representing $z_1 + z_2$. Once it was established that triangle $OQ_3P_2$ is equilateral the remainder of the solution was simple.
Geometrical intuition does not always feature so strongly in "S" level or in Scholarship questions, as the following examples suggest.

"Define the complex conjugate $z^*$ of a complex number $z$ and prove that $z \cdot z^* = |z|^2$.

In the Argand diagram, the numbers $z$, $z^*$ are represented by the points $P$, $Q$ respectively. Indicate on the diagram a length which is equal to $|z - z^*|$.

In the equation $z^5 - a^5 = 0$, $a$ is real.

(a) Write down in modulus and argument form the roots of this equation.

(b) Let $aw$ be the root of least positive argument; prove that the other complex roots are $aw^2$, $aw^3$, and $aw^4$ and that $1 + w + w^2 + w^3 + w^4 = 0$. 
(c) Let the roots be represented by the points A, B, C, D, E in the Argand diagram. Prove that
\[ PA^2 + PB^2 + PC^2 + PD^2 + PE^2 = 5(a^2 + |z_0|^2), \]
where P represents the complex number \( z_0 \).

The good candidate will recognise at once that he is not being asked anything so simple as the first two parts of the question for nothing. Whether the examiner intended the candidate to consider separately the two cases \( a > 0 \) and \( a < 0 \), or whether it was an oversight, does not alter the fact that parts (a) and (b) are more difficult than they appear (as the question stands). If the candidate realised that the first parts of the question had not yet been used, then part (c) would be simple.

"(a) If \( |z_1| = |z_2| \) where \( z_1 \) and \( z_2 \) are complex numbers, prove that the complex number \( (z_1 + z_2)/(z_1 - z_2) \) is purely imaginary.

(b) Give the modulus and argument of each root of the equation \( z^3 + 27 = 0 \) and show the three roots in an Argand diagram.

(c) Show that the points in the Argand diagram corresponding to complex numbers \( z \) satisfying the equation
\[ |z - 4| = 2 |z - 1| \]
all lie on the circle \( |z| = 2 \)."
"Sketch in an Argand diagram the locus of \( z \) for which
\[
(i) \quad |z - 1| = |z - 2|, \quad (ii) \quad |z - 1| = 2.
\]
The complex number \( w \) is derived from \( z \) by the formula
\[
z = \frac{w - 1}{w - 2}.
\]
Sketch in another Argand diagram the two loci of points \( w \) corresponding to the relations (i) and (ii) respectively."

Both these questions require some manipulative skill, but they also
require the candidate to choose an appropriate method. For example,
many candidates would choose to solve the first part of the first
question by multiplying numerator and denominator by \((z_1 - z_2)^*\). This
leads to some complicated algebra, although the result does follow. A
better approach is to use the fact that \(|z_1| = |z_2|\) to prove that
the points representing \( z_1, z_2 \) and \( z_1 + z_2 \) together with the origin form
a rhombus, and thence to show that the line joining the points represent-
ing \( z_1 + z_2 \) and \( z_1 - z_2 \) subtends a right angle at the origin. From this
it follows that \( z_1 + z_2 = kj(z_1 - z_2) \) for a suitable real constant \( k \).

The section continues with several further examples on various
topics which are regular contributions to the papers of the Board from
which they are taken (but not necessarily to other Boards' papers). No
comment is made about the difficulty of individual examples, but similar
Oxbridge questions are given after each.

"(a) Solve, subject to the conditions \( y = 1 \) and \( \frac{dy}{dx} = 0 \)
at \( x = 0 \), each of the differential equations
\[
(i) \quad \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 2y = 5 \cos x,
\]
(ii) \((1 + x) \frac{dy}{dx} - xy = x\).

(b) The variables \(x, y, z\) are connected by the relation \(x^2y + 3y^2z - 2z = 0\). If \(u = xy^2z^3\) and if \(x\) and \(y\) are taken as independent variables, find the value of \(\frac{du}{dx}\) when \(x = y = 1\) and \(z = -1\)." [M.E.I.]

"(i) Find the general solution of the differential equation

\[
\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 3y = 0.
\]

(ii) Find the solution of the differential equation

\[
\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 3y = x^2
\]

which passes through the point \((0, 0)\) and has derivative zero at that point." [C.S.]

"In a group \((G, \cdot)\), an element \(a\) of \(G\) is said to be of 'order n' if \(n\) is the smallest positive integer such that \(a^n = e\). (Thus the identity \(e\) is the only element of order 1.) Show that if \(a\) and \(b\) are distinct elements of order 2,

(i) \(a \cdot b\) is different from \(a, b,\) and \(e\);

(ii) \(a \cdot b \cdot a\) is different from \(a\) and \(e\);

(iii) either \(a \cdot b\) is of order 2 or \(a \cdot b \cdot a\) is different from \(b\)."
By considering the element \( a.b.a \), or otherwise, deduce that no group may have just two elements \( a,b \) of order 2. Is it possible for a group to have exactly three elements of order 2? Give your reason.

[S.M.P.]

"A group with multiplication as the binary operation contains distinct elements \( a \) and \( b \) for which \( ab = ba \) and \( a^2 = b^4 = e \) where \( e \) is the unit or neutral element of the group. Show that \( a^i b^j = b^j a^i \) and that \( a^i b^j = ab^k \) for some \( k = 1, 2 \) or 3 for all odd \( i \) and all \( j \) not a multiple of 4. Hence, or otherwise, show that the elements of the subgroup generated by \( a \) and \( b \) are \( e, a, b, b^2, b^3, ab, ab^2 \) and \( ab^3 \), assuming these elements are all distinct. Find the elements of three other subgroups."

[O.S.]

"A standard cell whose voltage is known to be 1.10 is used to test the accuracy of a voltmeter A. Eight independent readings of the voltage of the cell gave results: 1.11 1.15 1.14 1.10 1.09 1.11 1.13 1.15. Find 95 per cent confidence limits for the mean reading and discuss its difference from the standard.

Eight further readings were taken with a second voltmeter B and the same standard cell giving readings 1.12 1.06 1.02 1.08 1.11 1.05 1.06 1.08. Have the two voltmeters significantly different variance readings? Have they significantly different mean readings?"

[O & C]
"It is suspected that the foundations of a certain dam are moving. A surveyor takes a number of measurements of the bearing of a point on the dam from a fixed point on two occasions a week apart. The values he obtains differ only in seconds of arc, the figures being as follows:

<table>
<thead>
<tr>
<th>Number of observations</th>
<th>Mean reading</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>First set</td>
<td>6</td>
<td>72° 47' 36&quot;</td>
</tr>
<tr>
<td>Second set</td>
<td>8</td>
<td>72° 47' 34&quot;</td>
</tr>
</tbody>
</table>

Using the t-distribution, estimate the probability that the second mean differs from the first due to random errors in measurement. Comment accordingly on the significance of the two sets of measurements."

"Show that the line \( y = x - 1 \) touches the curve \( y = e^{kx} \) if \( k \) satisfies the equation

\[
k + 1 + \log_e k = 0.\]

Apply the Newton approximation formula to this equation and obtain the relation between two successive approximations to \( k \), denoted by \( k_n \) and \( k_{n+1} \). Supposing that \( \epsilon = k_n - k \) is small, expand \( k_{n+1} - k \) as a power series in \( \epsilon \). Neglecting terms of degree higher than 2, show that the error in \( k_{n+1} \) is approximately \( \frac{-\epsilon^2}{2k(k+1)} \)."

(J.M.B., Syllabus B)
"Find the largest root of the equation \( x = 2 + \ln x \) correct to 4 significant figures, using Newton's method and showing the details of your calculation. Illustrate the method by means of a suitable sketch.

By sketching graphs of \( y = x \) and \( y = 2 + \ln x \) show that the sequence of values defined by the recurrence relationship \( x_{n+1} = 2 + \ln x_n \) always converges to the root you have found, provided that the first value \( x_0 \) is large enough."

[C.S.]

The reader may have inferred from the examples that there is a one-to-one correspondence between "S" level and Oxbridge work. This is not so. Not all the Oxbridge questions reach "S" level standard. This is particularly true of questions in the Cambridge "Paper 910" and in the Oxford "Mathematics" paper, many of which are only "A" level in standard. Nor do all "S" level topics appear in Oxbridge Entrance Examinations. For instance, the method of pivotal condensation, which appeared as the theme of an MEI question in 1972, has not appeared in the 'single-subject' Oxbridge examinations. Nonetheless, there is a big overlap between "S" level and Oxbridge Entrance Examinations, and there appears to be a similar standard set.

The difficulty of an examination question depends on a number of factors. The concepts being tested may themselves be difficult to comprehend; that is, they may depend on a large number of "lower order" concepts. The question may involve a great deal of computational or manipulative work, where inaccuracy penalises a candidate. The question may require considerable intuition on behalf of the candidate. It will,
moreover, often require clear thinking from the candidate - reasoning ability of the 'if-then' type. The author believes that all the questions quoted illustrate some or all of these qualities.

To put it another way, the difficulty of a problem depends on the number of steps missing. Imagine trying to design and build even a small computer without the use of a reference manual! (A fifteen-year old boy I teach did exactly this.) One cannot pretend that this is easier than "S" level, but it does illustrate the difficulties which must be overcome - complete familiarity with concepts, accuracy in routine work, clear thinking and a liberal dose of intuition.

The human qualities necessary to overcome these difficulties may be thought of in three classes. First, the two domains of Bloom's taxonomy of educational objectives discussed in Chapter 2 are clearly relevant. One who is anxious to avoid the use of educational jargon might say an "S" level question requires a clear understanding of principles, an ability to think deeply, to reason, and an ability to carry through abstract manipulative work. In Bloom's terms the first three factors of the cognitive domain (knowledge, comprehension, and application) are certainly required and, depending on the particular question, so may the fourth. The corresponding factors in the affective domain must also be present.

A practising mathematics teacher will understand that these conditions alone are not sufficient to enable a candidate to answer "S" level questions. Necessary these conditions are, sufficient they are not! Another necessary quality is the ability to see which principles to apply. In a difficult question there are often a number of avenues along which a candidate can approach the solution, and a greater number along which he will get lost. What he must do is to tread a path which will lead him to that solution, and to choose (if possible) the most
elegant path. This ability I see as intuition, the creative aspect of mathematics. Even an examination question sometimes permits a mathematician to be creative!

There is, however, a third class of human qualities needed by a mathematician to succeed in "S" level and these come under the heading of personality. Mathematicians do not share many common personality traits, but determination is one 'they do.' Teachers will recognise that this quality is needed in some measure by "S" level students, not so much in the examination itself but in the preparation for it. This allied to self-confidence (at least in attempting mathematical solutions) are two qualities I see as important at "S" level.

Just as Bloom's taxonomy is to be thought of as finely interwoven, the author regards these three classes also as forming part of a whole. For instance, determination can lead to a student acquiring a wider and deeper understanding of mathematical concepts. This in turn may lead to an improvement in a student's mathematical intuition, which would undoubtedly increase his self-confidence. A greater self-confidence may, depending on other personality traits, either increase or decrease a student's determination.

The reader is invited to read the next section in the light of these remarks. An attempt will be made to outline their relevance to 'double-subject' questions in Pure Mathematics while, at the same time, some topics are examined in depth and comparisons made with the Oxbridge Scholarship papers.

4.3 The 'Double-subject' Syllabuses

The wide variety of topics examined in the Special papers titled "Pure Mathematics" has already been emphasised in Section 4.1. So too has the wide variety of rubrics. Although these factors could make a
comparison between Boards awkward, the author believes they can be overcome. First, the difference in rubrics, the author contends, implies more a difference in the time a candidate needs to write down the solution than a difference in difficulty of the questions. This point will be illustrated in due course. Second, while many topics are examined by only one or two Boards, there is a common core of material which makes comparison possible. For instance, co-ordinate geometry, integration, differentiation, complex numbers, roots of equations are all popular. To give some variety to the chapter we shall examine integration and co-ordinate geometry in some depth, rather than repeat the topics of complex numbers and differentiation.

Methods of integration are very popular at "S" level in Pure Mathematics. In particular, reduction formulae appear regularly on many of the papers. The first example illustrates very well the type of question that is asked.

" (i) If \( f \) is an even function of \( x \), prove that

\[
\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx .
\]

If \( g \) is an odd function of \( x \), prove that

\[
\int_{-a}^{a} g(x) \, dx = 0 .
\]

Evaluate \[
\int_{-1}^{1} (x^2 + x + 7) \sqrt{1 - x^2} \, dx .
\]

(ii) Defining \( I_n \) by \( I_n = \int_{0}^{\pi} \frac{\sin(nx)}{\sin x} \, dx \),

where \( n \) is a positive integer greater than 2, prove, by considering \( I_n - I_{n-2} \), that
\[ I_n = 0 \text{ when } n \text{ is even, and} \]
\[ I_n = \pi \text{ when } n \text{ is odd.} \]

The first part of the question contains sufficient clues to enable most candidates to answer it. While \((x^3 + x + 7)\sqrt{1 - x^2}\) is neither an even nor an odd function, it is not difficult to see that \((x^3 + x)\sqrt{1 - x^2}\) is an odd function while \(7\sqrt{1 - x^2}\) is an even function. Once this point is realised, the integration is standard "A" level work.

The second part of the question is more difficult than it first appears. It is easy to establish that \(I_n - I_{n-2} = 0\). What many candidates will overlook, however, is that \(I_n\) is defined only if \(n\) is greater than 2. Thus many candidates will evaluate the undefined \(I_1\) when \(n\) is odd and the undefined \(I_2\) when \(n\) is even, instead of the expected \(I_3\) and \(I_4\) (if the examiner intended his wording to be taken literally).

Such niceties are not present in all "S" level questions. The next example is more typical:

"If \(I_n = \int_0^{\pi/2} \sin^n \theta \cos \theta e^{-\sin \theta} \, d\theta\)

prove that \(I_n = nI_{n-1} - e^{-1}\).

Express the following integrals in a form suitable for the reduction formula to be applied.

(i) \(\int_0^{\pi/4} \sin^3 4\theta e^{-\sin 2\theta} \, d\theta\),

(ii) \(\int_0^{\pi/2} \sin^9 \theta \cos \theta e^{-\sin^2 \theta} \, d\theta\),
(iii) \( \int_0^1 t^8 e^{-t} \, dt \).

Evaluate (ii)."

[A.E.B.]

The reduction formula, like so many, is the result of using integration by parts. The substitutions required in the latter parts of the question are not difficult to spot. But this does not make the answer easy to obtain. For example, in part (i), when the substitution \( u = 2\theta \) has been made, the candidate has still to unscramble
\[
\frac{1}{2} \int_0^{\pi/2} \sin^3 2u \, e^{-\sin u} \, du.
\]
He may think of using the well-known formula
\[
\sin 2u = 2 \sin u \cos u \text{ which still leaves } 4 \int_0^{\pi/2} \sin^3 u \cos^3 u \, e^{-\sin u} \, du.
\]
This is not so bad and most candidates who are good enough to get here will think of \( \cos^2 u = 1 - \sin^2 u \).

The final part of the question is a simple test of recursion.

Some questions on reduction formulae are used to test a knowledge of other parts of mathematics. Consider this example.

"Given that, for \( n = 1, 2, 3, \ldots \)
\[
u_n = \int_0^1 \left( \frac{x - 1}{x + 1} \right)^n \, dx
\]
prove by integration by parts that
\[
\frac{1}{n} u_n - \frac{1}{n+1} u_{n+1} = (-1)^n \frac{1}{n(n + 1)}.
\]
Prove also that \( |u_n| < 1 \).

Use these two results to prove from first principles that the series
\[
\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \ldots
\]
is convergent and has the sum \( 2 \log_e 2 - 1 \)."
The first part of the question, although of a standard nature, may involve the candidate in a little trial-and-error experimentation before the correct result is obtained. Indeed some would be put off the entire question at this stage. A little care is needed over justifying the convergence of the series by showing that

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} = -u_1 + \frac{u_{N+1}}{N+1} - \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n(n+1)} \text{ for any positive integer } N.
\]

Since \(|u_n| < 1\) and \(\sum_{n=N}^{\infty} \frac{1}{n(n+1)} = O(N^{-1})\), it follows that \(\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = -u_1\), which is simple to evaluate.

This problem is certainly more difficult than the preceding examples. The reasons for this are that there are more possible approaches to consider together with a need for an understanding in mathematical terms of the difficult concept of limit. The candidate needs the ability to analyse the material (in Bloom's sense) and he needs to possess some intuitive feeling for the abstract process of integration by parts. Furthermore, if he is to complete this question he will need determination; it is very easy to give up on a question which requires a little more thought.

These qualities are required in even greater measure when Oxbridge questions are considered. Although the previous example would help in the solution of the Oxford Entrance Examination question quoted below, intuition is still required in large measure.

"For non-negative integers \(m, n\) let

\[
I_{m,n} = \int_{-1}^{1} (x - 1)^m (x + 1)^n \, dx
\]

Prove that
This question, like so many Oxbridge reduction formulae questions, involves two parameters $m, n$. (In general "S" level questions involve a single parameter.) The reduction formula is, however, very easy to obtain using the standard integration by parts.

The candidate will be seeking to use the formula in the second part of the question. It requires courage or experience to do so because, unlike some recursive formulae, one of the parameters increases while the other decreases. In fact, one obtains

$$I_{n,n} = \frac{(-1)^n (n!)^2 2^{2n+1}}{(2n+1)!}, \quad n \geq 0;$$

if the formula is applied $n$ times. $I_{2n,0}$ is simple to evaluate.

Most well-prepared candidates are looking for expressions of the type $a^n - b^n$. In particular, there will be few who do not write

$$(x^2 - 1)^n = (x - 1)^n (x + 1)^n,$$

especially in view of the earlier parts of this question. If the formula for the $n$th derivative of a product is correctly applied, then it is easy to show that

$$\int_{-1}^{1} \left\{ \frac{d^n}{dx^n} (x^2 - 1)^n \right\}^2 dx = \int_{-1}^{1} \left[ \sum_{r=0}^{n} \binom{n}{r} n! (x - 1)^{n-r}(x + 1)^r \right]^2 dx$$

$$= \int_{-1}^{1} (n!)^2 (2x)^{2n} dx$$

$$= (n!)^2 \frac{2^{2n+1}}{(2n + 1)},$$

from which the result follows. The difficulty here is that the
The candidate may be looking to express the required integral in terms of
\[ \int_{-1}^{1} (x + 1)^n (x - 1)^n \, dx \quad (= I_{n,n}) \], rather than evaluating it directly.

The abstract manipulation required by this problem is considerable, but is easy in comparison with the intuition needed. If the candidate knows how to tackle the question then the algebra (apart possibly from the binomial expansion in (iii)) is straightforward.

It has already been mentioned that the author considers the difficulty of a question to be directly proportional to the number of steps missing. (Perhaps this is why many teachers and pupils regard short questions as more difficult to answer.) The next example illustrates this point vividly.

"For positive \( Q \), evaluate the integrals
\[ I(Q) = \int_0^{\pi/2} \frac{\sin^3 \theta}{(1 + Q^2 \cos^2 \theta)} \, d\theta \quad , \quad J(Q) = \int_0^{\pi/2} \frac{\cos^2 \theta \sin \theta}{(1 + Q^2 \cos^2 \theta)} \, d\theta \]

and show that \( I(Q) > J(Q) \) when \( Q \) is sufficiently large."

[C.S.]
make the point clearer, the solution of the final parts are given following each question.

"Let n be a natural number, and let

\[ I_n = \int_0^1 \frac{dx}{(x^2 + 1)^n} \]

Express \( I_n \) in terms of \( I_{n-1} \).

Evaluate \( \int_{-1/2}^{(\sqrt{3}-1)/2} \frac{dx}{(x^2 + x + 1)^2} \).

\[ \int_{-1/2}^{(\sqrt{3}-1)/2} \frac{dx}{(x^2 + x + 1)^2} = \int_{-1/2}^{(\sqrt{3}-1)/2} \frac{dx}{((x + \frac{1}{2})^2 + \frac{3}{4})^2} \]

\[ = 4 \int_{-1/2}^{(\sqrt{3}-1)/2} \frac{dx}{((x + \frac{1}{2})^2 + 1)^2} \]

\[ = 4 \int_{-1/2}^{(\sqrt{3}-1)/2} \frac{2}{3\sqrt{3}} \frac{du}{(u^2 + 1)^2} \]

\[ = \frac{32}{9\sqrt{3}} \left\{ \left[ \frac{u}{2(u^2 + 1)} \right]_0^1 + \frac{1}{2} \int_0^1 \frac{du}{(u^2 + 1)} \right\} \]

(either directly using integration by parts or using the result of the first part)

\[ = \frac{32}{9\sqrt{3}} \left\{ \frac{1}{4} + \frac{\pi}{8} \right\} \]

\[ = \frac{4}{9\sqrt{3}} (\pi + 2) \]
"Prove that
\[ \int_{a}^{2a} f(x) \, dx = \int_{0}^{a} f(a + x) \, dx = \int_{0}^{a} f(2a - x) \, dx. \]

Evaluate \[ \int_{0}^{\pi} \frac{x^2 \sin x}{(8 + \sin^2 x)} \, dx. \]" 

\[ \int_{0}^{\pi} \frac{x^2 \sin x}{(8 + \sin^2 x)} \, dx = \int_{0}^{\pi} \frac{x^2 \sin x}{(8 + \sin^2 x)} \, dx + \int_{\pi}^{2\pi} \frac{x^2 \sin x}{(8 + \sin^2 x)} \, dx \]

\[ = \int_{0}^{\pi} \frac{x^2 \sin x}{(8 + \sin^2 x)} \, dx + \int_{0}^{\pi} \frac{(\pi + x)^2 \sin (\pi + x)}{(8 + \sin^2 (\pi + x))} \, dx \]

(Using first part)

\[ = -2\pi \int_{0}^{\pi} \frac{x \sin x}{(8 + \sin^2 x)} \, dx - \pi^2 \int_{0}^{\pi} \frac{\sin x}{(8 + \sin^2 x)} \, dx \]

\[ = -2\pi \left\{ \int_{0}^{\pi/2} \frac{x \sin x}{(8 + \sin^2 x)} \, dx + \int_{0}^{\pi/2} \frac{(\pi - x)\sin(\pi - x)}{(8 + \sin^2(\pi - x))} \, dx \right\} \]

\[ - \pi^2 \left\{ \int_{0}^{\pi/2} \frac{\sin x}{(8 + \sin^2 x)} \, dx + \int_{0}^{\pi/2} \frac{\sin(\pi - x)}{(8 + \sin^2(\pi - x))} \, dx \right\} \]

\[ = -4\pi^2 \int_{0}^{\pi/2} \frac{\sin x}{(8 + \sin^2 x)} \, dx \]

\[ = -4\pi^2 \int_{0}^{\pi/2} \frac{\sin x}{(9 - \cos^2 x)} \, dx \]
\[ = -4 \pi^2 \int_0^1 \frac{1}{(9 - u^2)} \, du \]

\[ = -4 \pi^2 \int_0^1 \frac{1}{6} \left[ \frac{1}{(3 - u)} + \frac{1}{(3 + u)} \right] \, du \]

\[ = -\frac{2}{3} \pi^2 \left[ \ln \left| \frac{3 + u}{3 - u} \right| \right]_0^1 \]

\[ = -\frac{2\pi^2}{3} \ln 2 . \]

"Let \( I_n = \int_0^\pi (a + b \cos x)^n \, dx \)

where \( n \) is an integer, and \( a \) and \( b \) are constants with

\( a \geq b > 0 \). By means of the substitution

\[ (a + b \cos x)(a + b \cos u) = a^2 - b^2 , \]

or otherwise, prove that

\[ I_{-n} = (a^2 - b^2)^{\frac{1}{2}} I_{n-1} . \]

Hence evaluate

\[ \int_0^\pi \frac{dx}{a + b \cos x} \quad \text{and} \quad \int_0^{\pi/2} (3 - 2 \sin^2 x)^{-3} \, dx . " \]

[M.E.I.]
\[
\int_0^{\pi} \frac{dx}{a + b \cos x} = (a^2 - b^2)^{\frac{1}{2}} I_o \quad \text{(from first part)}
\]

\[
= \pi / \sqrt{a^2 - b^2}.
\]

\[
\int_0^{\pi/2} (3 - 2 \sin^2 x)^{-3} \, dx = \int_0^{\pi/2} (2 + \cos 2x)^{-3} \, dx
\]

\[
= \frac{1}{2} \int_0^{\pi} (2 + \cos u)^{-3} \, du
\]

\[
= \frac{1}{2} (2^2 - 1^2)^{\frac{1}{2}-3} I_2
\]

\[
= \frac{1}{18 \sqrt{3}} \int_0^{\pi} (4 + 4 \cos u + \cos^2 u) \, du
\]

\[
= \frac{1}{18 \sqrt{3}} \int_0^{\pi} (\frac{9}{2} + 4 \cos u + \frac{1}{2} \cos 2u) \, du
\]

\[
= \pi / 4 \sqrt{3}.
\]

This last question is popular with examiners, and has appeared in various guises in several papers.
"Evaluate \( I = \int_{0}^{\pi/2} \frac{d\theta}{a + b \cos \theta} \),
where \( a \) and \( b \) are positive constants, for each of the cases \( a > b \), \( a = b \), and \( a < b \). Show that the values of \( I \) for \( a > b \) and \( a < b \) are each approximately equal to the value of \( I \) for \( a = b \) when \( a - b \) is small compared with \( a + b \)."

[J.M.B.]

In this form the question asks for some ability to deal with generalities. A general question such as this is also popular with Cambridge examiners. The following is not an isolated example.

"Evaluate the indefinite integral
\[
\int \frac{px + q}{rx^2 + 2sx + t} \, dx
\]
where \( p, q, r, s, t \) are real constants."

[C.S.]

The point about setting a general question such as the above is that it really does test a candidate's understanding - provided it is not such a regular feature that some candidates merely memorise the topic. But this particular example is difficult to memorise unless a student has a clear understanding of the principles involved. The manipulation required in the solution and the number of cases to be considered combine to make memorisation difficult.

Manipulative work also underlies our second set of examples on coordinate geometry. This subject is very popular with many Boards, and this is reflected in the Oxbridge Entrance Examinations.
"The circle $x^2 + y^2 - 2gx - 2fy = 0$ cuts the parabola
$x = at^2$, $y = 2at$ at the origin 0 and the points P, Q and
R. Obtain the cubic equation whose coefficients are
functions of $a$, $f$, $g$ and whose roots are the parameters
of P, Q and R.

Prove that the lines OP and QR are equally inclined to
the x-axis. If the circle touches the parabola at P and
passes through the parabola at 0 and R, show that the
centre of the circle lies on the curve whose equation is
$27ay^2 = 2(x - 2a)^3$.

Since it may be difficult for the reader to judge the difficulty
of this question without going into the details himself, a solution is
presented below.

Most candidates who start the question will have no trouble with
the first part. Let P, Q, R have parameters $p$, $q$, $r$, respectively.
Substitute $x = at^2$, $y = 2at$ into the equation of the circle

$$(at^2)^2 + (2at)^2 - 2g(at^2) - 2f(2at) = 0$$

$$a^2t^4 + 4a^2t^2 - 2gat^2 - 4fat = 0$$

Since P, Q, R all lie on the circle their parameters must satisfy this
equation. Further, since P, Q, R are all different from 0 we may divide
through by $t(\neq 0)$; we may also divide by $a (\neq 0)$. The result is the
required cubic

$$at^3 + 2(2a - g)t - 4f = 0 .$$

(1)

Since the roots of this cubic are $p$, $q$, $r$ and the coefficient of $t^2$ is
zero, it follows that

\[ p + q + r = 0. \]

Now the gradient of OP is \( 2ap/2p^2 \), that is \( 2/p \). The gradient of QR is \( 2a(r - q)/a(r^2 - q^2) \), i.e. \( 2/(q + r) \). But \( q + r = -p \). Hence the gradient of QR is numerically equal, but opposite in sign, to the gradient of OP. Thus OP and QR are equally inclined to the x-axis.

If the circle touches at P and also passes through O and R then \( q = p \), which implies that \( r = -2p \). Now \( p \) and \( r \) also satisfy (1).

Thus

\[ ap^3 + 2(2a - g)p - 4f = 0 \quad (2) \]

and

\[ ar^3 + 2(2a - g)r - 4f = 0. \]

Subtracting,

\[ a(p - r)(p^2 + pr + r^2) + 2(2a - g)(p - r) = 0 \]

Dividing by \( (p - r)(\neq 0) \) and substituting \( r = -2p \) gives

\[ 3ap^2 + 2(2a - g) = 0, \]

from which it follows that

\[ 3ap^2 = 2(g - 2a), \quad (3) \]

Substituting (3) into (2) yields

\[ ap^3 - 3ap^3 - 4f = 0, \]

that is,

\[ -ap^3 = 2f. \quad (4) \]

Taking the cubes of each side of (3) and the squares of each side of (4) we have
\[ 27a^3 p^6 = 8(g - 2a)^3 \]

and
\[ a^2 p^6 = 4f^2, \]

which together imply that
\[ 27a.4f^2 = 8(g - 2a)^3. \]

The result follows.

The second part of the question once again illustrates the role of intuition. The candidate may well obtain the gradients of $OP$ and $QR$ as $\frac{2}{p}$ and $\frac{2}{(q + r)}$ respectively, but whether he will realise that $p + q + r = 0$ is more open to doubt. If he does so, then the avenue is clear for solving the final part of the problem. Nonetheless there are still difficulties here and the candidate must be careful in using the best techniques to eliminate $r$ and $p$ from the equations. Otherwise he could find himself spending longer on the question than he intended.

The next example illustrates just the same points so that no time will be spent discussing it in detail.

"The normal at the point $P$ to the parabola $x^2 = 9y$ meets the $y$-axis at $Q$. Find the equation of the circle through $P$, $Q$ and the origin $O$. Determine the condition for this circle to cut the parabola again at two further real points $P_1$ and $P_2$, and prove that, in this case, the normals to the parabola at $P_1$ and $P_2$ both pass through $P$." 

[A.E.B.]

The art of choosing the best technique of manipulation depends largely on experience. With sufficient experience it is possible to
build up an intuitive feeling for which manipulative process is likely or unlikely to work.

There is once more a close link between "S" level and Oxbridge work.

"A, B, C, D are four points on a parabola. The lines through B and D parallel to the axis of the parabola meet CD and AB, respectively, in E and F. Prove that AC is parallel to EF."

[C.S.]

This question is much easier than the previous two examples provided the candidate works with the general parabola $y^2 = 4ax$ and uses parameters for the points A, B, C, D.

Not every co-ordinate geometry question is about parabolas, of course, nor does every question test manipulative skills. The next two examples show how other concepts can be introduced into co-ordinate geometry problems. First, an example on curvature:

"Find the equation of the circle of curvature of the hyperbola $xy = c^2$ at the point $P(ct, \frac{c}{t})$.

If the normal at P meets the hyperbola again at Q, show that the centre of curvature at P divides PQ externally in the ratio 1:3."

[O & C]

This example, which involves parametric differentiation, is straightforward in principle but once again the candidate must demonstrate good manipulative ability.
Ability to manipulate determinants is required in the following Oxford Scholarship question.

"Show that the area of the triangle with vertices at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is

$$\pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

For $i = 1, 2, 3$, $P_i$ is the point $(at_i^2, 2at_i)$ and $A_1$ is the area of the triangle $P_1, P_2, P_3$, $A_2$ the area of the triangle formed by the tangents at $P_1, P_2, P_3$, and $A_3$ the area of the triangle formed by the normals at $P_1, P_2, P_3$. Show that

$$A_1 : A_2 : A_3 = 2 : 1 : (t_1 + t_2 + t_3)^2.$$

Deduce a necessary and sufficient condition in terms of $t_1, t_2, t_3$ for the normals at $P_1, P_2, P_3$ to be concurrent."

[O.S.]

Where many candidates who get lost on this question make their mistakes is in trying to evaluate the determinant representing the area of triangle $P_1P_2P_3$ rather than demonstrating that the value of the determinant representing $A_2$ is half of that representing $A_1$.

If the reader is not yet convinced that there is a connection between "S" level mathematics and Oxbridge work in the 'double-subject'
as well as the 'single-subject' consider finally these two problems.

" (i) Find the equation of the normal to the rectangular hyperbola with equation \( xy = c^2 \) at the point \((ct, c/t)\).

Show that, if the normals to this hyperbola at the four distinct points \(P_1', P_2', P_3', P_4'\) are concurrent, then the chord joining any two of the points is perpendicular to the chord joining the other two.

(ii) . . . "

[ L ]

The second part of the question concerns a circle passing through points on a parabola (again)!

"\(A_1', A_2', A_3', A_4'\) are four points of the rectangular hyperbola whose general point is \((ct, c/t)\). If the normals at these points are concurrent, prove that each of the points is the orthocentre of the triangle formed by the other three.

Show also that the conic through \(A_1', A_2', A_3', A_4'\) and the centre of the hyperbola is a second rectangular hyperbola whose axes are parallel to the asymptotes of the first."

[C.S.]

The reader should not infer from these examples that such a close correspondence can always be shown! As with the 'single-subject' there are various "S" level topics which are rarely examined at Oxbridge and
there are various Scholarship questions which never appear at "S" level. However, the author feels he has made his point.

So far we have examined only two topics out of a total of more than thirty. To give some idea of the spread of work covered by the different syllabuses, one example is given from each course on a topic representative of that syllabus - a topic which is regularly examined by the Board.

"Show that the roots of the equation

\[(x + 1)^7 + (x - 1)^7 = 0\]

are \(\cot(\frac{2r + 1}{14})\pi\) \((r = 0, 1, 2, \ldots, 6)\).

Hence prove that

\[\sum_{r=0}^{6} \csc^2(\frac{2r + 1}{14})\pi = 49\]

and deduce that

\[\sec^2\frac{\pi}{7} + \sec^2\frac{2\pi}{7} + \sec^2\frac{3\pi}{7} = 24.\]

[0]

"Show that the substitution \(y = v \exp\left(\frac{1}{2} \int P \, dx\right)\)
transforms the differential equation

\[\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = 0,\]

where \(P\) and \(Q\) are functions of \(x\) only, into a differential equation with constant coefficients, provided that

\[4Q - P^2 - 2 \frac{dP}{dx}\]

is constant.
Solve completely the differential equation

\[ (1 - x^2)^2 \frac{d^2 y}{dx^2} + 2x(1 - x^2) \frac{dy}{dx} + (2 + x^4)y = 2(1 - x^2)^{5/2} \cos x. \]

\[ \text{[exp (z) is another way of writing } e^z \text{.]} \]

"Prove that, if \( a_i > 0 \) \( (i = 1, 2, \ldots, n) \), then\]

\[ \left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} \frac{1}{a_i} \right) > n^2. \]

Under what circumstances does equality hold?

If you prove the above by quoting a standard inequality then you should prove that inequality. "

"In the skew-symmetric matrix

\[ S = \begin{pmatrix} 0 & -n & m \\ n & 0 & -1 \\ -m & 1 & 0 \end{pmatrix} \]

the numbers \( l, m, n \) are such that \( l^2 + m^2 + n^2 = 1 \).

Show that \( S^2 \) is symmetric and that \( S^3 = -S \).

The matrix \( A(\theta) \) is defined by

\[ A(\theta) = I + S \sin \theta + S^2 \left( 1 - \cos \theta \right), \]

where \( I \) is the unit matrix of order 3. Show that
A(\(\Theta\)) A(\(\emptyset\)) = A(\(\Theta + \emptyset\)), and deduce that \( [A(\Theta)]^k = A(k\Theta) \) for any positive integer \( k \)."

"A curve is defined by the parametric equations

\[
\begin{align*}
  x &= a \left( u - \tanh u \right), \\
  y &= a \sech u, \quad u > 0.
\end{align*}
\]

If the tangent at any point \( P \) meets the \( x \)-axis at \( N \), prove that the length of \( PN \) is constant.

If the arc length from the point where \( u = 0 \) is denoted by \( s \), prove that \( y = a e^{-s/a} \). Prove further that the total surface area of the closed solid generated by the revolution about the \( x \)-axis of the above curve is \( 3\pi a^2 \)."

"(a) If \( u = \log_e \left( \frac{x^2 - y^2}{2xy} \right) \) show that

\[
\begin{align*}
  (i) \quad & \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{x + y} - \frac{1}{x^2} - \frac{1}{y^2}, \\
  (ii) \quad & \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \frac{1}{x^2} - \frac{1}{y^2}.
\end{align*}
\]

(b) \( f \) is a function of the real independent variables \( r \) and \( \Theta \). A change of variable is effected by the transformation \( x = r \cos \Theta \) and \( y = r \sin \Theta \).

Show that

\[
\frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \Theta^2} + \frac{1}{r} \frac{\partial f}{\partial r} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}
\]

where \( f(r, \Theta) = g(x, y) \)."
"Correct the following where necessary.

(i)  \[ \int \frac{dx}{1 + x^2} = \log(1 + x^2) \cdot \frac{2x}{2x} . \]

(ii)  \[ \int_{-1}^{1} \frac{dx}{x^2} = \left[ -\frac{1}{x} \right]_{-1}^{1} = -2 . \]

(iii)  \[ \lim_{t \to 0} (1 + t)^{1/t} = 1^\infty = 1 . \]

(iv)  \[ \lim_{n \to \infty} 3^{n(\pi - 22/7)} = 3^0 = 1 . " \]

"(i) Define fully the term field. Assuming the fundamental properties of the system of real numbers, construct the field of complex numbers.

(ii) Let \( G = (z_1, z_2) : z_1 \text{ and } z_2 \text{ complex numbers, } z_2 \neq 0 , \)

and let the binary operation \( o \) be defined on \( G \) as follows:

\[ (z_1, z_2) o (z_3, z_4) = (z_1 + z_2 z_3, z_2 z_4) . \]

Prove that \( G \) forms a group with respect to the operation \( o \). Show, by means of a counter-example, that the group is not Abelian."
"Let a be prime to n. Show that, if q and r are less than n and prime to n, then $qa \equiv ra \pmod{n}$ unless $q = r$. Hence show that if $s_1, s_2, \ldots, s_k$ are all the natural numbers (including 1) which are less than n and prime to n then the numbers $s_1a, s_2a, \ldots, s_ka$ are congruent (mod n) to $s_1, s_2, \ldots, s_k$ in some order. Deduce that $a^k \equiv 1 \pmod{n}$.

Hence, or otherwise, find the number $m$ between 0 and 13 inclusive for which $45^19 \equiv m \pmod{14}$.

[J.M.B.]

"A linear transformation $f$ on a three-dimensional vector space $V$ is called a projection if $f^2 = f$. Show that the following transformations are projections on $V$, and find their matrices with respect to the usual standard bases:

$$f_o : (x, y, z) \mapsto (x, y, 0)$$

$$g_o : (x, y, z) \mapsto \left(\frac{1}{5}(x + 2y), \frac{1}{5}(2x + 4y), z\right).$$

Interpret these projections geometrically.

Show also that if $f$ and $g$ are projections, then

(i) $1 - f$ and $1 - g$ are projections (where 1 denotes the identity transformation),

(ii) $f + g$ is a projection if and only if $fg + gf = 0$,

where 0 denotes the zero transformation.

Discuss these results in the case $f = f_o$, $g = g_o$.

What do (i) and (ii) tell us about the matrices representing $f$ and $g$?"

[M.E.I.]
Although some 'double-subject' questions are on topics outside a 'single-subject' syllabus, several of the problems cover work in such syllabuses. Yet I doubt that any would appear in a 'single-subject' examination. The reason is that a deeper appreciation of the subject is required. For example, consider the evaluation of \[ \int_0^{2\pi} \frac{x^2 \sin x}{(8 + \sin^2 x)} \, dx \], discussed earlier in the section. Even with the hints, this integral requires considerable manipulative ability and even intuition about the abstraction process. How many students of a 'single-subject' course would not use the wrong equality at some stage during the evaluation? (One may well ask also how many 'double-subject' students would not!) In Bloom's jargon the 'double-subject' requires analysis whereas the 'single-subject' is more at the level of application.

Also required in the 'double-subject' is more intuition about the techniques involved. Indeed I see this as the major distinguishing feature between the two courses, or, more particularly, between the two sets of candidates. This is not to say that 'single-subject' mathematicians are potentially less able than their 'double-subject' contemporaries. It is just that the latter have spent approximately twice the time on the subject and thus will have greater experience and will have had more opportunity to build up their intuitive processes.

The distinction between 'single-subject' mathematics and the 'double-subject' is that the former can be regarded as a test of mathematical competence whereas the second is more a test of mathematical ability. The former requires an understanding of basic concepts and a familiarity with techniques applied to relatively straightforward problems. Nonetheless the 'single-subject' "S" level is a stern examination. The 'double-subject' demands a much deeper understanding of concepts (as well as more of them); and a greater familiarity with techniques both applied to
problems which sometimes require ingenuity and almost always considerable manipulative skills.

The connection between "S" level work and the Oxbridge Entrance Examinations already remarked upon in Chapter 3 has, the author hopes, been highlighted in this chapter. But, while there is a great similarity in the content, standard and style of question asked in the examinations, it would be foolish to think that an "S" level course alone can provide the right preparation for the Oxbridge examinations.

Finally the author hopes he has demonstrated the wide range of material examined at "S" level as well as a similarity in standard of those topics which are to be found in most syllabuses.
5.1 Formal Teaching

There is no answer to the question "how should candidates be prepared for "S" level?". In Chapter 3 we saw that the examination was used for a wide range of purposes, particularly by those schools with most experience of the examination. The variety of topics examined at "S" level, as seen in Chapter 4, is so considerable that there can be no single answer to the question posed. In any case he would be an arrogant and stupid teacher who would claim to answer it.

Thus it is with very great hesitation and more than a share of apprehension that this section comes to be written. Its purpose is to show how candidates in one school were prepared for one particular part of one examination by one teacher - there is no presumption that what was done should be generally applied, or even that it was the best method of approach for those particular candidates. Yet it does illustrate the practical difficulties, some of which are probably unique to the school concerned, involved in preparing candidates for "S" level.

The school concerned has a tradition of "S" level entries in all subjects. About 20 entries are made each year in "S" level mathematics. The MEI syllabuses are used throughout the school. One feature of the MEI "S" level courses is that they permit a broader study of mathematics. In particular, the MEI Pure Mathematics examination contains a number of 'Topics' which concentrate on material outside the "A" level syllabus. One of these is "Topic II: Further Algebra and Number Theory". It is this which will be discussed in detail, but reference will also be made to "Topic I: Linear Algebra" and to relevant parts of the "A" level course.
There are a number of factors which can affect the treatment of a mathematics course. First, a class may be setted into an ability range. Allied to this many syllabuses presuppose a knowledge of 'lower order' concepts - a knowledge which is complete in only a few of the brightest students. At the school concerned, boys are entered for "S" level only if it is thought likely that they will obtain a good grade (A or B) at "A" level. Thus an "S" level class is naturally setted into an ability range (provided it has an existence separate from the "A" level class), and the treatment of the "S" level course is unlikely to be greatly hindered by a lack of knowledge of lower order concepts. A third factor which may affect the presentation of a mathematics course is that it may have to dove-tail with other subjects. For example, the use of logarithm tables or the slide rule may have to be taught for science colleagues before the mathematician can develop the concept of functions like $2^x$ or $10^x$ and their graphs. This factor is irrelevant in "S" level Pure Mathematics.

A fourth factor which can affect the treatment of a course is the number of students in a class. I think it is generally true to say that the smaller the class the more options there are for presenting the material. For instance, it is difficult to imagine an effective tutorial system operating in a class of thirty. Most "S" level classes, certainly the one being discussed, are not hindered by such restrictions. (There were four boys in the particular class involved.)

Finally, one factor which affects every class in every subject is that of time. If time is really short (it is always short) then this may restrict the options available. The author was fortunate in seeing the "A" level set for five periods per week and the "S" level subset for an additional three periods a week during the Lent and Summer terms.

The content of the "S" level topic consists of:
(i) group theory up to Lagrange's theorem;
(ii) some elementary number theory highlighted by Euclid's algorithm and work on congruences; and
(iii) some structural work on integral domains and fields.

In the "A" level syllabus, candidates are expected to be familiar with and to be able to apply the formal rules of composition in sets "as exemplified in concrete examples of groups, rings and fields". The actual words 'group', 'ring', and 'field' have not appeared in an MEI "A" level paper since the inception of the scheme in 1967.

The courses given to the "A" level and "S" level classes are summarised in Table 5.1.

<table>
<thead>
<tr>
<th>Number of weeks</th>
<th>&quot;A&quot; level course</th>
<th>&quot;S&quot; level course</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Completed work on numerical methods.</td>
<td>Primes, divisibility. Euclid's algorithm.</td>
</tr>
<tr>
<td>2</td>
<td>Properties of real and complex numbers.</td>
<td>Solubility of the Diophantine equation $ax + by = c$</td>
</tr>
<tr>
<td>3</td>
<td>Formal structure of fields and rings - construction of 'ring chart'.</td>
<td>Linear congruences.</td>
</tr>
<tr>
<td>4</td>
<td>Checking axioms. (END OF COURSE)</td>
<td>Quadratic congruences.</td>
</tr>
<tr>
<td>5</td>
<td>REVISION</td>
<td>Group axioms. Examples. Subgroups.</td>
</tr>
<tr>
<td>6</td>
<td>REVISION</td>
<td>Cosets. Lagrange's theorem.</td>
</tr>
<tr>
<td>7</td>
<td>REVISION</td>
<td>Rings and fields.</td>
</tr>
</tbody>
</table>

Table 5.1  Synopsis of "A" and "S" level courses, Lent term
The idea behind the "A" level course was first to initiate discussion in the whole set about fields with familiar examples on real and complex numbers. Students were asked to give as many elementary properties of real numbers as they could think of. These were written down on the blackboard. This process was helped by the fact that the words commutativity, associativity and distributivity were familiar to students from their "O" level days. Of course, the number of properties written down exceeded the number of axioms for a field. This was considered an advantage because the idea of an axiomatic structure could be given. When eleven of the properties were taken as axioms, the extra properties were proved to be a consequence of them.

Other examples of fields were sought - complex numbers and rational numbers were found to fit the bill whilst the integers did not. This led to a discussion of sets of residues (known to the class as 'finite' or 'clock' arithmetic). It was found that $J_3$ formed a field whilst $J_4$ did not. Further investigation soon yielded the conjecture that $J_n$ is a field if and only if $n$ is a prime. Matrices were then examined and it was obvious to the class that they did not form a field since everyone knew that matrices do not, in general, commute under multiplication.

In all the examples considered, a total of three field axioms were violated. This led quite naturally to the definition of a ring. Since some of the rings were non-commutative and others did not contain inverses, the examples led to consideration of different types of rings. A duplicated chart similar to that in Fig. 5.1 was handed out and the examples already covered were marked against the appropriate boxes. New examples were then sought. After suggesting the direct (or Cartesian) product $\mathbb{R} \times \mathbb{R}$ - the set of ordered pairs $(a, b)$ where $a, b \in \mathbb{R}$, the set of real numbers - the class considered many similar examples.
An interesting point came out of this process of marking down the examples on the chart. None was found of a division ring. This led to a discussion of quaternions and their applications in mechanics. (I suspect that many of the class remembered more of Hamilton's drinking problem than his quaternions!)
The theme of the "A" level work was the checking of axioms. This became the starting point for the algebraic content of the "S" level course. The group axioms were listed and three examples were given:

(i) residue classes, (ii) 2 x 2 matrices under addition, (iii) the group of symmetries of an equilateral triangle. The first two examples were familiar from the "A" level work while the third, although unfamiliar, was easy to check geometrically. Further, examples (i) and (iii) were used to illustrate the concept of a group table.

Some elementary properties of groups, e.g. \((ab)^{-1} = b^{-1} a^{-1}\), were proved. The idea of a subgroup was then presented and the three examples already quoted were used in illustrations. The theorem that a non-empty subset \(H\) of the group \(G\) is a subgroup of \(G\) if and only if \(ab^{-1} \in H\) whenever \(a, b \in H\) was then proved. Cyclic groups were introduced, again using the three examples as illustrations. All these concepts were tested by the issue of a work sheet containing problems on the groups of symmetries of the square and of the rectangle.

Cosets were introduced and Lagrange's theorem (that the order of a subgroup of a finite group divides the order of the group) was proved. Once again, examples (i) and (iii) were used in illustration.

The reader will recall that the class had already met, as part of the "A" level work, the axioms for a ring and for a field. These were restated and the ring chart was re-examined; a special note was made of the connection between integral domains and fields. The missing link of division was examined, first in the special case of integers and rationals, and later it was shown how any integral domain can be embedded in a field.

As revision, a work sheet was issued in which permutations were defined. Students were asked to prove that permutations formed a group under the given law of composition. Cycles were defined and students
were asked to calculate cycles of given permutations and to multiply cycles.

The number theoretic content of the syllabus could be taught before or after the algebraic material since it was independent of it. It was, however, more than more convenience which resulted in it being placed first. Work on residues would be done thus preparing the ground for an important example in group theory. A second and more important advantage is that the class becomes used to proving theorems - on work which is readily comprehensible - again preparing the way for more abstract work in group theory.

The distinction between the treatment of the "S" level material and the "A" level material is that the "S" level course was treated for structure and for the nature of proof, whereas the "A" level course was concerned solely with checking axioms.

It seems necessary to defend the unusual process of starting with the axioms for a field. The logical order of presentation - and the one adopted by most teachers and lecturers - is to begin with groups and then to consider rings and finally fields. In practice, however, precisely the reverse order was chosen and was found to work exceptionally well. The reason for this order was that the most familiar sets do, in fact, form fields or at least rings (real numbers, complex numbers, rational numbers, integers, matrices). It is not easy to think of simple and familiar examples of groups which cannot also be extended with an uncontrived operation to form fields or rings. Further, if one begins with fields, it is easy to give familiar examples of sets which do not belong to the system as well as of those which do. (It must be remembered that the aim of the "A" level course was the checking of axioms.)
Familiarity is the keynote of the above explanation. The "A" level course had as an aim (as the author saw it) merely knowledge, comprehension and perhaps application in Bloom's language. Knowledge and comprehension could be facilitated by studying the familiar. In the "S" level course, on the other hand, it was hoped that students could apply their knowledge to unfamiliar examples by examining the structure of a general group - groups of permutations, for instance. In Bloom's terms this would be analysis. Moreover, it was even hoped that some intuition about the subject would be generated. For example, all the students correctly formulated a conjecture about the order of a subgroup of a finite group and the order of the group before they encountered Lagrange's theorem.

Much time has been spent describing the treatment of the course without paying too much attention to the practical problems involved. First, the "A" level syllabus had to be completed - this would take about four weeks at five periods per week. Second, with regard to the whole of the "S" level syllabus, about seven weeks at three periods per week could be afforded to the Topic II syllabus. Third, since roughly four weeks were needed to conclude the "A" level course, and axiomatic structures had yet to be considered, it would be more convenient to delay the group theory in the Topic II syllabus until at least the fifth week. (This happened to fit in with other, more academic, considerations.) Fourth, the syllabus naturally divided into roughly two equal portions. The Special paper would contain one question on number theory and another on algebraic structure. Since the "A" level class would spend some time on groups, rings and fields, a 4:3 division of time in favour of number theory was thought to be appropriate.

These practical difficulties can affect the treatment of a course. For instance, since the number theory had to be covered in four weeks,
it would not be possible to discuss the law of quadratic reciprocity - the central result in the theory of quadratic residues. (Fortunately, no question on this law had been set previously by the examiners.)

Finally, it might appear that the Topic II syllabus contains two disjoint parts but, as has already been mentioned, these two parts have 'the nature of proof' as a linking theme. The coverage of the Topic II syllabus together with a knowledge of vectors could (and did) lead to an axiomatic treatment of vector spaces, part of the Topic I: Linear Algebra syllabus. Because the students were familiar with group, ring and field structures it was easy to form axioms for a vector space using a knowledge of vectors and their properties as a model. The development of the theory of vector spaces could be treated more quickly because the concept of "axiomatic structures" and indeed many of the proofs had been encountered in the Topic II syllabus. The concepts had 'floated down'; what would have taken a week to be assimilated now took only a lesson because the concepts had already been met in another context.

5.2 Enrichment

The preceding discussion has centred around a syllabus which contains a further and broader study of mathematics in a school which could provide students with additional time and teaching. Not all syllabuses contain a broader range of material; some papers are set entirely on the "A" level syllabus. Not all schools can find the extra time and manpower. How then should one proceed to prepare students for "S" level if no extra time is available?

The discerning reader may have distinguished two cases in which the answer to this question might be different. First there are those schools using syllabuses which contain no extra material. It could be argued that this causes no problem as the "S" level students could be
given harder questions to answer. Indeed, this may be an effective method of obtaining good results for the school, but the author believes it to be a sterile approach for the student.

The second case is more difficult. Some schools use syllabuses which contain a broader range of material but cannot provide the time for formal teaching. (This situation can occur for a variety of reasons — for example, the Head of Mathematics may like the "A" level syllabus and thus has to use that "S" level syllabus, or, worse, it may be a dictum of the Headmaster that a particular Board's examinations shall be used in all subjects.) Quite clearly if the student is to cover the "S" level syllabus then much of the work must be done on his own and in his own time. This is one form of 'enrichment'. Enrichment is defined to be the study of mathematics lying outside the normal school curriculum. It is usual, however, to refer to more lighthearted activities than the study of "S" level mathematics as enriching a student's curriculum.

It should not be forgotten that "S" level examinations occur at the end of a two year course. My belief is that enrichment can be used to broaden the outlook of "S" (and indeed "A") level students especially in the first year sixth before the pressure of examinations encourages a student to work. It is, I contend, of sound educational value to all "S" level students, whether they receive formal teaching or not, whether the syllabuses they study require them to learn more than the mere "A" level syllabuses.

To give the reader some idea of the scope of possibilities of enrichment, the following lengthy extract is taken from the American authors Johnson and Rising (1972, p. 362):
"Solve challenging mathematical problems like those found in periodicals such as *The Mathematics Student Journal* ...

Read about exciting mathematical topics in enrichment pamphlets such as *Exploring Mathematics on Your Own* ...

Learn new mathematical ideas from books such as *Mathematics and the Imagination* ... Study advanced topics such as limits, infinity, non-Euclidean geometry, finite mathematical systems, computer programming, symbolic logic, or game theory. Write creatively by preparing scripts, research reports, poems, plays or essays.

Enjoy recreational reading of science fiction based on mathematical ideas, as found in Fadiman's two books *Fantasia Mathematica* or *Mathematical Magpie*.

Learn to perform tricks, solve puzzles, or analyse paradoxes as described in books listed in Appendix D.

Participate in out-of-class activities such as mathematics clubs, seminars, contests and fairs.

Build models such as logic machines or tesseracts to represent mathematical ideas.

Find applications of mathematics in science, economics, industry, government, music or art as described in Kline's *Mathematics in Western Culture*.

Collect information about opportunities in mathematics and the work of contemporary mathematics.

Study more intensively and deeply the standard topics included in the accelerated course.

Investigate the history of mathematics and the biographies of mathematicians.

Prepare lessons or reports on unusual topics such as topology, transformations, symmetry or space travel ...

"
Some of these suggestions have been incorporated into the mathematical education of British pupils. For example, in the Midlands there are two mathematical societies for schools in the Birmingham and Coventry areas which organise annual competitions against each other, as well as providing interesting lectures and visits.

The author believes it is in the area of private study, publicly recognised, that students gain most. It is not uncommon for students to present lectures on topics which have interested them, although the practice is perhaps not common enough. The author has encouraged it in his own classes at a public school where, of course, Saturday morning lessons are a matter of routine. On Saturday mornings (perhaps the same is true on Friday afternoons in state schools) there is a general reluctance to knuckle down to routine school work. Boys in the first year sixth studying the 'double-subject' were given an opportunity to express their opinions about the desirability of preparing weekly lectures on topics outside the "A" level syllabus. There was agreement that the scheme was preferable to the normal routine, but it can only be said that the enthusiasm was less than rapturous!

No pressure was put on the three least able boys to prepare lectures since it was felt that they needed the time to spend on their "A" level work. They were not prohibited from participating, but they chose not to. A programme of lectures was drawn up for the term; the choice came largely from the boys themselves. There was a completely free choice as to who would speak on each topic.

The range of topics was very wide. A boy who was studying classics as part of his general studies course elected to give a talk on the importance of number in Pythagorean mathematics. Another proved the transcendence of e, while a third discussed the development of algebraic notation. One boy, who regarded himself as something of a mystic, gave
a mature talk on transfinite numbers. Other topics included logic circuits and topology.

The lecturer for the week was responsible for obtaining the necessary references which he begged, borrowed or otherwise acquired from a variety of sources. His teacher was usually able to provide some books or articles if the boy was stuck.

At the beginning of the following term the class was asked whether the scheme should continue. This time there was much greater enthusiasm. In fact, two of the three boys who had not given talks the previous term volunteered their services. (Indeed one of them later gave what was generally considered to be the most interesting, the most stimulating and the best-prepared lecture of the year.) The success of the scheme could be gauged by the sight of boys asking the lecturer for the week whether they might borrow his notes.

Not all the topics chosen have a direct bearing on "S" level work. Admittedly several of the lectures required either the manipulative skills or the type of thinking tested at "S" level. For instance, the boy who followed the proof of the transcendence of e carried through analytical ideas to well beyond "S" level. But most of the lectures did not provide such good practice. Nor was it the intention that they should since, in the author's school, separate provision is made for "S" level candidates in the second year of the sixth form. It would, of course, be possible to arrange a selection of lectures on "S" level topics from those syllabuses which contain other than "A" level material.

One of the shortcomings of both "A" level and "S" level mathematics is that there are necessarily many topics which interest students which remain unexamined. Some of the topics are indeed difficult to examine in the sixth form. But this should not preclude students from spending time and effort on topics such as transfinite numbers. It is regarded
as a shortcoming of the "S" level examination that such topics cannot be examined.

What the author hopes to have shown in this chapter are two possible methods of educating potential or actual "S" level candidates. Experienced teachers may think that they would not have approached the subject in the same way; if so, they could well be right. For there are many ways in which to make mistakes in the classroom. The author sincerely believes, however, that what has been described was successful. It would be most interesting to build up knowledge of how other teachers tackle "S" level work. But it could be achieved only after further very extensive research had been carried out.
CHAPTER 6

THE FUTURE OF "S" LEVEL

There are two questions which arise when discussing the future and their answers are rarely the same. What is likely to happen? What ought to happen? Predictions are never easy to make. Before they are made one must have a thorough and complete knowledge of the present position, a wide appreciation of factors external but relevant to the subject of the forecast - and good judgement. Once again the author hesitates!

The present position is not easy to ascertain, as the reader will no doubt concede. Part of the difficulty can be explained by the history of "S" level. First, in the latter half of the last century a great number of examining bodies were founded. This fact (whether or not it is desirable) makes assessment of an examination conducted by such a large number of different bodies a forbidding task. Second, the original Scholarship papers were introduced so that students could compete for financial awards. This consideration is invalid at "S" level since the abandonment of State Scholarships occurred before the first Special papers were set in 1963.

A third historical reason for the uncertainty of the present position is that at least one of the reasons for the inception of "S" level is much less important today. Provincial universities do not use the examination in their selection procedures.

Another difficulty in assessing the present position is that the G.C.E. Boards do not publish figures on how many or which types of schools enter candidates for the examination, and how many candidates are entered by those schools using the examination. The results of the questionnaire indicated a disturbing range in the practice of entering
candidates for "S" level. About one in four Comprehensive schools in the sample used the examination, whereas over half the Grammar schools, over half the Public schools and over half the Direct grant schools entered candidates. But it is in the numbers of candidates entered by each type of school that the situation becomes clearer. For every single candidate from a wholly maintained school (whether it is Grammar or Comprehensive) there are four from Public schools and three from Direct grant schools. Moreover, even within these groups there are wide differences. For instance, one Public school entered a total of 82 candidates in the years 1973-75 while another entered just 1. In the state system one Grammar school entered 24 candidates in the same period while three Comprehensives could only muster 1 each.

The reasons why teachers enter candidates for "S" level mathematics at least confirms one of the reasons for the existence of the examinations. There is a widespread belief that it is beneficial to subject more able "A" level students to harder questions and this was one of the reasons why many candidates were entered for the examination. But another reason why many candidates were entered (particularly by Heads of Mathematics with the most experience) is that the "S" level was thought to provide suitable practice for Oxbridge examinations. But there was little agreement on whether "S" level provided an extra qualification for university applicants, or whether candidates were entered for the examination to subject them to a broader range of material, or whether they were entered because the teacher could undertake more challenging work.

That "S" level is regarded with importance by the universities of Oxford and Cambridge was firmly established in correspondence with Tutors for Admissions. The connection between "S" level and Oxbridge examinations was also examined in Chapter 4 and it was found that there
was a great similarity in the questions appearing in the two sets of papers, both in the 'single-subject' and in the 'double-subject'.

It was also found that there is a wide range of topics examined in the different "S" level examinations, but where the topics overlapped a certain amount of consistency was discerned. The number of topics featured in "S" level papers made any comparisons almost impossible. Further, since some Special papers are restricted to examining topics in the "A" level syllabus while others contain new material, it is difficult to compare candidates with similar (let alone dissimilar) grades awarded by different Boards. Moreover, even where Special papers are set entirely on the "A" level syllabuses, it is difficult to compare two papers from different Boards since the "A" level syllabuses themselves may be vastly different. One may cite the Oxford Delegacy and the SMP examinations as an example.

The difficulty of comparison between Boards is again highlighted by an examination of the statistics published by the various Boards. The percentages of candidates being awarded supplementary grades at "S" level by different Boards varies between 30% and 60% of the "S" entry. This could have been due to a large number of inferior candidates sitting the papers of one Board. Amazingly it was found that there was a significant positive correlation between the proportion of the "A" level entry sitting the Special paper and the proportion of candidates being awarded a supplementary grade at "S" level amongst the Boards.

In the circumstances so described, it would be impossible to give any idea of the teaching methods used. But a personal account of how formal teaching and enrichment can be used in educating "S" level candidates was given. (From the replies to the questionnaire it was found that not all schools could make provision for formal teaching.)
Thus we see that it is difficult to describe a typical "S" level candidate. But he might well be a boy educated either in a Public school or in a Direct grant school. His mathematics teacher may well have entered him for the examination because it gave him harder questions to solve and because it provided suitable practice for Oxbridge examinations. He may have received some formal teaching, especially if he came from a school with a strong tradition of "S" level work.

Three factors are likely, more than others, to play major roles in the future of "S" level. First, there are political and economic trends. Direct grant schools are already in the process of being phased out. Many have decided to become fully independent of the state system; others have chosen to be integrated into that system. With no selection in a fully comprehensive system, the quality of pupils in terms of academic ability is bound to suffer. This means that the more able pupils will be dispersed among a number of schools instead of being concentrated in a single school. In particular, there will be fewer "S" level entries from schools which currently receive the direct grant. Further, I believe that there will be fewer "S" level entries altogether simply because of this action. For many of the most able pupils will go to schools with no tradition of, nor any belief in "S" level work.

If the Labour Party is returned to government in a time of economic prosperity, then the future of the Public schools is also threatened. Whatever the arguments for and against the abolition of the independent sector, it is certain that it would have a profound effect on current practices at "S" level. Indeed, I believe it likely that "S" level would disappear at the same time. The future is, however, by no means clear. Any of a number of conditions could alter the situation. Electoral reform, for example, might make the apparently inevitable absolutely impossible.
A second factor which could clearly affect the future of the "S" level examinations is the future of the Oxbridge Entrance Examinations. The discontinuance of the latter examinations would have an undoubted effect on the future of "S" level. For many candidates are entered for "S" level to give them practice for the Oxbridge examinations. However, it must be said that the Oxford and Cambridge examinations appear to have a secure future.

The third factor which could influence the role of "S" level is more tangible, and more likely, I suspect, to do so than the other two. It is the set of proposals made jointly by the Schools Council and the Standing Conference on University Entrance known as the N and F proposals (Schools Council, 1973). A summary of the recommendations, given in Appendix 2, is reproduced from the Working Paper (pp. 85-87).

The central recommendation is the replacement of the present "A" level structure by a two-tiered structure of Normal (N) and Further (F) levels. The Normal subjects would occupy a student for approximately half the time currently spent on an "A" level subject, while Further subjects would occupy him for three-quarters of the time of an "A" level subject. Students would be expected to take a combination of five Normal and Further subjects, with at most two being F levels.

The purpose of the N and F proposals is to broaden the sixth form curriculum. Their relevance to "S" level (implicit in that they replace "A" level) may be demonstrated by one simple quotation from the report (p. 73):

"We would not want Special papers, similar to those available in the present "A" level system, to be attached to the new structure."
Since the publication of the report, the Schools Council has circulated a questionnaire to all member interests of the Council, to some 300 organisations and institutions having interests in the field of sixth form examinations and to Council committees. The questionnaire was completed by only 70 of the 300 organisations. One was the Mathematical Association: "the mathematicians wanted three subjects out of five" ran the Digest of comments received (Schools Council, 1974).

A further development has been the setting-up of Syllabus Steering Committees, as part of a feasibility study, to draw up (if possible) aims and objectives for others to prepare detailed syllabuses, to consider who should prepare detailed syllabuses, and to discuss the target group for whom the N and F proposals are intended - bearing in mind several factors including the aims of N and F levels as set out in Working Paper 47 and working on the understanding that there will be no fundamental changes in current Higher and Further Education courses. These committees are not due to report until March, 1977. It is proposed that the new system would not be ready for implementation until after 1980.

While not wishing to anticipate the Mathematics Syllabus Steering Committee's report, it seems impossible to provide the necessary core of material for later university work in fewer than three subjects - unless there is a fundamental change in current university practice. Nonetheless, this does not alter my opinion that the N and F proposals will eventually be accepted; as I see it mathematics is very much the exceptional subject in these considerations. History, for example, can still be studied effectively by reducing the period of study; the reduction process does not affect the usefulness of the study as a preparation for university work since students can learn to analyse historical data and to form balanced value-judgments.
I am convinced that the N and P proposals will be accepted, and that mathematics will have to fit the pattern. How it will be made to fit is another matter. Whether universities will start their courses assuming less or whether there will be three possible choices in mathematics remains to be seen. I cannot see the latter possibility being permitted for it seems to contradict the whole basis of the proposals.

However one regards these developments it must be regarded as highly probable that "S" level as we know it is in the last ten years of its existence. The likely thing to happen is that "S" level will disappear at about the same time that N and P levels begin, that a broader curriculum at a lower level will replace the current G.C.E. "A" level examinations. As mentioned at the beginning of the chapter, it is quite a different question to ask what ought to happen.

One recurring theme in the Digest of replies to the Schools Council's questionnaire is concern over the plight of the most able students. There was a widespread belief in the beneficial effects of a broader curriculum on the student of average ability. There is a clear conflict between the needs of the gifted and the needs of the majority.

One of the most important principles in Western education is that children have a right to develop to their maximum. The guiding theme of comprehensive education in Britain is perhaps the politician's phrase 'equality of opportunity in a fair society'. Once more one wonders whether there is not a conflict in these aims. My belief is that gifted children are prevented from developing to the full in a Grammar school and in most Public schools. I cannot see how comprehensive education improves their lot, and I see the introduction of N and P levels in mathematics as a positive blow to their interests and to the morale of their teachers.
The basic problem concerning the teaching of the gifted is their ability to see relationships so much more quickly than their peers. Teachers are too often content to leave the most able students to their own devices (perhaps setting them more difficult or simply more problems) in order to pay more attention to other members of the class. I have been guilty of this practice, but I do not think it a mistake in the circumstances. In a democracy it is fair to satisfy the needs of the majority. Fortunately our society is humane enough to recognise its responsibility to make special provisions for the handicapped and the backward. Nonetheless, the gifted are still not catered for.

"S" level is one means of giving the most able students more challenging work in the sixth form. But the problem is that there are too many gifted students who are not entered for "S" level. If one is to believe the results of the questionnaire, then perhaps as many as half the schools in the country have such pupils. Further, the provision for "S" level candidates even within those schools which do use the examination appears to vary considerably. In schools with large numbers of such students - and the survey showed that there are several such schools - there is little to worry about. But in schools entering only a few candidates, separate classes are unlikely to exist and the teachers may have had little experience of the examination.

While some of the most able students are being well educated in schools, some with famous names and teachers, there are too many gifted pupils being overlooked. Even where enrichment is used as a method of providing for the most able, there are almost bound to be inadequacies. Many of the examples of enrichment, if they are to be thoroughly attempted, require considerable planning and supervision which would consume time unavailable to the majority of teachers. Further, they require a considerable number of teaching aids, finance for which is unavailable at most schools.
Another problem which emphasises the difficulty of educating these talented students is the quality of the teaching they receive. Many teachers enjoy teaching the gifted and some are good at it. But too many, I feel, are not aware of what the gifted child is capable and are satisfied if that child achieves the standard set for his less illustrious class-mates. It has to be said that there are some schools which simply do not possess staff of the right ability to cope with the problem.

What then should be done? The present situation is disturbing but the future looks bleaker for the most able students. Unfortunately one cannot speak of what should be done in mathematics in the final school year without also taking into account the educational provisions for the entire school life of a gifted child.

Some of the aims of educating the gifted listed at the end of Section 2.4 must have priority over others. Moreover some can be met only when others have been completely satisfied. In my view the first four aims are the most urgent. So much so that they must take precedence over the other aims no matter what conflicts of interest are aroused. Thus I believe that up to the age of about 15 the gifted should be educated with their peers. If educated separately, the gifted would not realise their own place in, and responsibility to society. Nor would the non-gifted. Further, there are practical problems in identifying the gifted child up to this age. After the age of 16, three-quarters of the age range has left school, and the objections to separate education are much less significant. Thus, if there is a need for special schooling at that age, then there can, in my opinion, be no strong objections on educational grounds.

The problem of what provisions should be made for the most able is so vast that a full-scale research investigation by a body such as the NFER or the Schools Council would be necessary. The Schools Council
has, in fact, conducted an investigation into the teaching of gifted children of primary age - see Ogilvie (op. cit., 1973); and the NFER has published a book by J. B. Shields (1968) entitled "The Gifted Child". But these contributions, important as they are, do not cover the whole age range. They simply highlight the need for further extensive research.

What does emerge from Ogilvie's work is the total lack of provision for the gifted at present. There are several ways in which more attention can be paid to the needs of the gifted. First, teachers could spend more time with their gifted pupils. Second, local education authorities could provide peripatetic teachers specially trained to meet the needs of the gifted. Third, local study centres could be set up to cater for children on whole day or half-day release from their schools.

Many teachers would be reluctant to devote less time to their students of average ability for reasons that I have previously outlined. The second alternative is a sound educational proposition but the gifted would be visited in their own classrooms probably with few special teaching aids and almost certainly lacking the company of other gifted children - at least in non-selective maintained schools. (By my definition one would expect to find at most two gifted children in every three unstreamed classes.) The third choice need not suffer from these disadvantages but might be more expensive to administer; on the other hand, it might also act as an incentive to the underachieving gifted child in a way that peripatetic teachers could not. (Underachievement is a major problem in the education of gifted children as Bridges (1975) has recently shown.)

Another method of providing the gifted with an education appropriate to their needs is to 'accelerate' them; that is, gifted children are promoted to a class of children a year or more older. This is a method
commonly in use in Public schools. In some of these schools there are sufficiently many gifted children to form an entire class — particularly in the lower school. This is an ideal situation from the strictly academic point of view since a treatment fitting the needs of the gifted may be given without pangs of conscience about the older, average student. But it is perhaps unsatisfactory from a more general viewpoint because the young gifted child is not in much contact with the less able, and conversely. In any event such is not the case in state education where acceleration is a rarer phenomenon.

I believe that most gifted children will be able to cope with a year's acceleration and could consequently enter the sixth form at 15. However, acceleration can be dangerous and great care is needed to ensure that children are not promoted too quickly. There are a whole range of personal and environmental factors, as well as intellectual ones, which can determine whether acceleration is desirable — see Pressey (1964, pp. 310-318) for an overall view.

On educational grounds I am in favour of a local study centre for gifted children. An experiment involving the use of such a centre was carried out at Brentwood College of Education from 1964-66 with 38 gifted children of primary age. The success of the experiment is reported in Bridges (1969); of particular interest is a chapter on mathematics (pp. 86-103) by D. A. Collins.

As I see it the provisions for gifted school children should consist of:

(a) educational experiences in common with other children up to the age of fifteen;

(b) half-day release in primary school, possibly increasing to day release during the early secondary school years, to attend a local study centre for the gifted;
(c) separate education after fifteen.

Clearly the course content for the release periods could not be planned in isolation from the syllabus for the entire school - a point also made by Ogilvie (op. cit., p. 192).

If my plan has any value then it needs to overcome many practical objections. First, gifted children are not a disturbing problem in society - if indeed they are regarded as a problem at all. The prevalent attitude is that the gifted can take care of themselves. I dismiss this argument. It may be that the gifted can easily reach the standards set for average children. But, by definition, they are far from average and their standards ought to be set much higher.

A second objection is that there is uncertainty about the state of knowledge of giftedness. Whether or not this is a valid objection, we shall learn little that is new if we are unprepared to experiment; and a well-designed experiment could hardly worsen the situation.

A lack of trained personnel is another objection to be overcome. I believe that suitable staff can be found at the right salaries and, with experience, it will be possible to train new personnel.

Finally, the cost of providing this education would be an additional financial burden either on local authorities or on central government. While there can be no argument that costs would rise, the amount might not be great since we are dealing with at most 2% of the school population and many resources currently duplicated could be pooled. However, a careful cost analysis would have to form part of comprehensive feasibility studies.

In the meantime, the problem of the gifted exists. For the sake of these children, and for the benefit of the nation, the problem needs solving. "S" level mathematics is an imperfect, seemingly obsolescent, means of providing an education appropriate to the needs of the
mathematically gifted. But it should not be abandoned until the problem of the gifted is solved.
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Appendix 1

The Schools Questionnaire on the Role of "S" Level Mathematics
Dear Head of Mathematics,

I am conducting some research into the role of S-level Mathematics. As far as I know, very little has been attempted in this subject, important to those interested in the mathematical education of sixth formers. Indeed it is difficult to ascertain even the aims and objectives of S-level syllabuses - the G.C.E. examination boards are somewhat reticent!

Since there is such a lack of knowledge, it seems important to find out from the schools themselves some of the relevant details. The short questionnaire enclosed is an attempt to gather some of this information "from the grass roots". As you will see, little time need be taken up in responding.

May I urge you to use the stamped addressed envelope provided - even if your school does not enter candidates for S-level? (One of the most serious disadvantages of using questionnaires is unreliability due to lack of response.) If you wish to be informed of the results of the questionnaire, then I shall be pleased to send you the information when it becomes available (see Question 8).

I thank you in anticipation of your valuable co-operation.

Yours sincerely,

P. J. STEPHENS, B.Sc., Ph.D., A.F.I.M.A.
The following notation is used in describing the schools forming the sample:

Size refers to the number of pupils

- C means comprehensive school
- Gr " grammar school
- Hi " high school
- Bi " bilateral school
- VI " sixth form college
- ? " type of school unknown
- P.S. " public school
- D.G. " direct grant school
- B " boys
- G " girls
- M " mixed
* denotes that a reply was received.
### Wholly Maintained Schools

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Appendix 2

Summary of Schools Council's N and F Proposals
The Curriculum and 18+ Examinations

1. The sixth-form curriculum and 18+ examinations in each subject should be on two levels, to be known as the 'Normal' and 'Further' levels.

2. The programmes for both Normal and Further subjects should be spread over both years of the sixth-form course, and should be suitable for those who have reached the standard of passes at O level (or CSE equivalent).

3. About 70 per cent of sixth formers' time should be spent on Normal and Further subjects, which should be five in number. If general studies were offered as an examination subject this should be additional to the other five.

4. The examination should be on a single-subject basis. Although it is expected most candidates would take it at age 18+, there should be no restriction as to the age of candidates, and the examination should be available to those studying privately or in further education colleges, as well as to those in schools.

5. The range of methods of assessment should be widened, so as to make full use of all the modern techniques that may be appropriate.

6. Schools should ensure that subjects chosen were such that the curriculum was broadly conceived for each sixth-former. Universities and polytechnics should encourage a broad sixth-form curriculum by paying attention to achievements in subjects outside the range of a student's intended specialism.
Syllabuses and Courses

7. Syllabuses should be prepared on the assumption that for a Normal subject the average student would require about one-half of the time at present devoted to the subject at A level and for a Further subject about three-quarters of that time.

8. Fresh thought should be given to the syllabuses in all subjects. The factual content should not be overloaded and the relation of one subject to another should be borne in mind.

9. Each Normal course should be within the competence of those not intending to study the subject in higher education as well as of those who do.

10. The number of Further subjects studied by any sixth-former should not usually exceed two.

11. Normal and Further candidates should be taught together for the first year in the sixth form although it might be necessary or desirable for them to be taught separately, for some or all of the time, in the second year.

Entrance Requirements for Degree Courses

12. General entrance requirements for universities and polytechnics should be framed in terms of achievements in five subjects in the 18+ examinations.

13. The naming of subjects in general entrance requirements should be kept to a minimum, but, where subjects are named as essential for general entrance purposes, achievements in the 16+ examinations should be accepted as an alternative to a Normal-level achievement
(although the number of subjects required in the 18+ examinations should remain at five).

14. Course requirements should demand not more than two Further subjects and not more than three named subjects, whether at Further or Normal level. Faculties should be encouraged to consider whether they might demand only one named Further subject, or perhaps none at all. Where the naming of two or three subjects is necessary for course requirements, faculties might in certain circumstances be willing to accept performance in the 16+ examination in a named subject.

15. Discussions should be held involving universities and polytechnics, the Standing Conference on University Entrance and the Council for National Academic Awards, to reach as much common ground as is possible in the framing of general entrance requirements. Similar discussions should be held with a view to securing greater uniformity in respect of course requirements (for degrees in the same subject) than prevails at present.

Theoretical and Operational Studies

16. A comprehensive programme of theoretical studies and experimental examinations should be mounted before a final decision is taken that the recommendations we have made should come into full-scale operation. These would require the active co-operation of the universities and polytechnics, other colleges (including the colleges of education), professional institutions and employers generally, the local education authorities and examining boards. The schools would have a special part to play and the support of
the Department of Education and Science would be essential. The approval of the Secretary of State would be necessary, so that the operational trials could be carried out during an experimental period, and the Schools Council should be prepared to commit itself to providing substantial financial backing for the studies, both theoretical and operational.