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Stochastic Perturbations of Intermittent Maps

by

Yuejiao Duan

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University

June 2013

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Abstract

This thesis studies statistical properties of intermittent maps. We obtain three new results. First we use an Ulam-type discretization scheme to provide pointwise approximations for invariant densities of interval maps with a neutral fixed point. We prove that the approximate invariant density converges pointwise to the true density at a rate $C^* \cdot \frac{\ln m}{m}$, where $C^*$ is a computable fixed constant and $\frac{1}{m}$ is the mesh size of the discretization.

We then study intermittent maps in a random setting. In particular, we study a random map $T$ which consists of intermittent maps $\{\tau_k\}_{k=1}^K$ and a position dependent probability distribution $\{p_{k,\epsilon}(x)\}_{k=1}^K$. We prove existence of a unique absolutely continuous invariant measure (ACIM) for the random map $T$. Moreover, we show that, as $\epsilon$ goes to zero, the invariant density of the random system $T$ converges in the $L^1$-norm to the invariant density of the deterministic intermittent map $\tau_1$. The outcome of Chapter 4 contains a first result on stochastic stability, in the strong sense, of intermittent maps.

Finally, we study the problem of correlation decay of random map built from finitely many intermittent maps with a common neutral fixed point. Using a Young-tower technique, we show that the map with the fastest relaxation rate dominates the asymptotics. In particular, we prove that the rate of correlation decay for the annealed dynamics of the random map is the same as the sharp rate of correlation decay for the map with the fastest relaxation rate.
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Introduction

Let \( \tau \) be a piecewise continuous transformation of \( X \) into itself. Let \((X, \mathcal{B}, \lambda)\) be a measure space. We say \( \mu \) is absolutely continuous \( \tau \)-invariant measure with respect to \( \lambda \) if 
\[
\mu(\tau^{-1}B) = \mu(B), \quad \text{for any } B \in \mathcal{B}(X), \quad \text{and} \quad \mu \text{ is absolutely continuous with respect to } \lambda.
\]
Absolutely continuous invariant measure (ACIM) can be used to describe the behavior of the map \( \tau \). In particular, absolutely continuous invariant measures (ACIMs) are main tools to study the statistical properties of the orbit \( \{\tau^n(x)\}_{n \geq 0} \) (see [13, 17] for an introduction and a comprehensive background on the importance of ACIMs in ergodic theory and dynamical systems).

There are many results on the existence of ACIMs for uniformly expanding interval map \( \tau \), that is \( |\tau'(x)| > 1 \) for all \( x \in [0, 1] \) (see [17]). In our work, we consider non-uniformly expanding maps. In particular, we study expanding maps of the interval which admit an indifferent fixed point. Such maps are good testing tools for physical systems with intermittent behaviour. Statistical properties of non-uniformly expanding maps were studied by Pianigiani in [47] who first proved existence of invariant densities of such maps. Later, it was independently proved in [32, 41, 54] that such maps exhibit polynomial decay of correlations. Then Gouëzel [25] showed that the rate obtained in [54] is sharp. The slow mixing behaviour that such maps exhibit has made them good testing tools for real and difficult physical problems. More recently, Hu and Vaienti generalized these results to general higher dimensional systems [33].

Finding the formula of an ACIM for a dynamical system is in general an impossible task. Therefore, approximation techniques are needed. In [40] it was shown that the original Ulam method [52] is remarkably successful in approximating isolated spectrum of transfer operators associated with piecewise expanding maps of the interval. In particular, it was shown that this method provides rigorous approximations in the \( L^1 \)-norm for invariant densities of Lasota-Yorke maps. This method has been partially successful \(^1\) in providing rigorous approximations for certain uniformly hyperbolic systems [23, 24]. Recently, Blank [14] and Murray [43] independently succeeded in applying the pure Ulam method in a non-uniformly hyperbolic setting. They obtained approximations in the \( L^1 \)-norm for invariant densities of certain non-uniformly expanding maps of the interval\(^2\).

We use an Ulam-type discretization scheme (see Chapter 3) to obtain a faster approximation rate in the \( L^1 \)-norm for invariant densities of certain non-uniformly expanding maps than those of Murray [43]. Although \( L^1 \) approximations provide significant information about the long-term statistics of the underlying system, they are not helpful when dealing with rare events in dynamical systems. In fact, when studying rare events in dynamical systems [2, 36] one often obtains probabilistic laws that depend on pointwise

\(^1\)See [15] for examples where the pure Ulam method provides fake spectra for certain hyperbolic systems.

\(^2\)In [43], in addition to proving convergence, Murray also obtained an upper bound on the rate of convergence.
information from the invariant density of the system. In particular, extreme value laws of interval maps with a neutral fixed point depend pointwise on the invariant density of the map [31]. In our result [7] (see Chapter 2), we use an Ulam-type discretization scheme to provide pointwise approximations for invariant densities of non-uniformly expanding maps.

In this thesis we also study random perturbations of intermittent maps. In particular when the indifferent fixed point persists under perturbations. Results on statistical stability of intermittent maps with perturbations of this type were obtained in [1, 4]. More recently results on metastability \(^3\) of intermittent maps where the neutral fixed point persists under deterministic perturbations were obtained in [12]. All the results of [1, 4, 12] are concerned with deterministic perturbations of intermittent maps. In [20] we are concerned with random perturbations of intermittent maps, i.e. a random system \(T\) defined by a collection of transformations \(\tau_1, \tau_2, \ldots, \tau_k\) from \(X\) to itself and \(\tau_k\) is chosen with position dependent probability \(p_k(x)\).

In [5] it was proved that intermittent maps of the type studied in [41] are stochastically stable in the weak sense. However, in [20] (see Chapter 4), we proved existence of a unique absolutely continuous invariant measure (ACIM) for the random system \(T\). Also, we obtained a result on the strong stochastic stability of such maps. We obtain our results by using a cone technique. This cone was also used in [45] to study Ulam approximations for deterministic intermittent map.

In this thesis we also study the problem of correlation decay of iid randomized compositions of two intermittent maps sharing a common indifferent fixed point. Obviously the annealed dynamics of the random process will also have a polynomial rate of correlation decay. However, we are interested in the following questions: How do the asymptotic of the random map relate to those of the original maps; in particular, the rate of correlation decay?

In our result [8] (see Chapter 5), we study a class of random transformations built over finitely many intermittent maps sharing a common indifferent fixed point. Using a Young-tower technique, we show that the map with the fastest relaxation rate dominates the asymptotics. In particular, we prove that the rate of correlation decay for the annealed dynamics of the random map is the same as the sharp rate of correlation decay for the map with the fastest relaxation rate.

In Chapter 1, we review some literature. For deep treatment of real and functional analysis, we refer to Dunford and Schwartz [21] and W.Rudin [48]. For ergodic theory and dynamical systems, we refer to [17, 53]. We also introduce Lai-Sang Young’s results [54] in this chapter. We mainly focus on results which will play important roles throughout this thesis. In particular, we discuss in depth Young’s tower for a class of circle maps studied in [18].

In Chapter 2, we use an Ulam-type discretization scheme to provide pointwise approximations for invariant densities of interval maps with a neutral fixed point. We prove that the approximate invariant density converges pointwise to the true density at a rate \(C^* \frac{\ln m}{m}\), where \(C^*\) is a computable fixed constant and \(\frac{1}{m}\) is the mesh size of the discretization. This chapter reflects our result in [7].

\(^3\)By a metastable system, we mean a system which initially has at least two ACIMs, but once it is perturbed it admits a unique ACIM. Such models were first studied in the expanding case in [27].
In Chapter 3, we obtain convergence in the $L^1$-norm. Although there are results, like Murray [43], our method has a faster rate.

In Chapter 4, we study a random map $T$ which consists of intermittent maps $\{\tau_k\}_{k=1}^K$ and a position dependent probability distribution $\{p_k(\epsilon)(x)\}_{k=1}^K$. We prove existence of a unique ACIM for the random map $T$. Moreover, we show that, as $\epsilon$ goes to zero, the invariant density of the random system $T$ converges in the $L^1$-norm to the invariant density of the deterministic intermittent map $\tau_1$. The outcome of this chapter contains a first result on stochastic stability, in the strong sense, of intermittent maps. This chapter follows our result in [20].

In Chapter 5, we study a class of random transformations built over finitely many intermittent maps sharing a common indifferent fixed point. Using a Young-tower technique, we show that the map with the fastest relaxation rate dominates the asymptotics. In particular, we prove that the rate of correlation decay for the annealed dynamics of the random map is the same as the sharp rate of correlation decay for the map with the fastest relaxation rate. This chapter reflects our result in [8].

In Chapter 6, we conclude and discuss our future direction of study.
Chapter 1

Preliminaries

This chapter contains a brief review of measure theory and probability theory, functional analysis, ergodic theory and dynamical systems. We mainly focus on results which will play important roles throughout this thesis. Some of the proofs in this chapter are routine exercises. We include them for the sake of completeness and because they are specific and cannot be found in classical text books. For deep treatment of real and functional analysis, we refer to Dunford and Schwartz [21] and W.Rudin [48]. For ergodic theory and dynamical systems, we refer to [17, 37, 53].

1.1 Measure Theory and Probability

1.1.1 Measure Theory

We recall some basic ideas from measure theory. Let $X$ be a set.

Definition 1.1.1 ($\sigma$-algebra). A $\sigma$-algebra of subsets of $X$ is a collection $\mathcal{B}$ of subsets of $X$ if and only if:

1. $X \in \mathcal{B}$;
2. for any $B \in \mathcal{B}$, $X \setminus B \in \mathcal{B}$;
3. if $B_n \in \mathcal{B}$, for $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$.

We then call the pair of $(X, \mathcal{B})$ a measurable space and elements of $\mathcal{B}$ are usually referred to as measurable sets.

Definition 1.1.2 (Measure). A function $\mu : \mathcal{B} \to \mathbb{R}^+$ is called a measure on $(X, \mathcal{B})$ if and only if:

1. $\mu(\emptyset) = 0$;
2. $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$, whenever $\{B_n\}_{n=1}^{\infty}$ is a sequence of members of $\mathcal{B}$ which are pairwise disjoint subsets of $X$.

A measure space is a triple $(X, \mathcal{B}, \mu)$ where $(X, \mathcal{B})$ a measurable space. We say $(X, \mathcal{B}, \mu)$ is a probability space, or a normalized measure space, if $\mu(X) = 1$. We then say $\mu$ is a probability measure.

Definition 1.1.3 (Algebra). A collection $\mathfrak{A}$ of subsets of $X$ is called an algebra if the following three conditions hold:

1. $\emptyset \in \mathfrak{A}$;
2. if $A \in \mathfrak{A}$, then $X \setminus A \in \mathfrak{A}$;
3. if $A, B \in \mathfrak{A}$ then $A \cup B \in \mathfrak{A}$. 
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Definition 1.1.4 (Borel σ-algebra). Let $X$ be a topological space. Let $\mathcal{D}$ denote a family of open sets of $X$. Then the σ-algebra $\mathcal{B} = \sigma(\mathcal{D})$ is called the Borel σ-algebra of $X$ and the elements of $\mathcal{B}$ are referred to Borel subsets of $X$.

Definition 1.1.5 (Measurable). Let $(X, \mathcal{B}, \mu)$ be a measure space. The function $f : X \to \mathbb{R}$ is said to be measurable if for all $c \in \mathbb{R}, f^{-1}(c, \infty) \in \mathcal{B}$, or, equivalently, if $f^{-1}(A) \in \mathcal{B}$ for any Borel set $A \subset \mathbb{R}$.

If $X$ is a topological space and $\mathcal{B}$ is the σ-algebra of Borel subsets $X$, then each continuous function $f : X \to \mathbb{R}$ is measurable.

Definition 1.1.6 (Absolutely Continuous). Let $\nu$ and $\mu$ be two measures on the same measure space $(X, \mathcal{B})$. We say that $\nu$ is absolutely continuous with respect to $\mu$ if for any $A \in \mathcal{B}$, such that $\mu(A) = 0$, it follows that $\nu(A) = 0$. We write $\nu \ll \mu$.

If $\nu \ll \mu$, then it is possible to represent $\nu$ in terms of $\mu$. This is the essence of the following theorem.

Theorem 1.1.1 (Radon-Nikodym). Let $(X, \mathcal{B})$ be a space and let $\nu$ and $\mu$ be two normalized measures on the space $(X, \mathcal{B})$. If $\nu \ll \mu$, then there exists a unique $f \in L^1(X, \mathcal{B}, \mu)$ such that for every $A \in \mathcal{B}$,

$$\nu(A) = \int_A f \, d\mu.$$ 

$f$ is called Radon-Nikodym derivative and is denoted by $\frac{d\nu}{d\mu}$.

1.1.2 Probability

Before starting this section probability, we firstly need to introduce two definitions.

Definition 1.1.7 (State space). A non-empty set $S$ is called state space if for each time $n$ there is a variable $\omega_n$ taking values in this set. We have

$$\mathbb{T} \ni n \mapsto \omega_n \in S.$$ 

Note that for every $n \in \mathbb{T}$ we use the same $S$.

Definition 1.1.8 (Configuration Space). The configuration space $\Omega$ for a finite time $n \in \mathbb{T}$ is the product space $\Omega_n := S^n = S \times S \times \ldots \times S$. Then for the full time we define

$$\Omega = S^\mathbb{T} \text{ called configuration space. An element } \omega = (\omega_n)_{n \in \mathbb{T}} \in \Omega \text{ is called a configuration with } \omega_n \in S.$$ 

In this section, we mainly refer to Shiryaev [49]. We denote $\Omega$ as a sample space with generic element $\omega$ called sample point and its Borel σ-algebra $\mathcal{F}$. A countable additive probability measure $\mathbb{P}$ is defined on $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Definition 1.1.9. A set in $\mathcal{F}$ of probability zero will be called a null set and we denote almost everywhere (a.e.) as all elements $\omega \in \Omega$ except a null set.

Definition 1.1.10 (Random Variables). Let $\mathbb{S}$ be a countable state space with state $i \in \mathbb{S}$. A random variable $X$ with values in $\mathbb{S}$ is a single-valued function $X : \Omega \to \mathbb{S}$ with $X(\omega) = i \in \mathbb{S}$. 

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1.1. MEASURE THEORY AND PROBABILITY

It follows that if $A$ is any Borel set of $\mathbb{S}$, then the set of elements $\omega$ for which $X(\omega) \in A$ is denoted by $\{ \omega : X(\omega) \in A \} \in \mathcal{F}$. The probability of this set will be defined as

$$P(\{ \omega : X(\omega) \in A \}).$$

We have that $P(\{ \omega : X(\omega) \in \mathbb{S} \}) = 1$.

**Definition 1.1.11 (Conditional Probability).** Let $A$ and $B$ be two sets in $\mathcal{F}$. We define the conditional probability of $A$ relative to $B$ by

$$P(A|B) = \frac{P(AB)}{P(B)},$$

where $P(B) > 0$. If $P(B) = 0$, then $P(A|B)$ is undefined.

**Example 1.1.2.** Consider a sequence of random variables $\{X_n\}_{n \geq 0}$ and states $i_n \in \mathbb{S}$. The conditional probability of the set $\{ \omega : X_1(\omega) = i_1 \}$ relative to the set $\bigcap_{n=2}^{3} \{ \omega : X_n(\omega) = i_n \}$ is written by

$$P\{X_1(\omega) = i_1|X_2(\omega) = i_2, X_3(\omega) = i_3 \}.$$ 

In particular, if the random variable $\{X_n\}$ are said to be independent, then

$$P\{\bigcap_n [X_n(\omega) = i_n]\} = \prod_n P\{X_n(\omega) = i_n\}.$$

On the probability triple $(\Omega, \mathcal{F}, P)$. For any sequence of random variables $\{X_n\}_{n \geq 0}$, a generated Borel $\sigma-$algebra for these random variables is denoted by $\mathcal{F}(\{X_n\}_{n \geq 0})$. In general this Borel $\sigma-$algebra is a subfield of $\mathcal{F}$.

**Definition 1.1.12 (Joint Probability).** If all random variables $X_n$ are discrete with the state space $\mathbb{S}$, then the probabilities of all sets in $\mathcal{F}(\{X_n\}_{n \geq 0})$ are completely determined by the joint probabilities

$$P\{X_0(\omega) = i_0, X_1(\omega) = i_1, ..., X_n(\omega) = i_n \},$$

for all $n \geq 0$ and $i_n \in \mathbb{S}$.

Moreover, above joint probability can be rewritten by

$$P\{X_0(\omega) = i_0 \} \prod_{t=1}^{n} P\{X_t(\omega) = i_t | X_s(\omega) = i_s, 0 \leq s \leq t \}.$$

and this is true for any stochastic process $\{X_n\}_{n \geq 0}$ with state space $\mathbb{S}$. In particular, this product of conditional probabilities could be reduced to a simple form if the the process is a Markov chain which is introduced in Definition 1.2.

**Definition 1.1.13 (Initial Distribution).** Let $\{X_n\}_{n \geq 0}$ be random variables with state $i_n \in \mathbb{S}$. Denote that

$$P\{X_0(\omega) = i_n \} = p_{i_n}.$$ 

Then $\{p_{i_n}, i_n \in \mathbb{S}\}$ is called initial distribution.

**Definition 1.1.14 (Transition Probability).** Let $\{X_n\}_{n \geq 0}$ be random variables with $i, j \in \mathbb{S}$, the state space. On probability space $(\Omega, \mathcal{F}, P)$, we call $p_{ij}$ is the (one-step) transition probability from state $i$ to $j$, if for all $n \geq 1$,

$$P\{X_n(\omega) = j | X_{n-1}(\omega) = i\} = p_{ij}.$$
1.1.3 Stochastic Matrices

For stochastic matrices, we mainly refer to [39]. The field here will always be the set $\mathbb{R}$ of all real numbers or the set $\mathbb{C}$ of all complex numbers. In the general definition, a general field $\mathcal{F}$ is introduced. The main reason for introducing this section is its relation to our work [7] which will be presented in Chapter 2.

**Definition 1.1.15 (Right Eigenvector and Eigenvalue).** Suppose that a matrix $A \in \mathcal{F}_{n \times n}$ and that a vector $x \in \mathcal{F}^n$. The vector $Ax$ is in $\mathcal{F}^n$ and is a member of the range of $A$. For those vectors $x \neq 0$, there exists a member $\lambda$ of $\mathcal{F}$ such that $Ax = \lambda x$. Such a nonzero vector $x$ is called a right eigenvector of $A$ and $\lambda$ is the corresponding eigenvalue.

The equation $Ax = \lambda x$ can also be written $(\lambda I - A)x = 0$ and applying [39] Theorem 1.16.1, we deduce that $\lambda$ is an eigenvalue of $A$ if and only if $\det(\lambda I - A) = 0$. By the definition of determinant, we know that $\lambda$ satisfies a polynomial equation with coefficients in $\mathcal{F}$ and this equation is known as the characteristic equation of $A$.

**Definition 1.1.16 (Characteristic Polynomial).** Let matrix $A \in \mathcal{F}_{n \times n}$ and $\lambda \in \mathcal{F}$. The polynomial

$$c(\lambda) \equiv \det(\lambda I - A) \quad (1.1.1)$$

is the characteristic polynomial of $A$.

It follows from the assumption that $\mathcal{F}$ is algebraically closed that the characteristic polynomial (1.1.1) can be factorized into a product of $n$ linear factors in the form

$$c(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i), \quad (1.1.2)$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of $A$. However, some of these eigenvalues are equal. Therefore we define the multiplicity of eigenvalues.

**Definition 1.1.17 (Multiplicity).** The multiplicity of the eigenvalue $\lambda_i$ of matrix $A \in \mathcal{F}_{n \times n}$ is the number of times the factor $\lambda - \lambda_i$ appears in the factorization (1.1.2) of the characteristic polynomial of $A$ into linear factors.

**Definition 1.1.18 (Dominant Eigenvalue).** Let $A$ be an square matrix. We say an eigenvalue $\lambda$ is dominant if $|\lambda| > |u|$ for any other eigenvalue $u$ of $A$. Also, this $\lambda$ is known as the spectral radius of $A$.

This definition for dominant eigenvalue implicitly includes repeated eigenvalues, so in particular a dominant eigenvalue must have multiplicity 1. As such, an eigenvector for a dominant eigenvalue will be called a dominant eigenvector.

**Example 1.1.3.** Let $A, B$ be square matrices. If $A$ has eigenvalues $-3, 1, 2$, then $-3$ is the dominant eigenvalue of $A$. If $B$ has eigenvalues $-3, 1, 3$ then $B$ has no dominant eigenvalue.

Recall that for each $n \times n$ matrix $A$ with elements $a_{ij}$ is the $n \times n$ matrix with elements $a_{ji}$. It is known as the transpose of $A$ and is denoted by $A'$. Clearly, $(A')' = A$. Also, $A$ and $A'$ have the same eigenvalues. Let $c$ represent a nonzero scalar, then we have the fact

$$A(cx) = cAx = c\lambda x = \lambda(cx)$$

which implies that there are infinitely many eigenvectors corresponding to a eigenvalue. In the following we introduce the Markov chain, then we compute a eigenvector of $A$ corresponding to the dominant eigenvalue $\lambda$. 


Definition 1.1.19 (Nonnegative Matrix). A matrix $A \in \mathbb{R}_{m \times n}$ is said to be nonnegative if and only if no element $a_{ij}$ of $A$ is negative.

If $A, B \in \mathbb{R}_{m \times n}$, we denote $A \geq B$ (or $A > B$) if for all elements $a_{ij} \geq b_{ij}$ (or $a_{ij} > b_{ij}$), $1 \leq i \leq m$ and $1 \leq j \leq n$. Then we say matrix $A$ is nonnegative if and only if $A \geq 0$. But $A \geq 0$ and $A \neq 0$ do not imply $A > 0$.

Let $e_j$ denote the unit column vector of length $n$ with one in the $j$th position and zero in every other position. Denote a permutation $\omega : \{1, 2, \ldots, n\} \to \{j_1, j_2, \ldots, j_n\}$. Then we have $e_{j_k}$ with entries are all zero except that in row $j_k$.

Definition 1.1.20 (Permutation Matrix). A matrix $E \in \mathbb{R}_{n \times n}$ is said to be a permutation matrix if it is of the form

$$E = \begin{bmatrix} e_{j_1} & e_{j_2} & \cdots & e_{j_n} \end{bmatrix},$$

where $j_1, j_2, \ldots, j_n$ is a permutation of $1, 2, \ldots, n$.

It is immediately verified that $E'AE = \begin{bmatrix} e_{j_1}' & e_{j_2}' & \cdots & e_{j_n}' \end{bmatrix} \begin{bmatrix} e_{j_1} & e_{j_2} & \cdots & e_{j_n} \end{bmatrix} = I_{n \times n}$.

Definition 1.1.21 (Reducible and Irreducible). A matrix $A \in \mathbb{R}_{n \times n}, n \geq 2$ is said to be reducible if and only if there exists a permutation matrix $E$ such that

$$E'AE = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

(1.1.3)

where $A_{11}, A_{22}$ are square matrices of order less than $n$. If no such $E$ exists then $A$ is irreducible.

From above definitions, we get that if there exists a permutation matrix $E$ such that $E'AE$ satisfies (1.1.3), then for any $k \geq 1$,

$$\underbrace{(E'AE) \cdot (E'AE) \cdots (E'AE)}_{k \text{ times}} = E' A^k E;$$

that is, matrix $A^k$ is reducible. If a matrix $A \in \mathbb{R}_{n \times n}, n \geq 2$ is a positive matrix, then matrix $A$ is irreducible (see details in [39]). Therefore, for a nonnegative matrix $A$, if there exists a positive number $k$ such that $A^k > 0$ then matrix $A$ is irreducible.

Definition 1.1.22 (Stochastic Matrix). A matrix $A \in \mathbb{R}_{n \times n}$ is said to be a stochastic matrix if and only if $A$ is nonnegative and \( \sum_{j=1}^{n} a_{ij} = 1, i = 1, 2, \ldots, n. \)

By Definition 1.1.14, we have transition matrix $P = (p_{ij})$ with

$$p_{ij} = \mathbb{P}\{X_{n+1}(\omega) = j | X_n(\omega) = i\},$$

for any process $\{X_n\}, n \geq 0$. Observe that $\sum_{j \in S} p_{ij} = 1$. Thus, transition matrix $P = (p_{ij})$ is also a stochastic matrix.

Definition 1.1.23 (Markov Chain). Consider a sequence of random variables $\{X_n\}_{n \geq 0}$ and states $\{i_n\}_{n \geq 0}$. Let $P = (p_{ij})$ be a transition matrix and $\{p_{ij}\}$ be initial distribution. We call that process $\{X_n\}_{n \geq 0}$ is a Markov chain if

1. $\mathbb{P}\{X_0 = i_0\} = p_{i_0}$, i.e., $X_0$ has initial distribution $\{p_{i_0}\}$;
2. $\mathbb{P}\{X_{n+1} = i_{n+1} | X_0 = i_0, \ldots, X_n = i_n\} = p_{i_ni_{n+1}}$, that is, $X_{n+1}$ is independent of $X_0, \ldots, X_{n-1}$. 

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Theorem 1.1.4. A discrete time random process \( \{ X_n \}_{0 \leq n \leq N} \) is a Markov chain with initial distribution \( \{ p_{i_0} \} \) and transition matrix \( P = (p_{ij}) \) if and only if the joint probability (in Definition 1.1.12)

\[
P\{ X_0 = i_0, X_1 = i_1, ..., X_N = i_N \} = p_{i_0} \cdot p_{i_0 i_1} \cdot \cdots \cdot p_{i_{N-1} i_N}.
\]

for all \( i_0, i_1, ..., i_N \in S \),

\[\begin{array}{cc}
\frac{1}{2} & \\
\frac{1}{2} & \\
\frac{1}{12} & \\
\frac{5}{12} & \\
3 & \\
\end{array}\]

Figure 1.1: A 3 \times 3 reducible matrix

Example 1.1.5. Let \( \{ X_n \}_{n \geq 0} \) be a Markov chain on state space \( \{ 1, 2, 3 \} \) with transition matrix (See Figure 1.2)

\[
P = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{5}{12} \\
0 & 0 & 1
\end{pmatrix}.
\]

The example is to illustrate the matrix \( P \) is reducible. Let \( E = I_{3 \times 3} \). Then we have

\[
E'PPE = P = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},
\]

where square matrices \( A_{11} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{12} \end{pmatrix} \) and \( A_{22} = (1)_{1 \times 1} \). We observe that if it is in state 3, it remains in the same state with probability 1. Thus, this matrix is reducible.

The Perron-Frobenius theorem concerns irreducible nonnegative matrices. We now state the Perron-Frobenius theorem.

Theorem 1.1.6 (Perron-Frobenius Theorem, 1907, 1912). If the matrix \( A \in \mathbb{R}^n \times n \) is nonnegative and irreducible then:

(1) A has a positive eigenvalue, \( \lambda \) equal to the spectral radius of \( A \).
(2) There is a positive right eigenvector associated with the eigenvalue \( \lambda \).
(3) The eigenvalue \( \lambda \) has algebraic multiplicity 1.

Example 1.1.7. Let \( \{ X_n \}_{n \geq 0} \) be a Markov chain on state space \( \{ 1, 2, 3 \} \) with transition matrix

\[
P = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} \\
1 & 0 & 0
\end{pmatrix}.
\]
Consider the three-state chain in Figure 1.2. The problem is to illustrate the Perron-Frobenius Theorem 1.1.6. It is obvious that matrix $P$ is nonnegative, then we check that $P$ is irreducible. If the Markov chain in any state $i \in \{1, 2, 3\}$, then it switches into one of other states or remains in the same state with probability non-zero. It is checked that for $k = 2$, $P^k$ is a positive matrix;

$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}$$

First we compute the eigenvalues of $P$ by writing down its characteristic equation

$$0 = \det(\lambda I - P) = \begin{vmatrix} \lambda - \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & \lambda - \frac{1}{2} & -\frac{1}{4} \\ -1 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda - 1).$$

The eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$ with multiplicities 1 and 2 respectively. Also, $\lambda_1 = 1$ is the dominant eigenvalue of $P$, or the spectral radius of $P$.

Next we compute the eigenvectors of $P$ corresponding to each eigenvalue. Suppose $v_1$ and $v_2$ are two non-zero column vectors of length 3. Set right vector $v_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $\lambda_1 \cdot v_1 = P \cdot v_1$. Then we have

$$v_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x}{2} + \frac{y+z}{4} \\ \frac{y+z}{4} \\ \frac{y+z}{4} \end{pmatrix}.$$ 

It implies that $x = y = z$. So we choose one vector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as a eigenvector corresponding to eigenvalue 1. Similarly, we compute eigenvector $v_2$ by

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{x}{2} + \frac{y+z}{4} \\ \frac{y+z}{4} \\ \frac{y+z}{4} \end{pmatrix}$$
which implies that \( x = 0, y = -z \). Thus, \( v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \) is one eigenvector corresponding to \( \lambda_2 = 0 \).

Therefore, these computations illustrate Theorem 1.1.6 for the nonnegative and irreducible matrix \( P \).

## 1.2 Functional Analysis

In this section, we recall the definition of a Banach space. We provide several examples of Banach spaces which will be used later in the thesis. Then we state a fixed point theorem which plays an important role in our result in Chapter 4.

Roughly speaking, a Banach space is coming from a vector space with complete metric. Here, the metric often comes from a norm:

**Definition 1.2.1 (Norm).** Let \( \mathcal{F} \) be a linear space. A function \( \| \cdot \| : \mathcal{F} \rightarrow \mathbb{R}^+ \) is called a norm if it has the following properties: for \( f, g \in \mathcal{F} \) and \( \alpha \in \mathbb{R} \),

\[
\begin{align*}
(1) \quad & \| f \| \geq 0, \text{ with } \| f \| = 0 \iff f \equiv 0; \\
(2) \quad & \| \alpha f \| = |\alpha| \| f \|, \text{ for any } \alpha \in \mathbb{C}; \\
(3) \quad & \| f + g \| \leq \| f \| + \| g \|.
\end{align*}
\]

The space \( \mathcal{F} \) endowed with a norm \( \| \cdot \| \) is called a normed linear space.

### 1.2.1 Examples of Banach Spaces

**Definition 1.2.2 (Cauchy Sequence).** Given a metric space \( (\mathcal{F}, \| \cdot \|) \), a sequence \( f_1, f_2, f_3, \ldots \) is Cauchy, if for every positive real number \( \varepsilon > 0 \) there is a positive integer \( N \) such that for all positive integer \( m, n > N \), the distance

\[
\| f_m - f_n \| < \varepsilon.
\]

**Definition 1.2.3 (Banach Space).** A normed linear space \( \mathcal{F} \) is complete if every Cauchy sequence converges, i.e., if for each Cauchy sequence \( \{f_n\} \) there exists \( f \in \mathcal{F} \) such that \( f_n \rightarrow f \). A complete normed space is called a Banach space.

**Example 1.2.1 (\( L^p \) Space).** Let \( 1 \leq p < \infty \) and \( (X, \mathcal{B}, \mu) \) be a positive measure space. \( L^p(X, \mathcal{B}, \mu, \mathcal{F}) \) is the space of \( \mu \)--measurable scalar functions \( f \) on \( X \) to \( \mathcal{F} \) for which the norm:

\[
\| f \|_p = \left\{ \int_X |f(x)|^p d\mu(x) \right\}^{1/p} < \infty.
\]

We observe a Cauchy sequence \( \{f_n\} \) of functions \( L^p(X, \mathcal{B}, \mu, \mathcal{F}) \), then there exists \( f \in L^p(X, \mathcal{B}, \mu, \mathcal{F}) \) such that \( \lim_{n \rightarrow \infty} \| f_n - f \|_p = 0 \). Thus, \( L^p(X, \mathcal{B}, \mu, \mathcal{F}) \) is a Banach space. See details of proof in [21] Theorem III 6.6.

Also, in Dunford and Schwartz [21], we can find the proofs of following several examples.

**Example 1.2.2 (\( C([a, b]) \) Space).** Let \( (X, \mathcal{B}, \lambda) \) denote the measure space and \( X = [a, b] \) be a bounded interval in \( \mathbb{R} \). \( C([a, b]) \) is the space of continuous functions on \( [a, b] \). \( C([a, b]) \) is a Banach space with sup-norm defined by

\[
\| f \|_\infty = \sup_{x \in [a, b]} |f(x)| < \infty.
\]
However $C([a, b])$ is a normed space, but not a Banach space equipped with norm

$$
\| f \|_1 = \int_a^b |f(x)| \, dx.
$$

**Example 1.2.3 ($C^{\text{Lip}}([a, b])$ Space).** Let $[a, b]$ be a bounded interval in $\mathbb{R}$. Let $C^{\text{Lip}}([a, b])$ denote the space of Lipschitz continuous functions on $[a, b]$ equipped with the norm

$$
\| f \|_{\text{Lip}} = \text{Lip}(f) + \| f \|_{\infty},
$$

where Lip$(f)$ means the Lipschitz constant of a function on $[a, b]$ and $\| f \|_{\infty}$ is the sup-norm defined above (or $L^\infty$ norm). The Arzelà-Ascoli theorem implies that $C^{\text{Lip}}([a, b])$ is a Banach space since the unit ball of $\| \cdot \|_{\text{Lip}}$ is $\| \cdot \|_{\infty}$-compact.

**Example 1.2.4 (Hölder Space).** Recall the Hölder continuity: A real or complex function $f$ is said to be Hölder continuous, if for real numbers $C \geq 0, \alpha \geq 0$,

$$
|f(x) - f(y)| \leq C|x - y|^{\alpha}.
$$

Let $(X, \mathfrak{B}, \lambda)$ denote the measure space and $X$ be a bounded set in $\mathbb{R}$ (or $\mathbb{C}$). Let $C^{0,\alpha}$ denote Hölder Space with the norm

$$
\| f \|_{C^{0,\alpha}} = \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty,
$$

where $0 \leq \alpha \leq 1$. Then $C^{0,\alpha}$ is a Banach space and we note that if $\alpha = 1$ this space is called the space of Lipschitz functions.

**Example 1.2.5 (Functions of Bounded Variation).** Let $[a, b] \in \mathbb{R}$ be a bounded interval and let $\lambda$ denote Lebesgue measure on $[a, b]$. Define a partition $P = \{[x_{i-1}, x_i] : i = 1, ..., n\}$ of $[a, b]$, for any sequence of points $a = x_0 < x_1 < ... < x_n = b, n \geq 1$. The points $x_0, x_1, ..., x_n$ are called endpoints of the partition $P$.

**Definition 1.2.4 (Bounded Variation).** Let $f : [a, b] \in \mathbb{R}$ and partition $P = P\{x_0, x_1, ..., x_n\}$ of $[a, b]$. If there exists a number $M > 0$ such that

$$
\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M,
$$

for all partitions $P$, then $f$ is said to be of bounded variation on $[a, b]$.

**Definition 1.2.5 (Total Variation).** Let $f : [a, b] \in \mathbb{R}$ be a function of bounded variation. The number

$$
\| f \|_{\text{BV}} = \inf_{f_1 = f \, (\text{a.e.})} \left\{ \int_a^b |f_1| \right\}
$$

is called the total variation or the variation of $f$ on $[a, b]$.

**Definition 1.2.6 (BV Normed Space).** Let space $BV([a, b]) = \{ f \in L^1 : \| f \|_{\text{BV}} = \| f \|_1 + \int_a^b f \}$

and for $f \in BV([a, b])$, we define a norm

$$
\| f \|_{\text{BV}} = \| f \|_1 + \int_a^b f.
$$
1.2. FUNCTIONAL ANALYSIS

When equipped with $\| \cdot \|_{BV}$, $BV([a,b])$ forms a Banach space (see [21]).

**Example 1.2.6 (\(\mathcal{B}-\) Normed Space).** Here we present an example of a Banach space which will play an important role in our result in Chapter 2. Let $0 < \alpha < 1$. Let

$$\mathcal{B} := \{ f \text{ is continuous on } (0,1] \text{ and } g(x):= x^{1+\alpha}f(x) \text{ is a bounded function on } (0,1] \}$$

equipped with norm

$$\| f \|_{\mathcal{B}} = \sup_{x \in (0,1]} |x^{1+\alpha}f(x)|.$$ 

Then we prove that \(\mathcal{B}\) is a Banach space. It follows form the fact that, let \(f \in \mathcal{B}\), then

$$\sup_{x \in (0,1]} |x^{1+\alpha}f(x)| = \sup_{x \in (0,1]} |g(x)| < \infty,$$

that is, \(g\) is an element of Banach space of bounded continuous functions on \((0,1]\), i.e. \((C_b(0,1], ||\cdot||_\infty)\). For completeness, let \(\{f_n\}\) be a Cauchy sequence of \(\mathcal{B}\), then \(\{g_n\}\) is a Cauchy sequence in \((C_b(0,1], ||\cdot||_\infty)\). Consequently, \(g_n(x)\) converges to \(g(x)\) in \(||\cdot||_\infty\), and also \(g(x)\) is continuous on \((0,1]\). Observe that \(f(x) = \frac{g(x)}{x^{1+\alpha}}\) is continuous on \((0,1]\) and \(x^{1+\alpha}f(x)\) is bounded on \((0,1]\). Moreover,

$$\lim_{n \to \infty} \|f_n - f\|_{\mathcal{B}} = \lim_{n \to \infty} \|g_n - g\|_{\infty} = 0.$$ 

Thus, \(\mathcal{B}\) is a Banach space.

1.2.2 A Fixed Point Theorem

For this part, we refer to [21] and [45]. If \(X\) is a linear vector space, a convex set \(E \subseteq X\) is called a cone with vertex \(v\), if \(v + x \in E\) implies that \(v + \rho x \in E\) for any \(\rho \geq 0\). The cone \(E\) with vertex \(v\) generated by \(B\) is the intersection of all cones with vertex \(v\) which contain the set \(B\). If \(B\) is convex, it is easy to see that \(E = \{ z \mid z = \rho(u - v) + v, \ u \in B, \rho \geq 0 \}\).

We give a simple notation of cone as follow.

**Definition 1.2.7 (Convex Set).** Let \(X\) be a linear vector space. A set \(E \subseteq X\) is convex if \(x, y \in E\) and \(0 \leq a \leq 1\), imply \(ax + (1-a)y \in E\).

**Definition 1.2.8.** Let \(\preceq\) denote the partial order on Banach space \((\mathcal{F}, ||\cdot||)\). We say \(||\cdot||\) respects the order if

$$0 \preceq f \preceq g \Rightarrow \| g \| = \| g - f \| + \| f \|.$$

**Definition 1.2.9 (Cone).** ([45]) An additive cone is a closed convex set \(C \subseteq \{ f \in \mathcal{F} : 0 \preceq f \}\) such that

1. \(f \in C \Rightarrow \alpha f \in C, \forall \alpha \geq 0\);
2. \(f + g \in C\), whenever \(f, g \in C\).

**Example 1.2.7.** Let \(\mathcal{F} = L^1[0,1]\) and norm be the \(L^1\) norm \(\| f \| = \int_{[0,1]} |f| d\lambda\). If the partial order is \(\preceq\), then the cone is

\[ C = \{ f \in L^1[0,1] : f \geq 0 \}. \]

**Definition 1.2.10 (Conditionally Compact).** A conditionally compact subset \(E\) of a topological space \(X\) is a subset whose closure is compact. Also known as a relatively compact set.
**Theorem 1.2.8.** Let $X$ be the real axis, $\mathcal{B}$ the field of Borel subsets of $X$, and $\lambda$ the Lebesgue measure of sets in $\mathcal{B}$. Suppose $1 \leq p < \infty$. Then a subset $E$ of $L_p(X, \mathcal{B}, \lambda)$ is conditionally compact if and only if it is bounded and

(1) \[ \lim_{x \to 0^+} \int_{-\infty}^{\infty} |f(x+y) - f(y)|^p dy = 0 \] uniformly for $f \in E$;

(2) \[ \lim_{A \to \infty} \left( \int_{-\infty}^{A} + \int_{A}^{+\infty} \right) |f(y)|^p dy = 0 \] uniformly for $f \in E$.

**Example 1.2.9.** We now present an example of a compact cone. This cone will play an important role in our result in chapter 4. For $A > 0$, $0 < \alpha < 1$, denote a setler $C_A = \{ f \in L^1 | f \geq 0, f \text{ is decreasing, } \int_0^x f d\lambda \leq Ax^{1-\alpha} \| f \|_1 \}$.

For any $f, g \in C_A$, and $r \geq 0$, it is easy to check that $rf$ and $f + g$ satisfy the first two conditions. Moreover,

\[ \int_0^x r f d\lambda = r \int_0^x f d\lambda \leq rAx^{1-\alpha} \| f \|_1 = Ax^{1-\alpha} \| rf \|_1 \]

and

\[ \int_0^x f + g d\lambda \leq Ax^{1-\alpha}(\| f \|_1 + \| g \|_1) = Ax^{1-\alpha} \| f + g \|_1. \]

Thus, the set $C_A$ is a cone. Also, the cone $C_A$ is compact. Then we know this closed compact cone has the fixed point property. For the compactness, we apply the Theorem 1.2.8.

**Definition 1.2.11 (Fixed Point Property).** Let $X$ be a topological space. If for every continuous mapping $\tau : X \rightarrow X$, there exists a point $p \in X$ with $p = \tau(p)$. We say $X$ has the fixed point property.

**Definition 1.2.12 (Locally Convex Space).** A topological vector space $V$ is locally convex if it has a base of topology consisting of convex open subsets. Equivalently, it is a vector space equipped with a gauge consisting of semi-norms.

**Theorem 1.2.10 (Schauder-Tychonoff, 1934).** If $K$ is a convex subset of a locally convex linear topological space $V$ and $\tau$ is a continuous mapping of $K$ into itself so that $\tau(K)$ is contained in a compact subset of $K$, then $\tau$ has a fixed point.

### 1.3 Ergodic Theory and Dynamical Systems

In this section we recall definitions and basic results from ergodic theory and dynamical systems. Our main references for this chapter are [17, 53].

#### 1.3.1 Measure Preserving Transformations

Let $(X, \mathcal{B}, \mu)$ be a normalized measure space.

**Definition 1.3.1 (Measurable Transformation).** A transformation $\tau : X \rightarrow X$ is measurable if for any $B \in \mathcal{B}, \tau^{-1}(B) \in \mathcal{B}$, where $\tau^{-1}(B) \triangleq \{ x \in X : \tau(x) \in B \}$. 
Definition 1.3.2 (Invariant Measure). We say the measurable transformation \( \tau : X \to X \) preserves a measure \( \mu \) or that \( \mu \) is \( \tau \)-invariant if \( \mu(\tau^{-1}(A)) = \mu(A) \) for all \( A \in \mathcal{B} \).

Let \( \mathcal{M}(X) \) denote the space of measures on \((X, \mathcal{B})\) and \( \tau : X \to X \) be a measurable transformation.

Definition 1.3.3 (Absolutely Continuous Invariant Measure). Let \( \mu, \lambda \in \mathcal{M}(X) \) be two probability measures. We say \( \mu \) is absolutely continuous invariant measure with respect to \( \lambda \) i.e., \( \mu \ll \lambda \) if \( \tau \)-invariant measure \( \mu \) satisfies \( \mu(A) = 0 \) whenever \( \lambda(A) = 0 \). Moreover, the measures \( \mu, \lambda \) are equivalent if \( \mu \ll \lambda \) and \( \lambda \ll \mu \).

![Figure 1.3: Lebesgue measure \( \lambda \) is \( \tau \)-invariant.](image)

Definition 1.3.4 (Nonsingular). Let \( \tau^* : \mathcal{M}(X) \to \mathcal{M}(X) \) be defined as

\[
(\tau^* \mu)(A) = \mu(\tau^{-1}A), \quad \text{for any } A, \tau^{-1}(A) \in \mathcal{B}.
\]

We say \( \tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) is nonsingular if and only if \( \tau^* \mu \ll \mu \), i.e., for any \( A \in \mathcal{B} \) such that \( \mu(A) = 0 \), we have \( \tau^* \mu(A) = \mu(\tau^{-1}A) = 0 \).

Example 1.3.1 (\( \lambda \) is \( \tau \)-invariant). Let \( \lambda \) denote Lebesgue measure on \((X, \mathcal{B})\). Suppose \( X = I = [0, 1] \) and the measurable transformation \( \tau : I \to I \),

\[
\tau(x) = \begin{cases} 
2x & , \text{for } 0 \leq x \leq \frac{1}{2} \\
2x - 1 & , \text{for } \frac{1}{2} < x \leq 1 
\end{cases}
\]

We will show that \( \lambda \) is \( \tau \)-invariant for any \( A \in \mathcal{B} \). See Figure 1.3. For all \( A \in \mathcal{B} = I \), we have

\[
\lambda(\tau^{-1}A) = \lambda(x_2, x_1) + \frac{1}{2} \lambda(A) + \frac{1}{2} \lambda(A) = \lambda(A).
\]

Hence, by the definition we know this transformation \( \tau \) preserves measure \( \lambda \).

Definition 1.3.5. Let \( \tau : X \to X \) preserve measure \( \mu \). Then \((X, \mathcal{B}, \mu, \tau)\) is called a measure preserving system.
1.3.2 Ergodicity

Let $\tau: X \rightarrow X$ be a transformation. In this thesis, we are interested in the statistical properties of the orbit $\{\tau^n(x)\}_{n \geq 0}$, where $\tau^n$ denote the $n$th iterate of $\tau$ by

$$\tau^n(x) = \tau \circ \ldots \circ \tau(x), \text{ for any } x \in X.$$ 

One of the earliest results which proved this area of research is due to Poincaré:

**Theorem 1.3.2 (Poincaré Recurrence Theorem, 1899).** Let $\tau$ be a measure-preserving transformation on a probability space $(X, \mathcal{B}, \mu)$. Let $K \in \mathcal{B}$ with $\mu(K) > 0$. Then almost all points of $K$ return infinitely to $K$ under the iterates of $\tau$, i.e., there exists $E \subset K$ with $\mu(E) = \mu(K)$ such that for each $x \in E$ there is a sequence $n_1 < n_2 < \ldots$ of natural numbers with $\tau^n(x) \in E$ for each $i$.

A very basic notation in this area of research is called “ergodicity”.

**Definition 1.3.6 (Ergodic).** Let $(X, \mathcal{B}, \mu)$ be a probability space. A measure-preserving transformation $\tau: (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is called ergodic if for any $B \in \mathcal{B}$ with $\tau^{-1}(B) = B$ satisfies $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

**Example 1.3.3 (Ergodicity of $\tau = 2x \pmod{1}$ ).** Using the same map of Example 1.3.1, $\tau: I \rightarrow I$,

$$\tau(x) = \begin{cases} 2x & \text{, for } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{, for } \frac{1}{2} < x \leq 1 \end{cases}.$$ 

Let $\lambda$ denote Lebesgue measure on $([0, 1], \mathcal{B})$ and $A = \tau^{-1}A$ be an invariant set. We will prove the ergodicity of $\tau$. By invariant set $A$, whenever $x \in A$ and $\tau(x) = \tau(y)$, then $y \in A$ as well. Since $\tau[0, \frac{1}{2}] = \tau[\frac{1}{2}, 1] = [0, 1]$, we have

$$\lambda(A) = \lambda(\tau(A \cap [0, 1])) = 2\lambda(A \cap [0, \frac{1}{2}]) = \frac{\lambda(A \cap [0, \frac{1}{2}])}{\lambda([0, \frac{1}{2}])}$$

or equivalently $\lambda(A \cap [0, \frac{1}{2}]) = \lambda(A)\lambda([0, \frac{1}{2}])$. Similarly, $\lambda(A \cap [\frac{1}{2}, 1]) = \lambda(A)\lambda([\frac{1}{2}, 1])$. For any $B \in \mathcal{B}$, let $B_1 = \tau^{-1}(B) \cap [0, \frac{1}{2}]$, $B_2 = \tau^{-1}(B) \cap [\frac{1}{2}, 1]$. Then,

$$\lambda(A \cap \tau^{-1}B) = 2\lambda(A \cap B_1) = 2\lambda(A \cap B_2).$$

By induction, we can show that $\lambda(A \cap E) = \lambda(A)\lambda(E)$ for any dyadic interval $E$. The set $A$ can be approximated arbitrarily closely by a union of dyadic intervals. For any $\varepsilon > 0$, $|\lambda(A \cap A) - \lambda(A)\lambda(A)| < \varepsilon$. Then, $\lambda(A) = \lambda^2(A)$ implies $\lambda(A) = 0$ or 1. Thus, $\tau$ is ergodic.

The symmetric difference of sets is denoted by the symbol $\Delta$:

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

**Definition 1.3.7 ($\tau$–invariant and Almost $\tau$–invariant Function).** Let $(X, \mathcal{B}, \mu, \tau)$ be a dynamical system. A set $B \in \mathcal{B}$ is a $\tau$–invariant if $B = \tau^{-1}B$ and almost $\tau$–invariant if $\mu(\tau^{-1}B \Delta B) = 0$. Similarly, a measurable function is called $\tau$–invariant if $f \circ \tau = f$ and almost $\tau$–invariant if $f \circ \tau = f \mu$–a.e.
1.3. ERGODIC THEORY AND DYNAMICAL SYSTEMS

Theorem 1.3.4. Let \( \tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) be a measure-preserving transformation. The following statements are equivalent:

1. \( \tau \) is ergodic.
2. For sets \( B \in \mathcal{B} \) such that \( \mu(\tau^{-1}B \triangle B) = 0 \) are those with \( \mu(B) = 0 \) or \( \mu(B) = 1 \).
3. For any \( A \in \mathcal{B} \) such that \( \mu(A) > 0 \), then \( \mu(\bigcup_{n=1}^{\infty} \tau^{-n}A) = 1 \).
4. For any \( A, B \in \mathcal{B} \) with \( \mu(A) > 0, \mu(B) > 0 \), there exists \( n > 0 \) such that
   \[
   \mu(\tau^{-n}A \cap B) > 0.
   \]

Definition 1.3.8 (Singular). Two positive measures \( \mu_1 \) and \( \mu_2 \) defined on a measurable space \( (X, \mathcal{B}) \) are called singular if there exist two disjoint sets \( A \) and \( B \) in \( \mathcal{B} \) whose union is \( X \) such that \( \mu_1 \) is zero on all measurable subsets of \( A \) while \( \mu_2 \) is zero on all measurable subsets of \( A \). This is denoted by \( \mu_1 \perp \mu_2 \).

Theorem 1.3.5. Let \( \mu_1 \) and \( \mu_2 \) are two different normalized \( \tau \)-ergodic measures, then \( \mu_1 \perp \mu_2 \) (\( \mu_1 \) and \( \mu_2 \) are mutually singular).

Theorem 1.3.6 (Birkhoff Ergodic Theorem, 1931). Let \( \tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) be a measure-preserving transformation, where \( (X, \mathcal{B}, \mu) \) is \( \sigma \)-finite \(^1\) and \( f \in L^1(\mu) \). Then there exists a function \( f^* \in L^1(\mu) \) such that
\[
\frac{1}{n} \sum_{i=0}^{n-1} f(\tau^i(x)) \to f^*, \quad \mu \text{ - a.e.}
\]
Also, \( f^* \circ \tau = f^* \), \( \mu \)-a.e. and if \( \mu(X) < \infty \), then \( \int_X f^* d\mu = \int_X f d\mu \).

Theorem 1.3.7. If \( \tau \) is a measure-preserving transformation of a probability space. Then \( \tau \) is ergodic if and only if for all \( A, B \in \mathcal{B} \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(\tau^{-i}A \cap B) = \mu(A)\mu(B).
\]

In stochastic process, an important notion is the independence of random variables. In ergodic theory, this is replaced by the notion of mixing, which is roughly speaking, independence in the limit.

Definition 1.3.9 (Mixing). Let \( \tau \) be a measure-preserving transformation of a probability space \( (X, \mathcal{B}, \mu) \).

1. We say \( \tau \) is weak-mixing if for any \( A, B \in \mathcal{B} \),
   \[
   \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(\tau^{-i}A \cap B) - \mu(A)\mu(B)| = 0.
   \]
2. We say \( \tau \) is strong-mixing if for any \( A, B \in \mathcal{B} \),
   \[
   \lim_{n \to \infty} \mu(\tau^{-n}A \cap B) = \mu(A)\mu(B).
   \]

Obviously every strong-mixing transformation is weak-mixing and every weak-mixing transformation is ergodic.

\(^1\)A positive (or signed) measure \( \mu \) defined on a \( \sigma \)-algebra \( \mathcal{B} \) of subsets of \( X \) is called finite if \( \mu(X) \) is a finite real number. The measure \( \mu \) is called \( \sigma \)-finite if \( X \) is the countable union of measurable sets with finite measure.
1.3.3 Perron-Frobenius Operator

In this thesis we are mainly interested in invariant measures which are absolutely continuous with respect to the ambient measure. A powerful tool in this direction is called the Perron-Frobenius operator.

Let \((X, \mathcal{B}, \lambda)\) be a probability measure space. If \(\tau\) is nonsingular and function \(f \in L^1\), for any measurable set \(A \in \mathcal{B}\), define \(\mu(A) = \int_{\tau^{-1}A} f d\lambda\). Then we have \(\mu \ll \lambda\), since \(\tau\) is nonsingular, \(\lambda(A) = \lambda(\tau^{-1}A) = 0\) implies \(\mu(A) = 0\). Hence, by the Radon-Nikodym theorem referring to Theorem 1.1.1, there exists a unique a.e. \(\varphi \in L^1\) such that \(\int_{\tau^{-1}A} f d\lambda = \int A \varphi d\lambda\). Set \(P_\tau f = \varphi\).

**Definition 1.3.10 (Perron-Frobenius Operator).** Let \(\tau : X \to X\) be a nonsingular transformation. We define Perron-Frobenius Operator \(P : L^1 \to L^1\) such that for \(f \in L^1\),

\[
\int_A P_\tau f d\lambda = \int_{\tau^{-1}A} f d\lambda,
\]

for any \(A \in \mathcal{B}\),

We recall the useful properties of \(P_\tau\), the classical Perron-Frobenius operator of a single deterministic map \(\tau\).

**Proposition 1.3.8.** \(P_\tau\) satisfies the properties as follows:

1. (Linearity) \(P_\tau : L^1 \to L^1\) is a linear operator.
2. (Positivity) Let \(f \in L^1\) and assume \(f \geq 0\), then \(P_\tau f \geq 0\).
3. (Preservation of integrals) \(\int_X P_\tau f d\lambda = \int_X f d\lambda\).
4. (Contraction) \(\|P_\tau f\| \leq \|f\|\) for any \(f \in L^1\).
5. (Composition) Let \(\tau_1, \tau_2 : I \to I\) be nonsingular.

\(P_{\tau_1 \circ \tau_2} f = P_{\tau_1} \circ P_{\tau_2} f\).

In particular, \(P_{\tau^n} f = P_{\tau^n} f\).

**Proof.** (1) Let \(\alpha, \beta\) be constants and \(f, g \in L^1\), then for any \(A \in \mathcal{B}\),

\[
\int_A P_\tau (\alpha f + \beta g) d\lambda = \alpha \int_{\tau^{-1}A} f d\lambda + \beta \int_{\tau^{-1}A} g d\lambda
\]

This proof is completed since for any measurable set \(A\), \(P_\tau (\alpha f + \beta g) = \alpha P_\tau f + \beta P_\tau g\), a.e.

(2) If \(f \in L^1\) and assume \(f \geq 0\), then for any measurable set \(A\),

\[
\int_A P_\tau f d\lambda = \int_{\tau^{-1}A} f d\lambda \geq 0.
\]
Hence, $P_\tau f \geq 0$ since $A \in \mathcal{B}$ is arbitrary.

(3) Since transformations $\tau : X \to X$, i.e. $\tau^{-1}X = X$,

$$\int \frac{P_\tau f}{\lambda} = \int \frac{f}{\tau^{-1}(X)} \frac{d\lambda}{X} = \int f d\lambda,$$

the result follows.

(4) By linearity property of $P_\tau f$ and $f \in L^1$, we have

$$P_\tau f = P_\tau (f^+ - f^-) = P_\tau f^+ - P_\tau f^-,$$

where $f^+ = \max(f, 0), f^- = -\min(0, f), f = f^+ - f^-$ and $|f| = f^+ + f^-$. Hence,

$$\|P_\tau f\|_1 = \int_X |P_\tau f| d\lambda \leq \int_X |P_\tau f^+| + |P_\tau f^-| d\lambda$$

$$= \int_X P_\tau f^+ + P_\tau f^- d\lambda = \int_X P_\tau (f^+ + f^-) d\lambda$$

$$= \int_X P_\tau |f| d\lambda = \int_X |f| d\lambda,$$

where we have used above property (3).

(5) Let $f \in L^1$. Since $\tau_1, \tau_2$ are nonsingular, there exists a function $P_{\tau_1 \circ \tau_2} f$ such that

$$\int_A P_{\tau_1 \circ \tau_2} f d\lambda = \int_{(\tau_1 \circ \tau_2)^{-1}(A)} f d\lambda.$$ 

Also we have

$$\int_A P_{\tau_1} \circ P_{\tau_2} f d\lambda = \int_{\tau_1^{-1}A} P_{\tau_2} f d\lambda = \int_{\tau_2^{-1}(\tau_1^{-1}A)} f d\lambda$$

and $(\tau_1 \circ \tau_2)^{-1}(A) = \tau_2^{-1}(\tau_1^{-1}A)$. Hence, $\int_A P_{\tau_1 \circ \tau_2} f d\lambda = \int_A P_{\tau_1} \circ P_{\tau_2} f d\lambda$. By induction, it follows that $P_{\tau^n} f = P_\tau^n f$. \qed

**Proposition 1.3.9.** Let $\tau : X \to X$ be nonsingular. Define $\mu = f^* \cdot \lambda$, i.e. $\mu(A) = \int_A f^* d\lambda$. Then $P_{\tau} f^* = f^*$ a.e., if and only if $\mu(\tau^{-1} A) = \mu(A)$ for all measurable sets $A$, where $f^* \geq 0, f^* \in L^1$ and $\|f^*\|_1 = 1$.

Now we introduce an extremely useful representation for the Perron-Frobenius operator for a large class of one-dimensional transformations. When $\tau$ is a map of the interval, a useful representation of the Perron-Frobenius operator is given by:

$$P_{\tau} f(x) = \sum_{y \in \tau^{-1}(x)} \frac{f(y)}{|\tau'(y)|}.$$  \hspace{1cm} (1.3.1)

**Remark 1.3.1.** The above representation of the Perron-Frobenius operator is not only valid on the unit interval. One can define an analogous formula for a non-singular map $\tau$ acting on any Riemannian manifold.
We give following two examples to illustrate the fixed point of Perron-Frobinus operator $P_{\tau}$.

**Example 1.3.10.** Recall Example 1.3.1 and Figure 1.3

\[ \tau : I \to I, \quad \tau = \begin{cases} 
2x, & 0 \leq x < \frac{1}{2} \\
2x - 1, & \frac{1}{2} \leq x \leq 1
\end{cases} \]

We prove that the constant function 1 is a fixed point of $P_{\tau}$.

First observe that there are two subintervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. Let $x_0 = 1, x_1 = \frac{1}{2}$ and $x_3 = 1$. Then $\tau_1 = \tau \mid (0, \frac{1}{2})$ and $\tau_2 = \tau \mid (\frac{1}{2}, 1)$. By (1.3.1), for any $x \in [0, 1]$

\[
P_{\tau}1(x) = \sum_{i=1}^{2} \frac{1(\tau_i^{-1}x)}{|\tau'(\tau_i^{-1}x)|} 1_{\tau[x_{i-1},x_i]}(x)
\]

\[
= \sum_{i=1}^{2} \frac{1(\tau_i^{-1}x)}{|\tau'(\tau_i^{-1}x)|}
= \frac{1(\tau_1^{-1}x)}{2} + \frac{1(\tau_2^{-1}x)}{2}
= \frac{1(x)}{2} + \frac{1(x)}{2}
= 1.
\]

Therefore, by Definition 1.2.11, we have 1 is the fixed point of Perron-Frobenius Operator under the map $\tau$. This is a restatement that the map $\tau = 2x \pmod{1}$ preserves Lebesgue measure.

**Figure 1.4:** A piecewise monotonic map $\tau$ on $[-1, 1]$

**Example 1.3.11.** Let piecewise monotonic transformation $\tau : [-1, 1] \to [-1, 1]$ be a circle map as shown in Figure 1.4. Define that

\[ \tau = \begin{cases} 
2\sqrt{x} - 1, & 0 \leq x \leq 1 \\
1 - 2\sqrt{-x}, & -1 \leq x < 0
\end{cases} \]
and denote that $\tau_1 = \tau|_{[0, 1]}, \tau_2 = \tau|_{[-1, 0]}$. Observe that $\tau_1' = x^{-\frac{1}{2}}$ and $\tau_2' = (-x)^{-\frac{1}{2}}$. Moreover, for each $x \in [-1, 1]$, there are two preimages:

$$y_1 = \tau_1^{-1}(x) = \left(\frac{1 + x}{2}\right)^2 \geq 0, \quad y_2 = \tau_2^{-1}(x) = -\left(\frac{1 - x}{2}\right)^2 \leq 0.$$ 

Now we show that constant function 1 is a fixed point of $P_\tau$ under this circle map $\tau$. By (1.3.1), for any $x \in [-1, 1]$

$$P_\tau 1(x) = \sum_{i=1}^{2} \frac{1(\tau_i^{-1}x)}{|\tau'(\tau_i^{-1}x)|} 1_{[\tau_i^{-1}x, \tau_i^{-1}x]}(x) = \sum_{i=1}^{2} \frac{1(y_i)}{|\tau'(y_i)|} = \frac{1}{y_1^2} + \frac{1}{(-y_2)^{-\frac{1}{2}}} = \frac{1}{\left(\frac{1+x}{2}\right)^{-1}} + \frac{1}{\left(\frac{1-x}{2}\right)^{-1}} = \frac{1}{2} + \frac{1}{2} = 1.$$ 

Therefore, by Definition 1.2.11, 1 is a fixed point of Perron-Frobenius Operator under circle map $\tau$.

### 1.4 Quasi-Compactness and The Spectral Approach

For certain systems, namely uniformly hyperbolic systems (see Katok, A. and Hasselblatt, B. [37] for more information on uniformly hyperbolic systems), one of the powerful techniques, to obtain statistical properties of a given dynamical system, is to find suitable Banach spaces $(B_1, \|\cdot\|)$ and $(B_2, |\cdot|)$ with $B_1 \subset B_2$ such that the Perron-Frobenius (or transfer) operator $\mathcal{L}$ from $B_1$ to $B_1$ is bounded with respect to both $\|\cdot\|$ and $|\cdot|_{B_1}$, the restriction of $|\cdot|$ to $B_1$, where

$$|\mathcal{L}|_{B_1} = \sup\{\frac{|\mathcal{L}f|}{|f|}, f \in B_1, f \neq 0\},$$

and,

1. If $f_n \in B_1, f \in B_2, \lim_{n \to \infty} |f_n - f| = 0$, and $\|f_n\| \leq C$ for all $n$, then $f \in B_1$ and $\|f\| \leq C$;
2. $H = \sup_{n \geq 0} |\mathcal{L}^n|_{B_1} < \infty$;
3. There exist $k \geq 1, 0 < r < 1$, and $R < \infty$ such that for $f \in B_1$,

$$\|\mathcal{L}^k f\| \leq r \|f\| + R |f|;$$
4. If $\mathcal{E}$ is a bounded subset of $(B_1, \|\cdot\|)$, then the closure of $\mathcal{L}^k \mathcal{E}$ is compact in $(B_2, |\cdot|)$. 

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The above scenario leads to the powerful result of Ionescu-Tulcea and Marinescu (find in [34]). For any complex number \( \eta \), let
\[
D(\eta) = \{ f \in \mathcal{B}_1 : \mathcal{L}f = \eta f, f \neq 0 \}.
\]
\( \eta \) is an eigenvalue of \( \mathcal{L} \) if and only if \( D(\eta) \neq \emptyset \). We now state the Ionescu-Tulcea and Marinescu Theorem.

**Theorem 1.4.1 (Ionescu-Tulcea and Marinescu, 1950).** Under conditions (1) – (4), the intersection of the spectrum of operator \( \mathcal{L} \) with the unit circle is a set \( G \) of eigenvalues of \( \mathcal{L} \) of modulus 1 which has only a finite number of elements. For each \( \eta \in G \), \( D(\eta) \) is finite-dimensional. Furthermore, there exist bounded linear operators \( Q_\eta, \eta \in G, \) and \( S \) on \( \mathcal{B}_1 \) such that
\[
\mathcal{L}^n = \sum_{\eta \in G} \eta^n Q_\eta + S^n,
\]
\( Q_\eta Q_{\eta'} = 0 \), if \( \eta \neq \eta' \), \( Q_\eta^2 = Q_\eta \),
\( Q_\eta S = SQ_\eta = 0 \),
\( Q_\eta \mathcal{B}_1 = D(\eta) \),
\( \varrho(S) < 1 \),
where \( \varrho(S) = \lim_{n \to \infty} ||S^n||^{\frac{1}{n}} \) is the spectral radius of \( S \).

**Remark 1.4.1.** The result of Ionescu-Tulcea and Marinescu implies that the transfer operator has a fixed point in \( \mathcal{B}_1 \). Moreover, if 1 is the only eigenvalue on the unit circle, it implies exponential decay of correlations. This tool was first used by [30, 35] for uniformly expanding one dimensional maps. Recent advances in this direction for uniformly hyperbolic systems has started with the work of [15].

### 1.5 Young Tower

In [54], Lai-Sang Young introduced a powerful tool which is called Young tower. It is a unified framework in which to study the statistical properties of both uniformly and non-uniformly expanding dynamical systems. In particular, it can be used to obtain rates of correlation decay for non-uniformly hyperbolic systems.

In this section, we first review the construction of Young’s tower and then present some results proved by Lai-Sang Young for general dynamical systems. Finally, a specific example is given to illustrate how the tower works step by step to get the desired properties.

#### 1.5.1 The Setup of Young Tower

Let us begin with a dynamical system \( \tau : M \to M \), a \( C^{1+\epsilon} \) diffeomorphism of a finite dimensional Riemannian manifold \( M \). In applications, \( \tau \) is allowed to be discontinuous or have some singularities. Also, we assume that \( \tau \) and \( \tau^{-1} \) are defined on all of \( M \).

Let \( \Delta_0 \subseteq M \) be chosen as a reference set of positive measure and \( \lambda \) denote the reference measure on \( \Delta_0 \). Let \( R : \Delta_0 \to \mathbb{Z}^+ \) be a return time function and \( \tau^R : \Delta_0 \to \Delta_0 \) be the return map. Precise definitions are given later.
A Young tower is a type of Markov extension $F : \Delta \to \Delta$ with the following representation. First step is to construct an extension of $\tau : \bigcup_{l \geq 0} \tau^l \Delta_0 \to \bigcup_{l \geq 0} \tau^l \Delta_0$ which has on it a natural Markov partition\(^2\) with a countable number of states. Generally, the Young tower is that extension of $\bigcup_{l \geq 0} \tau^l \Delta_0$ where the $l$th level of the tower corresponds to those $x \in \tau^l \Delta_0$ for which $l \leq R(x) - 1$. A formal definition of $\Delta$ is given by

$$
\Delta := \{ (x,n) : x \in \Delta_0 \text{ and } n = 0,1,...,R(x) - 1 \},
$$

and define map $F : \Delta \to \Delta$.

$$
F(x,l) = \begin{cases} 
(x,l+1), & \text{if } l < R(x) - 1, \\
(\tau^R(x),0), & \text{if } l = R(x) - 1.
\end{cases}
$$

We construct $\Delta$ as a tower and refer to $\Delta_l$ as the $l$th level of the tower. Clearly, $\Delta_l$ is a copy of $\{ x \in \Delta_0 : l < R(x) \}$ and $\Delta$ can be treated less formally as $\bigcup_{l \geq 0} \Delta_l$ in which $\{ \Delta_l \}$ are disjoint unions.

Let $\{ \Delta_{0,i} \}_{i \geq 1}$ be a partition of the basic object $\Delta_0$ such that $R_i = R \mid \Delta_{0,i} =$ constant for each $\Delta_{0,i}$. So $\Delta_{R_i - 1,i}$ is the top level of the part directly above $\Delta_{0,i}$. Here for simplicity, we assume that the greatest common divisor of $R_i$ is 1. We picture a simple tower in Figure 1.5 for understanding.

![Figure 1.5: A simple Young tower](image)

From here on, we will identify $\Delta_0$ with the corresponding subset of $\Delta$ covering for $\{ (x,l) : x \in \Delta_0 ; l = 0 \}$ and refer to points in $\Delta_0$ as $x$ rather than $(x,0)$. By definition of $F$, we know it sends $(x,0) \in \Delta_{0,i}$ to $(x,R_i - 1) \in \Delta_{R_i - 1,i}$ by $R_i - 1$ steps and maps each $\Delta_{R_i - 1,i}$ bijectively onto $\Delta_0$. Now we denote map $F^R : \Delta_0 \to \Delta_0$ by

$$
F^R(x) = F^R(x,0) = (\tau^R(x),0).
$$

Let $\mathcal{B}$ be a $\sigma$–algebra of subsets of $\Delta$ and $\mu$ be denoted as the reference measure on $(\Delta,\mathcal{B})$ with $\lambda(\Delta_0) < \infty$. We call $s(x,y)$ is separation time if for $x,y \in \Delta_0$, $s(x,y)$ is the

---

\(^2\)A Markov partition is a finite cover of the invariant set of the manifold by a set of curvilinear rectangles $\{ E_i \}_{i=1,2,...}$ such that

- For any pair of points $x,y \in E_i$, then $W_s(x) \cap W_u(y) \in E_i$, where $W_s(x)$ and $W_u(x)$ are the stable and unstable manifolds of $x$.
- $\text{Int} E_i \cap \text{Int} E_i = \emptyset$, for $i \neq j$.
- If $x \in \text{Int} E_i$ and $\tau(x) \in \text{Int} E_j$, then $\tau(W_s(x) \cap E_i) \subset W_s(\tau(x)) \cap E_j$ and $\tau(W_u(x) \cap E_i) \supset W_u(\tau(x)) \cap E_j$. 


 smallest $n \geq 0$ such that $(F^R)^n x$ and $(F^R)^n y$ lie in distinct $\Delta_{0,i}$. Observe that $s(x, y) \geq 0$ for any $x, y \in \Delta_0$ and $s(x, y) \geq 1$ for any $x, y \in \Delta_{0,i}$ etc. In order to proceed to describe the finer structures of $F : \Delta \to \Delta$ some assumptions are required as follows,

- $F$ and $(F \mid \Delta_{i,i})^{-1}$ are measurable,
- all sets mentioned above are $\mathcal{B}$-measurable,
- $F^R \mid \Delta_{0,i} : \Delta_{0,i} \to \Delta_0$ and its inverse are nonsingular with respect to $\lambda$ such that Jacobian $JF^R$ exists and positive $\lambda-$almost every,
- $\lambda \mid \Delta_{i,i}$ is carried to $\lambda \mid \Delta_{i+1,i}$ by $F$ for $l < R_i - 1$ so that $JF \equiv 1$ on $\Delta \setminus F^{-1}\Delta_0$,
- $\int Rd\lambda < \infty$,
- there exist a constant $C_{F,0} > 0$ and $0 < \theta < 1$ such that for any $x, y \in \Delta_{0,i}, \forall i$,
  \[
  \left| \frac{JF^R(x)}{JF^R(y)} - 1 \right| \leq C_{F,0} \cdot \theta^{s(F^R x, F^R y)} .
  \] (1.5.1)

Especially, the bottom assumption dictates the regularity of $F$ on top levels $\Delta_{R_i-1,i}$ which is called Hölder continuous condition.

### 1.5.2 Statements of Results by Young Tower

We first denote some function spaces which are compatible with the introduced Young tower. For simplicity, we refer to points in $\Delta$ as $x$ rather than a pair of coordinates. Let $0 < \theta < 1$ be as above and function $h(x) : \Delta \to \mathbb{R}$. Define that

\[
\begin{align*}
C_\theta(\Delta) & := \{ \exists H \text{ s.t. } |h(x) - h(y)| \leq H \cdot \theta^{s(F^R x, F^R y)}, \forall x, y \in \Delta \}, \\
C_\theta^+(\Delta) & := \{ h \in C_\theta(\Delta) \mid \text{ either } h(\Delta_{0,i}) \equiv 0 \text{ or } h(x) > 0, \exists H^+ \text{ s.t. } \left| \frac{h(x)}{h(y)} - 1 \right| \leq H^+ \cdot \theta^{s(x,y)}, \forall x, y \in \Delta \},
\end{align*}
\] (1.5.2)

where $H$ and $H^+$ depend on function $h(x)$. Denote $(F^*_n \mu)(E) := \mu(F^{-n}E)$ with $E \subseteq \Delta$ and reference measure $\mu$ on $\Delta$.

**Theorem 1.5.1** ([41]). Suppose all assumptions above are satisfied. Then

1. $F : \Delta \to \Delta$ admits an absolutely continuous invariant measure $\nu$ with respect to $\lambda$;
2. $\frac{d\nu}{d\lambda} \in C_\theta^+(\Delta)$ and $\frac{d\nu}{d\lambda} \geq c_0$ for some $c_0 > 0$;
3. $(F, \nu)$ is exact and then ergodic and mixing.

We introduce a new return time function $\hat{R} : \Delta \to \mathbb{Z}^+$ defined by

\[
\hat{R}(x) := \text{ the smallest integer } n \geq 0 \text{ such that } F^n(x) \in \Delta_0, \text{ for } x \in \Delta.
\]

Observe that $\lambda\{\hat{R} > n\} = \sum_{l \geq n} \lambda(\Delta_l)$. As $n \to \infty$, the asymptotics of $\lambda\{\hat{R} > n\}$ is an extremely important role in the following results.

**Theorem 1.5.2** ([41]). Suppose all assumptions above are satisfied.

1. **Lower bounds.** There exist (many but not all) probability measures $\nu$ on $\Delta$ with $\frac{d\nu}{d\lambda} \in C_\theta^+(\Delta)$ such that for some $c_1 = c_1(\nu) > 0$,
   \[
   |F^n_\ast \nu - \nu| \geq c_1 \cdot \lambda\{\hat{R} > n\}.
   \]
Upper bounds. For any $\nu$ with $\frac{d\nu}{d\lambda} \in C_{\theta}(\Delta)$, an upper bound for $|F_n^*\nu - \nu|$ is determined by the asymptotics of $\lambda\{\hat{R} > n\}$ and some certain decreasing exponential functions; see details in [54]. There are two special cases:

- if $\lambda\{\hat{R} > n\} = O(n^{-\alpha})$ for some $\alpha > 0$, then for all $\nu$ as above, $|F_n^*\nu - \nu| = O(n^{-\alpha})$;

- if $\lambda\{\hat{R} > n\} = O(\gamma n)$ for some $0 < \gamma < 1$, then there exists a $\tilde{\gamma} < 1$ such that for all $\nu$ as above, $|F_n^*\nu - \nu| = O(\tilde{\gamma} n)$.

Let $\Phi : \Delta \rightarrow \mathbb{R}$ be an observable on probability space $(\Delta, \nu)$. For random variables of the type $\{\Phi \circ F_n\}_{n=0,1,2,...}$, we denote the covariance of random variables with respect to $\nu$ as follow

$$
\text{Cov}(\Phi \circ F^n, \Psi) = \int (\Phi \circ F^n) \Psi d\nu - \int \Phi d\nu \int \Psi d\nu.
$$

The speed of correlation decay is close related to the rates of convergence. Replacing $|F_n^*\nu - \nu|$ by $|\text{Cov}(\Phi \circ F^n, \Psi)|$, we say that the Decay of Correlations Theorem holds for some $\Phi, \Psi$.

**Theorem 1.5.3 (Decay of Correlations,[41]).** Suppose all assumptions in case (2) of Theorem (1.5.2) are satisfied. Then for $\Phi \in L^\infty(\Delta, \lambda)$ and $\Psi \in C_\theta(\Delta)$,

- $\text{Cov}(\Phi \circ F^n, \Psi) = O(n^{-\alpha})$, if $\lambda\{\hat{R} > n\} = O(n^{-\alpha})$ for some $\alpha > 0$;

- $\text{Cov}(\Phi \circ F^n, \Psi) = O(\tilde{\gamma} n)$, if $\lambda\{\hat{R} > n\} = O(\gamma n)$ for some $0 < \gamma < 1$.

1.5.3 A Class of Circle Map and Distortion

This example was studied in [18]. We follow closely the presentation of [18] in this example.
In this section we introduce a family of one-parametric dynamical systems which mix at polynomial speeds and apply the Young tower to obtain the results as mentioned above. Now we give notations of the non-uniformly expanding circle map $\tau$.

Let $S^1 = [-1, 1]$ be a torus and $\tau : S^1 \to S^1$ be a map satisfying the properties:

- $\tau$ is $C^1$ on $S^1$;
- $\tau$ is $C^2$ on $S^1 \setminus \{0, \{\pm 1\}\}$;
- there are fixed points $p_i$ such that $\tau'(p_i) = 1$ and $|\tau'(x)| > 1$ on $S^1 \setminus \{p_i\}$;

Let $\tau$ be defined implicitly by the equations (see figure 1.6), for positive value of $x \in S^1$

$$x = \begin{cases} \frac{\alpha}{2} (1 + \tau(x))^\alpha, & \text{if } 0 \leq x \leq \frac{\alpha}{2} \\ \tau(x) + \frac{\alpha}{2} (1 - \tau(x))^\alpha, & \text{if } \frac{\alpha}{2} \leq x \leq 1 \end{cases}$$

(1.5.3)

and put $\tau(x) = -\tau(-x)$ for negative part of $S^1$. Observe that when $\alpha = 1$ it is the classical doubling map; that is $\tau(x) = \begin{cases} 2x - 1, & \text{if } x \in [-1, 0] \\ 2x + 1, & \text{if } x \in [0, 1] \end{cases}$. We assume that $0 < \alpha < 1$. Then the points 1, $-1$ are fixed points such that $\tau'(x) = 1$ while $|\tau'(x)|$ becomes infinity at the origin point. Actually, when $\alpha = \frac{1}{2}$ the circle map given by $\tau(x) = 1 - 2\sqrt{|x|}$ was studied by Hemmer in 1984 and he also proved that the invariant density is $\rho(x) = \frac{1}{2} (1 - x)$, but he just gave a slow decay of correlation. Following from technique of Young tower, we illustrate some statistical properties of this class of maps.

We notice that map $\tau$ preserves the Lebesgue measure $\lambda$ by checking that the Perron-Frobenius operator $P_\tau 1 = 1$ directly (shown in Example 1.3.11). Combining with distortion bound, we obtain the following consequence matched with established tower.

In the following, we study in detail the applications of Young tower to the circle version of these maps. We firstly focus on construction of Young tower and the distortion of maps which satisfy (1.5.1) the Hölder continuous condition. Then, a proof of decay of correlations will be shown in the next section.

![Figure 1.7: The first partition $\eta_1$ of $\tau$](image)

**Notations.** Given two sequences of points $\{a_n\}$ and $\{b_n\}$.  

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- $a_n \sim b_n$ means that \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1; \)
- $a_n \approx b_n$ means that there exists a constant $c$ such that $c^{-1}b_n \leq a_n \leq cb_n$, for $n \geq 1$;
- $a_n \lesssim b_n$, or equivalently $a_n = O(b_n)$, means that for non-negative $\{a_n\}$ and $\{b_n\}$, there exists a constant $c \geq 1$ such that $a_n \leq cb_n$, for any $n \geq 1$;
- $a_n = o(b_n)$ means that \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0; \)

Let map $\tau$ be defined as (1.5.3) with assumption $0 < \alpha < 1$ and set $\tau_+ = \tau|_{[0,1]}$ and $\tau_- = \tau|_{[-1,0]}$.

We introduce three different sequences of cylinders to complete the construction and induction. Firstly, a countable Markov partition $\eta_1 = \{I_n\}_{n=0,1,2,...}$ shown in Figure 1.7 is built on $(-1,0) \cup (0,1)$ as follows:

\[
I_n^+ = \begin{cases} (0,a_0) & \text{if } n = 0 \\ (a_{n-1},a_n) & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad I_n^- = \begin{cases} (a_{-0},0) & \text{if } n = 0 \\ (a_{-n},a_{-(n-1)}) & \text{if } n \geq 1 \end{cases}
\]

where

\[
a_n = \begin{cases} \frac{\alpha}{2} & \text{if } n = 0 \\ \tau_+^{-n}a_0 & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad a_{-n} = \begin{cases} -\frac{\alpha}{2} & \text{if } n = 0 \\ \tau_-^{-n}a_{-0} & \text{if } n \geq 1 \end{cases}
\]

We will see the first return map is Bernoulli on cylinders $\{I_n\}_{n=0,1,2,...}$. A distortion on these cylinders will be estimated but it is possible quite lengthy. We will proceed therefore in an easy way. So, in next step we induce over subsets $\{\tilde{I}_n\}$ where the first return map is mixing and has a nice topological structure. It turns out to be much easier to estimate the distortion on $\tilde{I}_n$ and such distortion persists over the cylinders $I_n^+ \subset \tilde{I}_n$.

![Figure 1.8: Points $\{b_{\pm n}\}$ in sets $\pm_{n,r}$](image)

Secondly, the structure $\eta_2 = \{\tilde{I}_n\}_{n=0,1,2,...}$ is also built on $(-1,0) \cup (0,1)$ but with new subsets $\{B_{n,r}\}$. Define that, for $n \geq 0$ and $r \geq 1$,

\[
\tilde{I}_n = \bigcup_{r \geq 1} B_{n,r}^\pm
\]

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where for any $n \geq 0$, 
\begin{align*}
B^{\pm}_{n,r} = \begin{cases} 
(b_{n+1}, a_n) & \text{if } r = 1 \\
(b_{n+r}, b_{n+r-1}) & \text{if } r \geq 2
\end{cases} \quad \text{and} \quad B^{-}_{n,r} = \begin{cases} 
(a_{-n}, b_{-(n+1)}) & \text{if } r = 1 \\
(b_{-(n+r-1)}, b_{-(n+r)}) & \text{if } r \geq 2
\end{cases}
\end{align*}

and points 
\begin{align*}
b_n = \tau^{-}_{+} a_{-(n-1)} \quad \text{and} \quad b_{-n} = \tau^{-}_{-} a_{n-1}.
\end{align*}

These new defined points are shown in Figure 1.8 and see structure $\eta_2 = \{\hat{I}_n\}_{n=0,1,2,\ldots}$ in Figure 1.9. Then we proceed to induce a map on the cylinder $\hat{I}_n$. Let the first return map be defined as 
\[ \tilde{\tau}_n = \tau^r : \hat{I}_n \to \hat{I}_n \]

acts on the sets $B^{\pm}_{n,r}$.

In particular, for $n = 0$, 
\[ \tau^r(B^+_{0,r}) = (a_{-0}, 0) \quad \text{and} \quad \tau^r(B^-_{0,r}) = (0, a_0), \quad \text{for } r \geq 1 \]

and for $n \geq 1$, 
\[ \tau^r(B^+_{n,r}) = \begin{cases} 
(a_{-n}, a_{n-1}) & \text{if } r = 1 \\
(a_{-n}, a_{-(n-1)}) & \text{if } r \geq 2
\end{cases} \quad \text{and} \quad \tau^r(B^-_{n,r}) = \begin{cases} 
(a_{-(n-1)}, a_n) & \text{if } r = 1 \\
(a_{n-1}, a_n) & \text{if } r \geq 2
\end{cases}. \]

By the first section of this chapter, we have that the induced first return map $\tilde{\tau}_n$ is uniformly expanding in the sense that there exists $\beta_0 > 1$ such that $|\tilde{\tau}'_n(x)| \geq \beta_0$, for any $x \in \hat{I}_n$ and for each $n$ and $r$. In the following, we give a bounded distortion proposition for this induced map which is required as the Hölder continuous condition (1.5.1) in Lai-Sang Young’s theory.

**Proposition 1.5.4.** Let $\tilde{\tau}_n$ be the map induced on $\hat{I}_n$. There exists a constant $C_1 = C_1(n) > 0$ such that for all $r$, we have 
\[ \frac{|\tilde{\tau}'_n(x)|}{|\tilde{\tau}'_n(y)|} = \left| \frac{D\tau^r(x)}{D\tau^r(y)} \right| \leq e^{C_1(|\tau^r(x) - \tau^r(y)|)} \leq e^{2C_1}, \]

where $x, y$ are in the same cylinder of the form $B^+_{n,r}$ or $B^-_{n,r}$.

By using proposition above we obtain the Alder’s condition mentioned in Theorem 1.6.2 as follows.

**Proposition 1.5.5.** Let $\tilde{\tau}_n$ be the map induced on $\hat{I}_n$. There exists a constant $C_2 = C_2(n) > 0$ such that for all $r$, we have 
\[ \left| \frac{\tilde{\tau}''_n(x)}{\tilde{\tau}'_n(x)} \right| \leq C_2, \]

for all $x$ in a cylinder of the form $B^+_{n,r}$ or $B^-_{n,r}$.

**Proof.** See details in [18].

It is important but is not enough to stress the distortion to work on the sets $\hat{I}_n$. Since the first return map is irreducible and non-aperiodic on $\hat{I}_0 = (a_{-0}, a_0)$. Thus, in the final step, we proceed to induce another first return map on interval $(a_{-0}, a_0)$ to get a bounded distortion estimate.
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Thirdly, we now return to the induction over the sets of partition \( \eta_1 \). For convenience, in the following the general forms \( I_n = (a_{n-1}, a_n), n \in \mathbb{Z} \) are used to cover for \( I_n^\pm, n = 0, 1, 2... \) of (1.5.4) and for \( n = 0 \) are intended to be \( I_0^+ = (0, a_0) \) and \( I_0^- = (a_0, 0) \). Then, for each \( n \in \mathbb{Z} \), we define a partition \( \hat{\eta}_n \) of \( I_n \) by

\[
\hat{\eta}_n = \{C_{n,1}, C_{n,2},..., C_{n,r},...\},
\]

where

\[
C_{n,r} = \{x \in I_n, R_{I_n}(x) = r\}
\]

and \( R_{I_n}(x) \) is the first return time of \( x \) back into \( I_n \). Let \( \hat{\tau}_n : I_n \to I_n \) be the first return map to \( I_n \). Each \( C_{n,r} \) is a disjoint union of subintervals of \( I_n \). Moreover, \( \hat{\tau}_n(C_{n,r}) = \tau^r(C_{n,r}) = I_n \) and the induction map \( \hat{\tau}_n \) is surjective and onto. Also, there exists \( \beta_0 > 1 \) (possibly different from the \( \beta_0 \) given for dynamics \( \tau_n \) ) such that \( |\hat{\tau}'_n(x)| \geq \beta_0 \), for any \( x \in I_n \) and for each \( n \).

We notice that the first return map \( \hat{\tau}_n \) induced by \( \tau \) on \( I_n \) coincides with the induced map \( \tilde{\tau}_n \) which is the first return map induced on \( \tilde{I}_n \) constructed in the second step. Then we conclude that the induced map satisfying the bounded distortion or Alder’s condition on sets \( B_{n,r}^+ \) satisfies the bounded distortion or Alder’s condition on sets \( C_{n,r} \) as well.

**Proposition 1.5.6.** Let \( \hat{\tau}_n \) be the map induced \( I_n \) and consider the cylinders \( I_n \subset \tilde{I}_n \). There exists constants \( \hat{C}_1 = \hat{C}_1(n) > 0 \) and \( \hat{C}_2 = \hat{C}_2(n) > 0 \) such that for all \( r \), we have

\[
\left| \frac{\hat{\tau}'_n(x)}{\hat{\tau}'_n(y)} \right| = \left| \frac{D\tau^r(x)}{D\tau^r(y)} \right| \leq e^{C_1|\tau^r(x) - \tau^r(y)|} \leq e^{2C_1},
\]

and

\[
\left| \frac{\hat{\tau}''_n(x)}{\hat{\tau}'_n(x)} \right| \leq \hat{C}_2,
\]

where \( x, y \) are in the component of \( C_{n,r} \).

In order to use Young tower technique to obtain some results, we will build the tower over the set \( I_0^+ = (0, a_0) \). Then from above, we have partition

\[
\hat{\eta}_0 = \{C_{0,1}, C_{0,2},..., C_{0,r},...\}
\]

and define the return time function as the first return time, that is, for all \( x \in I_0^+ \),

\[
R_{I_0^+}(x) = \min\{n \in \mathbb{N}^+ \text{ such that } \tau^n(x) \in I_0^+ \}.
\]
Now observe the set $C_{0,r}$, we have return time $r \geq 2$ and the precise form of set can be computed as follows:

- $r = 1$, $C_{0,1} = \emptyset$;
- $r = 2$, $C_{0,2} = (b_1, \tau_+b_{-1})$;
- $r = 3$, $C_{0,3} = (b_2, (\tau_+ \circ \tau_+)^{-1}b_{-1}) \cup (\tau_+^{-1}b_{-2}, \tau_+^{-2}b_{-2})$;
- $r = 4$, $C_{0,4} = (b_3, (\tau_+^2 \circ \tau_+)^{-1}b_{-1}) \cup ((\tau_- \circ \tau_+)^{-1}b_{-1}, (\tau_- \circ \tau_+)^{-1}b_{-2}) \cup (\tau_+^{-1}b_{-2}, \tau_+^{-3}b_{-3})$;

... $r = n$, $C_{0,n} = (b_{n-1}, (\tau_{-^2} \circ \tau_+)^{-1}b_{-1}) \cup ((\tau_{-^3} \circ \tau_+)^{-1}b_{-1}, (\tau_{-^3} \circ \tau_+)^{-1}b_{-2}) \cup ... \cup (\tau_+^{-1} \circ \tau_+)^{n-1}b_{-(n-1)}), \cup (\tau_+^0b_{-(n-2)}, (\tau_- \circ \tau_+)^{-1}b_{-(n-2)}) \cup (\tau_+^{-1}b_{-(n-2)}, \tau_+^{-1}b_{-(n-1)})$;

where $b_{\pm n}$ are points defined in (1.5.6) and $\tau^n(x) = \tau \circ \tau \circ ... \tau(x)$. For each $r \geq 2$, there are $r - 1$ disjoint components in $C_{0,r}$, where these components are subintervals of sets $\{B_{0,l}^+\}_{1 \leq l \leq r - 1}$ as defined in (1.5.5) such that

- if $r = 2$ , $(b_1, \tau_+^{-1}b_{-1}) \subset B_{0,1}^+ = (b_1, a_0)$;
- if $r \geq 3$ , $(\tau_+^{-1} \circ \tau_+)^{-1}b_{-(r-1)}, (\tau_+^{-1} \circ \tau_+)^{-1}b_{-(r-1)}) \subset B_{0,l}^+, \text{ for } 1 \leq l \leq r - 2);
  (b_{r-1}, (\tau_+^{-1} \circ \tau_+)^{-1}b_{-(r-1)}) \subset B_{0,r-1}^+, \text{ for } l = r - 1$.

Then we represent the cylinder set $C_{0,r}$ as a disjoint union of subintervals in the form:

$$C_{0,r} = \bigcup_{1 \leq l \leq r - 1} W_r^l, \text{ for } r \geq 2,$$

where for $1 \leq l \leq r - 2$,

$$W_r^l = ((\tau_+^{-1} \circ \tau_+)^{-1}b_{-(r-1)}, (\tau_+^{-1} \circ \tau_+)^{-1}b_{-(r-1)}) \text{ and } W_r^{r-1} = (b_{r-1}, (\tau_+^{-2} \circ \tau_+)^{-1}b_{-1}).$$

Noticing that $\tau_0 = \tau^r : W_r^0 \to I_0^+$ is surjective as usual

$$\tau^r(W_r^0) = \tau_+^{-1} \circ \tau_+ \circ \tau_+^{-1} \circ \tau_+ (W_r^0) = I_0^+$$

and for $W_r^{r-1}$, we have

$$\tau^r(W_r^{r-1}) = \tau_+ \circ \tau_+^{-2} \circ \tau_+ (W_r^{r-1}) = I_0^+.$$ 

Tracing the path of a point $x \in W_r^0 \subset (b_1, b_{-1})$, we find that $\tau_+(x) \in (a_{-(l-1)}, a_{-(l-2)})$ and $\tau_+^{-1} \circ \tau_+(x) \in (b_{-(r-1)}, b_{-(r-2)})$; then after the behaviour of $\tau_+, x$ goes into set $(a_{r-2}, a_{r-1})$; finally, it is into $I_0^+$ after $r - l - 1$ iterations of $\tau_+$.

The tower is defined by

$$\Delta := \{(x, l) : x \in I_0^+ \text{ and } l = 0, 1, ..., R_{00}(x) - 1\}.$$ 

Moreover, for $r = R_{00}(x) \geq 2$, the set $C_{0,r} \times r$ of $\Delta_{r-1}$ is the top level of the components directly above $I_0^+$ as shown in Figure 1.10.

Recall the map defined on the tower $T : \Delta \to \Delta$,

$$T(x, l) = \begin{cases} (x, l + 1), & \text{if } l < R_{00}(x) - 1, \\ (\tau_{R_{00}(x)}^r(x), 0), & \text{if } l = R_{00}(x) - 1. \end{cases}$$

According to Theorem 1.5.3, we have a decay of correlations for this class of circle maps.
1.5.4 Decay of Correlations

**Proposition 1.5.7.** Let \( \tau : S^1 \to S^1 \) be a circle map defined in (1.5.3) and \( \lambda \) be the Lebesgue measure. Then for all \( g \in L^{\infty}(S^1, \lambda) \) and \( f \in C_0(S^1) \) as defined in (1.5.2), map \( \tau \) enjoys polynomial decay of correlations with respect to the invariant measure \( \lambda \), that is,

\[
\left| \int (g \circ \tau^n) f \, d\lambda - \int g \, d\lambda \int f \, d\lambda \right| = O(n^{1 - \frac{1}{\alpha}}).
\]

Or equivalent, for \( \tau \) invariant measure \( \nu \) with density \( \rho = \frac{d\nu}{d\lambda} \in C^+_0(S^1) \), we have

\[
\left| \int (g \circ \tau^n) f \, d\nu - \int g \, d\nu \int f \, d\nu \right| = \left| \int g \cdot P^n_\tau(f \rho) \, d\lambda - \int g \rho \cdot (\int f \, d\nu) \, d\lambda \right| = O(n^{1 - \frac{1}{\alpha}}).
\]

**Proof.** The Young tower \( \Delta \) built over \( I_0^+ \) and dynamics \( T(x, l) \) are defined above. According to [54], we have that the decay of correlations is given by the asymptotics of \( \lambda \{ x \in I_0^+; R_{I_0^+} x > n \} \), where

\[
\lambda \{ x \in I_0^+; R_{I_0^+} x > n \} = \sum_{r=n+1}^{\infty} \lambda \{ x \in I_0^+; R_{I_0^+} x = r \} = \sum_{r=n+1}^{\infty} \lambda(C_0,r).
\]

Before computing this quantity, we require two inequalities as follows. We claim that

(I) there exist constants \( K_0 > 0 \) and \( 0 < \theta < 1 \) such that for any \( x, y \in C_{0,r}, r \geq 2 \), we have

\[
\frac{\hat{s}_0(x)}{\hat{s}_0(y)} \leq e^{K_0 \theta^{s(x,y)}}
\]

(1.5.7)

where \( s(x,y) \) is the separation time.

(II) \( \lambda(C_{0,r}) = O(\frac{1}{r^{1-\frac{1}{\alpha}}}) \).

First, Let us prove inequality (I). Review the separation time \( s(x,y) \); for \( x, y \in I_0^+ \), \( s(x,y) \) is the smallest \( n \geq 0 \) such that \( \hat{s}_0^n x \) and \( \hat{s}_0^n y \) lie in distinct components of \( C_{0,r}, r \geq 2 \). Now suppose that \( s(\hat{s}_0^n x, \hat{s}_0^n y) = n \), then under the action of \( \hat{s}_0 \), the orbits of these two points
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\( \hat{\tau}_0 x \) and \( \hat{\tau}_0 y \) will be in the same cylinder \( W^l_0 \) of type \( C_{0,r}, r \geq 2 \) up to time \( n-1 \). Moreover, on these cylinders \( \hat{\tau}_0 \) is monotone and uniformly expanding, that is, \( |\hat{\tau}_0^{-1}(x)| \geq \beta_0 > 1 \). We have

\[
|\hat{\tau}_0 x - \hat{\tau}_0 y| \leq \frac{1}{\beta_0^{n-1}} \leq \frac{1}{\beta_0^{n-1}}.
\]

By Proposition 1.5.6, we obtain that

\[
\left| \frac{\hat{\tau}_0^n(x) - \hat{\tau}_0^n(y)}{\hat{\tau}_0^n - \hat{\tau}_0^n(y)} \right| \leq e^{\hat{\mathcal{C}}_1 |\tau^n(x) - \tau^n(y)|} = e^{\hat{\mathcal{C}}_1 |\hat{\tau}_0 x - \hat{\tau}_0 y|}
\]

where \( \hat{\mathcal{C}}_1 = \hat{\mathcal{C}}_1 \cdot \beta_0 \) and \( \theta = \beta_0^{-1} \). This is the local Hölder condition for \( \log |\hat{\tau}_0^n(x)| \) with exponent \( \theta \) which is an important requirement of Young’s theory.

Next, we estimate \( \lambda(W^l_r) \), for \( r \geq 2, 1 \leq l \leq r-1 \), since we have \( \lambda(C_{0,1}) = 0 \). Thus, by

\[
C_{0,r} = \bigcup_{1 \leq l \leq r-1} W^l_r, \text{ for } r \geq 2,
\]

there are \( r-1 \) sets whose first return time in \( I^+_0 \) is \( r \) and by surjective of \( \hat{\tau}_0 = \tau^r : W^l_r \to I^+_0 \), we know that

\[
\lambda(W^l_r) = \frac{\lambda(I^+_0)}{|D\tau^r(z)|}, \text{ for some point } z \in W^l_r.
\]

If \( 1 \leq l \leq r-2 \), for \( z \in W^l_r \), we have

\[
\tau^r z = \tau^r_{+l-1} \circ \tau_{-l} \circ \tau_{+l-1} \circ \tau_{+}(z) \in I^+_0.
\]

If \( l = r-1 \), for \( z \in W^l_r \), we have

\[
\tau^r(z) = \tau_+ \circ \tau_{-l} \circ \tau_{+}(z) \in I^+_0.
\]

Then, we first compute for \( 1 \leq l \leq r-2 \),

\[
D\tau^r(z) = D\tau^r_{+l-1}(\tau_{-l} \circ \tau_{-l} \circ \tau_{+l-1} \circ \tau_{+l}) \cdot D\tau_{-l}(\tau_{-l} \circ \tau_{+l-1} \circ \tau_{+l}) \cdot D\tau^r_{l-1}(\tau_{+l} \circ \tau_{+l}) \cdot D\tau_{+}(z)
\]

\[
\geq D\tau^r_{+l-1}(a_{r-l-1} \cdot D\tau_{-l}(b_{r-l-1}) \cdot D\tau_{+l}(a_{r-l-1}) \cdot D\tau_{+l}(b_{r-l-1}) \cdot D\tau_{+l}(a_{r-l-1}) \cdot D\tau_{+l}(b_{r-l-1})
\]

\[
\geq \frac{a_1 - a_0}{a_{r-l} - a_{r-l-1}} \cdot D\tau_{+l}(b_{r-l-1}) \cdot \frac{a_1 - a_0}{a_l - a_{l-1}} \cdot D\tau_{+l}(b_{r-l-1}),
\]

since map \( \tau \) is centro-symmetric and \( \tau^r(x) > 1 \), for \( x \in I^+_0 \). In fact, for circle map \( \tau \) with \( 0 < \alpha < 1 \), we have

\[
\tau^r(x) = \alpha(\frac{2}{\alpha})^n x^{\alpha - 1} + o(x^{\alpha - 1}), \text{ when } x \to 0+;
\]

\[
a_n - a_{n-1} \sim \frac{\alpha}{2} (\frac{2}{1-\alpha})^{\frac{1}{1-\alpha}} \cdot n^{-\frac{1}{1-\alpha}}, \text{ for } n > 1;
\]

\[
b_n \sim \frac{\alpha}{2} (\frac{2}{1-\alpha})^{\frac{1}{1-\alpha}} \cdot n^{-\frac{1}{1-\alpha}}, \text{ for } n > 1.
\]

Therefore, our inequality (II) can be shown as follows: for \( 2 \leq l \leq r-2 \),

\[
\lambda(W^l_r) = \frac{\lambda(I^+_0)}{|D\tau^r(z)|}
\]

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\[ \leq \frac{a_1 - a_0}{a_{r-1} - a_{r-1}} \cdot D\tau_+ (b_{r-1}) \cdot \frac{a_1 - a_0}{a_{r-1} - a_{r-1}} \cdot D\tau_+ (b_{l-1}) \]

\[ = \frac{a_0}{(a_1 - a_0)^2} \cdot \frac{(a_{r-1} - a_{r-1}) \cdot (a_{l} - a_{l})}{D\tau_+ (b_{r-1}) \cdot D\tau_+ (b_{l-1})} \leq \frac{1}{((l-1)(r-l-1))^{1+\frac{1}{1-\alpha}}}. \]

Similarly, for \( l = 1 \) and \( l = r - 1 \), we have

\[ \lambda(W_r^1) \leq \frac{a_0}{D\tau_+ (a_{r-2}) \cdot D\tau_+ (b_{r-2})} \leq \frac{1}{(r-2)^{1+\frac{1}{1-\alpha}}}. \]

and

\[ \lambda(W_r^{r-1}) \leq \frac{a_0}{D\tau_+ (a_{r-2}) \cdot D\tau_+ (b_{r-2})} \leq \frac{1}{(r-2)^{1+\frac{1}{1-\alpha}}}. \]

Therefore, by disjoint sets \( \{W_t^i\}_t \),

\[ \lambda(C_{0,r}) = \lambda(\bigcup_{1 \leq t \leq r-1} W_t^i) \]

\[ = \sum_{l=1}^{r-1} \lambda(W_r^l) \]

\[ \leq \sum_{l=2}^{r-2} \frac{1}{((l-1)(r-l-1))^{1+\frac{1}{1-\alpha}}} + \frac{2}{(r-2)^{1+\frac{1}{1-\alpha}}} \]

\[ = \frac{1}{r^{1+\frac{1}{1-\alpha}}} \sum_{l=2}^{r-2} \frac{1}{((l-1)(1-\frac{l-1}{r}))^{1+\frac{1}{1-\alpha}}} + \frac{2}{(r-2)^{1+\frac{1}{1-\alpha}}} \]

\[ \leq \frac{1}{r^{1+\frac{1}{1-\alpha}}}, \]

since \( \sum_{l=2}^{r-2} \frac{1}{((l-1)(1-\frac{l-1}{r}))^{1+\frac{1}{1-\alpha}}} \) is bounded up as \( r \to \infty \).

Now we come back to estimate

\[ \lambda\{x \in I_0^+; R_{I_0^+}(x) > n\} = \sum_{r=n+1}^{\infty} \lambda(C_{0,r}). \]

According to inequality (II), we obtain that

\[ \sum_{r=n+1}^{\infty} \lambda(C_{0,r}) \leq \sum_{r=n+1}^{\infty} \frac{1}{r^{1+\frac{1}{1-\alpha}}} \leq \int_{r=n}^{\infty} \frac{1}{r^{1+\frac{1}{1-\alpha}}} dr = n^{-\frac{1}{1-\alpha}}. \]

Finally, the decay of correlation for \( \tau \) under Lebesgue measure \( \lambda \) is given by

\[ |\int (g \circ \tau^n) f d\lambda - \int g d\lambda \int f d\lambda| = O\{ \sum_{R \geq n+1} \lambda\{x \in I_0^+; R_{I_0^+}(x) > n\} \} \]

\[ = O(n \cdot n^{-\frac{1}{1-\alpha}}) \]

\[ = O(n^{1-\frac{1}{1-\alpha}}). \]
1.6 Pianigiani’s Results for One-dimensional Maps

1.6.1 First Return Map

For non-uniformly one-dimensional expanding maps, in particular when there are some points \( x_i \), such that \( |\tau'(x_i)| = 1 \), Pianigiani [47] provided a scheme to turn the non-expanding map into an expanding one. The first return map is the important tool in [47]. Let \( A \subset [0,1] \). Denote \( \hat{\tau} = \tau^n(x) : A \to A \) as the first return map where \( n(x) \) is the smallest positive integer such that \( \tau^n(x) \in A \).

We first introduce some important results for uniformly expanding map \( \hat{\tau} \) and it is absolutely continuous invariant measure \( \hat{\mu} \). Then we give a formula which relates the invariant density of \( \hat{\tau} \) to that of \( \tau \). We follow closely the work of [47] and [12].

1.6.2 A Piecewise Uniformly Expanding System

Given a map \( \tau \) on the unit interval \( I \), we study the first return map \( \hat{\tau} \) on a subset \( A \subset I \). Let \( (A, \mathcal{B}, \lambda) \) denote the measure space where \( A \) is an interval, \( \mathcal{B} \) is Borel \( \sigma \)-algebra and \( \lambda \) is normalized Lebesgue measure on \( A \). Let \( \hat{\tau} : A \to A \) be a measurable transformation. We assume that there exists a countable (or finite) partition \( \mathcal{P} \) of \( \Delta, \mathcal{P} = \{b_i\}_{i=0}^q \) such that \( \hat{\tau} : A \to A \) is countably (or finite) piecewise \( C^1 \) with finite image; that is

- for each integer \( i \geq 1 \), \( \hat{\tau}_i := \hat{\tau}|_{(b_{i-1}, b_i)} \) is a \( C^1 \) function;
- \( \bigvee_{A} \frac{1}{|\tau'|} < \infty \). We mean the variation obtained by the maximal partition related to \( \hat{\tau} \) is finite.
- there are only finitely many different intervals in the collection \( \{\hat{\tau}([b_{i-1}, b_i])\} \).

Obviously, \( \hat{\tau} \) has finite image if map \( \hat{\tau} : A \to A \) is finitely piecewise. In the following, we state the general results for this kind of piecewise uniformly expanding systems. See details in [47].

**Theorem 1.6.1** ([47]). Let \( \hat{\tau} : A \to A \) be countably (or finite) piecewise \( C^1 \) with finite image. Suppose that \( |\tau'|(x)\geq \beta_0 > 1 \) wherever \( \tau'(x) \) is defined. Then we have the Lasota-Yorke inequality

\[
\|P_n^\tau(f)\|_{BV} \leq A_0 r^n \|f\|_{BV} + B_0 |f|_1 , \text{ where } A_0, B_0 > 0, \text{ and } 0 < r < 1.
\]

In particular, \( \hat{\tau} \) admits an absolutely continuous invariant measure.

**Theorem 1.6.2** (in [47]). Let \( \hat{\tau} : A \to A \) be countably (or finite) piecewise \( C^1 \) with finite image. If we assume that

- \( \hat{\tau} \) is a piecewise onto map,
- \( |\hat{\tau}'(x)| \geq \beta_0 > 1 \) wherever \( \hat{\tau}'(x) \) is defined,
- \( \frac{|\hat{\tau}'|}{|\tau'|} \leq \gamma_0 < \infty \) wherever \( \hat{\tau}', \hat{\tau}'' \) are defined,

In particular, \( \hat{\tau} \) admits a unique absolutely continuous invariant measure with density function bounded away from zero.

Using these theorems on uniformly expanding maps \( \hat{\tau} \), we use the first return map to obtain results for the non-expanding system \( \tau \) to prove the existence of invariant measures and to obtain a formula of the invariant density of \( \tau \) in terms of the invariant density of \( \hat{\tau} \). Now we construct the non-uniformly expanding system.
1.6.3 The Non-uniformly Expanding System

Let $I = [0,1]$ be the unit interval, $\lambda$ be Lebesgue measure on $[0,1]$. Let $\tau : (I, \mathcal{B}, \lambda) \rightarrow (I, \mathcal{B}, \lambda)$ be a countably piecewise $C^1$ with finite image. Claim that $\tau^n(x), n \geq 1$ is countably piecewise $C^1$ with finite image. We assume that

- there exist finite many points $\{x_i\}$ such that $|\tau'(x_i)| = 1$,
- $|\tau'(x)| > 1$ wherever $\tau'(x)$ is defined for $x \in I \setminus \{x_i\}$.

Let $A \subset I$ be a positive measure set such that

$$A \subset \bigcup_{n=1}^{\infty} \tau^{-n}(A).$$

Then the induced map $\hat{\tau}$ on the set $A$ is well defined. Since for $x \in A, x \in \tau^{-n}(A)$ for some integer $n$, then we have $\tau^n(x) \in A$. Define that $\hat{\tau} : A \rightarrow A, \quad \hat{\tau} = \tau^n(x)$, where $n = n(x) = \text{the smallest positive integer such that } \tau^n(x) \in A$. Observe two sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$. Set $A^c = I \setminus A$,

$$A_n = \left\{ \begin{array}{ll} A_1, & n = 1 \\ A_{n-1} \cap \tau^{-(n-1)}(A^c), & n \geq 2 \end{array} \right., \quad B_n = A_n \setminus A_{n+1}, \quad n \geq 1. \quad (1.6.1)$$

Then we have disjoint sets $\{B_n\}$ such that $\bigcup B_n = A$ and for $x \in B_n, \hat{\tau}(x) = \tau^n(x), n \geq 1$.

**Theorem 1.6.3 ([47]).** Let $\tau : I \rightarrow I$ be a transformation and set $A \subset I$ satisfy $A \subset \bigcup_{n=1}^{\infty} \tau^{-n}(A)$. Let $\hat{\tau} : A \rightarrow A$ be a induced first return map and $\hat{\mu}$ be a probability measure invariant under $\hat{\tau}$. Then the measure $\mu$ defined by

$$\mu(E) = \sum_{n=1}^{\infty} \hat{\mu}(A_n \cap \tau^{-n}E) \quad (1.6.2)$$

is invariant under $\tau$, where $E \subset I$ and $\{A_n\}$ are denoted as above (1.6.1).

**Proof.** We observe that the sets $\{(A_n \cap \tau^{-n}E)\}_{n=1}^{\infty}$ are mutually disjoint and

$$\bigcup_{n=1}^{\infty} (A_n \cap \tau^{-n}E)$$

$$= (A_1 \cap \tau^{-1}E) \cup (A_2 \cap \tau^{-2}E) \cup ... \cup (A_n \cap \tau^{-n}E) \cup ...$$

$$= (A_1 \cap \tau^{-1}E) \cup (A \cap \tau^{-1}A^c \cap \tau^{-2}E) \cup (A \cap \tau^{-2}A^c \cap \tau^{-3}E) \cup ...$$

$$= (\tau^{-1}E \cap A) \cup [\tau^{-1}(\tau^{-1}E \cap A^c) \cap A^c] \cup [\tau^{-2}(\tau^{-1}E \cap A^c) \cap \tau^{-1}A^c \cap A] \cup ...$$

$$= (\tau^{-1}E \cap A) \cup [\tau^{-1}(\tau^{-1}E \cap A^c) \cap A_1] \cup [\tau^{-2}(\tau^{-1}E \cap A^c) \cap A_2] \cup ...$$

$$= (\tau^{-1}E \cap A) \cup \bigcup_{n=1}^{\infty} (\tau^{-n}(\tau^{-1}E \cap A^c) \cap A_n).$$

First suppose that $E \subset A$, then $\tau^{-1}E \cap A \subset B_1$, $\tau^{-n}(\tau^{-1}E \cap A^c) \cap A_n \subset B_{n+1}, n \geq 1$ which are mutually disjoint. So,

$$\mu(E) = \sum_{n=1}^{\infty} \hat{\mu}(A_n \cap \tau^{-n}E) = \hat{\mu}(\bigcup_{n=1}^{\infty} A_n \cap \tau^{-n}E) = \hat{\mu}(\tau^{-1}E) = \hat{\mu}(E); \quad (1.6.3)$$
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that is, \( \mu = \hat{\mu} \) restricted to set \( A \). Next, for the general set \( E \subseteq I \), we have

\[
\mu(\tau^{-1}E) = \mu(\tau^{-1}E \cap A) + \mu(\tau^{-1}E \cap A^c) = \hat{\mu}(\tau^{-1}E \cap A) + \mu(\tau^{-1}E \cap A^c).
\]

Here, replacing \( E \) by \( \tau^{-1}E \cap A^c \) in formula (1.6.2), then

\[
\mu(\tau^{-1}E \cap A^c) = \sum_{n=1}^{\infty} \hat{\mu} \left( A_n \cap \tau^{-n}(\tau^{-1}E \cap A^c) \right).
\]

Therefore, applying formula (1.6.2) again, we have

\[
\mu(\tau^{-1}E) = \hat{\mu}(\tau^{-1}E \cap A) + \sum_{n=1}^{\infty} \hat{\mu} \left( A_n \cap \tau^{-n}(\tau^{-1}E \cap A^c) \right)
\]

\[
= \hat{\mu}(\tau^{-1}E \cap A) + \sum_{n=1}^{\infty} \hat{\mu} \left( A_{n+1} \cap \tau^{-(n+1)}E \right)
\]

\[
= \hat{\mu}(\tau^{-1}E \cap A) + \sum_{n=2}^{\infty} \hat{\mu} \left( A_n \cap \tau^{-n}E \right)
\]

\[
= \sum_{n=1}^{\infty} \hat{\mu} \left( A_n \cap \tau^{-n}E \right) = \mu(E).
\]

\[ \square \]

Let \( \rho, \hat{\rho} \) be invariant density functions for measures \( \mu \) and \( \hat{\mu} \) respectively. In order to get the formula between \( \rho \) and \( \hat{\rho} \). We consider a normalizing constant \( c_R \) to relate these two densities. Let \( R_n \) denote the return time of \( x \in B_n \) and \( A_n, B_n \) are defined in (1.6.1).

**Corollary 1.6.4 ([47]).** Let \( \tau : I \to I \) be a transformation and \( \hat{\tau} : A \to A \) be a \( \tau \) induced first return map as defined above. If \( \hat{\tau} \) admits an invariant measure \( \hat{\mu} \) such that \( \hat{\mu}(A) = 1 \) and measure \( \mu \) is defined as

\[
\mu(E) = c_R \sum_{n=1}^{\infty} \sum_{j=0}^{R_n-1} \hat{\mu}(B_n \cap \tau^{-j}E),
\]

(1.6.4)

where \( c_R^{-1} = \sum_{n \geq 1} R_n \hat{\mu}(B_n) \). Then \( \mu \) is \( \tau \) invariant measure with \( \mu(I) = 1 \) and \( \mu(A) = c_R \).

**Proof.** First we claim that for any measurable set \( E \subseteq I \), \( c_R^{-1} = \sum_{n \geq 1} R_n \hat{\mu}(B_n) \)

\[
\mu(E) = c_R \sum_{n=1}^{\infty} \hat{\mu}(A_n \cap \tau^{-n}E),
\]

(1.6.5)

is \( \tau \) invariant measure and \( \mu(A) = c_R \). It is easy to check that \( \mu(E) = \mu(\tau^{-1}E) \) by Theorem 1.6.3 and \( \mu = c_R \hat{\mu} \) restricted to set \( A \). Recall that in (1.6.1), \( A_n \cap \tau^{-n}A = A_n \setminus (A_n \cap \tau^{-n}A) = A_n \setminus A_{n+1} \equiv B_n \). By formula (1.6.5) and assumption \( \hat{\mu}(A) = 1 \), we have

\[
\mu(A) = c_R \sum_{n=1}^{\infty} \hat{\mu}(A_n \cap \tau^{-n}A) = c_R \sum_{n=1}^{\infty} \hat{\mu}(B_n) = c_R \hat{\mu}(\bigcup_{n=1}^{\infty} B_n) = c_R \hat{\mu}(A) = c_R.
\]

Since \( \mu(E) \) defined in formula (1.6.5) is \( \tau \) invariant, then

\[
\mu(E) = \mu(\tau E) = c_R \sum_{n=1}^{\infty} \hat{\mu}(A_n \cap \tau^{-(n-1)}E).
\]
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Now we just need to prove that \( \sum_{n=1}^{\infty} \hat{\mu}(A_n \cap \tau^{-(n-1)}E) = \sum_{n=1}^{\infty} \sum_{j=0}^{R_n-1} \hat{\mu}(B_n \cap \tau^{-j}E) \). Observe that \( A_n = \bigcup_{j=n}^{\infty} B_j \) and \( \{B_n\} \) are pairwise disjoint sets, then for each \( n \geq 1 \), we have

\[
\hat{\mu}(A_n \cap \tau^{-(n-1)}E) = \hat{\mu} \left( \bigcup_{j=n}^{\infty} (B_j \cap \tau^{-(n-1)}E) \right) = \sum_{j=n}^{\infty} \hat{\mu}(B_j \cap \tau^{-(n-1)}E).
\]

Therefore,

\[
\sum_{n=1}^{\infty} \hat{\mu}(A_n \cap \tau^{-(n-1)}E) = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \hat{\mu}(B_j \cap \tau^{-(n-1)}E) = \hat{\mu}(B_1 \cap \tau^{-(n-1)}E) + \sum_{j=1}^{\infty} \hat{\mu}(B_2 \cap \tau^{-(n-1)}E) + \ldots = \hat{\mu}(B_1 \cap \tau^{-(n-1)}E) + \sum_{j=1}^{\infty} \hat{\mu}(B_2 \cap \tau^{-(n-1)}E) + \ldots = \sum_{n=1}^{\infty} \sum_{j=0}^{R_n-1} \hat{\mu}(B_n \cap \tau^{-j}E)
\]

since for each \( x \in B_n, R_n(x) = n \). Moreover,

\[
\mu(I) = cR \sum_{n=1}^{\infty} \sum_{j=0}^{R_n-1} \hat{\mu}(B_n \cap \tau^{-j}I) = cR \sum_{n=1}^{\infty} \sum_{j=0}^{R_n-1} \hat{\mu}(B_n) = cR \sum_{n=1}^{\infty} R_n \hat{\mu}(B_n) = 1.
\]

Using (1.6.4), we notice that for any measurable set \( E \subseteq A, \mu(E) = cR \hat{\mu}(E) \). Then passing to the densities \( \rho \) and \( \hat{\rho} \), we obtain that

\[
\rho(x) = cR \hat{\rho}(x), \quad \text{for almost all } x \in A.
\]

As for \( x \in A^c \), we can also give a formula for \( \rho \) extended by \( \hat{\rho} \). In the following example, we work with a particular transformation \( \tau \) to analyze the precise formula of densities.

**Example 1.6.5 (Formula of Densities).** This example follows closely Lemma 3.3 of [12]. Let \( I = [0,1] \) be the unit interval, \( \lambda \) be Lebesgue measure on \([0,1]\). We define a class of maps \( \tau : I \to I \) with a neutral fixed point 0. An example of this type map is shown in Figure 1.11. Let \( 0 < \alpha < 1 \) and we assume that

- \( \tau(0) = 0 \) and there is a \( x_0 \in (0,1) \) such that \( \tau_1 = \tau \big|_{[0,x_0)}, \tau_2 = \tau \big|_{[x_0,1]} \) and \( \tau_1 : [0,x_0) \onto [0,1), \tau_2 : [x_0,1] \onto [0,1]; \)

- \( \tau'(0) = 1 \) and \( \tau'(x) > 1 \) for \( x \in (0,x_0) ; \tau'(x) \geq \beta_0 > 1 \) for \( x \in (x_0,1) \);

- \( \tau_1 \) and \( \tau_1' \) have the form

\[
\tau_1(x) = x + x^{1+n} + x^{1+n} \delta_0(x), \\
\tau_1'(x) = 1 + (1+\alpha)x^n + x^n \delta_1(x),
\]

where \( \delta_i(x) \to 0 \) as \( x \to 0 \) for \( i = 0, 1 \) with \( \delta_i(x) \geq 0 \).
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Figure 1.11: The figure of induced map $\hat{\tau}$ for the value $\alpha = 0.5$

Figure 1.12: The figure of map $\tau$ for the value $\alpha = 0.5$
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Denote points and sets,

\[ x_{n+1} = \tau_1^{-1}x_n \in [0, x_0], \quad W_0 := (x_0, 1) \text{ and } W_n := (x_n, x_{n-1}), \text{ for } n \geq 1. \]

Then the induced map \( \hat{\tau} : A \to A \) with Figure 1.12 is well defined on set \( A = [x_0, 1] \) by

\[ \hat{\tau} = \tau^n(x), \quad \text{for } x \in B_n, n \geq 1, \]

where \( B_n \) are denoted such that

\[
B_n = \tau_2^{-1}W_{n-1} = \begin{cases} 
\tau_2^{-1}(x_0, 1), & \text{for } n = 1 \\
\tau_2^{-1}(x_{n-1}, x_{n-2}), & \text{for } n \geq 2
\end{cases},
\]

Observe that the return time \( R_n = R(B_n) = n \). Let \( \mu \) be a \( \tau \) invariant measure defined as in (1.6.4) and \( \hat{\mu} \) be a absolutely continuous invariant measure under \( \hat{\tau} \). Then, for the densities \( \rho \) and \( \hat{\rho} \) we have

\[
\rho(x) = \begin{cases} 
cR\hat{\rho}(x) & \text{for } x \in A \\
cR \sum_{n=1}^{\infty} \left( \frac{\hat{\rho}(\tau_2^{-1}x)}{|D\tau^{(n)}(\tau_2^{-1}x)|} \right) & \text{for } x \in A^c
\end{cases},
\]

More specifically, for each \( W_k, k \geq 1 \), \( \rho(x) \) can be represented by

\[
\rho(x) = cR \sum_{n=k+1}^{\infty} \left( \frac{\hat{\rho}(\tau_2^{-1}x)}{|D\tau^{(n-k)}(\tau_2^{-1}x)|} \right), \quad \text{for } x \in W_k.
\]

By (1.6.3) and Corollary 1.6.4, we easily check that \( \rho(x) = cR\hat{\rho}(x) \) for \( x \in A \). Now notice that for \( n \geq 1 \), \( B_n \cap \tau^{-1}W_{n-1} = B_n \) is the only non-empty set of \( \{B_n \cap \tau^{-j}W_{n-1}\} \), for \( 0 \leq j \leq n-1 \) and \( B_{n+p} \cap \tau^{-(1+p)}W_{n-1} = B_{n+p}, p \geq 1 \) as well. Suppose in (1.6.4) \( E = W_k \) for some \( k \geq 1 \). Then, we have

\[
\mu(W_k) = cR \sum_{n=1}^{\infty} \sum_{j=0}^{R_{n-1}} \hat{\mu}(B_n \cap \tau^{-j}W_k) = cR \left( \hat{\mu}(B_{k+1} \cap \tau^{-1}W_k) + \hat{\mu}(B_{k+2} \cap \tau^{-2}W_k) + ... + \hat{\mu}(B_{k+p} \cap \tau^{-(1+p)}W_k) + ...ight)
\]

\[
= cR \sum_{n=k+1}^{\infty} \hat{\mu}(B_n \cap \tau^{-1}W_k) = cR \sum_{n=k+1}^{\infty} \hat{\mu}(B_n).
\]

Thus, for any measurable set \( E \subseteq W_k \), we get the following formula

\[
\mu(E) = cR \sum_{n=k+1}^{\infty} \hat{\mu}(B_n \cap \tau^{-(n-k)}E).
\]

Or equivalently,

\[
\int_{E} \rho(x) d\lambda = cR \sum_{n=k+1}^{\infty} \int_{B_n \cap \tau^{-(n-k)}E} \rho(x) d\lambda = cR \sum_{n=k+1}^{\infty} \int_{\{x \in E \cap \tau^{-(n-k)}x \in B_n\}} P_{\tau^{-(n-k)}}\hat{\rho}(x) d\lambda,
\]

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where \( P_{\tau^{n-k}} \) is Perron-Frobenius Operator of map \( \tau^{n-k} \). For \( x \in E \subseteq W_k \subseteq A^c \), we only focus on the path of \( \tau^{-(n-k)}(x) \) who entries in \( B_n \) finally. It is firstly pushed backward \( n - k - 1 \) times with \( \tau_1^{-1} \) which is still in the set \( A^c \) and then it splits into two parts according to the behaviours of \( \tau_1^{-1} \) and \( \tau_2^{-1} \). But, the part from \( \tau_1^{-1} \) is not a subset of \( B_n \).

Let \( \eta \) denote the partition of map \( \tau^{n-k} \). By (1.3.1),

\[
P_{\tau^{n-k}} \hat{\rho}(x) = \sum_{z \in \{\tau^{-(n-k)}x\}} \frac{\hat{\rho}(z)}{|D\tau^{n-k}(z)|} 1_{\tau(n-k)\eta}(x),
\]

where \( D\tau^{n-k} \) is the derivative of \( \tau^{n-k} \). Thus,

\[
\int_{E} \rho(x) d\lambda = \int_{\{x \in \tau^{-(n-k)}x \in B_n\}} P_{\tau^{n-k}} \hat{\rho}(x) d\lambda = \int_{E} \frac{\hat{\rho}(\tau_2^{-1} \circ \tau_1^{-1}(n-k-1)x)}{|D\tau^{n-k}(\tau_2^{-1} \circ \tau_1^{-1}(n-k-1)x)|},
\]

where \( \tau^{(n-k)} = \tau_1^{n-k-1} \circ \tau_2 \) in above. Therefore, for almost all \( x \in A^c \), we have the expression

\[
\rho(x) = c_R \sum_{n=k+1}^{\infty} \frac{\hat{\rho}(\tau_2^{-1} \tau_1^{-1}(n-k-1)x)}{|D\tau^{n-k}(\tau_2^{-1} \tau_1^{-1}(n-k-1)x)|}, \quad \text{ (for some } k \geq 1 \text{ )}
\]

\[
= c_R \sum_{n=1}^{\infty} \frac{\hat{\rho}(\tau_2^{-1} \tau_1^{-1}(n-1)x)}{|D\tau^{n}(\tau_2^{-1} \tau_1^{-1}(n-1)x)|}, \quad \text{ ( replacing } n - k \text{ by } n \text{ )}.
\]
Chapter 2

Rigorous Pointwise Approximations for Invariant Densities of Nonuniformly Expanding Maps

2.1 Introduction

In this chapter, we provide pointwise approximation of invariant densities for intermittent maps that admit an absolutely continuous invariant probability measure.

Ulam-type discretization schemes provide rigorous approximations for dynamical invariants. Moreover, such discretizations are easily implementable on a computer. In [40] it was shown that the original Ulam method [52] is remarkably successful in approximating isolated spectrum of transfer operators associated with piecewise expanding maps of the interval. In particular, it was shown that this method provides rigorous approximations in the $L^1$ norm for invariant densities of Lasota-Yorke maps (see [40] and references therein). This method has been also successful when dealing with multi-dimensional piecewise expanding maps [44], and partially successful\(^1\) in providing rigorous approximations for certain uniformly hyperbolic systems [23, 24]. Recently, Blank [14] and Murray [43] independently succeeded in applying the pure Ulam method in a non-uniformly hyperbolic setting. They obtained approximations in the $L^1$ norm for invariant densities of certain non-uniformly expanding maps of the interval\(^2\).

Although $L^1$ approximations provide significant information about the long-term statistics of the underlying system, they are not helpful when dealing with rare events in dynamical systems. In fact, when studying rare events in dynamical systems [2, 36] one often obtains probabilistic laws that depend on pointwise information from the invariant density of the system. In particular, extreme value laws of interval maps with a neutral fixed point depend pointwise on the invariant density of the map [31].

The difficulties in obtaining pointwise approximations for invariant densities of interval maps with a neutral fixed point is two fold. Firstly, the transfer operator associated with such maps does not have a spectral gap in a classical Banach space. Therefore,

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\(^1\)See [15] for examples where the pure Ulam method provides fake spectra for certain hyperbolic systems.

\(^2\)In [43], in addition to proving convergence, Murray also obtained an upper bound on the rate of convergence.
2.2. PRELIMINARIES

powerful perturbation results \[38]³ are not directly available in this setting. Secondly, invariant densities of such maps are not \(L^\infty\) functions. Consequently, to provide pointwise approximation of such densities, one should first measure the approximations in a ‘properly weighted’ \(L^\infty\)-norm.

In this chapter we use a piecewise linear Ulam-type discretization scheme to provide pointwise approximations for invariant densities of nonuniformly expanding interval maps. We prove that the approximate invariant density converges pointwise to the true density at a rate \(C^* \cdot \ln m \cdot m^{-1}\), where \(C^*\) is a computable fixed constant and \(m^{-1}\) is the mesh size of the discretization. To overcome the spectral difficulties and the unboundedness of the densities which we discussed above, we first induce the map and obtain a uniformly piecewise, expanding and onto map. Then we perform our discretization on the induced space. After that we pull back, both the invariant density and the approximate one to the full space and measure their difference in a weighted \(L^\infty\)-norm. Full details of our strategy is given in subsection 2.3.2.

In section 2.2, we recall results on uniformly piecewise expanding and onto maps. Moreover, we introduce our discretization scheme and recall results about uniform approximations for invariant densities of uniformly piecewise expanding and onto maps. In section 2.3, we introduce our non-uniformly expanding system, set up our strategy, and state our main results, Theorem 2.3.1 and Corollary 2.3.2. Section 2.4 contains technical Lemmas and the proof of Theorem 2.3.1. This chapter is based on our work in [7].

2.2 Preliminaries

2.2.1 A Piecewise Expanding System

Let \((\Delta, \mathcal{B}, \hat{\lambda})\) denote the measure space where \(\Delta\) is an interval, \(\mathcal{B}\) is Borel \(\sigma\)-algebra and \(\hat{\lambda}\) is normalized Lebesgue measure on \(\Delta\). Let \(\hat{\tau} : \Delta \to \Delta\) be a measurable transformation. We assume that there exists a countable partition \(\mathcal{P}\) of \(\Delta\), which consists of a sequence of intervals, \(\mathcal{P} = \{I_i\}_{i=0}^{\infty}\), such that

1. for each \(i = 1, \ldots, \infty\), \(\hat{\tau}_i := \hat{\tau} |_{I_i}\) is monotone, \(C^2\) and it extends to a \(C^2\) function on \(\bar{I}_i\);
2. \(\hat{\tau}_i(I_i) = \Delta\); i.e., for each \(i = 1, \ldots, \infty\), \(\hat{\tau}_i\) is onto;
3. there exists a constant \(D > 0\) such that \(\sup_i \sup_{x \in I_i} |\hat{\tau}_i''(x)| \leq D\);
4. there exits a number \(\gamma\) such that \(\frac{1}{|I_i|} \leq \gamma < 1\).

Let \(\hat{P} : L^1 \to L^1\) denote the transfer operator (Perron-Frobenius) [13, 11] associated to \(\hat{\tau}\):

\[
\hat{P} f(x) = \sum_{y = \hat{\tau}^{-1} x} \frac{f(y)}{|\hat{\tau}'(y)|}.
\]

Under the above assumptions, among other ergodic properties, it is well known (see for instance [16]) \(\hat{\tau}\) admits a unique invariant density \(\hat{\rho}\); i.e. \(\hat{P} \hat{\rho} = \hat{\rho}\). Moreover, \(\hat{P}\) admits a

³See also [28] for another perturbation result, which also requires a spectral gap.
spectral gap when acting on the space of Lipschitz continuous functions over $\Delta$ \cite{6}\footnote{In \cite{6}, a Lasota-Yorke inequality was obtained for Markov interval maps with a finite partition. The proof carries over for piecewise onto maps with a countable number of branches satisfying assumptions of subsection 2.2.1.}. We will denote by $BV(\Delta)$ the space of functions of bounded variation defined on the interval $\Delta$. Set $|| \cdot ||_{BV(\Delta)} := V_{\Delta} + || \cdot ||_1$, where $V_{\Delta}$ denotes the one-dimensional variation over $\Delta$. Then it is well known that $(BV(\Delta), || \cdot ||_{BV(\Delta)})$ is a Banach space and $\hat{P}$ satisfies the following inequality (see Theorem 1.6.1 of Chapter 1 or \cite{47} for instance): there exists a constant $C_{LY} > 0$ such that for any $f \in BV(\Delta)$, we have

$$V_{\Delta} \hat{P} f \leq \gamma V_{\Delta} f + C_{LY} ||f||_1.$$  \hspace{1cm} (2.2.1)

Inequality (2.2.1) is called the Lasota-Yorke inequality.

### 2.2.2 Markov Discretization

We now introduce a discretization scheme which enables us to obtain rigorous uniform approximation of $\hat{\rho}$ the invariant density of $\hat{\tau}$. We use a piecewise linear approximations which was introduced by Ding and Li \cite{19}. Let $\eta = \{c_i\}_{i=0}^m$ be a partition of $I$ into intervals. Since uniform partitions are the first choice for numerical work, we set $c_i - c_{i-1} = \frac{1}{m}$. Everything we do can be easily modified for non-uniform partitions with only minor notational changes. Let

$$\varphi_i = \chi_{[c_{i-1}, c_i]} \text{ and } \phi_i(x) = m \int_0^x \varphi_i d\lambda.$$  \hspace{1cm} (2.2.2)

Let $\psi_i$ denote a set of hat functions over $\eta$:

$$\psi_0 := (1 - \phi_1), \psi_m := \phi_m \text{ and for } i = 1, \ldots, m - 1, \psi_i := (\phi_i - \phi_{i+1}).$$  \hspace{1cm} (2.2.2)

For $f \in L^1$, we set $I_i := [c_{i-1}, c_i]$ and

$$f_i := m \int_{I_i} f dx, \hspace{0.5cm} i = 1, 2, \ldots m,$$

the average of $f$ over the associated partition cell. For $f \in L^1$ we set

$$Q_m f := f_1 \psi_0 + \sum_{i=1}^{m-1} \frac{f_i + f_{i+1}}{2} \psi_i + f_m \psi_m$$

Obviously, the operator $Q_m$ retains good stochastic properties; i.e.,

- for $f \geq 0$, $Q_m f \geq 0$;
- $\int Q_m f = \int f$.

We now define a piecewise linear Markov discretization of $\hat{P}$ by

$$\mathbb{P}_m := Q_m \circ \hat{P}.$$  \hspace{1cm} (2.2.3)

Notice that $\mathbb{P}_m$ is a finite-rank Markov operator whose range is contained in the space of continuous, piecewise linear functions with respect to $\eta$. The matrix representation of $\mathbb{P}_m$...
2.3. POINTWISE APPROXIMATIONS FOR INVARIANT DENSITIES OF MAPS WITH A NEUTRAL FIXED POINT

restricted to this finite-dimensional space and with respect to the basis \( \{ \psi_i \} \) is a (row) stochastic matrix, with entries

\[ p_{ij} := m \int_{f_j} \hat{P} \psi_i \geq 0. \]

By the Perron-Frobenius Theorem for stochastic matrices [39], \( P^m \) has a left invariant density \( \hat{f}^m \); i.e.,

\[ \hat{f}^m = \hat{f}^m P^m. \]

The following theorem was proved in [6]:

**Theorem 2.2.1.** There exits a computable constant \( \hat{C} \) such that for any \( m \in \mathbb{N} \)

\[ ||\hat{\rho} - \hat{f}^m||_\infty \leq \frac{\hat{C} \ln m}{m}. \]

**Remark 2.2.1.** We recall that in [6] it was shown that the constant \( \hat{C} \), which is independent of \( m \), can be computed explicitly.

2.3 Pointwise Approximations for Invariant Densities of Maps with A Neutral Fixed Point

2.3.1 The Non-uniformly Expanding System

Let \( I = [0,1] \) be the unit interval, \( \lambda \) be Lebesgue measure on \([0,1]\). Let \( \tau : I \rightarrow I \) be a piecewise smooth map. We assume that

- \( \tau(0) = 0 \) and there is a \( x_0 \in (0,1) \) such that \( \tau_1 = \tau \mid_{[0,x_0]} \), \( \tau_2 = \tau \mid_{(x_0,1]} \) and \( \tau_1 : [0,x_0] \xrightarrow{onto} [0,1], \tau_2 : (x_0,1] \xrightarrow{onto} (0,1) \);

- \( \tau_1 \) is \( C^1 \) on \([0,x_0]\), \( \tau_1 \) is \( C^2 \) on \((0,x_0]\) and \( \tau_2 \) is \( C^2 \) on \([x_0,1]\);

- \( \tau'(0) = 1 \) and \( \tau'(x) > 1 \) for \( x \in (0,x_0] \); \( |\tau'(x)| \geq \beta > 1 \) for \( x \in (x_0,1); \)

- \( \tau_1 \) and \( \tau_1' \) have the form

\[ \tau_1(x) = x + x^{1+\alpha} + x^{1+\alpha}\delta_0(x), \]
\[ \tau_1'(x) = 1 + (1 + \alpha)x^\alpha + x^\alpha\delta_1(x), \]

where, \( 0 < \alpha < 1 \) and \( \delta_i(x) \rightarrow 0 \) as \( x \rightarrow 0 \) for \( i = 0,1 \) with \( \delta_0'(x) \geq 0 \).

It is well known that \( \tau \) admits a unique invariant density \( \rho \) [32, 41, 47, 54] and the system \((I, \tau, \rho \cdot \lambda)\) exhibits a polynomial mixing rate [32, 41, 54]. Moreover, it is well known [32, 41, 54] that the \( \tau \)-invariant density, \( \rho \), is not an \( L^\infty \)-function. In particular, near \( x = 0 \), \( \rho(x) \) behaves like \( x^{-\alpha} \). Despite this difficulty, we will show that, for any \( x \in (0,1], \)

one can obtain rigorous pointwise approximation of \( \rho(x) \).
2.3. POINTWISE APPROXIMATIONS FOR INVARIANT DENSITIES OF MAPS WITH A NEUTRAL FIXED POINT

2.3.2 Strategy

We first define a Banach space which is weighted $L^\infty$ space, but where $\rho$ has a finite norm. More precisely, let $B$ denote the set of continuous functions on $(0, 1]$ with the norm

$$\| f \|_B = \sup_{x \in [0, 1]} |x^{1+\alpha} f(x)|.$$ 

When equipped with the norm $\| \cdot \|_B$, referring to Definition 1.2.3 of Chapter 1, this $B$ is a Banach space. The fact that $\rho \in B$ follows from Lemma 3.3 of [32]. Our strategy for obtaining pointwise approximation $\rho$ consists of the following steps:

1. We first induce $\tau$ on $\Delta := [x_0, 1]$ and obtain a $\hat{\tau}$ which satisfies the assumptions of subsection 2.2.1.

2. On $\Delta$, we use Theorem 2.2.1 to say that $\hat{f}_m$, the invariant density of the discretized operator $P_m := Q_m \circ \hat{P}$, defined in equation (2.2.3), provides a uniform approximation of $\hat{\rho}$ the $\hat{\tau}$-invariant density.

3. Then we write $\rho$ in terms of $\hat{\rho}$, and define a function $f_m$ as the ‘pullback’ of $\hat{f}_m$.

4. We then use steps (2) and (3) to prove that $\| \rho - f_m \|_B \leq C^* m \ln m$, and deduce a pointwise approximation of $\rho$.

2.3.3 The Induced System

We induce $\tau$ on $\Delta := [x_0, 1]$. For $n \geq 0$ we define

$$x_{n+1} = \tau^{-1}_1(x_n).$$

Set

$$W_0 := (x_0, 1), \text{ and } W_n := (x_n, x_{n-1}), \text{ for } n \geq 1.$$ 

For $n \geq 1$, we define

$$Z_n := \tau^{-1}_2(W_{n-1}).$$

Then we define the induced map $\hat{\tau} : \Delta \to \Delta$ by

$$\hat{\tau}(x) = \tau^n(x) \text{ for } x \in Z_n. \tag{2.3.1}$$

Observe that

$$\tau(Z_n) = W_{n-1} \text{ and } R_{Z_n} = n,$$

where $\tau_{Z_n}$ is the first return time of $Z_n$ to $\Delta$. An example of the map $\tau$ and its induced counterpart $\hat{\tau}$ are shown in Figures 2.1 and 2.2 respectively. It is well known (see for instance [54]) that the $\hat{\tau}$ defined in (2.3.1) satisfies the assumptions of subsection 2.2.1, and, by Theorem 2.2.1, one can obtain a rigorous uniform approximation of its invariant density $\hat{\rho}$. Moreover, by Example 1.6.5 of Chapter 1 or Lemma 3.3 of [12], $\rho$, the invariant density of $\tau$, can be written in terms of $\hat{\rho}$:

$$\rho(x) = \begin{cases} 
  c_R \hat{\rho}(x) & \text{for } x \in \Delta \\
  c_R \sum_{n=1}^{\infty} \left( \frac{\hat{\rho}(x)}{|D^n(x_2^{-1}x_1^{-1}(n-1)x)|} \right) & \text{for } x \in I \setminus \Delta 
\end{cases} \tag{2.3.2},$$

where $\hat{\rho}$ is the $\hat{\tau}$-invariant density, $c_R^{-1} = \sum_{k=1}^{\infty} R_{Z_k} \hat{\mu}(Z_k)$, and $\hat{\mu} = \hat{\rho} \cdot \hat{\lambda}$. 

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Figure 2.1: A typical example of a map $\tau$ which belongs to the family defined in subsection 2.3.1.

Figure 2.2: This figure shows the induced map $\hat{\tau}$ corresponding to the map $\tau$ of Figure 2.1.
2.3.4 The Approximate Density and The Statement of The Main Result

Set
\[
  f_m(x) := \begin{cases} 
  c_{R,m} \hat{f}_m(x) & \text{for } x \in \Delta \\
  c_{R,m} \sum_{n=1}^{\infty} \left( \frac{f_m(\tau_{n-1}^{-1}(x))}{|D_{\tau(n)}(\tau_{n-1}^{-1}(x))|} \right) & \text{for } x \in I \setminus \Delta 
  \end{cases}
\]  \tag{2.3.3}

where \( \hat{f}_m = \mathbb{P}_m \hat{f}_m \), and \( \mathbb{P}_m \) is the Markov discretization of \( \hat{P} \) defined in (2.2.3), \( c_{R,m}^{-1} = \sum_{k=1}^{\infty} R_k \mu_m(Z_k) \), and \( \hat{\mu}_m = \hat{f}_m \cdot \hat{\lambda} \). We will show that the function \( f_m \) defined in (2.3.3) provides a rigorous pointwise approximation of \( \rho \).

**Theorem 2.3.1.** For any \( m \in \mathbb{N} \) we have
\[
  \| \rho - f_m \|_B \leq C^* \ln \frac{m}{m},
\]
where
\[
  C^* = \hat{C} \left( 1 + \frac{x_0^{1+\alpha}}{\beta} + M(1+\alpha) \right) C_4;
\]
in particular, \( \hat{C} \) is the computable constant of Theorem 2.2.1,
\[
  M := \frac{C_1^{1+\alpha} C_0 C_2}{\beta},
\]
\[
  C_0 := \frac{\alpha(1+\alpha)}{2} \left[ 1 + 2\delta_0(x_0) + \delta_0^2(x_0) \right], \quad C_1 := (2^{\frac{1}{\alpha}} - 1)^{1/\alpha},
\]
\[
  C_4 := 1 + C_3 \left( \frac{C_{LY} \gamma}{1 - \gamma} + \frac{1}{\Delta} \right), \quad C_3 := \frac{1}{\beta} + \frac{C_2}{\beta(1-x_0)} (\alpha - \frac{x_0^{1+\alpha}}{1-\alpha}),
\]
and
\[
  C_2 = \frac{1 - x_0^{1+\alpha}}{x_0^{1+\alpha}} 2^{1+\frac{1}{\alpha}} [2^{\frac{1}{\alpha}} - 1]^{1+\frac{1}{\alpha}}.
\]

As a direct consequence of the Theorem 2.3.1 we obtain a pointwise approximation of \( \rho \):

**Corollary 2.3.2.** For any \( x \in (0,1] \) we have
\[
  |\rho(x) - f_m(x)| \leq \frac{C^* \ln m}{x^{1+\alpha} \frac{m}{m}}.
\]

**Proof.** For \( x \in (0,1] \), we have
\[
  |\rho(x) - f_m(x)| = \frac{1}{x^{1+\alpha}} |x^{1+\alpha}(\rho(x) - f_m(x))| \leq \frac{1}{x^{1+\alpha}} \| \rho - f_m \|_B \leq \frac{1}{x^{1+\alpha}} C^* \ln m.
\]

2.4 Proof of Theorem 2.3.1

2.4.1 Technical Lemmas

We first introduce notation of certain functions which appear in the proof of Theorem 2.3.1. For \( x \in I \setminus \Delta \) set:
\[
  g(x) := \frac{(\frac{\pi x}{2})^{1+\alpha}}{\tau_1'(x)},
\]
\[
  G_1(x) := \frac{x^{1+\alpha}}{|\tau_1'\tau_2^{-1}(x)|}; \quad \text{and for } n \geq 2, \quad G_n(x) := \frac{x^{1+\alpha}}{|D_{\tau(n)}(\tau_2^{-1}(x))|}.
\]
2.4. PROOF OF THEOREM 2.3.1

Lemma 2.4.1. For $x \in I \setminus \Delta$, we have

$$[1 + x^\alpha + x^\alpha \delta_0(x)]^{1+\alpha} \leq 1 + (1 + \alpha)[x^\alpha + x^\alpha \delta_0(x)] + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2.$$  

Proof. Let

$$y_1(x) := [1 + x^\alpha + x^\alpha \delta_0(x)]^{1+\alpha}$$

and

$$y_2(x) := 1 + (1 + \alpha)[x^\alpha + x^\alpha \delta_0(x)] + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2.$$  

Note that $y_1(0) = y_2(0) = 1$. Therefore, to prove the lemma, it is enough to prove that $y_1'(x) \leq y_2'(x)$. We have:

$$y_1'(x) = (1 + \alpha)(1 + \xi(x))\xi'(x)$$

and

$$y_2'(x) = (1 + \alpha)(1 + \alpha \xi(x))\xi'(x),$$

where $\xi(x) := x^\alpha + x^\alpha \delta_0(x) \geq 0^5$. Notice that $\xi'(x) \geq 0$. Thus, we only need to show that

$$(1 + \alpha \xi(x))^\alpha \leq (1 + \alpha \xi(x)).$$  

(2.4.1)

Indeed, (2.4.1) holds because $(1 + \xi(0))^\alpha = (1 + \alpha \xi(0)) = 1$ and

$$[(1 + \xi(x))^\alpha]' = \frac{\alpha}{(1 + \xi(x))^{1-\alpha}} \xi'(x) \leq \alpha \xi'(x) = [1 + \alpha \xi(x)]'.$$

\□

Lemma 2.4.2. For $x \in I \setminus \Delta$, we have $g(x) \leq 1 + C_0 x^{2\alpha}$, where

$$C_0 = \frac{\alpha(1 + \alpha)}{2} [1 + 2 \delta_0(x_0) + \delta_0^2(x_0)].$$  

Proof. Using Lemma 2.4.1, we have:

$$g(x) = \frac{\tau_1^x [1 + \alpha \xi' + 1]}{\tau_1^x (x)} = \frac{[1 + x^\alpha + x^\alpha \delta_0(x)]^{1+\alpha}}{1 + (1 + \alpha)x^\alpha + x^\alpha \delta_1(x)}$$

$$\leq \frac{1 + (1 + \alpha)[x^\alpha + x^\alpha \delta_0(x)] + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2}{1 + (1 + \alpha)x^\alpha + x^\alpha \delta_1(x)}$$

$$= \frac{1 + (1 + \alpha)[x^\alpha + x^\alpha \delta_0(x)] + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2}{1 + (1 + \alpha)x^\alpha + x^\alpha \delta_1(x)}$$

$$\leq 1 + \frac{\alpha(1 + \alpha)}{2}[x^\alpha + x^\alpha \delta_0(x)]^2$$

$$= 1 + \frac{\alpha(1 + \alpha)}{2} [1 + 2 \delta_0(x) + \delta_0^2(x)] x^{2\alpha} \leq 1 + C_0 x^{2\alpha}.$$  

\□

Lemma 2.4.3. Let $x_n = \tau_1^{-n} x_0$. For $n \geq 1, x_n \leq C_1 n^{-\frac{1}{\alpha}}$, where $C_1 = (2[2^{\frac{1}{\alpha}} - 1])^{1/\alpha}$.

\footnote{It is obvious that $\xi(0) = 0$ and for $x > 0, \xi(x) > 0$.}
2.4. PROOF OF THEOREM 2.3.1

Proof. Observe that $C_1 > 1 \geq \tau_1^{-1}(x_0) = x_1$. Therefore, the lemma is true for $n = 1$. Next, for $n \geq 2$, we suppose that $x_{n-1} \leq C_1(n-1)^{-\frac{1}{n}}$, and prove that $x_n \leq C_1 n^{-\frac{1}{n}}$. If it is false, that is $x_n > C_1 n^{-\frac{1}{n}}$, then by our inductive statement on $x_{n-1}$, we have:

$$C_1(n-1)^{-\frac{1}{n}} \geq x_{n-1} = \tau_1(x_n) > C_1 n^{-\frac{1}{n}}[1 + C_1^\alpha n^{-1} + C_1^\alpha n^{-1} \delta_0(C_1 n^{-\frac{1}{n}})].$$

This is equivalent to

$$n[(1 + \frac{1}{n-1})^\frac{1}{n} - 1] > C_1^\alpha [1 + \delta_0(C_1 n^{-\frac{1}{n}})].$$

By convexity of the function $z^\frac{1}{n}$, it follows $\frac{n}{n-1}[2^\frac{1}{n} - 1] > C_1^\alpha [1 + \delta_0(C_1 n^{-\frac{1}{n}})]$, that is

$$C_1^\alpha < \frac{n}{n-1}[2^\frac{1}{n} - 1]/[1 + \delta_0(C_1 n^{-\frac{1}{n}})] < 2[2^\frac{1}{n} - 1] = C_1^\alpha.$$

A contradiction. Therefore, $x_n \leq C_1 n^{-\frac{1}{n}}$, and this completes the proof of the lemma. □

**Lemma 2.4.4.** For $x \in I \setminus \Delta$, we have

$$G_1(x) \leq \frac{x_0^{1+\alpha}}{\beta}.$$

and for $n \geq 2$,

$$G_n(x) \leq M(n - 1)^{-(1 + \frac{1}{n})},$$

where $M = \frac{C_1^{1+\alpha}c_0c_1^{2\alpha}}{\beta}$.

Proof. For $n = 1$, it is easy to see that

$$G_1(x) \leq \frac{x_0^{1+\alpha}}{\beta}.$$

For $n \geq 2$, we have

$$G_n(x) = \frac{x_1^{1+\alpha}}{|D\tau^{(n)}(\tau_1^{-2\alpha} x_1^{-1} x_1^{-n-1})|} x_1^{1+\alpha}$$

$$= \frac{x_1^{\frac{1}{1+\alpha}}(\tau_1^{-1} x_1 \tau_1^{-2} x_1 \cdots \tau_1^{-1} x_1^{-n-1}) |\tau_1^{-2\alpha} x_1^{-1} \tau_1^{-1} x_1^{-n-1}| \beta}{\tau_1^{-1} x_1, \tau_1^{-1} x_1^{-n-1} \tau_1^{-2\alpha} x_1^{-1} \tau_1^{-1} x_1^{-n-1}}$$

$$= g_1(\tau_1^{-1} x_1, \tau_1^{-1} x_1^{-n-1}) \cdot g_1(\tau_1^{-1} x_1, \tau_1^{-2} x_1^{-n-1}) \cdot g_1(\tau_1^{-1} x_1^{-n-1}) \cdot \frac{(\tau_1^{-1} x_1^{-n-1})^{1+\alpha}}{|\tau_1^{-1} x_1^{-n-1}|}$$

$$\leq g_1(\tau_1^{-1} x_1, \tau_1^{-1} x_1^{-n-1}) \cdot g_1(\tau_1^{-1} x_1^{-n-1}) \cdot \frac{(\tau_1^{-1} x_1^{-n-1})^{1+\alpha}}{|\tau_1^{-1} x_1^{-n-1}|}.$$

By Lemmas 2.4.2 and 2.4.3, for any $k \geq 1, x \in [0, x_0)$, we have

$$g_1(\tau_1^{-k} x_1) \leq 1 + C_0(\tau_1^{-k} x_1)^{2\alpha} \leq 1 + C_0(\tau_1^{-k} x_0)^{2\alpha}$$

$$= 1 + C_0(x_k)^{2\alpha} \leq 1 + C_0 C_1^{2\alpha} k^{-2}.$$
2.4. PROOF OF THEOREM 2.3.1

Therefore, using (2.4.2) and (2.4.3), for \( n \geq 2 \), we obtain:

\[
G_n(x) = \prod_{k=1}^{n-1} g(\tau_1^{-k}(x)) \cdot \frac{(\tau_1^{-[n-1]}(x))^{1+\alpha}}{\beta} \\
\leq \prod_{k=1}^{n-1} (1 + C_0 C_1^{2\alpha} k^{-2}) \cdot \frac{C_1^{1+\alpha}(n-1)^{-\left(1+\frac{1}{\alpha}\right)}}{\beta} \\
= \exp\left(\sum_{k=1}^{n-1} \ln(1 + C_0 C_1^{2\alpha} k^{-2})\right) \cdot \frac{C_1^{1+\alpha}(n-1)^{-\left(1+\frac{1}{\alpha}\right)}}{\beta} \\
\leq \exp\left(\sum_{k=1}^{n-1} C_0 C_1^{2\alpha} k^{-2}\right) \cdot \frac{C_1^{1+\alpha}(n-1)^{-\left(1+\frac{1}{\alpha}\right)}}{\beta} \\
\leq \exp(C_0 C_1^{2\alpha} (2 - \frac{1}{n-1})) \cdot \frac{C_1^{1+\alpha}(n-1)^{-\left(1+\frac{1}{\alpha}\right)}}{\beta} \\
\leq M(n-1)^{-\left(1+\frac{1}{\alpha}\right)}.
\]

Lemma 2.4.5.

\[
\sum_{n=1}^{\infty} n \cdot \hat{\lambda}(Z_n) \leq C_3,
\]

where \( C_3 = \frac{1}{\beta} + \frac{C_2}{\beta \alpha} (\alpha + \frac{2-\alpha}{1-\alpha}) \) and \( C_2 = \frac{1-x_0}{x_0} \alpha + 2^{1+\frac{1}{\alpha}} \left(2 \frac{1}{\alpha} - 1\right)^{1+\frac{1}{\alpha}}. \)

Proof. By Lemma 2.4.3, we have \( \lambda(W_n) = x_{n-1} - x_n = \tau_1(x_{n}) - x_n = \frac{1-x_0}{x_0} x_n^{1+\alpha} \leq \frac{1-x_0}{x_0} C_1^{1+\alpha} n^{-\left(1+\frac{1}{\alpha}\right)} = C_2 n^{-\left(1+\frac{1}{\alpha}\right)}. \) Since \( \tau_2(Z_n) = W_{n-1} \), we have

\[
\sum_{n=1}^{\infty} n \cdot \lambda(Z_n) \leq \sum_{n=1}^{\infty} \frac{\lambda(W_{n-1})}{\beta} \\
\leq \frac{1-x_0}{\beta} + \sum_{n=2}^{\infty} n \cdot \frac{(n-1)(x_{n-1} - x_n)}{\beta} \\
= \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} (n+1)(x_{n+1} - x_n) \\
= \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} \frac{n(x_{n-1} - x_n)}{\beta} + \sum_{n=1}^{\infty} \frac{x_{n-1} - x_n}{\beta} \\
\leq \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} \frac{C_2}{\beta} n^{-\frac{1}{\alpha}} + \sum_{n=1}^{\infty} \frac{C_2}{\beta} n^{-\left(1+\frac{1}{\alpha}\right)} \\
\leq \frac{1-x_0}{\beta} + \frac{C_2}{\beta} \left(1 + \int_1^{\infty} x^{-\frac{1}{\alpha}} dx\right) + \frac{C_2}{\beta} \left(1 + \int_1^{\infty} x^{-\left(1+\frac{1}{\alpha}\right)} dx\right) \\
= \frac{1-x_0}{\beta} + \frac{C_2}{\beta} \left(\alpha + \frac{2-\alpha}{1-\alpha}\right) = (1-x_0) \cdot C_3.
\]

This completes the proof of the lemma since \( \hat{\lambda}(\cdot) = \frac{\lambda(\cdot)}{1-x_0}. \)
2.4. PROOF OF THEOREM 2.3.1

**Lemma 2.4.6.** We have

\[ |c_{R,m} - c_R| \leq C_3 \cdot \hat{C} \frac{\ln m}{m}. \]

**Proof.** Using the fact that \( c_R \leq 1 \), \( c_{R,m} \leq 1 \) and Theorem 2.2.1, we have

\[
|c_{R,m} - c_R| \leq \left| \frac{1}{\sum_{k=1}^{\infty} R_{Z_k} \hat{\mu}_m (Z_k)} - \frac{1}{\sum_{k=1}^{\infty} R_{Z_k} \hat{\mu} (Z_k)} \right|
\]

\[
= \left| \frac{\sum_{k=1}^{\infty} k[\hat{\mu}(Z_k) - \hat{\mu}_m(Z_k)]}{\sum_{k=1}^{\infty} R_{Z_k} \hat{\mu}_m (Z_k) \cdot \sum_{k=1}^{\infty} R_{Z_k} \hat{\mu} (Z_k)} \right|
\]

\[
\leq \left( \sum_{k=1}^{\infty} k \int_{Z_k} |\hat{\rho} - \hat{f}_m| d\hat{\lambda} \right)
\]

\[
\leq ||\hat{\rho} - \hat{f}_m||_\infty \left( \sum_{k=1}^{\infty} k \hat{\lambda} (Z_k) \right)
\]

\[
\leq \hat{C} \frac{\ln m}{m} \cdot C_3.
\]

In the last estimate, we have used Lemma 2.4.5. \qed

We now have all our tools ready to prove Theorem 2.3.1.

**Proof. (of Theorem 2.3.1)** Using (2.3.2) and (2.3.3), we have

\[
||\rho - f_m||_B = \sup_{x \in (0,1]} |x^{1+\alpha}(\rho(x) - f_m(x))|
\]

\[
\leq \sup_{x \in I \setminus \Delta} |x^{1+\alpha}(\rho(x) - f_m(x))| + \sup_{x \in \Delta} |x^{1+\alpha}(\rho(x) - f_m(x))|
\]

\[
= \sup_{x \in I \setminus \Delta} \left| \sum_{n=1}^{\infty} \frac{x^{1+\alpha}}{D_{\tau(n)}(\tau^{-1}_{2} \tau^{-(n-1)}_{1} x)} |c_R \hat{\rho}(\tau^{-1}_{2} \tau^{-(n-1)}_{1} x) - c_{R,m} \hat{f}_m(\tau^{-1}_{2} \tau^{-(n-1)}_{1} x)| \right| (2.4.4)
\]

\[
+ \sup_{x \in \Delta} |x^{1+\alpha}(c_R \hat{\rho}(x) - c_{R,m} \hat{f}_m(x))|.
\]

Notice that for \( x \in I \setminus \Delta \), and \( n \geq 1, z_n := \tau^{-1}_{2} \tau^{-(n-1)}_{1} x \in \Delta \). Then using the fact that \( c_R \leq 1, c_{R,m} \leq 1 \), Theorem 2.2.1, Lemma 2.4.6, and (2.4.4), we obtain:
2.4. PROOF OF THEOREM 2.3.1

\[ \| \rho - f_m \|_B \leq \sup_{x \in I \setminus \Delta} \left| \sum_{n=1}^{\infty} \frac{x^n}{D_T(n)(\tau_2^{-1} - \tau_1^{-1})} \right| \cdot \sup_{z_n \in \Delta} |(c_R \hat{\rho}(z_n) - c_{R,m} \hat{f}_m(z_n)| \]

\[ + \sup_{x \in \Delta} |c_R \hat{\rho}(x) - c_{R,m} \hat{f}_m(x)| \]

\[ \leq \sup_{x \in I \setminus \Delta} \left| \sum_{n=1}^{\infty} \frac{x^n}{D_T(n)(\tau_2^{-1} - \tau_1^{-1})} \right| \times \]

\[ \left( \sup_{z_n \in \Delta} |\hat{\rho}(z_n) - \hat{f}_m(z_n)| + |c_R - c_{R,m}| \sup_{z_n \in \Delta} |\hat{\rho}(z_n)| \right) \]  

\[ + \sup_{x \in \Delta} |\hat{\rho}(x) - \hat{f}_m(x)| + |c_R - c_{R,m}| \sup_{x \in \Delta} |\hat{\rho}(x)| \]

\[ \leq C \ln m \left( \sup_{x \in I \setminus \Delta} \sum_{n=1}^{\infty} |G_n(x)| \left( 1 + C_3 \sup_{z_n \in \Delta} |\hat{\rho}(z_n)| \right) + (1 + C_3 \sup_{x \in \Delta} |\hat{\rho}(x)|) \right). \]

Since \( \hat{\rho} \in BV(\Delta) \), we have \( \sup_{x \in \Delta} |\hat{\rho}(x)| \leq V_{\Delta} \hat{\rho} + \frac{1}{1-x_0} \| \hat{\rho} \|_{1,\Delta} \). Therefore, using the Lasota-Yorke inequality (2.2.1), we obtain

\[ \sup_{x \in \Delta} |\hat{\rho}(x)| \leq \frac{C_{LY}}{1 - \gamma} + \frac{1}{1 - x_0}. \]  

(2.4.6)

Using Lemma 2.4.4 and (2.4.5), we obtain:

\[ \| \rho - f_m \|_B \leq C_4 \hat{C} \ln m \left( 1 + \frac{x_0^{1+\alpha}}{\beta} + M \sum_{n=2}^{\infty} (n-1)^{-\left(1+\frac{1}{\alpha}\right)} \right) \]

\[ = C_4 \hat{C} \ln m \left( 1 + \frac{x_0^{1+\alpha}}{\beta} + M \sum_{n=1}^{\infty} n^{-\left(1+\frac{1}{\alpha}\right)} \right) \]

\[ \leq C_4 \hat{C} \left( 1 + \frac{x_0^{1+\alpha}}{\beta} + M (1 + \alpha) \right) \cdot \frac{\ln m}{m}. \]

\[ \Box \]
Chapter 3

$L^1$–norm Approximations for Invariant Densities of Nonuniformly Expanding Maps

3.1 Introduction

In this chapter, we provide $L^1$–norm approximation of invariant densities for intermittent maps that admit an absolutely continuous invariant probability measure. Our method gives a faster rate than Murray’s result in [43].

3.2 Preliminaries

The following two sections are similar to the Preliminaries 2.2.

3.2.1 A Uniformly Expanding System

Let $(\Delta, \mathcal{B}, \hat{\lambda})$ denote the measure space where $\Delta$ is an interval, $\mathcal{B}$ is Borel $\sigma$-algebra and $\hat{\lambda}$ is normalized Lebesgue measure on $\Delta$. Let $\hat{\tau}: \Delta \to \Delta$ be a measurable transformation. We assume that there exists a countable partition $\mathcal{P}$ of $\Delta$, which consists of a sequence of intervals, $\mathcal{P} = \{I_i\}_{i=0}^\infty$, such that

1. for each $i = 1, \ldots, \infty$, $\hat{\tau}_i := \hat{\tau}_{|I_i}$ is monotone, $C^2$ and it extends to a $C^2$ function on $\overline{I_i}$;
2. $\hat{\tau}_i(I_i) = \Delta$; i.e., for each $i = 1, \ldots, \infty$, $\hat{\tau}_i$ is onto;
3. there exists a constant $D > 0$ such that $\sup_{x \in I_i} \frac{|\hat{\tau}^{\prime} \hat{\tau}^{\prime}(x)|}{|\hat{\tau}^{\prime}(x)|} \leq D$;
4. there exits a number $\gamma$ such that $\frac{1}{|\hat{\tau}_i|} \leq \gamma < 1$.

Let $\hat{P}: L^1 \to L^1$ denote the transfer operator (Perron-Frobenius) [13, 11] associated to $\hat{\tau}$:

$$\hat{P}f(x) = \sum_{y = \hat{\tau}^{-1}x} \frac{f(y)}{|\hat{\tau}^{\prime}(y)|}.$$
3.3. \textit{L}^1–\textit{NORM APPROXIMATIONS FOR INVARIANT DENSITIES OF MAPS WITH A NEUTRAL FIXED POINT}

3.2.2 Markov Discretization
Recall the linear Markov discretization of \( \hat{\mathcal{P}} \) by
\[
P_m := Q_m \circ \hat{\mathcal{P}}. \tag{3.2.1}
\]
which was introduced in Subsection 2.2.2.

3.3 \textit{L}^1–\textit{norm Approximations for Invariant Densities of Maps with A Neutral Fixed Point}

3.3.1 The Non-uniformly Expanding System
In this section, we introduce a class of non-uniformly expanding maps. They have the same first three assumptions of the non-uniformly expanding maps in Section 2.3.1 in Chapter 2, but for the last two conditions.

Let \( I = [0, 1] \) be the unit interval, \( \lambda \) be Lebesgue measure on \( [0, 1] \). Let \( \tau : I \to I \) be a map. We assume that
\begin{itemize}
  \item \( \tau(0) = 0 \) and there is a \( x_0 \in (0, 1) \) such that \( \tau_1 = \tau \mid_{[0,x_0]}, \tau_2 = \tau \mid_{(x_0,1]} \) and
    \[
    \tau_1 : [0,x_0] \overset{onto}{\to} [0,1], \tau_2 : (x_0,1] \overset{onto}{\to} (0,1];
    \]
  \item \( \tau_1 \) is \( C^1 \) on \( [0,x_0] \), \( \tau_1 \) is \( C^2 \) on \( (0,x_0) \) and \( \tau_2 \) is \( C^2 \) on \( [x_0,1] \);
  \item \( \tau'(0) = 1 \) and \( \tau'(x) > 1 \) for \( x \in (0,x_0) \); \( \tau'(x) \geq \beta > 1 \) for \( x \in (x_0,1) \);
  \item \( \tau_1 \) has the form
    \[
    \tau_1(x) = x + \delta x^{1+\alpha}.
    \]
    where, \( 0 < \alpha < 1 \) and \( \delta > 0 \) such that the following additional condition is satisfied;
\end{itemize}
\begin{itemize}
  \item for small enough \( \epsilon > 0 \), \( x_0 \geq \tau_1((\frac{1}{\delta(1+\alpha)})^{1/\alpha} + \epsilon] \).
\end{itemize}
The last condition will be used for Lemma 3.4.1, and the small \( \epsilon \) exists such that
\[
(\frac{1}{\delta(1+\alpha)})^{1/\alpha} + \epsilon \leq (\frac{1}{\delta \cdot \alpha})^{1/\alpha}.
\]
It is well known that \( \tau \) admits a unique invariant density \( \rho \) [32, 41, 47, 54] and the system \( (I, \tau, \rho \cdot \lambda) \) exhibits a polynomial mixing rate [32, 41, 54]. Moreover, it is well known [32, 41, 54] that the \( \tau \)-invariant density, \( \rho \), is not an \( L^\infty \)-function. In particular, near \( x = 0 \), \( \rho(x) \) behaves like \( x^{-\alpha} \). In [7], we show that, for any \( x \in (0,1] \), one can obtain rigorous pointwise approximation of \( \rho(x) \). Using the analogous strategy, we can prove that in \( L^1 \)-norm, the approximation of \( \rho(x) \) also has the rate \( \frac{\ln m}{m} \).

3.3.2 Strategy
Our strategy for obtaining \( L^1 \)- norm approximation \( \rho \) consists of the following steps:

1. We first induce \( \tau \) on \( \Delta \subset I \) and obtain a \( \hat{\tau} \) which satisfies the assumptions of subsection 3.2.1.
2. On $\Delta$, we use Theorem 2.2.1 to say that $\hat{f}_m$, the invariant density of the discretized operator $\mathbb{P}_m := Q_m \circ \hat{P}$, defined in equation (3.2.1), provides a uniform approximation of $\hat{\rho}$ the $\hat{\tau}$-invariant density.

3. Then we write $\rho$ in terms of $\hat{\rho}$, and define a function $f_m$ as the ‘pullback’ of $\hat{f}_m$.

4. We then obtain the result $||\rho - f_m||_1 \leq C L 1 \ln m \cdot m$, a faster $L^1$-norm approximation of $\rho$.

### 3.3.3 The Induced System and The Statement of Result

This is completely same as the induced map in Chapter 2. We induce $\tau$ on $\Delta := (x_0, 1]$. For $n \geq 1$ we define

$$x_{n+1} = \tau_1^{-1}(x_n) \text{ and } a_n = \tau_2^{-1}(x_n).$$

Then we define the induced map $\hat{T} : \Delta \to \Delta$ by

$$\hat{T}(x) = \tau^n(x) \text{ for } x \in Z_n, n \geq 1$$

(3.3.1)

where $Z_1 := (a_0, 1)$ and $Z_n := (a_{n-1}, a_{n-2}), n \geq 2$.

We now define the following sets:

$$W_0 := (x_0, 1) \text{ and } W_n := (x_n, x_{n-1}), n \geq 1.$$ 

Observe that

$$\tau(Z_n) = W_{n-1} \text{ and } R_{Z_n} = n,$$

where $R_{Z_n}$ is the first return time of $Z_n$ to $\Delta$. By Example 1.6.5 of Chapter 1 or Lemma 3.3 of [12], $\rho$, the invariant density of $\tau$, can be written in terms of $\hat{\rho}$:

$$\rho(x) = \begin{cases} c_R \hat{\rho}(x) & \text{ for } x \in \Delta \\ c_R \sum_{n=1}^{\infty} \frac{\hat{\rho}(\tau^{-1}_2(\tau^{-2}_1(n-1)x))}{|D\tau(n)(\tau^{-2}_1(n-1)x)|} & \text{ for } x \in I \setminus \Delta \end{cases}$$

(3.3.2)

where $\hat{\rho}$ is the $\hat{\tau}$-invariant density, $c_R^{-1} = \sum_{k=1}^{\infty} R_{Z_k} \hat{\mu}(Z_k)$, and $\hat{\mu} = \hat{\rho} \cdot \hat{\lambda}$. More specifically, for each $W_k$, $k \geq 1$, $\rho(x)$ can be represented by

$$\rho(x) = c_R \sum_{n=k+1}^{\infty} \frac{\hat{\rho}(\tau^{-1}_2(\tau^{-2}_1(n-k-1)x))}{|D\tau(n-k)(\tau^{-2}_1(n-k-1)x)|}, \text{ for } x \in W_k.$$ 

(3.3.3)

Moreover, we recall that

$$f_m(x) := \begin{cases} c_{R,m} \hat{f}_m(x) & \text{ for } x \in \Delta \\ c_{R,m} \sum_{n=1}^{\infty} \frac{\hat{f}_m(\tau^{-1}_2(\tau^{-2}_1(n-1)x))}{|D\tau(n)(\tau^{-2}_1(n-1)x)|} & \text{ for } x \in I \setminus \Delta \end{cases}$$

(3.3.4)

where $\hat{f}_m := \mathbb{P}_m \hat{f}_m$, and $\mathbb{P}_m$ is the Markov discretization of $\hat{P}$ defined in (2.2.3), $c_R^{-1} = \sum_{k=1}^{\infty} R_{Z_k} \hat{\mu}_m(Z_k)$, and $\hat{\mu}_m = \hat{f}_m \cdot \hat{\lambda}$.

Then, we give the following Theorem
Theorem 3.3.1. There exits a constant $C_{L^1}$ such that for any $m \in \mathbb{N}$

$$||\rho - f_m||_1 \leq C_{L^1} \frac{\ln m}{m},$$

where

$$C_{L^1} := \hat{C} \cdot [C_3 \left( \frac{CLY}{1-\gamma} + \frac{1}{\lambda(\Delta)} \right) + 1] \times [\lambda(\Delta) + \frac{x_0}{\beta} + \frac{1}{\beta} \cdot (M_I + M_{II})];$$

in particular, $\hat{C}$ is the computable constant of Theorem 2.2.1, $C_3$ a constant of Theorem 2.3.1, $M_I$ and $M_{II}$ are two computable constants given in the section 3.4.

### 3.4 Proof of Theorem 3.3.1

In order to prove this Theorem, we first list three results we need in the following. Statement of results:

1. (Theorem 2.2.1)  
   $$||\hat{\rho} - \hat{f}_m||_\infty \leq \hat{C} \frac{\ln m}{m}.$$

2. (Lemma 2.4.6)  
   $$|c_{R,m} - c_R| \leq C_3 \cdot \hat{C} \frac{\ln m}{m}.$$

3. (Inequality (2.4.6))  
   $$\sup_{x \in \Delta} |\hat{\rho}(x)| \leq C_{LY} \frac{1}{1-\gamma} + \frac{1}{\lambda(\Delta)}.$$

In $L^1$ space, we do some simple computation firstly. Then we need a series of technique lemmas. By representations (3.3.2) and (3.3.4) of $\rho$ and $f_m$, we have

$$||\rho - f_m||_1 = \int_{\Delta} |\rho(x) - f_m(x)| dx + \int_{\Delta} |\rho(x) - f_m(x)| dx$$

$$= \int_{\Delta} |c_R \hat{\rho}(x) - c_{R,m} \hat{f}_m(x)| dx$$

$$+ \int_{\Delta} \left| c_R \sum_{n=1}^{\infty} \frac{\hat{\rho}(\tau_2^{-1} \tau_1^{-(n-1)} x)}{|D\tau^{(n)}(\tau_2^{-1} \tau_1^{-(n-1)} x)|} - c_{R,m} \sum_{n=1}^{\infty} \frac{\hat{f}_m(\tau_2^{-1} \tau_1^{-(n-1)} x)}{|D\tau^{(n)}(\tau_2^{-1} \tau_1^{-(n-1)} x)|} \right| dx$$

$$= \int_{\Delta} |c_R \hat{\rho}(x) - c_{R,m} \hat{f}_m(x)| dx$$

$$+ \int_{\Delta} \left| \sum_{n=1}^{\infty} c_R \hat{\rho}(\tau_2^{-1} \tau_1^{-(n-1)} x) - c_{R,m} \hat{f}_m(\tau_2^{-1} \tau_1^{-(n-1)} x) \right| dx$$

$$\leq \int_{\Delta} \left( |c_R - c_{R,m}| \cdot |\hat{\rho}(x)| + c_{R,m} |\hat{\rho}(x) - \hat{f}_m(x)| \right) dx$$

$$+ \int_{\Delta} \left| \sum_{n=1}^{\infty} c_R - c_{R,m} \right| \cdot |\hat{\rho}(\tau_2^{-1} \tau_1^{-(n-1)} x)| dx.$$
3.4. Proof of Theorem 3.3.1

By above three results and \( c_{R,m} \leq 1 \), it follows that

\[
\| \rho - f_m \|_1 \leq [C_3 \cdot \hat{C} \frac{\ln m}{m} \cdot (\frac{C_{LY}}{1-\gamma} + \frac{1}{\lambda(\Delta)}) + \hat{C} \frac{\ln m}{m}] \cdot \lambda(\Delta)
\]

\[+ [C_3 \cdot \hat{C} \frac{\ln m}{m} \cdot (\frac{C_{LY}}{1-\gamma} + \frac{1}{\lambda(\Delta)}) + \hat{C} \frac{\ln m}{m}] \sum_{n=1}^{\infty} \frac{1}{|D_{\tau(n)}(\tau_2^{-1} \tau_1^{-(n-1)} x)|} \]

\[= \hat{C} \frac{\ln m}{m} \cdot [C_3(\frac{C_{LY}}{1-\gamma} + \frac{1}{\lambda(\Delta)}) + 1] \times \left( \lambda(\Delta) + \sum_{n=1}^{\infty} \frac{1}{|D_{\tau(n)}(\tau_2^{-1} \tau_1^{-(n-1)} x)|} \right) \]

Using the assumption \( \tau_2(x) \geq \beta > 1 \) for \( x \in (x_0, 1) \), we have for \( n \geq 2 \),

\[|D_{\tau(n)}(\tau_2^{-1} \tau_1^{-(n-1)} x)| = |D_{\tau_1(n-1)}(\tau_2^{-1} \tau_1^{-(n-1)} x) \cdot D_{\tau_2(n-1)}(\tau_2^{-1} \tau_1^{-(n-1)} x)| \leq \frac{1}{\beta} D_{\tau_1(n-1)}(\tau_2^{-1} \tau_1^{-(n-1)} x). \]

Then, by representation (3.3.3), we have

\[
\int_{I(\Delta)} \sum_{n=1}^{\infty} \frac{1}{|D_{\tau_1(n)}(\tau_2^{-1} \tau_1^{-(n-1)} x)|} dx \leq \int_{I(\Delta)} \left( \frac{1}{\beta} + \frac{1}{\beta} \sum_{n=2}^{\infty} \frac{1}{|D_{\tau_1(n)}(\tau_1^{-(n-1)} x)|} \right) dx
\]

\[= \frac{1}{\beta} \int_{I(\Delta)} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{|D_{\tau(n)}(\tau_1^{-(n-1)} x)|} \right) dx
\]

\[= \frac{1}{\beta} \int_{\bigcup_{k=1}^{\infty} W_k} \left( 1 + \sum_{n=k+1}^{\infty} \frac{1}{|D_{\tau_1(n-k)}(\tau_1^{-(n-k)} x)|} \right) dx
\]

\[= \frac{1}{\beta} \cdot \left( \lambda(I \setminus \Delta) + \int_{\bigcup_{k=1}^{\infty} W_k} \sum_{n=k+1}^{\infty} \frac{1}{|D_{\tau_1(n-k)}(\tau_1^{-(n-k)} x)|} dx \right)
\]

\[= \frac{x_0}{\beta} + \frac{1}{\beta} \sum_{k=1}^{\infty} \int_{W_k} \sum_{n=k+1}^{\infty} \frac{1}{|D_{\tau_1(n-k)}(\tau_1^{-(n-k)} x)|} dx.
\]

In the following, we prove a series of Lemmas to compute the part

\[
\sum_{k=1}^{\infty} \int_{W_k} \sum_{n=k+1}^{\infty} \frac{1}{|D_{\tau_1(n-k)}(\tau_1^{-(n-k)} x)|} dx.
\]

Lemma 3.4.1. Let \( x_n = \tau_1^{-n} x_0 \). Then,

(i) for any \( n \geq 1 \), we have \( x_n \geq c_1 n^{-\frac{\alpha}{\delta}} \), where

\[c_1 = \left( \frac{1}{\delta(1+\alpha)} \right)^{1/\alpha} + \epsilon \leq \left( \frac{1}{\delta \cdot \alpha} \right)^{1/\alpha},\]
for some small \( \epsilon > 0 \);
(ii) for any \( k \geq 1 \), we have \( \lambda(W_k) \leq C_1^{1+\alpha} \delta k^{-1-\frac{1}{\alpha}} \), where
\[
C_1 = \max\{ (2[2^{\frac{1}{\alpha}} - 1])^{1/\alpha}, \left[\frac{2[2^{\frac{1}{\alpha}} - 1]}{\delta}\right]\}^{1/\alpha},
\]
which is a little bit different from the \( C_1 \) of Lemma 2.4.3.

**Proof.** (i) It is true for \( n = 1 \), since we have the assumption \( x_0 \geq \tau_1[\left(\frac{1}{\delta(1+\alpha)}\right)^{1/\alpha} + \epsilon] \).
Next, for \( n \geq 2 \), we suppose that \( x_{n-1} \geq c_1(n-1)^{-\frac{1}{\alpha}} \), and prove that \( x_n \geq c_1 n^{-\frac{1}{\alpha}} \). If it is false, that is \( x_n < c_1 n^{-\frac{1}{\alpha}} \), then by our inductive statement on \( x_{n-1} \), we have:
\[
c_1(n-1)^{-\frac{1}{\alpha}} \leq x_{n-1} = \tau_1(x_n) < c_1 n^{-\frac{1}{\alpha}}[1 + \delta \cdot c_1^{\alpha} n^{-1}].
\]
This is equivalent to
\[
n[(1 + \frac{1}{n-1})^{\frac{1}{\alpha}} - 1] < c_1^{\alpha} \cdot \delta.
\]
By convexity of the function \( z^{\frac{1}{\alpha}} \), it follows \( \frac{n}{n-1} \cdot \frac{1}{\alpha} < c_1^{\alpha} \cdot \delta \), that is
\[
c_1 > \frac{n}{n-1} \cdot \frac{1}{\delta \cdot \alpha}.
\]
If we choose some small \( \epsilon > 0 \) such that
\[
c_1 = \left(\frac{1}{\delta(1+\alpha)}\right)^{1/\alpha} + \epsilon \leq \left(\frac{1}{\delta \cdot \alpha}\right)^{1/\alpha}.
\]
A contradiction. Therefore, \( x_n \geq c_1 n^{-\frac{1}{\alpha}} \), and this completes the proof of (i).

(ii) Similar to the proof of Lemma 2.4.3, we have
\[
x_n \leq C_1 n^{-\frac{1}{\alpha}}, \text{ for } n \geq 1,
\]
where \( C_1 = \max\{ (2[2^{\frac{1}{\alpha}} - 1])^{1/\alpha}, \left[\frac{2[2^{\frac{1}{\alpha}} - 1]}{\delta}\right]\}^{1/\alpha} \) > 1. Note that map \( \tau_1 \) in Chapter 2 has \( \delta = 1 \). Then, we have for any \( k \geq 1 \),
\[
\lambda(W_k) = x_{k-1} - x_k = \tau_1(x_k) - x_k = \delta \cdot x_k^{1+\alpha} \leq C_1^{1+\alpha} \delta k^{-1-\frac{1}{\alpha}}.
\]
\[\square\]

**Lemma 3.4.2.** For \( n \geq 1 \), we have \( \tau'_1 x_n \geq 1 + d \cdot n^{-1} \), where \( d = \delta(1+\alpha)c_1^{\alpha} > 1 \).

**Proof.** It is easy to check that, for \( n \geq 1 \),
\[
\tau'_1 x_n = 1 + (1+\alpha)\delta \cdot x_n^{\alpha}.
\]
By part (i) of Lemma 3.4.1, it follows
\[
\tau'_1 x_n \geq 1 + (1+\alpha)\delta \cdot c_1^{\alpha} n^{-1} = 1 + (1+\alpha)\delta \cdot \left[\frac{1}{\delta(1+\alpha)}\right]^{1/\alpha} + \epsilon]^{\alpha} n^{-1} = 1 + d \cdot n^{-1}.
\]
It is obvious that \( d > 1 \) since \( \epsilon > 0 \).
\[\square\]
Lemma 3.4.3. For each $k \geq 1$, if $x \in W_k = (x_k, x_{k-1})$, then we have
\[
Dr_{1}^{(n-k)}(\tau_1^{-(n-k)} x) \geq \left( \frac{n + 1}{k + 1} \right)^{\frac{d}{1 + \frac{k}{x+1}}} , \text{ for } n \geq k + 1.
\]

Proof. Given some $k \geq 1$, for any $x \in W_k = (x_k, x_{k-1})$, we have
\[
Dr_{1}^{(n-k)}(\tau_1^{-(n-k)} x) \geq Dr_{1}^{(n-k)}(\tau_1^{-(n-k)} x_k)
= Dr_1(\tau_1^{-(n-k)} x_k)
= Dr_{1}(x_{k+1}) \cdot Dr_1(x_{k+2}) \cdots Dr_1(x_n)
= \prod_{j=k+1}^{n} Dr_1(x_j).
\]

By Lemma 3.4.2, we have
\[
\prod_{j=k+1}^{n} Dr_1(x_j) \geq \prod_{j=k+1}^{n} (1 + dj^{-1}) = \exp\left\{ \sum_{j=k+1}^{n} \log(1 + dj^{-1}) \right\}.
\]

Using the fact that for $x \geq 0$,
\[
\frac{x}{1 + x} \leq \log(1 + x) \leq x.
\]

Therefore,
\[
\exp\left\{ \sum_{j=k+1}^{n} \log(1 + dj^{-1}) \right\} \geq \exp\left\{ \sum_{j=k+1}^{n} \frac{dj^{-1}}{1 + dj^{-1}} \right\}
\geq \exp\left\{ \sum_{j=k+1}^{n} \frac{dj^{-1}}{1 + d(k + 1)^{-1}} \right\}
\geq \exp\left\{ \frac{d}{1 + \frac{d}{k+1}} \sum_{j=k+1}^{n} \log(1 + j^{-1}) \right\}
\]
\[
= \exp\left\{ \frac{d}{1 + \frac{d}{k+1}} \log \prod_{j=k+1}^{n} \frac{1 + j}{j} \right\}
= \exp\left\{ \frac{d}{1 + \frac{d}{k+1}} \log \left( \frac{n + 1}{k + 1} \right) \right\}
= \left( \frac{n + 1}{k + 1} \right)^{\frac{d}{1 + \frac{k}{x+1}}}
\]

which completes the proof for any $x \in (x_k, x_{k-1})$, $n \geq k + 1$. \qed
3.4. PROOF OF THEOREM 3.3.1

Note that if we fix a big integer \( k' := \left\lceil \frac{d}{k+1} - 1 \right\rceil \), then for any \( k \geq k' \), we have the following:

\[
\int_{W_k} \sum_{n=k+1}^{\infty} \frac{1}{|D_{\tau_1}^{(n-k)}(\tau_1^{-(n-k)}x)|} \, dx \leq \int_{W_k} \sum_{n=k+1}^{\infty} \frac{n+1}{k+1} - \frac{d}{1 + \frac{k}{k+1}} \, dx
\]
\[
= \lambda(W_k) \sum_{n=k+1}^{\infty} \frac{n+1}{k+1} - \frac{d}{1 + \frac{k}{k+1}}
\]
\[
\leq \lambda(W_k) \int_k^{y+1} \frac{d}{k+1} \, dy
\]
\[
= (k+1)^{1+d/k+1} \lambda(W_k) \int_k^{y+1} \frac{d}{1 + \frac{k}{k+1}} - 1
\]
\[
= \frac{k+1}{1 + \frac{k}{k+1}} - 1 \lambda(W_k).
\]

By part (ii) of Lemma 3.4.1, we have

\[
\frac{k+1}{1 + \frac{k}{k+1}} - 1 \lambda(W_k) \leq \frac{k+1}{1 + \frac{k}{k+1}} - 1 C_1^{1+\alpha} \delta k - \frac{1}{\alpha}
\]
\[
= C_1^{1+\alpha} \delta \frac{k}{1 + \frac{k}{k+1}} - \frac{1}{\alpha} (k^{-\frac{1}{\alpha}} + k^{-\frac{1}{\alpha}}).
\]

Now, we will compute our main part (3.4.1). We write firstly

\[
\sum_{k=1}^{k' \leq 1} \int_{W_k} \sum_{n=k+1}^{\infty} \frac{1}{|D_{\tau_1}^{(n-k)}(\tau_1^{-(n-k)}x)|} \, dx
\]
\[
= \sum_{k=1}^{k' \leq 1} \int_{W_k} \sum_{n=k+1}^{\infty} \frac{1}{|D_{\tau_1}^{(n-k)}(\tau_1^{-(n-k)}x)|} \, dx + \sum_{k'=1}^{k' \leq 1} \int_{W_k} \sum_{n=k+1}^{\infty} \frac{1}{|D_{\tau_1}^{(n-k)}(\tau_1^{-(n-k)}x)|} \, dx
\]
\[
:= (I) + (II).
\]
3.4. PROOF OF THEOREM 3.3.1

For part (I), using assumption $\tau'_1(x) \geq 1$, for any $x \in (0, x_0)$ and fact $k < k'$, we have

$$\sum_{n=k+1}^{\infty} \frac{1}{|D\tau_1^{(n-k)}(\tau_1-(n-k)x)|} \leq \sum_{n=k+1}^{\infty} \frac{1}{|D\tau_1^{(n-k)}(\tau_1-(n-k)x_k)|} \leq k' - k + \sum_{j=k+1}^{k'+2} \frac{1}{|D\tau_1(x_j)|} + \prod_{j=k+1}^{\infty} 1$$

By proof of Lemma 3.4.3, we obtain that

$$\prod_{j=k+1}^{n} \frac{1}{|D\tau_1(x_j)|} \leq \frac{n}{(k'+1)} - \frac{d}{k'+1}.$$ 

It follows that

$$\sum_{n=k+1}^{\infty} \frac{1}{|D\tau_1^{(n-k)}(\tau_1-(n-k)x)|} \leq k' - k + \sum_{n=k'+1}^{\infty} \left( \frac{n+1}{k'+1} \right)^{-\frac{d}{k'+1}}$$

$$\leq k' - k + (k'+1)^{-\frac{d}{k'+1}} \int_{k'}^{\infty} \frac{(y+1)}{y} \cdot \frac{1}{y-1} dy$$

$$= k' - k + (k'+1)^{-\frac{d}{k'+1}} \left( \frac{k'+1}{1+i+\frac{d}{k'+1}} - 1 \right)$$

$$= 1 + \frac{d}{1+i+\frac{d}{k'+1}} - k := C_k.$$ 

Therefore, by part (ii) of Lemma 3.4.1, the term (I) is following:

$$(I) = \sum_{k=1}^{k'-1} \int_{W_k} \sum_{n=k+1}^{\infty} \frac{1}{|D\tau_1^{(n-k)}(\tau_1-(n-k)x)|} dx$$

$$\leq \sum_{k=1}^{k'-1} \lambda(W_k)(C_k - k)$$

$$\leq \sum_{k=1}^{k'-1} C_1^{1+\alpha} \delta k^{-1-\frac{1}{\alpha}} (C_k - k)$$

$$\leq C_1^{1+\alpha} \delta \left( (C_{k'} - 1) + \int_{1}^{k'-1} y^{-1-\frac{1}{\alpha}} (C_{k'} - y) dy \right) := M_I$$
3.4. PROOF OF THEOREM 3.3.1

Then for part (II), $k \geq k'$, using estimation (3.4.2), we have

\[
(II) = \sum_{k=k'}^{\infty} \int \sum_{n=k+1}^{\infty} \frac{1}{|D\tau_1^{(n-k)}(\tau_1^{-1}(n-k)x)|} dx
\]

\[
\leq \sum_{k=k'}^{\infty} \frac{k + 1}{d \frac{1}{1 + \frac{2}{\delta}} - 1} \lambda(W_k)
\]

\[
\leq \sum_{k=k'}^{\infty} \frac{C_1^{1+\alpha} \delta}{d \frac{1}{1 + \frac{\alpha}{\delta+1}} - 1} \cdot (k^{-1-\frac{1}{\alpha}} + k^{-\frac{1}{\alpha}})
\]

\[
\leq \frac{C_1^{1+\alpha} \delta}{d \frac{1}{1 + \frac{\delta}{\delta+1}} - 1} \cdot \int_{y=k'-1}^{\infty} (y^{-1-\frac{1}{\alpha}} + y^{-\frac{1}{\alpha}})
\]

\[:= M_{II}.\]
Chapter 4

ACIM of Random position dependent maps

4.1 Introduction

In this chapter, we are interested in perturbations of intermittent maps. In particular when the indifferent fixed point persists under perturbations. Results on statistical stability of intermittent maps with perturbations of this type were obtained in [1, 4]. More recently results on metastability \(^1\) of intermittent maps where the neutral fixed point persists under deterministic perturbations were obtained in [12]. All the results of [1, 4, 12] are concerned with deterministic perturbations of intermittent maps.

In the random setting, i.e., when a system is randomly perturbed, if the random system admits an ACIM \(\mu_R\) which converges in the weak-\(\ast\)–topology\(^2\) to an ACIM \(\mu\) of the initial system, then we say that the system is stochastically stable in the weak sense. In addition, if the density \(\rho_R\) of \(\mu_R\) converges to \(\rho\), the density of \(\mu\), in the \(L^1\)–norm, we say the system is stochastically stable in the strong sense.

In [5] it was proved that intermittent maps of the type studied in [41] are stochastically stable in the weak sense. However, there are no results on the strong stochastic stability of such maps.

In this chapter, we study a random map \(T\) which consists of a collection of intermittent maps \(\{\tau_k\}_{k=1}^K\) and a probability distribution \(\{p_{k,\varepsilon}(x)\}_{k=1}^K\). We prove existence of a unique ACIM for the random map \(T\). Moreover, we show that, as \(\varepsilon\) goes to zero, the invariant density of the random system \(T\) converges in the \(L^1\)-norm to the invariant density of \(\tau_1\). We obtain our results by using a cone technique. This cone was also used in [45] to study Ulam approximations for deterministic intermittent map.

In section 4.2, we present the setup of the problem. Section 4.3 contains the proof of the existence and uniqueness of the ACIM for the random map. Our main result in this section is Theorem 4.3.1. Section 4.4 contains an example of a random map which satisfy our conditions. In section 4.5, we show that our random maps give rise to an interesting family of 2-dimensional non-uniformly expanding maps which admit a unique

\(^1\)By a metastable system, we mean a system which initially has at least two ACIMs, but once it is perturbed it admits a unique ACIM. Such models were first studied in the expanding case in [27].

\(^2\)Let \(V\) be a topological vector space and \(V^*\) be a corresponding dual vector space consisting of all linear functionals on \(V\). Then the topology on \(V^*\) defined by seminorms is called weak–\(\ast\)–topology.
4.2. PRELIMINARIES

ACIM. Section 4.6 contains the stochastic stability result. Our main result in this section
is Theorem 4.6.1.

4.2 Preliminaries

4.2.1 Setup

Let \((I, \mathcal{B}(I), \lambda)\) be the measure space, where \(I = [0,1]\), \(\mathcal{B}(I)\) is Borel \(\sigma\)-algebra and \(\lambda\) is Lebesgue measure. To simplify the notation in the proofs, we consider a random map
which consists of two maps. The proofs for any finite number of maps is similar. We study
a position dependent random map

\[
T = \{\tau_1(x), \tau_2(x); p_1(x), p_2(x)\},
\]

where

\[
\tau_1 = \begin{cases} 
  x(1 + 2^\alpha x^\beta) & x \in [0, \frac{1}{2}), \\
  \tau_{1,2}(x) & x \in [\frac{1}{2}, 1].
\end{cases}
\]

\[
\tau_2 = \begin{cases} 
  x(1 + 2^\beta x^\beta) & x \in [0, \frac{1}{2}), \\
  \tau_{2,2}(x) & x \in [\frac{1}{2}, 1].
\end{cases}
\]

where \(0 < \beta < \alpha < 1\), \(\tau_{k,2}(\frac{1}{2}) = 0, \tau_{k,2}'(x) > 1, k = 1, 2\) and \(p_k : [0,1] \rightarrow [0,1]\) is a
measurable function such that \(p_1(x) + p_2(x) = 1\), i.e. \(p_1(x), p_2(x)\) are position dependent
probabilities. A position dependent random map is understood as a Markov process with
transition function

\[
P(x, A) = p_1(x)\chi_A(\tau_1(x)) + p_2(x)\chi_A(\tau_2(x)),
\]

where \(A\) is any measurable set in \(\mathcal{B}(I)\) and \(\chi_A\) is the characteristic function of the set \(A\).

4.2.2 Invariant Measures

The transition function \(P(x, A)\) induces an operator \(E_T\) on measures on \((I, \mathcal{B}(I))\) denoted by

\[
E_T\mu(A) = \int_I P(x, A)d\mu(x) = \int_I p_1(x)\chi_A(\tau_1(x)) + p_2(x)\chi_A(\tau_2(x))d\mu(x) = \int_{\tau_1^{-1}(A)} p_1(x)d\mu(x) + \int_{\tau_2^{-1}(A)} p_2(x)d\mu(x).
\]

We say that \(\mu\) is \(T\)-invariant if and only if

\[
E_T\mu(A) = \mu(A);
\]

\(3\)Note that the assumption that \(0 < \beta < \alpha < 1\) is essential for strong stochastic stability in the strong
sense (convergence in \(L^1\)). If for instance \(\alpha > 1\), then \(\tau_1\) will admit an infinite invariant measure, i.e. this
invariant measure does not have an \(L^1\)-density.

\(4\)The results of this paper hold for the following class of maps: Let \(0 < \alpha < 1\). \(\tau\) satisfying

- \(\tau(0) = 0\) and there is a \(t_0 \in (0, 1)\) such that \(\tau : [0, t_0) \rightarrow [0,1], \tau : [t_0, 1] \rightarrow [0,1]\).
- Each branch of \(\tau\) is increasing, convex and is \(C^1\); \(\tau'(0) = 1\) and \(\tau'(x) > 1\) for all \(x \in (0, t_0) \cup (t_0, 1)\).
- There is a constant \(C \in (0, \infty)\) such that \(\tau(x) \geq x + Cx^{1+\alpha}\) for \(x \in [0, t_0]\).

The convexity assumption is essential so that the transfer operator satisfies the cone condition \(\int_0^1 f d\lambda \leq A f^{1-\alpha}\lambda(f)\). We choose to work with a well known representative of this family. Namely, the model studied
in [41].
that is, for any measurable set \( A \),

\[
\mu(A) = \int_{\tau_1^{-1}(A)} p_1(x) d\mu(x) + \int_{\tau_2^{-1}(A)} p_2(x) d\mu(x).
\]

### 4.2. Transfer Operators

If \( \mu \) has a density function \( f \) with respect to \( \lambda \), then \( E_T \mu \) has also a density function which we call \( L_T f \). We obtain, for any measurable set \( A \),

\[
\int_A L_T f d\lambda(x) = E_T \mu(A) = \int_{\tau_1^{-1}(A)} p_1(x) d\mu(x) + \int_{\tau_2^{-1}(A)} p_2(x) d\mu(x)
\]

\[
= \int_{\tau_1^{-1}(A)} p_1(x) f d\lambda(x) + \int_{\tau_2^{-1}(A)} p_2(x) f d\lambda(x)
\]

\[
= \int_A P_{\tau_1}(p_1 f) d\lambda(x) + \int_A P_{\tau_2}(p_2 f) d\lambda(x)
\]

\[
= \int_A [P_{\tau_1}(p_1 f) + P_{\tau_2}(p_2 f)] d\lambda(x), \tag{4.2.1}
\]

where \( P_{\tau_1} \) and \( P_{\tau_2} \) are Perron-Frobenius operators \([17]\) associated with \( \tau_1 \) and \( \tau_2 \) respectively. Since (4.2.1) holds for any measurable set \( A \), we will get an almost everywhere equality:

\[
(L_T f)(x) = P_{\tau_1}(p_1 f)(x) + P_{\tau_2}(p_2 f)(x)
\]

\[
= \sum_{y \in \tau_1^{-1}(x)} \frac{(p_1 f)(y)}{|\tau_1'(y)|} + \sum_{y \in \tau_2^{-1}(x)} \frac{(p_2 f)(y)}{|\tau_2'(y)|}.
\]

We call \( L_T \) the Perron-Frobenius operator associated with the random map \( T \). The properties of \( L_T \) resemble the properties of the classical Perron-Frobenius operator associated with a single deterministic map. \( L_T \) satisfies the properties as follows (see Proposition 1.3.8 in Chapter 1 or \([11]\) Lemma 3.1):

(i)(Linearity) \( L_T : L^1 \to L^1 \) is a linear operator.

(ii)(Positivity) Let \( f \in L^1 \) and assume \( f \geq 0 \), then \( L_T f \geq 0 \).

(iii)(Preservation of integrals)

\[
\int_I L_T f dm(x) = \int_I f dm(x)
\]

(iv)(contraction) for any \( f \in L^1 \),

\[
\| L_T f \|_1 \leq \| f \|_1
\]

(v) \( L_T f = f \iff E_T \mu = \mu \), i.e measure \( \mu = f \cdot \lambda \) is \( T \)-invariant.

(vi)(composition)

\[
L_{T \circ R} f = L_T \circ L_R f
\]

In particular, \( L_{T^n} f = L_T^n f \).

\(^5\)Note that since \( p_1(x), p_2(x) \) are functions of \( x \), \( L_T \) is not a convex combination of \( p_1 \) and \( p_2 \).
### 4.3. Existence and Uniqueness of ACIM

#### 4.3.1 Sufficient Conditions for the Existence of a $T$–ACIM

For $k = 1, 2$, we assume

- (A) $\sum_{i=1}^{l} p_k(\tau_{k,i}^{-1}(x)) \leq 1$, $1 \leq l \leq 2$, is decreasing;
- (B) $\inf_{x \in I} p_k(x) \geq \delta > 0$.

**Theorem 4.3.1.** Under assumptions (A) and (B)

(i) The random map $T$ admits a unique ACIM $\mu$, $d\mu = \rho d\lambda$.

(ii) The invariant density $\rho$ is uniformly bounded below.

We first prove some technical lemmas. The proof of the Theorem 4.3.1 is at the end of this section.

**Lemma 4.3.2.** Let $f \in C_A$. Then, for $x \in (0, 1]$,

(i) $f(x) \leq Ax^{-\alpha} \lambda(f)$;

(ii) $f(x) \leq \frac{1}{2} \lambda(f)$, and in particular, $f(x)|_{x \in \left[\frac{1}{2}, y_2\right]} \leq 2\lambda(f)$;

(iii) $y_1 \geq \frac{x}{2}$, $y_2 \geq \frac{x}{2}$ and $x \geq y_*$;

(iv) $(1-x)^{1-\alpha} \leq 1 - (1-\alpha)x$;

(v) $x^{1-\alpha} - y_s^{1-\alpha} \geq \frac{1-\alpha}{2} x$.
4.3. EXISTENCE AND UNIQUENESS OF ACIM

Proof. (i) We have
\[ x f(x) = \int_0^x f(x) d\lambda(\xi) \leq \int_0^x f(\xi) d\lambda(\xi) \leq Ax^{1-\alpha} \lambda(f). \]

(ii) By \( f(x) \geq 0 \) and decreasing, we have
\[ x f(x) = \int_0^x f(x) d\lambda(\xi) \leq \int_0^x f(\xi) d\lambda(\xi) \leq \int_0^x f(\xi) d\lambda(\xi) + \int_x^\infty f(\xi) d\lambda(\xi) = \lambda(f). \]

So, \( f(x) \leq \frac{1}{x} \lambda(f) \) and in particular \( f(x) \leq 2\lambda(f) \), when \( x \in \left[ \frac{1}{2}, z_* \right) \).

(iii) For \( y_1, y_2 \leq \frac{1}{2}, 0 < \beta < \alpha < 1 \), we have
\[ x = \tau_1(y_1) = y_1(1 + 2^\alpha y_1^\alpha) \leq 2y_1 \quad \text{and} \quad x = \tau_2(y_2) = y_2(1 + 2^\alpha y_2^\beta) \leq 2y_2. \]

Also,
\[ x = \tau_1(y_1) = y_1(1 + 2^\alpha y_1^\alpha) \geq y_1 \quad \text{and} \quad x = \tau_2(y_2) = y_2(1 + 2^\alpha y_2^\beta) \geq y_2. \]

Therefore, \( y_1 \geq \frac{x}{2}, y_2 \geq \frac{x}{2} \) and \( x \geq y_* \).

(iv) Set
\[ g(x) = (1 - x)^{1-\alpha} - [1 - (1 - \alpha)x], \]
then \( g(0) = 1 - 1 = 0 \) and for \( x \in [0, 1] \),
\[ g'(x) = -(1 - \alpha)(1 - x)^{-\alpha} + (1 - \alpha) \cdot (1 - \alpha) \left[ 1 - \frac{1}{(1 - x)^\alpha} \right] \leq 0. \]
Therefore, \( g(x) \leq 0, x \in (0, 1] \), that is \( (1 - x)^{1-\alpha} \leq 1 - (1 - \alpha)x \).

(v) First write,
\[ x^{1-\alpha} - y_*^{1-\alpha} = x^{1-\alpha}[1 - \left( \frac{y_*}{x} \right)^{1-\alpha}] = x^{1-\alpha}[1 - (1 - \frac{x - y_*}{x})^{1-\alpha}]. \]

Let \( \zeta = \frac{x - y_*}{x} \).
In case \( y_* = y_1 \),
\[ x = \tau_1(y_1) = y_1(1 + 2^\alpha y_1^\alpha) > y_1 > 0, \quad x \leq 2y_1 \quad \text{and} \quad \zeta = \frac{x - y_1}{x} \in (0, 1]. \]
Thus,
\[ x^{1-\alpha} - y_*^{1-\alpha} = x^{1-\alpha}[1 - (1 - \zeta)^{1-\alpha}] \geq x^{1-\alpha}[1 - (1 - (1 - \alpha)\zeta)] = x^{1-\alpha}(1 - \alpha) \frac{x - y_1}{x} = x^{-\alpha}(1 - \alpha)(\tau_1(y_1) - y_1) = x^{-\alpha}(1 - \alpha)(2^\alpha y_1^{\alpha+1}) \geq (2y_1)^{-\alpha}(1 - \alpha)(2^\alpha y_1^{\alpha+1}) = (1 - \alpha)y_1. \]
In case $y_* = y_2$,

$$x = \tau_2(y_2) = y_2(1 + 2^\beta y_2^\beta) > y_2 > 0, \quad x \leq 2y_2 \quad \text{and} \quad \zeta = \frac{x - y_2}{x} \in (0, 1].$$

We have

$$x^{1-\alpha} - y_*^{1-\alpha} = x^{1-\alpha}[1 - (1 - \zeta)^{1-\alpha}]$$

$$\geq x^{1-\alpha}[1 - (1 - (1 - \alpha)\zeta)]$$

$$= x^{1-\alpha}(1 - \alpha)\frac{x - y_2}{x}$$

$$= x^{-\alpha}(1 - \alpha)(\tau_2(y_2) - y_2)$$

$$= x^{-\alpha}(1 - \alpha)(2^\beta y_2^\beta + 1)$$

$$\geq (2y_2)^{-\alpha}(1 - \alpha)(2^\beta y_2^\beta + 1)$$

$$= (1 - \alpha)y_2(2y_2)^{\beta - \alpha}.$$

Since $0 < \beta < \alpha < 1$ and $0 \leq 2y_2 \leq 1$, we get $(2y_2)^{\beta - \alpha} \geq 1$.

Thus,

$$x^{1-\alpha} - y_*^{1-\alpha} \geq (1 - \alpha)y_2 \geq (1 - \alpha)\frac{x}{2}.$$

**Lemma 4.3.3.** Let $f \geq 0$ be a decreasing function. Then $L_T f$ is also decreasing.

**Proof.** This proof follows closely Lemma 3.1 of [10].

Let $f$ be a positive decreasing function. Define $\tau_{k,i}^{-1}(x) = x_{k,i}$. Let $x < y$. Since $\tau_{k,i}$ is increasing and $\tau_k(a_{i-1}) = 0$, if $\chi_{\tau_{k,i}}(x) = 0$, then $\chi_{\tau_{k,i}}(y) = 0$. Thus, we consider the case when they are both nonzero and we have

$$(L_T f)(x) - (L_T f)(y) = \sum_{k=1}^{2} \sum_{i=1}^{2} \left[ P_{\tau_k}(p_k f)(x) - P_{\tau_k}(p_k f)(y) \right]$$

$$= \sum_{k=1}^{2} \sum_{i=1}^{2} \left[ \frac{(p_k f)(\tau_{k,i}^{-1}(x))}{\tau_k(\tau_{k,i}^{-1}(x))} - \frac{(p_k f)(\tau_{k,i}^{-1}(y))}{\tau_k(\tau_{k,i}^{-1}(y))} \right]$$

$$= \sum_{k=1}^{2} \sum_{i=1}^{2} \left[ \frac{(p_k f)(x_{k,i})}{\tau_k(x_{k,i})} - \frac{(p_k f)(y_{k,i})}{\tau_k(y_{k,i})} \right]$$

$$= \sum_{k=1}^{2} \sum_{i=1}^{2} \left[ \frac{(p_k f)(x_{k,i})}{\tau_k(x_{k,i})} - \frac{p_k(y_{k,i})f(x_{k,i}) + p_k(y_{k,i})f(x_{k,i})}{\tau_k(y_{k,i})} \right]$$

$$= \sum_{k=1}^{2} \sum_{i=1}^{2} \left[ \frac{(p_k f)(x_{k,i})}{\tau_k(x_{k,i})} - \frac{p_k(y_{k,i})f(x_{k,i})}{\tau_k(y_{k,i})} \right]$$

$$\geq \sum_{k=1}^{2} \sum_{i=1}^{2} \left[ \frac{p_k(x_{k,i})}{\tau_k(x_{k,i})} - \frac{p_k(y_{k,i})}{\tau_k(y_{k,i})} \right] f(x_{k,i})$$

---

[Note that this Lemma only requires assumption (A) to hold.]
4.3. EXISTENCE AND UNIQUENESS OF ACIM

since \( f \) is decreasing, \( x_{k,i} < y_{k,i} \) and \( p_k(y_{k,i})/\tau_k(y_{k,i}) > 0 \). Then the above equation implies that

\[ \mathcal{L}_T f \text{ is decreasing if } \sum_{i=1}^{2} \left[ \frac{p_k(y_{k,i})}{\tau_k(y_{k,i})} - \frac{p_k(y_{k,i})}{\tau_k(y_{k,i})} \right] f(x_{k,i}) \geq 0 \text{ for all } k. \]

In condition \( (A) \):

\[ \sum_{i=1}^{l} \frac{p_k(\tau_k^{-1}(x))}{\tau_k(\tau_k^{-1}(x))}, 1 \leq l \leq 2, \text{ is decreasing for all } k = 1, 2. \]

Then, \( \sum_{i=1}^{2} \left[ \frac{p_k(y_{k,i})}{\tau_k(y_{k,i})} - \frac{p_k(y_{k,i})}{\tau_k(y_{k,i})} \right] \geq 0 \). Also, positive function \( f(x_{k,i}) \geq 0 \). Therefore, \( \mathcal{L}_T f \) is also decreasing under the condition \( (A) \). \( \square \)

**Proposition 4.3.4.** For \( A \geq \frac{4}{1-\alpha} \) the cone \( C_A \) is invariant under the action of the operator \( \mathcal{L}_T \).

**Proof.** By Lemma 4.3.3, for \( f \in C_A \) we know that \( \mathcal{L}_T f \) is decreasing. Also, \( \mathcal{L}_T f \geq 0 \) and \( \lambda(\mathcal{L}_T f) = \lambda(f) \). Therefore we only need to prove that

\[ \int_{0}^{x} \mathcal{L}_T f d\lambda \leq Ax^{1-\alpha} \lambda(\mathcal{L}_T f) = Ax^{1-\alpha} \lambda(f), \]

when \( A \geq A_* = \frac{4}{1-\alpha} \). We have

\[ \int_{0}^{x} \mathcal{L}_T f d\lambda = \int_{0}^{y_1} P_{\tau_1}(p_1f) + P_{\tau_2}(p_2f) d\lambda + \int_{\tau_1^{-1}[0,x]} (p_1f) d\lambda + \int_{\tau_2^{-1}[0,x]} (p_2f) d\lambda \]

\[ = \left( \int_{0}^{\frac{y_1}{2}} + \int_{\frac{y_1}{2}}^{\frac{y_2}{2}} \right) (p_1f) d\lambda + \left( \int_{0}^{\frac{y_2}{2}} + \int_{\frac{y_2}{2}}^{\frac{z_2}{2}} \right) (p_2f) d\lambda \]

\[ \leq \left( \int_{0}^{\frac{y_*}{2}} + \int_{\frac{y_*}{2}}^{\frac{y_*}{2}} \right) (p_1f) d\lambda + \left( \int_{0}^{\frac{y_*}{2}} + \int_{\frac{y_*}{2}}^{\frac{z_2}{2}} \right) (p_2f) d\lambda \]

\[ = \int_{0}^{\frac{y_*}{2}} (p_1 + p_2) f d\lambda + \int_{\frac{z_2}{2}}^{\frac{z_2}{2}} (p_1 + p_2) f d\lambda \leq A y_*^{1-\alpha} \lambda(f) + \frac{z_2}{2} f d\lambda, \]

where \( y_* = \max\{y_1, y_2\} \in [0, \frac{1}{2}] \), \( z_* = \max\{z_1, z_2\} \in [\frac{1}{2}, 1] \). Since our transformations \( \tau_k(x) \) may not be piecewise onto, there are two cases to consider.

In the case 1, \( x \) has only one pre-image. By the Lemma 4.3.2 (iii), we get \( x \geq y_* \). So, \( y_*^{1-\alpha} \leq x^{1-\alpha} \), with \( 1-\alpha > 0 \). Therefore, \( \int_{0}^{x} \mathcal{L}_T f d\lambda \leq A y_*^{1-\alpha} \) for \( A > 0 \). In the case 2, \( x \) has two preimages. From Lemma 4.3.2, we have \( f(x) \leq 2\lambda(f), x \in [\frac{1}{2}, z_*] \). Then,

\[ \int_{\frac{1}{2}}^{\frac{z_2}{2}} f d\lambda \leq 2 \lambda(f) d\lambda = 2(z_* - \frac{1}{2}) \lambda(f). \]

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Moreover, we have $\tau'_{1,2}(x) > 1$, $\tau'_{2,2}(x) > 1$, then $x = \lambda[0, x] = \lambda \circ \tau_k[i, z_k] \geq z_k - \frac{1}{2}, k = 1, 2$ i.e. $x > z_\ast - \frac{1}{2}$. So,

$$\int_{\frac{1}{2}}^\infty f d\lambda < 2x\lambda(f).$$

By the result of Lemma 4.3.2 (iv), we obtain that $x \leq \frac{2}{1-\alpha}(x^{1-\alpha} - y_k^{1-\alpha})$. Then, for $A \geq A_\ast = \frac{4}{1-\alpha}$,

$$\int_0^x L_T f d\lambda < Ay_k^{1-\alpha} \lambda(f) + \frac{4}{1-\alpha}(x^{1-\alpha} - y_k^{1-\alpha})\lambda(f) \leq Ax^{1-\alpha} \lambda(f).$$

Therefore, $L_T f \in C_A$, for $f \in C_A$ and $A \geq \frac{4}{1-\alpha}$.

\[\square\]

Remark 4.3.1. Obviously, if $f \in C_A$ and $A \geq \frac{4}{1-\alpha}$, then $L^n_T f \in C_A, n \geq 1$.

Remark 4.3.2. Since $C_A$ is compact and convex, operator $L_T$ has a fixed point $f_\ast \in C_A$ by Proposition 4.3.4 and the Schauder-Tychonoff fixed point theorem of [21]. Thus, random map $T$ admits an ACIM.

Let $\mu$ be an ACIM for random map $T$. Each of the maps $\tau_k$ admits a unique ACIM (see Appendix). Let $\nu_1$ and $\nu_2$ be the unique ACIM for $\tau_1$ and $\tau_2$ respectively. Let $A_k = \text{supp}(\nu_k)$ and $U_k = \bigcup_{j=0}^{\infty} \tau_k^{-j} A_k$ be its basin. For $k = 1, 2$, we have $A_k = U_k = I$ (see Appendix).

Lemma 4.3.5. For $k = 1, 2, I = A_k \subseteq \text{supp}(\mu)$.

Proof. Since $A_k = U_k = I$ for $k = 1, 2$. Then $\mu(A_k) = \mu(U_k) > 0$.

Let $B = I \cap \text{supp}(\mu)$, then $B \neq \emptyset$ and $\mu(B) > 0$. Since $B$ is subset of $I = A_k$ and $A_k$ is an invariant set, then $\bigcup_{j=0}^{\infty} \tau_k^j B \subseteq A_k$.

Assume $A_k \nsubseteq \text{supp}(\mu)$. Then $\mu(A_k \setminus B) = 0$. Also,

$$\mu(A_k \setminus B) \geq \mu\left(\bigcup_{j=0}^{\infty} \tau_k^j B \setminus B\right) = \mu\left(\bigcup_{j=0}^{\infty} \tau_k^j B \setminus B\right) \geq \mu(\tau_k^j B), \quad j = 1, 2, \ldots$$

So, in this case, $\mu(\tau_k^j B) \leq 0, j = 1, 2, \ldots$. However this leads to a contradiction because, by condition (B),

$$\mu(\tau_1 B) = \int_{\tau_1^{-1}(\tau_1 B)} p_1 d\mu + \int_{\tau_2^{-1}(\tau_1 B)} p_2 d\mu \geq \inf_{x \in I} p_1(x) \mu(B) + \inf_{x \in I} p_2(x) \mu(\tau_2^{-1}(\tau_1 B)) > 0.$$

and

$$\mu(\tau_2 B) = \int_{\tau_1^{-1}(\tau_2 B)} p_1 d\mu + \int_{\tau_2^{-1}(\tau_2 B)} p_2 d\mu \geq \inf_{x \in I} p_1(x) \mu(\tau_1^{-1}(\tau_2 B)) + \inf_{x \in I} p_2(x) \mu(B) > 0.$$

Therefore, $I = A_k \subseteq \text{supp}(\mu)$.

\[\square\]
4.3. EXISTENCE AND UNIQUENESS OF ACIM

**Proposition 4.3.6.** Let $A \geq A_* = \frac{4}{1-\alpha}$ and $f \in \mathcal{C}_A$. There are $\gamma > 0$, $N \in \mathbb{Z}_+$ such that $\mathcal{L}_T^n f \geq \gamma \lambda(f)$, for all $n \geq N$, where $\gamma$ and $N$ depend only on $A$. In particular, if $\rho = \mathcal{L}_T \rho$ then $\mu = \rho m$ is equivalent to $\lambda$.

**Proof.** First by Proposition 4.3.4, if $A \geq \frac{4}{1-\alpha}$, $f \in \mathcal{C}_A$, then $\mathcal{L}_T^n f \in \mathcal{C}_A$. So, we have

$$\int_0^x f \, d\lambda \leq Ax^{1-\alpha} \lambda(f), \int_0^x \mathcal{L}_T^n f \, d\lambda \leq Ax^{1-\alpha} \lambda(\mathcal{L}_T^n f).$$

Without loss of generality, we suppose that $\lambda(f) = 1$. Then $\lambda(\mathcal{L}_T^n f) = \lambda(f) = 1$. Therefore we only need to prove $\mathcal{L}_T^n f \geq \gamma$. Fix a small number $0 < \sigma < \frac{1}{2}$, such that $A\sigma^{1-\alpha} = \frac{1}{2}$. Then,

$$\int_0^\sigma f \, d\lambda \leq A\sigma^{1-\alpha} = \frac{1}{2} \quad \text{and} \quad \int_0^1 f \, d\lambda = 1 - \int_0^\sigma f \, d\lambda \geq \frac{1}{2}.$$

When $x \in (0, \sigma)$, since $f(x)$ is a decreasing function, we have

$$f(x) \geq f(\sigma) = \frac{1}{\sigma} f(\sigma) \geq \frac{1}{\sigma} \int_0^1 f(x) \, d\lambda \geq \frac{1}{\sigma} \frac{1}{1-\sigma} \geq \frac{1}{2(1-\sigma)}.$$

Moreover, $\mathcal{L}_T^n f(x)$ is decreasing. Then it is enough to show that $\mathcal{L}_T^n f(1)$ is bounded below away from zero. By (vi) composition property of $\mathcal{L}_T$ we have $\mathcal{L}_T^n f(1) = \mathcal{L}_T f(1)$. We will show that $\mathcal{L}_T^n f(1) \geq \gamma > 0$.

Define $x_n = \tau_k^{-1}(x_{n-1}) \cap [0, \frac{1}{2}], n \geq 1$ and $x_0 = 1$. Obviously, $\{x_n\}$ is a strictly decreasing sequence and it converges to $0$. Since $\{x_n\}$ depends on $\omega_n$, we denote $\{x_{n,\omega_n}\} = \{x_n\}(\omega_n)$. With the fixed $\sigma$, we can find an $N$ such that $\{0, b_1, b_2, ..., b_q\}$, where $q = q(\omega_N)$, are critical points of $\mathcal{T}^N(x)$ and for all $\omega_N$,

$$\max_{\omega_N} x_{N-1,\omega_N} \leq \sigma.$$

For convenience, we set

$$\{b_1, b_2\} = \{\tau_k^{-1}(x_{N-1,\omega_N})\}.$$

Then, for all $\omega_N$, we have

$$\mathcal{L}_T f(x_{N-1,\omega_N}) \geq \mathcal{L}_T f(\sigma)$$

since $\mathcal{L}_T f(x)$ is decreasing.

Now we will estimate the lower bound of $\mathcal{L}_T^n f(1)$. By definition of critical points of map $\mathcal{T}^N(x)$, we have

$$\mathcal{L}_T f(1) = \sum_{\omega_N \in \{1, 2\}^N} \sum_{i=1}^q \frac{(p_{\omega_N}) f(b_i)}{DT^N(\omega_N)(b_i)} = \sum_{\omega_N \in \{1, 2\}^N} \sum_{i=1}^q \frac{p_{\omega_N} \tau_k^{-1}(\tau_k(b_i)) p_{\omega_N}(b_i) f(b_i)}{DT^{N-1} \tau_k^{-1}(\tau_k(b_i)) \tau_k'(b_i)} \geq \sum_{\omega_N \in \{1, 2\}^N} \sum_{i=1}^q \frac{2 p_{\omega_N} \tau_k^{-1}(\tau_k(b_i)) p_{\omega_N}(b_i) f(b_i)}{DT^{N-1} \tau_k^{-1}(\tau_k(b_i)) \tau_k'(b_i)}.$$
Thus, for all \( n \in \mathbb{N} \) and \( \mu \),

\[
\therefore, \text{for all } n \in \mathbb{N} \text{ and } \mu \text{, we obtain that }
\]

\[
\mathcal{L}_N f(1) = \sum_{\omega_N \in \{1,2\} \mathbb{N}} \sum_{i=1}^{2 \mu \omega_N} \frac{p_{\omega_N-1}(x_{N-1},\omega_N)}{\Delta T_{\omega_N-1}(x_{N-1},\omega_N)} \tau_k^i(\tau_{k+1}^{-1}x_{N-1},\omega_N)
\]

\[
= \sum_{\omega_N \in \{1,2\} \mathbb{N}} \sum_{k=1}^{2 \mu \omega_N} \frac{p_{\omega_N-1}(x_{N-1},\omega_N)}{\Delta T_{\omega_N-1}(x_{N-1},\omega_N)} \tau_k^i(\tau_{k+1}^{-1}x_{N-1},\omega_N)
\]

\[
= \sum_{\omega_N \in \{1,2\} \mathbb{N}} \frac{p_{\omega_N-1}(x_{N-1},\omega_N)}{\Delta T_{\omega_N-1}(x_{N-1},\omega_N)} \left[ \mathcal{L}_T f(x_{N-1},\omega_N) \right]
\]

\[
\geq \sum_{\omega_N \in \{1,2\} \mathbb{N}} \frac{p_{\omega_N-1}(x_{N-1},\omega_N)}{\Delta T_{\omega_N-1}(x_{N-1},\omega_N)} \left[ \mathcal{L}_T f(\sigma) \right].
\]

(4.3.1)

Note that

\[
\max_{k \in \{1,2\}, x \in [0,1]} \tau_k^i(x) = \tau_1^i(\frac{1}{2}) = 2 + \alpha
\]

and \( f(\tau_{k+1}^{-1}\sigma) > f(\sigma) \geq \frac{1}{2(1-\sigma)} \). Also, by condition (B):

\[
\inf_{\Delta k} \rho(\Delta k) \geq \delta > 0.
\]

Therefore, from (4.3.1) it remains to show that \( \mathcal{L}_T f(\sigma) > 0. \) Indeed,

\[
\mathcal{L}_T f(\sigma) \geq \frac{p_1(\tau_{1,1}^{-1}\sigma) f(\tau_{1,1}^{-1}\sigma)}{\tau_1^i(\tau_{1,1}^{-1}\sigma)} + \frac{p_2(\tau_{2,1}^{-1}\sigma) f(\tau_{2,1}^{-1}\sigma)}{\tau_1^i(\tau_{2,1}^{-1}\sigma)} \\
\geq \frac{p_1(\tau_{1,1}^{-1}\sigma) f(\tau_{1,1}^{-1}\sigma)}{\tau_1^i(\tau_{1,1}^{-1}\sigma)} \geq \frac{\delta}{2(1-\sigma)(2+\alpha)} > 0.
\]

Therefore,

\[
\mathcal{L}_N^N f(x) \geq \mathcal{L}_T f(1) \geq \gamma > 0,
\]

where \( \gamma = \frac{\delta}{2(1-\sigma)(2+\alpha)} \) \( \sum_{\omega_N \in \{1,2\} \mathbb{N}} \frac{p_{\omega_N-1}(x_{N-1},\omega_N)}{\Delta T_{\omega_N-1}(x_{N-1},\omega_N)} \) with \( N, \sigma \) depending only on \( A \).

Moreover, for \( n > N \), we set \( h(x) = \mathcal{L}_N^{n-N} f(x) \). Then \( h(x) \in \mathcal{C}_A \), and

\[
\mathcal{L}_T^N f(x) = \mathcal{L}_N^N \left( \mathcal{L}_T^{n-N} f(x) \right) = \mathcal{L}_N^N h(x) \geq \gamma.
\]

Thus, for all \( n \geq N, \mathcal{L}_T^N f(x) \geq \gamma > 0 \). For last part of the proposition, suppose that \( \rho = \mathcal{L}_T^N \rho \in \mathcal{C}_A \). Clearly, if set \( E \) such that \( \lambda(E) = 0 \), it follows that \( \mu(E) = \int_E \rho d\lambda = 0 \).

Conversely, \( \mu(E) = 0 \). \( \rho = \mathcal{L}_T^N \rho \) implies that \( 0 = \mu(E) = \int_E \rho d\lambda = \int_E \mathcal{L}_T^N \rho d\lambda \geq \gamma \lambda(E) \).

Hence, if \( \rho = \mathcal{L}_T^N \rho \) then \( \mu = \rho \lambda \) is equivalent to \( \lambda \). □
4.4. **Example**

We present an example of a random map $T$ which satisfies assumptions (A) and (B). Consequently, by Theorem 4.3.1 this random map has a unique ACIM.

**Example 4.4.1.** Let random map $T = \{\tau_1(x), \tau_2(x); p_1(x), p_2(x)\}$, for $0 < \beta < \alpha < 1$,

\[
\begin{align*}
\tau_1(x) &= \begin{cases} 
    x(1 + 2^\alpha x^\alpha) & x \in [0, \frac{1}{2}), \\
    2x - 1 & x \in [\frac{1}{2}, 1].
\end{cases} \\
\tau_2(x) &= \begin{cases} 
    x(1 + 2^\beta x^\beta) & x \in [0, \frac{1}{2}), \\
    \frac{3}{2}x - \frac{3}{4} & x \in [\frac{1}{2}, 1].
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
p_1(x) &= \begin{cases} 
    \frac{1 + x^\alpha}{3} & x \in [0, \frac{1}{2}), \\
    \frac{1}{3} & x \in [\frac{1}{2}, 1].
\end{cases} \\
p_2(x) &= \begin{cases} 
    \frac{2 - x^\alpha}{3} & x \in [0, \frac{1}{2}), \\
    \frac{2}{3} & x \in [\frac{1}{2}, 1].
\end{cases}
\end{align*}
\]

We have $p_1(x), p_2(x) \in [0, 1], p_1(x) + p_2(x) = 1, \forall x \in [0, 1]$ and $\inf_{x \in I} p_k(x) \geq \frac{1}{3} > 0$. Thus, condition (B) is satisfied. We now check condition (A). First, for $x \in [0, \frac{1}{2})$, $\frac{p_2(x)}{\tau_2(x)}$ is obviously decreasing. For $\frac{p_1(x)}{\tau_1(x)}$, we show

\[
p_1'(x)\tau_1'(x) - p_1(x)\tau_1''(x) \leq 0, \forall x \in [0, \frac{1}{2}).
\]

\[
p_1'(x)\tau_1'(x) - p_1(x)\tau_1''(x) = \frac{1}{3} \alpha x^{\alpha-1} [1 + (1 + \alpha) 2^\alpha x^\alpha] - \frac{1 + x^\alpha}{3} [\alpha 2^\alpha (1 + \alpha) x^{\alpha-1}] = \frac{\alpha x^{\alpha-1}}{3} [1 + (1 + \alpha) 2^\alpha x^\alpha - (1 + x^\alpha) 2^\alpha (1 + \alpha)] = \frac{\alpha x^{\alpha-1}}{3} [1 - 2^\alpha (1 + \alpha)].
\]

The term in square bracket is negative, i.e. $1 < 2^\alpha (1 + \alpha), \forall \alpha \in (0, 1)$.

So, $\frac{p_1(x)}{\tau_1(x)}$ is decreasing since

\[
\frac{p_1(x)}{\tau_1(x)} = \frac{p_1'(x)\tau_1'(x) - p_1(x)\tau_1''(x)}{\tau_1'(x)^2} \leq 0.
\]

For $x \in [\frac{1}{2}, 1]$, $\frac{p_1(x)}{\tau_1(x)} = \frac{1}{6} \frac{p_2(x)}{\tau_2(x)} = \frac{4}{9}$. Therefore, $\sum_{i=1}^{l} \frac{p_k(\tau_{k,i}^{-1}(x))}{\tau_{k,i}^{-1}(x)^2}, 1 \leq l \leq 2$, is decreasing for all $k = 1, 2$, since $x \mapsto \tau_{1,i}^{-1}x$ and $x \mapsto T_{2,i}^{-1}x$ are increasing. This random map preserves a unique ACIM.
4.5 Two Dimensional Non-uniformly Expanding Map

In this section we use the skew product representation of [9] and show that our random maps give rise to an interesting family of 2-dimensional non-uniformly expanding maps which admit a unique ACIM. This family could serve as a good testing tool for the analysis of 2-dimensional systems with slow mixing.

Let $S(x,\omega): I^2 \to I^2$ be

$$S(x,\omega) = (\tau_k(x), \varphi_{k,x}(\omega)), \quad \text{where} \quad \begin{cases} \varphi_{1,x}(\omega) = \frac{\omega}{p_1(x)} , & \omega \in [0, p_1(x)) , \\ \varphi_{2,x}(\omega) = \frac{\omega-p_1(x)}{p_2(x)} , & \omega \in [p_1(x), 1] . \end{cases}$$

Define $S_i = S|_{U_i}, i = 1, 2, 3, 4$.

- $S_1 = (\tau_{1,1}(x), \varphi_{1,x}(\omega)) = (x(1+2^\alpha x^\alpha), \frac{\omega}{p_1(x)}), \quad U_1 = [0, \frac{1}{2}) \times [0, p_1(x))$,
- $S_2 = (\tau_{1,2}(x), \varphi_{1,x}(\omega)) = (\tau_{1,2}(x), \frac{\omega}{p_1(x)}), \quad U_2 = [\frac{1}{2}, 1] \times [0, p_1(x))$,
- $S_3 = (\tau_{2,2}(x), \varphi_{2,x}(\omega)) = (\tau_{2,2}(x), \frac{\omega-p_1(x)}{p_2(x)}), \quad U_3 = [\frac{1}{2}, 1] \times [p_1(x), 1]$,
- $S_4 = (\tau_{2,1}(x), \varphi_{2,x}(\omega)) = (x(1+2^\beta x^{\beta}), \frac{\omega-p_1(x)}{p_2(x)}), \quad U_4 = [0, \frac{1}{2}) \times [p_1(x), 1]$.

One can easily check that $(0,0)$ is a fixed point of $S$. Moreover, the lyapunov exponent in the horizontal direction has value zero at $(0,0)$. Therefore, $S$ is nonuniformly expanding map. Moreover, under conditions (A) and (B), since $T$ has unique ACIM, by [9], $S$ has a unique ACIM too. A version of this skew product will be used to study the decay of correlations for random intermittent map (see Chapter 5 and our result [8]).

4.6 Stochastic Stability

In this section we study stochastic stability of random intermittent maps. For this purpose, we write, for $\varepsilon > 0, 0 < \alpha - \varepsilon < 1$,

$$T_\varepsilon = \{ \tau_1(x), \tau_1, \varepsilon(x); p_1, \varepsilon(x), p_2, \varepsilon(x) \},$$

where

$$\tau_1 = \begin{cases} x(1+2^\alpha x^\alpha) & x \in [0, \frac{1}{2}) , \\ g_1(x) & x \in [\frac{1}{2}, 1] , \end{cases} \quad \tau_1, \varepsilon = \begin{cases} x(1+2^{\alpha-\varepsilon} x^{\alpha-\varepsilon}) & x \in [0, \frac{1}{2}) , \\ g_1(x) & x \in [\frac{1}{2}, 1] . \end{cases}$$

Our main result in this section is the following theorem.

**Theorem 4.6.1.** Let $\rho_0$ be the unique invariant density of $T_\varepsilon$. Let $\rho_1$ be the unique invariant density of $\tau_1$. If $\lim_{\varepsilon \to 0} \sup_x p_2, \varepsilon(x) = 0$, then $\lim_{\varepsilon \to 0} \| \rho_\varepsilon - \rho_1 \|_1 = 0$.

**Proof.** Let $\mathcal{L}_{T_\varepsilon}$ be the Perron-Frobenius operator associated with the random map $T_\varepsilon$. By Theorem 4.3.1, there exist a fixed point $\rho_0$ of $\mathcal{L}_{T_\varepsilon}$ and $\rho_1 \in \mathcal{C}_A$, for some $A \geq \frac{1}{1-\alpha}$. Since $\mathcal{C}_A$ is a compact set, there exists a subsequence $\{ f_{\varepsilon_k} \}_{\varepsilon_k > 0}$ of $\{ \rho_0 \}_{\varepsilon > 0}$ such that

$$f_{\varepsilon_k} \xrightarrow{L} f^* \in \mathcal{C}_A, \text{ as } \varepsilon_k \to 0 .$$

We have

$$\| f^* - P_{\tau_1} f^* \|_1 \leq \| f^* - f_{\varepsilon_k} \| + \| f_{\varepsilon_k} - P_{\tau_1} f_{\varepsilon_k} \| + \| P_{\tau_1} f_{\varepsilon_k} - P_{\tau_1} f^* \|_1 \leq \| f^* - f_{\varepsilon_k} \| + \| f_{\varepsilon_k} - P_{\tau_1} f_{\varepsilon_k} \| + \| f_{\varepsilon_k} - f^* \|_1 = 2 \| f^* - f_{\varepsilon_k} \| + \| \mathcal{L}_{T_{\varepsilon_k}} f_{\varepsilon_k} - P_{\tau_1} f_{\varepsilon_k} \|_1 .$$

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4.7. COMPARISON

The first term on the right converges to 0 as $\varepsilon_k \to 0$ by the choice of subsequence. Moreover, we have

$$
\|L_{\tau_k}f_{\varepsilon_k} - P_{\tau_k} f_{\varepsilon_k}\|_1 = \|P_{\tau_k}(p_{1,\varepsilon_k}f_{\varepsilon_k}) + P_{\tau_k}(p_{2,\varepsilon_k}f_{\varepsilon_k}) - P_{\tau_k} f_{\varepsilon_k}\|_1
\leq 2\|p_{2,\varepsilon_k}f_{\varepsilon_k}\|_1 \leq 2\sup_x p_{2,\varepsilon_k}\|f_{\varepsilon_k}\|_1 \to 0, \text{ as } \varepsilon_k \to 0.
$$

Thus, $f^* = P_{\tau_k} f^* \mod a.e.$

By the uniqueness of $\tau_1$ invariant density $\rho_1$, all subsequences $\{f_{\varepsilon_k}\}_{\varepsilon_k > 0}$ of $\{\rho_\varepsilon\}_{\varepsilon > 0}$ have $\rho_1$ as their common limit point. Hence, $\|\rho_\varepsilon - \rho_1\|_1 \to 0$, as $\varepsilon \to 0$. \[ \square \]

4.7. Comparison

After publishing our work [20], Shen and van Strien [50] studied the problem of stochastic stability, where all the intermittent maps are perturbed by additive noise. Moreover, the measure on the noise space in [50] is Lebesgue measure. Our result in [20] does not fit in the setting of [50], since we do not consider additive noise, and we work with a singular measure on the noise space.

4.8. Appendix

Let

$$
\tau(x) = \begin{cases} 
  x(1 + 2^nx^n) & x \in [0, \frac{1}{2}), \\
  g(x) & x \in [\frac{1}{2}, 1].
\end{cases}
$$

We study a deterministic map $\tau : I \to I$, with partition $\mathcal{P} = \{I_1, I_2\}, I_1 = [0, \frac{1}{2}], I_2 = [\frac{1}{2}, 1], g(\frac{1}{2}) = 0, g'(x) > 1$.

**Lemma 4.8.1.** Let $\nu$ be a $\tau$ ACIM. Then the support of $\nu$ is $I$.

**Proof.** If $g(x)$ is onto, the uniqueness of the $\tau -$ACIM is well known (see [41]). We only consider the case, when $g(1) < 1$. We have $\tau(0, \frac{1}{2}) = [0, 1)$. We need to show that for any interval $J \subset I$, there exists an $n \geq 1$ such that $\tau^n(J) \supseteq [0, \frac{1}{2}]$. If $J \supseteq I_k$, $k = 1, 2$, then obviously $\tau(J) \supseteq [0, \frac{1}{2}]$. If $J \subset I_k$. Since $\lambda(\tau(J)) > \lambda(J)$, there exists a $j \geq 1$ such that $\tau^j(J)$ contains $\frac{1}{2}$ in its interior.

Since $\tau^j(J)$ contains the partition point $\frac{1}{2}$ in its interior, i.e. $\tau^j(J) \supseteq (t_1, t_2)$ with $\frac{1}{2} \in (t_1, t_2)$. Then $\tau[\frac{1}{2}, t_2] = [g(\frac{1}{2}), g(t_2)] = [0, g(t_2)]$, which contains the point 0. Then obviously there exists a $l \geq 1$ such that $\tau^{j+l}(J) \supseteq [0, \frac{1}{2}]$.

Let $A$ denote the support of $\nu$. Since $A$ contains an interval $J$ and $A$ is an invariant set $\tau^n(J) \subseteq A, n \geq 1$. Then, $[0, 1) \subset A$. Consequently (by invariance) $A$ must contain $I$. Moreover, $A \subset I$. Therefore, the support of $\nu$ is $I$. \[ \square \]
Chapter 5

Decay of Correlation for Random Intermittent Maps

5.1 Introduction

In this chapter we are interested in studying i.i.d. randomized compositions of two intermittent maps sharing a common indifferent fixed point. It is intuitively clear that the annealed\(^1\) dynamics of the random process will also have a polynomial rate of correlation decay. However, we are interested in the following question: How do the asymptotics of the random map relate to those of the original maps; in particular, the rate of correlation decay?

We show that the map with the fast relaxation rate dominates the asymptotics (see Theorem 5.2.2 for a precise statement). Interestingly, in our setting, the map with slow relaxation rate is allowed to be of ‘boundary-type’, and consequently admit an infinite (\(\sigma\)-finite) invariant measure, but the random system will always admit an absolutely continuous invariant probability measure. We obtain our result by using a version of the skew product representation studied in [9] and a Young-tower technique [54].

In Section 5.2 we introduce our random system and its skew product representation. The statement of our main result Theorem 5.2.2 is also in Section 5.2. In Section 5.3 we build a Young-tower for the skew product representation. Proofs, including the proof of Theorem 5.2.2, are in Section 5.4.

5.2 Setup and Statement of The Main Result

5.2.1 A Random Dynamical System

Let \((I, \mathcal{B}(I), \lambda)\) be the measure space, with \(I = [0,1]\), \(\mathcal{B}(I)\) the Borel \(\sigma\)-algebra and \(\lambda\) being Lebesgue measure. By a Liverani-Saussol-Vaienti (LSV)-map we mean a member of the parameterized family of maps on \(I\) given by

\[
\tau_\alpha(x) = \begin{cases} 
  x(1 + 2^\alpha x^\alpha) & x \in [0, \frac{1}{2}] \\
  2x - 1 & x \in (\frac{1}{2}, 1]
\end{cases}.
\]  

Annealed dynamics refers to the randomized dynamics, averaged over the randomizing space, see Subsection 5.2.2 and Theorem 5.2.2. This should be contrasted with the notion of quenched dynamics, the behaviour of the system with one random choice of the randomizing sequence. The term almost sure dynamics is also used to refer to quenched dynamics.

\(^1\)Annealed dynamics refers to the randomized dynamics, averaged over the randomizing space, see Subsection 5.2.2 and Theorem 5.2.2. This should be contrasted with the notion of quenched dynamics, the behaviour of the system with one random choice of the randomizing sequence. The term almost sure dynamics is also used to refer to quenched dynamics.
5.2. SETUP AND STATEMENT OF THE MAIN RESULT

Here the parameter $\alpha \in (0, \infty)$. Each LSV map has a neutral fixed point at $x = 0$. For $0 < \alpha < 1$, $\tau_\alpha$ admits a finite, absolutely continuous invariant measure while for $\alpha \geq 1$ the absolutely continuous invariant measure is $\sigma$–finite. See [47] and [51] for some of the earliest results of this type.

Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq 1$. We consider a random map $T$ which is given by:

$$T(x) := \{\tau_{\alpha_1}(x), \tau_{\alpha_2}(x), \ldots, \tau_{\alpha_r}(x); p_1, p_2, \ldots, p_r\},$$

(5.2.2)

where $p_i > 0$ and $\sum p_i = 1$. Note that all the individual maps share a single common neutral fixed point at $x = 0$.

Assumption 5.2.1. Since nothing we will do in the sequel depends on $r$, the number of maps making up the random map, we will restrict the discussion to the case $r = 2$ and denote the parameters $0 < \alpha < \beta \leq 1$.

At the same time, this will simplify our notation:

$$T(x) := \{\tau_{\alpha}(x), \tau_{\beta}(x); p_1, p_2\}.$$  

(5.2.3)

The random map $T$ in (5.2.3) maybe viewed as a Markov process with transition function

$$P(x, A) = p_1 1_A(\tau_{\alpha}(x)) + p_2 1_A(\tau_{\beta}(x))$$

of a point $x \in I$ into a set $A \in \mathcal{B}(I)$, where $1_A$ is the indicator function of a subset $A$ of $I$ defined as $1_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$.

The transition function induces an operator, $E_T$, acting on measures; i.e., if $\mu$ is a measure on $(I, \mathcal{B})$,

$$(E_T \mu)(A) = p_1 \mu(\tau_{\alpha}^{-1}(A)) + p_2 \mu(\tau_{\beta}^{-1}(A)).$$

A measure $\mu$ is said to be $T$-invariant if

$$\mu = E_T \mu,$$

and $\mu$ is said to be an absolutely continuous invariant measure if $d\mu = f^* d\lambda$, $\int_I f^* d\lambda = 1$.

To study absolutely continuous invariant measures, we introduce the transfer operator (Perron-Frobenius) of the random map $T$:

$$(L_T f)(x) = p_1 P_{\tau_{\alpha}}(f)(x) + p_2 P_{\tau_{\beta}}(f)(x),$$

where $P_{\tau_{\alpha}}, P_{\tau_{\beta}}$ denote the transfer operators associated with the $\tau_{\alpha}, \tau_{\beta}$ respectively. Then it is a straightforward computation to show that a measure $\mu = f^* \cdot \lambda$ is absolutely continuous invariant measure if

$$L_T f^* = f^*.$$

5.2.2 Skew Product Representation

By the annealed dynamics of the random map we mean the statistics of the random dynamical system averaged over the randomizing space (see [3] for a general treatment of annealed versus quenched interpretation). Probabilistic aspects of $T$, in particular the correlation decay of the annealed dynamics, are frequently studied through a skew-product representation of $T$. Since our strategy for obtaining correlation decay rates is based on a Young-tower technique, which requires a manifold structure and a single map, we are
5.2. SETUP AND STATEMENT OF THE MAIN RESULT

going to use a version\(^2\) of the skew product representation which was studied in [9]. Define
the skew product transformation \(S(x, \omega) : I \times I \to I \times I\) by
\[
S(x, \omega) = (\tau_{\alpha(\omega)}, \varphi(\omega)),
\]
where
\[
\alpha(\omega) = \begin{cases} 
\alpha, & \omega \in [0, p_1) \\
\beta, & \omega \in [p_1, 1] 
\end{cases}; \quad \varphi(\omega) = \begin{cases} 
\frac{\omega}{p_1}, & \omega \in [0, p_1) \\
\frac{\omega - p_1 p_2}{p_2}, & \omega \in [p_1, 1] 
\end{cases}.
\]

We denote the transfer operator associated with \(S\) by \(L_S\): for \(g \in L^1(I \times I)\) and measurable
\(A \subseteq I \times I\),
\[
\int_{S^{-1}A} g d(\lambda \times \lambda)(x, \omega) = \int_A L_S g d(\lambda \times \lambda)(x, \omega).
\]
Then a measure \(\nu\), such that \(d\nu = g^* d(\lambda \times \lambda)\) and \(\int_{I \times I} g^* d(\lambda \times \lambda) = 1\), is an absolutely
continuous \(S\)-invariant measure if
\[
L_S g^* = g^*.
\]
In [9], Theorem 5.2 it is shown that if \(g \in L^1(\lambda \times \lambda)\) and \(L_S g = \rho g\) with \(|\rho| = 1\), then
\[
g(x, \omega) = f(x) \cdot 1(\omega)
\]
and \(L_T f = g f\), that is, \(g\) depends only on the spatial coordinate \(x\) and as a function of
\(x\) only, is also an eigenfunction for \(L_T\). Setting \(\rho = 1\) we obtain \(L_S g^* = g^*\) if and only if
\(g^*(x, \omega) = f^*(x)\) with \(L_T f^* = f^*\). Consequently there is a one to one correspondence
between invariant densities for \(S\) and invariant densities for \(T\). Moreover, dynamical
properties such as ergodicity, number of ergodic components or weak-mixing, properties
that are determined by peripheral\(^3\) eigenfunctions, can be determined via either system.

On the other hand, properties like correlation decay (or even strong mixing) cannot
be established by peripheral spectrum alone.

**Definition 5.2.1.** Suppose \(\tau : X \to X\) preserves the measure \(\mu\) on \(X\). For \(f \in L^\infty(X, \mu)\)
and \(g \in L^1(X, \mu)\) denote by
\[
Cor_n(f, g) = Cor_{n, \tau}(f, g) := \int_X f \circ \tau^n \cdot g d\mu - \int_X f d\mu \int_X g d\mu.
\]

Normally we will simply write \(Cor_n(f, g)\) when the map being applied is understood.

Estimates on correlation decay are known in many dynamical settings. For example,
it was shown in [54] that for \(f \in L^\infty\), \(g\) Hölder continuous on \(I\) and \(\tau = \tau_\alpha, \ 0 < \alpha < 1\) an
LSV-map, \(|Cor_{n, \tau_\alpha}(f, g)| = O(n^{1-\frac{\alpha}{2}})\). Gouëzel in [25] proved that this rate is sharp.

Our main result, Theorem 5.2.2, establishes exactly the same rate of correlation decay
for the random map.

\(^2\)The results obtained in [9] are valid for any class of measurable non-singular maps on \(\mathbb{R}^q\), without any
regularity assumptions. Moreover in [9], the probability distribution on the noise space is allowed to be
place-dependent.

\(^3\)The peripheral spectrum of an operator is defined as a set of points in its spectrum which have modulus
equal to its spectral radius.
5.2. SETUP AND STATEMENT OF THE MAIN RESULT

5.2.3 Statement of The Main Result

Theorem 5.2.2. Let \(0 < \alpha < \beta \leq 1\) and \(S\) be as defined in Subsection 5.2.2. Then

1. \(S\) admits a unique absolutely continuous invariant probability measure \(\nu\);
2. \((S, \nu)\) is mixing;
3. for \(\phi \in L^\infty(I \times I, m \times m)\) and \(\psi\) a Hölder continuous function on \(I \times I\) we have
   \[ |\text{Cor}_{n,S}(\phi, \psi)| = O(n^{1-\frac{1}{\alpha}}); \]

Remark 5.2.1. Our main goal in Theorem 5.2.2 is not so much to show that \(S\) has polynomial rate of correlation decay, but to discover how the correlation decay for \(S\) relates to those of the original maps. Indeed, if \(0 < \alpha < \beta < 1\), and without any further conditions on \(\alpha\) and \(\beta\), one can easily obtain, by just using the rough estimates contained in Lemma 5.4.4 and the Young tower construction detailed in the next two sections, an upper bound on the rate of order \(O(n^{1-\frac{1}{\beta}})\); that is, the rate of decay is at least as fast as the slowest escape rate map. What we have shown in Theorem 5.2.2 is that the actual decay rate of the random map is completely determined by the faster escape rate of the map \(\tau_\alpha\).

Remark 5.2.2. It is worth noting that in Theorem 5.2.2, \(\beta \leq 1\). The case when \(\beta = 1\) is interesting on its own since in this case the map \(\tau_\beta\) admits only an infinite (\(\sigma\)-finite) absolutely continuous invariant measure, but Theorem 5.2.2 shows that the skew product \(S\), and hence the random map \(T\), admits a unique absolutely continuous invariant probability measure.

Remark 5.2.3. Limit theorems for the following related skew product were studied by Gouëzel in [26]:

\[ S(x, \omega) = (\tau_\alpha(\omega), 4\omega), \]
with \(\omega \in S^1\) and \(\tau_\alpha(\omega)\) being a random choice of LSV-map from Equation 5.2.1. For the randomizing process, it is further assumed that

1. \(\alpha(\omega)\) is \(C^2\);
2. \(0 < \alpha_{\min} < \alpha_{\max} < 1\);
3. \(\alpha(\omega)\) takes the value \(\alpha_{\min}\) at a unique point \(\omega_0 \in S^1\), with \(\alpha''(\omega_0) > 0\);
4. \(\alpha_{\max} < \frac{3}{2} \alpha_{\min}\).

Under the above conditions, using a result of Pène [46] (see [26] Theorem B.1), Gouëzel ([26], Theorem 4.1) obtained asymptotics that would lead to a correlation decay rate of order \(O(\sqrt{\log n} \cdot n^{1-\frac{1}{\alpha_{\min}}})\), which is larger than the sharp rate of \(T_{\alpha_{\min}}\) by a \(\sqrt{\log n}\) factor.

Gouëzel suggests in [26] that the conditions \(\alpha_{\max} < 1\) and \(\alpha_{\max} < \frac{3}{2} \alpha_{\min}\) may be technical constraints, arising from the method of proof.

In our setting we do not need to assume \(\alpha_{\max} < \frac{3}{2} \alpha_{\min}\) and \(\alpha_{\max} = 1\) is allowed, lending support to Gouëzel’s conjecture. It is possible that our condition \(\beta \leq 1\) is also a purely technical constraint.

Furthermore, note that we do not obtain the multiplicative factor of \(\sqrt{\log n}\) as in [26] but, rather, exactly the correlation decay rate of the fastest mixing map \(\tau_\alpha\). Finally, our proof is quite different from that in [26], relying on relatively simple (and classical) estimates on large deviations for i.i.d. randomizers.
5.3 A Young-tower for $S$

5.3.1 Notation

Set

$$T^n_\omega(x) := \tau_{\alpha(\varphi^{n-1}\omega)} \circ \cdots \circ \tau_{\alpha(\varphi\omega)} \circ \tau_{\alpha(\omega)}(x).$$

Then

$$S^n(x,\omega) = (T^n_\omega(x),\varphi^n(\omega)).$$

Also, set

$$P^n_\omega := p_{\alpha(\varphi^{n-1}\omega)} \times \cdots \times p_{\alpha(\varphi\omega)} \times p_{\alpha(\omega)},$$

where $p_{\alpha(\omega)} = p_1$, for $\alpha(\omega) = \alpha$ and $p_{\alpha(\omega)} = p_2$, for $\alpha(\omega) = \beta$. We define two sequences of random points $\{x_n(\omega)\}$ and $\{x'_n(\omega)\}$ in $[0,1]$ which will be useful in the construction of a suitable Young tower. The points $x_n(\omega)$ lie in $[0,\frac{1}{2}]$. Set

$$x_1(\omega) \equiv \frac{1}{2} \text{ and } x_n(\omega) = \tau_{\alpha(\omega)}^{-1} |_{[0,\frac{1}{2}]} [x_{n-1}(\varphi\omega)], n \geq 2. \quad (5.3.1)$$

Observe that with this notation,

$$S(x_n(\omega),\omega) = (\tau_{\alpha(\omega)}(x_n(\omega)),\varphi\omega) = (x_{n-1}(\varphi\omega),\varphi\omega).$$

The points $\{x'_n(\omega)\}$ lie in $(\frac{1}{2},1]$, defined by

$$x'_0(\omega) \equiv 1, x'_1(\omega) \equiv \frac{3}{4} \text{ and } x'_n(\omega) = \frac{x_n(\varphi\omega) + 1}{2}, n \geq 2, \quad (5.3.2)$$

that is, $\{x'_n(\omega)\}$ are preimages of $\{x_n(\omega)\}$ in $(\frac{1}{2},1]$ under the right branch $2x - 1$.

5.3.2 A Tower for $S$

Let $\Delta_0 = (\frac{1}{2},1] \times [0,1]$. Let $R : \Delta_0 \to \mathbb{Z}^+$ be the first return time function and $S^R : \Delta_0 \to \Delta_0$ be the return map. $\Delta_0$ is referred to as the base of the tower $\Delta$ which is given by

$$\Delta := \{(z,n) : z \in \Delta_0 \text{ and } n = 0,1,...,R(z)-1\}.$$ 

Let $F : \Delta \to \Delta$ be the map acting on the tower as follows:

$$F(z,l) = \begin{cases} (z,l+1), & \text{if } l < R(z)-1, \\ (S^R(z),0), & \text{if } l = R(z)-1. \end{cases}$$

We refer to $\Delta_l := \Delta \cap \{n = l\}$ as the $l$th level of the tower. For $n \geq 1$, set $I_n(\omega) := (x_{n+1}(\omega),x_n(\omega)]$ and $J_n(\omega) := (x'_n(\omega),x'_{n-1}(\omega)]$. Observe that any $x \in J_n(\omega)$ will return to $\Delta_0$ in $n$ steps as follows:

$$J_n(\omega) \to I_{n-1}(\varphi\omega) \to I_{n-2}(\varphi^2\omega) \to \cdots \to I_1(\varphi^{n-1}\omega) \to (\frac{1}{2},1].$$

We now introduce elements of the partition of $\Delta_0$, which will be denoted by $\Delta^j_{0,i}$, where $j = 1,2,...,2^i$ and $i = 1,2,...$. For example, in the case $i = 2$, there are four sets $\Delta^j_{0,2}$ such that $S^R$ maps each set bijectively to $\Delta_0$:

$$\Delta^j_{0,2} = \begin{cases} J_2(\omega) \times [0,p_1^2), & \text{if } j = 1, \\ J_2(\omega) \times [p_1^2,p_1), & \text{if } j = 2, \\ J_2(\omega) \times [p_1,p_1+p_1 \cdot p_2), & \text{if } j = 3, \\ J_2(\omega) \times [p_1+p_1 \cdot p_2,1), & \text{if } j = 4. \end{cases}$$
5.3. A YOUNG-TOWER FOR $S$

We partition $\Delta_0$ using $\{\Delta^j_{0,i}\}_{i=1,2,...,j=1,2,...,2^i}$. Then, for all $j = 1,2,...,2^i$, $R|_{\Delta^j_{0,i}} = i$

and the tower $\Delta$ is given:

$$\Delta = \bigcup_{i=1}^{\infty} \bigcup_{l=0}^{2^i} (\bigcup_{j=1}^{2^i} \Delta^j_{0,i}).$$

We also set:

$$\Delta_{0,i} := \bigcup_{j=1}^{2^i} \Delta^j_{0,i}.$$

An example of the base of the tower is presented in Figure 5.1.

![Figure 5.1](image.png)

Figure 5.1: An example of the base of the tower when $\alpha = 0.5$, $\beta = 0.7$, $p_1 = 0.6$.

5.3.3 Using Young’s Technique to Prove Theorem 5.2.2

We say $s(z_1,z_2)$ is a separation time for $z_1,z_2 \in \Delta_0$ if $s$ is the smallest $n \geq 0$ such that $(FR)^n(z_1)$ and $(FR)^n(z_2)$ lie in distinct $\Delta^j_{0,i}$. Also let $\hat{R} : \Delta \to \mathbb{Z}$ be the function defined.
by
\[ \hat{R}(x, \omega) = \text{the smallest integer } n \geq 0 \text{ s.t. } F^n(x, \omega) \in \Delta_0. \]

To prove Theorem 2.3.1, we have to:

(A) Prove that \( \int_{I \times I} Rd(\lambda \times \lambda) < \infty \), and establish the asymptotic estimate \( (\lambda \times \lambda)\{\hat{R} > n\} = O(n^{1-\frac{1}{\alpha}}) \).

(B) Establish the bounded distortion conditions on the return map: there exists \( 0 < \theta < 1 \) and \( C(F) > 0 \) such that
\[
\left| \frac{DF^R(z_1)}{DF^R(z_2)} - 1 \right| \leq C(F) \cdot \theta^{\theta(F^R(z_1),F^R(z_2))}, \forall i = 1, 2, ..., \forall j = 1, ..., 2^i, \forall z_1, z_2 \in \Delta_{i,j}^1.
\]

(C) Confirm that the return times are aperiodic.

(A) is established by Proposition 5.4.1 in Subsection 5.4.1 while (B) is the content of Proposition 5.4.9 in Subsection 5.4.2. Since we have all possible integer return times, (C) is immediate. It is interesting to note that the upper bound constraint \( \beta \leq 1 \) specified in our main result, Theorem 5.2.2, is only used in Proposition 5.4.9, so the tower asymptotics detailed in (A) hold for all pairs \( 0 < \alpha < \beta < \infty \).

5.4 Proofs

5.4.1 Estimates on The Return Sets

Throughout this section we will adopt the notation \( E_\omega(\cdot) = \int_I \cdot(\omega)d\omega \) for expectation with respect to the randomizing variable. Also, we write \( a_n \sim b_n \) if there is a constant \( C > 1 \) such that \( C^{-1}b_n \leq a_n \leqCb_n \) for all \( n \).

**Proposition 5.4.1.** For all \( 0 < \alpha < \beta < \infty \) we have

1. \( E_\omega(x_n(\omega)) \sim n^{-\frac{1}{\alpha}}; \)
2. \( E_\omega(x'_n(\omega) - \frac{1}{2}) \sim n^{-\frac{1}{\alpha}}; \)
3. \( \lambda \times \lambda\{\hat{R} > n\} \sim n^{1-\frac{1}{\alpha}}. \)

Before proving this result, we gather some estimates in a sequence of lemmas.

**Lemma 5.4.2.** For all \( x \in [0, \frac{1}{2}] \) \( \tau_\alpha(x) \geq \tau_\beta(x) \) with strict inequality on the open interval \((0, \frac{1}{2})\).

**Proof.** This is a straightforward calculation.

**Corollary 5.4.3.** For \( 0 \leq x \leq y < \frac{1}{2} \) we have \( \tau_\alpha(y) \geq \tau_\beta(x) \) with strict inequality in either situation: \( 0 < x \leq y < \frac{1}{2} \) or \( 0 \leq x < y \leq \frac{1}{2} \).
5.4. PROOFS

We will estimate the position of $x_n(\omega)$ by comparing to the sequence of non-random backwards iterates constructed with only one map; either always choosing $\tau_{\alpha}|_{[0,\frac{1}{2})}$ or $\tau_{\beta}|_{[0,\frac{1}{2})}$ in place of $\tau_{\alpha}(\omega)|_{[0,\frac{1}{2})}$ in equation (5.3.1). Denote these non-random iterates by $x_n^\alpha$ and $x_n^\beta$ respectively. It is immediate from Lemma 5.4.2 that for every $n$, $x_n^\alpha \leq x_n^\beta$. Furthermore, it is well-known that $x_n^\alpha \sim n^{-\frac{1}{2}}$ with similar estimates for the parameter $\beta$. (See, for example, estimates at the beginning of Section 6.2 of [54].)

We begin with a very rough (but intuitively obvious) estimate on $x_n(\omega)$.

**Lemma 5.4.4.** For all $n \geq 1$ and for all $\omega$

$$x_n^\alpha \leq x_n(\omega) \leq x_n^\beta.$$  

**Proof.** Suppose, to the contrary, $x_n(\omega) < x_n^\alpha$ for some $n, \omega$. Note that if $\alpha(\varphi^k(\omega)) = \alpha$ for all $k$ then $x_n(\omega) = x_n^\alpha$, contradicting our assumption. Let $k \in \{0,1,\ldots,n-1\}$ be smallest integer such that $\alpha(\varphi^k(\omega)) = \beta$. Then $x_{n-k}(\varphi^k(\omega)) < x_n^\alpha$ since $\tau_{\alpha}$ is increasing and $x_{n-k-1}(\varphi^{k+1}(\omega)) = \tau_{\beta}(x_{n-k}(\varphi^k(\omega))) < \tau_{\alpha}(x_{n-k}(\varphi^k(\omega))) = x_n^\alpha$.

Here we have invoked Corollary 5.4.3. Iterating this argument for each index where $\alpha(\varphi^j(\omega)) = \beta$ gives

$$\frac{1}{2} = x_1(\varphi^{n-1}(\omega)) < x_1^\alpha = \frac{1}{2},$$

which is again a contradiction. A similar argument shows $x_n(\omega) \leq x_n^\beta$ for all $n, \omega$. \qed

**Lemma 5.4.5.** Suppose $n$ is given and $K_0 \in [0,n-1]$ is fixed. Suppose $\omega \in [0,1]$ is such that

$$\# \{j \in \{0,1,\ldots,n-1\} \mid \alpha(\varphi^j(\omega)) = \alpha\} > K_0.$$  

Then, $x_n^\alpha \leq x_n(\omega) \leq x_n^{\alpha_{[K_0]}}$.

**Proof.** The left-hand inequality is given by Lemma 5.4.4. For the other side, suppose $x_n(\omega) > x_n^{\alpha_{[K_0]}}$. Consider the following iteration of points

$$y_{n-i} = G^i(x_n(\omega)),$$

where

$$G(x_n(\omega)) := \begin{cases} 
\tau_{\alpha}(x_n(\omega)) & \text{if } \alpha(\omega) = \alpha, \\
\tau_{\beta}(x_n(\omega)) & \text{if } \alpha(\omega) = \beta.
\end{cases}$$

For each $i = 1,2,\ldots,n$ define $K(i) := \# \{j \in \{0,1,\ldots,i-1\} \mid \alpha(\varphi^j(\omega)) = \alpha\}$. For example, $K(1) = 0$ or $1$, $K(i) \leq i$ and $K(n) > |K_0|$ by hypothesis.

By an argument similar to the proof of Lemma 5.4.4, using $\tau_{\beta}$ compared to the identity map we have

$$y_{n-i} \leq x_{n-i}(\omega),$$

for all $i = 1,2,\ldots,n$. On the other hand, comparing $\tau_{\alpha}$ to the identity map and applying Lemma 5.4.2 gives

$$x_{\alpha_{[K_0]}}^{K(i)-1} < y_{n-i},$$

for all $i = 1,2,\ldots,n$.

Pick $i_0$ so that $K(i_0) = |K_0| - 1$. Note that $i_0 \leq n - 1$. Then

$$\frac{1}{2} = x_1^\alpha = x_{\alpha_{[K_0]}}^{\alpha_{[K_0]}} = x_{\alpha_{[K_0]}}^{\alpha_{[K_0]}-1} < y_{n-i_0} \leq x_{n-i_0}(\omega)$$

which contradicts the hitting time of $x_n(\omega)$ to the interval $[\frac{1}{2},1]$.

\qed
5.4. PROOFS

Pick any $0 < p_0 < p_1$, fix $n > 1$ and let $K_0 := np_0$. There are many standard large deviation estimates for i.i.d. random variables that will ensure that most $\omega$ encounter at least $K_0$ instances of $\alpha(\varphi^l\omega) = \alpha$ in their first $n$ iterates. As we are aiming for exponential decay in the tail estimate, we invoke a classical result due to Hoeffding [29] that works especially well for our case of Bernoulli random variables. It is precisely at this point that we avoid generating an upper bound constraint on $\beta$ as was the case in Gouëzel [26]. Indeed, if we use the well-known estimates from the Berry-Esseen Theorem (e.g. Theorem 1, Section XVI.5 in [22]) we obtain power law decay in the tail leading to the condition $\beta < \frac{3}{2}\alpha$ in order to complete the proof.

Lemma 5.4.6. For every $n \geq 1$

$$\Pr\{\omega \mid \#\{j \in \{0, 1, \ldots n - 1\} \mid \alpha(\varphi^j \omega) = \alpha\} \leq K_0\} \leq \exp[-2n(p_1 - p_0)^2].$$

Proof. Let $S_n$ count the number of times the value $\beta$ occurs in the first $n$ iterates. Observe that

$$\Pr\{\omega \mid \#\{j \in \{0, 1, \ldots n - 1\} \mid \alpha(\varphi^j \omega) = \alpha\} \leq K_0\} \leq \Pr\{\omega \mid \#\{j \in \{0, 1, \ldots n - 1\} \mid \alpha(\varphi^j \omega) = \beta\} \geq n - K_0\}$$

In Theorem 1 of [29] let $\mu = p_2$ and let $t = p_1 - p_0 < p_1 = 1 - \mu$. Then the bottom probability in equation (5.4.1) equals

$$\Pr\{S_n - \mu n \geq (1 - p_0 - p_2)n\} = \Pr\{\frac{S_n}{n} - \mu \geq t\}.$$

The exponential estimate now follows from (2.3) in Theorem 1 of [29].

Proof. (Of Proposition 5.4.1)

(1) For fixed $n$, with $p_0 < p_1$ as above, let $K_0 = np_0$. Set

$$G_n = \{\omega \mid \{\omega \mid \#\{j \in \{0, 1, \ldots n - 1\} \mid \alpha(\varphi^j \omega) = \alpha\} > K_0\}.\$$

Lemma 5.4.6 estimates $\Pr(I \setminus G_n) \leq \exp[-2n(p_1 - p_0)^2]$.

Now

$$E_\omega(x_n(\omega)) = \int_{G_n} x_n(\omega) \, d\omega + \int_{I \setminus G_n} x_n(\omega) \, d\omega \leq x_{K_0}^\alpha + \frac{1}{2} \Pr(I \setminus G_n)$$

where we have used Lemma 5.4.5 for the first term and the fact that $x_n(\omega) \leq \frac{1}{2}$ for the second term. Now $x_{K_0}^\alpha \leq C_1(K_0^{-\frac{3}{2}}) \leq C_2(n^{-\frac{1}{2}})$ when $K_0 = np_0 n$. On the other hand, the second term tends to zero exponentially fast. Since $x_n(\omega) \geq x_{n}^\alpha \geq C_3 n^{-\frac{1}{2}}$ by Lemma 5.4.4 and the fact that $x_n^\alpha \sim n^{-\frac{1}{2}}$ we have established the required estimate on the expectation.

(2) This follows from part (1) immediately, since both maps have the same linear second branch.

(3) We have to get an estimate on

$$(\lambda \times \lambda)\{\hat{R} > n\} = \sum_{l \geq n+1} (\lambda \times \lambda)(\Delta_l) = \sum_{l \geq n+1} \sum_{i \geq l+1} (\lambda \times \lambda)(\Delta_{l,i}).$$
First, observe that
\[ (\lambda \times \lambda)(\Delta_{0,i}) = \sum_{j=1}^{2^i} (\lambda \times \lambda)(\Delta_{0,i}^j) = \int_0^1 J_i(\omega) d\omega \]
\[ = \int_0^1 [x_{i-1}^j(\omega) - x_i^j(\omega)] d\omega \]
\[ = E[x_{i-1}^j(\omega)] - E[x_i^j(\omega)] \]
\[ = \frac{1}{2} [E(x_{i-1}(\omega)) - E(x_i(\omega))]. \]

Therefore, by equation (5.4.2) and part (1) of this proposition we have
\[ (\lambda \times \lambda)\{\hat{R} > n\} = \frac{1}{2} \sum_{l \geq n+1} \sum_{i \geq l+1} [E(x_{i-1}(\omega)) - E(x_i(\omega))] \]
\[ = \frac{1}{2} \cdot [(Ex_{n+1}(\omega) - Ex_{n+2}(\omega)) + 2(Ex_{n+2}(\omega) - Ex_{n+3}(\omega)) \]
\[ + 3(Ex_{n+3}(\omega) - Ex_{n+4}(\omega)) + ...] \]
\[ = \frac{1}{2} \cdot [E(x_{n+1}(\omega)) + E(x_{n+2}(\omega)) + E(x_{n+3}(\omega)) + ...] \]
\[ \sim \sum_{k>n} k^{-\frac{1}{\alpha}} \]
\[ \sim n^{1-\frac{1}{\alpha}}. \]

\[ \square \]

5.4.2 Distortion

**Lemma 5.4.7.** If \((x, \cdot), (y, \cdot) \in \Delta_0\) and \(s((x, \cdot), (y, \cdot)) = n\), then \(|x - y| \leq \theta^n\).

**Proof.** Set \(\theta := \frac{1}{2} < 1\) and observe that on \(\Delta_0\), \(DT^R_\omega \geq 2 = \theta^{-1}\). Thus, if \((x, \cdot), (y, \cdot)\) lie in a common atom \(\Delta_{0,i}\) such that \(x, y \in J_i(\omega) \subseteq (T^R_\omega)^{-1}(\frac{1}{2}, 1)\), then

\[ \min DT^R_\omega \leq \left| \frac{T^R_\omega x - T^R_\omega y}{x - y} \right| \leq \frac{1}{|x - y|}. \]

Therefore, \(|x - y| \leq \theta\) and the result follows by induction on \(k \leq n\). \(\square\)

**Lemma 5.4.8.** There exists a constant \(C > 0\) such that for \((x, \cdot), (y, \cdot) \in \Delta_{0,i}\),

\[ |\log \frac{DT^R_\omega(x)}{DT^R_\omega(y)}| \leq C |T^R_\omega(x) - T^R_\omega(y)| \leq C. \]

**Proof.** It is trivial for \(J_i, i = 1\) since \(\tau_0(x) = 2x - 1\). We apply the Koebe principle to prove the result for \(J_i, i \geq 2\).

Recall the Schwarzian derivative of a function \(f \in C^3\) is given by:

\[ (Sf)(x) = \frac{f''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2. \]
5.4. PROOFS

It is also well known that the Schwarzian derivative of the composition of two functions \( h, f \in C^3 \) satisfies

\[
S(h \circ f)(x) = (Sh)(f(x)) \times (f'(x))^2 + (Sf)(x).
\]

Consequently, Schwarzian derivative of the composition is negative if both functions have negative Schwarzian derivatives. Let \( g \) denote the composition of the left branches of \( \tau_{\alpha(\varphi^{-1}(\omega)}, ..., \tau_{\alpha(\varphi\omega)} \) and the right branch of \( \tau_{\alpha(\omega)} \). Notice that on \( J_i(\omega), i \geq 2 \), we have \( g(x) = T_{\omega}^R(x) \). Since \( 0 < \alpha < \beta \leq 1 \), we have for the left branch \( \tau'_{\alpha(\omega)} > 0, \tau''_{\alpha(\omega)} > 0 \) and \( \tau'''_{\alpha(\omega)} \leq 0 \); in particular, \( \tau'''_{\alpha(\omega)} = 0 \) if and only if \( \alpha(\omega) = \beta = 1 \). Thus, \( Sg < 0 \).

For each \( J_i(\omega), i \geq 2 \), let \( J = [x_{i+1}(\omega), 2] \). Note that \( g(x_{i+1}(\omega)) < \frac{1}{2} \). Set \( \kappa := \frac{1}{2} - \sup_{\omega} g(x_{i+1}(\omega)) > 0 \). Then \( J_i(\omega) \subset J \) and \( g(J_i(\omega)) = (\frac{1}{2}, 1) \subset (\frac{1}{2} - \kappa, 2] \subset g(J) \). This means \( g(J) \) contains a \( \kappa \)-scaled neighborhood of \( g(J_i(\omega)) \) with constant \( \kappa \). Therefore, by Koebe principle [42] there exists a constant \( C(\kappa) > 0 \) such that

\[
\left| \log \frac{g'(x)}{g'(y)} \right| \leq C(\kappa) |x - y| \leq C(\kappa), \quad \forall x, y \in J_i(\omega),
\]

and consequently,

\[
\frac{|g'(x)|}{|g'(y)|} \leq e^{C(\kappa)}.
\]

It follows that

\[
\left| \frac{x - y}{J_i(\omega)} \right| \leq e^{C(\kappa)} \cdot \left| \frac{g(x) - g(y)}{g(J_i(\omega))} \right|.
\]

Hence, \( \left| \log \frac{g'(x)}{g'(y)} \right| \leq C(\kappa) \cdot e^{C(\kappa)} \cdot \left| \frac{g(x) - g(y)}{g(J_i(\omega))} \right| \), which completes the proof. \( \square \)

**Proposition 5.4.9.** There exists a constant \( C(F) > 0 \) such that for \( z_1, z_2 \in \Delta^j_{0,i} \),

\[
\frac{|DF^R(z_1)|}{|DF^R(z_2)|} - 1 \leq C(F) \cdot g^s(F^R(z_1), F^R(z_2)).
\]

**Proof.** Let \( z_1 = (x_1, \omega_1), z_2 = (x_2, \omega_2) \in \Delta^j_{0,i} \). Then they have same realization

\[
(\alpha(\omega_1), \alpha(\varphi\omega_1), ..., \alpha(\varphi^{R-1}\omega_1)),
\]

for \( j = 1, 2 \). Using this fact and \( F^R(z_l) = S^R(z_l) \), for \( z_l \in \Delta_{0,i}, l = 1, 2 \), we have:

\[
\frac{DF^R(z_1)}{DF^R(z_2)} = \frac{DS^R(x_1, \omega_1)}{DS^R(x_2, \omega_2)} = \left| \begin{array}{cc}
DT^R_{\omega_1}(x_1) & \frac{\partial T^R_{\omega_1}}{\partial \omega_1}(z_1) \\
0 & D_{\omega_1}^R(\omega_1)
\end{array} \right| \bigg| \begin{array}{cc}
DT^R_{\omega_2}(x_2) & \frac{\partial T^R_{\omega_2}}{\partial \omega_2}(z_2) \\
0 & D_{\omega_2}^R(\omega_2)
\end{array} \bigg|
\]

\[
= DT^R_{\omega_1}(x_1) \cdot \frac{1}{\frac{1}{2} \omega_1} = DT^R_{\omega_2}(x_2) \cdot \frac{1}{\frac{1}{2} \omega_2}
\]

\[
= DT^R_{\omega_1}(x_1) = DT^R_{\omega_2}(x_2) = DT^R_{\omega_1}(x_1) = DT^R_{\omega_2}(x_2).
\]

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for any \( \omega \in \Delta_{0,i}^j \). By using Lemma 5.4.7, Lemma 5.4.8 and the following inequality:

\[
|x - 1| \leq \frac{e^{-1}}{C} |\log x|, \text{ if } |\log x| \leq C,
\]

we obtain

\[
\left| \frac{DT_R^\omega(x_1)}{DT_R^\omega(x_2)} - 1 \right| \leq \frac{e^{-1}}{C} |\log \frac{DT_R^\omega(x_1)}{DT_R^\omega(x_2)}| \\
\leq \frac{e^{-1}}{C} \times C |T_R^\omega(x) - T_R^\omega(y)| \\
\leq C(F) \cdot \theta^s(F_R(z_1), F_R(z_2)).
\]
Chapter 6

Conclusion and Future Research

6.1 Results

This thesis studies statistical properties of intermittent maps. We obtain four new results. First in Chapter 2, we use an Ulam-type discretization scheme to provide pointwise approximations for invariant densities of interval maps with a neutral fixed point. We prove that the approximate invariant density converges pointwise to the true density at a rate \( C^* \cdot \frac{\ln m}{m} \), where \( C^* \) is a computable fixed constant and \( \frac{1}{m} \) is the mesh size of the discretization. This chapter reflects our result in [7].

Then in Chapter 3, we obtain convergence in the \( L^1 \)-norm. Although there are results, like Murray [43], our method has a faster rate.

Thirdly, in Chapter 4, we study a random map \( T \) which consists of intermittent maps \( \{\tau_k\}_{k=1}^K \) and a position dependent probability distribution \( \{p_{k,\varepsilon}(x)\}_{k=1}^K \). We prove existence of a unique absolutely continuous invariant measure (ACIM) for the random map \( T \). Moreover, we show that, as \( \varepsilon \) goes to zero, the invariant density of the random system \( T \) converges in the \( L^1 \)-norm to the invariant density of the deterministic intermittent map \( \tau_1 \). The outcome of this chapter contains a first result on stochastic stability, in the strong sense, of intermittent maps. This chapter follows my result in [20].

Finally, in Chapter 5, we study a class of random transformations built over finitely many intermittent maps sharing a common indifferent fixed point. Using a Young-tower technique, we show that the map with the fastest relaxation rate dominates the asymptotics. In particular, we prove that the rate of correlation decay for the annealed dynamics of the random map is the same as the sharp rate of correlation decay for the map with the fastest relaxation rate. This chapter reflects our result in [8].

6.2 Future Research

In my plan, firstly, I want to study limit theorems, similar to what Gouëzel studied, using our result in Chapter 5. Then, I would like to generalize all the results of my thesis to the higher dimensional systems, similar to those studied by Hu and Vaienti in [33]. Finally, it will also be interesting to obtain limit theorems for higher dimensional systems.
Bibliography


[7] Bahsoun, W., Bose, C. and Duan, Y.: Rigorous pointwise approximations for invariant densities of nonuniformly expanding maps. Accepted (subject to minor corrections) in Ergodic Theory and Dynamical Systems. Available at arxiv.org/pdf/1301.4033.


