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B-Spline Surfaces Over An Irregular Topology By Recursive Subdivision

by

David J.T. Storry.

A doctoral thesis submitted in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy of Loughborough University of Technology.

December 1984.

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The technique of recursive subdivision can be visualised, loosely, as successively chopping off the corners of a polyhedron to make it less pointed. If the polyhedron is represented as a mesh of points connected by edges, repeated application of the subdivision results in progressively finer meshes tending in the limit to a surface. The subdivision is determined by the weightings given to the respective points and their neighbours.

In this thesis, the author considers recursive subdivision over arbitrary meshes where the subdivision weightings are chosen so that, if the mesh is rectangular, a standard B-spline bi-cubic surface is obtained. The resulting surface is $C^2$, except at the so-called extraordinary points where the refined mesh is not locally rectangular. The technique is potentially useful in CAD since it has all the advantages of B-spline surfaces and also accommodates non-rectangular topologies, in particular the occurrence of 3 and 5-sided patches.

The problem is to analyse the surface at the extraordinary points and to determine the optimal subdivision weightings. An extraordinary point and its neighbours at successive subdivisions can be related by a transformation matrix and a detailed analysis of eigenproperties is carried out. A necessary and sufficient condition for $C^1$ continuity is derived.

Using a discrete Fourier transform technique, the $C^1$ continuity results are confirmed. This method is more compact and more easily applied to an investigation of curvature. The nature of $C^2$ continuity is related to subdivision surfaces and a totally general analysis is completed which enables the optimisation of the subdivision. In this case, for a 3-node the maximum disparity of curvature is less than 10%; for a 5-node it is less than 5%.
Acknowledgements

I am deeply indebted to Dr. A.A. Ball who provided not only the ideas upon which the work described here is based, but also help, direction and encouragement at every stage. Without this contribution, I surely would not even have contemplated the problems I attempt to tackle herein, far less pursued them for the time required.

I must thank the Department of Engineering Mathematics at Loughborough University, under Professor A.C. Bajpai, for employing me and making available the facilities required for my work. The Science and Engineering Research Council also deserve my thanks for funding a research project to develop and implement the technique of recursive subdivision for Computer-Aided Design. The figures herein were produced on an Imlac Series II terminal bought with these funds.

If reading this thesis is the notational nightmare that writing it was, the rigours undergone by the typist can readily be appreciated. My thanks then to Mrs. B. Wright who has made such a fine job of this painstaking work.

Finally, any errors are those of the author alone.
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Methods of Surface Design and the Subdivision Algorithm

The problem of designing a doubly curved surface over an arbitrary topology of patches has not yet been fully solved. Partial solutions, however, exist in abundance, largely due to the pioneering work of Coons [10], for rectangular and Barnhill [1] for triangular patches. From the former, there have been developed several techniques for the piecewise representation of a surface by rectangular patches, defined mathematically as biparametric vector-valued functions. Three important examples, based on the theory of approximation and interpolation, are

1) The B-spline method.
2) The Bézier patch.
3) The Hermite patch.

Each of the three methods of generating a rectangular surface patch (or in case(1) an assembly of these) is an extension of a curve design technique. We therefore consider them, initially, in this latter form.

1) The B-spline segment.

The theory of mathematical splines has been developed substantially since 1946 from a seminal paper by Schoenberg [25]. A detailed exposition of the B-spline basis functions and their application to curve design is given by Gordon and Riesenfeld [18].
We confine ourselves here to a brief outline of their properties.

The B-spline segment of degree $n$ is controlled by a polygon $P$ consisting of $n+1$ points, $P_0, P_1, \ldots, P_n$. This governs the shape of the curve.

The uniform B-spline basis functions are given inductively as follows:

$$
N_{0,0}(u) = 1 \\
N_{i,0}(u) = 0 \quad \text{otherwise} \\
N_{i,k}(u) = k^{-1}\left[(k-i+u)N_{i-1,k-1}(u) + (i+1-u)N_{i,k-1}(u)\right] \\
\text{for } k \in \mathbb{N}, \ i \in \mathbb{Z} , \ 0 \leq u \leq 1
$$

It can easily be shown that the following properties hold.

a) $N_{i,n} \equiv 0$ unless $i = 0, 1, \ldots, n$

b) $N_{i,n} > 0$ on $[0, 1]$

c) $N_{i,n}$ is a polynomial of degree $n$

d) $\sum_{i=0}^{n} N_{i,n} \equiv 1$

e) $N_{i,n}^{(j)}(1) = N_{i-1,n}^{(j)}(0)$

$$
N_{0,n}^{(j)}(1) = N_{n,n}^{(j)}(0) = 0 \quad \text{being the derivative index.}
$$

f) $N_{n-i,n}(u) = N_{i,n}(1-u)$

The B-spline curve of degree $n$ is then given by

$$
P_N(u) = \sum_{i=0}^{n} N_{i,n}(u) P_i \quad (0 \leq u \leq 1)
$$
The curve will reflect the gross features of the polygon and, by properties b) and d), lie in its convex hull. In general, the curve will have continuous non-zero derivatives up to, and including, order n.

Loosely, the curve will be located in the middle span of the polygon (or between the centres of the two middle spans if n is even). Suppose, then, the polygon is extended to include another point \( p_{n+1} \).

Consider the curve \( p^*_N(u) = \sum_{i=0}^{n} N_{i,n}(u) p_{i+1} \)

For \( j = 0, 1, \ldots, n-1 \)

\[
\begin{align*}
p^*_N(j)(0) &= \sum_{i=0}^{n} N_{i,n}(0) p_{i+1} \\
&= \sum_{i=0}^{n} N_{i+1,n}(1) p_{i+1} \quad \text{by e)} \\
&= \sum_{i=1}^{n} N_{i,n}(1) p_i \quad \text{by a)}
\end{align*}
\]

But \( p_N(j)(1) = \sum_{i=0}^{n} N_{i,n}(1) p_i \)

\[
\begin{align*}
p_N(j)(1) &= \sum_{i=1}^{n} N_{i,n}(1) p_i \quad \text{by e)} \\
&= p^*_N(j)(0)
\end{align*}
\]

Thus if we extend the curve a further span by adding a point to the polygon, there is \( C^{n-1} \) continuity at the knot. As a consequence, a B-spline segment of degree n may be extended indefinitely in both directions by the addition of more control
The advantages of this technique lie in the global design of free-form curves. Given a series of \( m \) control points, curves may be derived which simulate the shape of the polygon, with continuity of up to order \( m-2 \).

In engineering applications, it is, in general, sufficient to have \( C^2 \) continuity, and thus the B-spline cubic segment has found favour.

For \( n = 3 \) the basis functions are:

\[
N_{0,3}(u) = \frac{1}{6} (1-u)^3
\]

\[
N_{1,3}(u) = \frac{1}{6} (4 - 6u^2 + 3u^3)
\]

\[
N_{2,3}(u) = \frac{1}{6} (1 + 3u + 3u^2 - 3u^3)
\]

\[
N_{3,3}(u) = \frac{1}{6} u^3
\]

With control points \( P_0, P_1, P_2, P_3 \):

\[
P_N(u) = [U] [M_N] [P] \quad (0 \leq u \leq 1)
\]

where

\[
[U] = [u^3, u^2, u, 1]
\]

\[
[M_N] = \frac{1}{6} \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{bmatrix}
\]

\[
[P] = [P_0, P_1, P_2, P_3]^T
\]

A B-spline cubic segment and its control points are illustrated in Fig. 1.1.
Fig. 1.1 The B-spline Cubic Segment and its Control Points

Fig. 1.2 The Bézier Cubic Segment and its Control Points

Fig. 1.3 The Hermite Cubic Segment, its Endpoints and 1st Derivative Vectors
2) The Bézier segment.

The properties of Bernstein polynomials [19] have been successfully exploited in curve and surface design by P. Bézier of Régie Renault [6]. The Bernstein polynomial approximation of degree \( n \) to a function \( f \), defined on \([0, 1]\) is given by

\[
[B_n(f)](u) = \sum_{i=0}^{n} B_{i,n} (u) f(i/n)
\]  

(1.4)

where the \( B_{i,n} \)'s are the discrete binomial probability density functions, i.e.

\[
B_{i,n} (u) = \binom{n}{i} u^i (1-u)^{n-i}, \quad i = 0, 1, 2, \ldots, n
\]  

(1.5)

The concept of the Bézier segment is very similar, except that, instead of the values of a function at \( i/n \), a control polygon \( P_0, P_1, \ldots, P_n \) is used (see Gordon and Riesenfeld [19])

\[
P_B (u) = \sum_{i=0}^{n} B_{i,n} (u) P_i \quad (0 \leq u \leq 1)
\]  

(1.6)

With the exception of e), the Bernstein basis functions have similar properties to their B-spline counterparts. In particular, unlike the B-spline segment, we find that \( P_B (0) = P_0 \) and \( P_B (1) = P_n \) and so the segment extends over the entire range of its control polygon. Therefore, although we have the same continuity properties within a segment as encountered in the B-spline case, the end derivative properties enabling us to extend the polygon are absent; we can, of course, extend the parameter range but, in doing so, we lose the convex hull property.

Nonetheless, since \( P_B'(0) = P_0' \) and \( P_B'(1) = P_n' \), \( P_B'(0) = n(P_n - P_0) \)
and \( P'_B(1) = n(P_n - P_{n-1}) \), the curve not only has as its endpoints the endpoints of the polygon, but is tangent to its first and last sides here. These qualities render this method well-suited to local curve design. Requirements of continuity of derivatives across endpoints of a segment will impose constraints on the polygons which control neighbouring segments [12, p.133].

It is found that, for most engineering applications, the cubic segment provides sufficient flexibility, although this can be increased by raising the degree or the insertion of more control points [15]. The advantage of using higher order Bézier curves is that higher order continuity can be achieved between segments. The disadvantage lies in the progressively weaker relationship between curve and polygon.

The Bézier cubic segment, with control points \( P_0, P_1, P_2, P_3 \), is given by

\[
P_B(u) = [U][M_B][P] \quad (0 \leq u \leq 1)
\]

where

\[
[M_B] = \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 3 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

and \([U]\) and \([P]\) are as before.

A Bézier cubic segment and its control points are illustrated in Fig. 1.2.

3) The Hermite segment.

The classical interpolation method of Hermite has been
adapted to the "blending function" notion of Coons [10] suitable for curve and surface concatenation. Unlike the two techniques mentioned above, the governing factor is not a control polygon, but the position and derivative values at either end of the segment. Given these values for order up to \( n \), we can obtain a curve segment which is \( C^n \) continuous over the knots at either end, in addition to its span. Since this involves \( 2(n+1) \) items of data, the blending polynomials will be of degree \( 2n+1 \).

The Hermite polynomials are defined as follows:

\[
H_{i,2n+1}(u) = \frac{1}{i!} (1-u)^{n+1} u \sum_{k=0}^{n-i} \binom{n+k}{k} u^k \\
(0 \leq u \leq 1) \quad (1.8)
\]

\[
K_{i,2n+1}(u) = (-1)^i H_{i,2n+1}(1-u) \quad \text{for } i = 0,1,\ldots,n
\]

The following properties can easily be demonstrated

a) \( H_{0,2n+1} + K_{0,2n+1} = 1 \)

b) \( \begin{align*}
H_{i,2n+1}^{(j)}(0) &= \delta_{ij} \\
K_{i,2n+1}^{(j)}(0) &= 0 \\
H_{i,2n+1}^{(j)}(1) &= 0 \\
K_{i,2n+1}^{(j)}(1) &= \delta_{ij}
\end{align*} \)

for \( i,j \leq n \)

where \( \delta_{ij} \) is the Kronecker delta

\[ \delta_{ij} = 1 \text{ if } i = j \]

\[ = 0 \text{ otherwise.} \]

Let the \( i \)th vector-valued derivative at the beginning of the segment be \( p_0^{(i)} \) and at the end \( p_{-1}^{(i)} \) (\( i = 0,1,\ldots,n \)). Then
the Hermite segment of degree $2n+1$ is given by:

$$p_{H}(u) = \sum_{i=0}^{n} \left\{ H_{i,2n+1}(u) \ p^{(i)}_{0} + K_{i,2n+1}(u) \ p^{(i)}_{1} \right\} .$$  \hspace{1cm} (1.9)

It follows from property b) that

$$p^{(j)}_{H}(0) = p^{(j)}_{0} \quad \text{and} \quad p^{(j)}_{H}(1) = p^{(j)}_{1} \quad (j = 0, 1, \ldots, n)$$

If no derivatives are furnished, property a) shows that the resulting segment is simply the straight line connecting $p^{(0)}_{0}$ and $p^{(0)}_{1}$, the end position vectors.

This method has advantages in local curve design, where the derivatives at either end of the putative segment are known. However, unless the adjoining segments are of uniform parametrization, the results may not be satisfactory. Because the design data is vector-valued, and the designer is unlikely to have a clear idea of the direction and magnitude of suitable derivatives beyond the first, for $n > 1$ the extra freedom may be used by the system for shape adjustment.

In the cubic case, i.e. $n = 1$, the Hermite approach is widely used to generate $C^1$ curves.

$$H_{0,3}(u) = 2u^3 - 3u^2 + 1$$

$$K_{0,3}(u) = -2u^3 + 3u^2$$

$$H_{1,3}(u) = u^3 - 2u^2 + u$$

$$K_{1,3}(u) = u^3 - u^2$$
Thus the Hermite cubic can be represented as:

\[ p^H(u) = [U][H][Q] \]

where \[ [H] = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ [Q] = \begin{bmatrix} P_0(0), P_1(0), P_0(1), P_1(1) \end{bmatrix}^t \]

and \[ [U] \] is as above.

A Hermite cubic segment and its 1st derivatives are illustrated in Fig. 1.3.

The above techniques only give an indication of the range of basis functions for curve design. Faux and Pratt [12] consider also the Ferguson curve segment, rational parametric cubic curves and non-linear splines.

As remarked earlier, each of our three methods of curve design may be generalized to one of surface design. This is effected by taking the tensor product of each of the defining formulae. Thus the biparametric B-spline patch of degree \( n \) is governed by a control polyhedron, given by the \( n+1 \) order square matrix of vectors

\[ [P_{ij}]_{i,j = 0,1,...,n} \]

the surface patch, \( S \), being defined:

\[ S_N(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{n} N_{i,n}(u) N_{j,n}(v) P_{ij}. \]
For the Bézier patch, we again require such an array of control points and the biparametric surface is defined by:

\[
S_B(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{n} B_i(u) B_j(v) P_{ij}
\]

(1.12)

In both these cases, the nature of the surface obtained and the relationship between the surface and its control points is analogous to what is found in curve design. The patch will lie in the convex hull of its control points and reflect the shape of the polyhedron they form. The B-spline patch will have approximate dimensions of one polyhedron span square and be located in the centre of the array of points while the Bézier patch will have its vertices at \( \mathbf{P}_{00}, \mathbf{P}_{0n}, \mathbf{P}_{n0} \), and \( \mathbf{P}_{nn} \).

The Hermite patch may be generalised in two different ways from the definition of the segment. Data may be supplied either in a zerovariate or univariate manner, that is the provision of derivative data at the four corners of the patch, or in the form of the boundary curves and their cross-derivatives. This distinction is made by Forrest [14]. In the former case, let \( \mathbf{P}_{ab}^{u(i)v(j)} \) represent the \( i,j \)th partial derivative with respect to \( u \) and \( v \) respectively at \( u = a, v = b \).

Then the zerovariate Hermite patch of degree \( 2n+1 \) is given by the formula:

\[
S_H(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{n} \begin{bmatrix} H_i,u,2n+1 \left( u \right),K_i,v,2n+1 \left( u \right) \end{bmatrix} \begin{bmatrix} P_{ij}^{u(i)v(j)} & P_{ij}^{u(i)v(j)} \\ P_{ij}^{u(i)v(j)} & P_{ij}^{u(i)v(j)} \end{bmatrix} \begin{bmatrix} H_j,u,2n+1 \left( v \right) \\ K_j,v,2n+1 \left( v \right) \end{bmatrix}
\]
for \( 0 \leq u, v \leq 1 \).  

(1.13)

It can be shown that this patch has the appropriate 

derivatives at each corner and, if the surrounding patches are 
Hermite patches of the same type, it is \( C^n \) within and across its 
boundaries.

Suppose, on the other hand, the boundary curves and their 
cross-derivatives are known. We shall call the curves themselves 
\( F(u, 0), F(u, 1), F(0, v) \) and \( F(1, v) \) and let, for example, 
\( F^v(j)(u_0, 1) \) represent the \( j^{th} \) partial derivative with respect 
to \( v \) at the boundary point given by \( u = u_0, v = 1 \). Following 
Gordon [17], we then apply the so-called Projector functions 
to \( F \). These are defined as follows:

Let \( P_1[F](u,v) = \)

\[
\sum_{i=0}^{n} \left\{ H_{i,2n+1}(u) F^u(i)(0,v) + K_{i,2n+1}(u) F^u(i)(1,v) \right\}
\]

(1.14)

and \( P_2[F](u,v) = \)

\[
\sum_{j=0}^{n} \left\{ H_{j,2n+1}(v) F^v(j)(u,0) + K_{j,2n+1}(v) F^v(j)(u,1) \right\}
\]

Defining \( P \) as the Boolean sum of \( P_1 \) and \( P_2 \), i.e.

\[
P = P_1 \oplus P_2 = P_1 + P_2 - P_1 P_2
\]

(not noting that \( P_1 P_2 = P_2 P_1 \))

then \( P[F] \) interpolates \( F \) and its derivatives up to order \( n \) on 
the boundary of the patch now defined.
As in the case of curve design, it is found that high order approximation, for practical implementation, because of the opacity inherent in vector-valued derivatives and anomalies in parametrization, does not find wide favour. For each technique, the cubic case is in most general use for surface design in engineering, providing an acceptable balance of approximation features, flexibility, stability and desirable surface properties.

The B-spline bicubic patch is defined by:

\[ S_N(u,v) = [U][M_N][P][M_N]^t[V]^t \quad (0 \leq u, v \leq 1) \]

where \([U], [M_N]\) are as before,

\[ [V] = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix} \]

and

\[ [P] = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} \]

The Bézier patch is defined by:

\[ S_B(u,v) = [U][M_B][P][M_B]^t[V]^t \quad (0 \leq u, v \leq 1) \]

and the Hermite patch by:

\[ S_H(u,v) = [U][M_H][Q][M_H]^t[V]^t \quad (0 \leq u, v \leq 1) \]

where

\[ [Q] = \begin{bmatrix} P_{00} & P_{01} & P^v_{00} & P^v_{01} \\ P_{10} & P_{11} & P^v_{10} & P^v_{11} \\ P^u_{00} & P^u_{01} & P^{uv}_{00} & P^{uv}_{01} \\ P^u_{10} & P^u_{11} & P^{uv}_{10} & P^{uv}_{11} \end{bmatrix} \]
Fig. 1.4 The B-spline Bicubic Patch and its Control Polyhedron

Fig. 1.5 The Bézier Bicubic Patch and its Control Polyhedron

Fig. 1.6 The Hermite Bicubic Patch showing Corner Positions and Derivatives
Examples of these three types of surface patch are shown in Figs. 1.4 - 1.6.

In this, the cubic case, only the B-spline method can guarantee $C^2$ continuity of the design surface across patch boundaries and, while the relative merits of each approach may be assessed differently with respect to other priorities such as concatenation, stability or interpolative qualities, in the creation of global free-form doubly curved surfaces this technique is clearly most appropriate. We shall, therefore, proceed to investigate another property of the B-spline surface which will establish the utility of this approach for free-form design not only in the metric sense, but over an arbitrary topology, for it must be remembered that the three techniques described above generate only four-sided patches.

It will be instructive, then, to return to the B-spline cubic segment. From (1.3) we have

$$ P(u) = [U][M][P] \quad (0 \leq u \leq 1) $$

dropping the subscript $N$ for the sake of simplicity.

Consider the subsegment

$$ P_{[\lambda_0, \lambda_1]}(u) = [U][M][P] \quad (0 \leq \lambda_0 < \lambda_1 \leq 1) $$

$$ \lambda_0 \leq u \leq \lambda_1 \quad (1.18) $$

We show this is a B-spline cubic segment in its own right with control points $P_0$, $P_1$, $P_2$, $P_3$ represented by the tensor $[P']$

i.e.

$$ P_{[\lambda_0, \lambda_1]}(u) = [U][M][P'] \quad (0 \leq u \leq 1). \quad (1.19) $$
For, since \([0, 1]\) is mapped onto \([\lambda_0, \lambda_1]\) by the isomorphism

\[
\phi : u \rightarrow \lambda_0 + u(\lambda_1 - \lambda_0)
\]

we can express \(P_{[\lambda_0, \lambda_1]}\) in another form, parametrized for

\[
0 \leq u \leq 1.
\]

\[
\phi : [u^3, u^2, u, 1] \rightarrow \\
[(\lambda_0 + u(\lambda_1 - \lambda_0))^3, (\lambda_0 + u(\lambda_1 - \lambda_0))^2, \lambda_0 + u(\lambda_1 - \lambda_0), 1]
\]

\[
= [u, u^2, u, 1] \begin{bmatrix}
(\lambda_1 - \lambda_0)^3 & 0 & 0 & 0 \\
3\lambda_0 (\lambda_1 - \lambda_0)^2 & (\lambda_1 - \lambda_0)^2 & 0 & 0 \\
3\lambda_0^2 (\lambda_1 - \lambda_0) & 2\lambda_0 (\lambda_1 - \lambda_0) & \lambda_1 - \lambda_0 & 0 \\
\lambda_0^3 & \lambda_0^2 & \lambda_0 & 1
\end{bmatrix}
\]

\[
= [U][H_{[\lambda_0, \lambda_1]}] \text{ say}.
\]

Thus

\[
P_{[\lambda_0, \lambda_1]}(u) = [U][H_{[\lambda_0, \lambda_1]}][M][P] \quad (0 \leq u \leq 1)
\]

We can now determine \([P']\).

\[
[U][M][P'] \equiv [U][H_{[\lambda_0, \lambda_1]}][M][P] \quad (0 \leq u \leq 1)
\]

\[
\Rightarrow [M][P'] = [H_{[\lambda_0, \lambda_1]}][M][P]
\]

\[
\Rightarrow [P'] = [M]^{-1}[H_{[\lambda_0, \lambda_1]}][M][P]. \quad (1.20)
\]

Now we can establish a relation between this process and the curve segment itself since, as \(\lambda_1 \rightarrow \lambda_0\)
\[
\begin{bmatrix}
H_{[\lambda_o, \lambda_1]} + 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\lambda_o^3 & \lambda_o^2 & \lambda_o & 1 \\
\end{bmatrix} & = & H_{\lambda_o} \\
\end{bmatrix}
\]

\[
[M]^{-1} = \frac{1}{3} 
\begin{bmatrix}
0 & 2 & -3 & 3 \\
0 & -1 & 0 & 3 \\
0 & 2 & 3 & 3 \\
18 & 11 & 6 & 3 \\
\end{bmatrix}
\]

and so \[
[M]^{-1}[H_{\lambda}][M]
\]

\[
= \begin{bmatrix}
\lambda^3 & \lambda^2 & \lambda & 1 \\
\lambda^3 & \lambda^2 & \lambda & 1 \\
\lambda^3 & \lambda^2 & \lambda & 1 \\
\lambda^3 & \lambda^2 & \lambda & 1 \\
\end{bmatrix} \quad [M] = [\Lambda][M] \text{ say.}
\]

Thus, as \( \lambda_1 \to \lambda_o \)

\[
[P'] + [\Lambda_o][M][P] = [P(\lambda_o)]
\]

This result ensures that the control points for progressively smaller nested subsegments tending to zero length, converge to the point on the curve whose parametric value is the intersection of the nested intervals.

Applying this subdivision to the interval \([0, \frac{1}{2}]\) we have:

\[
[P'] = [M]^{-1}[H_{[0, \frac{1}{2}]})[M][P]
\]
\[
[M]^{-1} = \begin{bmatrix}
\frac{1}{8} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8}
\end{bmatrix}
\]

i.e. \( P'_0 \) is the midpoint of \( P_0 \) and \( P_1 \)

\( P'_2 \) is the midpoint of \( P_1 \) and \( P_2 \)

and if \( Q'_0 \) is the midpoint of \( P'_0 \) and \( P'_1 \)

\( Q'_1 \) is the midpoint of \( P'_1 \) and \( P'_2 \)

and \( Q'_2 \) is the midpoint of \( P'_2 \) and \( P'_3 \)

then \( P'_1 \) is the midpoint of \( Q'_0 \) and \( Q'_1 \)

and \( P'_3 \) is the midpoint of \( Q'_1 \) and \( Q'_2 \)

Similarly, we can establish the control points for \( p_{\frac{1}{4}, 1}(u) \) which turn out to be:

\( P'_1, P'_2, P'_3, \) and \( P'_4, \) where \( P'_1, P'_2, P'_3 \) are as above and

\[ P'_4 = \frac{1}{2} (P_2 + P_3). \]

The B-spline curve, its control points and these new subsegment control points are illustrated in Fig. 1.7.
Fig. 1.7 The B-spline Curve, Control Points and Subdivided Control Points

Fig. 1.8 The B-spline Control Polyhedron and the Subdivided Array
This technique can be regarded as a subdivision algorithm on the control points of the original segment. These have been replaced by a new set consisting of the midpoints of each span and, adjacent to each of the old internal points, a new point which is a weighted average of the old point and its immediate neighbours on either side. This procedure can be repeated to generate a set of seven points \( \{ P^n_i \} \), of which

\[
\begin{align*}
[P^n_i]_{i=0,\ldots,3} & \text{ will be the control points for } P_{[0,1/4]} \\
[P^n_i]_{i=1,\ldots,4} & \text{ for } P_{[1/4,1/2]} \\
[P^n_i]_{i=2,\ldots,5} & \text{ for } P_{[1/2,1/4]} \\
[P^n_i]_{i=3,\ldots,6} & \text{ for } P_{[3/4,1]} 
\end{align*}
\]

Since one subdivision generates \( 2m-3 \) points from \( m \), after \( n \) subdivisions there will be \( 2^n + 3 \) points which, taken in successive groups of four will be the control points for subsegments with parametric range \( [i2^{-n}, (i+1)2^{-n}] \). As a consequence of the observation made above with regard to the control points for diminishing subsegments, it is clear that this polygon, as the number of subdivisions increases, will converge, in the uniform norm, to the segment itself.

Catmull and Clark [8] have implemented this technique of subdivision for B-spline bicubic surfaces. The mathematics as we shall see, is just a generalisation of that set out above.

Let \( \[ R \] = \{ R_{i,j} \}_{i,j=0,\ldots,3} \) as in (1.15). Then the B-spline
bicubic subpatch is given by:

\[ s_{[0,1]}(u,v) = \frac{\mathcal{U}}{[0,1]} \]

\[ [R][H_{[0,1]}][M][R][H_{[0,1]}]^{T}[V]^{T} \]

\[ = [R][M][R'][M]^{T}[V]^{T} \]

Thus \([M][R'][M]^{T} = [H_{[0,1]}][M][R][M]^{T}[H_{[0,1]}]^{T} \]

\[ \leftrightarrow [R'] = [M]^{-1}[H_{[0,1]}][M][R][M]^{-1}[H_{[0,1]}][M] \]

\[ = \begin{bmatrix}
\frac{1}{2} & 1 & 2 & 0 & 0 \\
1 & \frac{3}{4} & 1 & 8 & 0 \\
0 & \frac{1}{2} & 1 & 2 & 0 \\
0 & \frac{1}{8} & 3 & 1 & 8 \\
\end{bmatrix} \begin{bmatrix}
R_{00} & R_{01} & R_{02} & R_{03} \\
R_{10} & R_{11} & R_{12} & R_{13} \\
R_{20} & R_{21} & R_{22} & R_{23} \\
R_{30} & R_{31} & R_{32} & R_{33} \\
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} & 1 & 2 & 0 & 0 \\
1 & \frac{3}{4} & 1 & 8 & 0 \\
0 & \frac{1}{2} & 1 & 2 & 0 \\
0 & \frac{1}{8} & 3 & 1 & 8 \\
\end{bmatrix} \]

(1.21)

Multiplying this out gives:

\[ R'_{00} = \frac{1}{4} \left( R_{00} + R_{01} + R_{10} + R_{11} \right) \]

\[ R'_{02} = \frac{1}{4} \left( R_{01} + R_{02} + R_{11} + R_{12} \right) \]

\[ R'_{20} = \frac{1}{4} \left( R_{10} + R_{11} + R_{20} + R_{21} \right) \]

\[ R'_{22} = \frac{1}{4} \left( R_{11} + R_{12} + R_{21} + R_{22} \right) \]

\[ R'_{01} = \frac{3}{8} \left( R_{01} + R_{11} \right) + \frac{1}{16} \left( R_{00} + R_{10} + R_{02} + R_{12} \right) \]
This new array of control points can be obtained from the old-
by applying to them the subdivision algorithm for the cubic segment
in the \(i\) and \(j\) (i.e. \(u\) and \(v\)) directions successively. By similar
consideration of the other three subpatches, \(S^{[1,1]}\), \(S^{[0,1]}\) and
\(S^{[1,1]}\), we obtain the subdivided 5 \(\times\) 5 control polyhedron
\([R^i_{ij}]_{i,j=0,\ldots,4}\). This is shown in Fig. 1.B.

It is clear that the subdivision creates three types of points,
namely those which lie in the centre of the faces of the original
polyhedron, which we shall call face points, those adjacent to the
midpoint of an edge, or edge points, and those adjacent to the vertices of the polyhedron, or vertex points. We look more closely at the way in which these points are defined.

1) The new face point is defined as the average of the four original points incident upon the face. In the context of Fig. 1.9 we have

\[ F = \frac{1}{4} (P_1 + P_2 + P_3 + P_4) \]

2) The new edge point is defined as the average of the two new face points on the faces meeting on the edge and the two old points at either end of the edge. i.e. (see Fig. 1.10).

\[ E = \frac{1}{8} \left( P_3 + P_4 + \frac{1}{4} (P_1 + P_2 + P_3 + P_4) \right) \]

\[ = \frac{3}{8} (P_3 + P_4) + \frac{1}{16} (P_1 + P_2 + P_3 + P_4) \]

3) The new vertex point is defined as follows:

Let \( G \) be the average of the new face points on the four surrounding faces, i.e. (see Fig. 1.11).

\[ G = \frac{1}{4} (F_1 + F_2 + F_3 + F_4) \]

Let \( Q \) be the average of the points connected to \( P_5 \), the old vertex point, by an edge

\[ Q = \frac{1}{4} (P_2 + P_4 + P_6 + P_8) \]
Fig. 1.9 The New Face Point

Fig. 1.10 The New Edge Point

Fig. 1.11 The New Vertex Point
then \[ V = \frac{1}{4} (G + 2P_5 + Q) \]

which, consistent with the entries in [R'] resolves to:

\[
\frac{9}{16} P_5 + \frac{3}{32} (P_2 + P_4 + P_6 + P_8) + \frac{1}{64} (P_1 + P_3 + P_7 + P_9).
\]

Applying this procedure repeatedly, the polyhedron, consisting after \( n \) subdivisions of \( (2^n+3) \times (2^n+3) \) points, will converge to the bicubic patch \( S \). More generally, starting with an \( k \times \ell \) rectangular array of points, the method will yield the bicubic (and thus \( C^2 \)) surface governed by these control points taken in neighbouring \( 4 \times 4 \) groups, (see Fig. 1.12).

In Chapter 2 we shall consider how this approach can be applied to non-rectangular topologies.
Fig. 1.12 A $5 \times 5$ Array after 4 Subdivisions
Chapter 2

Non-Rectangular Topologies

We have seen that, for a B-spline surface, there is both an explicit biparametric definition, in terms of the control polyhedron, and a subdivision algorithm applicable to these points which converges to it. In the former case, however, we are confined to a rectangular array of points, that is each of the polyhedral faces is quadrilateral and each of the internal vertices is the junction of four edges, i.e. a 4-node.

In the generation of free-form smooth surfaces for engineering applications it is often the case that a topology occurs containing a non-rectangular patch or, in the context of control points, which cannot be modelled solely with 4-nodes. This would arise, for example, at the corner of a cuboid (a 3-node), the interface of two cuboids (a 5-node) or the junction of three (a 6-node), (see Figs. 2.1-3). Important practical examples of this phenomenon are at the windscreen post of a car, the heat exchanger on an aircraft and certain parts of a ship's hull. To deal with the problems which arise here it is necessary to find some technique which, while distinct from bicubic patches, can be blended with them.

Attention has, in the main, been directed to the problem of the triangular patch. To regard the solution of this problem as a priority is quite understandable since, as in a standard proof of Cauchy's integral theorem for analytic functions round a closed
Fig. 2.1 A 3-node at the Corner of a Cuboid

Fig. 2.2 A 5-node at the Interface of Two Cuboids

Fig. 2.3 A 6-node at the Junction of Three Cuboids
complex curve [26, p.143], the fact that every domain can be triangulated to within specified tolerance implies that such a solution would serve as a basis for generalisation to \(N\)-sided patches.

The obvious solution to this problem would be to define a triangular patch as a degenerate rectangular patch, coalescing two vertices. This method has proved to be unsatisfactory not only from the intuitive point of view of inherent asymmetry, for this may be marginalised as but mathematical whim, but also, if a knot, cusp or ridge is to be avoided in the surface, severe restrictions must be imposed on the tangency conditions. In a practical context, these are unacceptable. The direct use of interpolants over triangles has been explored by Sabin [24] and Barnhill [1, 2, 3, 4], the latter using a modification of the Projector functions of (1.14), to create a vector-valued scheme compatible with the bicubic Hermite patch of Chapter 1. Nonetheless, in both the zero-variate and univariate cases described above problems arise with mixed partial derivatives at the vertices of the triangle which, although Gregory [20] provides a partial solution, have not, in the general case, been resolved.

Gregory and Charrot [21] have suggested an approach which is suitable for the design of a triangular patch at the corner of a rectangular patch system to guarantee \(C^1\) continuity and, by virtue of its relative simplicity, is amenable to application in Computer Aided Design. Nonetheless the more powerful techniques analysed by Charrot [9] designed to yield continuity of order greater than 1, would appear, on account of the instability of
the blending functions involved and attendant restrictions on
derivatives, not to be suitable for the numerical description
of surfaces.

These approaches, however, adopt the Hermite approach to
surface design, insofar as the data required is in either the
zero-variate form or in terms of boundary values and cross
derivatives. Furthermore, with the exception of Charrot, a
virtual monopoly is given to the 3-sided patch to the exclusion
of a more uniform approach. Although in practical terms such an
approach is not strictly necessary, since, with Charrot's partition
of a surface by X, Y and Z lines [9], each set of curves being
mutually skew, a patch can be at most bounded by two of each such
lines, i.e. 6-sided, its generality will be attractive.

Let us therefore return to the concept of a control polyhedron
consisting of points (not necessarily 4-nodes) connected by edges
enclosing faces (not necessarily rectangular). Catmull and Clark
[8] have shown that, given an arbitrary topology of this type,
the subdivision rules of Chapter 1 can be generalised so that
after one application, joining face points to edge points and
edge points to vertex points, all the faces of the new polyhedron
are quadrilateral (see Fig. 2.4). Consequently, since the algorithm
creates only 4-nodes from a topology of quadrilaterals, the problem,
beyond this stage, reduces to that of the rectangular grid, except
at a fixed number of non-4-nodes, or extraordinary points. This
number is the sum of the original number of extraordinary points
and the number of non-quadrilateral faces. Further subdivision
will generate more points to extend the carpet of patches over the
Fig. 2.4 The Subdivision of a Triangular Face yielding a 3-node and Quadrilateral Faces

Fig. 2.5 An Example of a Non-Rectangular Topology

Fig. 2.6 The Surface generated by Locally Rectangular Arrays of Control Points

Fig. 2.7 The Subdivided Points near the Non-4-nodes and the Further Surface generated
entire configuration apart from ever-decreasing areas round the extraordinary points. This is illustrated in Figs. 2.5 - 2.7.

Since these points become isolated from each other by the creation of new 4-nodes, the resulting surface will be $C^2$ smooth except at the limiting positions of such points. $C^2$ continuity almost everywhere, however, does not guarantee even $C^1$ continuity at a singular point, as Fig. 2.8 illustrates.

Explicitly, the subdivision algorithm is given as follows.

1) The new face point is defined as the average of all the points incident upon the face.

2) The new edge point is defined as the average of the two new face points on either side and the two old points at either end of the edge.

3) The new vertex point $v'$ is given by

$$v' = \frac{1}{N} \left( \bar{G} + (N-2)\bar{v} + \bar{Q} \right)$$

(2.1)

where $N$ is the order of the node, $\bar{v}$ is the old vertex point, $ar{G}$ is the average of the new face points surrounding the node and $\bar{Q}$ is the average of the old points connected to $\bar{v}$ by an edge.

This definition is equivalent to both that given by Catmull and Clark [8] and Doo and Sabin [11]. The former have implemented this algorithm, using a computer graphic technique, to investigate the behaviour of the limiting surface round the extraordinary points. The results would seem to indicate that there is at least continuity of tangent plane there.
The Function $z = \exp\left\{-\left( x + y^2 \right)^{\frac{1}{4}} \right\}$ which is $C^2$ continuous everywhere but $x = y = 0$ where it is not $C^1$ continuous.
Doo and Sabin have, using this formulation, mathematically demonstrated $C^1$ continuity. Nonetheless we may reasonably ask if some redefinition of the vertex point, not necessarily that already suggested, will reduce the limiting surface, on each of the $N$ faces round the node, to a bicubic patch, governed by some assembly of control points which may be expressed in terms of the original array. We investigate the ostensibly simplest case, that of the 3-node. The configuration of points round the node are shown in Fig. 2.9.

The vertex point, the 3-node, is denoted by $V$, the edge points by $E_i$ and the face points by $F_i$, $i = 1, 2, 3$. The rectangular subdivision algorithm determines all the redefined points except $V'$. In particular

$$E'_j = \frac{3}{8} (V + E_j) + \frac{1}{16} (E_{j-1} + F_{j-1} + F_j + E_{j+1})$$

where the subscript $j$ is cyclic over $(1, 2, 3)$.

According to Catmull's definition

$$V' = \frac{5}{12} V + \frac{1}{6} (E_1 + E_2 + E_3) + \frac{1}{36} (F_1 + F_2 + F_3)$$

However, to test the hypothesis that the limiting surface consists of 3 bicubic patches we consider the consequences of using

$$V' = A V + \frac{1}{3} B (E_1 + E_2 + E_3) + \frac{1}{3} C (F_1 + F_2 + F_3)$$

where $A, B, C \geq 0$ and $A + B + C = 1$. 
Fig. 2.9 The Configuration of Points round a 3-node

Fig. 2.10 The Subdivided Points and the Corresponding Subpatch
This is the most general formulation since the new vertex point must lie in the convex hull of the old one and its neighbours and, by symmetry, the weightings of the edge points and the face points in the subdivision must be the same.

Thus we have the relation

\[ [R'] = [T][R] \]

where

\[ [R] = [v, e_1, f_1, e_2, f_2, e_3, f_3]^T \]

and

\[
[T] = \begin{bmatrix}
A & \frac{1}{3}B & \frac{1}{3}C & \frac{1}{3}B & \frac{1}{3}C & \frac{1}{3}B & \frac{1}{3}C \\
\frac{3}{8} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{16} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
\frac{3}{8} & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 \\
\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\
\frac{3}{8} & \frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4}
\end{bmatrix}
\]

After the \( n \)th subdivision we have

\[ [R^{(n)}] = [T]^n[R] \]

Now \([T]\) is a stochastic matrix, i.e. its rows each sum to unity, and so \([T]^n\) has a limit \([X]\), also a stochastic matrix, all of whose rows are the same [13, p.357]. The entries in \([X]\) will give us the weightings of the components of \([R]\) in the limit point to which \(v^{(n)}\) tends.
By considerations of symmetry, the weightings of the $E_{\frac{1}{2}}$ and the $F_{\frac{1}{2}}$ must be the same in the limit also. Hence $\{X\}$ has each row of the form:

$$\left[ X, \frac{1}{3} Y, \frac{1}{3} Z, \frac{1}{3} Y, \frac{1}{3} Z, \frac{1}{3} Y, \frac{1}{3} Z \right].$$

Since the limiting point of $\{R'\}$ is the same as that of $\{R\}$ we have

$$\{X\}[R] = \{X\}[R']$$

$$\iff \{X\}[R] = \{X\}[T][R]$$

$$\iff \{X\} = \{X\}[T]$$

Multiplying this out we obtain the three independent equations

$$\begin{align*}
AX + \frac{3}{8} Y + \frac{1}{4} Z &= X \\
\frac{1}{3} BX + \frac{1}{6} Y + \frac{1}{6} Z &= \frac{1}{3} Y \\
\frac{1}{3} CX + \frac{1}{24} Y + \frac{1}{12} Z &= \frac{1}{3} Z
\end{align*}$$

$$\begin{align*}
8(A-1)X + 3Y + 2Z &= 0 \\
2BX - Y + Z &= 0 \\
8CX + Y - 6Z &= 0
\end{align*}$$

Now $C = 1 - A - B$

and $Z = 1 - X - Y$

so, making these substitutions, we have the two independent equations

$$\begin{align*}
2(4A-5)X + Y &= -2 \\
(2B-1)X - 2Y &= -1
\end{align*}$$

giving the solutions
\[ X = \frac{5}{21 - 16A - 2B} \]

\[ Y = \frac{8 - 8A + 4B}{21 - 16A - 2B} \]

and \[ Z = \frac{8 - 8A - 6B}{21 - 16A - 2B} \]

Thus \[ \lim_{n \to \infty} \frac{V^{(n)}}{n} = \]

\[ \frac{1}{21 - 16A - 2B} \left\{ \frac{5V}{3} + \frac{1}{3} (8 - 8A + 4B) \sum_{i=1}^{3} E_i \right. \]

\[ + \frac{1}{3} (8 - 8A - 6B) \sum_{i=1}^{3} F_i \} \] \hspace{1cm} (2.2)

In Fig. 2.10, the shaded patch, given by the equation

\[ S^*(u, v) = [U][M][R^*][M^t][V]^t \]

\[ (0 \leq u, v \leq 1) \]

where \[ [R^*] = \begin{bmatrix}
V^t & E^t_1 & R^*_0 & R^*_3 \\
E^t_1 & R^*_1 & 0 & 0 \\
0 & 0 & R^*_2 & 0 \\
0 & 0 & 0 & R^*_3 \\
\end{bmatrix} \]

must correspond, if our hypothesis is correct, to the subpatch given by \([\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \). If we denote by \([R]\) the set of points governing the entire patch, we have, as in (1.21):

\[ [R^*] = [M]^{-1}[H_{[\frac{1}{2}, 1]}][M][R] \left\{ [M]^{-1}[H_{[\frac{1}{2}, 1]}][M] \right\}^t \]

\[ \iff [R] = [M]^{-1}[H_{[\frac{1}{2}, 1]}]^{-1}[M][R^*] \left\{ [M]^{-1}[H_{[\frac{1}{2}, 1]}]^{-1}[M] \right\}^t \]
The whole patch, therefore, is given by:

\[ S(u,v) = [U][M][R][M]^T[V]^t \]

\[ = [U][H_{\frac{1}{4},1}]^{-1}[M][R]^*[M]^t[H_{\frac{1}{4},1}]^{-1}[V]^t \]

\[ = [U][H_{\frac{1}{4},1}]^{-1}[M][R]^*[H_{\frac{1}{4},1}]^{-1}[M]^t[V]^t \]  \hspace{1cm} (2.3)

Now \( H_{\frac{1}{4},1} \) =

\[
\begin{bmatrix}
\frac{1}{8} & 0 & 0 & 0 \\
\frac{3}{8} & \frac{1}{4} & 0 & 0 \\
\frac{3}{8} & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \\
\end{bmatrix}
\]

\( [H_{\frac{1}{4},1}]^{-1} =
\[
\begin{bmatrix}
8 & 0 & 0 & 0 \\
-12 & 4 & 0 & 0 \\
6 & -4 & 2 & 0 \\
-1 & 1 & -1 & 1 \\
\end{bmatrix}
\]

and \( H_{\frac{1}{4},1}^{-1}[M] =
\[
\begin{bmatrix}
-\frac{4}{3} & 4 & -4 & \frac{4}{3} \\
4 & -10 & 8 & -2 \\
-4 & 7 & -4 & 1 \\
\frac{4}{3} & \frac{5}{6} & \frac{2}{3} & -\frac{1}{6} \\
\end{bmatrix}
\]

According to our supposition, we must have

\[ S(0,0) = \lim_{n \to \infty} v^{(n)} \]
But $\mathbf{S}(0, 0)$, from (2.3), is given by:

\[
\begin{bmatrix}
0, 0, 0, 1
\end{bmatrix}
\begin{bmatrix}
-\frac{4}{3} & 4 & -4 & \frac{4}{3} \\
4 & -10 & 8 & -2 \\
-4 & 7 & -4 & 1 \\
\frac{4}{5} & -\frac{5}{6} & \frac{2}{3} & -\frac{1}{6}
\end{bmatrix}
\begin{bmatrix}
R^* \\
R^* \\
R^* \\
R^*
\end{bmatrix}
\begin{bmatrix}
-\frac{4}{3} & 4 & -4 & \frac{4}{3} \\
4 & -10 & 7 & -\frac{5}{6} \\
-4 & 8 & -4 & \frac{2}{3} \\
\frac{4}{3} & -2 & 1 & -\frac{1}{6}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{4}{3}, -\frac{5}{6}, \frac{2}{3}, -\frac{1}{6}
\end{bmatrix}
\begin{bmatrix}
\frac{4}{3}, -\frac{5}{6}, \frac{2}{3}, -\frac{1}{6}
\end{bmatrix}
\]

\[
\frac{16}{9} \mathbf{V}' - \frac{10}{9} (E_1' + E_2') + \frac{8}{9} (R_{02}^* + R_{20}^*) - \frac{2}{9} (R_{03}^* + R_{30}^*) \\
+ \frac{25}{36} F_1' - \frac{5}{9} (R_{12}^* + R_{21}^*) + \frac{5}{36} (R_{13}^* + R_{31}^*) + \frac{4}{9} R_{22}^* \\
- \frac{1}{9} (R_{23}^* + R_{32}^*) + \frac{1}{36} R_{33}^*
\]

(2.4)

We can now express this in terms of the original set of vectors (shown in Fig. 2.9).

\[
\mathbf{V}' = A \mathbf{V} + \frac{1}{3} B (E_1' + E_2' + E_3') + \frac{1}{3} C (E_1' + E_2' + E_3')
\]

\[
E_1' = \frac{3}{8} (V + E_1') + \frac{1}{16} (E_3' + F_1' + E_1' + E_2')
\]

\[
E_2' = \frac{3}{8} (V + E_2') + \frac{1}{16} (E_1' + F_1' + E_2' + E_3')
\]

\[
F_1' = \frac{1}{4} (V + E_1' + F_1' + E_2')
\]
\( R_{02}^* = \frac{9}{16} E_2 + \frac{3}{32} (V + F_1 + F_2 + I_2) + \frac{1}{64} (E_1 + E_3 + I_1 + J_2) \)

\( R_{03}^* = \frac{3}{8} (E_2 + I_2) + \frac{1}{16} (F_1 + L_1 + J_2 + F_2) \)

\( R_{12}^* = \frac{3}{8} (E_2 + F_1) + \frac{1}{16} (V + E_1 + L_1 + I_2) \)

\( R_{13}^* = \frac{1}{4} (E_2 + F_1 + L_1 + I_2) \)

and so on.

(2.4) then resolves to:

\[
\left( \frac{16}{9} A - \frac{5}{9} \right) V + \left( \frac{16}{27} B - \frac{1}{18} \right) E_1 + E_2 + \left( \frac{16}{27} B - \frac{1}{9} \right) E_3 + \frac{16}{27} C (F_1 + F_2 + F_3)
\]

(2.5)

with the other vectors cancelling out.

This is of a different form to (2.2) since the weighting of the opposite edge point \( E_3 \) and the other two, \( E_1 \) and \( E_2 \) are necessarily different. It could be argued that choosing the subpatch to have parametric range \([\frac{1}{4}, 1] \times [\frac{1}{4}, 1] \) is an unwarranted assumption, but in fact any other choice will produce the same result.

Consequently, the hypothesis that the areas round the extraordinary point can be reduced to a bicubic patch is not tenable.

To generate a smooth surface here some significant development of this approach is required.
Chapter 3

Formulation of the Smoothness Problem

We saw in the previous chapter that no subdivision algorithm applied to the configuration of points round a 3-node will generate a surface which can be represented by 3 bicubic patches. Nonetheless it has been shown that such a procedure will produce a progressively finer mesh of points forming a rectangular grid, with the exception of the extraordinary point itself, and consequently will cover the surface with a tiling of patches, diminishing in size to zero as we approach the node.

We recall that the subdivision weightings, A, B and C of Chapter 2 provide us with two degrees of freedom. Our purpose will be to optimize the continuity properties of the surface at the extraordinary point and establish the corresponding constraints on these weightings. By virtue of the surrounding concatenation of patches we have $C^2$ continuity everywhere else.

We thus start from a configuration of points centred upon an $N$-node, as shown in Fig. 3.1. This consists of the points

$$\{v\} \cup \{E_j, F_j, I_{j1}, I_{j2}, I_{j3}, I_{j4}, O_{j1}, O_{j2}, O_{j3}, O_{j4}, O_{j5}, O_{j6}\}$$

for each $j$ from 1 to $N$, i.e. the vertex point $v$ and, for each $j$ from 1 to $N$, the edge and face points $E_j$ and $F_j$ respectively, a set of 4 points $I_{ji}$ ($i = 1, 2, 3, 4$) which we call the inner ring, and a set of 6 points $O_{ji}$ ($i = 1, 2, \ldots, 6$) which we call the outer ring. Fig. 3.1 also shows the collection of subdivided points.
Fig. 3.1 The Configuration of Points and Subdivided Points round an \( N \)-node

Fig. 3.2 The Three Types of Patch and the Effect of one Subdivision
Fig. 3.2 shows that between the \( j \text{th} \) and the \( (j+1) \text{th} \) edge, we have three types of patch, which are labelled 1, 2 and 3. We note here that patches 1 and 3 are distinguished only by the cyclic orientation of \( j \). We shall consider convergence to the node as patch 1, shaded in the diagram, is generated by each subdivision. The result of the first subdivision is shown.

The control points for patch 1 are given by:

\[
[R] = \begin{bmatrix}
E_{j-1} & V & E_{j+1} & I_{j+1,2} \\
F_{j-1} & E_j & F_j & I_{j+1,1} \\
I_{j1} & I_{j2} & I_{j3} & I_{j4} \\
0_{j2} & 0_{j3} & 0_{j4} & 0_{j5} 
\end{bmatrix}
\]

and for the patch obtained after one subdivision by:

\[
[R'] = \begin{bmatrix}
E'_{j-1} & V' & E'_{j+1} & I'_{j+1,2} \\
F'_{j-1} & E'_{j} & F'_{j} & I'_{j+1,1} \\
I'_{j1} & I'_{j2} & I'_{j3} & I'_{j4} \\
0'_{j2} & 0'_{j3} & 0'_{j4} & 0'_{j5} 
\end{bmatrix}
\]

We proceed in this way towards the node, generating smaller bicubic patches from the continually finer rectangular grids adjacent to the vertex point. The criterion of \( C^1 \) smoothness at the node is that the direction of the normal to the patch at \((0, 0)\) tends to a limit which is symmetric in the edge and face points etc., i.e. independent of \( j \), the edge along which we converge.
Let \([R^{(n)}]\) denote the control points after the \(n^{th}\) subdivision, \(S^{(n)}\) the resultant patch and \(S^u(n)\) and \(S^v(n)\) the partial derivatives.

We must evaluate

\[
\lim_{n \to \infty} \left\{ S^u(n)(0, 0) \times S^v(n)(0, 0) \right\}
\]

From Chapter 1 we have

\[
S(u, v) = [U][M][R][M][V]^t
\]

where

\[
[U] = [u^3, u^2, u, 1]
\]

\[
[V] = [v^3, v^2, v, 1]
\]

\[
[M] = \frac{1}{6} \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{bmatrix}
\]

and \([R]\) is given in (3.1).

Then

\[
S^u(0, 0) = \frac{1}{36} [-3, 0, 3, 0][R][1, 4, 1, 0]^t
\]

\[
= \frac{1}{12} \left\{ (I_{j1} - E_{j-1}) + 4(I_{j2} - V) + (I_{j3} - E_{j+1}) \right\}
\]

and

\[
S^v(0, 0) = \frac{1}{36} [1, 4, 1, 0][R][-3, 0, 3, 0]^t
\]

\[
= \frac{1}{12} \left\{ (E_{j+1} - E_{j-1}) + 4(F_j - F_{j-1}) + (I_{j3} - I_{j1}) \right\}
\]

Hence

\[
S^u(n)(0, 0) = \frac{1}{12} \left\{ (I_{j1}^{(n)} - E_{j-1}^{(n)}) + 4(I_{j2}^{(n)} - V(n)) + (I_{j3}^{(n)} - E_{j+1}^{(n)}) \right\}
\]

\[
S^v(n)(0, 0) = \frac{1}{12} \left\{ (E_{j+1}^{(n)} - E_{j-1}^{(n)}) + 4(F_j^{(n)} - F_{j-1}^{(n)}) + (I_{j3}^{(n)} - I_{j1}^{(n)}) \right\}
\]

(3.2)
From our work in Chapter 2 on subdivision we have

\[
\begin{align*}
V' &= A V + B \frac{1}{N} \sum_{i=1}^{N} E_i + C \frac{1}{N} \sum_{i=1}^{N} F_i \\
E_j' &= \frac{3}{8} (V + E_j) + \frac{1}{16} (E_{j-1} + F_{j-1} + F_j + E_{j+1}) \\
F_j' &= \frac{1}{4} (V + E_j + F_j + E_{j+1}) \\
I_{j1}' &= \frac{3}{8} (F_{j-1} + E_j) + \frac{1}{16} (E_{j-1} + I_{j1} + I_{j2} + V) \\
I_{j2}' &= \frac{9}{16} E_j + \frac{3}{32} (V + F_{j-1} + F_j + I_{j2}) + \frac{1}{64} (E_{j-1} + I_{j1} + I_{j3} + E_{j+1}) \\
I_{j3}' &= \frac{3}{8} (E_j + F_j) + \frac{1}{16} (V + E_{j+1} + I_{j2} + I_{j3})
\end{align*}
\]

Putting \( a = A \), \( b = \frac{B}{N} \) and \( c = \frac{C}{N} \)
we set these equations in matrix form

\[
[R_j'] = [T][R_j]
\]

where \([R_j'] = [V, E_j, F_j, E_{j+1}, F_{j+1}, E_{j+2}, \ldots, E_{j-1}, F_{j-1}, I_{j1}, I_{j2}, I_{j3}]^t\)

and \([T] = \begin{bmatrix}
T_1 & 0 \\
\vdots & \vdots \\
T_3 & T_2
\end{bmatrix}\)

\([T_1] \) is the transformation matrix, of order \( 2N+1 \), from the vertex point and its associated edge and face points to the redefined points.
More precisely, let \( [T_1] = \{ t_{ij} \} \) \( i,j = 1,2,\ldots,2N+1 \).

Then \( t_{11} = a \)

and for \( k = 1,2,\ldots,N \)

\[
\begin{align*}
t_{1,2k} &= b, \quad t_{1,2k+1} = c, \quad t_{2k,1} = \frac{3}{8}, \quad t_{2k+1,1} = \frac{1}{4} \\
t_{2k,2k} &= \frac{3}{8} \\
t_{2k,2k-2} &= t_{2k,2k-1} = t_{2k,2k+1} = t_{2k,2k+2} = \frac{1}{16} \\
t_{2k,j} &= 0 \quad \text{otherwise} \\
t_{2k+1,2k} &= t_{2k+1,2k+1} = t_{2k+1,2k+2} = \frac{1}{4} \\
t_{2k+1,j} &= 0 \quad \text{otherwise}
\end{align*}
\]
where the second subscript ranges cyclically over the set
\((2, 3, 4, \ldots, 2N+1)\)

\[
[T_2] = \begin{bmatrix}
\frac{1}{16} & \frac{1}{16} & 0 \\
\frac{1}{64} & \frac{3}{32} & \frac{1}{64} \\
0 & \frac{1}{16} & \frac{1}{16}
\end{bmatrix}
\]

and \([T_3]\) is a \(3 \times (2N+1)\) matrix given by

\[
\begin{bmatrix}
\frac{1}{16} & \frac{3}{8} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{16} & \frac{3}{8} \\
\frac{3}{32} & \frac{9}{16} & \frac{3}{32} & \frac{1}{64} & 0 & \cdots & 0 & \frac{1}{64} & \frac{3}{32} \\
\frac{1}{16} & \frac{3}{8} & \frac{3}{8} & \frac{1}{16} & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

We are concerned with the limit of the normal vector at \((0, 0)\) as we approach the extraordinary point. From (3.2) we may formulate this as:

\[
\lim_{n \to \infty} \begin{bmatrix}
\frac{1}{12} & -4 & 0 & \cdots & 0 \\
0 & \frac{1}{12} & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{12} & \cdots & 0 \\
-1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots \\
-1 & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & \cdots & \cdots \\
4 & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\times
\begin{bmatrix}
\gamma(n) \\
E_j(n) \\
E_j(n) \\
E_{j+1}(n) \\
E_j(n) \\
E_{j+1}(n) \\
E_{j+1}(n) \\
I_{j1}(n) \\
I_{j2}(n) \\
I_{j3}(n)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{12} & 0 & \cdots & 0 \\
0 & 4 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & \cdots \\
-1 & \cdots & \cdots & \cdots \\
-4 & \cdots & \cdots & \cdots \\
-1 & \cdots & \cdots & \cdots \\
-1 & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & \cdots \\
\end{bmatrix}
\times
\begin{bmatrix}
\gamma(n) \\
E_j(n) \\
E_j(n) \\
E_{j+1}(n) \\
E_j(n) \\
E_{j+1}(n) \\
E_{j+1}(n) \\
I_{j1}(n) \\
I_{j2}(n) \\
I_{j3}(n)
\end{bmatrix}
\]
Let \([D_x^i]\) denote the numerical vector representing the partial derivative with respect to \(x\) (= u or v) on patch \(i\) (= 1, 2 or 3). Then we may abbreviate this expression to

\[
\lim_{n \to \infty} \left\{ [D_u^1][R_{i_j}^{(n)}] \times [D_v^1][R_{i_j}^{(n)}] \right\}
\]

\[
= \lim_{n \to \infty} \left\{ [D_u^1][T]^n[R_{i_j}] \times [D_v^1][T]^n[R_{i_j}] \right\}
\]

In fact, since intuitively the spatial range of the parameters \(u\) and \(v\) is diminishing at each stage of the process, we shall have to modify the transformation by a scale factor equal to the reciprocal of its largest non-unit eigenvalue, so that the limit is non-zero.

Our first purpose, then, is to establish the eigenproperties of \([T]\) and, wherever possible, diagonalize it to facilitate calculation of the limit. In the next chapter we evaluate the eigenvalues of \([T]\) in the general case.
Chapter 4

Analysis of the Subdivision Transformation

4.1 THE DECOMPOSITION OF THE CHARACTERISTIC DETERMINANT.

We recall that $[T]$, the matrix representing the subdivision transformation is of the form:

$$
\begin{bmatrix}
T_1 & 0 \\
T_3 & T_2
\end{bmatrix}
$$

where $[T_1]$ is of order $(2N+1) \times (2N+1)$, $[T_2] = 3 \times 3$, and $[T_3] = 3 \times (2N+1)$.

It is clear that the set of eigenvalues of $[T]$ is the union of the sets containing those of $[T_1]$ and $[T_2]$.

$$
|T_2 - \lambda I| = 
\begin{vmatrix}
\frac{1}{16} & -\lambda & \frac{1}{16} & 0 \\
\frac{1}{64} & \frac{3}{32} & -\lambda & \frac{1}{64} \\
0 & \frac{1}{16} & \frac{1}{16} & -\lambda
\end{vmatrix}
= 
\begin{vmatrix}
\frac{1}{16} & -\lambda & \frac{1}{16} & 0 \\
\frac{1}{64} & \frac{3}{32} & -\lambda & \frac{1}{64} \\
\lambda - \frac{1}{16} & 0 & \frac{1}{16} & -\lambda
\end{vmatrix}
= R3' = R3 - R1
$$

$$
= 
\begin{vmatrix}
\frac{1}{16} & -\lambda & \frac{1}{16} & 0 \\
\frac{1}{32} & \frac{3}{32} & -\lambda & \frac{1}{64} \\
0 & 0 & \frac{1}{16} & -\lambda
\end{vmatrix}
= C1' = C1 + C3
$$

$$
= (\frac{1}{16} - \lambda) 
\begin{vmatrix}
\frac{1}{16} & -\lambda & \frac{1}{16} \\
\frac{1}{32} & \frac{3}{32} & -\lambda
\end{vmatrix}
$$

expanding by $R3$
\[ = \left( \frac{1}{8} - \lambda \right) \left( \frac{1}{16} - \lambda \right) \left( \frac{1}{32} - \lambda \right) \]

so \([T_2]\) has eigenvalues of \(\frac{1}{8}\), \(\frac{1}{16}\) and \(\frac{1}{32}\).

We now proceed to \([T_1]\) =

\[
\begin{bmatrix}
  a & b & c & b & c & b & c & \ldots & c & b & c & b & c \\
  \frac{3}{8} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} \\
  \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
  \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
  \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
  \frac{3}{8} & 0 & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \ldots & 0 & 0 & 0 & 0 & 0 \\
  \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \ldots & 0 & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\
  \frac{3}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{16} & \frac{1}{16} & 0 & \frac{3}{16} \\
  \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\
\end{bmatrix}
\]

Let \([P] = [T_1] - \lambda I\)

\(|P| = 0\) determines a polynomial of degree \(2N+1\) in \(\lambda\), whose roots are the eigenvalues of \([T_1]\).

We perform two operations to simplify \(|P|\). Denoting, for example, column 6 by \(C_6\) and row \(j\) by \(R_j\), these are given as follows:

**Operation I.**  
\(C(N + 2 + \ell)' = C(N + 2 + \ell) - C(N + 2 - \ell)\)

\(C(N + 3 - \ell)' = C(N + 3 - \ell) - C(N + 1 - \ell)\)

for \(\ell = 1, 2, \ldots, N-1\).
Operation II. \( R_{2}' = \sum_{s=0}^{N-1} R(2 + 2s) \)

\[
R(N + 2 - k)' = \sum_{s=0}^{k} R(N + 2 - k + 2s)
\]

for \( k = 1, 2, \ldots, N-1 \).

The efficacy of these operations in reducing \(|P|\) to the desired form, for general \( N \), is proved rigorously in Appendix A. We confine ourselves here to consideration of the cases \( N = 3 \) and \( N = 4 \).

| If \( N = 3 \) | \(|P| = a - \lambda \ b \ c \ b \ c \ b \ c \) |
| 3/8 | 3/8 - \lambda | 1/16 | 1/16 | 0 | 1/16 | 1/16 |
| 1/4 | 1/4 | 1/4 - \lambda | 1/4 | 0 | 0 | 0 |
| 3/8 | 1/16 | 1/16 | 3/8 - \lambda | 1/16 | 1/16 | 0 |
| 1/4 | 0 | 0 | 1/4 | 1/4 - \lambda | 1/4 | 0 |
| 3/8 | 1/16 | 0 | 1/16 | 1/16 | 3/8 - \lambda | 1/16 |
| 1/4 | 1/4 | 0 | 0 | 0 | 1/4 | 1/4 - \lambda |

Operation I involves \( C7' = C7 - C3 \)

\( C6' = C6 - C4 \)

\( C5' = C5 - C3 \)

\( C4' = C4 - C2 \)

which yields
We note that the top row, beyond the 3rd entry, is now zero.

Operation II involves

\[ R_2' = R_2 + R_4 + R_6 \]
\[ R_3' = R_3 + R_5 + R_7 \]
\[ R_4' = R_4 + R_6 \]

giving

\[
\begin{vmatrix}
| \rho'' | = \\
\begin{array}{cccccc}
9/8 & 1/2 - \lambda & 1/8 & 0 & 0 & 0 \\
3/4 & 1/2 & 1/4 - \lambda & 0 & 0 & 0 \\
3/4 & 1/8 & 1/16 & 5/16 - \lambda & 1/16 & 0 \\
1/4 & 0 & 0 & 1/4 & 1/4 - \lambda & 0 \\
3/8 & 1/16 & 0 & 0 & 1/16 & 5/16 - \lambda & 1/16 \\
1/4 & 1/4 & 0 & -1/4 & 0 & 1/4 & 1/4 - \lambda \\
\end{array}
\end{vmatrix}
\]
If $N = 4$

$$|p| = \begin{vmatrix}
  a - \lambda & b & c & b & c & b & c \\
  \frac{3}{8} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} \\
  \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
  \frac{3}{8} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 \\
  \frac{1}{4} & 0 & 0 & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 \\
  \frac{3}{8} & 0 & 0 & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 \\
  \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 \\
  \frac{3}{8} & 0 & 0 & 0 & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} \\
  \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda
\end{vmatrix}$$

Operation I : 

$C_9' = C_9 - C_3$

$C_8' = C_8 - C_4$

$C_7' = C_7 - C_5$

$C_6' = C_6 - C_4$

$C_5' = C_5 - C_3$

$C_4' = C_4 - C_2$

giving
\[ |P'| = \begin{vmatrix} a - \lambda & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{16} & \lambda & -\frac{5}{16} & -\frac{1}{16} & -\frac{1}{16} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \lambda - \frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & \lambda - \frac{1}{4} \\ \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & \frac{5}{16} & 0 & \lambda & -\frac{5}{16} & -\frac{1}{16} & \lambda - \frac{3}{8} & -\frac{1}{16} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\lambda & 0 & \lambda - \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{3}{8} & 0 & 0 & \frac{1}{16} & \frac{1}{16} & \frac{5}{16} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \lambda & \frac{1}{4} & 0 \\ \frac{3}{8} & \frac{1}{16} & 0 & -\frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & -\lambda & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\lambda \end{vmatrix} \]

Operation II:
\[ R2' = R2 + R4 + R6 + R8 \]
\[ R3' = R3 + R5 + R7 + R9 \]
\[ R4' = R4 + R6 + R8 \]
\[ R5' = R5 + R7 \]

giving
\[ |P''| = \begin{vmatrix} a - \lambda & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \lambda & 0 & 0 & 0 & 0 & 0 \\ \frac{9}{8} & \frac{1}{8} & \frac{1}{16} & \frac{5}{16} & -\lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\lambda & \frac{1}{4} & 0 & 0 \\ \frac{3}{8} & 0 & 0 & \frac{1}{16} & \frac{1}{16} & \frac{5}{16} & -\lambda & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & -\lambda & \frac{1}{4} & 0 \\ \frac{3}{8} & \frac{1}{16} & 0 & -\frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & -\lambda & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\lambda \end{vmatrix} \]
Appendix A shows that, in general, Operations I & II will reduce $|P|$ to the form

$$
\begin{bmatrix}
\Delta_a & \Delta_N & \Delta'_N
\end{bmatrix}
$$

where $\Delta_a$ is a $3 \times 3$ determinant and $\Delta_N$, $\Delta'_N$ are both of order $N-1$.

This is achieved, as demonstrated above, by reducing blocks of the determinant to zero.

More specifically

$$
\Delta_a = \begin{bmatrix}
a - \lambda & b & c \\
\frac{3}{8} N & \frac{1}{2} - \lambda & \frac{1}{8} \\
\frac{1}{4} N & \frac{1}{2} & \frac{1}{4} - \lambda
\end{bmatrix}
$$

and for odd $N$

$$
\Delta_N = \Delta'_N
$$

$$
= \begin{bmatrix}
\frac{5}{16} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & \ldots & 0 \\
\frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & \ldots & 0 \\
\frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & \ldots & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
$$

(4.1)
It is simple to evaluate $\Lambda_a$ directly

\[
\Lambda_a = \begin{vmatrix}
a - \lambda & b & c \\
\frac{3}{8}N & \frac{1}{2} - \lambda & \frac{1}{8} \\
\frac{1}{4}N & \frac{1}{2} & \frac{1}{4} - \lambda
\end{vmatrix}
\]
\begin{align*}
\begin{vmatrix}
1 - \lambda & b & \frac{1 - a - Nb}{N} \\
N - N\lambda & \frac{1}{2} - \lambda & \frac{1}{8} \\
N - N\lambda & \frac{1}{2} & \frac{1}{4} - \lambda
\end{vmatrix}
\end{align*}
= C1' = C1 + NC2 + NC3

\begin{align*}
\begin{vmatrix}
1 - \lambda & b & \frac{1 - a - Nb}{N} \\
0 & \frac{1}{2} - Nb - \lambda & a + Nb - \frac{7}{8} \\
0 & \lambda & \frac{1}{8} - \lambda
\end{vmatrix}
\end{align*}
= R2' = R2 - NR1

\begin{align*}
\begin{vmatrix}
1 - \lambda & b & \frac{1 - a - Nb}{N} \\
0 & \frac{1}{2} - Nb - \lambda & a + Nb - \frac{7}{8} \\
0 & \lambda & \frac{1}{8} - \lambda
\end{vmatrix}
\end{align*}
= R3' = R3 - R2

\begin{align*}
0.43
\end{align*}

\begin{align*}
= (1 - \lambda) \left\{ \lambda^2 - (a - \frac{1}{4})\lambda + \left( \frac{1}{16} - \frac{N}{8} b \right) \right\}
\end{align*}

\begin{align*}
= (1 - \lambda) \left\{ \lambda^2 - (\Lambda - \frac{1}{4})\lambda + \left( \frac{1}{16} - \frac{1}{8} B \right) \right\}
\end{align*}

\begin{align*}
\text{recalling that } A = a \\
B = Nb
\end{align*}

\begin{align*}
= (1 - \lambda) Q_0(\lambda) \text{ say.}
\end{align*}

(4.3)

At this point we note that, since \( \Lambda_N \) and \( \Lambda'_N \) are independent of \( A \) and \( B_N \), only 2 eigenvalues of \([T]\) in the total of \( 2N+4 \) are affected by the choice of subdivision.

The characteristic equation of \([T_1]\) is given by

\begin{align*}
(1 - \lambda) Q_0(\lambda) \Lambda_N \Lambda'_N
\end{align*}

(4.4)

where \( \Lambda'_N = \Lambda_N \) if \( N \) is odd.

This undesirable asymmetry of the odd and even case is rectified by the following lemma.
Lemma 4.1: Let \([A]\) and \([B]\) be \((2m+1) \times (2m+1)\) matrices \((m \geq 1)\) of the following form

\[
[A] = \begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
[B] = \begin{bmatrix}
\frac{5}{16} & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & 0 \\
\end{bmatrix}
\]
Then \([A]\) and \([B]\) are similar, i.e. \(\exists [Q]\) s.t.

\[
[B] = [Q]^{-1} [A][Q]
\]

The proof is not inherently interesting but may be found in Appendix B.

If \([A]\) and \([B]\) are two similar matrices then they have the same characteristic equation [23, p.199] and so, recalling (4.2), for even \(N\), \(\Delta_N^{'} = \Delta_N^{''}\).

From (4.4), for all \(N\), the characteristic equation of \([T_1]\) is given by

\[
(1-\lambda) Q(\lambda) \Delta_N^2
\]

where

\[
\Delta_N = \begin{vmatrix}
\frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & \ldots & 0 \\
\frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & \ldots & 0 \\
0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} \\
0 & \ldots & \ldots & 0 & \frac{1}{4} & \frac{1}{4} - \lambda \\
\end{vmatrix}
\]

\((N \text{ even})\)

\[
\Delta_N = \begin{vmatrix}
\frac{5}{16} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 & \ldots & 0 \\
\frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & \ldots & 0 \\
\frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} \\
0 & \ldots & \ldots & 0 & 0 & \ldots & \frac{1}{4} & \frac{1}{4} - \lambda \\
\end{vmatrix}
\]

\((N \text{ odd})\)
\( \Delta_N \) is a determinant of order \( N-1 \). Solving the equation \( \Delta_N = 0 \) will yield the natural spectrum of the subdivision transformation round the \( N \)-node. To this end, we investigate the properties of \( \Delta_N \).

### 4.2 The Natural Spectrum of the Subdivision Transformation.

We saw in the preceding section that the characteristic equation of \([T]\), the matrix representing subdivision round an \( N \)-node, is given by

\[
|T - \lambda I| = (1-\lambda)(\frac{1}{16} - \lambda)(\frac{1}{32} - \lambda) Q_0(\lambda) \Delta_N^2
\]

where \( Q_0(\lambda) \) is a quadratic dependent on the subdivision weightings and \( \Delta_N \) is a determinant of order \( N-1 \) independent of them. Our purpose now is to solve \( \Delta_N = 0 \).

We show that a recursive relation for \( \Delta_N \) can be derived by virtue of the systematic way in which the determinant is built up.

\[
\begin{array}{c|c|c|c|c|c|c}
\Delta_N & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Delta_{N-2} & 0 & 0 & \frac{1}{16} & 0 & \frac{1}{4} & 0 \\
\frac{1}{16} & 0 & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{4} & \frac{1}{4} - \lambda \\
\frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{4} & \frac{1}{4} - \lambda \\
\end{array}
\]
At this stage we compute $\Delta_3$ and $\Delta_4$:

$$
\Delta_3 = \begin{vmatrix}
\frac{5}{16} - \lambda & \frac{1}{16} \\
\frac{1}{4} & \frac{1}{4} - \lambda
\end{vmatrix} = \frac{1}{16} - \frac{9}{16} \lambda + \lambda^2
$$
\[ \Delta_4 = \begin{vmatrix} \frac{1}{4} - \lambda & \frac{1}{4} & 0 \\ \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} \\ 0 & \frac{1}{4} & \frac{1}{4} - \lambda \end{vmatrix} \]

\[ = \begin{vmatrix} \frac{1}{4} - \lambda & \frac{1}{4} & 0 \\ \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} \\ \lambda - \frac{1}{4} & 0 & \frac{1}{4} - \lambda \end{vmatrix} \]

\[ R3' = R3 - R1 \]

\[ = \begin{vmatrix} \frac{1}{4} - \lambda & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} - \lambda & \frac{1}{16} \\ 0 & 0 & \frac{1}{4} - \lambda \end{vmatrix} \]

\[ C1' = C1 + C3 \]

\[ = (\frac{1}{2} - \lambda)(\frac{1}{4} - \lambda)(\frac{1}{8} - \lambda) \]

expanding by R3 and simplifying.

Taking \( \Delta_0 = 0 \) and \( \Delta_1 = 1 \)

we have \( \Delta_2 = \frac{1}{4} - \lambda \) and \( \Delta_3 \) and \( \Delta_4 \) as above. This will be convenient when we proceed to establish an explicit expression for \( \Delta_N \).

Even at this stage, however, with a recursive definition for \( \Delta_N \), we can show it has certain properties.

Devising a program to calculate and plot the roots (Table 4.1, Fig. 4.1), we observe several phenomena. As we would expect, all the roots have modulus less than 1.0 [16, p.100]. In addition they
are all real and lie in the interval \([0.095, 0.655]\). The most striking features illustrated by Fig. 4.1, which shows the roots for \(N = 2, 3, \ldots, 25\), can be summarized as follows.

**Property 1.**

The roots fan out within the above interval. More precisely, if \(\Delta_N\) has roots

\[
0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_{N-1}
\]

then \(\Delta_{N+1}\) has roots

\[
0 < \mu_1 < \mu_2 < \mu_3 < \ldots < \mu_{N-1} < \mu_N
\]

such that

\[
\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots < \mu_{N-1} < \lambda_{N-1} < \mu_N.
\]

**Property 2.**

If \(m \nmid n\) and \(\lambda\) is a root of \(\Delta_m\)

then it is a root of \(\Delta_n\)

i.e. \(\Delta_m \setminus \Delta_n\).

**Property 3.**

The roots occur in pairs whose product is \(\frac{1}{16}\); if the number of roots is odd, the unpaired root is \(\frac{1}{4} = (\frac{1}{16})^{\frac{1}{2}}\).

**Theorem 4.2** Property 1 holds.

**Proof:** This is a constructive proof by induction. The table
### Table 4.1: The Roots of $\Delta_n$

<table>
<thead>
<tr>
<th>$N=2$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2500</td>
<td>0.4101</td>
<td>0.5000</td>
<td>0.3401</td>
<td>0.2500</td>
<td>0.1838</td>
<td>0.1250</td>
<td>0.1136</td>
</tr>
<tr>
<td>0.1524</td>
<td>0.2500</td>
<td>0.1524</td>
<td>0.2500</td>
<td>0.1838</td>
<td>0.1524</td>
<td>0.1353</td>
<td>0.1250</td>
</tr>
<tr>
<td>0.1250</td>
<td>0.1524</td>
<td>0.1250</td>
<td>0.1838</td>
<td>0.1524</td>
<td>0.1353</td>
<td>0.1250</td>
<td>0.1183</td>
</tr>
<tr>
<td>0.1136</td>
<td>0.1524</td>
<td>0.1250</td>
<td>0.1838</td>
<td>0.1524</td>
<td>0.1353</td>
<td>0.1250</td>
<td>0.1183</td>
</tr>
<tr>
<td>0.1078</td>
<td>0.1524</td>
<td>0.1250</td>
<td>0.1838</td>
<td>0.1524</td>
<td>0.1353</td>
<td>0.1250</td>
<td>0.1183</td>
</tr>
<tr>
<td>0.1044</td>
<td>0.1524</td>
<td>0.1250</td>
<td>0.1838</td>
<td>0.1524</td>
<td>0.1353</td>
<td>0.1250</td>
<td>0.1183</td>
</tr>
<tr>
<td>0.1023</td>
<td>0.1524</td>
<td>0.1250</td>
<td>0.1838</td>
<td>0.1524</td>
<td>0.1353</td>
<td>0.1250</td>
<td>0.1183</td>
</tr>
<tr>
<td>0.1008</td>
<td>0.1524</td>
<td>0.1250</td>
<td>0.1838</td>
<td>0.1524</td>
<td>0.1353</td>
<td>0.1250</td>
<td>0.1183</td>
</tr>
</tbody>
</table>

**Fig. 4.1** The Roots of $\Delta_n$ for $N = 2, 3, 4, \ldots, 25$

**Fig. 4.2** The Interlacing of Roots of $\Delta$

$K_{i-1}, \lambda_i, K_i, K_{i+1}$ roots of $\Delta_{n-1}$

$\lambda_i, \lambda_{i+1}$ roots of $\Delta_n$
provides our inductive base for the first few values of N.

We first show that if the least root of $\Delta_N$ is $\lambda_1$ and the least root of $\Delta_{N-1}$ is $K_1$ with $0 < \lambda_1 < K_1$ then $\Delta_{N+1}$ has a root $\mu_1$ with $0 < \mu_1 < \lambda_1$.

$$\Delta_1(0) = 1$$
$$\Delta_2(0) = \frac{1}{4}$$

and in general $\Delta_{N+1}(0) = \frac{1}{4} \Delta_N(0)$.

Hence $\Delta_k(0) = \left(\frac{1}{4}\right)^{k-1}$ for all $k$.

So $\Delta_k(0) > 0$.

Now $\Delta_{N+1}(\lambda_1) = \left(\frac{1}{4} - \lambda_1\right) \Delta_N(\lambda_1) - \frac{1}{16} \lambda_1 \Delta_{N-1}(\lambda_1)$

from (4.6)

$$= -\frac{1}{16} \lambda_1 \Delta_{N-1}(\lambda_1)$$

since $\lambda_1$ is a root of $\Delta_N$.

Thus $\Delta_{N+1}$ has the opposite sign to $\Delta_{N-1}$ at $\lambda_1$.

Since $K_1 > \lambda_1$ is the least root of $\Delta_{N-1}$,

$$\Delta_{N-1}(\lambda_1) > 0$$

and so $\Delta_{N+1}(\lambda_1) < 0$.

But $\Delta_{N+1}(0) > 0$, and so $\Delta_{N+1}$ has a root $\mu_1$
such that $0 < \mu_1 < \lambda_1$.

A similar argument shows that $\Delta_{N+1}$ has a root greater than the greatest root of $\Delta_N$.

We now have two of the roots of $\Delta_{N+1}$ accounted for, namely

$\mu_1 < \lambda_1$ and $\mu_N > \lambda_{N-1}$. 

From the inductive hypothesis, we assume that between each adjacent pair of roots of $\Delta_{N-1}$ lies just one root of $\Delta_N$, as shown in Fig. 4.2.

Suppose $K_{i-1}$, $K_i$, $K_{i+1}$ are roots of $\Delta_{N-1}$ and $\lambda_i, \lambda_{i+1}$ are roots of $\Delta_N$ with $K_{i-1} < \lambda_i < K_i < \lambda_{i+1} < K_{i+1}$

$$\Delta_{N+1}(\lambda_j) = \left(\frac{1}{4} - \lambda_j\right) \Delta_N(\lambda_j) - \frac{1}{16} \lambda \ \Delta_{N-1}(\lambda_j)$$

$$= - \frac{1}{16} \lambda_j \Delta_{N-1}(\lambda_j)$$

for $j = i, i+1$.

$\Delta_{N+1}$ has the opposite sign to $\Delta_{N-1}$ at $\lambda_j$; but $\Delta_{N-1}$ changes sign just once between $\lambda_i$ and $\lambda_{i+1}$ so $\Delta_{N+1}$ must change sign an odd number of times. We have $N-2$ roots to locate and $N-2$ intervals of the form $(\lambda_i, \lambda_{i+1})$ so there is just one root of $\Delta_{N+1}$ in each. Consequently the roots interlace and fan out in the manner described; in addition, they are all real.

To establish properties 2) and 3) we need a more explicit formulation for $\Delta_N$.

Lemma 4.3:

For a sequence $\{u_n\}$ with the recursive relation

$$u_{n+1} = au_n + bu_{n-1}$$

$$u_{n+m} = \left\{ \sum_{r=0}^{m-1} \binom{m-1}{r} a^{m-2r} b^r \right\} u_n$$

$$+ b\left\{ \sum_{r=0}^{m-1} \binom{m-1-r}{r} a^{m-1-2r} b^r \right\} u_{n-1}$$
Proof by induction on $m$.

$$m = 1 : \sum_{r=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m-1-r}{r} a^{m-1-2r} b^r = 1$$

$$\sum_{r=0}^{\left\lceil \frac{m}{2} \right\rceil} \binom{m-r}{r} a^{m-2r} b^r = a$$

so the proposition holds.

Assume it holds for $m < k$.

We have

$$u_{n+k+1} = au_{n+k} + bu_{n+k-1}$$

$$= a \left\{ \sum_{r=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-r}{r} a^{k-2r} b^r \right\} u_n + b \left\{ \sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} a^{k-1-2r} b^r \right\} u_{n-1}$$

$$+ b \left\{ \sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} a^{k-1-2r} b^r \right\} u_n + b \left\{ \sum_{r=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \binom{k-2-r}{r} a^{k-2-2r} b^r \right\} u_{n-1}$$

by our inductive hypothesis: this gives

$$\left\{ \sum_{r=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-r}{r} a^{k+1-2r} b^r + \sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} a^{k-1-2r} b^{r+1} \right\} u_n$$

$$+ b \left\{ \sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} a^{k-2r} b^r + \sum_{r=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \binom{k-2-r}{r} a^{k-2-2r} b^{r+1} \right\} u_{n-1}$$

$$= \left\{ \sum_{r=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-r}{r} a^{k+1-2r} b^r + \sum_{r=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \binom{k-r}{r} a^{k+1-2r} b^r \right\} u_n + \left\{ \sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rceil} \binom{k-1-r}{r} a^{k-1-2r} b^r + \sum_{r=1}^{\left\lfloor \frac{k-1}{2} \right\rceil} \binom{k-r}{r-1} a^{k-1-2r} b^r \right\} u_{n-1}$$
\[
+ b \left\{ \frac{k}{2} \sum_{r=0}^{\frac{k}{2}} \binom{k-1-r}{r} a^{k-2r} b^r \right\} u_{n} + \frac{k}{2} \sum_{r=1}^{\frac{k}{2}} \binom{k-1-r}{r-1} a^{k-2r} b^r u_{n-1} .
\]

Now \( \binom{k-r}{r} + \binom{k-r}{r-1} = \binom{k+1-r}{r} \)

and \( \binom{k-1-r}{r} + \binom{k-1-r}{r-1} = \binom{k-r}{r} \)

Also if \( k \) is even \( \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 = \frac{k}{2} \)

so the expression gives

\[
\left\{ \binom{k+1}{2} a^{k+1} + \frac{k}{2} \sum_{r=1}^{\frac{k}{2}} \binom{k+1-r}{r} a^{k+1-2r} b^r \right\} u_{n}
\]

\[
+ b \left\{ \binom{k}{2} a^k + \sum_{r=1}^{\frac{k}{2}} \binom{k-r}{r} a^{k-2r} b^r + \frac{k}{2} \right\} u_{n-1}
\]

\[
= \left\{ \frac{k+1}{2} \sum_{r=0}^{\frac{k}{2}} \binom{k+1-r}{r} a^{k+1-2r} b^r \right\} u_{n} + b \left\{ \frac{k}{2} \sum_{r=0}^{\frac{k}{2}} \binom{k-r}{r} a^{k-2r} b^r \right\} u_{n-1}
\]

If \( k \) is odd \( \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor = \left\lfloor \frac{k+1}{2} \right\rfloor - 1 = \frac{k-1}{2} \)

so we obtain

\[
\left\{ \binom{k}{2} a^k + \frac{k}{2} \sum_{r=1}^{\frac{k}{2}} \binom{k-1-r}{r} a^{k-2r} b^r + \frac{k+1}{2} \right\} u_{n}
\]

\[
+ b \left\{ \binom{k}{2} a^k + \sum_{r=1}^{\frac{k}{2}} \binom{k-r}{r} a^{k-2r} b^r \right\} u_{n-1}
\]

\[
= \left\{ \sum_{r=0}^{\frac{k}{2}} \binom{k+1-r}{r} a^{k+1-2r} b^r \right\} u_{n} + b \left\{ \sum_{r=0}^{\frac{k}{2}} \binom{k-r}{r} a^{k-2r} b^r \right\} u_{n-1}
\]

Hence, in either case, the results holds for \( m = k+1 \)

\[ \square \]
Corollary 4.4  
If \( u_0 = 0, \ u_1 = 1, \ u_{n+1} = au_n + bu_{n-1} \)

then
\[
\begin{align*}
\sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-r}{r} a^{n-2r} b^r
\end{align*}
\]

Proof:  In the lemma take \( n = 1, \ m = n-1. \)

We can now establish Properties 2) and 3).

Theorem 4.5  
If \( m \nmid n \) then \( \Delta_m \nmid \Delta_n \)

Proof:  
\[
\begin{align*}
\Delta_n = \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-r}{r} a^{n-2r} b^r
\end{align*}
\]

where
\[
\begin{align*}
a &= \frac{1}{4} - \lambda \\
b &= -\frac{1}{16} \lambda
\end{align*}
\]

Suppose \( m \nmid n \); then \( n = km \) and we proceed by induction on \( k \).

Let \( k = 2. \)

Then
\[
\begin{align*}
\Delta_n &= \Delta_{m+m} \\
&= \left\{ \sum_{r=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{m-r}{r} a^{-2r} b^r \right\} \Delta_m + b \left\{ \sum_{r=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m-1-r}{r} a^{-2r} b^r \right\} \Delta_{m-1}
\end{align*}
\]

by lemma 4.3.

By Corollary 4.4 this is equal to
\[
\begin{align*}
\Delta_{m+1} \Delta_m + b \Delta_m \Delta_{m-1}
\end{align*}
\]

\[
\begin{align*}
= (\Delta_{m+1} + b \Delta_{m-1}) \Delta_m
\end{align*}
\]
Now assume the proposition for arbitrary \( k \)

i.e. \( \Delta_{km} = P_k \Delta_m \) where \( P_k \) is some polynomial.

\[
\Delta_{(k+1)m} = \Delta_{m+km}
\]

\[
= \sum_{r=0}^{\frac{km}{2}} \binom{km-r}{r} \binom{km-2r}{r} a^r b^{km-r} \Delta_m + b \sum_{r=0}^{\frac{km-1}{2}} \binom{km-1-r}{r} a^{km-1-2r} b^r \Delta_{m-1}
\]

by lemma 4.3.

\[
= \Delta_{km+1} \Delta_m + b \Delta_{km} \Delta_{m-1}
\]

by Corollary 4.4

\[
= \Delta_{km+1} \Delta_m + b P_k \Delta_m \Delta_{m-1}
\]

by inductive hypothesis

\[
= (\Delta_{km+1} + b P_k \Delta_{m-1}) \Delta_m
\]

The coefficient of \( \Delta_m \) is a polynomial and so our result holds \( \square \)

Theorem 4.6. \( \Delta_n(\lambda) = 0 \Leftrightarrow \Delta_n \left( \frac{1}{16\lambda} \right) = 0 \).

Proof:

\[
\Delta_n \left( \frac{1}{16\lambda} \right) = \sum_{r=0}^{\frac{n-1}{2}} \binom{n-1-r}{r} \left( \frac{1}{4} - \frac{1}{16\lambda} \right)^{n-1-2r} \left( -\frac{1}{16} + \frac{1}{16\lambda} \right)^r
\]

\[
= \sum_{r=0}^{\frac{n-1}{2}} \binom{n-1-r}{r} \left( \frac{1}{4} - \frac{1}{4\lambda} \right)^{n-1-2r} \left( -\frac{1}{256\lambda} \right)^r
\]

\[
= \sum_{r=0}^{\frac{n-1}{2}} \binom{n-1-r}{r} \left( \frac{1}{4} - \frac{1}{4\lambda} \right)^{n-1-2r} \left( -4\lambda \right)^{2r+1-n} \left( -\frac{1}{256\lambda} \right)^r
\]
Thus

\[
\Delta_n \left( \frac{1}{16\lambda} \right) = 0 \iff \Delta_n (\lambda) = 0
\]

If there are an odd number of roots, \( n \) is even and so

\[
\left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n}{2} - 1.
\]

Hence

\[
\Delta_n = \sum_{r=0}^{\frac{n}{2} - 1} \binom{n-1-r}{r} \left( \frac{1}{4} - \lambda \right)^{n-1-2r} \left( -\frac{1}{16} \lambda \right)^r
\]

\[
= \left( \frac{1}{4} - \lambda \right) \sum_{r=0}^{\frac{n}{2} - 1} \binom{n-1-r}{r} \left( \frac{1}{4} - \lambda \right)^{n-2(r+1)} \left( -\frac{1}{16} \lambda \right)^r
\]

and so \( \lambda = \frac{1}{4} \) is a root.

The next step must be to obtain an explicit formulation of the roots. To do this, we require another expression for \( \Delta_n \).

**Theorem 4.7.** For a sequence \( \{u_n\} \) with \( u_0 = 0 \), \( u_1 = 1 \) and

\[
u_{n+1} = a u_n + b u_{n-1}
\]

\[
u_n = (a^2 + 4b)^{-\frac{1}{4}} \left\{ \left[ \frac{a + (a^2 + 4b)^{\frac{1}{2}}}{2} \right]^n - \left[ \frac{a - (a^2 + 4b)^{\frac{1}{2}}}{2} \right]^n \right\}
\]
Proof: by induction on $n$.

We first note that $x_1 = \frac{a + (a^2 + 4b)^{\frac{1}{2}}}{2}$

and $x_2 = \frac{a - (a^2 + 4b)^{\frac{1}{2}}}{2}$

are the two solutions of $x^2 = ax + b$

and $x_1 + x_2 = a$

$x_1 - x_2 = (a^2 + 4b)^{\frac{1}{2}}$

$u_1 = (a^2 + 4b)^{-\frac{1}{2}}(x_1 - x_2) = 1$

$u_2 = (a^2 + 4b)^{-\frac{1}{2}}(x_1^2 - x_2^2) = (a^2 + 4b)^{-\frac{1}{2}}(x_1 - x_2)(x_1 + x_2)$

$= a$

Assume the proposition holds for $m \leq k$

$u_{k+1} = au_k + bu_{k-1}$

$= a(a^2 + 4b)^{-\frac{1}{2}}(x_1^k - x_2^k) + b(a^2 + 4b)^{-\frac{1}{2}}(x_1^{k-1} + x_2^{k-1})$

$= (a^2 + 4b)^{-\frac{1}{2}} \left\{ x_1^{k-1}(ax_1 + b) - x_2^{k-1}(ax_2 + b) \right\}$

$= (a^2 + 4b)^{-\frac{1}{2}} \left\{ x_1^{k+1} - x_2^{k+1} \right\}$

Thus it is true for $m = k+1$.

In our case

$a = \frac{1}{4} - \lambda$

$b = -\frac{1}{16} \lambda$

so $a^2 + 4b = \lambda^2 - \frac{3}{4} \lambda + \frac{1}{16}$. 
Thus
\[ \Delta_n = (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16})^{-\frac{1}{2}} \]

\[ \left\{ \left[ \frac{\left(\frac{1}{4} - \lambda\right) + (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16})^{\frac{1}{2}}}{2} \right]^n - \left[ \frac{\left(\frac{1}{4} - \lambda\right) - (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16})^{\frac{1}{2}}}{2} \right]^n \right\} \]

\[ \Delta_n(\lambda) = 0 \]

\[ \left[ \frac{(\frac{1}{4} - \lambda) + (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16})^{\frac{1}{2}}}{2} \right]^n = \left[ \frac{(\frac{1}{4} - \lambda) - (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16})^{\frac{1}{2}}}{2} \right]^n \]

\[ \left(\frac{1}{4} - \lambda\right) + (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16})^{\frac{1}{2}} = e^{\frac{2\pi ki}{n}} \left[ (\frac{1}{4} - \lambda) - (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16})^{\frac{1}{2}} \right] \]

for some \( k = 1, 2, 3, \ldots, n-1 \)

\[ \left( e^{\frac{2\pi ki}{n}} + 1 \right) (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16})^{\frac{1}{2}} = \left( e^{\frac{2\pi ki}{n}} - 1 \right) \left( \frac{1}{4} - \lambda \right) \]

Squaring, we obtain

\[ (e^{\frac{4\pi ki}{n}} + 2e^{\frac{2\pi ki}{n}} + 1) (\lambda^2 - \frac{3}{4}\lambda + \frac{1}{16}) \]

\[ = (e^{\frac{4\pi ki}{n}} - 2e^{\frac{2\pi ki}{n}} + 1) (\lambda^2 - \frac{1}{2}\lambda + \frac{1}{16}) \]

\[ 4e^{\frac{2\pi ki}{n}} (\lambda^2 - \frac{5}{8}\lambda + \frac{1}{16}) = (e^{\frac{4\pi ki}{n}} + 1) \frac{1}{4} \lambda \]

\[ \lambda^2 - \frac{5}{8}\lambda + \frac{1}{16} = \frac{1}{16}\lambda (e^{\frac{4\pi ki}{n}} + e^{\frac{2\pi ki}{n}}) \]

\[ = \frac{1}{8} \cos \frac{2\pi k}{n} \lambda \]

\[ \lambda^2 - \left( \frac{5}{8} + \frac{1}{8} \cos \frac{2\pi k}{n} \right) \lambda + \frac{1}{16} = 0 \]

(4.7)
Thus the roots of $\Delta_n$ are given by these quadratics for $k = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$, together with $\frac{1}{4}$ if $n$ is even.

Noting that $\frac{1}{4}$ is the double root obtained when $k = \frac{n}{2}$ we observe that

$$\Delta_n^2 = \prod_{k=1}^{n-1} \left[ \lambda^2 - \left( \frac{5}{8} + \frac{1}{8} \cos \frac{2\pi k}{n} \right) \lambda + \frac{1}{16} \right] = \prod_{k=1}^{n-1} Q_k(\lambda) \text{ say.}$$

Returning to $[T_1]$ in our partition of the transformation matrix, we find it has characteristic equation

$$(1-\lambda) \left[ \lambda^2 - \left( A - \frac{1}{4} \right) \lambda + \left( \frac{1}{16} - \frac{1}{8} B \right) \right] \prod_{k=1}^{N-1} \left[ \lambda^2 - \left( \frac{5}{8} + \frac{1}{8} \cos \frac{2\pi k}{n} \right) \lambda + \frac{1}{16} \right]$$

where $N$ is the node order and $A$ and $B$ are the weights assigned to the vertex and edge points respectively. This may be abbreviated to

$$(1-\lambda) \prod_{k=0}^{N-1} Q_k(\lambda) .$$

Note that as $N \to \infty$ $Q_k(\lambda) \to \lambda^2 - \frac{3}{4} \lambda + \frac{1}{16}$ for any fixed $k$.

This has roots $\frac{3 \pm \sqrt{5}}{8}$, the two asymptotes in Fig. 4.1.
Chapter 5

The Eigenvectors of the Subdivision Transformation

In Chapter 4 we obtained all the eigenvalues for the matrix $[T]$, representing the subdivision transformation. Our purpose, we recall, is to diagonalise $[T]$ and so we now require to calculate the eigenvectors.

$$[T] = \begin{bmatrix}
T_1 & 0 \\
T_3 & T_2
\end{bmatrix}$$

It is clear that if $\mathbf{v}$ is an eigenvector for $[T_2]$ then $\mathbf{v}' = \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix}$ is an eigenvector for $[T]$ since

$$[T]\mathbf{v}' = \begin{bmatrix}
T_1 & 0 \\
T_3 & T_2
\end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} [T_1] \mathbf{0} \\ [T_2] \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \mathbf{v} \end{bmatrix} = \lambda \mathbf{v}'$$

Now $[T_2] = \begin{bmatrix}
\frac{1}{16} & \frac{1}{16} & 0 \\
\frac{1}{64} & \frac{1}{32} & \frac{1}{64} \\
0 & \frac{1}{16} & \frac{1}{16}
\end{bmatrix}$ which has eigenvalues

$$1, \frac{1}{16}, \frac{1}{32}$$

We introduce the following notation:

Corresponding to $\frac{1}{8}$ we have $\mathbf{v}_1 = [1, 1, 1]^t$

$$\frac{1}{16} \mathbf{v}_1 = [1, 0, -1]^t$$

$$\frac{1}{32} \mathbf{v}_1 = [2, -1, 2]^t$$
Now if \( v \) is an eigenvector for \([T_1]\) corresponding to the eigenvalue \( \lambda \), we show that
\[
v' = \begin{bmatrix} v \\ w \end{bmatrix}
\]
is an eigenvector for \([T]\) where
\[
w = - [T_2 - \lambda I]^{-1}[T_3]v
\]

\[
[T]v' = \begin{bmatrix} T_1 & 0 \\ T_3 & T_2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} [T_1]v \\ [T_3]v + [T_2]w \end{bmatrix}
\]

\[
= \begin{bmatrix} [T_1]v \\ \{[I] - [T_2][T_2 - \lambda I]^{-1}\} [T_3]v \end{bmatrix}
\]

\[
= \begin{bmatrix} [T_1]v \\ \{[T_2 - \lambda I] - [T_2]\} [T_2 - \lambda I]^{-1}[T_3]v \end{bmatrix}
\]

\[
= \begin{bmatrix} [T_1]v \\ - \lambda[T_2 - \lambda I]^{-1}[T_3]v \end{bmatrix} = \begin{bmatrix} \lambda v \\ \lambda w \end{bmatrix} = \lambda v'
\]

It is thus necessary to establish the eigenvectors for \([T_1]\).

We recall that it has eigenvalues of 1,
\[
\lambda_a, \lambda_b, \text{ the roots of } Q_0(\lambda) \quad \text{[see (4.3)]}
\]
and the roots of \( Q_k(\lambda) \) for \( k = 1, 2, \ldots, N-1 \) [see (4.7)].

Let \([T_1] = \{t_{ij}\}\) as defined in (3.3).

Corresponding to \( \lambda = 1 \), since \([T_1]\) is a stochastic matrix,
\[
2N+1
\]
i.e. \( \sum_{j=1}^{2N+1} t_{ij} = 1 \), it is clear that
\( \mathbf{v}_0 = [1, 1, 1, 1, \ldots, 1]^T \) is an eigenvector.

Corresponding to \( \lambda = \lambda_a \) and \( \lambda_b \), we show that the eigenvectors are of the form

\[ \mathbf{v} = [x, y, z, y, z, y, z, \ldots, y, z]^T \]

where
\[ x = 16\lambda^2 - 12\lambda + 1 \]
\[ y = 6\lambda - 1 \]
\[ z = 4\lambda + 1 \]

We must show that
\[ \sum_{j=1}^{2N+1} t_{1j} \mathbf{v}_j = \lambda x \] (5.1)

and for \( k = 1, \ldots, N \)
\[ \sum_{j=1}^{2N+1} t_{2k,j} \mathbf{v}_j = \lambda y \] (5.2)
\[ \sum_{j=1}^{2N+1} t_{2k+1,j} \mathbf{v}_j = \lambda z \] (5.3)

Now for \( \lambda = \lambda_a, \lambda_b \)

\[ \lambda^2 - (A - \frac{1}{4})\lambda + \left( \frac{1}{16} - \frac{1}{8} B \right) = 0 \]

so
\[ \lambda_a + \lambda_b = A - \frac{1}{4} \]
\[ \lambda_a \lambda_b = \frac{1}{16} - \frac{1}{8} B \]

i.e.
\[ A = \lambda_a + \lambda_b + \frac{1}{4} \]
\[ B = \frac{1}{2} - 8\lambda_a \lambda_b \]

and
\[ C = 1 - A - B = \frac{1}{4} - (\lambda_a + \lambda_b) + 8\lambda_a \lambda_b \]

\[ \sum_{j=1}^{2N+1} t_{1j} \mathbf{v}_j = ax + Nby + Ncz \]
\[ \quad = Ax + By + Cz \]
Since A, B and C are symmetric in $\lambda_a$ and $\lambda_b$ it suffices to show for $\lambda = \lambda_a$. Thus our expression becomes

$$\begin{align*}
[\lambda_a + \lambda_b + \frac{1}{4}] & (16\lambda_a^2 - 8\lambda_a + 1) + \left[ \frac{1}{2} - 8\lambda_a \lambda_b \right] (6\lambda_a - 1) \\
+ \left[ \frac{1}{4} - (\lambda_a + \lambda_b) + 8\lambda_a \lambda_b \right] (4\lambda_a + 1) \\
= 16\lambda_a^3 + 16\lambda_a^2 \lambda_b + 4\lambda_a^2 - 12\lambda_a^2 - 12\lambda_a \lambda_b - 3\lambda_a \\
+ \lambda_a + \lambda_b + \frac{1}{4} + 3\lambda_a - 48\lambda_a^2 \lambda_b - \frac{1}{2} + 8\lambda_a \lambda_b \\
+ \lambda_a - 4\lambda_a^2 - 4\lambda_a \lambda_b + 32\lambda_a^2 \lambda_b + \frac{1}{4} - \lambda_a - \lambda_b \\
+ 8\lambda_a \lambda_b \\
= 16\lambda_a^3 - 12\lambda_a^2 + \lambda_a \\
= \lambda x
\end{align*}$$

Thus (5.1) holds

$$\sum_{j=1}^{2N+1} t_{2k, j} v_j = \frac{3}{8} v_1 + \frac{1}{16} v_{2k-2} + \frac{1}{16} v_{2k-1} + \frac{3}{8} v_{2k} + \frac{1}{16} v_{2k+1} + \frac{1}{16} v_{2k+2}$$

$$\begin{align*}
= \frac{3}{8} x + \frac{1}{2} y + \frac{1}{8} z \\
= 6\lambda_a^2 - \frac{9}{2} \lambda + \frac{3}{8} + 3\lambda - \frac{1}{2} + \frac{1}{2} \lambda + \frac{1}{8} \\
= 6\lambda_a^2 - \lambda \\
= \lambda y. \quad \text{so (5.2) holds}
\end{align*}$$
\[ \sum_{j=1}^{2N+1} \frac{1}{4} v_j = \frac{1}{4} v_1 + \frac{1}{4} v_{2k} + \frac{1}{4} v_{2k+1} + \frac{1}{4} v_{2k+2} \]
\[ = \frac{1}{4} x + \frac{1}{2} y + \frac{1}{4} z \]
\[ = 4\lambda^2 - 3\lambda + \frac{1}{4} + 3\lambda - \frac{1}{2} + \lambda + \frac{1}{4} \]
\[ = 4\lambda^2 + \lambda \]
\[ = \lambda z \quad \text{so (5.3) holds.} \]

Let \( v_a \) and \( v_b \) correspond to \( \lambda_a \) and \( \lambda_b \) respectively. The case where \( \lambda_a = \lambda_b \) will not concern us for the present. We shall return to this in Chapter 6.

We now consider the two eigenvalues which are roots of

\[ Q_k(\lambda) = \lambda^2 - \left( \frac{5}{8} + \frac{1}{8} c_k \right) \lambda + \frac{1}{16} \]

where \( c_k = \cos \frac{2\pi k}{N} \).

Let these be \( \lambda_k \) and \( \lambda_k' \) with \( \lambda_k \geq \lambda_k' \).

We show that the corresponding eigenvectors are respectively

\[ u = [u_1, u_2, u_3, \ldots, u_{2N+1}]^t \]

where \( u_1 = 0 \)

\[ u_{2j} = f_k c(j-1)k \]

\[ u_{2j+1} = c(j-1)k + c_{jk} \]

and

\[ v = [v_1, v_2, v_3, \ldots, v_{2N+1}]^t \]
where \( v_1 = 0 \)

\[
v_{2j} = f_k s(j-1)k
\]

\[
v_{2j+1} = s(j-1)k + s_{jk}
\]

where \( c_i \) is as above

\[
s_i = \sin \frac{2\pi i}{N}
\]

and \( f_k = 4\lambda - 1 \) for \( \lambda = \lambda_k, \lambda'_{k} \) \( (k \neq \frac{N}{2}) \)

The following equations need to be proved.

\[
\sum_{j=1}^{2N+1} t_{1j} u_j = \sum_{j=1}^{2N+1} t_{1j} v_j = 0 \tag{5.4}
\]

\[
\sum_{j=1}^{2N+1} t_{2\ell, j} u_j = \lambda u_{2\ell} \tag{5.5}
\]

\[
\sum_{j=1}^{2N+1} t_{2\ell, j} v_j = \lambda v_{2\ell} \tag{5.6}
\]

\[
\sum_{j=1}^{2N+1} t_{2\ell+1, j} u_j = \lambda u_{2\ell+1} \tag{5.7}
\]

\[
\sum_{j=1}^{2N+1} t_{2\ell+1, j} v_j = \lambda v_{2\ell+1} \tag{5.8}
\]

for \( \ell = 1, 2, \ldots, N \).
To establish these, we first need the following lemma.

Lemma 5.1:

\[ \sum_{j=0}^{N-1} c \omega^j = 0 \quad \text{for } \omega = 1, 2, \ldots, N-1 \]

\[ \sum_{j=0}^{N-1} \omega^j = N \quad \text{for } \omega = 0 \]

\[ \sum_{j=0}^{N-1} s \omega^j = 0 \quad \text{for } \omega = 0, 1, \ldots, N-1. \]

Proof: Let \( a_\omega = e^{2\pi i \omega/N} = c_\omega + i s_\omega \).

Then

\[ \sum_{j=0}^{N-1} a_\omega \omega^j = \sum_{j=0}^{N-1} c_\omega \omega^j + i \sum_{j=0}^{N-1} s_\omega \omega^j \]

But \( \sum_{j=0}^{N-1} a_\omega \omega^j \) is a geometric progression and so is equal to

\[ \frac{a_{\omega N} - 1}{a_\omega - 1} = 0 \quad \text{since } a_{\omega N} = 1 \]

unless \( \omega = 0 \), in which case \( a_\omega = 1 \)

and so

\[ \sum_{j=0}^{N-1} a_\omega \omega^j = N. \]

Taking real and imaginary parts, the result is proved. \( \Box \)

Then

\[ \sum_{j=1}^{2N+1} t_{1j} u_j = a u_1 + b \sum_{j=1}^{N} u_{2j} + c \sum_{j=1}^{N} u_{2j+1} \]

\[ = b \sum_{k=1}^{N} c (j-1)k + c \sum_{k=1}^{N} c (j-1)k + c \sum_{k=1}^{N} c jk \]

\[ = 0 \]
\[ \sum_{j=1}^{2N+1} t_{1j} v_j = a v_1 + b \sum_{j=1}^N v_{2j} + c \sum_{j=1}^N v_{2j+1} \]

\[ = b \sum_{j=1}^N s(j-1)k + c \left\{ \sum_{j=1}^N s(j-1)k + \sum_{j=1}^N s_{jk} \right\} \]

\[ = 0 \quad \text{so (5.4) holds.} \]

\[ \sum_{j=1}^{2N+1} t_{2\xi, j} u_j = \frac{3}{8} u_1 + \frac{3}{8} u_{2\xi} + \frac{1}{16} \left\{ u_{2\xi-2} + u_{2\xi-1} + u_{2\xi+1} + u_{2\xi+2} \right\} \]

\[ = \frac{3}{8} f_k c(\xi-1)k + \frac{1}{16} \left\{ f_k c(\xi-2)k + c(\xi-2)k + c(\xi-1)k + c(\xi-1)k \right\} \]

\[ + c_{\xi k} \right\} \]

\[ = \frac{3}{8} f_k c(\xi-1)k + \frac{1}{16} \left\{ 2c(\xi-1)k + 2(f_k + 1) c_k c(\xi-1)k \right\} \]

\[ = \frac{1}{8} c(\xi-1)k \left\{ 3f_k + 1 + (f_k + 1) c_k \right\} \quad (5.9) \]

Now

\[ 3f_k + 1 + (f_k + 1) c_k \]

\[ = 12\lambda - 2 + 4c_k \lambda \]

\[ = 2(6\lambda - 1 + 2c_k \lambda) \]

\[ = 2(6\lambda - 1 + 16\lambda^2 - 10\lambda + 1) \]

\[ \{\text{since } Q_k(\lambda) = 0 \}

\[ \Rightarrow 2c_k \lambda = 16\lambda^2 - 10\lambda + 1 \} \]
\[= 2(16\lambda^2 - 4\lambda)\]
\[= 8\lambda f_k\]

Hence (5.9) gives
\[
\lambda f_k c_{(\ell-1)k} = \lambda u_{2\ell}
\]
and (5.5) holds.

A similar argument will prove (5.6)

\[
\sum_{j=1}^{2N+1} u_{j+2\ell+1} = \frac{1}{4} (u_1 + u_{2\ell} + u_{2\ell+1} + u_{2\ell+2})
\]
\[= \frac{1}{4} (f_k c_{(\ell-1)k} + c_{(\ell-1)k} + c_{\ell k} + f_k c_{\ell k})
\]
\[= \frac{1}{4} (f_k + 1)(c_{(\ell-1)k} + c_{\ell k})
\]
\[= \lambda (c_{(\ell-1)k} + c_{\ell k})
\]
\[= \lambda u_{2\ell+1}
\]
so (5.7) holds and similarly (5.8).

Thus for \(k = 1, 2, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \) we have established the eigenvectors \(u_k, v_k\) and \(u_k', v_k'\) corresponding to \(\lambda_k\) and \(\lambda'_k\) respectively.

If \(N\) is odd, we have accounted for all the eigenvectors of \(T_1\). If \(N\) is even, corresponding to \(k = \frac{N}{2}\) we have a double eigenvalue of \(\frac{1}{4}\) and it is easily shown that two eigenvectors are

\[
u_{N/2} = [0, 0, 1, 0, -1, 0, 1, \ldots, 0, -1, 0]^t
\]
and
\[
u_{N/2} = [0, 0, 1, 0, -1, 0, 1, \ldots, 1, 0, -1]^t
\]
It now remains only to extend these eigenvectors to eigenvectors for \([T]\).

Let \( u_k \) have extension \( u_k(T) = \begin{bmatrix} u_k \\ w_k \end{bmatrix} \)

and \( v_k \) have extension \( v_k(T) = \begin{bmatrix} v_k \\ x_k \end{bmatrix} \)

Then \( w_k = -[T_2 - \lambda_k I]^{-1}[T_3] u_k \)

and \( x_k = -[T_2 - \lambda_k I]^{-1}[T_3] v_k \)

Now \(-[T_2 - \lambda_k I]^{-1}\)

\[
= \frac{4}{(8\lambda-1)(32\lambda-1)} \begin{bmatrix}
\frac{1024\lambda^2-160\lambda+5}{16\lambda-1} & 4 & \frac{1}{16\lambda-1} \\
1 & 4(16\lambda-1) & 1 \\
\frac{1}{16\lambda-1} & 4 & \frac{1024\lambda^2-160\lambda+5}{16\lambda-1}
\end{bmatrix}
\]

and \([T_3][v_0, u_k, v_k, u_N, v_N] = \]

\[
\begin{bmatrix}
\frac{1}{16} & 3 & 8 & 0 & 0 \\
\frac{3}{32} & 9 & \frac{3}{16} & \frac{1}{64} & 0 \\
\frac{1}{16} & 3 & 8 & \frac{1}{16} & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{16} & 3 \\
\frac{3}{32} & \frac{9}{16} \\
\frac{1}{16} & 3 \\
\end{bmatrix}
\]

\[
= \frac{1}{64} \begin{bmatrix}
\frac{1}{64} & 3 & \frac{3}{32} \\
\frac{3}{64} & \frac{3}{32} & 1 \\
\frac{3}{64} & \frac{3}{32} & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{16} & 3 \\
\frac{3}{8} & \frac{3}{8} \\
\frac{1}{16} & 0 \\
\end{bmatrix}
\]

\[
\times \begin{bmatrix}
\frac{1}{16} & 3 \\
\frac{3}{8} & \frac{3}{8} \\
\frac{1}{16} & 0 \\
\end{bmatrix}
\]
$$
\begin{bmatrix}
16\lambda_a^2 - 12\lambda_a + 1 & 0 & 0 & 0 & 0 \\
6\lambda_a - 1 & f_k c_0 & f_k s_0 & 1 & 0 \\
4\lambda_a + 1 & c_0 c_k & s_0 + s_k & 0 & 1 \\
6\lambda_a - 1 & f_k c_k & f_k s_k & -1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
6\lambda_a - 1 & f_k c(N-1)k & f_k s(N-1)k & -1 & 0 \\
4\lambda_a + 1 & c(N-1)k c_k & s(N-1)k + s_k & 0 & -1 \\
\end{bmatrix}
$$

\[
\begin{bmatrix}
\frac{1}{8} \lambda_a (8\lambda_a + 27) & \frac{3}{2} \lambda_k + \frac{1}{16}(4\lambda_k + 5)c_k & -\frac{1}{16}(4\lambda_k + 5)s_k \\
\frac{1}{16}(24\lambda_a^2 + 51\lambda_a - 5) & \frac{3}{8}(6\lambda_k - 1) + \frac{1}{32}(4\lambda_k + 5)c_k & 0 \\
\frac{1}{8} \lambda_a (8\lambda_a + 27) & \frac{3}{2} \lambda_k + \frac{1}{16}(4\lambda_k + 5)c_k & \frac{1}{16}(4\lambda_k + 5)s_k \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{5}{16} & -\frac{3}{8} \\
\frac{17}{32} & 0 \\
\frac{5}{16} & \frac{3}{8} \\
\end{bmatrix}
\]

Hence

\[
\begin{bmatrix}
x_a, w_k, x_k, w_N, x_N \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
256\lambda_a^3 + 864\lambda_a^2 - 30\lambda_a - 5 \\
(8\lambda_a - 1)(32\lambda_a - 1) \\
384\lambda_a^3 + 800\lambda_a^2 - 104\lambda_a + 5 \\
(8\lambda_a - 1)(32\lambda_a - 1) \\
256\lambda_a^3 + 864\lambda_a^2 - 30\lambda_a - 5 \\
(8\lambda_a - 1)(32\lambda_a - 1) \\
2(4\lambda_k - 1)(4\lambda_k + 13) \\
(32\lambda_k - 1) \\
2(4\lambda_k - 1)(4\lambda_k + 13) \\
(32\lambda_k - 1) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{2(400\lambda_k^2 + 36\lambda_k - 13)}{(16\lambda_k - 1)(32\lambda_k - 1)} & \frac{4\lambda_k + 5}{16\lambda_k - 1}s_k & 3 & -2 \\
\frac{2(400\lambda_k^2 + 36\lambda_k - 13)}{(16\lambda_k - 1)(32\lambda_k - 1)} & \frac{4\lambda_k + 5}{16\lambda_k - 1}s_k & 0 & 4 \\
\frac{2(400\lambda_k^2 + 36\lambda_k - 13)}{(16\lambda_k - 1)(32\lambda_k - 1)} & \frac{4\lambda_k + 5}{16\lambda_k - 1}s_k & 3 & 2 \\
\end{bmatrix}
\]
Clearly $x_b$, the extension of $v_b$, is equal to $x_a$ with $\lambda_b$ instead of $\lambda_a$ and since $[T]$ is stochastic

$$v_0(T) = [1, 1, 1, \ldots, 1]^T$$

We have now calculated all the eigenvectors of $[T]$; we will show that these form a linearly independent set, assuming $\lambda_a \neq \lambda_b$.

It now remains for us to diagonalize $[T]$.

Let $[\Lambda]$ be the diagonal matrix of eigenvalues in the following order as we go down the leading diagonal

$$1, \lambda_a, \lambda_b, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \frac{\lambda_{N-1}}{2}, \frac{\lambda_{N-1}}{2}$$

followed by

$$\frac{1}{4}, \frac{1}{4} \text{ if } N \text{ is even, then}$$

$$\lambda_1', \lambda_1', \lambda_2', \lambda_2', \ldots, \frac{\lambda_{N-1}'}{2}, \frac{\lambda_{N-1}'}{2}$$

and finally

$$\frac{1}{8}, \frac{1}{16}, \frac{1}{32} \ldots$$

Let $[V]$ be the matrix of corresponding eigenvectors

i.e.

$$[V] = \begin{bmatrix}
v_0 & v_a & v_b & u_1 & v_1 & \cdots & \frac{u_1}{2} & \frac{v_1}{2} & 0 \\
v_0 & v_a & v_b & \cdots & \frac{u_1}{2} & \frac{v_1}{2} & \cdots & \frac{u_1}{2} & \frac{v_1}{2} \\
x_0 & x_a & x_b & x_1 & \cdots & \frac{x_1}{2} & \frac{x_1}{2} & \cdots & \frac{x_1}{2} & \frac{x_1}{2}
\end{bmatrix}$$

To diagonalize $[T]$ we must find $[V]^{-1}$.

Let $[V] = \begin{bmatrix} v_1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ v_3 & 1 & v_2 \end{bmatrix}$
Then \[ [V]^{-1} = \begin{bmatrix} v_1^{-1} & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ x & v_2^{-1} \end{bmatrix} \]

where \( [X] = -[V_2]^{-1}[V_3][V_1]^{-1} \).

Although \( [V_1] \) is not an orthogonal matrix, it will be instructive to consider \( [V_1]^T[V_1] \).

To evaluate this matrix product, we must calculate the scalar product of all possible pairs of eigenvectors. To do this, in turn, we shall require the following group of identities, each of which invoke Lemma 5.1 or (4.7).

In all these equalities \( k \) and \( \ell \) may range over \( \{1, 2, \ldots, \left[ \frac{N}{2} \right] \} \) while \( j \) may in addition be zero.

\[
\sum_{i=0}^{N-1} c_{ik} c_{(i+j)k} = \frac{1}{2} \sum_{i=0}^{N-1} [c_{(2i+j)k} + c_{jk}] = \frac{N}{2} c_{jk} \quad (5.10a)
\]

\[
\sum_{i=0}^{N-1} s_{ik} s_{(i+j)k} = \frac{1}{2} \sum_{i=0}^{N-1} [c_{jk} - c_{(2i+j)k}] = \frac{N}{2} c_{jk} \quad (5.10b)
\]

\[
\sum_{i=0}^{N-1} c_{ik} c_{(i+j)\ell} = \frac{1}{2} \sum_{i=0}^{N-1} [c_{i(k+\ell) + j\ell} + c_{i(k-\ell) - j\ell}] = 0 \quad (k \neq \ell) \quad (5.11a)
\]
\[\sum_{i=0}^{N-1} s_{ik} s(i+j)x = \frac{1}{2} \sum_{i=0}^{N-1} (c_i(k-x) - c_{i-(k-x)+j} x)\]

\[= 0 \quad (k \neq x) \quad (5.11b)\]

\[\sum_{i=0}^{N-1} c_{ik} s(i+j)x = \frac{1}{2} \sum_{i=0}^{N-1} (s_{i(k+x)} + s_{i(k-x)+j} x)\]

\[= 0 \quad (5.12)\]

\[\xi_k^2 = (4\lambda_k - 1)^2\]

\[= 16\lambda_k^2 - 8\lambda_k + 1\]

\[= 16\lambda_k^2 - (10 + 2c_k)\lambda_k + 1 + 2(1 + c_k)\lambda_k\]

\[= 2(1 + c_k)\lambda_k \quad (5.13)\]

\[\xi_k \xi_k' = (4\lambda_k - 1)(4\lambda_k' - 1)\]

\[= 16\lambda_k \lambda_k' - 4(\lambda_k + \lambda_k') + 1\]

\[= 2 - 4\left(\frac{5}{8} + \frac{1}{8} c_k\right)\]

\[= -\frac{1}{2} (1 + c_k) \quad (5.14)\]

We may now show the following identities

For \(k, l\) in \(\{1, 2, \ldots, \lfloor\frac{N-1}{2}\rfloor\}\)

\[u_k \cdot u_k = v_k \cdot v_k = N(1 + c_k)(1 + \lambda_k) \quad (5.15)\]

\[u_k' \cdot u_k' = v_k' \cdot v_k' = N(1 + c_k)(1 + \lambda_k') \quad (5.16)\]
\[ u_k \cdot u'_k = v_k \cdot v'_k = \frac{3}{4} N(1+c_k) \]  \hspace{1cm} (5.17)

\[ u_k \cdot v'_k = u'_k \cdot v_k = 0 \]  \hspace{1cm} (5.18)

\[ u_k \cdot v'_k = v_k \cdot u'_k = 0 \]  \hspace{1cm} (5.19)

If \( k \neq \ell \) then

\[ u_k \cdot u'_\ell = v_k \cdot v'_\ell = 0 \]  \hspace{1cm} (5.20)

\[ u_k \cdot u'_\ell = v_k \cdot v'_\ell = 0 \]  \hspace{1cm} (5.21)

In addition

\[ u_k \cdot v_0 = u_k \cdot v_a = u_k \cdot v_b = 0 \]  \hspace{1cm} (5.22)

\[ v_k \cdot v_0 = v_k \cdot v_a = v_k \cdot v_b = 0 \]  \hspace{1cm} (5.23)

\[ u'_k \cdot v_0 = u'_k \cdot v_a = u'_k \cdot v_b = 0 \]  \hspace{1cm} (5.24)

\[ v'_k \cdot v_0 = v'_k \cdot v_a = v'_k \cdot v_b = 0 \]  \hspace{1cm} (5.25)

and if \( N \) is even

\[ \frac{u_N}{2} \cdot \frac{v_N}{2} = \frac{v_N}{2} \cdot \frac{u_N}{2} = N \]  \hspace{1cm} (5.26)

\[ \frac{u_N}{2} \cdot \frac{v_N}{2} = 0 \]  \hspace{1cm} (5.27)

and they are perpendicular to all the other eigenvectors.

\[ u_k \cdot u_k = \sum_{i=0}^{N-1} \left[ c_{ik} + c_{(i+1)k} \right]^2 + \frac{2}{k} \sum_{i=0}^{N-1} c_{ik}^2 \]
\[ \begin{align*}
&= (2 + f_k^2) \sum_{i=0}^{N-1} c_{ik}^2 + 2 \sum_{i=0}^{N-1} c_{ik} c_{(i+1)k} \\
&= \left\{ 2 + 2(1+c_k)\lambda_k \right\} \frac{N}{2} + 2 \frac{N}{2} c_k \\
&= N \left\{ 1 + c_k + (1+c_k)\lambda_k \right\} \\
&= N(1+c_k)(1+\lambda_k)
\end{align*} \]

\[ \begin{align*}
\nu_k \cdot \nu_k &= \sum_{i=0}^{N-1} \left[ s_{ik} + s_{(i+1)k} \right]^2 + f_k^2 \sum_{i=0}^{N-1} s_{ik}^2 \\
&= (2 + f_k^2) \sum_{i=0}^{N-1} s_{ik}^2 + 2 \sum_{i=0}^{N-1} s_{ik} s_{(i+1)k} \\
&= N(1+c_k)(1+\lambda_k) \\
&\text{as above.}
\end{align*} \]

Similarly,

\[ \begin{align*}
\frac{\partial u^i}{\partial k} \cdot \frac{\partial u^i}{\partial k} &= \frac{\partial v^i}{\partial k} \cdot \frac{\partial v^i}{\partial k} = N(1+c_k)(1+\lambda_k'')
\end{align*} \]

\[ \begin{align*}
\frac{\partial u^i}{\partial k} \cdot \frac{\partial u^i}{\partial k} &= \sum_{i=0}^{N-1} \left[ c_{ik} + c_{(i+1)k} \right]^2 + f_k f_k' \sum_{i=0}^{N-1} c_{ik}^2 \\
&= (2 + f_k f_k') \sum_{i=0}^{N-1} c_{ik}^2 + 2 \sum_{i=0}^{N-1} c_{ik} c_{(i+1)k} \\
&= \left\{ 2 - \frac{1}{2} \left( 1+c_k \right) \right\} \frac{N}{2} + 2 \frac{N}{2} c_k \\
&= N \left\{ \frac{3}{4} + c_k - \frac{1}{4} c_k \right\} \\
&= \frac{3}{4} N(1+c_k)
\end{align*} \]
Similarly

\[ v_k \cdot v'_k = \frac{3}{4} N(1 + c_k) \]

\[ u_k \cdot v'_k = \sum_{i=0}^{N-1} \left[ c_{ik} + c_{(i+1)k} \right] \left[ s_{ik} + s_{(i+1)k} \right] \]

\[ + f_k f_{k\ell} \sum_{i=0}^{N-1} c_{ik} s_{ik} \]

\[ = 0 \text{ from (5.12)}. \]

Similarly

\[ u'_k \cdot v'_k = 0 \]

\[ u_k \cdot v'_k = 0 \]

\[ v_k \cdot u'_k = 0 \]

If \( k \neq \ell \)

\[ u_k \cdot u'_k = \sum_{i=0}^{N-1} \left[ c_{ik} + c_{(i+1)k} \right] \left[ c_{ik} + c_{(i+1)k} \right] \]

\[ + f_k f_{k\ell} \sum_{i=0}^{N-1} c_{ik} c_{ik} \]

\[ = 0 \text{ from (5.11a)}. \]

Similarly

\[ v_k \cdot v'_k = 0 \]

\[ u_k \cdot u'_k = 0 \]

\[ v_k \cdot v'_k = 0 \]

\[ u_k \cdot v_0 = \sum_{i=0}^{N-1} \left[ c_{ik} + c_{(i+1)k} \right] + f_k \sum_{i=0}^{N-1} c_{ik} \]

\[ = 0 \text{ from Lemma 5.1}. \]
\[ u_k \cdot \mathbf{a} = (6 \lambda a - 1) f k \sum_{i=0}^{N-1} c_{ik} + (4 \lambda a + 1) \sum_{i=0}^{N-1} [c_{ik} + c_{(i+1)k}] = 0. \]

Similarly
\[ u_k \cdot \mathbf{b} = 0. \]

(5.23) (5.24) and (5.25) show similar properties for \( v_k, u_k' \) and \( v_k' \) with respect to \( v_0, v_a \) and \( v_b \).

(5.26) and (5.27) are easily proved given the simple structure of \( u \frac{N}{2} \) and \( v \frac{N}{2} \).

Thus \( [V_1]^T [V_1] = \)

\[
\begin{bmatrix}
\mathbf{x}_1 & \mathbf{x}_1' \\
\mathbf{x}_2 & \mathbf{x}_2' \\
\mathbf{y}_1 & \mathbf{y}_1' \\
\mathbf{y}_N-1 & \mathbf{y}_N-1' \\
\end{bmatrix}
\]

(N odd)
and

\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & y_1 \\
0 & 0 \\
0 & y_1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

where

\[ x_k = N(1+c_k)(1+\lambda_k) \]

\[ y_k = \frac{3}{4} N(1+c_k) \]

and

\[
[Q] = 
\begin{bmatrix}
\nu_0 \cdot \nu_0 & \nu_0 \cdot \nu_a & \nu_0 \cdot \nu_b \\
\nu_0 \cdot \nu_a & \nu_a \cdot \nu_a & \nu_a \cdot \nu_b \\
\nu_0 \cdot \nu_b & \nu_a \cdot \nu_b & \nu_b \cdot \nu_b \\
\end{bmatrix}
\]

We can now find the inverse of \([V_1]^T [V_1]\) and since

\[ [V_1]^{-1} = \left\{ [V_1]^T [V_1] \right\}^{-1} [V_1]^T \]

also that of \([V_1]\).
\[
\{[v_1]^t[v_1]\}^{-1} = 
\]

\[
\begin{array}{c|c|c}
 & \mathbf{Q}^{-1} & 0 \\
\hline
\mathbf{r}_1 & p_1 & 0 \\
\mathbf{0} & 0 & \frac{p_{N-1}}{2} \\
\mathbf{r}_1 & 0 & \frac{r_{N-1}}{2} \\
\hline
\mathbf{r}_1 & 0 & \frac{r_{N-1}}{2} \\
\mathbf{0} & \frac{r_{N-1}}{2} & \frac{r_{N-1}}{2} \\
\mathbf{0} & \frac{r_{N-1}}{2} & \frac{r_{N-1}}{2} \\
\mathbf{1} & 0 & 0 \\
\end{array}
\]

(N odd)

and

\[
\begin{array}{c|c|c}
 & \mathbf{Q}^{-1} & 0 \\
\hline
\mathbf{r}_1 & p_1 & 0 \\
\mathbf{0} & 0 & \frac{p_{N-1}}{2} \\
\mathbf{r}_1 & 0 & \frac{r_{N-1}}{2} \\
\hline
\mathbf{r}_1 & 0 & \frac{r_{N-1}}{2} \\
\mathbf{0} & \frac{r_{N-1}}{2} & \frac{r_{N-1}}{2} \\
\mathbf{0} & \frac{r_{N-1}}{2} & \frac{r_{N-1}}{2} \\
\mathbf{0} & \frac{1}{N} & 0 \\
\end{array}
\]

(N even)
where  
\[ p_k = \frac{1 + \lambda_k'}{8N(\lambda_k' - \lambda_k')^2} \]  
(5.28)

\[ p_k' = \frac{1 + \lambda_k'}{8N(\lambda_k' - \lambda_k')^2} \]  
(5.29)

and  
\[ r_k = -\frac{3}{32N(\lambda_k' - \lambda_k')^2} \]  
(5.30)

which is easily checked.

We are now in a position to evaluate \([V_1]^{-1}\).

The first three rows are given by

\[ [Q]^{-1} = \begin{bmatrix} v_0^t \\ v_a^t \\ v_b^t \end{bmatrix} \]

Let  
\[ x(\lambda) = 16\lambda^2 - 12\lambda + 1 \]
\[ y(\lambda) = 6\lambda - 1 \]
\[ z(\lambda) = 4\lambda + 1 \]

as in (5.1) - (5.3)

Then \([Q] = \)

\[ \begin{bmatrix} 1 & \sqrt{N} & \sqrt{N} \\ x(\lambda_a) & y(\lambda_a) & \sqrt{N} z(\lambda_a) \\ x(\lambda_b) & y(\lambda_b) & \sqrt{N} z(\lambda_b) \end{bmatrix} \begin{bmatrix} 1 & x(\lambda_a) & x(\lambda_b) \\ \sqrt{N} & \sqrt{N} y(\lambda_a) & \sqrt{N} y(\lambda_b) \\ \sqrt{N} & \sqrt{N} z(\lambda_a) & \sqrt{N} z(\lambda_b) \end{bmatrix} \]

Hence  
\[ |Q| = N^2 \begin{vmatrix} 1 & 1 & 1 \\ x(\lambda_a) & y(\lambda_a) & z(\lambda_a) \\ x(\lambda_b) & y(\lambda_b) & z(\lambda_b) \end{vmatrix}^2 \]
\[ [Q]^{-1} = |Q|^{-1} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \]

where \( c_{ij} \) is the \( ij \)th cofactor of \([Q]\)

\[ c_{11} = 4N\left\{3328\lambda_a^2 - 256(\lambda_a + \lambda_b)\lambda_a \lambda_b + 128(\lambda_a^2 + \lambda_b^2) + 32\lambda_a \lambda_b - 176(\lambda_a + \lambda_b) + 25N + 73\right\} \]

\[(\lambda_a - \lambda_b)^2\]

\[ c_{12} = c_{21} \]

\[ = -4N\left\{640\lambda_a \lambda_b^2 - 48\lambda_a \lambda_b + 8\lambda_a - 32\lambda_b + 5N - 3\right\} \]

\[(\lambda_a - \lambda_b)(1 - \lambda_b)\]

\[ c_{13} = c_{31} \]

\[ = 4N\left\{640\lambda_a^2 \lambda_b - 48\lambda_a \lambda_b^2 - 32\lambda_a + 8\lambda_b + 5N - 3\right\} \]

\[(\lambda_a - \lambda_b)(1 - \lambda_a)\]

\[ c_{22} = 4N\left\{128\lambda_b^2 - 16\lambda_b + N + 1\right\} \]

\[(1 - \lambda_b)^2\]

\[ c_{23} = c_{32} \]

\[ = 4N\left\{128\lambda_a \lambda_b - 8(\lambda_a + \lambda_b) + N + 1\right\} \]

\[(1 - \lambda_a)(1 - \lambda_b)\]
where \( \alpha_1 = |Q|^{-1} \left\{ c_{11} + c_{21} x(\lambda_a) + c_{31} x(\lambda_b) \right\} \)

\[ = \frac{5}{16(1-\lambda_a)(1-\lambda_b)} \]

\( \beta_1 = |Q|^{-1} \left\{ c_{11} + c_{21} y(\lambda_a) + c_{31} y(\lambda_b) \right\} \)

\[ = \frac{1 - (\lambda_a + \lambda_b) - 4\lambda_a \lambda_b}{2N(1-\lambda_a)(1-\lambda_b)} \]
\[
\gamma_1 = |Q|^{-1} \left\{ c_{11} + c_{21} z(\lambda_a) + c_{31} z(\lambda_b) \right\} \\
= \frac{3 - 8(\lambda_a + \lambda_b) + 48\lambda_a \lambda_b}{16N(1-\lambda_a)(1-\lambda_b)}
\]

\[
\alpha_2 = |Q|^{-1} \left\{ c_{12} + c_{22} x(\lambda_a) + c_{32} x(\lambda_b) \right\} \\
= \frac{-1}{16(\lambda_a-\lambda_b)(1-\lambda_a)}
\]

\[
\beta_2 = |Q|^{-1} \left\{ c_{12} + c_{22} y(\lambda_a) + c_{32} y(\lambda_b) \right\} \\
= \frac{\lambda_b}{2N(\lambda_a-\lambda_b)(1-\lambda_a)}
\]

\[
\gamma_2 = |Q|^{-1} \left\{ c_{12} + c_{22} z(\lambda_a) + c_{32} z(\lambda_b) \right\} \\
= \frac{- (8\lambda_b - 1)}{16N(\lambda_a-\lambda_b)(1-\lambda_a)}
\]

\[
\alpha_3 = |Q|^{-1} \left\{ c_{13} + c_{23} x(\lambda_a) + c_{33} x(\lambda_b) \right\} \\
= \frac{1}{16(\lambda_a-\lambda_b)(1-\lambda_b)}
\]

\[
\beta_3 = |Q|^{-1} \left\{ c_{13} + c_{23} y(\lambda_a) + c_{33} y(\lambda_b) \right\} \\
= \frac{- \lambda_a}{2N(\lambda_a-\lambda_b)(1-\lambda_b)}
\]
\[ \gamma_3 = |Q|^{-1} \left\{ c_{13} + c_{23} z(\lambda_a) + c_{33} z(\lambda_b) \right\} \]

\[ 8 \lambda_a^{-1} = \frac{16N(\lambda_a - \lambda_b)(1-\lambda_b)}{16N(\lambda_a - \lambda_b)(1-\lambda_b)} \]

Returning to our definitions of \( p_k, p_k' \) and \( r_k \) in (5.28) - (5.30) and noting the structure of \([V_1^t [V_1]]^{-1}\) and \([V_1]^t\), we can evaluate the other rows from 4 onwards.

For \( k = 1, 2, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \)

row \( 2k + 2 \) of \([V_1]^{-1}\) is given by

\[ p_k u_{k-1} + r_k u_{k-1}' \]

Thus the 1st entry on this row is 0:

the 2jth entry is given by

\[ p_k f_k c(j-1)k + r_k f_k' c(j-1)k \]

\[ = \left\{ \frac{1 + \lambda_k^*}{8N(\lambda_k^* - \lambda_k)^2} (4\lambda_k^* - 1) - \frac{3}{32N(\lambda_k^* - \lambda_k)^2} (4\lambda_k^* - 1) \right\} c(j-1)k \]

\[ = \frac{1}{32N(\lambda_k^* - \lambda_k)^2} \left\{ 4(1+\lambda_k^*) (4\lambda_k^* - 1) - 3(4\lambda_k^* - 1) \right\} c(j-1)k \]

\[ = \frac{1}{32N(\lambda_k^* - \lambda_k)^2} \left\{ 16\lambda_k^* - 4 + 1 - 4\lambda_k^* - 12\lambda_k^* + 3 \right\} c(j-1)k \]

\[ = \frac{1}{2N(\lambda_k^* - \lambda_k)} c(j-1)k \]

(5.31)
The \((2j+1)^{th}\) entry is given by

\[ p_k [c_{(j-1)k} + c_{jk}] + r_k [c_{(j-1)k} + c_{jk}] \]

\[ = (p_k + r_k) [c_{(j-1)k} + c_{jk}] \]

\[ = \frac{4\lambda_k' + 1}{32N(\lambda_k - \lambda_k')} [c_{(j-1)k} + c_{jk}] \]

Now \((4\lambda_k' + 1)(4\lambda_k' - 1) = 4(\lambda_k - \lambda_k')\)

so this is equal to

\[ \frac{1}{8N(4\lambda_k - 1)(\lambda_k - \lambda_k')} [c_{(j-1)k} + c_{jk}] \quad (5.32) \]

Similarly the \((2k+3)^{th}\) row is given by

\[ p_k v_k + r_k v_k' \]

whose 1st entry is 0.

\(2j^{th}\) entry is

\[ \frac{1}{2N(\lambda_k - \lambda_k')} s_{(j-1)k} \quad (5.33) \]

and \((2j+1)^{th}\) entry is

\[ \frac{1}{8N(4\lambda_k - 1)(\lambda_k - \lambda_k')} [s_{(j-1)k} + s_{jk}] \quad (5.34) \]

If \(N\) is even the next two rows are given by

\[ \frac{1}{N} \frac{u_N}{2} \quad \text{and} \quad \frac{1}{N} \frac{v_N}{2} \]
\[
\begin{bmatrix}
0, \frac{1}{N}, 0, -\frac{1}{N}, 0, \ldots, 0, -\frac{1}{N}, 0
\end{bmatrix}^t
\]

and
\[
\begin{bmatrix}
0, 0, \frac{1}{N}, 0, -\frac{1}{N}, 0, \ldots, \frac{1}{N}, 0, -\frac{1}{N}
\end{bmatrix}^t
\]

respectively.

The remaining rows are given by, alternately,

\[p_k' u_k' + r_k u_k\]

and

\[p_k' v_k' + r_k v_k\]

For \(p_k' u_k' + r_k u_k\)

the 1st entry is 0;

the 2\(j\)th entry is

\[-\frac{1}{2N(\lambda_k - \lambda_k')} c(j-1)k\]

and the (2\(j+1\))th entry

\[-\frac{1}{8N(4\lambda_k' - 1)(\lambda_k - \lambda_k')} [c(j-1)k + c_{jk}]\]

Similarly for \(p_k' v_k' + r_k v_k\)

the 1st entry is 0;

the 2\(j\)th entry is

\[-\frac{1}{2N(\lambda_k' - \lambda_k)} s(j-1)k\]

and the (2\(j+1\))th entry

\[-\frac{1}{8N(4\lambda_k' - 1)(\lambda_k - \lambda_k')} [s(j-1)k + s_{jk}]\]
We have now evaluated \([V_1]^{-1}\)

\[
[V_2] = \begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & -1 \\
1 & -1 & 2 \\
\end{bmatrix}
\]

\[
[V_2]^{-1} = \begin{bmatrix}
\frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
\frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\
\end{bmatrix}
\]

\[
[X] = -[V_2]^{-1}[V_3][V_1]^{-1}
\]

We shall see in the next chapter it is unnecessary to calculate this.
Chapter 6

Evaluation of the Surface Normal

In the preceding chapter we obtained explicitly the matrix of eigenvectors \([V]\) and its inverse \([V]^{-1}\). We now return to our original expression, constructed in Chapter 3, for the limiting value of the normal vector at \(u = v = 0\) on edge \(j\), patch 1, as our progressively smaller patches approach the vertex point.

We recall that this was

\[
\lim_{n\to\infty} \left\{ [D^{u}]^{n}[R_{j}] \times [D^{v}]^{n}[R_{j}] \right\}
\]

where

\[
[D^{u}] = \frac{1}{12} [-4, 0, 0, -1, 0 \ldots 0, -1, 0 1, 4, 1]
\]

\[
[D^{v}] = \frac{1}{12} [0, 0, 4, 1, 0 \ldots 0, -1, -4 -1, 0, 1]
\]

In the general case, where \(\lambda_{a} \neq \lambda_{b}\), we can diagonalize \([T]\)

i.e.

\[
[T] = [V][\Lambda][V]^{-1}
\]

\[
[T]^{n} = [V][\Lambda]^{n}[V]^{-1}
\]

\[
[D^{X}]^{n}[R_{j}] = [D^{X}]^{n}[V][\Lambda]^{n}[V]^{-1}[R_{j}]
\]

\[
= [D^{X}]^{n} [v_{0}(T), v_{a}(T), v_{b}(T), v_{1}(T), v_{1}(T), \ldots]
\]

\[
\left[\begin{array}{c}
\frac{u_{1}}{16} (T), \frac{v_{1}}{16} (T) \\
\frac{v_{1}}{32} (T)
\end{array} \right] \times \left[\begin{array}{c}
v_{1}(T), v_{1}(T), v_{1}(T)
\end{array} \right]
\]
in the notation of the previous chapter, i.e. \( \tilde{v}_k(T) \) represents the extension of the eigenvector \( \tilde{v}_k \) of \( [T_1] \) to \( [T] \).

If \( [D^{x_1}] = \{d^{x_1}_i\} \) we note

\[
\sum_{i=1}^{2N+4} d^{x_1}_i = 0.
\]

Hence

\[
[D^{x_1}] \tilde{v}_0(T) = 0.
\]

Thus (6.1) gives

\[
[D^{x_1}][0, \tilde{v}_a(T), \tilde{v}_b(T), u_1(T), \tilde{v}_1(T), \ldots] = [v]^{-1}[R_j] (6.2)
\]
Thus the effect of the eigenvalue of 1 is eliminated.

Let us assume that $\lambda_1 > \lambda_a > \lambda_b$.

To evaluate the direction of (6.2) in the limit we must scale at each stage by $\lambda_1^{-1}$.

The limit then becomes

$$[D^x^1][0, v_a(T), v_b(T), u_1(T), v_1(T), \ldots, \ldots]$$

$$= [D^x^1] u_1(T), [D^x^1] v_1(T), [r_4] [R_j], [r_5]$$

where $r_i$ represents the $i^{th}$ row of $[V]^{-1}$

$$= [D^x^1] u_1(T) r_4[R_j] + [D^x^1] v_1(T) r_5[R_j]$$

Now from (5.31) and (5.32) $r_4[R_j]$

$$= \frac{N}{\sum_{i=1}^{\infty} \frac{1}{2N(\lambda_1 - \lambda_i)} c_i E_{j+i}}$$

$$+ \frac{N}{\sum_{i=1}^{\infty} \frac{1}{8N(4\lambda_1 - 1)(\lambda_i - \lambda_1)} (c_i + c_{i+1}) E_{j+i}}$$
\[
= \frac{1}{2N(\lambda'_1 - \lambda'_1)} \left\{ \sum_{i=1}^{N} c_i \frac{E_{j+i}}{i+j} + \frac{1}{4} \left(4\lambda_1 - 1\right)^{-1} \sum_{i=1}^{N} \left(c_i + c_{i+1}\right) E_{j+i} \right\}
\]

= \frac{C_j}{c_j} \text{ say.}

Similarly from (5.33) and (6.34) \[ R_j \]

\[
= \frac{1}{2N(\lambda'_1 - \lambda'_1)} \left\{ \sum_{i=1}^{N} s_i \frac{E_{j+i}}{i+j} + \frac{1}{4} \left(4\lambda_1 - 1\right)^{-1} \sum_{i=1}^{N} \left(s_i + s_{i+1}\right) E_{j+i} \right\}
\]

= \frac{S_j}{s_j} \text{ say.}

Let \[ C_1 = C_1^N \]

\[ S_1 = S_1^N \]

\[ i.e. \quad \left\{ \begin{array}{l} C_1 = \frac{1}{2N(\lambda'_1 - \lambda'_1)} \left\{ \sum_{i=1}^{N} c_i \frac{E_{i}}{i} + \frac{1}{4} \left(4\lambda_1 - 1\right)^{-1} \sum_{i=1}^{N} \left(c_i + c_{i+1}\right) E_{i} \right\} \\
S_1 = \frac{1}{2N(\lambda'_1 - \lambda'_1)} \left\{ \sum_{i=1}^{N} s_i \frac{E_{i}}{i} + \frac{1}{4} \left(4\lambda_1 - 1\right)^{-1} \sum_{i=1}^{N} \left(s_i + s_{i+1}\right) E_{i} \right\} \end{array} \right. \]

(6.3)

Consider \[ c_j C_1 + s_j S_1 \]

The summations become

\[
\sum_{i=1}^{N} \left(c_j c_i + s_j s_i\right) E_i
\]

\[
= \sum_{i=1}^{N} c_{(i-j)} E_i
\]
\[
\begin{align*}
\sum_{i=1}^{N} \left[ c_i \left( c_i + c_{i+1} \right) + s_i \left( s_i + s_{i+1} \right) \right] F_i \\
= \sum_{i=1}^{N} \left[ c_i c_{i-1} + s_i s_{i-1} + c_i c_{i+1} \right. \\
\left. + s_i s_{i+1} \right] F_i \\
= \sum_{i=1}^{N} \left[ c_{i-j} + c_{i-j+1} \right] F_i \\
= \sum_{i=1}^{N} \left( c_i + c_{i+1} \right) F_{j+i}
\end{align*}
\]

Thus

\[
\frac{c_i}{c_{i-1}} = c_j \frac{c_i}{c_{i-1}} + s_j \frac{s_i}{s_{i-1}}
\]

Similarly \( \frac{s_i}{s_{i-1}} = -s_j \frac{c_i}{c_{i-1}} + c_j \frac{s_i}{s_{i-1}} \)

Let

\[
[D^{x_1}] u_1(T) = a_1^{x_1}
\]

\[
[D^{x_1}] v_1(T) = b_1^{x_1}
\]

Thus, scaling at each stage by \( \lambda_1^{-1} \), the surface normal is given by

\[
\lim_{n \to \infty} \left\{ S_{(n)}^{u} \times S_{(n)}^{v} \right\}
\]

\[
= \left\{ [D^{u_1}] u_1(T) \frac{c_i}{c_{i-1}} + [D^{u_1}] v_1(T) \frac{s_i}{s_{i-1}} \right\}
\]

\[
\times \left\{ [D^{v_1}] u_1(T) \frac{c_i}{c_{i-1}} + [D^{v_1}] v_1(T) \frac{s_i}{s_{i-1}} \right\}
\]

\[
= \left\{ a^u c_j c_1 + s_j s_1 \right\} + \beta^u \left\{ - s_j c_1 + c_j s_1 \right\}
\]
\[
\times \left\{ a^v c_j c_1 + s_j s_1 \right\} + \beta^v \left\{ - s_j c_1 + c_j s_1 \right\}
\]
\[
= \left| \begin{array}{c}
\alpha^u \\
\alpha^v
\end{array} \right| 
\left| \begin{array}{c}
\beta^u \\
\beta^v
\end{array} \right| 
\left\{ [c_j c_1 + s_j s_1] \times [- s_j c_1 + c_j s_1] \right\}
\]
\[
= \left| \begin{array}{c}
\alpha^u \\
\alpha^v
\end{array} \right| \left( c_1 \times s_1 \right)
\]
\[
(6.4)
\]
Now
\[
\mathbf{u}_1(T) = \begin{bmatrix}
0 \\
(4\lambda_1 - 1)c_0 \\
c_0 + c_1 \\
(4\lambda_1 - 1)c_1 \\
c_1 + c_2 \\
(4\lambda_1 - 1)c_2 \\
\vdots \\
\vdots \\
(4\lambda_1 - 1)c_{N-1} \\
c_{N-1} + c_0 \\
\end{bmatrix}
\]
\[
\mathbf{v}_1(T) = \begin{bmatrix}
0 \\
(4\lambda_1 - 1)s_0 \\
s_0 + s_1 \\
(4\lambda_1 - 1)s_1 \\
s_1 + s_2 \\
(4\lambda_1 - 1)s_2 \\
\vdots \\
\vdots \\
(4\lambda_1 - 1)s_{N-1} \\
s_{N-1} + s_0 \\
\end{bmatrix}
\]
Then
\[ \alpha_u^1 = [D^{u1}] \frac{u_1(T)}{(1 - \lambda_1)(4\lambda_1 - 1)(4\lambda_1 + 1)} \frac{3\lambda_1}{3} \]

\[ \beta_u^1 = [D^{u1}] \frac{v_1(T)}{0} \]

\[ \alpha_v^1 = [D^{v1}] \frac{u_1(T)}{0} \]

\[ \beta_v^1 = [D^{v1}] \frac{v_1(T)}{\frac{(32\lambda_1^2 + 24\lambda_1 + 1)}{3(16\lambda_1 - 1)} s_1} \]

We note that the surface normal is independent of \( j \), the edge at which we evaluate it. We may, of course, calculate the limit on any of the patches 1, 2 or 3, and at any values of \( u \) and \( v \). Assuming that \( \lambda_1 \) is the dominant non-unit eigenvalue, it is clear from the above calculations that, scaling by \( \lambda_1^{-1} \) at each subdivision, each of the tangent vectors \( S_u \) and \( S_v \) will have as its limit some linear combination of \( C_j^1 \) and \( S_j^1 \). Thus, taking the vector product, the normal will be a multiple of \( C_j^1 \times S_j^1 = C_1^1 \times S_1^1 \). This vector, furthermore, is independent of \( V \), the vertex point.

From this we conclude that, if \( \lambda_1 > \lambda_a^1 \), the limiting surface will be \( C^1 \) smooth.

Suppose that \( \lambda_1 < \lambda_a^1 \).

Returning to equation (6.2) the limit, scaling by \( \lambda_a^{-1} \) at each stage to obtain a finite non-zero result, becomes

\[
[D^{x1}][0, \frac{v_a(T)}{0}, \frac{v_b(T)}{0}, \frac{u_1(T)}{0}, \frac{v_1(T)}{0}, \ldots \ldots ]
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & \ldots & 0 \\
\end{bmatrix} [V]^{-1}[R_j]
\]
\[ [D^{x_1}] v_a(T) r_2[R_j] \]

where \( r_2 \) is the 2nd row of \([V]^{-1}\)

\[ r_2[R_j] \]

\[ = \frac{1}{16(\lambda_a - \lambda_b)(1-\lambda_a)} \sum_{i=1}^{N} F_i \]

\[ + \frac{\lambda_b}{2N(\lambda_a - \lambda_b)(1-\lambda_a)} \sum_{i=1}^{N} E_i \]

\[ - \frac{(8\lambda_b - 1)}{16N(\lambda_a - \lambda_b)(1-\lambda_b)} \sum_{i=1}^{N} F_i \]

\[ = \frac{1}{16(\lambda_a - \lambda_b)(1-\lambda_b)} \left\{ \frac{1}{N} \sum_{i=1}^{N} E_i - (8\lambda_b - 1) \frac{1}{N} \sum_{i=1}^{N} F_i \right\} \]

\[ = C_0 \text{ say.} \]

Now both \( \lim_{n \to \infty} [u(n)] \) and \( \lim_{n \to \infty} [v(n)] \) are multiples of the same vector \( C_0 \) and so the vector product is 0. We can envisage this as the pinching of the surface into a peak. Thus if \( \lambda_1 < \lambda_a \) the surface is not \( C^1 \) continuous.

We define

\[ a^u_0 = [D^{u_1}] v_a(T) = \frac{2(1-\lambda_a)(16\lambda - 1)(4\lambda_a + 1)}{3(8\lambda_a - 1)} \]

and \( a^v_0 = [D^{v_1}] v_a(T) = 0 \).

If \( \lambda_a = \lambda_1 \) (6.2) will give, with the same scaling

\[ \left\{ a^u_0 C_0 + a^u_1 C_1 + \beta^u_1 S_1 \right\} \times \left\{ a^v_1 C_1 + \beta^v_1 S_1 \right\} \]
As we shall see in Chapter 8, \( \mathcal{C}_0 \) does not in general, lie in the plane spanned by \( \mathcal{C}_1 \) and \( \mathcal{S}_1 \). Thus the expression is dependent on \( j \), and so the surface is not smooth.

We now consider the implications of these results for our choice of \( \mathbf{A} \) and \( \mathbf{B} \).

We recall that \( \lambda_a \) and \( \lambda_b \) are given by the roots of the quadratic

\[
Q_0(\lambda) = \lambda^2 - \left[ A - \frac{1}{4} \right] \lambda + \left[ \frac{1}{16} - \frac{1}{8} B \right]
\]

where \( A, B > 0 \) \( A + B \leq 1 \).

Suppose first that the roots of \( Q_0 \) are real: this is the case iff

\[
(A - \frac{1}{4})^2 \geq 4\left( \frac{1}{16} - \frac{1}{8} B \right)
\]

\[
\Rightarrow B \geq 2\left[ \frac{1}{4} - (A - \frac{1}{4})^2 \right]
\]

This area is shown in Fig. 6.1.

We find the extremal values of \( \lambda \)

\[
\lambda = \frac{1}{2} \left\{ (A - \frac{1}{4}) \pm \left[ (A - \frac{1}{4})^2 - 4\left( \frac{1}{16} - \frac{1}{8} B \right) \right]^{\frac{1}{2}} \right\}
\]
So \[
\frac{\partial \lambda}{\partial A} = \frac{1}{2} \left\{ 1 \pm \left( A - \frac{1}{4} \right) \left[ \left( A - \frac{1}{4} \right)^2 - 4 \left( \frac{1}{16} - \frac{1}{8} B \right) \right]^{-\frac{1}{2}} \right\}
\]

\[
\frac{\partial \lambda}{\partial B} = \pm \frac{1}{8} \left[ \left( A - \frac{1}{4} \right)^2 - 4 \left( \frac{1}{16} - \frac{1}{8} B \right) \right]^{-\frac{1}{2}}
\]

\[
\frac{\partial \lambda}{\partial B} \neq 0
\]

so there are no local maxima or minima.

Consider the boundary of the area where \( \lambda \) is real. We establish whether any local extrema occur here, examining separately the four parts (see Fig. 6.1).

i) When \( B = 0 \).

\[
\lambda = \frac{1}{2} \left\{ \left( A - \frac{1}{4} \right) \pm \left[ \left( A - \frac{1}{4} \right)^2 - \frac{1}{4} \right]^{\frac{1}{2}} \right\}
\]

When \( A = \frac{3}{4} \)

\[
\lambda_a = \lambda = \frac{1}{4}
\]

When \( A = 1 \)

\[
\lambda_a = \frac{3+\sqrt{5}}{8}, \quad \lambda_b = \frac{3-\sqrt{5}}{8}
\]

\[
\frac{d\lambda}{dA} = \frac{1}{2} \left\{ 1 \pm \left( A - \frac{1}{4} \right) \left[ \left( A - \frac{1}{4} \right)^2 - \frac{1}{4} \right]^{-\frac{1}{2}} \right\} = 0
\]

\[
\Rightarrow \left( A - \frac{1}{4} \right) \left[ \left( A - \frac{1}{4} \right)^2 - \frac{1}{4} \right]^{-\frac{1}{2}} = \pm 1
\]

\[
\Rightarrow \left( A - \frac{1}{4} \right)^2 = \left( A - \frac{1}{4} \right)^2 - \frac{1}{4}
\]

which is impossible, so we have no local extrema on this part of the boundary.

ii) When \( A + B = 1 \).

\[
\lambda = \frac{1}{2} \left\{ \left( A - \frac{1}{4} \right) \pm \left( A^2 - A + \frac{5}{16} \right)^{\frac{1}{2}} \right\}
\]
Fig. 6.1 The Parabola \( B = 2\left[\frac{1}{4} - (A - \frac{1}{4})^2\right] \) and the Shaded Area where \( \lambda_a, \lambda_b \) are real, showing the Four Parts of its Boundary.

Fig. 6.2 The Line \( 8\lambda, A + B = (c_1 + 7)\lambda, \) for \( N = 3, 4, \ldots, 8 \)
When $A = 1$
\[ \lambda_a = \frac{3 + \sqrt{5}}{8} \quad \lambda_b = \frac{3 - \sqrt{5}}{8} \]

When $A = 0$
\[ \lambda_a = \frac{-1 + \sqrt{5}}{8} \quad \lambda_b = \frac{-1 + \sqrt{5}}{8} \]
\[ \frac{d\lambda}{dA} = \frac{1}{2} \left\{ 1 \pm (A - 0.5) \left[ A^2 - A + \frac{5}{16} \right]^{-\frac{1}{2}} \right\} = 0 \]

\[ \Rightarrow (A - 0.5) \left[ A^2 - A + \frac{5}{16} \right]^{-\frac{1}{2}} = \pm 1 \]

\[ \Rightarrow (A - 0.5)^2 = (A - 0.5)^2 + \frac{1}{16} \]

which is impossible.

iii) When $A = 0$
\[ \lambda = \frac{1}{2} \left\{ -\frac{1}{4} \pm \left[ \frac{1}{2} B - \frac{3}{16} \right]^{-\frac{1}{2}} \right\} \]

When $B = 1$
\[ \lambda_a = \frac{-1 + \sqrt{5}}{8} \quad \lambda_b = \frac{-1 + \sqrt{5}}{8} \]

When $B = \frac{3}{8}$
\[ \lambda_a = \lambda_b = -\frac{1}{8} \]
\[ \frac{d\lambda}{dB} = \pm \frac{1}{8} \left[ \frac{1}{2} B - \frac{3}{16} \right]^{-\frac{1}{2}} \quad \text{which is never zero.} \]

iv) When $B = 2 \left[ \frac{1}{4} - (A - \frac{1}{4})^2 \right]$
\[ \lambda = \frac{1}{2} (A - \frac{1}{4}) \]

Hence $\lambda_a = \lambda_b$ which increases from $-\frac{1}{8}$ at $A = 0$

to $\frac{1}{4}$ at $A = \frac{3}{4}$. 
The value of the larger root of $Q_0$ has no local extrema either within the area where it is real or on the boundary. The maximum value attained is $\lambda_a = \frac{3+\sqrt{5}}{8} \approx 0.6545$, when $A = 1$, $B = 0$.

Now $\lambda_1 = \frac{1}{16} \left\{ c_1 + 5 + \left[ (c_1 + 1)(c_1 + 9) \right]^{\frac{1}{4}} \right\}$

< $\frac{1}{16} \left\{ 6 + \sqrt{20} \right\}$ = $\frac{3+\sqrt{5}}{8}$

so taking $A = 1$ will produce a non-$C^1$ surface for any value of $N$.

Complex roots will not concern us because, if $\lambda$, $\lambda^*$ are two such roots

$$|\lambda|^2 = |\lambda\lambda^*| = \left| \frac{1}{16} - \frac{1}{8} B \right| \leq \frac{1}{16}$$

so $|\lambda| \leq \frac{1}{4} < \lambda_1$.

In addition it is impossible to have two roots larger than $\lambda_1$ since

if $|\lambda_a| > |\lambda_b| > \frac{1}{4}$

$$|\lambda_a\lambda_b| = \left| \frac{1}{16} - \frac{1}{8} B \right| > \frac{1}{16}$$

i.e. $B < 0$ or $B > 1$.

Problems, therefore, in terms of continuity of tangent plane, only arise where the larger root $\lambda_a$ is real and exceeds or equals $\lambda_1$.

We now formulate this criterion in terms of the subdivision weightings $A$ and $B$ and represent it graphically

$$\lambda_a = \lambda_1$$

$$\Rightarrow (A - \frac{1}{4}) + \left[ (A - \frac{1}{4})^2 - 4 (\frac{1}{16} - \frac{1}{8} B) \right]^{\frac{1}{4}} = 2\lambda_1$$
\[ \left( A - \frac{1}{4} \right)^2 - 4 \left( \frac{1}{16} - \frac{1}{8} B \right) \frac{1}{4} = 2\lambda_1 - \left( A - \frac{1}{4} \right) \]

\[ (A - \frac{1}{4})^2 - 4 \left( \frac{1}{16} - \frac{1}{8} B \right) = 4\lambda_1^2 - 4\lambda_1 \left( A - \frac{1}{4} \right) + \left( A - \frac{1}{4} \right)^2 \]

\[ 4\lambda_1^2 - 4\lambda_1 \left( A - \frac{1}{4} \right) + \frac{1}{4} - \frac{1}{2} B = 0 \]

\[ 4\lambda_1 A + \frac{1}{2} B = 4\lambda_1^2 + \lambda_1 + \frac{1}{4} \]

But \[ 4\lambda_1^2 = \left( \frac{5}{2} + \frac{1}{2} c_1 \right) \lambda_1 - \frac{1}{4} \]

So \[ 4\lambda_1 A + \frac{1}{2} B = \left( \frac{7}{2} + \frac{1}{2} c_1 \right) \lambda_1 \]

\[ 8\lambda_1 A + B = (c_1 + 7)\lambda_1 \]

The area where a \( C^1 \) surface will result, i.e. where \( 8\lambda_1 A + B < (c_1 + 7)\lambda_1 \), is shown in Fig. 6.2 for several values of \( N \).

We are now in a position to answer some of the questions posed by our original problem. We were given an arbitrary topology of nodes connected by edges, and by means of a subdivision algorithm, well-defined in the areas where the configuration is rectangular, we generate a progressively finer mesh of points which converges to a surface. After the first subdivision, each face in the mesh is quadrilateral and, as we proceed, the surface is gradually constructed of bicubic patches. The non-4-nodes become isolated from one another and the remaining area is \( C^2 \) smooth. The only problems arise at the limiting positions of these extraordinary points.
In the subdivision, we redefine an N-node as follows:

\[ \mathbf{v}' = \mathbf{A} \mathbf{v} + \mathbf{B} \sum_{i=1}^{N} \mathbf{e}_i + \mathbf{C} \sum_{i=1}^{N} \mathbf{f}_i \]

where \( A, B, C \geq 0 \), \( A + B + C = 1 \).

Taking into account other points upon which derivatives depend, the subdivision transformation is represented in the foregoing work by a \( 2N + 4 \) matrix. \( 2N + 2 \) eigenvalues of this matrix are fixed and 2 depend on our choice of \( A, B \) and \( C \).

In the preceding chapters we have established:

1) If the modulus of both eigenvalues arising from \( A \) and \( B \) is less than \( \lambda_1 \), the largest non-unit eigenvalue in the natural spectrum, the resulting surface is \( C^1 \) smooth.

2) If one eigenvalue exceeds or equals \( \lambda_1 \), the surface is not \( C^1 \).

These are the only possibilities which may occur. If \( \lambda_a \) and \( \lambda_b \) are complex their modulus is at most \( \frac{1}{4} \), less than \( \lambda_1 \), and the surface is smooth. If \( \lambda_a = \lambda_b \) again the modulus is at most \( \frac{1}{4} \) and the surface is smooth.

The tangent plane, where it exists, is independent of \( \mathbf{v} \), the position of the original \( N \)-node, and symmetric in the edge and face points surrounding it. In addition it is independent of the subdivision weightings.

Limiting surfaces either side of the borderline case, \( \lambda_a = \lambda_1 \), are shown in Figs. 6.3 and 6.4 for 3 and 5 node configurations.
Fig. 6.3 Either Side of the Borderline Case when \( N = 3 \)

Fig. 6.4 Either Side of the Borderline Case when \( N = 5 \)
Chapter 7

The Discrete Fourier Transform Method

As we have seen in the preceding chapters, the direct matrix method enables us to determine sufficient conditions on the subdivision weightings so that $C^1$ continuity of the surface is obtained. The matrix $[T]$ representing the subdivision is, however, of order $2N+4$ and consequently the calculations involved are extremely cumbersome, if only in terms of notation. Doo and Sabin [11] devised a method, analogous to the treatment of time series data in terms of components of different frequencies, which makes the analysis radically more compact and more easily manipulated. This involves the use of the discrete Fourier transform to exploit the cyclic symmetry inherent in the topological configuration round an $N$-node. The very compactness of this technique will provide further insights.

7.1 THE TRANSFORM, THE SUBDIVISION AND PARTIAL DERIVATIVES.

As in Lemma 5.1 we let

$$a_\omega = e^{\frac{2\pi i}{N}} = c_\omega + is_\omega$$

$$a^*_\omega = e^{-\frac{2\pi i}{N}} = c_\omega - is_\omega$$

Consider the set of points

$$\{v\} \cup \{E_j, F_j, I_{j1}, I_{j2}, I_{j3}, I_{j4}\}_{j=1,2,\ldots,N}$$
Corresponding to \( E_j, F_j \) and \( I_{jk} \), we introduce the complex vectors \( e_{-\omega}, f_{-\omega} \) and \( i_{-\omega k} \) by means of the discrete Fourier transform.

We have the following definitions:

\[
\begin{align*}
e_{-\omega} &= \frac{1}{N} \sum_{j=1}^{N} a^* \omega_j \ E_j \quad \Rightarrow \quad E_j = \sum_{\omega=0}^{N-1} a_{\omega j} e_{-\omega} \\
f_{-\omega} &= \frac{1}{N} \sum_{j=1}^{N} a^* \omega_j \ F_j \quad \Rightarrow \quad F_j = \sum_{\omega=0}^{N-1} a_{\omega j} f_{-\omega} \\
i_{-\omega k} &= \frac{1}{N} \sum_{j=1}^{N} a^* \omega_j \ I_{jk} \quad \Rightarrow \quad I_{jk} = \sum_{\omega=0}^{N-1} a_{\omega j} i_{-\omega k}
\end{align*}
\]

\[(7.1)\]

for \( \omega = 0, 1, \ldots, N-1 \)

\( j = 1, 2, \ldots, N \)

Consistently, and reflecting the invariance of \( V \) under rotation

\[ v_{-\omega} = V, \quad v_{\omega} = 0 \text{ otherwise} \]

We now consider, instead of the overall transformation of the \( E_j \)'s, \( F_j \)'s etc., the transformation at each frequency \( \omega = 0, 1, \ldots, N-1 \) of the \( e_{-\omega} \)'s, \( f_{-\omega} \)'s etc.

Rewriting the equations of (3.3) we have

\[
\begin{align*}
v' &= A \ nu + B \frac{1}{N} \sum_{i=1}^{N} E_{-i} + C \frac{1}{N} \sum_{i=1}^{N} F_{-i} \\

\Rightarrow \quad v'_{-0} &= A \ v_{-0} + B \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{\omega=0}^{N-1} a_{\omega i} e_{-\omega} \right) + C \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{\omega=0}^{N-1} a_{\omega i} f_{-\omega} \right)
\end{align*}
\]

\[
= A \ v_{-0} + B \frac{1}{N} \sum_{\omega=0}^{N-1} \left( \sum_{i=1}^{N} a_{\omega i} \right) e_{-\omega} + C \frac{1}{N} \sum_{\omega=0}^{N-1} \left( \sum_{i=1}^{N} a_{\omega i} \right) f_{-\omega}
\]
\[ \Rightarrow v'_0 = A v_0 + B e_0 + C f_0 \]  

(7.2)

by the proof of Lemma 5.1.

\[ E'_j = \frac{3}{8} (v + E_j) + \frac{1}{16} (E_{j-1} + F_{j-1} + F_j + E_{j+1}) \]

\[ \Rightarrow \sum_{\omega=0}^{N-1} a_{\omega j} e'_\omega = \frac{3}{8} (v_0 + \sum_{\omega=0}^{N-1} a_{\omega j} e_\omega) \]

\[ + \frac{1}{16} \left( \sum_{\omega=0}^{N-1} a_{\omega(j-1)} e_\omega + \sum_{\omega=0}^{N-1} a_{\omega(j-1)} f_\omega + \sum_{\omega=0}^{N-1} a_{\omega j} f_\omega + \sum_{\omega=0}^{N-1} a_{\omega(j+1)} e_\omega \right) \]

\[ = \frac{3}{8} v_0 + \sum_{\omega=0}^{N-1} a_{\omega j} \left[ \frac{3}{8} e_\omega + \frac{1}{16} a_{\omega j} e_\omega + \frac{1}{16} a_{\omega j} f_\omega + \frac{1}{16} f_\omega + \frac{1}{16} a_{\omega j} e_\omega \right] \]

\[ = \frac{3}{8} v_0 + \sum_{\omega=0}^{N-1} a_{\omega j} \left[ \frac{1}{8} (3+c_{\omega j}) e_\omega + \frac{1}{16} (1+a_{\omega j}^*) f_\omega \right] \]

Separating frequencies we have

\[ e'_0 = \frac{3}{8} v_0 + \frac{1}{2} e_0 + \frac{1}{8} f_0 \]

\[ e'_\omega = \frac{1}{8} (3+c_{\omega j}) e_\omega + \frac{1}{16} (1+a_{\omega j}^*) f_\omega \]

(7.3)

for \( \omega = 1, 2, \ldots, N-1 \)

\[ F'_j = \frac{1}{4} (v + E_j + F_j + E_{j+1}) \]

\[ \Rightarrow \sum_{\omega=0}^{N-1} a_{\omega j} \hat{f}'_\omega = \frac{1}{4} (v_0 + \sum_{\omega=0}^{N-1} a_{\omega j} e_\omega + \sum_{\omega=0}^{N-1} a_{\omega j} f_\omega + \sum_{\omega=0}^{N-1} a_{\omega(j+1)} e_\omega) \]
Thus
\[
F^0 = \frac{1}{4} v_0 + \frac{1}{2} e_0 + \frac{1}{4} f_0
\]
\[
F^\omega = \frac{1}{4} (1+a_\omega) e_\omega + \frac{1}{4} f_\omega
\]

Similarly
\[
I^1_{j1} = \frac{3}{8} (F_{j-1} + E_j) + \frac{1}{16} (V + E_{j-1} + I_{j1} + I_{j2})
\]
\[
I^1_{j2} = \frac{9}{16} E_j + \frac{3}{32} (V + F_{j-1} + F_j + I_{j2}) + \frac{1}{64} (E_{j-1} + I_{j1} + I_{j3} + E_{j+1})
\]
\[
I^1_{j3} = \frac{3}{8} (E_j + F_j) + \frac{1}{16} (V + E_{j+1} + I_{j2} + I_{j3})
\]
\[
\begin{align*}
\begin{cases}
i_{03}^r &= \frac{1}{16} v_0 + \frac{7}{16} e_0 + \frac{3}{8} f_0 + \frac{1}{16} i_{02} + \frac{1}{16} i_{03} \\
i_{03}^\omega &= \frac{1}{16} (6+\alpha_w) e_\omega + \frac{3}{8} f_\omega + \frac{1}{16} i_{\omega 2} + \frac{1}{16} i_{\omega 3}
\end{cases}
\end{align*}
\]
(7.7)

Also
\[
I_{-j4}^r = \frac{9}{16} f_{j-1} + \frac{3}{32} (E_j + E_{j+1} + I_{-j3} + I_{-j+1}) + \frac{1}{64} (v_{j-2} + i_{-j2} + I_{-j4} + I_{-j+2})
\]
(7.8)

In matrix notation we have:

\[
\begin{bmatrix}
v_0^r \\
e_0^r \\
f_0^r \\
i_{01}^r \\
i_{02}^r \\
i_{03}^r \\
i_{04}^r
\end{bmatrix} =
\begin{bmatrix}
A & B & C & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_0^r \\
e_0^r \\
f_0^r \\
i_{01}^r \\
i_{02}^r \\
i_{03}^r \\
i_{04}^r
\end{bmatrix}
\]

i.e. \([R_0^r] = [T_0][R_0]\)
and for $\omega = 1, 2, \ldots, N-1$

\[
\begin{bmatrix}
e_{\omega}^t \\
f_{\omega}^t \\
i_{\omega1}^t \\
i_{\omega2}^t \\
i_{\omega3}^t \\
i_{\omega4}^t
\end{bmatrix} = \begin{bmatrix}
\frac{1}{8}(3+c_{\omega}) & \frac{1}{16}(1+a_{\omega}) & 0 & 0 & 0 & 0 \\
\frac{1}{4}(1+a_{\omega}) & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\frac{1}{16}(6+a_{\omega}) & \frac{3}{8} a_{\omega} & \frac{1}{16} & \frac{1}{16} & 0 & 0 \\
\frac{1}{32}(18+c_{\omega}) & \frac{3}{32}(1+a_{\omega}) & \frac{1}{64} & \frac{3}{64} & \frac{1}{64} & 0 \\
\frac{1}{16}(6+a_{\omega}) & \frac{3}{8} & 0 & \frac{1}{16} & \frac{1}{16} & 0 \\
\frac{3}{32}(1+a_{\omega}) & \frac{9}{16} & \frac{3}{32} a_{\omega} & \frac{1}{64}(1+a_{\omega}) & \frac{3}{32} & \frac{1}{64}
\end{bmatrix} \begin{bmatrix}
e_{\omega} \\
f_{\omega} \\
i_{\omega1} \\
i_{\omega2} \\
i_{\omega3} \\
i_{\omega4}
\end{bmatrix}
\]

i.e. $[R_{\omega}^t] = [T_{\omega}][R_{\omega}]$

We note here that $[R_{N-\omega}^t] = [R_{\omega}]^*$

and $[T_{N-\omega}^t] = [T_{\omega}]^*$

Representing the subdivision in this more compact form, we investigate the characteristics of the limiting surface. To do this we must calculate the partial derivatives.

As Fig. 3.2 illustrates, between $E_j$ and $E_{j+1}$, i.e. on the $j^{th}$ face, there are three patches for each subdivision.

The control points for patches 1, 2 and 3 are

\[
\begin{bmatrix}
E_{j-1} & V & E_{j+1} & I_{j+12} \\
F_{j-1} & E_j & F_j & I_{j+11} \\
I_{j-1} & I_{j2} & I_{j3} & I_{j4} \\
0_{j2} & 0_{j3} & 0_{j4} & 0_{j5}
\end{bmatrix}, \quad \begin{bmatrix}
V & E_{j+1} & I_{j+12} & 0_{j+13} \\
E_j & F_j & I_{j+11} & 0_{j+12} \\
I_{j2} & I_{j3} & I_{j4} & 0_{j+11} \\
0_{j3} & 0_{j4} & 0_{j5} & 0_{j6}
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
E_{j+2} & E_{j+1} & I_{j+13} & 0_{j+14} \\
V & E_{j+1} & I_{j+12} & 0_{j+13} \\
E_{j} & E_{j} & I_{j+11} & 0_{j+12} \\
I_{j2} & I_{j3} & I_{j4} & 0_{j+11}
\end{bmatrix}
\]

respectively.

We denote the partial derivatives on face \(j\), patch \(i\), (with respect to \(x(=u,v,uu,uv,vv)\) at the parametric values \(u = v = 0\) by \(S^{xji}\). This is illustrated in Fig. 7.1.

For patch 1

a) \(S^{uj1} = \frac{1}{12} \left\{ -4 V - E_{j-1} - E_{j+1} + I_{j1} + 4 I_{j2} + I_{j3} \right\} \)

\[
= \frac{1}{12} \left\{ -4 V - \sum_{\omega=0}^{N-1} a_{\omega(j-1)} e_{\omega} - \sum_{\omega=0}^{N-1} a_{\omega(j+1)} e_{\omega} + \sum_{\omega=0}^{N-1} a_{\omega j} i_{\omega 1} + 4 \sum_{\omega=0}^{N-1} a_{\omega j} i_{\omega 2} + \sum_{\omega=0}^{N-1} a_{\omega j} i_{\omega 3} \right\}
\]

\[
= \frac{1}{12} \left\{ -4 V + \sum_{\omega=0}^{N-1} a_{\omega j} \left[ -2 c_{\omega} e_{\omega} + i_{\omega 1} + 4 i_{\omega 2} + i_{\omega 3} \right] \right\}
\]

\[
= \frac{1}{12} [-4, -2, 0, 1, 4, 1, 0][R_0]
\]

\[
+ \sum_{\omega=1}^{N-1} \frac{1}{12} a_{\omega j} [-2 c_{\omega}, 0, 1, 4, 1, 0][R_{\omega}]
\]

\[
= \sum_{\omega=0}^{N-1} a_{\omega j} [D^{u1}_\omega][R_{\omega}]
\]

say

(7.9)
Fig. 7.1 The First-Order Partial Derivatives on the $j^{th}$ Face
where \[ D_{0}^{u1} = \frac{1}{12} [-4, -2, 0, 1, 4, 1, 0] \]
\[ D_{0}^{u1} = \frac{1}{12} [-2c_{0}, 0, 1, 4, 1, 0] \quad (\omega \neq 0) \]

Similarly
\[ s_{vj}^{1} = \frac{1}{12} \left\{ -E_{j-1} + E_{j+1} - 4E_{j-1} - 4E_{j} - I_{j1} + I_{j3} \right\} \]
\[ = \sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{v1}][R_{\omega}] \quad (7.10) \]

where \[ D_{0}^{v1} = \frac{1}{12} [0, 0, 0, -1, 0, 1, 0] \]
\[ D_{0}^{v1} = \frac{1}{12} [(a_{\omega} - a_{\omega}^{*}), 4(1 - a_{\omega}^{*}), -1, 0, 1, 0] \quad (\omega \neq 0) \]

\[ s_{uj}^{1} = \frac{1}{6} \left\{ 4V + E_{j-1} - 8E_{j} + E_{j+1} - 2F_{j-1} - 2F_{j} + I_{j1} + 4I_{j2} + I_{j3} \right\} \]
\[ = \sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{u1}][R_{\omega}] \quad (7.11) \]

where \[ D_{0}^{u1} = \frac{1}{6} [4, -6, -4, 1, 4, 1, 0] \]
\[ D_{0}^{u1} = \frac{1}{6} [2c_{0} - 8, -2(1 + a_{\omega}^{*}), 1, 4, 1, 0] \quad (\omega \neq 0) \]

\[ s_{uvj}^{1} = \frac{1}{4} \left\{ E_{j-1} - E_{j+1} - I_{j1} + I_{j3} \right\} \]
\[ = \sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{uv1}][R_{\omega}] \quad (7.12) \]
where \[ [D_{uv1}^0] = \frac{1}{4} [0, 0, 0, -1, 0, 1, 0] \]

\[ [D_{uv1}^w] = \frac{1}{4} [(a_w^* - a_w), 0, -1, 0, 1, 0] \quad (w \neq 0) \]

e) \[ S_{vv}^{v1} = \frac{1}{6} \left\{ -2V + E_{j-1} - 8E_j + E_{j+1} + 4F_{j-1} + 4F_j + I_{j+1} - 2I_{j+2} + I_{j+3} \right\} \]

\[ = \sum_{w=0}^{N-1} a_{w,j} [D_{vv1}^w][R_w] \] (7.13)

where \[ [D_{vv1}^0] = \frac{1}{6} [-2, -6, 8, 1, -2, 1, 0] \]

\[ [D_{vv1}^w] = \frac{1}{6} [2c_w - 8, 4(1 + a_w^*), 1, -2, 1, 0] \quad (w \neq 0) \]

For patch 2

a) \[ S_{u}^{u2} = \frac{1}{12} \left\{ -V - 4E_{j+1} + I_{j+1} - I_{j+2} + 4I_{j+3} + I_{j+4} \right\} \]

\[ = \sum_{w=0}^{N-1} a_{w,j} [D_{u2}^w][R_w] \] (7.14)

where \[ [D_{u2}^0] = \frac{1}{12} [-1, -4, 0, 0, 0, 4, 1] \]

\[ [D_{u2}^w] = \frac{1}{12} [-4a_w, 0, 0, 1-a_w, 4, 1] \quad (w \neq 0) \]

b) \[ S_{v}^{v2} = \frac{1}{12} \left\{ -V - 4E_j + 4I_{j+1} - I_{j+2} + I_{j+11} + I_{j+12} + I_{j+14} \right\} \]

\[ = \sum_{w=0}^{N-1} a_{w,j} [D_{v2}^w][R_w] \] (7.15)
where  \[ D_{v0}^2 = \frac{1}{12} [-1, -4, 0, 4, 0, 0, 1] \]

\[ D_{v0}^2 = \frac{1}{12} [-4, 0, 4a_\omega, a_\omega - 1, 0, 1] \]

c) \[ S_{uvj2} = \frac{1}{6} \left\{ v - 2E_j + 4E_{j+1} - 8E_j - 2I_j + I_{j+1} + I_{j+2} + 4I_{j+3} + I_{j+4} \right\} \]

\[ = \sum_{\omega=0}^{N-1} a_{\omega j} \left[ D_{v2}^{uu} \left[ R_\omega \right] \right] \quad (7.16) \]

where  \[ D_{v2}^{uu} = \frac{1}{6} [1, 2, -8, -2, 2, 4, 1] \]

\[ D_{v2}^{uu} = \frac{1}{6} [4a_\omega - 2, -8, -2a_\omega, 1 + a_\omega, 4, 1] \quad (\omega \neq 0) \]

d) \[ S_{uvj2} = \frac{1}{4} \left\{ v - I_{j+2} - I_{j+12} + I_{j+4} \right\} \]

\[ = \sum_{\omega=0}^{N-1} a_{\omega j} \left[ D_{v2}^{uv} \left[ R_\omega \right] \right] \quad (7.17) \]

where  \[ D_{v2}^{uv} = \frac{1}{4} [1, 0, 0, 0, -2, 0, 1] \]

\[ D_{v2}^{uv} = \frac{1}{4} [0, 0, 0, -(1 + a_\omega), 0, 1] \quad (\omega \neq 0) \]

e) \[ S_{vvj2} = \frac{1}{6} \left\{ v + 4E_j - 2E_{j+1} - 8E_j + 4I_{j+11} + I_{j+2} + I_{j+12} - 2I_{j+3} + I_{j+4} \right\} \]

\[ = \sum_{\omega=0}^{N-1} a_{\omega j} \left[ D_{v2}^{vv} \left[ R_\omega \right] \right] \quad (7.18) \]
where

$$[D_{0}^{V2}] = \frac{1}{6} [1, 2, -8, 4, 2, -2, 1]$$

$$[D_{\omega}^{V2}] = \frac{1}{6} [4 - 2a_{\omega}, -8, 4a_{\omega}, 1 + a_{\omega}, -2, 1] \quad (\omega \neq 0)$$

We note $$[D_{N-\omega}^{x_i}] = [D_{\omega}^{x_i}]^*$$.

As we observed in Chapter 3, patch 3 is distinguished from patch 1 only by the cyclic orientation of j, so it is not necessary to evaluate derivatives here.

7.2 THE EIGENPROPERTIES OF $$[T_{\omega}]$$.

We have now separated the components of the subdivision transformation and the partial derivatives; this format will render more simple the task of establishing the conditions for tangent plane continuity. We are interested in the values of the partial derivatives on the limiting surface so we must consider

$$\lim_{n \to \infty} S^{x_{ji}(n)}$$

where n denotes the number of subdivisions performed.

$$\sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{x_i}][R_{\omega}^{(n)}]$$

and $$[R_{\omega}^{(n)}] = [T_{\omega}]^{n}[R_{\omega}]$$

so it will be fruitful to examine the eigenproperties of $$[T_{\omega}]$$.

We first consider $$[T_{0}]$$.

This has characteristic equation

$$(1-\lambda)\left[\lambda^{2} - (A - \frac{1}{4})\lambda + \left(\frac{1}{16} - \frac{1}{8} B\right)\right] = 0$$

$$\lambda = \frac{1}{16} \frac{1}{32} - \lambda$$

$$\frac{1}{64} - \lambda$$
with roots
\[ \lambda_a, \lambda_b, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64} \]  \hspace{1cm} (7.20)

The matrix of eigenvectors is

\[
[v_0] = \begin{bmatrix}
\frac{1}{16}v_0^a & \frac{1}{32}v_0^b & \frac{1}{64}v_0^c & v_0^d & v_0^e & v_0^f
\end{bmatrix}
\]

where
\[ v_0^1 = [1, 1, 1, 1, 1, 1]^T \]
\[ \frac{1}{8}v_0^1 = [0, 0, 0, 1, 1, 1, 2]^T \]
\[ \frac{1}{16}v_0^2 = [0, 0, 0, 1, 0, -1, 0]^T \]
\[ \frac{1}{32}v_0^3 = [0, 0, 0, 2, -1, 2, 22]^T \]
\[ \frac{1}{64}v_0^4 = [0, 0, 0, 0, 0, 0, 1]^T \]

and
\[
v_0^\gamma = \begin{bmatrix}
16\lambda^2 - 12\lambda + 1 \\
6\lambda - 1 \\
4\lambda + 1 \\
\frac{256\lambda^3 + 864\lambda^2 - 30\lambda - 5}{(8\lambda - 1)(32\lambda - 1)} \\
\frac{384\lambda^3 + 800\lambda^2 - 104\lambda + 5}{(8\lambda - 1)(32\lambda - 1)} \\
\frac{256\lambda^3 + 864\lambda^2 - 30\lambda - 5}{(8\lambda - 1)(32\lambda - 1)} \\
\frac{4096\lambda^4 + 55424\lambda^3 + 10224\lambda^2 - 1364\lambda - 25}{(8\lambda - 1)(32\lambda - 1)(64\lambda - 1)}
\end{bmatrix}
\]
where $\gamma = a, b$

$\lambda = \lambda_a, \lambda_b$

We note the similarity to the eigenvectors of $[T]$ in Chapter 5.

$$[v_0]^{-1} = \begin{bmatrix}
M_0 & 0 \\
-\frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
-\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} & 0 \\
-4 & 6 & -4 & 1 \\
\end{bmatrix}$$ (7.21)

where $[M_0] =$

$$\begin{bmatrix}
\frac{5}{16(1-\lambda_a)(1-\lambda_b)} & \frac{1-(\lambda_a+\lambda_b)-4\lambda_a\lambda_b}{2(1-\lambda_a)(1-\lambda_b)} & \frac{3-8(\lambda_a+\lambda_b)+48\lambda_a\lambda_b}{16(1-\lambda_a)(1-\lambda_b)} \\
\frac{-1}{16(\lambda_a-\lambda_b)(1-\lambda_a)} & \frac{\lambda_a}{2(\lambda_a-\lambda_b)(1-\lambda_a)} & \frac{-(8\lambda_a-1)}{16(\lambda_a-\lambda_b)(1-\lambda_a)} \\
\frac{-1}{16(\lambda_b-\lambda_a)(1-\lambda_b)} & \frac{\lambda_a}{2(\lambda_b-\lambda_a)(1-\lambda_b)} & \frac{-(8\lambda_a-1)}{16(\lambda_b-\lambda_a)(1-\lambda_b)} \\
\end{bmatrix}$$

Thus $[T_0] = [v_0][A_0][v_0]^{-1}$
where  
\[ \Lambda_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{64} \end{bmatrix} \]

For \( \omega = 1, 2, \ldots, N-1 \), \( [T_{\omega}] \) has characteristic equation

\[ Q_k(\lambda) = \left( \frac{1}{8} - \lambda \right) \left( \frac{1}{16} - \lambda \right) \left( \frac{1}{32} - \lambda \right) \left( \frac{1}{64} - \lambda \right) \]

where  
\[ Q_k(\lambda) = \lambda^2 - \left( \frac{5}{8} + \frac{1}{8} c_{\omega} \right) \lambda + \frac{1}{16} \] as in Chapter 4.  \( (7.22) \)

The roots are

\[ \lambda_{\omega}, \lambda_{\omega}^* = \frac{1}{16} \left\{ (c_{\omega} + 5) \pm \left[ (c_{\omega} + 1)(c_{\omega} + 9) \right]^{1/2} \right\} \]

\[ \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64} \].

The matrix of eigenvectors is

\[ [V_{\omega}] = \begin{bmatrix} v_{\omega} & v_{\omega}' & v_{\omega} \frac{1}{8} & v_{\omega} \frac{1}{16} & v_{\omega} \frac{1}{32} & v_{\omega} \frac{1}{64} \end{bmatrix} \]

where  
\[ \frac{1}{8} v_{\omega} = [0, 0, 1, 1, 1, 1 + a_{\omega}]^t \]

\[ \frac{1}{16} v_{\omega} = [0, 0, 1, 0, -1, 2(a_{\omega} - 1)]^t \]

\[ \frac{1}{32} v_{\omega} = [0, 0, 2, -1, 2, 11(1 + a_{\omega})]^t \]

\[ \frac{1}{64} v_{\omega} = [0, 0, 0, 0, 0, 1]^t \]
and \[ v_\omega = \begin{bmatrix} 4\lambda - 1 \\ 1 + a_\omega \\ \frac{2(400\lambda^2 + 36\lambda - 13)}{(16\lambda - 1)(32\lambda - 1)} + \frac{4\lambda + 5}{16\lambda - 1} a_\omega \\ \frac{2(4\lambda - 1)(4\lambda + 13)}{(32\lambda - 1)} \\ \frac{2(400\lambda^2 + 36\lambda - 13)}{(16\lambda - 1)(32\lambda - 1)} + \frac{4\lambda + 5}{16\lambda - 1} a_\omega \\ \frac{10(1280\lambda^3 + 2128\lambda^2 - 56\lambda - 13)}{(16\lambda - 1)(32\lambda - 1)(64\lambda - 1)} (1 + a_\omega) \end{bmatrix} \]

where \( \lambda = \lambda_\omega \)

\( v_\omega' \) is given by a similar vector, \( \lambda_\omega \) being changed to \( \lambda_\omega' \).

\[
[V_\omega]^{-1} = \begin{bmatrix}
M_\omega & 0 \\
\cdots & \cdots \\
L_\omega & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-(1 + 3a_\omega) & 3(1 + a_\omega) & -(3 + a_\omega) & 1 \\
\end{bmatrix}
\]

where \[
[M_\omega] = \frac{1}{4(\lambda_\omega - \lambda_\omega')} \begin{bmatrix} 1 & -\frac{4\lambda_\omega'}{1 + a_\omega} \\ -\frac{4\lambda_\omega}{1 + a_\omega} & 1 \end{bmatrix} \]

(7.23)

Thus \( [T_\omega] = [V_\omega][\Lambda_\omega][V_\omega]^{-1} \)
where 

\[
\begin{bmatrix}
\lambda & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 1/8 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/16 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/32 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/64
\end{bmatrix}
\]

We note: 

\[
[V_{N-\omega}] = [V_{\omega}]^* 
\]

and 

\[
[V_{N-\omega}]^{-1} = \{[V_{\omega}]^{-1}\}^* 
\]

For \( \omega = 0, 1, \ldots, N-1 \)

\[
[T_{\omega}]^n = [V_{\omega}][\Lambda_{\omega}]^n[V_{\omega}]^{-1} 
\]

At this point, we must register the possibility that complications may, in certain circumstances, occur. Firstly, with regard to the diagonalization of \([T_0]\), we may have \(\lambda_a = \lambda_b\) or \(\lambda_a, \lambda_b\) complex.

Our immediate concern, however, is \(C^1\) continuity and since neither case can jeopardize this, the maximum modulus being \(1/4\), as we showed in Chapter 6, we may put this to one side for the moment. The use of Jordan canonical form [7, p.333] will, in any case, obviate the need for a separate approach when \(\lambda_a = \lambda_b\).

Secondly if \(\omega = \frac{N}{2}\) we have \(c_{\omega} = \cos \pi = -1\) and \(\lambda_{\omega} = \lambda_{\omega}' = \frac{1}{4}\). In this case
\[ [V_\omega] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 1 & 2 \\ 4 & 0 & 1 & 0 & -1 \\ 3 & 2 & 1 & -1 & 2 \\ 0 & 4 & 0 & -4 & 0 \\ 1 \end{bmatrix} \]

\[ [V_\omega]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{11}{3} & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 2 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ 0 & 4 & 2 & 0 & -2 \end{bmatrix} \] (7.24)

In considerations of \( C^1 \) continuity this will play no part but, when we come to investigate curvature, this will be relevant when \( N = 4 \).

7.3 TANGENT PLANE CONTINUITY.

We now evaluate the limits of the partial derivatives under the subdivision. We recall from (7.19) that

\[
\lim_{n \to \infty} \sum_{j=1}^{N} \omega j \frac{\partial x_i(n)}{\partial \omega} = \lim_{n \to \infty} \sum_{\omega=0}^{N-1} \omega_j [D^\omega x_i[R_\omega]]
\]

\[
= \lim_{n \to \infty} \sum_{\omega=0}^{N-1} \omega_j [D^\omega x_i[R_\omega]]
\]
\[ (7.25) \]

Since 
\[ a_{(N-\omega)j} [D_{N-\omega}^{x_i}](V_{N-\omega})[\Lambda_{N-\omega}]^{n}[V_{N-\omega}]^{-1}[R_{N-\omega}] \]

\[ = \{ a_{\omega j} [D_{\omega}^{x_i}](V_{\omega})[\Lambda_{\omega}]^{n}[V_{\omega}]^{-1}[R_{\omega}] \}^* \]

(7.25) gives
\[
\lim_{n \to \infty} \left\{ [D_0^{x_i}](V_0)[\Lambda_0]^{n}[V_0]^{-1}[R_0] \right. \\
+ 2 \sum_{\omega=1}^{\frac{N-1}{2}} \Re\left\{ a_{\omega j} [D_{\omega}^{x_i}](V_{\omega})[\Lambda_{\omega}]^{n}[V_{\omega}]^{-1}[R_{\omega}] \right\} \\
\left. + (-1)^j [D_{N}^{x_i}](V_{N})[\Lambda_{N}]^{n}[V_{N}]^{-1}[R_{N}] \quad \text{if } N \text{ even} \right\}. \quad (7.26)
\]

Now as \( n \to \infty \) the patches whose partial derivatives we are considering tend in size to zero and so, therefore, do these partial derivatives. Consequently, to evaluate the limit direction, we must incorporate some scaling factor at each stage. This is borne out by the following:

Consider \([D_0^{x_i}](V_0)[\Lambda_0]^{n}\).

All eigenvalues are strictly less than unity except in the case \( \omega = 0 \). Here the entry in \([D_0^{x_i}](V_0)\) corresponding to the eigenvalue 1 is \([D_0^{x_i}](V_0)^{1}\). We note from (7.9) to (7.18) that the entries in \([D_0^{x_i}]\) sum to zero and \(v_0^{1} = [1, 1, 1, 1, 1, 1, 1]^T\). Hence \([D_0^{x_i}] v_0^{1} = 0\). The effect of the eigenvalue 1 is lost,
therefore, and we must consider the next largest which is either
\( \lambda_1 \) or \( \lambda_a \).

Suppose \( \lambda_1 \) is the larger. So that our answer is non-zero we must scale by \( \lambda_1^{-1} \) at each subdivision. It is clear that

\[
[A_w / \lambda_1]^n \to [0] \quad \text{for } 1 < w \leq \frac{N}{2}
\]

and \([D^x_0][V_0][A_0/\lambda_1]^n \to [0]\)

\([A_1/\lambda_1]^n \to [H]\)

where \( h_{11} = 1 \)

\( h_{ij} = 0 \quad \text{otherwise.} \)

Returning to (7.26), dividing by \( \lambda_1^n \) throughout, we obtain

\[
2\Re \left\{ a_j [D^x_1][V_1][H][V_1]^{-1}[R_1]\right\}
\]

(7.27)

for the limit direction of the partial derivative.

Now in the first place we are interested in the \( C^1 \) properties at the limit point, so we take

\([D^x_1] = [D^u_1] \quad \text{and} \quad [D^v_1]\)

i.e. we evaluate the limiting \( u \) and \( v \) derivatives on patch 1.

We introduce the following notation

\[
[D^u_1]_w \frac{v}{\omega} = \frac{(1-\lambda_w)(4\lambda_w-1)(4\lambda_w+1)}{3\lambda_w} = A^u_1
\]

(7.28)

\[
[D^v_1]_w \frac{v}{\omega} = \frac{32\lambda_w^2 + 24\lambda_w + 1}{3(16\lambda_w-1)} = A^v_1
\]

for \( 0 < \omega < \frac{N}{2} \)
Let \( [V_\omega]^{-1} [R_\omega] = [Z_\omega] = [z_{\omega 1}, z_{\omega 2}, \ldots] \).

Then
\[
\lim_{n \to \infty} S_{\omega j}^{11}(n) = 2 \Re \left\{ a_{j} A_{1}^{u1} z_{11} \right\}
\]

\[
\lim_{n \to \infty} S_{\omega j}^{v1}(n) = 2 \Re \left\{ a_{j} A_{1}^{v1} i z_{11} \right\}
\]

Now for \( 1 \leq \omega < \frac{N}{2} \), from (7.23), we have

\[
z_{\omega 1} = \frac{1}{4(\lambda_{\omega} - \lambda'_{\omega})} \left\{ e_{\omega} - \frac{4\lambda'_{\omega} - 1}{1 + a_{\omega}} \sum_{i=1}^{N} a_{\omega i} E_{i} - \frac{4\lambda'_{\omega} - 1}{1 + a_{\omega}} \sum_{i=1}^{N} a_{\omega i} F_{i} \right\}
\]

\[
(7.29)
\]

Now
\[
\frac{4\lambda'_{\omega} - 1}{1 + a_{\omega}} a_{\omega i} = \left( \frac{4\lambda'_{\omega} - 1}{1 + a_{\omega}} \right) \frac{a_{\omega i} + a_{\omega(i+1)}^{*}}{1 + a_{\omega}}
\]

\[
(7.30)
\]

But \((1 + a_{\omega})(1 + a_{\omega}^{*}) = 2(1 + c_{\omega}) = -4(4\lambda_{\omega} - 1)(4\lambda'_{\omega} - 1)\) using (5.14).

Thus from (7.30)

\[
\frac{4\lambda'_{\omega} - 1}{1 + a_{\omega}} a_{\omega i} = -\frac{(4\lambda'_{\omega} - 1)(a_{\omega i} + a_{\omega(i+1)}^{*})}{4(4\lambda_{\omega} - 1)(4\lambda'_{\omega} - 1)} = -\frac{a_{\omega i} + a_{\omega(i+1)}^{*}}{4(4\lambda_{\omega} - 1)}
\]

and from (7.29)

\[
z_{\omega 1} = \frac{1}{4(\lambda_{\omega} - \lambda'_{\omega}) N} \left\{ \sum_{i=1}^{N} a_{\omega i} E_{i} + \left( \frac{R_{\omega} - 1}{4(4\lambda_{\omega} - 1)} \right) \left( \sum_{i=1}^{N} a_{\omega i} + a_{\omega(i+1)}^{*} \right) F_{i} \right\}
\]

\[
= \frac{1}{2} \left\{ C_{\omega} - i S_{\omega} \right\}
\]
where, as in (6.3)

\[
\begin{align*}
    C_\omega &= \frac{1}{2(\lambda - \lambda') N} \left\{ \sum_{i=1}^{N} \omega_i E_i + \frac{1}{4} (4\lambda - 1)^{-1} \sum_{i=1}^{N} (c_{\omega i} + c_{\omega(i+1)}) F_i \right\} \\
    S_\omega &= \frac{1}{2(\lambda - \lambda') N} \left\{ \sum_{i=1}^{N} \omega_i E_i + \frac{1}{4} (4\lambda - 1)^{-1} \sum_{i=1}^{N} (s_{\omega i} + s_{\omega(i+1)}) F_i \right\}
\end{align*}
\]

(7.31)

Returning to our derivatives

\[
\lim_{n \to \infty} \sum_{j=1}^{u} j(n) = 2 \text{Re} \left\{ a_j A^u_1 \bar{z}_{11} \right\}
\]

\[
= \text{Re} \left\{ A^u_1 \left( c_j + i s_j \right) \left( C_1 - i S_1 \right) \right\}
\]

\[
= A^u_1 \left[ c_j C_1 + s_j S_1 \right]
\]

\[
\lim_{n \to \infty} \sum_{j=1}^{v} j(n) = 2 \text{Re} \left\{ a_j A^v_1 i \bar{z}_{11} \right\}
\]

\[
= \text{Re} \left\{ A^v_1 i \left( c_j + i s_j \right) \left( C_1 - i S_1 \right) \right\}
\]

\[
= A^v_1 \left[ -s_j C_1 + c_j S_1 \right]
\]

Defining

\[
\begin{align*}
    C^j_\omega &= \omega_j C_\omega + s_j S_\omega \\
    S^j_\omega &= -s_j C_\omega + c_j S_\omega
\end{align*}
\]

we have

\[
\lim_{n \to \infty} \sum_{j=1}^{u} j(n) \to A^u_1 C^j_1
\]

\[
\lim_{n \to \infty} \sum_{j=1}^{v} j(n) \to A^v_1 S^j_1
\]

(7.33)
Thus the normal vector $\mathbf{n} \times \mathbf{n}$ has limit

$$A_1^{u1} A_1^{v1} (c_j^i \times s_j^i)$$

(7.34)

which, as in Chapter 6, is independent of $j$ since

$$c_j^i \times s_j^i = [c_j^i - c_j^i + s_j^i - s_j^i] \times [-s_j^i c_j^i + c_j^i s_j^i]$$

$$= (c_j^2 + s_j^2) (c_j^i \times s_j^i)$$

$$= c_j^i \times s_j^i.$$

The method used above calculates the limiting derivatives at $u = v = 0$ on patch 1. Naturally we would expect to obtain a normal vector in the same direction as $c_j^i \times s_j^i$ by taking the limit of the vector product of the $u$ and $v$ derivatives at any value of $u$ and $v$ and on any patch, as the entire configuration is shrinking towards the limit point. We confirm this by following through the above mathematics.

We denote the partial derivative with respect to $x(= u, v)$ on face $j$, patch $i$ at $(u, v)$ by $\mathbf{s}^{xj}(u, v)$.

As above we separate into components to give

$$\lim_{n \to \infty} \mathbf{s}^{xj}(u,v) = \lim_{n \to \infty} \sum_{\omega=0}^{N-1} a_{\omega j} [D^x_{\omega}(u,v)](R^{(n)})$$

$$= \lim_{n \to \infty} \sum_{\omega=0}^{N-1} a_{\omega j} [D^x_{\omega}(u,v)][V_{\omega}]^{\omega \eta \omega^{-1}}(R_{\omega})$$

Scaling by $\lambda_1^{-1}$ at each subdivision we have
\[
\lim_{n \to \infty} s^x_{ji}(n)(u,v) = 2 \text{Re} \left\{ a_j[D^x_i(u,v)]_{v_1} z_{11} \right\}
\]

We found above that \([D^u_1(0,0)]_{v_1}\) was real

and \([D^v_1(0,0)]_{v_1}\) imaginary.

Hence

\[
\lim_{n \to \infty} s^u_{ji}(n)(0,0) \text{ was a multiple of } \frac{c_j}{\overline{z}_1}
\]

\[
\lim_{n \to \infty} s^v_{ji}(n)(0,0) \text{ was a multiple of } \frac{s_j}{\overline{z}_1}.
\]

This is not in general the case.

Let \([D^x_i(u,v)]_{v_1} = a^x_i + i \beta^x_i\)

Then

\[
2 \text{Re} \left\{ a_j[D^x_i(u,v)]_{v_1} z_{11} \right\}
\]

\[
= 2 \text{Re} \left\{ (c_j + i s_j)(a^x_i + i \beta^x_i) \frac{1}{2} (c_{11} - i s_{11}) \right\}
\]

\[
= \text{Re} \left\{ (a^x_i + i \beta^x_i)(c_{11}^{\overline{1}} - i s_{11}^{\overline{1}}) \right\}
\]

\[
= a^x_i c_{11}^{\overline{1}} + \beta^x_i s_{11}^{\overline{1}}
\]

(7.35)

Hence

\[
\lim_{n \to \infty} \left\{ s^u_{ji}(n)(u,v) \times s^v_{ji}(n)(u,v) \right\}
\]

\[
= [a^u_i \overline{c}_1 + \beta^u_i \overline{s}_1] \times [a^v_i \overline{c}_1 + \beta^v_i \overline{s}_1]
\]

\[
= \begin{vmatrix}
  a^u_i & \beta^u_i \\
  a^v_i & \beta^v_i
\end{vmatrix}

(\overline{c}_1 \times \overline{s}_1)
\]

(7.36)
To recapitulate then, using this Fourier transform technique, we have shown that, if \( \lambda_1 > \lambda_a \), we have a well-defined tangent plane at the limit point, spanned by \( \mathbf{c}_1 \) and \( \mathbf{s}_1 \), so the surface is \( C^1 \) smooth.

We now investigate the case where \( \lambda_a > \lambda_1 \).

We return to the formula
\[
\lim_{n \to \infty} S^{xj}(n) = \lim_{n \to \infty} \sum_{\omega=0}^{N-1} a_{\omega j} [D^x{\omega}] [V_\omega] [\Lambda_\omega]^n [V_\omega]^{-1} [R_\omega]
\]

**Case 1 \( \lambda_a > \lambda_1 \)**

To obtain a non-zero answer we must scale at each stage by \( \frac{1}{\lambda_a} \).

Then \( [A_\omega / \lambda_a]^n \to [0] \) for \( \omega \geq 1 \)

and \( [D^x{\omega}] [V_\omega] [A_\omega / \lambda_a]^n \to [0, [D^x{\omega}] v_0, 0, 0, 0, 0] \)

\( [V_\omega]^{-1} [R_\omega] = [z_0] = [z_{01}, z_{02}, \ldots, t] \)

so \( \lim_{n \to \infty} S^{xj}(n) \to [D^x{\omega}] v_0 z_{02} \).

From (7.21)
\[
z_{02} = -\frac{1}{16(\lambda_a - \lambda_b)(1 - \lambda_a)} \left\{ \frac{N}{N} \sum_{i=1}^{N} E_i + (8\lambda_b - 1) \frac{1}{N} \sum_{i=1}^{N} E_i \right\}
\]

\[
= \mathbf{c}_0.
\] (7.37)
As in Chapter 6 we find both partial derivatives have a limit in this direction so the surface is being squeezed into a peak.

Case 2. \( \lambda_a = \lambda_1 \).

We must scale by \( \lambda_a^{-1} = \lambda_1^{-1} \).

Now

\[
[D^u_0] \nu^a = \frac{2(1-\lambda_1)(16\lambda_1-1)(4\lambda_1+1)}{3(8\lambda_1-1)} = A^u_0 \text{ say}
\]

\[
[D^v_0] \nu^a = 0
\]

Hence

\[
\lim_{n \to -\infty} S^u j(n) = A^u_1 \mathcal{C}_1^j + A^u_0 \mathcal{C}_0
\]

\[
\lim_{n \to -\infty} S^v j(n) = A^v_1 \mathcal{S}_1^j
\]

Thus the normal is

\[
A^u_1 A^v_1 (\mathcal{C}_1 \times \mathcal{S}_1) + A^u_0 A^v_1 (\mathcal{C}_0 \times \mathcal{S}_1^j)
\]

which is dependent on \( j \), as we noted in Chapter 6.

In both these cases then, the surface is not \( C^1 \) smooth.

We have now duplicated the results summarized in Chapter 6, indeed obtained them in a much more economical way. In the next chapter we consolidate what we have so far discovered. In particular we must interpret the eigenvalues, eigenvectors and the implications of our results for the configuration of points \( V_j, E_j, F_j, \) etc. We shall also compare with the 4-node case where we can evaluate directly.
Chapter 8

Interpretation of the Fourier Method

The results obtained in the preceding chapter, as we have noted, correspond to those reached by the matrix method. Indeed, as we shall see, the Fourier method will prove more fruitful than the manipulation of very large matrices. However, because of the use of complex numbers, some interpretation of the expressions which emerge is necessary. We have seen that the tangent plane at the extraordinary point, assuming $\lambda_a$ to be suppressed beneath $\lambda_1$, is spanned by $C_1$ and $S_1$, but what do these vectors represent? In general, what are $C_\omega$ and $S_\omega$? How are we to interpret the eigenvalues $\lambda_\omega$ and eigenvectors $v_\omega$? Does the case $N = 4$ tally with what we can evaluate directly? In this chapter we shall shed some light on these problems.

Firstly we consider

$$
C_\omega = \frac{1}{2(\lambda - \lambda')N} \left\{ \sum_{i=1}^{N} c_{\omega i} E_i + \frac{1}{4} (4\lambda_\omega - 1)^{-1} \sum_{i=1}^{N} (c_{\omega i} + c_{\omega(i+1)}) F_i \right\}
$$

$$
S_\omega = \frac{1}{2(\lambda - \lambda')N} \left\{ \sum_{i=1}^{N} s_{\omega i} E_i + \frac{1}{4} (4\lambda_\omega - 1)^{-1} \sum_{i=1}^{N} (s_{\omega i} + s_{\omega(i+1)}) F_i \right\}
$$

where $0 < \omega < \frac{N}{2}$.

We note immediately that both are independent of $V$ and the subdivision weightings; in particular $C_1$ and $S_1$ and the normal vector at the limit point are independent of $V$, the original vertex point.
To interpret $C_\omega$ and $S_\omega$ in any way, we must make some assumption about the configuration of points \( \{ \mathcal{W} \} \cup \{ E_i, F_i \} \), \( i = 1, \ldots, N \). The most natural assumption to make is that of rotational symmetry in both the edge and face points. In fact, by the very labelling of the points $E_1, E_2$ etc., it is implicit that there is no radical deviation from this situation.

To formulate this, we can exploit the fact that the parametric bicubic patch is independent of the coordinate system. Thus we may choose $\mathcal{W}$ at the origin, and $k$, the axis of notation, in the $z$ direction. Thus $\mathcal{W} = 0$,

\[
\begin{bmatrix}
  k \\
  E_i \\
  F_i
\end{bmatrix}
= \begin{bmatrix}
  r & c_i \\
  r & s_i \\
  z_E
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  E_i \\
  F_i
\end{bmatrix}
= \begin{bmatrix}
  t_1 & c_i + t_2 & c_{i+1} \\
  t_1 & s_i + t_2 & s_{i+1}
\end{bmatrix}
\]

where $z_E, z_F, r, t_1, t_2$ are constants and, taking $E_i$ to be, in some sense, between $E_i$ and $E_{i+1}$, $t_1$ and $t_2$ are positive.

We recall the identities

\[
\sum_{i=1}^{N} c_{\omega i} = \begin{cases} 
N & \text{if } \omega = mN \quad (m \in \mathbb{Z}) \\
0 & \text{otherwise}
\end{cases}
\]

\[
\sum_{i=1}^{N} s_{\omega i} = 0 \quad (\omega \in \mathbb{Z})
\]

Then

\[
\sum_{i=1}^{N} c_{\omega i} E_i = \begin{bmatrix}
  r \sum_{i=1}^{N} c_{\omega i} c_i \\
  r \sum_{i=1}^{N} c_{\omega i} s_i \\
  z_E \sum_{i=1}^{N} c_{\omega i}
\end{bmatrix}
\]
\[ r \begin{bmatrix} N/2 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \omega = 1 \]
\[ = 0 \quad \text{otherwise } \quad (0 < \omega < N/2) \quad \text{(8.1)} \]

Similarly
\[ \sum_{i=1}^{N} s_{wi} E_i = \begin{bmatrix} \sum_{i=1}^{N} s_{wi} c_i \\ \sum_{i=1}^{N} s_{wi} s_i \\ z_E \sum_{i=1}^{N} s_{wi} \end{bmatrix} \]
\[ = \begin{bmatrix} 0 \\ r \frac{N}{2} \\ 0 \end{bmatrix} \quad \text{for } \omega = 1 \]
\[ = 0 \quad \text{otherwise } \quad (0 < \omega < N/2) \quad \text{(8.2)} \]

Also
\[ \sum_{i=1}^{N} (c_{\omega i} + c_{\omega(i+1)}) F_i \]
\[ = \begin{bmatrix} \sum_{i=1}^{N} (c_{\omega i} + c_{\omega(i+1)}) (t_1 c_i + t_2 c_{i+1}) \\ \sum_{i=1}^{N} (c_{\omega i} + c_{\omega(i+1)}) (t_1 s_i + t_2 s_{i+1}) \\ z_F \sum_{i=1}^{N} (c_{\omega i} + c_{\omega(i+1)}) \end{bmatrix} \]
\[\begin{align*}
&\left[ (t_1 + t_2) \sum_{i=1}^{N} c_{\omega i} c_i + t_1 \sum_{i=1}^{N} c_{\omega(i+1)} c_i + t_2 \sum_{i=1}^{N} c_{\omega i} c_{i+1} \\
&\left[ (t_1 + t_2) \sum_{i=1}^{N} c_{\omega i} s_i + t_1 \sum_{i=1}^{N} c_{\omega(i+1)} s_i + t_2 \sum_{i=1}^{N} c_{\omega i} s_{i+1} \\
&\left[ 2z_F \sum_{i=1}^{N} c_{\omega i} \right]
\right]
\end{align*}\]

\[\begin{align*}
&= \begin{cases}
(t_1 + t_2)(1 + c_1) \frac{N}{2} & \text{if } \omega = 1 \\
0 & \text{otherwise (}0 < \omega < \frac{N}{2}\text{)}
\end{cases}
\end{align*}\] (8.3)

and similarly

\[\begin{align*}
&\sum_{i=1}^{N} (s_{\omega i} + s_{\omega(i+1)}) F_i \\
&= \begin{cases}
0 & \text{if } \omega = 1 \\
(t_1 + t_2)(1 + c_1) \frac{N}{2} & \text{otherwise (}0 < \omega < \frac{N}{2}\text{)}
\end{cases}
\end{align*}\] (8.4)

From (8.1) - (8.4), we deduce that, for \(1 < \omega < \frac{N}{2}\)

\[C_{\omega} = S_{\omega} = 0 .\] (8.5)
\[
C_1 = \frac{1}{2(\lambda_1 - \lambda_1')} N \left\{ \sum_{i=1}^{N} c_i E_i + \frac{1}{4} (4\lambda_1 - 1)^{-1} \sum_{i=1}^{N} (c_i + c_{i+1}) E_i \right\}
\]

\[
= \frac{1}{2(\lambda_1 - \lambda_1')} N \left\{ \begin{bmatrix} N/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{4} (4\lambda_1 - 1)^{-1} \frac{N}{2} \begin{bmatrix} (t_1 + t_2)(1 + c_1) \\ 0 \\ 0 \end{bmatrix} \right\}
\]

\[
= \frac{1}{4(\lambda_1 - \lambda_1')} \begin{bmatrix} r + \frac{1 + c_1}{4(4\lambda_1 - 1)} (t_1 + t_2) \\ 0 \\ 0 \end{bmatrix}
\]

\[
= \frac{1}{2(16\lambda_1^2 - 1)} \begin{bmatrix} 8\lambda_1 r + (4\lambda_1 - 1)(t_1 + t_2) \\ 0 \\ 0 \end{bmatrix}
\]

since \( \frac{1 + c_1}{4(4\lambda_1 - 1)} = \frac{4\lambda_1 - 1}{8\lambda_1} \) by (5.13)

Similarly \( S_1 = \frac{1}{2(16\lambda_1^2 - 1)} \begin{bmatrix} 0 \\ 8\lambda_1 r + (4\lambda_1 - 1)(t_1 + t_2) \\ 0 \end{bmatrix} \)

(8.6)

If follows that \( k \), the axis of rotation, is normal to both \( C_1 \) and \( S_1 \) and hence the tangent plane. This comes as no surprise, being apparent from considerations of symmetry. In addition \( C_1 \)
and $S_1$ are mutually orthogonal and the same length.

We must now account for $C_0$ and, if $N$ is even, $C_{\frac{N}{2}}$ and $S_{\frac{N}{2}}$.

$$C_0 = \frac{1}{16(\lambda_a - \lambda_b)(1-\lambda_a)} \left\{ \bar{V} - 8\lambda_b \frac{1}{N} \sum_{i=1}^{N} E_i + (8\lambda_b - 1) \frac{1}{N} \sum_{i=1}^{N} F_i \right\}$$

Now

$$\sum_{i=1}^{N} E_i = \begin{bmatrix} r \sum_{i=1}^{N} c_i \\ r \sum_{i=1}^{N} s_i \\ \sum_{i=1}^{N} z_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ N z_E \end{bmatrix}$$

and

$$\sum_{i=1}^{N} F_i = \begin{bmatrix} (t_1 + t_2) \sum_{i=1}^{N} c_i \\ (t_1 + t_2) \sum_{i=1}^{N} s_i \\ \sum_{i=1}^{N} z_F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ N z_F \end{bmatrix}$$

Hence

$$C_0 = \frac{1}{16(\lambda_a - \lambda_b)(1-\lambda_a)} \begin{bmatrix} 0 \\ 0 \\ 8\lambda_b z_E + (1-8\lambda_b)z_F \end{bmatrix}$$  \hfill (8.8)

$C_0$ is therefore in the direction of the normal. This is the component out of the tangent plane when the surface is not smooth.
If $N$ is even, we have

$$\left[ \frac{z_N}{2} \right] = \left[ V_N \right]^{-1} \left[ R_N \right] = \left[ \frac{z_N}{2}, \frac{z_N}{2}, \ldots \right]^t$$

and since, from Chapter 7, section 2

$$[A_N] = \begin{bmatrix}
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{16} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{32} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{64}
\end{bmatrix}$$

the dominant eigenvalue, $\frac{1}{4}$, will pick up both $\frac{z_N}{2}$ and $\frac{z_N}{2}^2$.

From (7.24)

$$\frac{z_N}{2} = e_N = \frac{1}{N} \sum_{i=1}^{N} a_N^* \frac{E_i}{2}$$

$$= \frac{1}{N} \sum_{i=1}^{N} (-1)^i E_i$$

$$= \frac{C}{\frac{N}{2}} \text{ say.} \quad (8.9)$$

$$\frac{z_N}{2}^2 = f_N = \frac{1}{N} \sum_{i=1}^{N} a_N^* \frac{F_i}{2}$$

$$= \frac{1}{N} \sum_{i=1}^{N} (-1)^i F_i$$

$$= \frac{S}{\frac{N}{2}} \text{ say.} \quad (8.10)$$
Now if $N$ is even

$$\sum_{j=1}^{N} (-1)^j c_j + i \sum_{j=1}^{N} (-1)^j s_j = \sum_{j=1}^{N} (-1)^j a_j$$

$$= 0$$

Hence

$$\sum_{i=1}^{N} (-1)^i c_i = \sum_{i=1}^{N} (-1)^i s_i = 0$$

and so

$$\frac{C_N}{2} = \frac{S_N}{2} = 0.$$  

Collecting these results together therefore, if we have the rotational symmetry specified above, the only non-zero vectors to emerge are $C_1$, $S_1$, and $C_0$ which form, in the order given, a right-handed orthogonal set, with $C_0$ in the direction of the axis of rotation.

Since

$$C_1^j = c_j C_1 + s_j S_1$$

$$S_1^j = -s_j C_1 + c_j S_1$$

(and as we would expect from considerations of symmetry), $C_1^j$ and $S_1^j$ (the directions, we recall of the $u$ and $v$ partial derivatives on edge $j$, patch 1 in the limit) are simply $C_1$ and $S_1$ rotated through $\frac{2\pi j}{N}$ about $k$.

We may glean further insight into the situation by consideration of the eigenvector $v_1$ corresponding to $\lambda_1$ (as do $C_1$ and $S_1$).

From Chapter 7, section 2
\[
\mathbf{v}_1 = \begin{bmatrix}
4\lambda - 1 \\
1 + a_1 \\
\frac{2(400\lambda^2 + 36\lambda - 13)}{(16\lambda - 1)(32\lambda - 1)} + \frac{4\lambda + 5}{16\lambda - 1} a_1 \\
\frac{2(4\lambda - 1)(4\lambda + 13)}{32\lambda - 1} \\
\frac{2(400\lambda^2 + 36\lambda - 13)}{(16\lambda - 1)(32\lambda - 1)} + \frac{4\lambda + 5}{16\lambda - 1} a_1 \\
\frac{10(1280\lambda^3 + 2128\lambda^2 - 56\lambda - 13)}{(16\lambda - 1)(32\lambda - 1)(64\lambda - 1)} (1 + a_1)
\end{bmatrix}
\]

Taking \( N = 4 \), and thus \( \lambda_1 = \frac{1}{2}, \quad a_1 = i \)

\[
\mathbf{v}_1 = [1, 1+i, 2-i, 2, 2+i, 2(1+i)]^T
\]

Now we know that this vector relates to the points

\[
[E_j, F_j, I_{j1}, I_{j2}, I_{j3}, I_{j4}]^T
\]

Fig. 8.1 shows the configuration of these points in the complex plane. It is clear from this that a natural configuration is thus represented, or more correctly a natural projection on the tangent plane. In the complex plane the complete arrangement of points for \( j = 1, \ldots, N \) will be given by

\[
\begin{bmatrix}
a_j \\
\mathbf{v}_1
\end{bmatrix}_{j=1, \ldots, N}
\]

In fact we can extend \( \mathbf{v}_1 \) by considering the transformation of the set of vectors

\[
[E_j, F_j, I_{j1}, I_{j2}, I_{j3}, I_{j4}, 0_{j1}, 0_{j2}, 0_{j3}, 0_{j4}, 0_{j5}, 0_{j6}]^T
\]

at the frequency \( \omega = 1 \).
Fig. 8.1 The Argand Diagram showing the Configuration of Points on an Edge for $N = 4$, $\omega = 1$
This gives

\[
\mathbf{\nu}^E_{-1} = \left[ \begin{array}{c}
\frac{5(2048\lambda^3 + 7168\lambda^2 + 28\lambda - 109)}{(16\lambda - 1)(32\lambda - 1)(64\lambda - 1) a_1} + \frac{5(3872\lambda^3 + 784\lambda^2 - 56\lambda - 1)}{\lambda(16\lambda - 1)(32\lambda - 1)(64\lambda - 1) a_1} \\
\frac{640\lambda^2 + 528\lambda - 109}{(16\lambda - 1)(32\lambda - 1)} + \frac{5\lambda + 1}{\lambda(16\lambda - 1) a_1} \\
\frac{(4\lambda - 1)(100\lambda^2 + 42\lambda - 1)}{4\lambda^2(32\lambda - 1)} \\
\frac{640\lambda^2 + 528\lambda - 109}{(16\lambda - 1)(32\lambda - 1)} + \frac{5\lambda + 1}{\lambda(16\lambda - 1) a_1} \\
\frac{5(2048\lambda^3 + 7168\lambda^2 + 28\lambda - 109)}{(16\lambda - 1)(32\lambda - 1)(64\lambda - 1) a_1} + \frac{5(3872\lambda^3 + 784\lambda^2 - 56\lambda - 1)}{\lambda(16\lambda - 1)(32\lambda - 1)(64\lambda - 1) a_1} \\
\frac{5(5248\lambda^3 + 1568\lambda^2 - 133\lambda - 5)}{\lambda(16\lambda - 1)(32\lambda - 1)(64\lambda - 1)} (1+a_1)
\end{array} \right]
\]

evaluated at \( \lambda = \lambda_1 \).

With \( N = 4 \), \( \lambda_1 = \frac{1}{2} \)

\[
\mathbf{\nu}^E_{-1} = \{1, 1+i, 2-i, 2, 2+i, 2(1+i), 3-2i, 3-i, 3, 3+i, 3+2i, 3(1+i)\}^t
\]

In the general \( N \)-node case, the set of points \( \{a_j \mathbf{\nu}^E_{-1}\}_{j=1,\ldots,N} \) gives the natural configuration round the node. Figs. 8.2 a) - g) illustrate this for \( N = 3,4,5,\ldots,9 \).

Relating \( \mathbf{\nu}_{-1} \) to our earlier choice of \( \mathbf{E}_{-1}, \mathbf{F}_{-1} \) we have

\[
r = 4\lambda_1 - 1
\]

and \( t_1 = t_2 = 1 \).

Thus, from (8.6) we have
Fig. 8.2 The Natural Configurations for $N = 3, 4, \ldots, 9$
\[ C_1 = \frac{1}{2(16\lambda_1^2 - 1)} \begin{bmatrix} 8\lambda_1(4\lambda_1 - 1) + 2(4\lambda_1 - 1) \\ 0 \\ 0 \end{bmatrix} \]

= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}

and similarly, from \((8.7)\)

\[ S_{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]

As before \( C_\omega = S_\omega = 0 \) for \( 2 \leq \omega \leq \frac{N}{2} \).

It is important that we stress here certain implications which cannot be drawn from the above results. We might suppose, for example, that in the limit the natural configuration is attained. This, however, is not, in general, the case. Were it so, we would expect that the angle subtended by the vectors

\[ \frac{E_j(n) - v(n)}{E_{j+1}(n) - v(n)} \]

would tend to \( \frac{2\pi}{N} \) as \( n \to \infty \).

By following through the mathematics of the previous chapter we have

\[ \lim_{n \to \infty} \left\{ \frac{E_j(n) - v(n)}{E_{j+1}(n) - v(n)} \right\} = (4\lambda_1 - 1) C_{-1}^j. \]
We choose the following assembly of points, far from pathological in distribution.

Put $N = 3, \quad v = 0$

\[
E_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

and $F_i = E_i + E_{i+1}$ i.e.

\[
F_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 1 - \frac{\sqrt{2}}{2} \end{bmatrix} \quad F_2 = \begin{bmatrix} 1 - \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad F_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

This configuration is illustrated in Fig. 8.3.

\[
C_1 = \frac{1}{6(\lambda_1 - \lambda_1)} \left\{ \sum_{i=1}^{3} c_i E_i + \frac{1}{4} (4\lambda_1 - 1)^{-1} \sum_{i=1}^{3} (c_i + c_{i+1}) F_i \right\}
\]

\[
S_1 = \frac{1}{6(\lambda_1 - \lambda_1)} \left\{ \sum_{i=1}^{3} s_i E_i + \frac{1}{4} (4\lambda_1 - 1)^{-1} \sum_{i=1}^{3} (s_i + s_{i+1}) F_i \right\}
\]

\[
c_1 = -\frac{1}{2} \quad c_2 = -\frac{1}{2} \quad c_3 = 1
\]

\[
s_1 = \frac{\sqrt{3}}{2} \quad s_2 = -\frac{\sqrt{3}}{2} \quad s_3 = 0
\]

Thus

\[
C_1 = \frac{1}{6(\lambda_1 - \lambda_1)} \left\{ \begin{bmatrix} 1 + \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix} + \frac{1}{4} (4\lambda_1 - 1)^{-1} \begin{bmatrix} 1 + \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix} \right\}
\]
Fig. 8.3 A Non-symmetric Configuration for $N = 3$
\[
S_1 = \frac{16\lambda_1^{-3}}{24(\lambda_1 - \lambda_1^1)(4\lambda_1 - 1)} \left[ 1 + \frac{\sqrt{2}}{4} \right]
\]

\[
S_1 = \frac{16\lambda_1^{-3}}{96(\lambda_1 - \lambda_1^1)(4\lambda_1 - 1)} \left[ 4 + \sqrt{2} \right]
\]

\[
S_1 = \frac{16\lambda_1^{-3}}{96(\lambda_1 - \lambda_1^1)(4\lambda_1 - 1)} \left[ \frac{\sqrt{2} + \sqrt{3}}{4} \right] + \frac{1}{4} (4\lambda_1 - 1)^{-1} \left[ \frac{\sqrt{2} + \sqrt{3}}{4} \right]
\]

Put \( K = \frac{16\lambda_1^{-3}}{96(\lambda_1 - \lambda_1^1)(4\lambda_1 - 1)} \)

Then \( C_1 = K \begin{bmatrix} 4 + \sqrt{2} \\ \sqrt{2} - 2 \end{bmatrix} \)

\( S_1 = K \begin{bmatrix} \sqrt{2} + \sqrt{3} \\ (2 + \sqrt{2})\sqrt{3} \end{bmatrix} \)

Then \( C_1^3 = C_1 \)

\( C_1^1 = c_1 C_1 + s_1 S_1 \)

\( = K \begin{bmatrix} \sqrt{2} - 2 \\ 4 + \sqrt{2} \end{bmatrix} \)

\( C_1^2 = c_2 C_1 + s_2 S_1 \)

\( = K \begin{bmatrix} -2(1 + \sqrt{2}) \\ -2(1 + \sqrt{2}) \end{bmatrix} \)
We note that the bilateral symmetry in \( y = x \) is preserved, so

\[ |C_1^1| = |C_1^3|, \text{ but that } |C_1^1| \neq |C_1^2|. \]

Furthermore the angle between \( C_1^3 \) and \( C_1^1 \) is 102.35° but that between \( C_1^1 \) and \( C_1^2 \) and also \( C_1^2 \) and \( C_1^3 \) is 128.825°.

Thus, in the limit, the natural configuration is not attained, since otherwise the angles in this example would each be 120°.

Although we have shown that, given a rotationally symmetric configuration \( C_2 = S_2 = 0 \) (\( N \geq 4 \)) the converse does not hold.

**Theorem 8.1.** If \( N > 4 \), and \( C_2 = S_2 = 0 \), the configuration is not necessarily symmetric of order \( N \).

**Proof.** It is sufficient to consider the simple case when

\[
N \sum_{i=1}^{N} c_{2i} E_i = \sum_{i=1}^{N} s_{2i} E_i = 0
\]

and

\[
F_{E_i} = \alpha [E_i + E_{i+1}]\]

Thus

\[
N \sum_{i=1}^{N} \left(c_{2i} + c_{2(i+1)}\right) F_{E_i} = 2\alpha (1 + c_2) \sum_{i=1}^{N} c_{2i} E_i = 0
\]

\[
N \sum_{i=1}^{N} \left(s_{2i} + s_{2(i+1)}\right) F_{E_i} = 2\alpha (1 + c_2) \sum_{i=1}^{N} s_{2i} E_i = 0
\]

Suppose we move \( E_j \) by \( \delta E \)

i.e. \( E'_j = E_j + \delta E \)
a) If \( N \) is even, putting \( E'_{j+\frac{N}{2}} = E_{j+\frac{N}{2}} - \delta E \) will compensate since

\[
\sum_{i=1}^{N} c_{2i} E'_i = \sum_{i=1}^{N} c_{2i} E_i + c_{2j} \delta E + c \]

\[
\frac{1}{2(j+N/2)} (-\delta E)
\]

\[
= 0 + c_{2j}[\delta E - \delta E]
\]

\[
= 0
\]

\[
\sum_{i=1}^{N} s_{2i} E'_i = \sum_{i=1}^{N} s_{2i} E_i + s_{2j} \delta E + s \]

\[
\frac{1}{2(j+N/2)} (-\delta E)
\]

\[
= 0 + s_{2j}[\delta E - \delta E]
\]

\[
= 0
\]

b) If \( N \) is odd put \( E'_{j+\frac{N-1}{2}} = E_{j+\frac{N-1}{2}} - (2c_1)^{-1} \delta E \)

\[
E'_{j+\frac{N+1}{2}} = E_{j+\frac{N+1}{2}} - (2c_1)^{-1} \delta E
\]

\[
\sum_{i=1}^{N} c_{2i} E'_i = \sum_{i=1}^{N} c_{2i} E_i + c_{2j} \delta E
\]

\[
- \left[ c \frac{1}{2(j+N/2)} + c \frac{1}{2(j+N/2)} \right] (2c_1)^{-1} \delta E
\]

\[
= 0 + \left[ c_{2j} - 2c_{2j} c_1 (2c_1)^{-1} \right] \delta E
\]

\[
= 0
\]
\[
\frac{N}{2} \sum_{i=1}^{N} s_{2i} E_i' = \frac{N}{2} \sum_{i=1}^{N} s_{2j} E_i + s_{2j} \delta E
\]

\[
- \left[ s \left( \frac{2(j + N-1)}{2} + 2 \left( \frac{j + N+1}{2} \right) \right) \right] (2c_1)^{-1} \delta E
\]

\[
= 0 + \left[ s_{2j} - 2s_{2j} c_1 (2c_1)^{-1} \right] \delta E
\]

\[
= 0
\]

Hence in both cases we still have \( C_2 = S_2 = 0 \).

Consequently we may deform a rotationally symmetric configuration while preserving this property.

An example of such a situation would be the classic "three box" 6-node topology illustrated in Fig. 2.3. Here we have

\[
E_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}
\]

and

\[
E_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

with\( E_i = E_i + E_{i+1} \)

It is easily shown that, if \( E_i = E_i + E_{i+1} \)

\[
C_\omega = \frac{2(8\lambda - 1)}{(4\lambda - 1)(4\lambda + 1)N} \sum_{i=1}^{N} c_{\omega i} E_i
\]

\[
S_\omega = \frac{2(8\lambda - 1)}{(4\lambda - 1)(4\lambda + 1)N} \sum_{i=1}^{N} s_{\omega i} E_i
\]

for \( 1 \leq \omega < \frac{N}{2} \)
and, if $N$ is even, 
\[ \frac{S_N}{2} = \sum_{i=1}^{N} (-1)^i E_i = 0. \]

Thus, 
\[ C_1 = 0.277 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad S_1 = 0.277 \begin{bmatrix} 0 \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix} \]

and so 
\[ C_1 \cdot S_1 = 0, \quad |C_1| = |S_1| = 0.277\sqrt{6} \]

\[ C_2 = S_2 = 0. \quad Also \quad C_0 = 0 \]

and 
\[ C_3 = \sum_{i=1}^{6} (-1)^i E_i = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \quad S_3 = 0. \]

Nor should it be thought that, if we have a "sensible" configuration of points, consistent with some idea of rotational symmetry, $C_j^i$ and $S_j^i$ are necessarily negligible with respect to $C_1^j$ and $S_1^j$.

Consider the classic "two-box" 5-node case illustrated in Fig. 2.2.

\[ E_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

and 
\[ F_i = E_i + E_{i+1} \]

From (8.11)
\[
C_1 = \begin{bmatrix}
0.39598 \\
0.39598 \\
0.35418
\end{bmatrix} \quad S_1 = \begin{bmatrix}
-0.54502 \\
0.54502 \\
0
\end{bmatrix}
\]

and

\[
C_2 = \begin{bmatrix}
-0.90458 \\
-0.90458 \\
0.80908
\end{bmatrix} \quad S_2 = \begin{bmatrix}
0.29392 \\
-0.29392 \\
0
\end{bmatrix}
\]

We note \( C_1 \cdot S_1 = C_2 \cdot S_2 = 0 \).

However \( |C_\omega| \neq |S_\omega| \) and so \( C_j^j \) describes an ellipse as \( j \) increases; the areas of the respective ellipses will serve as an indicator of the relative sizes of \( C_2^j \) and \( C_1^j \). This area is given by

\[
A_\omega = |C_\omega| |S_\omega| \pi
\]

In this example \( A_1 = 0.5107\pi \)

\( A_2 = 0.6292\pi \)

Consequently \( C_2^j \) cannot be regarded as insignificant. This will have repercussions in the study of curvature.

Having established these provisos we now relate our results to the 4-node case. We consider the tangent plane which, in the general case, we showed, in (7.34) to have the normal vector

\[
\lim_{n \to \infty} S^{uj1(n)} \times \lim_{n \to \infty} S^{vj1(n)} = A^{u1}_1 \cdot C_1 \otimes A^{v1}_1 \otimes S_1
\]

where

\[
A^{u1}_1 = \frac{(1-\lambda_1)(4\lambda_1-1)(4\lambda_1+1)}{3\lambda_1}
\]
and
\[ A_1^{u1} = \frac{(32\lambda_1^2 + 24\lambda_1 + 1)}{3(16\lambda_1 - 1)} \cdot s_1. \]

If \( N = 4 \), \( \lambda_1 = \frac{1}{2} \) and \( A_1^{u1} = A_1^{v1} = 1 \).

Thus
\[ C_1 = \frac{1}{3} (E_4 - E_2) + \frac{1}{12} (F_4 - F_1 - F_2 + F_3) \]
\[ S_1 = \frac{1}{3} (E_1 - E_3) + \frac{1}{12} (F_4 + F_1 - F_2 - F_3) \]

Where \( N = 4 \) we can evaluate directly as we have the set of control points

\[ [R] = \begin{bmatrix} E_2 & E_2 & F_1 & I_{13} \\ E_3 & V & E_1 & I_{12} \\ F_3 & E_4 & F_4 & I_{11} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix} \]

Then
\[ S^u(0,0) = \frac{1}{36} [-3, 0, 3, 0][R][1, 4, 1, 0]^T \]
\[ = \frac{1}{3} (E_4 - E_2) + \frac{1}{12} (F_4 - F_1 - F_2 + F_3) \]

and
\[ S^v(0,0) = \frac{1}{36} [1, 4, 1, 0][R][-3, 0, 3, 0]^T \]
\[ = \frac{1}{3} (E_1 - E_3) + \frac{1}{12} (F_4 + F_1 - F_2 - F_3) \]

As expected, the evaluated derivatives correspond to the general result for \( N = 4 \).

Furthermore, in the 4-node case, it can be shown that
\[
\lim_{n \to \infty} s^u_{ji}(n)(u, v) = \frac{c^j_i}{l_{1}}
\]

\[
\lim_{n \to \infty} s^v_{ji}(n)(u, v) = \frac{s^j_i}{l_{1}}
\]

for \( j = 1, 2, 3, 4 \)

\( i = 1, 2, 3 \)

and \( 0 \leq u, v \leq 1 \).

Thus the limiting partial derivatives are independent of the patch and the parametric values thereon. In the general case, this is not true and so the limit normal varies in size according to which direction we approach the extraordinary point. The repercussions of this will manifest themselves in the next chapter.
Chapter 9

Curvature Properties at the N-node

9.1 CURVATURE AND DIFFERENTIAL GEOMETRY.

Having established the conditions under which we have a $C^1$ surface at the N-node we now investigate whether further constraints will yield a $C^2$ surface. We must first make explicit precisely what we mean by continuity of curvature on a surface. This may be summarised as follows.

A surface is said to have continuity of curvature at a point $P$ if every plane curve on the surface passing through $P$ has continuity of curvature there. In fact, since it is the curvature along any tangent which concerns us, it suffices to confine our attention to those curves which lie in a plane containing the normal to the surface at $P$ [22, p.123].

We shall now introduce some important results in differential geometry by considering the vector-valued function $\mathbf{r}(t)$, representing a space curve. The following treatment is similar to that of Faux and Pratt [12].

Let $\dot{\mathbf{r}}$ denote differentiation with respect to $t$ and $\mathbf{r}'$ denote differentiation with respect to $s$, the distance along the curve.

Thus, since the change in arc length $\delta s$ and that in chord length $|\delta \mathbf{r}|$ (see Fig. 9.1) become equal in the limit, $\mathbf{r}'$ is the unit tangent vector.

Hence $\mathbf{r}' = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$.

But also $\mathbf{r}' = \frac{\dot{\mathbf{r}}}{\dot{s}}$, since $t' \dot{s} = 1$. 

Fig. 9.1 The Curve $x(t)$

As $\delta t \to 0$, $\frac{\delta x}{\delta s} \to 1$
So \( \dot{s} = |\dot{t}| = (\dot{\mathbf{r}} \cdot \dot{t})^{\frac{1}{2}} \)

\[
\dddot{\mathbf{r}} = \frac{d}{ds} \left[ \frac{\dot{\mathbf{r}}}{\dot{s}} \right] = \frac{\dddot{\mathbf{r}} \cdot \dot{t} - \dddot{\mathbf{r}} \dddot{s} \dot{t}}{\dot{s}^2}
\]

\[= \frac{\dddot{\mathbf{r}} \cdot \dot{t} - \dddot{s} \dddot{t}}{\dot{s}^3} \]

But \( \dddot{s} = \frac{d}{dt} \left\{ (\dot{\mathbf{r}} \cdot \dot{t})^{\frac{1}{2}} \right\} \)

\[= \frac{1}{2} (\dot{\mathbf{r}} \cdot \dot{t})^{-\frac{1}{2}} \cdot 2(\dddot{\mathbf{r}} \cdot \dot{t}) \]

\[= \frac{\dddot{\mathbf{r}} \cdot \dot{t}}{\dot{s}} \]

Hence \( \dddot{\mathbf{r}} = \frac{(\dot{s})^2 \dddot{\mathbf{r}} - (\dot{\mathbf{r}} \cdot \dot{t}) \dddot{t}}{(\dot{s})^4} \)

\[= \frac{(\dot{\mathbf{r}} \cdot \dot{t}) \dddot{\mathbf{r}} - (\dddot{\mathbf{r}} \cdot \dot{t}) \dddot{t}}{(\dot{\mathbf{r}} \cdot \dot{t})^2} \]

\[= \frac{\dddot{t} \times (\dddot{\mathbf{r}} \times \dddot{t})}{(\dot{\mathbf{r}} \cdot \dot{t})^2} \quad (9.1) \]

assuming, as we are, a curve in three dimensions.

Then the curvature \( K = |\dddot{\mathbf{r}}| \)

We may derive an alternative expression for \( K \) by consideration of a surface such as we have, at least piecewise, parametrised in terms of \( u \) and \( v \).
Let \( \mathbf{r}(u, v) \) be a biperametric vector-valued function representing a surface. A curve on that surface is given by \( u = u(t) \) and \( v = v(t) \), or, more compactly,

\[
\mathbf{u} = u(t).
\]

Denoting the partial derivative with respect to \( x(=u,v) \) by \( \mathbf{r}^x \), we have

\[
\dot{\mathbf{r}} = \dot{u} \mathbf{r}^u + \dot{v} \mathbf{r}^v.
\]

The length of this tangent vector is given by \( \dot{s} \) where

\[
\dot{s}^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = (\dot{u} \mathbf{r}^u + \dot{v} \mathbf{r}^v). (\dot{u} \mathbf{r}^u + \dot{v} \mathbf{r}^v)
\]

\[
= \dot{u}^T \mathbf{G} \dot{u}
\]

with \( \dot{u} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} \) and \( \mathbf{G} = \begin{bmatrix} \mathbf{r}^u \cdot \mathbf{r}^u & \mathbf{r}^u \cdot \mathbf{r}^v \\ \mathbf{r}^v \cdot \mathbf{r}^u & \mathbf{r}^v \cdot \mathbf{r}^v \end{bmatrix} \)

\( \mathbf{G} \) is the first fundamental matrix of the surface.

Now

\[
\ddot{\mathbf{r}} = (\ddot{u} \mathbf{r}^{uu} + \ddot{v} \mathbf{r}^{uv}) \dot{u} + \dddot{u} \mathbf{r}^u
\]

\[
+ (\ddot{u} \mathbf{r}^{uv} + \ddot{v} \mathbf{r}^{vv}) \dot{v} + \dddot{v} \mathbf{r}^v
\]

\[
= (\dddot{u})^2 \mathbf{r}^{uu} + 2 \ddot{u} \dot{v} \mathbf{r}^{uv} + (\dddot{v})^2 \mathbf{r}^{vv} + \dddot{u} \mathbf{r}^u + \dddot{v} \mathbf{r}^v
\]

(9.3)

\( \mathbf{r}^u \cdot \mathbf{r}^u = 1 \) and so, differentiating, we have

\( \mathbf{r}^{uv} \cdot \mathbf{r}^u = 0 \).

But if \( \mathbf{r} \) is a plane curve both \( \mathbf{r}' \) and \( \mathbf{r}'' \) lie in the plane.

Hence if this plane contains \( \mathbf{n} \), the unit surface normal, then \( \mathbf{r}'' \) is parallel to \( \mathbf{n} \) and so

\[
K = |\mathbf{r}''| = \mathbf{r}'' \cdot \mathbf{n}.
\]
In any case, we define the normal curvature

\[ K = \mathbf{r}'' \cdot \hat{n} \]

\[ = \left[ \frac{(\mathbf{\dot{r}} \cdot \mathbf{\dot{r}}) \mathbf{r} - (\mathbf{\ddot{r}} \cdot \mathbf{\dot{r}}) \mathbf{\dot{r}}}{(\mathbf{\dot{r}} \cdot \mathbf{\dot{r}})^2} \right] \cdot \hat{n} \]

\[ = \frac{\mathbf{r} \cdot \mathbf{\dot{n}}}{\mathbf{\dot{r}} \cdot \mathbf{\dot{r}}} \tag{9.4} \]

Since \( \mathbf{r}^u \cdot \hat{n} = \mathbf{r}^v \cdot \hat{n} = 0 \), we have, from (9.3)

\[ \mathbf{r} \cdot \hat{n} = (\dot{u})^2 \mathbf{r}^{uu} \cdot \hat{n} + 2 \dot{u} \dot{v} \mathbf{r}^{uv} \cdot \hat{n} + (\dot{v})^2 \mathbf{r}^{vv} \cdot \hat{n} \]

\[ = \dot{u}^T \lbrack D \rbrack \dot{u} \tag{9.5} \]

where \( \dot{u} \) is given in (9.2) and

\[ \lbrack D \rbrack = \begin{bmatrix} \mathbf{r}^{uu} \cdot \hat{n} & \mathbf{r}^{uv} \cdot \hat{n} \\ \mathbf{r}^{uv} \cdot \hat{n} & \mathbf{r}^{vv} \cdot \hat{n} \end{bmatrix} \]

is the second fundamental matrix of the surface.

From (9.2), (9.4) and (9.5)

\[ K = \frac{\dot{u}^T \lbrack D \rbrack \dot{u}}{\dot{u}^T \lbrack G \rbrack \dot{u}}. \tag{9.6} \]

Now the angle \( \theta \) between two tangents \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) is given by

\[ \cos \theta = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{(\mathbf{r}_1 \cdot \mathbf{r}_1)^{\frac{1}{2}}(\mathbf{r}_2 \cdot \mathbf{r}_2)^{\frac{1}{2}}} \]

Let \( \mathbf{r}_i = \dot{u}_i \mathbf{r}^u + \dot{v}_i \mathbf{r}^v \).
Then \( \mathbf{r}_i \cdot \mathbf{r}_j = [u_i, \mathbf{v}_i] \mathbf{G} \begin{bmatrix} \dot{u}_j \\ \dot{v}_j \end{bmatrix} \)

\[ = \dot{u}_i^t \mathbf{G} \dot{u}_j \]

Hence \( \cos \Theta = \frac{\dot{u}_1^t \mathbf{G} \dot{u}_2}{(\dot{u}_1^t \mathbf{G} \dot{u}_1)^{1/2}(\dot{u}_2^t \mathbf{G} \dot{u}_2)^{1/2}} \) \hspace{1cm} (9.7)

If the surface is \( C^2 \), the normal curvature attains a maximum and minimum value with respect to variation of \( \mathbf{u} \). These are called the principal curvatures.

Differentiating (9.6) with respect to \( \mathbf{u} \) and \( \mathbf{v} \) we obtain

\[
K \mathbf{u} = \left\{ ([1, 0][D] \mathbf{u} + \dot{u}^t[D][1, 0]^t) \mathbf{u} - ([1, 0][G] \mathbf{u} + \dot{u}^t[G][1, 0]^t) \mathbf{u}^t[D] \mathbf{u} \right\} / (\mathbf{u}^t[G] \mathbf{u})^2
\]

\[ = 2([1, 0][D] \mathbf{u} \mathbf{u}^t[G] \mathbf{u} - 2[1, 0][G] \mathbf{u} \mathbf{u}^t[D] \mathbf{u}) / (\mathbf{u}^t[G] \mathbf{u})^2
\]

since both \([G]\) and \([D]\) are symmetric.

\[ = 2 \left\{ \frac{[1, 0][D] \mathbf{u} - K[1, 0][G] \mathbf{u}}{\mathbf{u}^t[G] \mathbf{u}} \right\}
\]

Similarly

\[ K \mathbf{v} = 2 \left\{ \frac{[0, 1][D] \mathbf{u} - K[0, 1][G] \mathbf{u}}{\mathbf{u}^t[G] \mathbf{u}} \right\}
\]

Setting \( K \mathbf{u} \) and \( K \mathbf{v} \) to zero, the principal curvatures \( K_1, K_2 \)
in the directions \( \hat{u}_1, \hat{u}_2 \) (or more correctly \( \hat{r}_1 \) and \( \hat{r}_2 \)) are given by

\[
\begin{cases}
[1, 0][D - K_i G] \hat{u}_i = 0 & (i = 1, 2) \\
[0, 1][D - K_i G] \hat{u}_1 = 0
\end{cases}
\]

\( \Rightarrow [D - K_i G] \hat{u}_i = 0 \)

\( \Rightarrow [D] \hat{u}_i = K_i [G] \hat{u}_i \)

\( \Rightarrow [G]^{-1}[D] \hat{u}_i = K_i \hat{u}_i \) \hspace{1cm} (9.8)

Hence \( K_1, K_2 \) are the eigenvalues of \([G]^{-1}[D]\) with corresponding eigenvectors \( \hat{u}_1, \hat{u}_2 \).

The angle between the corresponding tangents \( \hat{r}_1 \) and \( \hat{r}_2 \) is given by (9.7).

But

\( [D] \hat{u}_1 = K_1 [G] \hat{u}_1 \)

\( [D] \hat{u}_2 = K_2 [G] \hat{u}_2 \)

and so

\( \hat{u}_2^T [D] \hat{u}_1 = K_1 \hat{u}_2^T [G] \hat{u}_1 \) \hspace{1cm} (9.9)

\( \hat{u}_1^T [D] \hat{u}_2 = K_2 \hat{u}_1^T [G] \hat{u}_2 \) \hspace{1cm} (9.10)

Transposing (9.9) and comparing with (9.10) we have

\[
\begin{cases}
\hat{u}_1^T [D] \hat{u}_2 = K_1 \hat{u}_1^T [G] \hat{u}_2 \\
\hat{u}_1^T [D] \hat{u}_2 = K_2 \hat{u}_1^T [G] \hat{u}_2
\end{cases}
\]

\( = K_2 \hat{u}_1^T [G] \hat{u}_2 \) \hspace{1cm} (9.11)

Thus either \( K_1 = K_2 \) or \( \hat{u}_1^T [D] \hat{u}_2 = \hat{u}_1^T [G] \hat{u}_2 = 0 \).
If $K_1 = K_2$ the curvature is constant in all directions and the surface is locally spherical.

Otherwise $\hat{\mathbf{u}}^t_1 [G] \hat{\mathbf{u}}_2 = 0$ and so by (9.7) the directions of principal curvature are orthogonal. In this case we can determine the distribution of curvature as we move round the tangent plane.

Suppose $\hat{\mathbf{i}}_1$ and $\hat{\mathbf{i}}_2$ are unit vectors in the principal curvature directions and $\hat{\mathbf{i}}_3$ is a unit vector given by

$$\hat{\mathbf{i}}_3 = \cos \theta \hat{\mathbf{i}}_1 + \sin \theta \hat{\mathbf{i}}_2$$

Then

$$\hat{\mathbf{i}}_3 = [\hat{\mathbf{u}}_3, \hat{\mathbf{v}}_3] \begin{bmatrix} \mathbf{r}^u \\ \mathbf{r}^v \end{bmatrix}$$

$$= \left\{ \cos \theta [\hat{\mathbf{u}}_1, \hat{\mathbf{v}}_1] + \sin \theta [\hat{\mathbf{u}}_2, \hat{\mathbf{v}}_2] \right\} \begin{bmatrix} \mathbf{r}^u \\ \mathbf{r}^v \end{bmatrix}$$

Hence

$$\hat{\mathbf{u}}_3 = \cos \theta \hat{\mathbf{u}}_1 + \sin \theta \hat{\mathbf{u}}_2$$

Now

$$K_3 = \frac{\hat{\mathbf{u}}^t_3 [D] \hat{\mathbf{u}}_3}{\hat{\mathbf{u}}^t_3 [G] \hat{\mathbf{u}}_3}$$

$$= \frac{[\cos \theta \hat{\mathbf{u}}_1 + \sin \theta \hat{\mathbf{u}}_2]^t [D] [\cos \theta \hat{\mathbf{u}}_1 + \sin \theta \hat{\mathbf{u}}_2]}{\hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_3}$$

$$= \cos^2 \theta \hat{\mathbf{u}}^t_1 [D] \hat{\mathbf{u}}_1 + 2 \sin \theta \cos \theta \hat{\mathbf{u}}^t_1 [D] \hat{\mathbf{u}}_2 + \sin^2 \theta \hat{\mathbf{u}}^t_2 [D] \hat{\mathbf{u}}_2$$

$$= K_1 \cos^2 \theta + K_2 \sin^2 \theta$$

(9.12)
Since $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are orthogonal unit vectors spanning the
tangent plane, the parameter $\theta$ represents the physical angle
subtended by $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_3$, taking $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ as the x and y axes
respectively. Thus, on a $C^2$ surface, we may derive the normal
curvature in any direction from the principal curvatures and
their directions.

In the context of subdivision surfaces, these quantities
are not available and so we must derive a more appropriate form
of the result.

Theorem 9.1. Let $\hat{\mathbf{e}}_i$ ($i = 1,2,3$) be tangent vectors at a
point $P$ of a locally $C^2$ surface with $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \neq 0$ for $i \neq j$.
Let $K_i$ be the corresponding normal curvatures.

If $\hat{\mathbf{r}}$ is another tangent vector with associated normal curvature
$K$, then

$$K = (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})^{-1} \sum_{i=1}^{3} \frac{(\hat{\mathbf{r}} \cdot \hat{\mathbf{e}}_i) (\hat{\mathbf{e}}_{i-1} \times \hat{\mathbf{e}}_i) \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{e}}_{i+1})}{(\hat{\mathbf{e}}_{i-1} \times \hat{\mathbf{e}}_i) \cdot (\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_{i+1})} K_i$$

where the subscript ranges cyclically over the ordered set $(1,2,3)$.

For the proof of this theorem we shall require the following
two trigonometric identities.

Lemma 9.2:

a) $\sin X \sin Y \sin (X-Y) + \sin Y \sin Z \sin (Y-Z)$

+ $\sin Z \sin X \sin (Z-X)$

= $\sin (Y-X) \sin (Z-Y) \sin (X-Z)$. 
b) \[ \sin (Y-\theta) \sin (Z-Y) \sin (\theta-Z) \cos 2X \]
\[ + \sin (\theta-X) \sin (Z-\theta) \sin (X-Z) \cos 2Y \]
\[ + \sin (Y-X) \sin (\theta-Y) \sin (X-\theta) \cos 2Z \]
\[ = \sin (Y-X) \sin (Z-Y) \sin (X-Z) \cos 2\theta. \]

The proof is given in Appendix C.

**Proof of Theorem 9.1.**

We assume for the moment that \( \hat{r}_i \) (\( i = 1,2,3 \)) and \( \hat{r} \) are unit vectors.

Let \( K_A \) and \( K_B \) be the principal curvatures in the directions given by the unit vectors \( \hat{r}_A \) and \( \hat{r}_B \). Then \( \hat{r}_A \perp \hat{r}_B \) and
\[ \hat{r} = \cos \theta \hat{r}_A + \sin \theta \hat{r}_B \] say.

From (9.12)
\[ K = K_A \cos^2 \theta + K_B \sin^2 \theta \]
\[ = \frac{1}{2} (K_A + K_B) + \frac{1}{2} (K_A - K_B) \cos 2\theta \]

Let \( \hat{r}_i = \cos \theta_i \hat{r}_A + \sin \theta_i \hat{r}_B \)

and \[ C = \frac{1}{2} (K_A + K_B) \]
\[ D = \frac{1}{2} (K_A - K_B) \]

Then \[ K_i = C + D \cos 2\theta_i \] (\( i = 1,2,3 \))

\[ K = C + D \cos 2\theta \]  \hspace{1cm} (9.13)
Hence

\[
\sin(\theta_2 - \theta) \sin(\theta_3 - \theta_2) \sin(\theta - \theta_3) K_1 \\
+ \sin(\theta - \theta_1) \sin(\theta_3 - \theta) \sin(\theta_1 - \theta_3) K_2 \\
+ \sin(\theta_2 - \theta_1) \sin(\theta - \theta_2) \sin(\theta_1 - \theta) K_3
\]

(9.14)

\[
= \left\{ \sin(\theta_2 - \theta) \sin(\theta_3 - \theta_2) \sin(\theta - \theta_3) \\
+ \sin(\theta - \theta_1) \sin(\theta_3 - \theta) \sin(\theta_1 - \theta_3) \\
+ \sin(\theta_2 - \theta_1) \sin(\theta - \theta_2) \sin(\theta_1 - \theta) \right\} C
\]

+ \left\{ \sin(\theta_2 - \theta) \sin(\theta_3 - \theta_2) \sin(\theta - \theta_3) \cos 2\theta_1 \\
+ \sin(\theta - \theta_1) \sin(\theta_3 - \theta) \sin(\theta_1 - \theta_3) \cos 2\theta_2 \\
+ \sin(\theta_2 - \theta_1) \sin(\theta - \theta_2) \sin(\theta_1 - \theta) \cos 2\theta_3 \right\} D. \quad (9.15)

Consider the coefficient of \( C \) with

\[
\begin{align*}
X &= \theta_1 - \theta \\
Y &= \theta_2 - \theta \\
Z &= \theta_3 - \theta
\end{align*}
\]

We have

\[
\sin Y \sin(Z-Y) \sin(-Z) + \sin(-X) \sin Z \sin(X-Z) \\
+ \sin(Y-X) \sin(-Y) \sin X
\]

\[
= \sin X \sin Y \sin(X-Y) + \sin Y \sin Z \sin(Y-Z) \\
+ \sin Z \sin X \sin(Z-X)
\]

\[
= \sin(Y-X) \sin(Z-Y) \sin(X-Z) \text{ by Lemma 9.2 a)}
\]
= \sin(\theta_2 - \theta_1) \sin(\theta_3 - \theta_2) \sin(\theta_1 - \theta_3) \tag{9.16}

By Lemma 9.2 b), putting \(X = \theta_1, \ Y = \theta_2, \ Z = \theta_3\) in the
coefficient of D it becomes

\[\sin(\theta_2 - \theta_1) \sin(\theta_3 - \theta_2) \sin(\theta_1 - \theta_3) \cos 2\theta \tag{9.17}\]

By (9.16) and (9.17), (9.15) gives

\[\sin(\theta_2 - \theta_1) \sin(\theta_3 - \theta_2) \sin(\theta_1 - \theta_3) [C + D \cos 2\theta] \]

= \sin(\theta_2 - \theta_1) \sin(\theta_3 - \theta_2) \sin(\theta_1 - \theta_3) K \tag{9.18}

Comparing (9.14) and (9.18) we have

\[K = \frac{3}{i=1} \frac{\sin(\theta - \theta_{i-1}) \sin(\theta_{i+1} - \theta)}{\sin(\theta_{i-1} - \theta_{i-1}) \sin(\theta_{i+1} - \theta_{i+1})} K_i \tag{9.19}\]

with cyclic subscript.

But

\[\frac{(\hat{r}_i \times \hat{r}_{i-1}) (\hat{r}_i \times \hat{r}_{i-1})}{(\hat{r}_i \times \hat{r}_{i-1}) (\hat{r}_i \times \hat{r}_{i-1})}\]

\[= \frac{(\hat{r}_i \times \hat{r}_{i-1}) (\hat{r}_i \times \hat{r}_{i+1}) (\hat{r}_i \times \hat{r}_{i+1})}{(\hat{r}_i \times \hat{r}_{i+1}) (\hat{r}_i \times \hat{r}_{i+1}) (\hat{r}_i \times \hat{r}_{i+1})}\]

\[= \frac{\sin(\theta - \theta_{i-1}) \sin(\theta_{i+1} - \theta)}{\sin(\theta_{i-1} - \theta_{i-1}) \sin(\theta_{i+1} - \theta_{i+1})}\]

so (9.19) gives, as required

\[K = (\hat{r} \cdot \hat{r})^{-1} \sum_{i=1}^{3} \frac{(\hat{r}_i \times \hat{r}_{i+1}) (\hat{r}_i \times \hat{r}_{i-1}) (\hat{r}_i \times \hat{r}_{i+1})}{(\hat{r}_i \times \hat{r}_{i-1}) (\hat{r}_i \times \hat{r}_{i+1})} K_i\]

\[\square\]
Corollary 9.3  If the normal curvatures in any three distinct directions on a $C^2$ surface are equal the surface is locally spherical.

Proof: Direct from Theorem 9.1.

9.2 CURVATURE ON TANGENT-CONTINUOUS LOCI.

We now proceed to examine the implications of these results for the subdivision surface generated round an $N$-node. To do this we must evaluate the expression

$$\frac{[\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}})]}{(\dot{\vec{r}} \cdot \dot{\vec{r}})^2}$$

for curves through the limiting vertex point.

The most convenient such curves are those given by

a) $U \sum_{n=0}^{\infty} \left\{ s_j^1(n)(u,0); \quad 0 \leq u \leq 1 \right\}$

that is, the aggregate of edges of patch 1 on face $j$, where $v = 0$

and

b) $U \sum_{n=0}^{\infty} \left\{ s_j^2(n)(u,u); \quad 0 \leq u \leq 1 \right\}$

the $u = v$ diagonals on patch 2, face $j$.

These loci are shown in Fig. 9.2.

It is clear that if $N$ is odd, loci of type a) will not align and so we must distinguish between the even and odd node cases.

On the edge loci a) the tangent at the vertex, as we saw in (7.33), is given by
Fig. 9.2 Edge Loci a) and Face Loci b) on the Surface generated round a 5-node
\[ \hat{r} = \lim_{n \to \infty} S^{u_j1(n)} = A^{u_1}_1 C_j^1. \]

If \( N \) is even then
\[
C_1^{(j + \frac{N}{2})} = c^{(j + \frac{N}{2})} C_1^{(j + \frac{N}{2})} + s^{(j + \frac{N}{2})} S_1^{(j + \frac{N}{2})}
\]
\[ = - [c_j C_1 + s_j S_1] \]
\[ = - C_1^j \quad (9.20) \]

and so the edge loci are aligned in pairs.

On the face loci b) we have
\[ u = t \quad \text{and so} \quad \hat{u} = \hat{v} = 1 \]
\[ v = \hat{t} \]
\[ \hat{r} = \hat{u} \hat{r}^u + \hat{v} \hat{r}^v = \hat{r}^u + \hat{r}^v \]
\[ = S^{u_j2} + S^{v_j2} \quad (9.21) \]

Hence the tangent at the vertex is given by
\[ \hat{r} = \lim_{n \to \infty} \left\{ S^{u_j2(n)} + S^{v_j2(n)} \right\} \]
\[ = \lim_{n \to \infty} \sum_{\omega=0}^{N-1} a_{\omega j} \left[ D^{u(j)}_\omega + D^{v(j)}_\omega \right] R(n) \]
\[ = \lim_{n \to \infty} \sum_{\omega=0}^{N-1} a_{\omega j} \left[ D^{t(j)}_\omega \right] R(n) \] say
\[ = 2 \text{Re} \left\{ a_j \left[ D^{t(j)}_1 \right] \right\} \]
\[ = a C_1^j + \beta S_1^j \]
where \[\alpha = \text{Re}[D_1^{t2}] \nu_1\] \[\beta = \text{Im}[D_1^{t2}] \nu_1\] by (7.35)

From (7.14) and (7.15), for \(0 < \omega < \frac{N}{2}\)

\[\begin{align*}
[D_{t2}] &= \frac{1}{6} [-2(1 + a_\omega), 0, 2a_\omega, 0, 2, 1] \\
[D_{t2}] \nu_\omega &= \nonumber \\
&= \frac{(131072\lambda_\omega^4 - 112896\lambda_\omega^3 - 20272\lambda_\omega^2 + 2376\lambda_\omega + 35)}{3(16\lambda_\omega - 1)(32\lambda_\omega - 1)(64\lambda_\omega - 1)} (1 + a_\omega) \\
&= B_{t2} (1 + a_\omega) \quad \text{say.} \\
\text{Thus } \hat{x} &= B_{t2} \left\{ (1 + c_{\frac{1}{2}}) \frac{C_1^j}{S_1^j} + s_1 \frac{S_1^j}{S_1^j} \right\} \\
\text{If } N \text{ is even } \frac{j + N}{2} &= -s \left( j + \frac{N}{2} \right) \frac{C_1}{S_1} + c \left( j + \frac{N}{2} \right) S_1 \\
&= s \frac{C_1}{S_1} - c \frac{S_1}{S_1} \\
&= - \frac{S_1^j}{S_1^j} \\
\end{align*}\]

and so the face loci are also aligned.

If \(N\) is odd we show that the \(j\)th edge locus aligns with the \((j + \frac{N-1}{2})\)th face locus.

\textbf{Theorem 9.4} \( (1 + c_\omega) \frac{(j + \frac{N-1}{2})}{S_\omega} + s_\omega \frac{(j + \frac{N-1}{2})}{S_\omega} = (-1)^\omega 2c_\omega \frac{C_\omega^j}{S_\omega^j} \)
Proof: Since \[ C_j^\omega = c_{\omega j} \frac{C_\omega}{\omega} + s_{\omega j} \frac{S_\omega}{\omega}, \]
\[ S_j^\omega = -s_{\omega j} \frac{C_\omega}{\omega} + c_{\omega j} \frac{S_\omega}{\omega}, \]

it suffices to show

\[
\begin{align*}
\left(1 + c_\omega\right) c_{\omega j} \omega(j + \frac{N-1}{2}) - s_\omega s_{\omega j} \omega(j + \frac{N-1}{2}) &= (-1)\omega \frac{2c_\omega}{2} c_{\omega j} \\
\left(1 + c_\omega\right) s_{\omega j} \omega(j + \frac{N-1}{2}) + s_\omega c_{\omega j} \omega(j + \frac{N-1}{2}) &= (-1)\omega \frac{2c_\omega}{2} s_{\omega j}
\end{align*}
\]

Multiplying the second equation by \(i = \sqrt{-1}\) and adding gives

\[
\left(1 + c_\omega + i s_\omega\right) a_{\omega j} \omega(j + \frac{N-1}{2}) = (-1)\omega \frac{2c_\omega}{2} a_{\omega j}
\]

and this is proved as follows

\[
\left(1 + a_\omega\right) a_{\omega j} \omega(j + \frac{N-1}{2})
\]

\[
= a_{\omega j} \omega(j + \frac{N-1}{2}) + a_{\omega j} \omega(j + \frac{N+1}{2})
\]

\[
= a_{\omega j} \omega(j + \frac{N}{2}) \left[\frac{a_\omega}{2} \omega(j + \frac{N}{2}) + \frac{a_\omega}{2} \right]
\]

\[
= (-1)\omega \frac{2c_\omega}{2} a_{\omega j}
\]

\[\square\]

Corollary 9.5 The \(j\)th edge locus aligns with the \((j + \frac{N-1}{2})\)th face locus.

Proof: The \((j + \frac{N-1}{2})\)th face locus is given by
\[
B_{1}^{c_{2}} \left\{ (1 + c_{1}) \frac{(j + \frac{N-1}{2})}{s_{1} S_{1}} + s_{1} S_{1} \right\}
\]

\[
= -2B_{1}^{c_{2}} c_{1} \frac{C_{j}^{i}}{\frac{N-1}{2}}
\]

which is parallel to \(A_{1}^{u_{1}} C_{j}^{i}\), the \(j^{th}\) edge locus.

We note here that, in general, although they have the same sign

\[
2B_{1}^{c_{2}} c_{1} \neq A_{1}^{u_{1}}
\]

and so the aligned tangents are not of the same magnitude.

Curvature, however, is a geometrical invariant and thus is independent of parametrization so this inequality will not affect our results. In addition, the problem of scaling must be considered in this light. To obtain a non-zero tangent we were obliged to scale by \(\lambda_{1}^{-1}\). As we shall now see, this determines the scale factor applied to the denominator of the curvature vector, and consequently, due to the invariance noted, the numerator also.

We consider first the limiting curvature vector on an edge locus. This is given by

\[
\lim_{n \to \infty} \frac{S_{u_{1}j}(n) \times \left\{ S_{u_{1}j}(n) \times S_{u_{1}j}(n) \right\}}{\left\{ S_{u_{1}j}(n) \times S_{u_{1}j}(n) \right\}^2}
\]

Now \(S_{u_{1}j}(n)\) must be scaled by \(\lambda_{1}^{-1}\) at each subdivision and so the denominator as a whole is scaled by \(\lambda_{1}^{-4}\). Consequently the product

\[
S_{u_{1}j}(n) \times S_{u_{1}j}(n)
\]

must be scaled by \(\lambda_{1}^{-3}\).
Now \( \frac{\partial^{uu} j_1(n)}{\partial j_1(n)} \)

\[ = \left\{ \sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{uu}] [v_{\omega}] [\Lambda_{\omega}]^{n} [v_{\omega}]^{-1} [R_{\omega}] \right\} \]

\[ \times \left\{ \sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{u}] [v_{\omega}] [\Lambda_{\omega}]^{n} [v_{\omega}]^{-1} [R_{\omega}] \right\} \]

In the limit, only the larger values of \( \lambda \) will have influence so, ignoring values of \( \lambda \) less than \( \min \{ \lambda_a, \lambda_2 \} \), this becomes

\[ \left\{ [D_{0}^{u}] v_{a} z_{02} \lambda_{a}^{n} + 2 \text{Re} \left\{ [D_{1}^{uu}] v_{1} a_{j} z_{11} \right\} \lambda_{1}^{n} \right. \]

\[ + 2 \text{Re} \left\{ [D_{2}^{uu}] v_{2} a_{2j} z_{21} \right\} \lambda_{2}^{n} \}

\[ \times \left\{ [D_{0}^{u}] v_{a} z_{02} \lambda_{a}^{n} + 2 \text{Re} \left\{ [D_{1}^{u}] v_{1} a_{j} z_{11} \right\} \lambda_{1}^{n} \right. \]

\[ + 2 \text{Re} \left\{ [D_{2}^{u}] v_{2} a_{2j} z_{21} \right\} \lambda_{2}^{n} \}. \quad (9.25) \]

As we saw in (7.29) and (7.38)

\[ [D_{0}^{u}] v_{a} = A_{0}^{u} \]

\[ [D_{\omega}^{u}] v_{-\omega} = A_{\omega}^{u} \quad \text{for } 0 < \omega < \frac{N}{2}. \]

We now extend this notation to give

\[ [D_{N}^{u}] v_{-\frac{N}{2}} = 2 = A_{\frac{N}{2}}^{u} \]

\[ [D_{N}^{u}] v_{\frac{N}{2}} = 0 \]
\[ [D_{uu}^{1}] v^a = \frac{8(\lambda_\alpha - 1)(2\lambda_\alpha - 1)(16\lambda_\alpha - 1)}{3(8\lambda_\alpha - 1)} = A_{uu}^1 \]

\[ [D_{\omega}^{uu}] v^\omega = \frac{4(\omega_\omega - 1)(2\omega_\omega - 1)(4\omega_\omega - 1)}{3\omega_\omega} = A_{\omega}^{uu} \]

for \( 0 < \omega < \frac{N}{2} \)

\[ [D_{N}^{uu}] \frac{v_N}{2} = 2 = A_{N}^{uu} \]

\[ [D_{N}^{uu}] \frac{v'_N}{2} = 0 \]

where \( v_N = \frac{[1, 0, 3, 3, 0]^T}{2} \)

and \( v'_N = \frac{[0, 1, -2, 0, 2, 4]^T}{2} \)

Thus expression (9.25) can be rewritten as

\[ \left\{ A_0^{uu} C_0^\omega \lambda_\alpha^n + A_1^{uu} C_1^\omega \lambda_1^n + A_2^{uu} C_2^\omega \lambda_2^n \right\} \]

\[ \times \left\{ A_0^{uu} C_0^i \lambda_a^n + A_1^{uu} C_1^i \lambda_1^n + A_2^{uu} C_2^i \lambda_2^n \right\} \]

\[ = \begin{vmatrix} A_0^{uu} & A_1^{uu} \\ A_0^{uu} & A_1^{uu} \end{vmatrix} \left( C_0^i \times C_1^i \right) \lambda_a^n \lambda_1^n \]

\[ + \begin{vmatrix} A_0^{uu} & A_2^{uu} \\ A_0^{uu} & A_2^{uu} \end{vmatrix} \left( C_0^i \times C_2^i \right) \lambda_a^n \lambda_2^n \]

\[ + \begin{vmatrix} A_2^{uu} & A_1^{uu} \\ A_2^{uu} & A_1^{uu} \end{vmatrix} \left( C_1^i \times C_2^i \right) \lambda_1^n \lambda_2^n \]
\[ \begin{align*}
\Delta^E_{0,1} &\quad (C_0 \times c^0_1)^n \lambda_1^n + \Delta^E_{0,2} (C_0 \times c^0_2)^n \lambda_a^n \lambda_2^n \\
+ \Delta^E_{2,1} (C^{-1}_2 \times c^0_1)^n \lambda_1^n \lambda_2^n 
\end{align*} \] 

Say.

Scaling at each stage by \( \lambda_1^{-3} \) gives

\[ \begin{align*}
\Delta^E_{0,1} &\quad (C_0 \times c^0_1) (\lambda_a/\lambda_1^2)^n + \Delta^E_{0,2} (C_0 \times c^0_2) (\lambda_a \lambda_2/\lambda_1^3)^n \\
+ \Delta^E_{2,1} (C^0_2 \times c^0_1)^n (\lambda_2/\lambda_1^2)^n 
\end{align*} \] 

(9.26)

We can compare this expression to what may be evaluated directly in the 4-node case.

If \( N = 4 \)

\[ \lambda_1 = \frac{1}{2} \]

\[ \lambda_2 = \lambda_a = \frac{1}{4} \]

and so

\[ \frac{\lambda_a/\lambda_1^2}{\lambda_2/\lambda_1^2} = 1 \]

\[ \frac{\lambda_a \lambda_2/\lambda_1^3}{\lambda_1} = \frac{1}{2} \]

Hence, in the limit, the second term in (9.26) will disappear.

\[ A^{u1}_{0} = 3 \quad A^{u1}_{1} = 1 \quad A^{u1}_{2} = A^{u1}_{N/2} = 2 \]

\[ A^{uu1}_{0} = 3 \quad A^{uu1}_{1} = 0 \quad A^{uu1}_{2} = A^{uu1}_{N/2} = 2 \]

so

\[ \Delta^E_{0,1} = 3 \quad \Delta^E_{2,1} = 2. \]
Then, according to (9.26)

$$\lim_{n \to \infty} \left\{ S_{u}^{a} u_{j}^{1}(n) \times S_{u}^{a} u_{j}^{1}(n) \right\} = \left\{ 3 \ C_{0} + 2 \ C_{j}^{1} \right\} \times C_{j}^{1} \quad (9.27)$$

where 

$$C_{0} = -\frac{4}{9} \ V + \frac{1}{18} \sum_{i=1}^{4} E_{i} + \frac{1}{18} \sum_{i=1}^{4} F_{i}$$

and, taking \( j = 4 \)

$$C_{j}^{1} = C_{4}^{1} = \frac{1}{3} \left( E_{4} - E_{2} \right) + \frac{1}{12} \left( F_{4} - F_{1} - F_{2} + F_{3} \right)$$

$$C_{j}^{2} = C_{2}^{1} = \frac{1}{4} \sum_{i=1}^{4} (-1)^{i} E_{i}$$

Evaluating directly, we have the set of control points

$$[R] = \begin{bmatrix} E_{2} & E_{2} & E_{1} & I_{13} \\ E_{3} & V & E_{1} & I_{12} \\ F_{3} & E_{4} & F_{4} & I_{11} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}$$

and 

$$S^{u} = [0, 0, 1, 0][M][R][M]^T[0, 0, 0, 1]^T$$

where \([M]\) is given in (1.3)

$$= \frac{1}{3} \left( E_{4} - E_{2} \right) + \frac{1}{12} \left( F_{4} - F_{1} - F_{2} + F_{3} \right)$$

$$= C_{j}^{1} \text{ as in (8.12)}.$$
\[ S_{uu} = [0, 2, 0, 0][M][R][M]^t[0, 0, 0, 1]^t \]

\[ = \frac{1}{6} [1, -2, 0, 0][R][1, 4, 1, 0]^t \]

\[ = -\frac{4}{3} V + \frac{1}{6} (-2E_1 + 4E_2 - 2E_3 + 4E_4) + \frac{1}{6} \sum_{i=1}^{4} F_i \]

\[ = 3 \sum_{i=1}^{4} E_i = 3 C_0 + \frac{1}{2} \sum_{i=1}^{4} E_i \]

\[ = 3 C_0 + 2 C_2 \]

Thus, verifying (9.26) for \( N = 4 \)

\[ S_{uu} \times S_u = \{3 C_0 + 2 C_2\} \times C_1 \]

It will be convenient to divide the other cases into \( N = 3 \) and \( N > 4 \).

**Case I: \( N = 3 \)**

When \( N = 3 \), \( \lambda_2 < \lambda_1^2 \) and so the third term of (9.26) disappears.

For the surface to be \( C^1 \) smooth we must have \( \lambda_a < \lambda_1 \), so

\[ \lambda_a \lambda_2/\lambda_1^3 < \lambda_2/\lambda_1^2 < 1 \]

and the second term also disappears.

If we choose \( \lambda_a < \lambda_1^2 \) the first term, moreover, will vanish and so the curvature will be zero.

If we choose \( \lambda_a > \lambda_1^2 \) the curvature will become infinite.
If \( \lambda_a = \lambda_1^2 \), the limiting curvature vector will be, from (7.33) and (9.26)

\[
\begin{align*}
\frac{A_1^{u^1} c_j \times \Delta_0^E (c_0 \times c_1^j)}{(A_1^{u^1})^4 (c_1^j \cdot c_1^j)^2} \\
= (A_1^{u^1})^{-3} \Delta_0^E c_1^j \times (c_0 \times c_1^j) (c_1^j \cdot c_1^j)^2 
\end{align*}
\]

Thus the curvature on the \( j \)th edge is given by

\[
K^j(E) = (A_1^{u^1})^{-3} \Delta_0^E |c_0 \times c_1^j| |c_1^j|^{-3}
\]

**Case II : \( N > 4 \)**

When \( N > 4 \), \( \lambda_2 > \lambda_1^2 \) and so the third term of (9.26) tends to infinity, unless of course \( c_2^j \times c_1^j = 0 \); thus, in general, the curvature on the edge locus is infinite. However, as we saw in Chapter 8, with reasonable assumptions of rotational symmetry, \( c_2^j \) is unlikely to be significant. This unwelcome property will be, as we shall show, in most cases very local.

Since \( \lambda_2 / \lambda_1 < 1 \), then

\[
\frac{\lambda_a}{\lambda_1} \frac{\lambda_2}{\lambda_1^3} < \frac{\lambda_a}{\lambda_1^2}
\]

and so the second term can be ignored.

According as \( \lambda_a \) or \( \lambda_2 \) is the larger, one or both of the terms

\[
\Delta_0^E (c_0 \times c_1^j)(\lambda_a / \lambda_1^2)^n
\]
and 
\[ \Lambda_{2,1}^E \left( C_2^j \times C_1^j \right) \left( \lambda_2 / \lambda_1^2 \right)^n \]

will dominate.

These will yield curvature vectors

\[ \lim_{n \to \infty} (\Lambda_{1}^u)^{-3} \Delta_0,1 \frac{C_1^j \times \left( C_0^j \times C_1^j \right)}{\left( C_1^j \cdot C_1^j \right)^2} \left( \lambda_a / \lambda_1^2 \right)^n \quad (9.30) \]

\[ \lim_{n \to \infty} (\Lambda_{1}^u)^{-3} \Delta_2,1 \frac{C_1^j \times \left( C_2^j \times C_1^j \right)}{\left( C_1^j \cdot C_1^j \right)^2} \left( \lambda_2 / \lambda_1^2 \right)^n \quad (9.31) \]

respectively.

We recall that, in the even node case, opposite edge loci align as do opposite face loci, but that in the odd node case edge loci align with opposite face loci. We consider, then, the vector

\[ \frac{\hat{\epsilon} \times \left( \hat{\epsilon} \times \hat{\epsilon} \right)}{(\hat{\epsilon} \cdot \hat{\epsilon})^2} \]

evaluated on the face loci.

Now we have from (9.21) and (9.23), putting \( u = v = t \), that

\[ \hat{\epsilon} = S^{uj^2} + S^{vj^2} \]

\[ = B_{e^2} \left\{ \left( 1 + c_1 \right) C_1^j + s_1 S_1^j \right\} \]

\[ = S^{e^2 j^2} \text{ say.} \]

From (9.3)

\[ \ddot{r} = (\ddot{u})^2 \dot{r}^{uu} + 2 \ddot{u} \dot{v} \dot{r}^{uv} + (\ddot{v})^2 \dot{r}^{vv} + \dddot{u} \dot{r}^u + \dddot{v} \dot{r}^v \]

\[ = \dddot{r}^{uu} + 2 \dddot{r}^{uv} + \dddot{r}^{vv} \]
\[
\begin{align*}
\mathcal{S}^{ttj^2} &= \mathcal{S}^{uuj^2} + 2 \mathcal{S}^{uvj^2} + \mathcal{S}^{vvj^2} \\
&= \mathcal{S}^{ttj^2} \text{ say.}
\end{align*}
\]

\[
\begin{align*}
\mathcal{S}^{ttj^2} &= \mathcal{S}^{uuj^2} + 2 \mathcal{S}^{uvj^2} + \mathcal{S}^{vvj^2} \\
&= \sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{uu} + 2 D_{\omega}^{uv} + D_{\omega}^{vv}] [R_{\omega}] \\
&= \sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{tt}] [R_{\omega}] \text{ say.} \\
&= \sum_{\omega=0}^{N-1} a_{\omega j} [D_{\omega}^{tt^2}] [R_{\omega}] \text{ say.} \quad (9.32)
\end{align*}
\]

Then \([D_0^{tt^2}] = [D_0^{uu} + 2 D_0^{uv} + D_0^{vv}] \]

\[
\begin{align*}
&= \frac{1}{6} [5, 4, -16, 2, -2, 2, 5] \quad (9.33)
\end{align*}
\]

\[
\begin{align*}
[D_{\omega}^{tt^2}] &= [D_{\omega}^{uu} + 2 D_{\omega}^{uv} + D_{\omega}^{vv}] \\
&= \frac{1}{6} [2(1 + a_{\omega}), -16, 2a_{\omega}, -(1 + a_{\omega}), 2, 5] \quad (9.34)
\end{align*}
\]

for \(\omega \neq 0\).

Proceeding as on the edge loci we have

\[
\begin{align*}
[D_0^{tt^2}] v_0 &= 0 \\
[D_0^{tt^2}] v^1 &= \frac{8(\lambda - 1)(81920\lambda^4 - 32256\lambda^3 + 4768\lambda^2 + 648\lambda + 5)}{3(8\lambda - 1)(32\lambda^2 - 1)(64\lambda^2 - 1)} \\
&= \theta_0^{tt^2} \text{ say.} \quad (9.35)
\end{align*}
\]

\[
\begin{align*}
[D_{\omega}^{tt^2}] v_{-\omega} &= \frac{16(2\lambda - 1)(3584\lambda^3 - 6512\lambda^2 + 391\lambda + 17)}{3(16\lambda - 1)(32\lambda^2 - 1)(64\lambda^2 - 1)} (1 + a_{-\omega}) \\
&= \theta_{\omega}^{tt^2} (1 + a_{-\omega}) \text{ say.} \quad (9.36)
\end{align*}
\]

for \(0 < \omega < \frac{N}{2}\).
\[ [D_{N}^{t2}] \frac{v}{2} = 0 \]

\[ [D_{N}^{t2}] \frac{v'}{2} = 2 = B_{N}^{t2} \text{ say.} \]

Now \[ [D_{N}^{t2}] = \frac{1}{6} \begin{bmatrix} -1, -4, 0, 2, 0, 2, 1 \end{bmatrix} \]

and \[ [D_{\omega}^{t2}] = \frac{1}{6} \begin{bmatrix} -2(1 + \alpha), 0, 2\alpha, 0, 2, 1 \end{bmatrix} \]

for \( \omega \neq 0 \)

as we saw earlier.

\[ [D_{0}^{t2}] \frac{v}{0} = 0 \]

\[ [D_{0}^{t2}] \frac{v}{0} = \frac{4(1-\lambda_{a})(16\lambda +1)(2048\lambda_{a}^{3} + 2560\lambda_{a}^{2} - 260\lambda_{a} -1)}{3(8\lambda_{a} -1)(32\lambda_{a} -1)(64\lambda_{a} -1)} \]

\[ = B_{0}^{t2} \text{ say.} \quad (9.37) \]

From (9.22) \[ [D_{\omega}^{t2}] \frac{v}{-\omega} = B_{\omega}^{t2}(1 + \alpha \omega) \]

for \( 0 < \omega < \frac{N}{2} \).

\[ [D_{N}^{t2}] \frac{v}{2} = 0 \]

\[ [D_{N}^{t2}] \frac{v'}{2} = 2 = B_{N}^{t2} \text{ say.} \]

We can now evaluate the vector product \( \vec{r} \times \vec{r} \).
\[ S_{ttj2(n)} = \sum_{n=0}^{\infty} \left( \left[ D_{1}^{tt2} \right] v_{0} a_{0} z_{0j} \lambda_{n}^{a} + 2 \text{Re} \left\{ \left[ D_{1}^{tt2} \right] v_{1} a_{j} z_{1j} \lambda_{1}^{a} \right\} \lambda_{1}^{n} \right) \]

\[ + 2 \text{Re} \left\{ \left[ D_{2}^{tt2} \right] v_{2} a_{2j} z_{2j} \lambda_{2}^{a} \right\} \lambda_{2}^{n} \]

\[ \times \left\{ \left[ D_{0}^{tt2} \right] v_{0} a_{0} z_{0j} \lambda_{n}^{a} + 2 \text{Re} \left\{ \left[ D_{1}^{tt2} \right] v_{1} a_{j} z_{1j} \lambda_{1}^{a} \right\} \lambda_{1}^{n} \right\} \lambda_{1}^{n} \]

\[ + 2 \text{Re} \left\{ \left[ D_{2}^{tt2} \right] v_{2} a_{2j} z_{2j} \lambda_{2}^{a} \right\} \lambda_{2}^{n} \]

as in (9.25)

\[ = \left\{ b_{0}^{tt2} c_{0} \lambda_{n}^{a} + b_{1}^{tt2} \left[ (1 + c_{1}) c_{0}^{j} + s_{1} s_{0}^{j} \right] \lambda_{1}^{a} \lambda_{1}^{n} \right\} \]

\[ + b_{2}^{tt2} \left[ (1 + c_{2}) c_{0}^{j} + s_{2} s_{0}^{j} \right] \lambda_{2}^{a} \lambda_{2}^{n} \]

\[ \times \left\{ b_{0}^{tt2} c_{0} \lambda_{n}^{a} + b_{1}^{tt2} \left[ (1 + c_{1}) c_{0}^{j} + s_{1} s_{0}^{j} \right] \lambda_{1}^{a} \lambda_{1}^{n} \right\} \lambda_{1}^{n} \]

\[ + b_{2}^{tt2} \left[ (1 + c_{2}) c_{0}^{j} + s_{2} s_{0}^{j} \right] \lambda_{2}^{a} \lambda_{2}^{n} \]

\[ = \left| \begin{array}{cc} b_{0}^{tt2} & b_{1}^{tt2} \\ b_{0}^{tt2} & b_{1}^{tt2} \end{array} \right| \left\{ b_{0}^{tt2} c_{0} \lambda_{n}^{a} + b_{1}^{tt2} \left[ (1 + c_{1}) c_{0}^{j} + s_{1} s_{0}^{j} \right] \lambda_{1}^{a} \lambda_{1}^{n} \right\} \lambda_{1}^{n} \lambda_{1}^{n} \]

\[ + \left| \begin{array}{cc} b_{0}^{tt2} & b_{2}^{tt2} \\ b_{0}^{tt2} & b_{2}^{tt2} \end{array} \right| \left\{ b_{0}^{tt2} c_{0} \lambda_{n}^{a} + b_{2}^{tt2} \left[ (1 + c_{2}) c_{0}^{j} + s_{2} s_{0}^{j} \right] \lambda_{2}^{a} \lambda_{2}^{n} \right\} \lambda_{2}^{n} \lambda_{2}^{n} \]
and scaling by $\lambda_1^{-3}$ as before gives

$$\Delta_{0,1}^F \left\{ C_0 \times [(1 + c_1) C^i + s_1 s^j] \right\} \left( \frac{\lambda_a}{\lambda_1^2} \right)^n$$

$$+ \Delta_{0,2}^F \left\{ C_0 \times [(1 + c_2) C^j + s_2 s^j] \right\} \left( \frac{\lambda_a}{\lambda_2^3} \right)^n$$

$$+ \Delta_{2,1}^F \left\{ [(1 + c_2) C^j + s_2 s^j] \times [(1 + c_1) C^j + s_1 s^j] \right\} \left( \frac{\lambda_2}{\lambda_1^2} \right)^n$$

(9.38)

As with the case of the edge loci, we can compare this to $N = 4$, where we can evaluate directly. As before the second term disappears in the limit and $\lambda_a/\lambda_1^2 = \lambda_2/\lambda_1^2 = 1$

$$B_0^{ttt} = 6 \quad B_1^{ttt} = 1 \quad B_2^{ttt} = \left[ D_N^{ttt} \right] \frac{v'_N}{N} = 2$$

$$B_0^{ttt} = 6 \quad B_1^{ttt} = 0 \quad B_2^{ttt} = \left[ D_N^{ttt} \right] \frac{v'_N}{N} = 2$$

$$(1 + c_1) C^j + s_1 s^j = C^j + s^j$$

and $(1 + c_2) C^j + s_2 s^j$ is replaced by $s^j$

$$\Delta_{0,1}^F = 6 \quad \Delta_{2,1}^F = 2.$$

Hence

$$\lim_{n \to \infty} \left\{ \frac{C^{tij2(n)}}{s^{tij2(n)}} \times \frac{s^{tij2(n)}}{s^{tij2(n)}} \right\}$$

$$= \left\{ 6 C_0 + 2 s^j \right\} \times \left\{ C^j + s^j \right\}$$

(9.39)
Taking $j = 4$ and evaluating directly we have

$$\frac{s_j}{s_2} = \frac{1}{4} \sum_{i=1}^{4} (-1)^i F_i$$

$$C_j^1 + \frac{s_j}{s_1} = \frac{1}{3} (E_4 + E_1 - E_2 - E_3) + \frac{1}{6} (F_4 - F_2)$$

$$s_{ttj}^2 = C_j^1 + \frac{s_j}{s_1}$$

$$s_{ttj}^2 = s_{uu} + 2 s_{uv} + s_{vv}$$

$$= \frac{1}{6} [1, -2, 1, 0][R][1, 4, 1, 0]^T$$

$$+ \frac{1}{2} [-1, 0, 1, 0][R][-1, 0, 1, 0]^T$$

$$+ \frac{1}{6} [1, 4, 1, 0][R][1, -2, 1, 0]^T$$

$$= -\frac{8}{3} v + \frac{1}{3} \sum_{i=1}^{4} E_i + \frac{1}{6} (5 F_4 - F_1 + 5 F_2 - F_3)$$

$$= 6 C_0^j + 2 \frac{s_j}{s_2}.$$  

Thus (9.39) is verified and so (9.38) for $N = 4$.

Returning to our earlier categorisation, we have

**Case I : $N = 3$**

We have seen that, to obtain finite non-zero curvature we must choose $\lambda = \lambda_1^2$. In the limit, from (9.23) and (9.38), the curvature vector will be
Thus the curvature on the \( j \)th face is given by

\[
K^j(F) = (B_1^{t2})^{-3} \Delta^F_{0,1}[C_0 \times [(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}]] \frac{(\lambda_a / \lambda_1^2)^n}{\left\{[(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}].[(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}]\right\}^2}
\]

(9.40)

Case II: \( N > 4 \)

When \( N > 4 \), \( \left(\lambda_2 / \lambda_1^2\right)^n \to \infty \) and so the third term in (9.38) will dominate, unless \( \frac{C_j}{C_1} \) and \( \frac{S_j}{S_1} \) are insignificant. In any case one or both of the terms

\[
\Delta^F_{0,1}\left\{C_0 \times [(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}]\right\} \frac{(\lambda_a / \lambda_1^2)^n}{\left\{[(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}].[(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}]\right\}^2}
\]

\[
\Delta^F_{2,1}\left\{[(1 + c_2) \frac{C_j}{C_2} + s_2 \frac{S_j}{S_2}] \times [(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}]\right\} \frac{(\lambda_2 / \lambda_1^2)^n}{\left\{[(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}].[(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}]\right\}^2}
\]

will dominate, according to which of \( \lambda_a, \lambda_2 \) is larger, giving curvature vectors

\[
\lim_{n \to \infty} (B_1^{t2})^{-3} \Delta^F_{0,1}\left\{C_0 \times [(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}]\right\} \frac{(\lambda_a / \lambda_1^2)^n}{\left\{[(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}].[(1 + c_1) \frac{C_j}{C_1} + s_1 \frac{S_j}{S_1}]\right\}^2}
\]

(9.41)
We note, from (9.31) and (9.42), that unless \( \bar{c}_2 \) and \( \bar{s}_2 \) are negligible both the edge and face curvature vectors will become infinite in the limit. It will be instructive, then, to examine the implications of this, and the other formulae we have derived, in more detail, in particular to compute the ratio of curvatures on opposite loci at the limit point. We will have continuity of curvature here, as of course we have everywhere else, only if this ratio is unity. From Corollary 9.5 it will be necessary to subdivide Case II, where \( N > 4 \), and consider \( N \) odd and \( N \) even separately.

**Case I : \( N = 3 \)**

\( N \) is odd so we must compare the curvatures on the \( j^{th} \) edge and \( (j + \frac{N-1}{2})^{th} \) face loci. We have seen that, to obtain finite non-zero curvature, we must take \( \lambda_a = \lambda_1^2 \).

By convention, in the odd node case, we shall evaluate the ratio \( K(F)/K(E) \).

When \( N = 3 \), from (9.29)

\[
K^j(E) = (A_1^{u1})^{-3} \Delta_0,1 \frac{E}{C_0} \times \frac{C_j}{C_1} \frac{|C_j|^{-3}}{|C_1|} \quad (9.43)
\]
On the opposite face locus we have

\[
(1 + c^1_1) \frac{j + N-1}{2} + s_1 \frac{j + N-1}{2}
\]

\[
= \quad -2 c^j_1 \frac{c^j_1}{c^j_1 - 1} \quad \text{by Theorem 9.4.}
\]

Thus, from (9.41)

\[
k^n = (B^t_1)^{-3} F \frac{(-2 c^j_1 c^j_1) \times \left\{c^0_0 \times (-2 c^j_1 c^j_1) \right\}}{\left\{ (-2 c^j_1 c^j_1). (-2 c^j_1 c^j_1) \right\}^2}
\]

\[
= \frac{1}{4} c^j_1 \quad (B^t_1)^{-3} F \frac{c^j_1 \times (c^0_0 \times c^j_1)}{(c^j_1 c^j_1)^2}
\]

Hence \( k^{(F)} = \frac{1}{4} c^j_1 (B^t_1)^{-3} F \frac{c^j_1 \times (c^0_0 \times c^j_1) \delta_{0,1}^F}{c^j_1 c^j_1 - 1} \)  

(9.44)

Comparing (9.43) and (9.44) we have

\[
K(F)/K(E) = \frac{1}{4} c^j_1 (B^t_1)^{-3} F \frac{(A^u_1)^{-3} \delta_{0,1}^E}{(B^t_1)^{-3} F \delta_{0,1}^E}
\]

\[
= \frac{(A^u_1)^{-3} \delta_{0,1}^F}{4 c^2_1 (B^t_1)^{-3} F \delta_{0,1}^E}
\]

(9.45)

which is, of course, independent of \( j \).

Substituting in the expressions given in (7.28), (7.38), (9.22), (9.35), (9.36) and (9.38) yields, after some cancellation

\[
\frac{(1 - \lambda)(4\lambda + 1)^2 (16\lambda - 1)^2 (32\lambda - 1)^2 (64\lambda - 1)^2 P_{12}(\lambda)}{2(4\lambda - 1)(8\lambda - 1)(8\lambda + 1)(32\lambda^2 - 1)^2 \lambda^2 [P_4(\lambda)]^3}
\]

(9.46)
where \( P_{12}(\lambda) = \)

\[
2952790016\lambda^{12} - 1631584256\lambda^{11} - 3422289920\lambda^{10} + 3214671872\lambda^9 - 793862144\lambda^8 + 123965440\lambda^7 + 9850608\lambda^6 - 41908480\lambda^5 + 1099040\lambda^4 + 18016\lambda^3 - 19288\lambda^2 + 3008\lambda + 13
\]

\( P_4(\lambda) = 131072\lambda^4 - 112896\lambda^3 - 20272\lambda^2 + 2376\lambda + 35 \)

and \( \lambda = \lambda_1 \).

When \( N = 3 \) \( \lambda_1 = \frac{9 + \sqrt{17}}{32} = 0.410097 \)

and \( \lambda_a = \frac{\lambda_1^2}{\lambda_1} = 49 + 9\sqrt{17} \over 512 = 0.168180 \).

The ratio in (9.46) becomes 0.905772.

Consequently we cannot obtain non-trivial \( C^2 \) continuity here since

a) Choosing \( \lambda_a = \lambda_1^2 \) to obtain finite non-zero curvature we find that the edge and face curvatures differ by 10%.

b) Choosing \( \lambda_a < \lambda_1^2 \), since \( (\lambda_a / \lambda_1^2)^n \) and \( (\lambda_2 / \lambda_1^2)^n \) tend to zero the surface will be locally flat.

c) If \( \lambda_a > \lambda_1^2 \) the surface will have infinite curvature at the limit point.

Catmull and Clark [8] choose the subdivision weightings
A = 1 \quad 7
\frac{-}{4N}

\begin{align*}
B &= \frac{3}{2N} \quad , \\
C &= \frac{1}{4N}
\end{align*}

which give

\[ \lambda_a, \lambda_b = \frac{3N - 7 \pm (5N^2 - 30N + 49)^{1/2}}{8N} \quad (9.47) \]

If \( N = 3 \)

\[ \lambda_a = \frac{1}{6} < \lambda_1^2 \]

\[ \lambda_b = 0 \]

and thus the surface generated will be locally flat at the extraordinary point.

Originally they had chosen

\begin{align*}
A &= \frac{9}{16} \\
B &= \frac{3}{8} \\
C &= \frac{1}{16}
\end{align*}

as in the 4-node case but found the resulting surface "too pointy".

We may now appreciate why since

\[ \lambda_a = \frac{1}{4} > \lambda_1^2 \]

\[ \lambda_b = \frac{1}{16} \quad . \]

Case II: \( N > 4 \)

From (9.31) and (9.42), the curvature at the limit point will be infinite unless \( \frac{C_2}{S_2} = \frac{S_2}{C_2} = 0 \). There is no contradiction in having a \( C^1 \) continuous surface which has infinite curvature.
at some point. Indeed we find examples of plane curves with this property, for example

\[ y = x^{4/3} \text{ at } x = 0. \]

We must ascertain how significant this phenomenon is, in terms of the breakdown in continuity of curvature. The inescapable factor which brings about this infinite curvature is \( \lambda_2 / \lambda_1^2 \). Table 9.1 shows how this varies with \( N \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_2 / \lambda_1^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.41010</td>
<td>0.15240</td>
<td>0.90619</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>0.25</td>
<td>1.0</td>
</tr>
<tr>
<td>5</td>
<td>0.54999</td>
<td>0.34011</td>
<td>1.12437</td>
</tr>
<tr>
<td>6</td>
<td>0.57968</td>
<td>0.41010</td>
<td>1.22042</td>
</tr>
<tr>
<td>7</td>
<td>0.59851</td>
<td>0.46186</td>
<td>1.28935</td>
</tr>
<tr>
<td>8</td>
<td>0.61112</td>
<td>0.5</td>
<td>1.33882</td>
</tr>
<tr>
<td>9</td>
<td>0.61994</td>
<td>0.52843</td>
<td>1.37496</td>
</tr>
<tr>
<td>10</td>
<td>0.62634</td>
<td>0.54999</td>
<td>1.40195</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0.65451</td>
<td>0.65451</td>
<td>1.52786</td>
</tr>
</tbody>
</table>

**Table 9.1 : Eigenvalues and Ratios**

It can be seen that the rate at which the curvature increases with each subdivision grows larger with \( N \). With regard to the ratio of curvatures on either side of the extraordinary point we must, as remarked above, distinguish between the odd and even nodes.
a) N even.

$C^2$ continuity is, in general, precluded since $\lambda_2/\lambda_1^2 > 1$. Nevertheless it will still be productive to consider the ratio of the curvatures on either side of the extraordinary point on the edge and face loci which we evaluated earlier.

With regard to the edge locus we must consider (9.30) and (9.31) for opposite edges, that is for $j$ and $j + \frac{N}{2}$. It was shown in (9.20) that

$$\frac{(j + \frac{N}{2})}{C_1} = -\frac{c_j}{C_1};$$

since $C_0$ is independent of $j$ and

$$\frac{(j + \frac{N}{2})}{C_2} = \frac{c_j}{C_2};$$

it is immediately apparent that the curvature ratio in the limit is unity. In addition,

$$\frac{(j + \frac{N}{2})}{S_1} = -\frac{s_j}{S_1}$$

and

$$\frac{(j + \frac{N}{2})}{S_2} = \frac{s_j}{S_2};$$

so it follows from (9.41) and (9.42) that this is also the case on the face loci.

In fact, we can extend this equality further. At each subdivision, the boundary of the surface and the hole which remains consists of, for each $j$, the $u = 0$ edge of patch 1 and the $v = 0$ edge of patch 3, (see Fig. 9.3). We call these the leading edges of the subdivision surface. For even $N$, they line up with each
other as expected, e.g. on patch 1, edge $j$, the locus at the point with parameter values $u = 0$, $v = v_0$, ($P_j$ say) in the direction of $\nu$, the vertex point, will, in the limit, align with a similar locus on patch 1, edge $j + \frac{N}{2}$, at the point with the same parameter values.

In this terminology

$$\lim_{n \to \infty} \left( P_j^{(n)} - V_j^{(n)} \right) = - \lim_{n \to \infty} \left( P_j^{(n)} - V_j^{(n)} \right)$$

scaling at each subdivision by $\lambda_1^{-1}$, or in our earlier terminology

$$\frac{\dot{x}_j}{j + \frac{N}{2}} = - \frac{\dot{x}_j}{j + \frac{N}{2}} \quad \text{(9.48)}$$

Moreover, since the relevant components of the second derivative, i.e. those corresponding to $\lambda_1^n$ and $\lambda_2^n$, are the same in the limit on either side

$$\frac{\ddot{x}_j}{j + \frac{N}{2}} \times \frac{\dot{x}_j}{j + \frac{N}{2}} = - (\ddot{x}_j \times \dot{x}_j) \quad \text{(9.49)}$$

Thus

$$K_{j + \frac{N}{2}} = \left| \frac{\ddot{x}_j}{j + \frac{N}{2}} \times \left( \frac{\ddot{x}_j}{j + \frac{N}{2}} \times \frac{\dot{x}_j}{j + \frac{N}{2}} \right) \right|$$

$$= \left| \frac{\dot{x}_j \times (\ddot{x}_j \times \dot{x}_j)}{(\dot{x}_j \cdot \dot{x}_j)^2} \right|$$

$$= K_j.$$
Fig. 9.3 A Pencil of Corresponding Loci across a 6-node
Hence if the curvatures are finite they are equal. This will occur only if $C_2 = S_2 = 0$ and $\lambda_1 \leq \lambda^2_a$; of course if $\lambda_a < \lambda^2_1$ the surface will have a local flat spot.

In the case of rotational symmetry, the normal curvature on each of the edge loci must be equal. Since there are $N/2 \geq 3$ of these in different (but not opposite) directions and since $C_2 = S_2 = 0$, choosing $\lambda_a = \lambda^2_1$ will give a $C^2$ surface with (by Corollary 9.3) equal finite curvature in all directions. Thus we have a local spherical spot.

In certain circumstances it is possible to generate a doubly curved $C^2$ surface. These arise where there is no rotational symmetry but $C_2 = S_2 = 0$ and $\lambda_a = \lambda^2_1$. It must be stressed, however, that this is not generally the case since it is contingent on the configuration of control points and not simply the subdivision weightings.

On a more positive note, Charrot [9] remarks that the most common node of this type is of order 6. In this case the ratio $\lambda_2/\lambda^2_1$ is closest to unity and so the effect of infinite curvature is least likely to be significantly felt, irrespective of the configuration of points, in the number of subdivisions required for the numerical description of the surface.

In the "three-box" case, $C_0 = C_2 = S_2 = 0$ and so, whatever values we choose for $A$, $B$ and $C$, the surface will be locally flat. In fact since

$$\sum_{i=1}^{6} E_i = \sum_{i=1}^{6} F_i = 0,$$

we are bound by symmetry to put $\sum = \sum$ and so the resulting surface is uniquely determined.
b) N odd.

Here we begin by comparing curvatures on edge and face loci, which, from Theorem 9.4, (9.30), (9.31), (9.41) and (9.42) we know to be aligned in both 1st and 2nd derivatives. We suppose first that $C_2$ and $S_2$ are negligible. As is the 3-node case we take $\lambda_a = \lambda_1^2$ to give the curvature ratio in (9.45). Table 9.2 shows these values for odd $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$K(F)/K(E)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.90577</td>
</tr>
<tr>
<td>5</td>
<td>1.04744</td>
</tr>
<tr>
<td>7</td>
<td>1.11594</td>
</tr>
<tr>
<td>9</td>
<td>1.15927</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1.25757</td>
</tr>
</tbody>
</table>

**TABLE 9.2: CURVATURE RATIOS**

Thus we see, for example, in the 5-node case, the disparity in curvature across the extraordinary point from face locus to edge locus is less than 5%. In this instance, the curvatures are finite and so this figure represents a direct comparison between their magnitudes on either side. However, as we noted in Chapter 8 with the "two-box" configuration, it is quite possible that $C_2$ and $S_2$ will be significant and thus the curvature will not be finite. Under these circumstances we adopt a different approach.

From Theorem 9.4 we have the identities
\begin{equation}
(1 + c_1) \frac{j + N-1}{2} + s_1 \frac{j + N-1}{2} = -2 c_1 \frac{c_j}{c_{j-1}}
\end{equation}

\begin{equation}
(1 + c_2) \frac{j + N-1}{2} + s_2 \frac{j + N-1}{2} = 2 c_1 \frac{c_j}{c_{j-1}}
\end{equation}

Thus (9.42), the expression for the face curvature vector, gives

\begin{equation}
\frac{1}{2} c_4 c_1 (B_1^t)^{-3} \Delta F \frac{c_j^j \times (c_j^j \times c_j^j)}{(c_j^j \cdot c_j^j)^2} (\lambda_2 / \lambda_1)^n
\end{equation}

From (9.31) the edge curvature vector is

\begin{equation}
(A_1^{u_1})^{-3} \Delta E \frac{c_j^j \times (c_j^j \times c_j^j)}{(c_j^j \cdot c_j^j)^2} (\lambda_2 / \lambda_1)^n
\end{equation}

n being the number of subdivisions performed.

Now let \( C(E) = (A_1^{u_1})^{-3} \Delta E \frac{|c_j^j \cdot c_j^j|}{|c_j^j|^3} \)

and \( C(F) = \frac{1}{2} c_4 c_1 (B_1^t)^{-3} \Delta F \frac{|c_j^j \cdot c_j^j|}{|c_j^j|^3} \)

Our initial estimation of the ratio would be \( C(F)/C(E) \); these values are given in Table 9.3

<table>
<thead>
<tr>
<th>N</th>
<th>( C(F)/C(E) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.93657</td>
</tr>
<tr>
<td>5</td>
<td>1.02742</td>
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<td>7</td>
<td>1.13915</td>
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<tr>
<td>9</td>
<td>1.25241</td>
</tr>
<tr>
<td>\infty</td>
<td>1.64302</td>
</tr>
</tbody>
</table>

\textbf{TABLE 9.3 : ESTIMATED RATIOS}
However, it must be borne in mind that we are comparing quantities which tend, in the limit, to infinity and a high degree of circumspection is required in such circumstances. The ratios shown above can rightly be interpreted as a comparison of curvatures on opposite sides of the "hole" on the edge/face locus after a given number of subdivisions. In the limit, we must ensure that we are comparing like with like in the sense of curvatures an equal distance from the extraordinary point.

The relative distances of the edge and face points from the extraordinary point are given by the relative magnitudes of the tangent vectors. We noted, in the remarks following Corollary 9.5, that these are

\[ 2 \frac{B^2}{C} \] for the face locus

\[ \frac{A^1}{N} \] for the edge locus

and that these values are not equal.

Now in the limit, one subdivision scales the distances by a factor of \( \lambda_1 \) and the curvatures by a factor of \( \lambda_2/\lambda_1^2 \). We find an exponential function which interpolates this behaviour.

Let \( a, C, \lambda \) and \( \mu \) be constants with \( a > 0, \ C > 0, \ \lambda < 1 \) and \( \mu > 1 \). The exponential function which satisfies

\[ f(\lambda^n a) = C \mu^n \]

is given by

\[ f(x) = C \mu^{\frac{\log x - \log a}{\log \lambda}}. \] (9.51)

Let \( f \) represent the curvature on either side of the extraordinary point. We take \( 0 \) to represent this point, \( f'(x) \) to
represent the curvature at a distance $x$ along the edge curve and $f^-(x)$ similarly along the face curve. The corrected ratio in the limit is given therefore by

$$\lim_{x \to 0} \frac{f^-(x)}{f^+(x)}.$$

But on the edge curve we have

$$f^+(\lambda_1^a) = C(E)(\lambda_2/\lambda_1)^n$$

where $a = A_{1u}^1$.

Thus from (9.51)

$$f^+(x) = C(E)(\lambda_2/\lambda_1^2).$$

Similarly on the face curve

$$f^-(\lambda_1^b) = C(F)(\lambda_2/\lambda_1)^n$$

where $b = 2 B_1^{t2} c_1$

and so

$$f^-(x) = C(F)(\lambda_2/\lambda_1^2).$$

Thus

$$\lim_{x \to 0} \frac{f^-(x)}{f^+(x)} = \frac{C(F)}{C(E)} \left(\frac{\lambda_2}{\lambda_1^2}\right).$$

and so for a corrected estimate of the curvature ratio we must modify the figures in Table 9.3 by a factor of

$$\frac{\log a - \log b}{\log \lambda_1} \left(\frac{\lambda_2}{\lambda_1^2}\right)$$

these are given in Table 9.4.
<table>
<thead>
<tr>
<th>N</th>
<th>K(F)/K(E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.89134</td>
</tr>
<tr>
<td>5</td>
<td>1.08919</td>
</tr>
<tr>
<td>7</td>
<td>1.29194</td>
</tr>
<tr>
<td>9</td>
<td>1.46564</td>
</tr>
<tr>
<td>∞</td>
<td>2.02293</td>
</tr>
</tbody>
</table>

**TABLE 9.4: CORRECTED RATIOS**

As we might expect, these results are in all cases worse than those given in Table 9.2.

We note, nonetheless, that in the 5-node case, the most likely in practice, the discrepancy is still only 9%.

Now we have only calculated the curvature ratios for odd N on the edge/face locus; although we can evaluate the curvature itself along the leading edge in the direction of the extraordinary point, there is no simple algebraic formula to match the loci on \( u = 0 \) on patch 1 with the opposite loci on the corresponding patch 3 (see Fig. 9.4). Nor can we use the elegant results from differential geometry developed earlier in the chapter, as we have shown that the surface is not, in general, \( C^2 \).

It is, however, important to establish that the ratios we have calculated represent the maximum disparity. It suffices to show that the curvature along this leading edge in the direction of the extraordinary point attains its extrema at either end. The author has written a program, with user definition of the configuration of control points, which evaluates these curvatures
Fig. 9.4 A Pencil of Corresponding Loci across a 5-node
and it is found that something stronger emerges, namely that the variation is monotonic. Examples of this for $N = 3, 5, 7$ are shown in Figs. 9.5 - 7.

If $N$ is odd, then, the maximum discrepancy in curvature occurs on the edge/face loci. By choosing $A$, $B$ and $C$ so that $\lambda_a = \lambda_1^2$ we can reduce this to 9%, at worst, in the 5-node case and less than 5% if $C_2$ and $S_2$ are negligible. As $N$ increases, the situation deteriorates due primarily to the predominance of $(\lambda_2/\lambda_1^2)^n$.

9.3 CONCLUSION.

We are now in a position to summarize our results and assess them against some general demands of surface definition. It has been remarked that, where the configuration of the surface is rectangular, several techniques, with their applicability and limitations, are well known. Our purpose here, then, has been to develop a method for the generation of a surface over a locally non-rectangular topology; Catmull and Clark [8] have shown how, in the context of B-splines, such topological irregularities can be localised by the technique of subdivision.

In essence, then, the problem was distilled to one of subdivision and how the continuity properties of the resulting surface could be optimised by the choice of subdivision weightings in the definition of a subdivided non-4-node. The transformation representing such a subdivision was set up and the eigenproperties of this studied. It was discovered that, for each $N$, there is a natural spectrum of eigenvalues, independent of $A$, $B$ and $C$, the subdivision weightings.
Fig. 9.5
Curvature Variation
along the Leading
Edge for $N = 3$

Fig. 9.6
Curvature Variation
along the Leading
Edge for $N = 5$

Fig. 9.7
Curvature Variation
along the Leading
Edge for $N = 7$
However there remain two degrees of freedom which govern the
dependent eigenvalues $\lambda_a$ and $\lambda_b$. The eigenvalues of the natural
spectrum are all real and occur in pairs whose product is $\frac{1}{16}$; by convention we take $\lambda_1 > \lambda_2 > \ldots > \lambda_n > \lambda_1'$. Within our
area of interest, the two dependent eigenvalues are also real; we take $\lambda_a > \lambda_b$.

The inherent properties of the uniform B-spline bicubic.
patch system guarantee that the subdivision surface is $C^2$ everywhere
except at the limiting position of an irregular node. We established
in Chapters 6 and 7 that the surface is $C^1$ at this point if and only
if the subdivision weightings are chosen so that $\lambda_a < \lambda_1$. A formula
for the surface normal is obtained.

The Discrete Fourier Transform method employed in Chapter 7
proves to be a substantially more compact means of establishing
continuity conditions. In this chapter, therefore, this technique
was used to investigate the optimization of curvature properties at
the extraordinary point. This resolved into 3 distinct cases, according to $N$, the order of the node.

i) $N = 3$.

We cannot, in general, achieve non-trivial $C^2$ continuity of
the surface. If we choose the subdivision weightings such that
$\lambda_a < \lambda_1^2$, there is local flatness at the limit vertex point.
If $\lambda_a > \lambda_1^2$ the surface has infinite curvature here. However if
$\lambda_a = \lambda_1^2$ the curvature is finite but not continuous. The disparity,
irrespective of the configuration of control points, is, at most,
10% across the extraordinary point.
ii) $N > 4$.

If $N > 4$ then $\lambda_2 > \lambda_1^2$ and so, if no restrictions are imposed on the configuration of control points, the curvature will be infinite. This phenomenon will be least evident where $N = 5$ or 6, the cases of most practical significance. Moreover, if the vectors $\underline{C}_2$ and $\underline{S}_2$, given in (7.31), are close to zero, as happens when the configuration is close to rotational symmetry, by taking $\lambda_a = \lambda_1^2$ we obtain finite curvature. In any case we evaluate the maximum discrepancy in curvature across the extraordinary point.

a) If $N$ is even we find that the ratio of curvatures either side of the point is unity. Thus, if $\underline{C}_2 = \underline{S}_2 = 0$, choosing $\lambda_a = \lambda_1^2$ will generate a $C^2$ surface. Even if the size of these vectors is significant, the behaviour of a curve through the point will be the same either side.

b) If $N$ is odd, the ratio of curvatures either side will not, in general, be unity. In the 5-node case, if $\underline{C}_2$ and $\underline{S}_2$ are close to zero, choosing $\lambda_a = \lambda_1^2$ reduces the maximum disparity to less than 5%. Even if the curvature is infinite, this difference rises to only 9%. As $N$ increases, the situation deteriorates as Tables 9.2 and 9.4 show.

In general, for $N > 4$, Catmull and Clark's formula (9.47) gives $\lambda_1^2 < \lambda_a < \lambda_2$ and so the surface will not be $C^2$. In this case, the figures in Table 9.4 will apply.

The continuity properties of the subdivision technique, then, can be stated thus. The generation of a $C^0$ surface is immediate by the nature of the process itself. A $C^1$ surface follows if the
constraint of choosing \( A \) and \( B \) such that \( \lambda_a < \lambda_1 \) is observed. This is not particularly stringent as the relative size of the proscribed area in Fig. 6.2 shows.

A \( C^2 \) surface cannot, in general, be obtained at an extraordinary point. Nonetheless evaluation of curvatures here shows that, even in the worst case, by choosing \( \lambda_a = \lambda_1^2 \), we can reduce the disparity for the 3 and 5 nodes to under 10%. Examples of the 3 and 5 sided patches resulting from such an optimal choice of subdivision weightings are shown in Fig. 9.8 and 9.9.

The properties of this technique for surface definition in terms of local control, stability and concatenation are largely inherited from the uniform B-spline method described in Chapter 1. We note, however, that, setting \( \lambda_a = \lambda_1^2 \) to optimise the continuity properties of the surface, we are still left with a degree of freedom in \( \lambda_b \). This constitutes a further degree of control for the surface designer which can be used to influence the shape or fullness of the final surface. It is expected that the flexibility and local control intrinsic to this approach to areas of irregular topology will prove attractive to the designer of smooth surfaces.
Fig. 9.8 Surface round a 3-node with Optimal Curvature Properties interrogated by Planar and Cylindrical Intersects

Fig. 9.9 Surface round a 5-node with Optimal Curvature Properties interrogated by Planar and Cylindrical Intersects
Appendix A

The Technique for Simplifying $|T_1 - \lambda I|$  

$[T_1]$ is the $(2N+1) \times (2N+1)$ matrix

\[
\begin{bmatrix}
  a & b & c & b & c & \ldots & c & b & c & b & c \\
  3/8 & 3/8 & 1/16 & 1/16 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1/16 & 1/16 \\
  1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  3/8 & 1/16 & 1/16 & 3/8 & 1/16 & 1/16 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
  1/4 & 0 & 0 & 1/4 & 1/4 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
  3/8 & 0 & 0 & 1/16 & 1/16 & 3/8 & 1/16 & \ldots & 0 & 0 & 0 & 0 \\
  1/4 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & \ldots & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  1/4 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1/4 & 1/4 & 1/4 & 0 \\
  3/8 & 1/16 & 0 & 0 & 0 & 0 & \ldots & 0 & 1/16 & 1/16 & 3/8 & 1/16 \\
  1/4 & 1/4 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1/4 & 1/4 & 0 \\
\end{bmatrix}
\]

We note the rotational symmetry of the lower right hand matrix in the above partition, i.e.

\[
t_{2N+2-i,2N+2-j} = t_{i+2,j+2}
\]  \hspace{1cm} (1)

for $i, j = 1, 2, \ldots, 2N-1$.

Let $[P] = [T_1 - \lambda I]$ so $p_{ij} = t_{ij} - \lambda \delta_{ij}$

where $\delta_{ij}$ is the Kronecker delta.
Consider the following operations applied to \([P]\).

**Operation I:**
\[
C(N + 2 + \ell) = C(N + 2 + \ell) - C(N + 2 - \ell)
\]
\[
C(N + 3 - \ell) = C(N + 3 - \ell) - C(N + 1 - \ell)
\]
for \(\ell = 1, 2, \ldots, N-1\)

to give \([P']\), followed by

**Operation II:**
\[
R(2 + 2s) = \sum_{s=0}^{N-1} R(2 + 2s)
\]
\[
R(N + 2 - k) = \sum_{s=0}^{k} R(N + 2 - k + 2s)
\]
for \(k = 1, 2, \ldots, N-1\)

to give \([P'']\).

We will show that \([P'']\) is of the form

\[
\begin{bmatrix}
\Delta_0 & 0 & 0 \\
0 & \Delta_0' & 0 \\
\Delta_N & 0 & \Delta_N'
\end{bmatrix}
\]

where \(\Delta_0\) is of order \(3 \times 3\)

and \(\Delta_N, \Delta_N'\) are of order \((N-1) \times (N-1)\).

Thus
\[
|T_1 - \lambda I| = |P| = |P'| = |P''| = \Delta_0 \Delta_N \Delta_N'
\]
I) \( C(N + 2 + \xi)' = C(N + 2 + \xi) - C(N + 2 - \xi) \)

and \( C(N + 3 - \xi)' = C(N + 3 - \xi) - C(N + 1 - \xi) \)

for \( \xi = 1, 2, \ldots, N-1 \)

to give \([P'] = \{p_{ij}^l\} \).

Clearly \( |P'| = |P| \)

The first 3 columns of \([P']\) are just those of \([P]\)

\( p_{ij}^l = 0 \) for \( j \geq 4 \)

\( p_{2j}^l = 0 \) for \( j \geq \min\{2N, 7\} \)

From (1), taking \( i = N, \ j = N-\xi \) we have

\[ p_{N+2,N+2+\xi} = p_{N+2,N+2-\xi} \]

so

\[ p_{N+2,N+2+\xi} = 0 \quad (\xi = 1, 2, \ldots, N-1) \]  \hspace{1cm} (2)

Also

\[ p_{N+2-k,N+2+\xi} = p_{N+2-k,N+2-\xi} = p_{N+2-k,N+2+\xi} - p_{N+2+k,N+2+\xi} \]

(taking \( i = N-k, \ j = N+\xi \) in (1) )

\begin{align*}
&= - (p_{N+2+k,N+2+\xi} - p_{N+2+k,N+2-\xi}) \\
&= - p_{N+2+k,N+2+\xi} \\
&(k, \ \xi = 1, 2, \ldots, N-1) \hspace{1cm} (3)
\end{align*}

Since we are making use of the symmetry round \( p_{N+2,N+2} \) we must, at this point, distinguish between the odd and even case.
If \( N \) is even then \( P_{N+2,N+2} = \frac{3}{8} - \lambda \)

and the surrounding block is of the form

\[
\begin{array}{cccccc}
\frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 \\
\frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} \\
0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} \\
0 & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda
\end{array}
\]

\( \text{R}(N+2) \)

\( \text{C}(N+2) \)

i.e. \( P_{N+2+k,N+2-k} = 0 \)

\[ P_{N+2+k,N+2+k} = P_{N+2+k,N+2+k} \quad (4) \]

If \( N \) is odd then \( P_{N+2,N+2} = \frac{1}{4} - \lambda \)

and we have

\[
\begin{array}{cccccc}
\frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 \\
\frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 \\
0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 \\
0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda
\end{array}
\]

\( \text{R}(N+2) \)

\( \text{C}(N+2) \)
Thus \( P_{N+2+k,N+2-\lambda} = 0 \) unless \( k = \lambda = 1 \)

where \( P_{N+3,N+1} = \frac{1}{16} \)

Hence \( P'_{N+3,N+3} = \frac{5}{16} - \lambda \)

and \( P'_{N+2+k,N+2+\lambda} = P_{N+2+k,N+2+\lambda} \) otherwise \( (k,\lambda = 1,2,\ldots,N-1) \)  

\[
\begin{align*}
\text{II) } R_2' &= \sum_{s=0}^{N-1} R(2 + 2s) \\
R(N + 2 - k)' &= \sum_{s=0}^{k} R(N + 2 - k + 2s)
\end{align*}
\]

for \( k = 1,2,\ldots,N-1 \)

i.e. \( R_2' = R_2 + R_4 + R_6 + \ldots + R(2N) \)

\( R_3' = R_3 + R_5 + R_7 + \ldots + R(2N+1) \)

\( R_4' = R_4 + R_6 + R_8 + \ldots + R(2N) \)

\( R_5' = R_5 + R_7 + R_9 + \ldots + R(2N-1) \)

\( R_6' = R_6 + R_8 + R_{10+} + \ldots + R(2N-2) \)

\( \vdots \)

\( R(N)' = R(N) + R(N+2) + R(N+4) \)

\( R(N+1)' = R(N+1) + R(N+3) \)

\[ [P''] = \{p''_{ij}\} \]

where \( |P''| = |P'| = |P| \)

We show \( p''_{i,N+2+\lambda} = 0 \) for \( i = 1,2,\ldots,N+2 \)

\( \lambda = 1,2,\ldots,N-1 \)
Now \[ p_{1, N+2+l}'' = p_{1, N+2+l}' = 0 \]
\[ p_{N+2, N+2+l}'' = p_{N+2, N+2+l}' = 0 \]
\[ p_{2, N+2+l}'' = p_{2, N+2+l}' + \sum_{s=0}^{N-2} p_{4+2s, N+2+l}' \]
\[ = 0 + \sum_{s=0}^{N-2} p_{4+2s, N+2+l}' \]
since \[ N + 2 + l \geq N + 3 \geq \min\{2N, 7\} \]
\[ = p_{4, N+2+l}' \] (7)

For \( k = 1, 2, \ldots, N-1 \).

\[ p_{N+2-k, N+2+l}'' = \sum_{s=0}^{k} p_{N+2-k+2s, N+2+l}' \]

a) If \( k \) is even
\[ \left. \begin{array}{c}
= p_{N+2, N+2+l}'' + \sum_{s=0}^{\frac{k}{2} - 1} (p_{N+2-k+2s, N+2+l}' + p_{N+2+k-2s, N+2+l}') \\
= 0 + \sum_{s=0}^{\frac{k}{2} - 1} (p_{N+2+(k-2s), N+2+l}'' + p_{N+2-(k-2s), N+2+l}')
\end{array} \right\}
\]
\[ = \sum_{s=0}^{\frac{k}{2} - 1} (p_{N+2+(k-2s), N+2+l}'' - p_{N+2+(k-2s), N+2+l}') \]
\[ = 0. \] from (3)
b) If $k$ is odd

$$\frac{k-1}{2} \sum_{s=0}^{k-1} \left( p_{N+2-(k-2s),N+2+\ell}^r + p_{N+2+(k-2s),N+2+\ell}^r \right)$$

$$= 0 \quad \text{from (3) similarly.}$$

So $p_{N+2-k,N+2+\ell}^r = 0$ for $k = 1, 2, \ldots, N-1$

and from (7) $p_{2,N+2+\ell}^r = 0$

Hence (6) is proved.

We now show

$$p_{1j}^r = p_{2j}^r = p_{3j}^r = 0 \quad \text{for } j = 4, 5, \ldots, N+2$$

$$p_{1j}^r = p_{1j}^l = 0 \quad \text{follows immediately.}$$

Now

$$p_{2j}^r = \sum_{k=0}^{N-1} p_{2+2k,j}^l = \sum_{k=0}^{N-1} \left( p_{2+2k,j} - p_{2+2k,j-2} \right) \quad (8)$$

$$\quad \text{for } j = 4, 5, \ldots, N+2 \ .$$

We note that for $i, j \geq 4$

$$p_{ij}^l = p_{i-2,j-2} \quad (9)$$

so (8) gives

$$p_{2,j} - p_{2,j-2} + \sum_{k=1}^{N-1} \left( p_{2+2k,j} - p_{2+2k,j-2} \right)$$

$$= p_{2,j} - p_{2,j-2} + \sum_{k=1}^{N-1} \left( p_{2k,j-2} - p_{2k+2,j-2} \right)$$
\[
= p_{2,j} - p_{2,j-2} \\
+ p_{2,j-2} - p_{4,j-2} \\
+ p_{4,j-2} - p_{6,j-2} \\
+ \ldots \\
+ p_{2N-2,j-2} - p_{2N,j-2}
\]

For \( j = 4 \)

\[
p_{24} - p_{2N,2} = \frac{1}{16} - \frac{1}{16} = 0
\]

For \( j = 5, 6, \ldots, N+2 \)

\[
p_{2,j} - p_{2N,j-2} = 0 - 0 = 0.
\]

Hence

\[
p_{2,j}'' = 0 \quad (j = 4, 5, \ldots, N+2)
\]

Similarly

\[
p_{3j}'' = \sum_{k=0}^{N-1} p_{3+2k,j}
\]

\[
= \sum_{k=0}^{N-1} (p_{3+2k,j} - p_{3+2k,j-2})
\]

\[
= p_{3,j} - p_{3,j-2} + \sum_{k=1}^{N-1} (p_{1+2k,j-2} - p_{3+2k,j-2})
\]

\[
= p_{3,j} - p_{3,j-2} \\
+ p_{3,j-2} - p_{5,j-2} \\
+ p_{5,j-2} - p_{7,j-2} \\
+ \ldots \\
+ p_{2N-1,j-2} - p_{2N+1,j-2}
\]

\[
= p_{3,j} - p_{2N+1,j-2}
\]
For \( j = 4 \)
\[ P_{34} - P_{2N+1,2} = \frac{1}{4} - \frac{1}{4} = 0 \]

For \( j = 5, 6, \ldots, N+2 \)
\[ P_{3,j} - P_{2N+1,j-2} = 0 - 0 = 0 \]

Hence \( p_{3,j}^n = 0 \) \((j = 4, 5, \ldots, N+2)\)

Thus \( |P^n| \) is of the following form

\[
\begin{array}{ccc}
3 & \leftarrow & N-1 \\
\uparrow & & \uparrow \\
\Delta_0 & & \Delta_0 \\
\downarrow & & \downarrow \\
3 & \rightarrow & N-1 \\
\end{array}
\]

i.e. the characteristic equation of \([T_1]\) is of the form

\[ \Delta_0 \Delta_N \Delta_N' \]

where \( \Delta_0 \) is a cubic

and \( \Delta_N, \Delta_N' \) are both of degree \( N-1 \)
It is easy to determine $\Delta_0$ since in Operation I we did not alter the first three columns and in II the first row was left unchanged while the second and third were replaced but the sum of the even and odd rows (excluding R1) respectively.

Thus

$$\Delta_0 = \begin{vmatrix} a - \lambda & b & c \\ \frac{3}{8} N & \frac{1}{2} - \lambda & \frac{1}{8} \\ \frac{1}{4} N & \frac{1}{2} & \frac{1}{4} - \lambda \end{vmatrix}$$

We now consider $\Delta_N'$. Since the rows beyond $R(N+1)$ are unaffected by Operation II we have

$$p''_{N+2+k,N+2+i} = p'_{N+2+k,N+2+i} = p_{N+2+k,N+2+i}$$

unless $N$ is odd, $k = \lambda = 1$, when, as we saw from (5)

$$p''_{N+3,N+3} = p'_{N+3,N+3} = \frac{5}{16} - \lambda$$

Thus if $N$ is even

$$\Delta_N = \begin{vmatrix} \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda \end{vmatrix}$$
and if $N$ is odd

$$
\Delta_N = \begin{bmatrix}
\frac{5}{16} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

To establish $\Delta'_N$, we must consider $p''_{i,j} \quad (i,j = 4, 5, \ldots, N+2)$

$$
p''_{i,j} = \sum_{s=0}^{N+2-i} p'_{i+2s, j} = \sum_{s=0}^{N+2-i} (p_{i+2s, j} - p_{i+2s, j-2})
$$

$$
= \sum_{s=0}^{N+2-i} (p_{i+2s-2, j-2} - p_{i+2s, j-2})
$$

$$
= p_{i-2, j-2} - p_{i, j-2} \\
+ p_{i, j-2} - p_{i+2, j-2} \\
+ p_{i+2, j-2} - p_{i+4, j-2} \\
+ \cdots \\
+ p_{2N+2-i, j-2} - p_{2N+4-i, j-2}
$$

$$
= p_{i-2, j-2} - p_{2N+4-i, j-2} \\
= p_{ij} - p_{2N+4-i, j-2}
$$
For our range of $i$ and $j$ \( p_{2N+4-i,j-2} = 0 \)

unless \( 2N + 4 - i = 2N \) and \( j - 2 = 2 \)

i.e. \( i = j = 4 \) in which case

\[ p_{2N,2} = \frac{1}{16} \]

so \[ p''_{44} = \left( \frac{3}{8} - \lambda \right) - \frac{1}{16} = \frac{5}{16} - \lambda \]

OR as may be seen from (4) and (5), $N$ is even and

\[ 2N + 4 - i = N + 2 \]

\( j - 2 = N \)

i.e. \( i = j = N+2 \) in which case

\[ p_{N+2,N} = \frac{1}{16} \]

so \[ p''_{N+2,N+2} = \left( \frac{3}{8} - \lambda \right) - \frac{1}{16} = \frac{5}{16} - \lambda \]

Thus \[ p''_{ij} = p_{ij} \quad i,j = 4,5,...,N+2 \]

unless \( i = j = 4 \) in which case \( p''_{44} = \frac{5}{16} - \lambda \)

and if $N$ is even

\( i = j = N+2 \) in which case \( p''_{N+2,N+2} = \frac{5}{16} - \lambda \).
Thus, if $N$ is even

$$
\Delta'_N = \begin{bmatrix}
\frac{5}{16} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & \ldots & 0 \\
\frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & \ldots & 0 \\
\frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & \ldots & 0 \\
\vdots \\
0 & \ldots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} \\
0 & \ldots & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} \\
0 & \ldots & 0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & \frac{5}{16} - \lambda
\end{bmatrix}
$$

and if $N$ is odd

$$
\Delta'_N = \begin{bmatrix}
\frac{5}{16} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & \ldots & 0 \\
\frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & 0 & 0 & 0 & \ldots & 0 \\
\frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} & 0 & \ldots & 0 \\
0 & \ldots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} - \lambda & \frac{1}{16} & \frac{1}{16} \\
0 & \ldots & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4}
\end{bmatrix}
$$

We note that when $N$ is odd \[ \Delta'_N = \Delta_N \].

\[ |T_1 - \lambda I| \] is now in the desired form.
Appendix B

Lemma 4.1  Let $[A]$ and $[B]$ be $(2m+1) \times (2m+1)$ matrices $(m \geq 1)$ of the following form.

$$[A] = \begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
\end{bmatrix}$$

and

$$[B] = \begin{bmatrix}
\frac{1}{5} & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{16} & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{16} & \frac{1}{16} & \frac{3}{8} & \frac{1}{16} & \cdots & 0 \\
\end{bmatrix}$$
Then \([A]\) and \([B]\) are similar, i.e. \(\exists [Q] \text{ s.t. } [B] = [Q]^{-1}[A][Q]\).

**Proof:** Let \([A] = \{a_{ij}\}\)

\([B] = \{b_{ij}\}\)

and \([Q] = \{q_{ij}\}\)

\([A]\) is defined by:–

\[a_{11} = a_{12} = \frac{1}{4}; \ a_{1j} = 0 \text{ otherwise} \]

\[a_{22} = \frac{3}{8}; \ a_{21} = a_{23} = a_{24} = \frac{1}{16}; \ a_{2j} = 0 \text{ otherwise} \]

\[a_{2m,2m} = \frac{3}{8}; \ a_{2m,2m-2} = a_{2m,2m-1} = a_{2m,2m+1} = \frac{1}{16}; \ a_{2m,j} = 0 \text{ otherwise} \]

\[a_{2m+1,2m} = a_{2m+1,2m+1} = \frac{1}{4}; \ a_{2m+1,j} = 0 \text{ otherwise} \]

For \(i = 1,2,\ldots,m-1\)

\[a_{2i+1,2i} = a_{2i+1,2i+1} = a_{2i+1,2i+3} = \frac{1}{4}; \ a_{2i+1,j} = 0 \text{ otherwise} \]

For \(i = 2,3,\ldots,m-1\)

\[a_{2i,2i} = \frac{3}{8}; \ a_{2i,2i-2} = a_{2i,2i-1} = a_{2i,2i+1} = a_{2i,2i+2} = \frac{1}{16}; \ a_{2i,j} = 0 \text{ otherwise} \]

\([B]\) is defined by:–

\[b_{11} = \frac{5}{16}; \ b_{12} = b_{13} = \frac{1}{16}; \ b_{1j} = 0 \text{ otherwise} \]

\[b_{2m+1,2m+1} = \frac{5}{16}; \ b_{2m+1,2m-1} = b_{2m+1,2m} = \frac{1}{16}; \ b_{2m+1,j} = 0 \text{ otherwise} \]
For \( i = 1,2,\ldots, m \)

\[ b_{2i,2i-1} = b_{2i,2i} = b_{2i,2i+1} = \frac{1}{4} ; \quad b_{2i,j} = 0 \quad \text{otherwise} \]

For \( i = 1,2,\ldots, m-1 \)

\[ b_{2i+1,2i+1} = \frac{3}{8} ; \quad b_{2i+1,2i-1} = b_{2i+1,2i} = b_{2i+1,2i+2} = b_{2i+1,2i+3} = \frac{1}{16} ; \]

\[ b_{2i+1,j} = 0 \quad \text{otherwise} \]

We define \([Q]\) as follows:

For \( i = 1,2,\ldots, m+1 \)

\[ q_{2i-1,j} = \delta_{2i-1,j} \quad \text{the standard Kronecker delta} \]

For \( i = 1,2,\ldots, m \)

\[ q_{2i,2i-1} = q_{2i,2i} = q_{2i,2i+1} = \frac{1}{4} ; \quad q_{2i,j} = 0 \quad \text{otherwise} \]

Firstly we show \([Q]^{-1} = [R] = \{r_{ij}\}\) where

For \( i = 1,2,\ldots, m+1 \)

\[ r_{2i-1,j} = \delta_{2i-1,j} \]

For \( i = 1,2,\ldots, m \)

\[ r_{2i,2i} = 4 ; \quad r_{2i,2i-1} = r_{2i,2i+1} = -1 ; \quad r_{2i,j} = 0 \quad \text{otherwise} \]

Let \([S] = [R][Q] = \{s_{ij}\}\)

Then \( s_{ij} = \sum_{k=1}^{2m+1} r_{ik} q_{kj} \)
For $i = 1, 2, \ldots, m+1$

$$s_{2i-1,j} = \sum_{k=1}^{2m+1} r_{2i-1,k} q_{k,j} = \sum_{k=1}^{2m+1} \delta_{2i-1,k} q_{k,j}$$

$$= \delta_{2i-1,2i-1} q_{2i-1,j}$$

$$= q_{2i-1,j} = \delta_{2i-1,j}$$

For $i = 1, 2, \ldots, m$

$$s_{2i,j} = \sum_{k=1}^{2m+1} r_{2i,k} q_{k,j}$$

$$= r_{2i,2i-1} q_{2i-1,j} + r_{2i,2i} q_{2i,j} + r_{2i,2i+1} q_{2i+1,j}$$

$$= (-1)^{i-1} q_{2i,j} + 4q_{2i,j} + (-1)^{i+1} q_{2i,j}$$

If $j \neq 2i-1, 2i, 2i+1$ then $s_{2i,j} = 0$

$$s_{2i,2i-1} = (-1)^{i-1} + 4 \cdot \frac{1}{4} + (-1)^{i-1} = 0$$

$$s_{2i,2i} = (-1)^{i-1} + 4 \cdot \frac{1}{4} + (-1)^{i-1} = 1$$

$$s_{2i,2i+1} = (-1)^{i-1} + 4 \cdot \frac{1}{4} + (-1)^{i-1} = 0$$

Thus $s_{ij} = \delta_{ij}$ for $i = 1, 2, \ldots, 2m+1$

$$[R][Q] = [I]$$

and so $[R] = [Q]^{-1}$

Let $[C] = [Q]^{-1}[A][Q]$

Then $c_{ij} = \sum_{k=1}^{2m+1} r_{ik} \left( \sum_{l=1}^{2m+1} a_{kl} q_{lj} \right)$
\[ c_{1j} = \sum_{\ell=1}^{2m+1} \sum_{k=1}^{2m+1} r_{\ell k} a_{\ell k} q_{\ell j} = \sum_{\ell=1}^{2m+1} \sum_{k=1}^{2m+1} \delta_{\ell k} a_{\ell k} q_{\ell j} \]

\[ c_{11} = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{1}{4} = \frac{5}{16} \]

\[ c_{12} = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \]

\[ c_{13} = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \]

\[ c_{1j} = 0 \quad \text{otherwise} \]

\[ c_{2m+1,j} = \sum_{\ell=1}^{2m+1} \sum_{k=1}^{2m+1} r_{2m+1,k} a_{\ell k} q_{\ell j} = \sum_{\ell=1}^{2m+1} \sum_{k=1}^{2m+1} \delta_{2m+1,k} a_{\ell k} q_{\ell j} \]

\[ c_{2m+1,j} = \sum_{\ell=1}^{2m+1} a_{2m+1,\ell} q_{\ell j} = \frac{1}{4} q_{2m,j} + \frac{1}{4} q_{2m+1,j} \]

So \( c_{2m+1,2m+1} = \frac{5}{16} \); \( c_{2m+1,2m-1} = c_{2m+1,2m} = \frac{1}{16} \); \( c_{2m+1,j} = 0 \quad \text{otherwise} \)

For \( i = 1, 2, \ldots, m-1 \)

\[ c_{2i+1,j} = \sum_{\ell=1}^{2m+1} a_{2i+1,\ell} q_{\ell j} = \frac{1}{4} q_{2i,j} + \frac{1}{4} q_{2i+1,j} + \frac{1}{4} q_{2i+2,j} \]
\[ c_{2i+1,2i-1} = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 = \frac{1}{16} \]

\[ c_{2i+1,2i} = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 = \frac{1}{16} \]

\[ c_{2i+1,2i+1} = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{8} \]

\[ c_{2i+1,2i+2} = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \]

\[ c_{2i+1,2i+3} = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \]

\[ c_{2i+1,j} = 0 \text{ otherwise.} \]

For \( i = 1, 2, \ldots, m \)

\[
c_{2i,j} = \sum_{\ell=1}^{2m+1} \left( -a_{2i-1,\ell} q_{\ell j} + 4a_{2i,\ell} q_{\ell j} - a_{2i+1,\ell} q_{\ell j} \right)
\]

\[
= \sum_{\ell=1}^{2m+1} q_{\ell j} \left( 4a_{2i,\ell} - a_{2i-1,\ell} - a_{2i+1,\ell} \right)
\]

\[
= q_{2i-2,j} \left( 4a_{2i,2i-2} - a_{2i-1,2i-2} - a_{2i+1,2i-2} \right)
+ q_{2i-1,j} \left( 4a_{2i,2i-1} - a_{2i-1,2i-1} - a_{2i+1,2i-1} \right)
+ q_{2i,\ell} \left( 4a_{2i,2i} - a_{2i-1,2i} - a_{2i+1,2i} \right)
+ q_{2i+1,\ell} \left( 4a_{2i,2i+1} - a_{2i-1,2i+1} - a_{2i+1,2i+1} \right)
+ q_{2i+2,\ell} \left( 4a_{2i,2i+2} - a_{2i-1,2i+2} - a_{2i+1,2i+2} \right)
\]

\[
= q_{2i-2,j} \left( \frac{1}{4} - \frac{1}{4} - 0 \right) + q_{2i-1,j} \left( \frac{1}{4} - \frac{1}{4} - 0 \right)
+ q_{2i,j} \left( \frac{3}{2} - \frac{1}{4} - \frac{1}{4} \right) + q_{2i+1,j} \left( \frac{1}{4} - 0 - \frac{1}{4} \right)
+ q_{2i+2,j} \left( \frac{1}{4} - 0 - \frac{1}{4} \right)
\]

\[ = q_{2i,j} \]
So \( c_{2i,2i-1} = c_{2i,2i} = c_{2i,2i+1} = \frac{1}{4} \); \( c_{2i,j} = 0 \) otherwise.

Hence \( c_{i,j} = b_{i,j} \) for \( i,j = 1,2,\ldots,2m+1 \)

and so \( [B] = [Q]^{-1}[A][Q] \). \( \square \)
We prove Lemma 9.2.

a) \( \sin X \sin Y \sin(X-Y) + \sin Y \sin Z \sin(Y-Z) \)
\[ + \sin Z \sin X \sin(Z-X) \]
\[ = \sin(Y-X) \sin(Z-Y) \sin(X-Z) \]

b) \( \sin(Y-\theta) \sin(Z-Y) \sin(\theta-Z) \cos 2X \)
\[ + \sin(\theta-X) \sin(Z-\theta) \sin(X-Z) \cos 2Y \]
\[ + \sin(Y-X) \sin(\theta-Y) \sin(X-\theta) \cos 2Z \]
\[ = \sin(Y-X) \sin(Z-Y) \sin(X-Z) \cos 2\theta. \]

It will be helpful to show first the following identities.

If \( A + B + C = 0 \) then
\[ \sin 2A + \sin 2B + \sin 2C = -4 \sin A \sin B \sin C \quad (1) \]
For all \( \alpha, \beta, \gamma \)
\[ \cos \alpha \sin(\beta-\gamma) + \cos \beta \sin(\gamma-\alpha) + \cos \gamma \sin(\alpha-\beta) = 0 \quad (2) \]
For (1), \( C = -(A+B) \) so
\[ \sin 2A + \sin 2B + \sin 2C \]
\[ = \sin 2A + \sin 2B - \sin 2(A+B) \]
\[ = \sin 2A + \sin 2B - \sin 2A \cos 2B - \cos 2A \sin 2B \]
\[ = \sin 2A(1 - \cos 2B) + \sin 2B(1 - \cos 2A) \]
\[ = 2 \sin 2A \sin^2 B + 2 \sin 2B \sin^2 A \]
For (2)
\[ \cos X \sin(Y-Z) + \cos Y \sin(Z-X) + \cos Z \sin(X-Y) \]
= \[ \cos X [\sin Y \cos Z - \cos Y \sin Z] \]
+ \cos Y [\sin Z \cos X - \cos Z \sin X]
+ \cos Z [\sin X \cos Y - \cos X \sin Y]
= 0.

Now
\[ \sin X \sin Y \sin(X-Y) + \sin Y \sin Z \sin(Y-Z) \]
+ \sin Z \sin X \sin(Z-X)
= \[ -\frac{1}{2} [\cos(X+Y) - \cos(X-Y)] \sin(X-Y) \]
- \[ \frac{1}{2} [\cos(Y+Z) - \cos(Y-Z)] \sin(Y-Z) \]
- \[ \frac{1}{2} [\cos(Z+X) - \cos(Z-X)] \sin(Z-X) \]
= \[ -\frac{1}{4} [\sin 2X - \sin 2Y] + \frac{1}{4} \sin 2(X-Y) \]
- \[ \frac{1}{4} [\sin 2Y - \sin 2Z] + \frac{1}{4} \sin 2(Y-Z) \]
- \[ \frac{1}{4} [\sin 2Z - \sin 2X] + \frac{1}{4} \sin 2(Z-X) \]
= \[ \frac{1}{4} [\sin 2(X-Y) + \sin 2(Y-Z) + \sin 2(Z-X)] \]
= \[ \frac{1}{4} [-4 \sin(X-Y) \sin(Y-Z) \sin(Z-X)] \] from (1)
= \[ \sin(Y-X) \sin(Z-Y) \sin(X-Z) \]
and a) holds.
\[
\sin(Y-\theta) \sin(Z-Y) \sin(\theta-Z) \cos 2X \\
+ \sin(\theta-X) \sin(Z-\theta) \sin(X-Z) \cos 2Y \\
+ \sin(Y-X) \sin(\theta-Y) \sin(X-\theta) \cos 2Z \\
= \frac{1}{2} \left[ \cos(2\theta-Y-Z) - \cos(Y-Z) \right] \sin(Z-Y) \cos 2X \\
+ \frac{1}{2} \left[ \cos(2\theta-Z-X) - \cos(Z-X) \right] \sin(X-Z) \cos 2Y \\
+ \frac{1}{2} \left[ \cos(2\theta-X-Y) - \cos(X-Y) \right] \sin(Y-X) \cos 2Z
\]

Combining the two factors in each term containing \( \theta \).

\[
= \frac{1}{2} \left[ \cos 2\theta \cos(Y+Z) + \sin 2\theta \sin(Y+Z) - \cos(Y-Z) \right] \sin(Z-Y) \cos 2X \\
+ \frac{1}{2} \left[ \cos 2\theta \cos(Z+X) + \sin 2\theta \sin(Z+X) - \cos(Z-X) \right] \sin(X-Z) \cos 2Y \\
+ \frac{1}{2} \left[ \cos 2\theta \cos(X+Y) + \sin 2\theta \sin(X+Y) - \cos(X-Y) \right] \sin(Y-X) \cos 2Z
\]

\[
= \frac{1}{2} \cos 2\theta \left[ \cos(Y+Z) \sin(Z-Y) \cos 2X \\
+ \cos(Z+X) \sin(X-Z) \cos 2Y \\
+ \cos(X+Y) \sin(Y-X) \cos 2Z \right] \\
+ \frac{1}{2} \sin 2\theta \left[ \sin(Y+Z) \sin(Z-Y) \cos 2X \\
+ \sin(Z+X) \sin(X-Z) \cos 2Y \\
+ \sin(X+Y) \sin(Y-X) \cos 2Z \right] \\
- \frac{1}{2} \left[ \cos(Z-Y) \sin(Z-Y) \cos 2X \\
+ \cos(X-Z) \sin(X-Z) \cos 2Y \\
+ \cos(Y-X) \sin(Y-X) \cos 2Z \right]
\]

Collecting terms in \( \theta \).
\[
\sin 2Y = 2 \cos 2X \sin 2Z - \cos 2X \sin 2Y - \cos 2Y \sin 2Z \]

We note that the coefficient of \( \sin 2 \theta \) is 0.

Putting \( \alpha = 2X \)
\( \beta = 2Y \)
\( \gamma = 2Z \)

in the constant term we have

\[
\frac{1}{4} \left[ \cos \alpha \sin(\beta - \gamma) + \cos \beta \sin(\alpha - \gamma) + \cos \gamma \sin(\beta - \alpha) \right]
\]

\[
= \frac{1}{4} \left[ \cos \alpha \sin(\beta - \gamma) + \cos \beta \sin(\gamma - \alpha) + \cos \gamma \sin(\alpha - \beta) \right]
\]

\[
= 0 \quad \text{by (2).}
\]

Hence (3) gives

\[
\frac{1}{4} \cos 2 \theta \left[ \sin 2Z \cos 2X - \cos 2Z \sin 2X \\
+ \sin 2X \cos 2Y - \cos 2X \sin 2Y \\
+ \sin 2Y \cos 2Z - \cos 2Y \sin 2Z \right]
\]
= \frac{1}{4} \cos 2\theta \left[ \sin 2(Z-X) + \sin 2(X-Y) + \sin 2(Y-Z) \right]

= \frac{1}{4} \cos 2\theta \left[ -4 \sin(Z-X) \sin(X-Y) \sin(Y-Z) \right]

\text{by (1)}

= \sin(Y-X) \sin(Z-Y) \sin(X-Z) \cos 2\theta

\text{and b) is proved.}
References


