Examination of optical fibre inhomogeneities and their effects on fibre coupling

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EXAMINATION OF OPTICAL FIBRE INHOMOGENEITIES
AND
THEIR EFFECTS ON FIBRE COUPLING

by
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A Doctoral Thesis
submitted in partial fulfilment of the requirements
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C. Tariq A.K. Al-Jumailly, 1984
To my wife Muntaha
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ABSTRACT

Coupling effects from source to optical fibre or from fibre to fibre are very important in the design of communication systems using such fibres, since both the amount of guided power and the output pulse shape depend on them.

This work is an investigation into such effects.

Various fibre inhomogeneities have been studied analytically. Internally, these are the refractive index difference $\Delta$, the core ellipticity and the difference in numerical apertures; externally, the factors considered are fibre separation, offset and tilt.

The problem is treated using geometrical optics and is applied to various profiles of different refractive index exponent ($\alpha$). Fibres produced currently by the (MCVD) technique normally have a dip in the refractive index profile at the centre of the core. This dip in the refractive profile affects the launching efficiency and causes pulse broadening in multimode graded index fibres and birefringence in single mode graded index elliptical fibres. The effect of the dip on these parameters has been investigated.

Finally experiments have been carried out to test the validity of some of the theoretical derivations.
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CHAPTER I

INTRODUCTION

1.1 Review of Optical Fibres

The advent of high quality optical fibres added new dimensions to the field of communications. A great deal of attention is being devoted to this technology since bandwidths approaching GHz, and attenuations below 1 dB/km have been achieved \(^{1,2,3}\).

The transfer of signal from the source to the fibre, and from fibre to fibre involves fine geometrical and mechanical accuracy. Therefore research has been oriented towards reducing coupling losses and achieving acceptable coupling efficiencies.

An optical fibre waveguide consists of two concentric cylinders, the inner known as the core, and the outer known as the cladding. If the refractive index of the core is higher than that of the cladding, light is guided along the waveguide core, while the cladding provides protection against oxidation and mechanical stresses. Optical fibres are classified into two groups according to their refractive index profile: step index and graded index.

After light has been launched into the fibre, it is collected and carried by the fibre in the form of modes. The ability of the fibre to collect as much light as possible from the source is a function of the launching conditions. Modes that propagate in the fibre do so with different velocities, hence the refractive index in the core is graded in such a way that mode delay differences are equalized.
The sources of dispersion in fibres are intermodal and intramodal\(^{(4)}\).

The first part vanishes if mode delay differences are equalised. The second part represents an average of the pulse broadening within each mode and is the only term that is experienced in single mode fibres.

Intramodal dispersion arises from two distinct causes.

1. **Material dispersion** - This arises from the fact that the light sources do not radiate at a single frequency but comprise a finite spectrum of frequencies and because the refractive index of the material is frequency dependent, it ascribes to each frequency component a different velocity of propagation.

2. **Waveguide dispersion** - This is due to the dispersive nature of the types of waveguides to which optical fibres belong. Compared with step index, graded index fibres have a much higher bandwidth and are less stringent to alignment requirements.

1.2 **Formulation of the problem**

Efficient power coupling between source and fibre or fibre and fibre represents a fundamental problem in optical communications, for both the amount of guided power and the output pulse shape depend on it.

The solid angle at which light is emitted from the end face of a fibre is determined by the numerical aperture of the fibre, the launching angle of the source side and the degree of homogeneity in the fibre.

Efficient coupling is a function of intrinsic parameters and extrinsic parameters. The intrinsic parameters are\(^{(5)}\): maximum index of refraction difference between the refractive indices of the core and the cladding (\(\Delta\)), the index of refraction profile parameter (\(\alpha\)), the radius and the core ellipticity. The extrinsic parameters are\(^{(6)}\): separation, displacement and misalignment.
As light is launched into the fibre, refracting rays would be extracted immediately leaving only guided and leaky rays. Leaky rays would gradually radiate from the cladding at a rate depending on their skewness. The output at the other end of the fibre will be a combination of trapped and leaky rays. For multimode step index fibres these rays can be identified on the far field radiation pattern as inner and outer regions respectively. The amount of optical power accepted by the fibre depends, ultimately, for a source of given radiance and size and for a fibre of given numerical aperture and size, on their coupling condition. The best coupling of power occurs if the two surfaces are coaxially joined together. Such a configuration also gives the best coupling even if the source area is smaller than the core area\(^7\). Any deviation from this condition, whether intrinsic or extrinsic, causes a loss in the coupled power. The effects the various parameters have on the coupled power have been investigated in some detail.

Fibres produced currently by the chemical vapour decomposition (CVD) technique, normally have a dip in the profile\(^8\) at the centre of the core. This is due to the out-diffusion of doping material during the collapsing process owing to the evaporation of the dopant. The shape of this dip has been assumed to be gaussian\(^8\) and its effect on propagation characteristics has been investigated\(^9\).

It is however possible to assume that this dip can have the shape of an inverse $\alpha$-power profile\(^{10}\). An investigation has been made to determine how such a dip affects the launching efficiency, the coupling loss and pulse broadening in multimode optical fibres and birefringence in single mode fibres of elliptical cross section.
CHAPTER 2

COUPLING THEORY

2.1 Introduction

The electromagnetic theory of launching energy into optical fibres is based upon Maxwell's equations. However, the mathematical complexity of the resulting description, led to an asymptotic theory (1), the main component of which is the geometrical theory (2,3), which describes ray trajectories or paths of energy flow (4). The validity of geometric optical analysis applies only when the dimensions involved are much larger than the wavelength (5). If the dimensions are of the same order as a wavelength, ray analysis can no longer be used to describe the propagation in fibres adequately, and mode theory must be used.

In the following analysis, the spatial coherence characteristics of the emitted radiation are neglected, because in geometrical theory, such properties have little meaning (6).
2.2 Elementary Analysis

A general ray striking the core-cladding interface is characterized by its angle of incidence \( \theta_o \) and the angle \( \theta_n \) which the projection of the incident ray makes with the tangent to the circular cross-section as shown in Figure 2.1. The angle \( \theta_n \) represents the inclination of the incident ray to the normal OP in the plane of the cross-section. This angle is related to \( \theta \) and \( \phi \) by the following equation:

\[
\sin \theta \sin \phi = \cos \theta_n \tag{2.1}
\]

Meridional rays are defined as those rays having \( \phi = \frac{\pi}{2} \). Otherwise, rays are defined as skew rays and \( \phi \) becomes a measure of their skewness. The behaviour of rays in the fibre depends upon the angle of incidence, and the critical angle may be expressed as:

\[
\theta_c = \cos^{-1} \frac{n_2}{n_1} \tag{2.2}
\]

where \( n_1 \) and \( n_2 \) denote the refractive index values for the core and cladding respectively.

In the simplest case when \( \phi = \frac{\pi}{2} \), a ray will be trapped inside the core providing that

\[
\theta < \theta_c \tag{2.3}
\]

and when

\[
\theta > \theta_c \quad \text{the rays will be refracted.} \tag{2.4}
\]

With skew rays

\[
\theta < \theta_c \quad \text{represents trapped rays, but for total refraction}
\]

\[
\theta > \theta_c \quad \text{and} \quad \theta_n > \frac{\pi}{2} - \theta_c \tag{2.5}
\]

and the ray is partially trapped, i.e. leaky.
When \( \theta > \theta_c \) and \( \frac{\pi}{2} - \theta_c \) the ray is completely refracted.

The numerical apertures (N.A.) for various fibres are as follows:

For step index fibres:

\[
N.A. = n_1 \sin \theta_c = (n_1^2 - n_2^2)^{\frac{1}{2}} \text{ for meridional rays} \tag{2.6}
\]

\[
N.A. = (n_1^2 - n_2^2)^{\frac{1}{2}} \left[ 1 + \left( \frac{r_o \sin \delta}{a} \right)^2 \right]^{\frac{1}{2}} \text{ for skew rays} \tag{2.7}
\]

where \( \delta = \cos^{-1} \left( \frac{r_o}{a} \cos \phi \right) + \phi \), and \( r_o \) is the radial distance from the core centre.

For graded index fibres

\[
N.A. = \left[ n_1^2 (r_o) - n_2^2 \right]^{\frac{1}{2}} \text{ for meridional rays} \tag{2.8}
\]

\[
N.A. = \left[ \frac{n_1^2 (r_o) - n_2^2}{1 - \left( \frac{r_o}{a} \right)^2 \cos^2 \phi} \right]^{\frac{1}{2}} \text{ for all rays} \tag{2.9}
\]

The fibre is assumed to have a profile of the form

\[
n^2(r) = n_1^2 \left[ 1 - 2\Delta \left( \frac{r}{a} \right)^2 \right] \quad r < a \tag{2.10}
\]

\[
n(r)^2 = n_1^2 \left[ 1 - 2\Delta \right] \quad r > a \tag{2.10}
\]

The graphs in Figures 2.2 and 2.3 show the variation of the numerical aperture as a function of the displacement \( \left( \frac{r}{a} \right) \) and the projection angle \( \phi \).

The error (E) made when only meridional rays are considered to be accepted by the fibre is:

\[
E = \frac{N.A._{M+S} - N.A.}{N.A._{M+S}} \tag{2.11}
\]
a) RAY ON FIBRE FRONT FACE

b) CROSS SECTIONAL VIEW OF THE INCIDENT RAY AT THE BOUNDARY

Fig.(2.1)
Substitution gives

$$E = 1 - \left[ 1 - \left( \frac{\xi}{a} \right)^2 \cos^2 \phi \right]^{\frac{1}{2}}$$  \hspace{1cm} (2.12)$$

and this is independent of the index profile parameter ($a$). Fig. (2.4) shows the variation of this error with the displacement ($\frac{\xi}{a}$) for various values of $\phi$. As anticipated, the correction factor approaches zero as $\phi \rightarrow \frac{\pi}{2}$ which is the region for meridional rays only.
Dif. (2.3) Variation of the N.A. with the oblique angle $\phi$ for different displacements
Fig. (2.4) Variation of the error with the displacement \( \frac{e}{a} \) and the projection angle \( \phi \).
2.3 Launching Efficiency

An important quantity that is used to evaluate the quality of source-fibre coupling is the launching efficiency $\eta$. It is defined as the ratio of the power guided by the fibre $P_0$ to the whole power emitted by the source $P$.

Consider a circular emitting surface $S_e$ of radius $b$ facing a fibre surface $S_f$ of radius $a$. Fig (2.5) with the source very close to the fibre input surface, and the gap filled with an index matching liquid (7) of refractive index $n''$, then following the analysis carried out in reference 8 it is possible to determine analytic expressions for the total power emitted by the source and the total power accepted and guided by the fibre.

The radiance (9) of the source in the medium of refractive index $n'$ is $R'(r', \psi', \theta', \phi')$ and is a function of the position polar coordinates $(r', \psi')$ of an emitting point on the source and is a function of the angular polar coordinates $(\theta', \phi')$ of a ray emitted by that point.

$\theta'$ is the angle the ray forms with the source axis. $\phi'$ is the azimuthal angle with respect to the radius vector.

Since the two surfaces are concentric and very close then; $r' = r_0,$
$\psi' = \psi_0,$ $\theta' = \theta_0',$ $\phi' = \phi_0$.

The total power emitted by the source is (10)

$$P = \int_{0}^{b} r' \, dr' \int_{0}^{2\pi} d\psi' \int_{0}^{2\pi} d\phi' \int_{0}^{1/2} R'(r', \psi', \theta', \phi') \sin \theta' \cos \theta' \, d\theta'$$

(2.13)
FIG.(2.5) COUPLING GEOMETRY BETWEEN LIGHT SOURCE AND A FIBRE
and the total power accepted and guided by the fibre is

\[ P_o = \int_0^{a'} r_o \, dr_o \int_0^{2\pi} d\psi_o \int_0^{2\pi} d\phi_o \int_0^{\theta_m} R(r_o, \psi_o, \theta_o, \phi_o) \times T(r_o, \theta_o) \sin \theta_o \cos \theta_o \, d\theta_o \quad (2.14) \]

where \( a' \) is the minimum between \( a \) and \( b \) and \( T(r_o, \theta_o) \) is the transmittance \( (11) \) at the interface between the fibre and the medium of refractive index \( n' \) and is given by \( (12) \):

\[ i.e. \ T(r_o, \theta_o) = T_o = \frac{4n' n_o}{(n' + n_o)^2} \quad (2.15) \]

For sources whose radiance is given by

\[ R'(\theta') = R_o \cos \ell \theta' \quad \ell \geq 0 \quad (2.16) \]

\( \ell = 0 \) represents a Lambertian distribution and as \( \ell \) increases the radiation pattern becomes narrower and finally becomes unidirectional as \( \ell \to \infty \). From the definition of the launching efficiency given previously, and for an (LED) whose radiation pattern is given by:

\[ R'(\theta') = R_o \quad (2.17) \]

the following launching efficiencies are obtained:

\[ n = \frac{2 T_o}{b^2 n'^2} \int_0^{a'} \left[ n'(r_o^2 - n_o^2) \right] \frac{r_o}{1 - \left( \frac{r_o}{a} \right)^2 \cos^2 \phi} \, dr_o \quad (2.18) \]

for all rays (bound and skew)

and
for bound rays only.

For the proof of these equations see Appendix A.

For a source whose radiation pattern is given by:

\[ R'(\theta') = R_o \cos \ell \theta' \quad \ell > 0 \] (2.20)

The launching efficiencies are (10)

\[
\eta = \frac{T_o}{\pi b^2} \int_0^{a'} \int_0^{2\pi} \left[ 1 - \left( 1 - \frac{n(r_o^2) - n_2^2}{n_1^2(1 - \frac{r_o^2}{a^2} \cos^2 \phi)} \right)^{\frac{\ell + 2}{2}} \right] r_o \, dr_o \, d\phi \tag{2.21}
\]

for all rays (bound + skew)

\[
\text{and } \eta' = \frac{2 T_o}{b^2} \int_0^{a'} \left[ 1 - \left( 1 - \frac{n(r_o^2) - n_1^2}{n_2^2} \right)^{\frac{\ell + 2}{2}} \right] r_o \, dr_o \tag{2.22}
\]

for bound rays only.

The variations of the launching efficiencies \( \eta \) and \( \eta' \) with the refractive index profile (\( n \)) are shown in Figures (2.6) and (2.7), with the source directivity \( '\ell' \) as a parameter. Both \( \eta \) and \( \eta' \) increase with increasing \( '\ell' \), i.e. more directional source. Also both of them increase with increasing \( (a) \); i.e. larger angular acceptance.

Fig. (2.8) shows a comparison between the two launching efficiencies.
Fig. (2.6) Variation of Launching Efficiency ($\eta$) with the index profile parameter ($a$)
Fig. (2.7) Variation of Launching Efficiency $n'$ with the index profile parameter $(\alpha)$.
Fig. (2.8) Comparison between the launching efficiencies $\eta$ & $\eta'$
2.4 Application to LED Sources

For an LED source, the launching efficiencies are given by equations (2.18) and (2.19) of the previous section.

Applying these results to fibres having power law profiles of the form

\[ n(r^2) = n_o^2 \left[ 1 - 2\Delta (\frac{r}{a})^\alpha \right] \quad \text{for } r \leq a \]

and

\[ n(r^2) = n_o^2 \left[ 1 - 2\Delta \right] \quad \text{for } r > a \]

(2.23)

analytical expressions were obtained relating the launching efficiency to various source and fibre parameters.

Consider first the case of bound rays only, i.e. equation (2.19).

Here, two alternatives are to be considered

i) \( b \geq a \) (source radius larger than the fibre radius)

   In this case the launching efficiency is given by:

   \[
   \eta' = \frac{4 T_o n_o^2 \Delta}{n_i^2} \left( \frac{a}{b} \right)^2 \left[ \frac{1}{2} - \frac{1}{\alpha + 2} \right] \quad (2.24)
   \]

ii) \( b < a \)

   In this case, direct coupling does not give the maximum possible power input to the fibre since the light does not fill the numerical aperture cone.

\[
\therefore \phantom{0} \eta' = \frac{4 T_o n_o^2 \Delta}{n_i^2} \left[ \frac{1}{2} - \frac{1}{\alpha + 2} \left( \frac{b}{a} \right)^\alpha \right] \quad (2.25)
\]

and these two expressions match when \( b = a \).
For the proof of equations 2.24 and 2.25 see Appendix B.

These results were plotted in Fig (2.9). It can be seen from the graph that as the index profile parameter $a$ increases, the launching efficiency increases for given values of $b/a$ and approaches that of the step index fibre as $a \to \infty$.

Secondly, when considering all rays (bound + skew) the launching efficiency for the above two conditions is given by:

i) $b \geq a$

$$\eta = \frac{2 T_o n_o^2 \Delta}{\pi b^2} \int_0^a \int_0^{2\pi} \left[ \frac{1 - \left(\frac{r}{a}\right)^a}{1 - \left(\frac{r}{a}\right)^2 \cos^2 \phi} \right] \ r \, dr \, d\phi$$

(2.26)

ii) $b < a$

$$\eta = \frac{2 T_o n_o^2 \Delta}{\pi a^2} \int_0^b \int_0^{2\pi} \left[ \frac{1 - \left(\frac{r}{a}\right)^a}{1 - \left(\frac{r}{a}\right)^2 \cos^2 \phi} \right] \ r \, dr \, d\phi$$

(2.27)

These results are plotted in Fig. (2.10).

Figure (2.11) shows a comparison between the two launching efficiencies.

For the region where $b > a$, the launching efficiency decreases as $b$ increases due to the fact that the fibre is not capturing all the light that is emitted from the source. For $b < a$, the launching efficiency increases as $b$ decreases, due to the fact that the angular acceptance of the fibre is maximum at the centre and decreases towards the edge of the fibre.

Higher launching efficiencies are obtained for higher values of the index profile parameter ($a$) and reach a maximum value for $a \to \infty$ (step index fibre).
Fig. (2.9) Launching Efficiency $V_{\alpha}(a/b)$ for different index profiles (α) For bound rays only
Fig. (2.10) Launching efficiency (bound + skew) $V_{\alpha \rightarrow b/a}$ for different $\alpha$ values.
Fig. (2.11) Launching efficiencies of $\eta$ & $\eta'$ versus $\left( \frac{b}{a} \right)$ for different values of the profile index $\alpha$.
Also an improvement in the launching efficiency is obtained when the effects of skew rays are taken into account.

Coupling loss, which is defined as $L_c = -10 \log \eta$ is shown in Fig. (2.12) and (2.13).

Fig. (2.14) shows a comparison between coupling losses for the two types of launching efficiencies. Lower coupling loss is obtained when the contribution of skew rays is being taken into consideration. This is due to the increase in the value of $\eta$ for the bound + skew ray case.
Fig. (2.12) Coupling Loss $V_{c} \left( \frac{b}{a} \right)$ for different $\alpha$ values for bound rays only
Fig. (2.13) Coupling Loss $V_b(\frac{b}{a})$ for bound + skew rays
Fig. (2.14) Comparison between coupling losses $\nu^2$ for various values $\alpha$.
2.5 Inverse $\alpha$-power law profiles

Inverse $\alpha$-power law profiles \((13, 14, 15)\) have a refractive index distribution of the form:

\[
n(r)^2 = n_o^2 \left[ 1 - 2\delta (1 - \frac{r}{a})^\alpha \right] \quad \text{for } 0 < r < a
\]

and

\[
n(r)^2 = n_o^2 \left[ 1 - 2\delta \right] = n_2^2 \quad \text{for } r \geq a
\]

as shown in Fig. (2.15b) where all the parameters are as defined previously and $\delta$ is the relative refractive difference between the edge and the centre of the core. It can be considered as a measure of the dip height \((13)\).

Using equation (2.19 and following the same reasoning as explained in section 2.4, the launching efficiency for bound rays is given by:

i) $b > a$

\[
\eta' = \frac{4 n_o^2 T_o}{n'^2} \left( \frac{a}{b} \right)^2 \left[ \frac{a}{2} - \delta \left( \frac{1}{2} - \frac{a}{3} + \frac{a(a - 1)}{8} \right) \right] \quad (2.29)
\]

ii) $b < a$

\[
\eta' = \frac{4 n_o^2 T_o}{n'^2} \left[ \frac{a}{2} - \delta \left( \frac{1}{2} - \frac{a}{3} \left( \frac{b}{a} \right) + \frac{a(a - 1)}{8} \left( \frac{b}{a} \right)^2 \right) \right] \quad (2.30)
\]
a) $\alpha$ - POWER LAW PROFILES

b) INVERSE $\alpha$ - POWER LAW PROFILES
From equations 2.29 and 2.30 graphs of $\eta'$ against $b/a$ are plotted for various values of $\delta$. These are given in Figures 2.16 and 2.17.

For the region $b > a$, these fibres behave exactly the same as power law fibres. For the region $b < a$, the launching efficiency is a minimum at the centre and increases towards the core-cladding interface. This is due to the angular acceptance of such fibres increasing to a maximum value at the core-cladding interface.

It is evident from the graphs that the smaller the value of $\delta$, the higher the launching efficiency and the maximum launching efficiency is obtained when $\delta$ is zero (step index fibre).

For a fixed value of $\delta$, the launching efficiency improves greatly as the profile parameter $\alpha$ is increased, so higher values of $\delta$ which tend to reduce the launching efficiency could be compensated for by an increase in the profile parameter $\alpha$.

Unlike power law profile fibres, which have maximum launching efficiency at the centre and when the ratio of the source area to the fibre area is minimum these inverse profile law fibres have maximum launching efficiency at the core-cladding interface when the source area is equal to the cross-sectional area of the fibre.

Coupling losses were plotted in Figures (2.18) and (2.19) for different values of the index profile parameter $\alpha$. From the graphs it is clear that these fibres have a maximum coupling efficiency which is less than that of the power law fibres, but because of the nature of their angular acceptance, these fibres can best be operated by light sources whose surface areas are as large as the cross-sectional area of the fibre. This is not the case with power law fibres.
Launching Efficiency $\eta$ for different dip heights ($\delta$) and different index profile parameter ($a$).

- $\delta = 0.002235$ → e and f
- $\delta = 0.004465$ → c and d
- $\delta = 0.0066975$ → a and b

Fig. (2.16) Launching Efficiency $V(\frac{b}{a})$ for different dip heights ($\delta$) and different index $\frac{b}{a} \times 10^{-1}$.
Launching Efficiency $n'$ for different dip height ($a$) and different index ($b$)

Profile parameters ($a$) and ($b$):

$a = 0.002235 + e$ and $f = 0.0004465 + c$ and $d$

$e = 0.0006975 + a$ and $b$
Fig. (2.18) Coupling loss $V_c(\delta)$ for different dip heights ($\delta$) and different index profile parameters ($\alpha$)
Fig. (2.19): Coupling loss $V_c(\delta)$ for different dip heights ($\delta$) and different index profile parameter $\alpha$. 

$\delta = 0.002235 \rightarrow e$ and $f$

$\delta = 0.004465 \rightarrow c$ and $d$

$\delta = 0.0066975 \rightarrow a$ and $b$
2.6 Elliptical fibres

In the previous analysis, it has been assumed that the refractive index profile within the core is of the form \( n = n(r) \), which means that the contours of constant index are circles. In practice, however, the fibre manufacturing process often introduces geometrical or dielectric imperfections\(^\text{(16)}\). Barrel and Pask\(^\text{(17)}\) have shown that fibres need not have circular symmetry in order to retain the desirable properties of the circular power law fibres, since the power accepted and guided by the fibre is a function of the degree of homogeneity of the refractive index exponent \( \alpha \) and not of the shape parameter. Petermann\(^\text{(18)}\), has shown that the introduction of a small ellipticity in the index profile, leads to a considerable change of leaky mode behaviour in the case of square-law fibres. Ramskov and Adams\(^\text{(19)}\) have shown that the presence of small ellipticity in parabolic index fibres \( \alpha \approx 2 \) leads to a large increase in the minimum attenuation of leaky modes. This can be of importance for coupling purposes and could lead to an improvement in the launching efficiency of optical fibres.

In polar co-ordinates \((r, \theta)\), the refractive index profile, for parabolic index fibres \((\alpha = 2)\), can be written in the form\(^\text{(21)}\):

\[
n^2(r, \theta) = n_0^2 \left[ 1 - 2 \Delta \frac{r^2}{y^2} (1 - e^2 \cos^2 \theta) \right]
\]

where \( e \) is the ellipticity which is given by

\[
e^2 = 1 - \frac{y^2}{x^2}
\]

where \( x \) and \( y \) are the major and minor axis of the ellipse respectively.
Fig. (2.20) Variation of the launching efficiency as a function of $\left(\frac{b}{a}\right)$ for different ellipticities for parabolic graded index fibre.
Fig. (2.21) Ray domains ($\beta - L$) for parabolic graded index fibre
Following the analysis carried out in Section (2.3) for circular fibres, the launching efficiency, taking into consideration bound rays only, was calculated for different values of \( e \).

Fig. 2.20 shows that the launching efficiency improves with increasing ellipticity. It has been suggested by P.F. Checcacci\(^{(22)}\) that the number of bound modes increases with the introduction of a small ellipticity in the fibre core. This can be explained from fig (2.21) which shows the ray domains B-L for graded index circularly symmetric fibres (\( \alpha = 2 \)).

In elliptical co-ordinates \((\varepsilon, \Lambda)\) the refractive index profile is given by\(^{(19)}\)

\[
n^2(\varepsilon, \Lambda) = n_0^2 - \frac{g(\varepsilon) + h(\Lambda)}{\sinh^2 \varepsilon + \sin^2 \Lambda}
\]  

(2.32)

where \( n_0 \) is the index at the core centre, and

\[
g(\varepsilon) = (n_0^2 - n_2^2)(\frac{f}{r})^\alpha \sinh^2 \varepsilon \cosh \varepsilon
\]

\[
h(\Lambda) = (n_0^2 - n_2^2)(\frac{\Lambda}{r})^\alpha \sin^2 \Lambda \cos \Lambda
\]

\( f \) is the distance of the focus from the centre of the core and \( \frac{f}{r} = e \)
\( r \) is the distance from the core centre to the core cladding interface along the x-axis.

For the parabolic index elliptical fibre (\( \alpha = 2 \))

\[
g(\varepsilon) = (n_0^2 - n_2^2) e^2 \sinh^2 \varepsilon \cosh^2 \varepsilon
\]

(2.33)

and

\[
h(\Lambda) = (n_0^2 - n_2^2) e^2 \sin^2 \Lambda \cos^2 \Lambda
\]

(2.34)
The scalar wave equation for each mode $E(x,y,z)$ is

$$\left[ \nabla^2 + k^2 n^2(x,y) \right] E = 0 \quad (2.35)$$

where $n(x,y)$ is the refractive index and $k = \frac{2\pi}{\lambda}$ and $\lambda$ is the free space wavelength.

Taking $E(x,y,z) = \phi(x,y)e^{-j\beta z}$

$$\left[ \nabla^2 + (k^2 n^2 - \beta^2) \right] \phi = 0 \quad (2.36)$$

where

$$\nabla^2 \phi \equiv \phi^2 - \frac{\partial^2}{\partial \varepsilon^2}$$

In elliptical co-ordinates:

$$\nabla^2 \phi = \frac{1}{e^2 r^2 (\sinh^2 \varepsilon + \sin^2 \Lambda)} \left( \frac{\partial^2 \phi}{\partial \varepsilon^2} + \frac{\partial^2 \phi}{\partial \Lambda^2} \right)$$

letting $\phi = Q(\varepsilon) S(\Lambda)$ and following the same analysis carried out in (23)

$$\frac{1}{e^2 r^2 (\sinh^2 \varepsilon + \sin^2 \Lambda)} \left[ \frac{1}{Q} \frac{d^2 Q}{d \varepsilon^2} + \frac{1}{S} \frac{d^2 S}{d \Lambda^2} + k^2 n^2(\varepsilon, \Lambda) - \beta^2 \right] = 0 \quad (2.37)$$

But $n^2(\varepsilon, \Lambda)$ is given by eq. (2.32)

$$\frac{1}{Q} \frac{d^2 Q}{d \varepsilon^2} + \frac{1}{S} \frac{d^2 S}{d \Lambda^2} + e^2 r^2 (k^2 n_0^2 - \beta^2) (\sinh^2 \varepsilon + \sin^2 \Lambda)$$

$$- k^2 e^2 r^2 \left[ g(\varepsilon) + h(\Lambda) \right] = 0 \quad (2.38)$$

This equation can be separated into

$$\frac{d^2 Q}{d \varepsilon^2} + k^2 G(\varepsilon) Q(\varepsilon) = 0 \quad (2.39)$$

and

$$\frac{d^2 S}{d \Lambda^2} + k^2 H(\Lambda) S(\Lambda) = 0 \quad (2.40)$$
where

\[ G(\varepsilon) = e^{2r^2 \frac{n_o}{L_o} } - \beta^2 \sinh^2 \varepsilon - g(\varepsilon) \] - \left( n_o^2 - n_2^2 \right) r^2 \hat{\mu} \quad (2.41) \]

and

\[ H(\Lambda) = e^{2r^2 \left[ n_o^2 - \beta^2 \sin^2 \Lambda - h(\Lambda) \right]} + \left( n_o^2 - n_2^2 \right) r^2 \hat{\mu} \quad (2.42) \]

where \( \hat{\beta} \) and \( \hat{\mu} \) are the ray invariants \( ^{(24)} \).

It has been shown \( ^{(16, 23)} \) that elliptical fibres support two types of modes, hyperbolic or H-type and elliptic or E-type depending on whether \( \hat{\mu} \) is greater or less than zero. In the H type, the mode is confined to the regions:

\[ 0 < \varepsilon < \varepsilon_{tp} \]
\[ \Lambda_{min} < \Lambda < \pi/2 \]

and the E-type mode is confined to the region

\[ \varepsilon_{min} < \varepsilon < \varepsilon_{tp} \]
\[ 0 < \Lambda < \pi/2 \]

It has been shown \( ^{(16)} \) that the H-type mode can only be bound whereas an E-type mode can be bound or tunnelling.

Substitution of equation (2.33) into equation (2.41) and with the mathematical manipulation shown in Appendix C gives:

\[ 2 \sinh^2 \varepsilon_{tp} = \frac{1}{e^{2\gamma^2}} (n_o^2 - \beta^2) - 1 + \left[ \frac{1}{\gamma^2 e^2} (n_o^2 - \beta^2) - 1 \right]^2 = \frac{4\nu}{e^4} \]

\( \delta \)
and similarly

\[ 2 \sinh^2 e_{\text{min}} = \frac{1}{e^2} \left( n_o^2 - \beta^2 \right) - 1 - \left( \frac{1}{\gamma^2 e^2} \left( n_o^2 - \beta^2 \right) - 1 \right)^2 - \frac{4\mu}{e^4} \] (2.44)

where \( \gamma^2 = (n_o^2 - n_2^2) \)

In E-type modes, the inner and outer caustics are ellipses (see Fig. 2 of Reference 16).

Let the distance from the centre of the core to the outer caustic be \( r_{tp} \), then (23):

\[ \left( \frac{r_{tp}}{r} \right) = e^2 \left[ \cosh^2 e_{tp} - \sin^2 \Lambda_{\text{min}} \right] \] (2.45)

But the E-type mode is confined to the region where

\[ \Lambda_{\text{min}} = 0 \]

\[ \therefore \left( \frac{r_{tp}}{r} \right)^2 = e^2 \cosh^2 e_{tp} \]

\[ = e^2 \left[ 1 + \sinh^2 e_{tp} \right] \] (2.46)

Substitution of equation (2.46) into (2.43) gives

\[ \left( \frac{r_{tp}}{r} \right)^2 = e^2 + \frac{1}{2\gamma^2} \left( n_o^2 - \beta^2 \right) - \frac{e^2}{2} + \frac{e^2}{2} \left[ \frac{1}{\gamma^2} \left( n_o^2 - \beta^2 \right)^2 - 1 \right] - \frac{4\mu}{e^4} \] (2.47)

\[ \frac{r_{tp}}{r} = \frac{1}{2} \left[ \frac{1}{\gamma^2} \left( n_o^2 - \beta^2 \right) + e^2 + \left\{ \frac{1}{\gamma^2} \left( n_o^2 - \beta^2 \right)^2 \right\} - \frac{4\mu}{e^4} \right]^\frac{1}{2} \] (2.48)

\( \frac{r_{tp}}{r} \) is a real quantity, therefore the term under the square root must be positive, implying

\[ \frac{1}{\gamma^2} \left( n_o^2 - \beta^2 \right) - e^2 > \frac{2\mu^2}{e^2} \] (2.49)
also \( \left( \frac{r_{tp}}{r} \right) \) should be less than unity, for if \( r_{tp} \) is outside the core, this type of mode is not possible. \( r_{tp} \) must be within the core to ensure guided propagation, therefore for \( \frac{1}{\gamma^2} (n_o^2 - \beta^2) > 1 \) and to ensure that

\[
\frac{r_{tp}}{r} < 1
\]

the following condition holds (23):

\[
\left[ \frac{1}{\gamma^2} (n_o^2 - \beta^2) - 1 \right] \left[ 1 - e^2 \right] < \frac{n}{e^2}
\]  

(2.50)

From equations (2.49) and (2.50)

\[
\frac{1}{\gamma^2} (n_o^2 - \beta^2) + e^2 = 2
\]  

(2.51)

\[
\therefore (n_o^2 - \beta^2) = \gamma^2(2 - e^2)
\]  

(2.52)

\[
\therefore \beta^2 = n_o^2 - \gamma^2(2 - e^2)
\]  

(2.53)

Equation (2.53) shows that the presence of ellipticity in the fibre core tends to shift the propagation constant towards the leaky mode region, in other words the presence of ellipticity reduces the ratio of tunnelling rays to bound rays that the fibre can support.
2.7 **Coupling Errors**

Coupling errors are classified as follows (see Fig. 2.22):

1. **Separation**: when the source surface and the fibre surface have the same axis but are separated by a gap ($s$),

2. **Displacement**: when the two surfaces have their axes parallel and separated by a distance ($u$),

3. **Misalignment**: when the surfaces axes are at an angle ($\phi$) to one another.

In the analysis carried out in Section 2.3 it was assumed that the fibre and the source are concentric and very close. In the following analysis the following assumptions were made:

1. that the space between the source and the fibre is filled with a medium which has the same refractive index as the fibre core so that power loss at the fibre surface due to reflection need not be considered.

2. multimode operation is assumed, justifying use of the geometrical optic approach.

The power emitted by the course is given by equation (2.13) and the power accepted and guided by the fibre is given by equation (2.14). The power emitted by the source is not affected by the presence of any of the coupling errors but the power accepted and guided by the fibre is, and the existence of small gap ($s$) in Fig. 2.22 requires a transfer of coordinates from the source surface to the fibre input surface, on other words, if the radiance distribution on the emitting surface is $R'(r', \phi', \theta', \phi')$, this should be transformed to the radiance at the fibre input surface which
Fig. (2.22.) REPRESENTATION OF COUPLING ERRORS
is $R(r,\psi,\theta,\phi)$, this is done by relating each variable $r', \psi', \theta', \phi'$ on the source surface to the variables $(r,\psi,\theta,\phi)$ on the fibre surface. This was done in the same way as that carried out in reference (8) to obtain

$$r' = r'(r_o, \psi_o, \theta_o, \phi_o), \quad \psi' = \psi'(r_o, \psi_o, \theta_o, \phi_o), \quad \theta' = \theta'(r_o, \psi_o, \theta_o, \phi_o), \quad \phi' = \phi'(r_o, \psi_o, \theta_o, \phi_o)$$

so that:

$$R(r_o, \psi_o, \theta_o, \phi_o) = R'[r'(r_o, \psi_o, \theta_o, \phi_o), \quad \psi'(r_o, \psi_o, \theta_o, \phi_o), \quad \theta'(r_o, \psi_o, \theta_o, \phi_o), \quad \phi'(r_o, \psi_o, \theta_o, \phi_o)].$$

and if the interposed medium is lossy then $R'$ should be multiplied by the transmittivity $T_o(r_o, \theta_o')$ which takes into account fresnel losses at the fibre input surface. The mathematical transformation can be found in (8,9) (no analytical formula could be obtained) and only the results which were obtained by numerical techniques, are presented here.
Fig. (2.23) Launching Efficiency v. separation for bound rays only.
Fig. (2.24) Launching Efficiency versus separation for (bound and skew) rays
Fig. (2.25) Launching Efficiency v. separation for bound rays only

\[ \frac{b}{a} = \frac{1}{2} \]
Fig. (2.26) Coupling Efficiency v. separation for (bound + skew) rays

\[ \frac{b}{a} = \frac{1}{2}. \]
Fig. (2.27) Comparison between Launching Efficiencies v. Separation

1, 2, 3 \( \frac{b}{a} = 1 \)
4, 5, 6 \( \frac{b}{a} = \frac{1}{2} \)
1, 4 \( \alpha = 1 \)
2, 5 \( \alpha = 2 \)
3, 6 \( \alpha = 3 \)
Fig. (2.28) Coupling loss v. separation considering bound rays only

$\frac{b}{a} = 1$
Fig. (2.29) Coupling loss v. separation considering bound rays only

\[
\frac{b}{a} = \frac{1}{2}
\]
Fig. (2.30) Coupling loss v. separation considering (bound + skew) rays

- Coupling Loss $L_c$ (dB)
- $a = 1$
- $b/a = 1$
- $a = 2$
- $a = 2.5$
- $a = 3$
Fig. (2.31) Coupling loss v. separation considering (bound + skew) rays

\[ \frac{b}{a} = \frac{1}{2} \]
Fig. (2.32) Coupling loss v. separation for (bound + skew) rays for different ratio of \( \frac{b}{a} \)
Fig. (2.33) Variation of the launching efficiency with displacement for different index profile parameter ($\alpha$)
Fig. (2.34) Variation of the launching efficiency $\eta'$ with displacement for different index profile parameter $\alpha$
Fig. (2.35) Comparison between the variation of the launching efficiencies $n'$ and $\alpha'$ with displacement
Fig. (2.36) Variation of the launching efficiency \( \eta \) with misalignment for different index profile parameter \((\alpha)\)
Fig. (2.37) Variation of the launching efficiency $\eta'$ with misalignment for different index profile parameter ($\alpha$)
Fig. (2.38) Comparison between the variation of the launching efficiencies $\eta$ and $\eta'$ with misalignment
2.8 Discussion

Simple analytical expressions have been obtained for calculating the launching efficiency and coupling loss of power law optical fibres. A similar approach has been used for inverse -α profile fibres.

The analysis shows that the launching efficiencies for both power law profiles and inverse α-profiles approach that of the step index fibre as the limiting case. Power law profile fibres are more efficient in power coupling than inverse α-profile fibres. This is due to the variation of the numerical aperture and the acceptance angle, for in the power-law fibres they are a maximum at the centre and decrease gradually towards the periphery while in the case of inverse -α profile fibres they are a function of the dip at the centre of the core.

Elliptical fibres, in which the contours of constant refractive index are concentric ellipses have been considered. It has been shown (25) that ellipticity in single mode fibres establishes preferred fast and slow axes of propagation and produces a retardance which depends on the degree of ellipticity. In multimode optical fibres, the presence of small ellipticity in the fibre core, improves the launching efficiency of parabolic index (α = 2) fibres. This is due to the fact that a small ellipticity tends to shift the propagation constant to any intermediate value between \( \hat{\beta} = \hat{\beta}_{\text{max}} \) and \( \hat{\beta} = \mathsf{n}_{\text{cladding}} \) (Fig. (2.21). This represents a reduction in the ratio of tunnelling to bound modes. This shows that ellipticity can be employed to improve the launching efficiency of parabolic index optical fibres. But it has been shown by Schlosser (26) that the ellipticity should be less than 10% otherwise it will greatly reduce the bandwidth.
Since this improvement in the launching efficiency is significant only in parabolic index fibres ($\alpha = 2$), an optimum value of ellipticity can be found that improves the launching efficiency without affecting the fibre bandwidth. This is of practical importance since the values of $\alpha$ in the region $\alpha = 2$ are the values of main interest as the optimum profile for minimum pulse dispersion is close to parabolic.

When coupling errors are introduced, the graphs show that the launching efficiency improves for higher values of the index profile exponent ($\alpha$) and therefore the coupling loss is reduced. The graphs also show that the launching efficiency is less sensitive to small separation values than misalignment and displacement.
CHAPTER 3

INFLUENCE OF THE CENTRAL DIP IN THE REFRACTIVE INDEX PROFILE ON THE 
LAUNCHING EFFICIENCY OF GRADED INDEX OPTICAL FIBRES

3.1 Introduction

The best optical fibres available to date are usually prepared by the chemical vapour deposition (CVD) method \(^{(1-3)}\). However, one problem in this process which has not been completely overcome is the volatilisation of the dopant from the innermost layers during the high temperature collapse, thereby yielding a refractive index dip on the fibre axis \(^{(4,5)}\). This effect is commonly observed in fibres having binary glass cores in which one component is more volatile than the other. Gambling \(^{(7)}\) suggested that this effect can be minimized by increasing the concentration of the more volatile component in the innermost layers and adding a less volatile component to the last layer.

The effect of this dip on propagation characteristics and pulse broadening has been investigated by several authors \(^{(6-8)}\). In their analysis, the shape of this central dip has always been assumed to be Gaussian. It is equally possible, however, to assume that the central dip can have an inverse \(n\)-power law profile, as shown in fig. (3.1). The effect of this dip on the launching efficiency is investigated in the present chapter.
Fig (3.1) ILLUSTRATION OF A CENTRAL DIP IN OPTICAL FIBRES.
3.2 Theory

Referring to fig. (3.1), the refractive index distribution can be divided into three regions.

In region 1, the refractive index variation is given by:

\[ n^2(r) = n_0^2 \left[ 1 - 2\delta (1 - R_1^{\alpha_1}) \right] \quad 0 \leq R_1 < a \tag{3.1} \]

where

\[ R_1 = \frac{r_1}{A_a} \]

\( a \) is the fibre core radius

\( A \) is a fraction of the fibre radius \((a)\), \( A << 1 \)

\( n_0 \) is the refractive index at the edge of the core

\( \delta \) is the relative refractive index difference between the edge and the centre of the core, and denotes the dip height

\( \alpha_1 \) is the index profile parameter in region 1.

In region 2, it is expressed as:

\[ n^2(r) = n_0^2 \left[ 1 - 2\delta(R_2^{\alpha_2}) \right] \quad A_a \leq R_2 < a \tag{3.2} \]

where

\[ R_2 = \frac{r_2}{a - A_a} = \frac{r_2}{a(1 - A)} \]

and \( \alpha_2 \) is the index profile parameter in region 2.

In region 3,

\[ n^2(r) = n_0^2 \left[ 1 - 2\delta \right] = n_2^2 \quad r_2 > a \]

where \( n_2 \) is the cladding refractive index.
Consider the case of bound rays only, i.e. equation (2.19). Here as before, two alternatives are to be considered.

1. \( b > a \) (source radius larger than the fibre radius)

In this case the launching efficiency is given by:

\[
\eta' = \frac{2 T_0}{b^2} \int_0^a \left[ n'(r) - n_2^2 \right] r \, dr
\]  

(3.3)

Since \( n(r) \) is expressed by two different functions in the two regions of fig (3.1), the above integral can be split into two parts and the launching efficiency evaluated for the two separate regions as follows:

In region 2: the integral \( I_2 \) is given by:

\[
I_2 = \frac{2 T_0}{b^2} \int_{A a}^{a} \left[ n_2^2(r) - n_2^2 \right] r \, dr  
\]

As \( A a < r < a \) (3.4)

But \( n_2^2(r) = n_0^2 \left[ 1 - 2\Delta(R_2)^2 \right] \)

and \( R_2 = \frac{r_2}{a(1 - \Delta)} \)

\[
I_2 = \frac{2 T_0}{b^2} \int_{A a}^{a} \left[ n_0^2 \left[ 1 - 2\Delta(R_2)^2 \right] - n_2^2 \right] r_2 \, dr_2
\]

(3.5)

\[
= \frac{2 T_0}{b^2} \int_{A a}^{a} n_o^2 \left\{ \left( \frac{n_o^2}{n_2^2} - 2\Delta \left[ \frac{r_2}{a(1 - \Delta)} \right]^2 \right) \right\} r_2 \, dr_2
\]

(3.6)
\[ I_2 = \frac{4 T o n_o^2 \Delta}{b^2} \int_{Aa}^{a} \left[ 1 - \frac{r_2}{a(1 - A)} \right] r_2 \, dr_2 \]  

(3.7)

where \( \frac{n_o^2 - n_2^2}{n_o^2} = 2\Delta \)

\[ I_2 = \frac{4 T o n_o^2 \Delta}{b^2} \left[ \frac{r_2^2}{2} - \frac{r_2^{a_2 + 2}}{a^2(a_2 + 2)(1 - A)a} \right] \]

(3.8)

Substituting and neglecting high order terms of \( A \) since it has been assumed that \( A << 1 \)

\[ I_2 = \frac{4 T o n_o^2 \Delta}{b^2} \left( \frac{a}{b} \right)^2 \left[ \frac{1}{2} - \frac{1}{(a_2 + 2)(1 - A)a} \right] \]

(3.9)

Similarly in region 1, the integral \( I_1 \) is given by:

\[ I_1 = \frac{2 T o}{b^2} \int_{0}^{Aa} \left\{ n_o^2 \left[ 1 - 2\Delta (1 - R_1)^{a_1} \right] - n_o^2 \right\} r_1 \, dr_1 \]

(3.10)

where \( R_1 = \frac{r_1}{Aa} \)

\[ I_1 = -\frac{4 T o n_o^2 \delta}{b^2} \int_{0}^{Aa} \left[ r_1^2 - \frac{\alpha_1 r_1^2}{A a} + \frac{\alpha_1 (\alpha_1 - 1)}{2} \right]\frac{r_1^3}{A^2 a^2} - \frac{\alpha_1 (\alpha_1 - 1) (\alpha_1 - 2)}{6} \frac{r_1^4}{A^3 a^3} \, dr_1 \]

(3.11)

\[ = -\frac{4 T o n_o^2 \delta}{b^2} \left[ \frac{r_1^2}{2} - \frac{\alpha_1 r_1^3}{3 A a} + \frac{\alpha_1 (\alpha_1 - 1)}{8} \right]\frac{r_1^4}{A^2 a^2} - \frac{\alpha_1 (\alpha_1 - 1) (\alpha_1 - 2)}{30} \frac{r_1^5}{A^3 a^3} \]

(3.12)
The launching efficiency is the sum of the two integrals $I_1$ and $I_2$:

$$
\therefore \quad I_1 = -4T_0 n_o^2 A^2 \delta (\frac{a}{b})^2 \left[ \frac{1}{2} - \frac{a_1}{3} + \frac{a_1(a_1 - 1)}{8} - \frac{a_1(a_1 - 1)(a_1 - 2)}{30} \right] \quad (3.13)
$$

It can be seen that if $\delta = 0$ implying that $A = 0$ as well, the above expression reduces to equation (2.24) which gives the launching efficiency for $b > a$ when no dip is present at the centre of the fibre.

2. $b < a$

This is the case where the source radius is less than the fibre core radius and in this case the acceptance angle of the fibre is given by:

$$
\sin \theta_m = \frac{a}{b} (n_1^2(r) - n_2^2)^{1/2}
$$

and the launching efficiency is expressed as:

$$
\eta' = \frac{2T_0}{b^2} \int_{0}^{b} \left[ n_1^2(r) - n_2^2 \right] rdr \quad (3.15)
$$

note that the limits of integration are determined by the smaller of $a$ or $b$. 
Evaluating equation (3.15) for the two regions as before; in region 2
the integral $I_2$ is

$$I_2 = \frac{2 \frac{T_0}{b^2}}{b^2} \int_0^b \left( n_o^2 \left[ 1 - 2\Delta \frac{A^2}{r_2} \right] - n_2^2 \right) r_2 \, dr_2$$

(3.16)

$$= \frac{4 \frac{T_0}{b^2} n_o^2 \Delta}{b^2} \int_0^b \left\{ 1 - \left[ \frac{r_2}{a(1 - \Delta)} \right]^2 \right\} r_2 \, dr_2$$

(3.17)

$$= \frac{4 \frac{T_0}{b^2} n_o^2 \Delta}{b^2} \left[ \frac{r_2^2}{2} - \frac{r}{a^2(a^2 + 2)(1 - \Delta)} \right]_0^b$$

(3.18)

$$\therefore \quad I_2 = 4 \frac{T_0}{b^2} n_o^2 \Delta \left[ \frac{1}{2} - \frac{1}{(a^2 + 2)(1 - \Delta)} \cdot \left( \frac{b}{a} \right)^2 \right]$$

(3.19)

Similarly in region 1, the integral $I_1$ is:

$$I_1 = \frac{2 \frac{T_0}{b^2}}{b^2} \int_0^b A^2 \left( n_o^2 \left[ 1 - 2\delta (1 - R) \frac{A^2}{r_1} \right] - n_o^2 \right) r_1 \, dr_1$$

(3.20)

$$= -\frac{4 \frac{T_0}{b^2} n_o^2 A^2 \delta}{b^2} \int_0^b \left[ 1 - \alpha_1 R_1 + \frac{\alpha_1(a_1 - 1)}{2} R_1^2 - \frac{\alpha_1(a_1 - 1)(a_1 - 2)}{6} R_1^3 \right] r_1 \, dr_1$$

(3.21)

where $R_1 = \frac{r_1}{Aa}$
After some manipulation the above integral yields:

\[
I_1 = -4 \int_0^\infty d^2 A^2 \delta \left[ \frac{1}{2} - \frac{a_1}{3} \left( \frac{b}{a} \right) + \frac{a_1(a_1-1)}{8} \left( \frac{b}{a} \right)^2 - \frac{a_1(a_1-1)(a_1-2)}{30} \left( \frac{b}{a} \right)^3 \right] \tag{3.22}
\]

The launching efficiency for this case is therefore

\[
\eta' = 4 \int_0^\infty d^2 A^2 \delta \left[ \frac{1}{2} - \frac{1}{(a_2 + 2)(1 - A)} \left( \frac{b}{a} \right)^2 \right] \tag{3.23}
\]

It is also evident in this case that when the dip height \( \delta \) is zero the above expression reduces to equation (2.25) which gives the launching efficiency for \( b < a \) when no dip is present at the centre of the fibre.

Equations (3.14) and (3.23) when plotted give Figs. (3.2 - 3.8) for different values of dip height \( \delta \) and dip width \( A \) for a fibre with the following parameters:

\[
\begin{align*}
n &= 1.47028 \\
n_2 &= 1.45799 \\
2\Delta &= 0.01794 \\
a &= 25 \times 10^{-6} \text{ meters}
\end{align*}
\]

and a refractive index variation which is parabolic with a dip at the centre.

It is clear from the graphs that an increase in the dip height \( \delta \) or the dip width \( A \) represents a reduction in the launching efficiency. For example a dip width of 20% of the fibre radius, for \( \delta = \frac{A}{4} \) and \( a_1 = a_2 = 1 \), represents...
a 5\% reduction while for $\delta = \Delta$ the reduction increase to 8\%. This reduction in the launching efficiency represents an increase in the coupling loss as shown in Fig (3.9).
Fig. (3.2) Launching Efficiency $V \cdot (\frac{b}{a})$
Fig. (3.3) Launching Efficiency $V. \left( \frac{b}{a} \right)$

\[
\delta = \frac{\Delta}{2} \\
1 - A = 1\%a \\
2 - A = 10\%a \\
3 - A = 20\%a
\]

for $\alpha_1 = \alpha_2 = 1$
Fig. (3.4) Launching Efficiency $V \left( \frac{b}{a} \right)$

\[
\text{for } a_2 = a_1 = 1
\]

\[
\begin{align*}
\delta &= \Delta \\
1 - A &= 1\%a \\
2 - A &= 10\%a \\
3 - A &= 20\%a
\end{align*}
\]
Fig. (3.5) Launching Efficiency $v. \left( \frac{a}{b} \right)$

\[ \delta = \frac{\Delta}{4} \]

1. $A = 1\%a$
2. $A = 10\%a$
3. $A = 20\%a$

for $a_1 = a_2 = 2$
Fig. (3.6) Launching Efficiency $V. \left( \frac{\delta}{a} \right)$

\[
\delta = \frac{A}{2}
\]

1. $A = 1\%a$
2. $A = 10\%a$
3. $A = 20\%a$

for $a_1 = a_2 = 2$
Fig. (3.7) Launching Efficiency $V_b\left(\frac{b}{a}\right)$

\[
\delta = \Delta \\
1 - A = 1\%a \\
2 - A = 10\%a \\
3 - A = 20\%a
\]

for $a_1 = a_2 = 2$
Fig. (3.8) Comparison between launching efficiencies $\gamma_{_{\frac{b}{a}}}$

$\delta = \frac{A}{4}$

$a, b, c$, for $a_1 = a_2 = 2$

d, e, f, for $a_1 = a_2 = 1$
Fig. (3.9) Coupling loss $V \left( \frac{b}{a} \right)$ for various values of $\delta$ and $A$.
3.3 Discussion

The fact that pulse spread can be reduced dramatically by quadratic grading of the refractive index profile led to the expectation that if the refractive index profiles were optimised, a further reduction in intermodal pulses spread might be achieved. Thus the investigation of fibres having arbitrary shaped refractive index profiles has been prompted by this expectation.

The manufacturing process of optical fibres involves the presence of two dips (valleys) in the refractive index profile, one is at the core-cladding boundary and the other at the centre of the fibre core. The former is made deliberately during the manufacturing process and serves two purposes:

1. Reduction of internal mechanical stresses due to the gradient of the dopant concentration,
2. The difference between group velocities of modes is reduced to a minimum, hence pulse dispersion is reduced.

The latter is unavoidable and its presence is due to the high temperatures involved during the manufacturing process.

Its effect on the launching efficiency of optical fibres has been investigated and analytical expressions have been obtained. It has been suggested by P. Di Vita (13) that the presence of a dip in the refractive index profile in the centre of the core does not produce appreciable differences in the launching efficiency. However, the present analysis shows that it does, and a reduction in the launching efficiency of 8% has been demonstrated. It has also been found that, for a certain dip height and dip width,
increasing the refractive index profile $\alpha_1$ improves the launching efficiency. Therefore it would appear that a minimisation of the central dip effect on the launching efficiency of an optical fibre may be obtained by making the refractive index variation in the dip region an $\alpha$-power law profile.
CHAPTER 4

EFFECT OF CENTRAL DIP IN THE REFRACTIVE INDEX PROFILE ON BIREFRINGENCE IN GRADED INDEX SINGLE MODE FIBRES WITH ELLIPTICAL CROSS SECTION

4.1 Introduction

Single mode optical fibres offer greater transmission bandwidth than all other guiding media\(^{(1)}\). Their bandwidth is limited by material dispersion and waveguide structure. These limitations may be reduced or eliminated by choosing an optimum wavelength such that the material dispersion either vanishes \((\frac{d^2n}{d\lambda^2} = 0)\), or compensates for mode dispersion\(^{(2,3)}\).

If circularly polarized light is launched into such a fibre, the state of polarization of the light changes along the fibre length. This is due to the different propagation velocities of the polarization components of the light with respect to the axes of the ellipse, i.e. the fibre acts as a birefringent medium\(^{(1)}\). This birefringence is the result of a difference in the values of \(\beta\) for the propagation constants of the fundamental mode. This difference in the values of \(\beta\) causes the fibre to exhibit a linear retardation, the value of which depends upon the fibre length. The resulting polarization state varies periodically along the fibre with a characteristic length "\(L\)" where

\[
L = \frac{2\pi}{\delta \beta}
\]

Typical single mode fibres were found to have a period of a few centimeters\(^{(4)}\).
The effect of different central dip heights \( \delta \) in the refractive index profile of graded index single mode fibres with elliptical cross-section is investigated.

4.2 Theoretical background

Several calculations of the birefringence due to core ellipticity have been performed using vector (5) and scalar (6) perturbation methods.

For weakly guiding \((\Delta \ll 1)\), slightly elliptical \((e^2 \ll 1)\), step index fibres, the difference in phase velocity \((\delta \beta)\) between the two polarization states of the fundamental mode is given by (6);

\[
\delta \beta = \frac{\epsilon^2 (2\Delta)^3 m^2}{4aV^3} \left[ U^2 + (U^2 - W^2) \left\{ \frac{J_0(U)}{J_1(U)} \right\} + U W^2 \left\{ \frac{J_0(U)}{J_1(U)} \right\} \right]^{3/2}
\] (4.1)

where

\[
U = a(k_o^2 n_o^2 - \beta^2)^{1/2}
\] (4.2)

\[
W = a\beta^2 - k_o^2 n_2^2 \left( \frac{1}{2} \right)
\] (4.3)

\[
V = a k_o (n_o^2 - n_2^2)^{1/2}
\] (4.4)

\[
V^2 = (U^2 + W^2)
\] (4.5)

\(a\) is the fibre radius, \(J_0\) and \(J_1\) are Bessel functions of the 1st and 2nd kind respectively.

\[k_o = \frac{2\pi}{\lambda}\] is the wave number in free space

\(n_o \& n_2\) are defined previously
A monomode graded index elliptical fibre whose major and minor axes are
a and b respectively has a core refractive index of the form

\[ n^2(r, \theta) = n_o^2 \left[ 1 - 2f(r, \theta) \right] \]  \hspace{1cm} (4.6)

where

\[ f(r, \theta) = f \left[ R(1 - e^2 \cos^2 \theta)^{1/2} \right] = f \left[ R(1 - \frac{1}{2} e^2 \cos^2 \theta) \right] \]  \hspace{1cm} (4.7)

and a cladding refractive index of the form

\[ n^2(r, \theta) = n_o \left[ 1 - 2\Delta \right]^{1/2} \]

and R is the normalised radius \( \frac{r}{a} \).

For fibres having such refractive index profiles, the phase difference
(delta \( \beta \)) is given by (7)

\[ \delta \beta = \frac{e^2(2\Delta)^{3/2}}{4 V N_o} \left[ \left[ 1 - f(1) \right] \left[ \psi'_2 + 2\psi_2 + \psi'_o \psi_2 + \frac{1}{4}(\psi'_o^2 + \psi''_o - \psi'_o) \right] \right]_{\text{cladding}} \]

\[ + \int_0^1 \int R f' \left[ \psi_o(\psi'_2 + \frac{2}{R} \psi_2 - \frac{1}{4} \psi'_o) + \psi'_o \psi_2 + \frac{R}{4}(\psi'_o^2 + \psi' o \psi''_o) \right] dR. \]  \hspace{1cm} (4.8)

where the prime indicates differentiation with respect to R and

\[ N_o = \int_0^\infty \psi_o^2 R \, dR \]

\( \psi_o \) is the scalar field in circular fibres and is found by solving the scalar
field equation (8).

\( f(1) \) indicates that the evaluation of the fields is being carried out
in the cladding side of the boundary.

For step index fibres \( f(r, \theta) = 0 \), and equation (4.8) reduces to:
\[
\delta \beta = \frac{e^2 (2\Delta)^{3/2}}{4aV \frac{\psi_o}{N_0}} \left\{ \psi'_2 + 2\psi_2 + \psi'_o \psi_2 + \frac{1}{4} (\psi''_o + \psi'' - \psi'_o) \right\} \tag{4.9}
\]

with \( \psi_o \) and \( \psi_2 \) given by (6): i.e.

\[
\psi_o = \frac{J_o(UR)}{J_o} - \frac{e^2}{2J_o} \left[ cJ_o(UR) - \frac{U}{2} \left( \frac{K_o}{K_1} \right)^2 RJ_1(UR) - \frac{U^2}{4} \frac{K_oK_2}{K_1^2} J_2(UR) \cos \theta \right] \tag{4.10}
\]

\[
\psi_2 = \frac{K_o(WR)}{K_o} - \frac{e^2}{2K_o} \left[ dK_o(WR) + \frac{W}{2} \left( \frac{J_o}{J_1} \right)^2 R K_1(WR) - \frac{W^2}{4} \frac{K_oK_2}{K_1^2} K_2(WR) \cos \theta \right] \tag{4.11}
\]

where \( R \) is the normalized radius \( \left( \frac{r}{a} \right) \)

\( \theta \) is the azimuthal angle

\( U \) and \( W \) are the arguments of the Bessel and modified Hankel functions.

d and \( c \) are normalization parameters and are given by (9):

\[
d = \frac{1}{2} \left( \frac{U}{\psi'_2} \right) \left[ \frac{W^2}{U^2} - 1 + W^2 \left( \frac{1}{\psi'_o} + \frac{\psi'_o}{U^2} \right) \right] \tag{4.12}
\]

and

\[
c - d = \frac{V^2 J_o}{2 U J_1} \tag{4.13}
\]

If the function \( f \) in equation (4.7) is expressed as

\[
f = R a^2 + 2\theta (1-R) a^1 \tag{4.14}
\]

where the first term represents the normal \( \alpha \)-profile refractive index

and the second term represents the dip at the centre of the fibre.

Then, following the same analysis given in (7), \( \psi_o \) is obtained in the same manner as before. \( \psi_2 \) in the cladding is given by (4.11) and in the
cladding is given by (4.11) and in the core $\psi_2$ is the solution of the equation
\[
\psi''_2 + \frac{1}{R} \psi'_2 + \left( \frac{V^2}{R^2} - \frac{4}{R^2} - V^2 f \right) \psi_2 + \frac{1}{4} V^2 R f \psi_0 = 0
\] (4.15)

A series of numerical calculations were performed to calculate $\psi_0$ and $\psi_2$ and these values were substituted in equation (4.8) to calculate delta for different values of dip height.
Fig. (4.1) Normalized phase difference \( \delta \phi \) as a function of \( v \)
Fig. (4.2) Normalized phase difference $\Delta \beta$ as a function of $V$
Fig. (4.3) Normalized phase difference $\delta \theta$ as a function of $V$ calculated from equations (4.1) and (4.9).
Fig. (4.4) Normalized phase difference delta $\beta$ as a function of $V$ for various dip index profile ($\alpha_1$)
Fig. (4.5) Normalized phase difference delta $\delta$ as a function of V for various dip profile parameter ($\alpha_1$)
4.3 Discussion

The origins of birefringence are twofold.

1. Ellipticity in the fibre core establishes preferred fast and slow axes of propagation and produces a retardance which depends on the degree of ellipticity\(^{(3)}\).

2. The presence of an asymmetrical residual stress distribution within the fibre, results in stress birefringence\(^{(10)}\).

A number of investigations into the effects of core ellipticity on the operation of single mode fibres have been carried out both in telecommunication systems\(^{(11)}\) and in devices such as Faraday-effect fibre current transducers\(^{(7,8)}\).

Fig. (4.1) show the variation of the phase difference (\(\delta S\)) as a function of \((V)\) for a step index fibre (Equation 4.9). This was compared with values obtained for (\(\delta S\)) using equation (4.1). There is a slight difference between the results obtained. This may be due either to approximation error during the computation process, or to the approximation involved in the derivation of the scalar modes by solving the scalar wave equation in elliptical co-ordinates and then correcting the scalar propagation constants to include the effect of the difference between a circle and an ellipse in the refractive index term. In this extra region the fields obtained by considering a circular system give an insufficient approximation for the case of an elliptical system, because the radial components of the two fields are discontinuous along the two different boundaries. It can be seen from the graph that the circularity required to produce low birefringence for current-transducer application is highest near the
preferred operating point for a single mode fibre. Lowering $V$ or increasing its value towards the multimode region relaxes the tolerance on circularity. The expression obtained for calculating the phase difference suggests that larger values of ellipticity can be tolerated in fibres having small $\Delta$, but again the index difference $(\Delta)$ cannot be reduced too much because radiation loss due to fibre bending becomes intolerable when $(\Delta)$ is excessively small.

The presence of a dip at the centre shows that the higher the dip height $(\delta)$ the lower the maximum value of the birefringence. Making the refractive index profile in the dip region an $\alpha$ profile does not appear to have much effect on the birefringence compared with the effect on the launching efficiency as explained in the previous chapter, i.e. the shape of the dip profile is not important but the depth $\delta$ of the dip is the governing factor.
CHAPTER 5

EFFECT OF THE CENTRAL DIP IN THE REFRACTIVE INDEX PROFILE ON PULSE BROADENING OF MULTIMODE OPTICAL FIBRES

5.1 Introduction

Use of optimum refractive index profile can reduce the intermodal delay differences in multimode optical fibres (1, 2) so that pure material dispersion becomes the dominant source of pulse broadening. Thus multimode fibres with optimum or near optimum refractive index profiles represent an attractive transmission medium for high bandwidth optical communication systems.

Modal delay distortions occur because the many different modes travel at different group velocities. This results in spreading an impulse response over a time interval that is equal to the difference between the arrival times of the slowest and fastest modes. This pulse spreading is accompanied by a reduction in signal bandwidths (3).

Since it is difficult to make index profiles of optimum shape, it is important to know how any departure from the optimum affects the impulse response of an optical fibre.

One of the problems encountered in fibres produced by the Modified Chemical Vapour Decomposition (MCVD) method is the drop-off of the index value (step distortion) near the core cladding interface as shown in Fig (5.1). This represents a deviation from the optimum index profile.
A second deviation that may be encountered in fibres produced by the same method is the depression in the index distribution on the fibre axis as shown in Fig. (5.2). This is due to the evaporation of the dopant material from the innermost layers during the collapsing process and is caused by the high temperatures involved in the manufacturing process.

Calculations of impulse response and profile dispersion of optical fibres have been carried out by many authors. R. Hanson and Nicolaisen\(^{(4)}\) tried to predict the impulse response by fitting the refractive index profile by an \(\alpha\)-profile, but have indicated that this gives rather inaccurate results. A more general approach has been used by Hansen\(^{(5)}\) and Marcuse\(^{(6)}\). The group delay of each mode is calculated using the WKB(1) approach, and the impulse response is obtained by counting the number of modes arriving at the receiving end of the fibre in a given interval of time. These methods involve the calculation of two caustics for each propagating mode. Chu\(^{(7)}\) has proposed a method of calculating the impulse response, without the need of calculating the caustics. In the following, a review of the theory proposed in Reference (7) is given and its application to arbitrary refractive index profiles with deviations from the optimum is investigated. The effect of these deviations in the impulse response of the optical fibre is demonstrated.
Fig (5.1)  ILLUSTRATION OF A STEP DISTORTION IN
THE INDEX PROFILE OF OPTICAL FIBRES
Fig (5.2)  ILLUSTRATION OF CENTRAL DIP IN THE REFRACTIVE INDEX PROFILE OF OPTICAL DIBRES
5.2 Theoretical review:

The refractive index profile of an optical fibre is given by:

\[ n^2(r) = n_o^2 [1 - F(\lambda, r)] \]  \hspace{1cm} (5.1)

where \( F(\lambda, r) \) is any function of radius \( r \) and wavelength \( \lambda \) and \( n_o \) is the refractive index on the fibre axis.

The propagation constant \( \beta \) for the mode \((\mu, \nu)\) can be expressed as

\[ \beta^2 = n_o^2 k^2 (1 - p) \]  \hspace{1cm} (5.2)

where \( p \) is the mode parameter which varies between zero for the lowest order mode and \( 2\Delta \) for the modes whose phase velocities coincide with that of a plane wave in the cladding(8).

\[ k = \frac{2\pi}{\lambda} \] is the free space propagation constant.

The wave propagation in an optical fibre is given by(9):

\[ (2\mu + \nu + 1) \frac{\pi}{2} = \int_0^{r_1} \sqrt{p - F(\lambda, r)} \, dr \]  \hspace{1cm} (5.3)

where \( r_1 \) is the caustic that makes the radical zero, i.e. is the solution of the equation

\[ p - F(\lambda, r) = 0 \]  \hspace{1cm} (5.4)
Let the principal mode number be $m$ then,

$$\begin{align*}
  m &= (2\mu + \nu + 1)
  
\end{align*}$$

and equation (5.3) becomes

$$\begin{align*}
  m &= \frac{2}{\pi} \int_0^{\Gamma_1} N_0 k \sqrt{P - F(\lambda, r)} \, dr
  
\end{align*}$$

(5.6)

In reference (7), it has been shown that equation (5.6) can be evaluated without the necessity of computing the caustic point $r_1$. This was achieved by obtaining $G(\lambda, F)$ which is the inverse of the profile function $F(\lambda, r)$.

Obtaining the inverse profile function $G(\lambda, F)$ and changing the variables, equation (5.6) becomes

$$\begin{align*}
  m &= \frac{4}{\pi} N_0 k \int_0^P \sqrt{P - F} \, G'(F) \, dF
  
\end{align*}$$

(5.7)

where $G'(F)$ is the derivative of the inverse profile function with respect to $F$.

The above integral can be expressed as the convolution of two functions; $\sqrt{P}$ and $G'(P)$, where $G'(P)$ is the derivative of the profile function.

$$\begin{align*}
  \therefore m &= \frac{4}{\pi} N_0 k \sqrt{P} \ast G'(P)
  
\end{align*}$$

(5.8)

where $\ast$ denotes the convolution operation.
The normalized group delay $\tau$ for the modes of principal mode number $m$ can be obtained by differentiating equation (5.6) with respect to $k$ and substituting for $m$ from equation (5.8). This gives

$$\tau = \frac{1}{\sqrt{1-P}} \frac{1}{G'(P)^*} - 1$$

(5.9)

where:

$$\phi(P) = \left[1 + D_1 - \frac{D_2}{n_0^2} - P(1 + D_1)\right] G'(P)$$

(5.10)

$$D_1 = -\frac{\lambda}{n_0} \frac{d n_0}{d\lambda} - \frac{\lambda}{2\Delta} \frac{d\Delta}{d\lambda}$$

(5.11)

$$D_2 = -\frac{n_0^2 \lambda}{2\Delta} \frac{d\Delta}{d\lambda}$$

(5.12)

$D_1$ and $D_2$ are the dispersion parameters.

The impulse response $f(\tau)$ is given by (7):

$$f(\tau) = \frac{4}{M} \frac{d^2}{d\tau} \frac{dm}{dP}$$

(5.13)

where $M$ is the largest principal mode number given by (10):

$$M = \left[\frac{\alpha}{a + 2} - \frac{2}{n_0^2} \Delta\right]^{1/2}$$

(5.14)
For fibres having refractive index profiles of the form shown in Fig (5.1) which are composed of a power law and a linear function i.e.

\[
F(\lambda, r) = 2\Delta (\frac{r}{a})^\alpha \quad 0 \leq r \leq r_0
\]

and

\[
F(\lambda, r) = 2\Delta \frac{(r - r_0) - (a - r)(\frac{r_0}{a})^\alpha}{(r_0 - a)} \quad r_0 \leq r \leq a
\]  

(5.15)

The impulse response has been reported previously\(^{(3,7)}\). It has been shown that the step distortion produces a pulse tail of low amplitude, and if the step extends deeper into the core, the normalized group delay \(\tau\) becomes larger, the tail becomes longer and the main part of the impulse, which is determined by the power law region, becomes narrower. This justifies the presence of the valley in the index profile near the core cladding interface mentioned in Chapter 3, which serves to minimize this effect.

Profiles of the form shown in Fig (5.2) are composed of a linear function in the depression region and a power law as follows

\[
F(\lambda, r) = 2\Delta (\frac{r}{a})^\alpha \quad Aa \leq r \leq a
\]

\[
F(\lambda, r) = 1 - \frac{n_3^2}{n_0^2} - \frac{n_1^2 - n_3^2}{n_0^2} (\frac{r}{AA}) \quad 0 \leq r \leq aA
\]

(5.16)

where \(A\) is as defined previously.
In the above definition the depth height \( \delta \) is determined by the difference between \( n_0 \) and \( n_3 \) while the one considered in Chapter 3 was the difference between \( n_1 \) and \( n_3 \). The reason for that is that in the present analysis the aim is to extend the power law profile to the fibre axis and not to reach a step index as was mentioned in the previous chapter.

For such a profile given by equation (5.16), the principal mode number \( m \) and the normalized group delay are calculated for each region as follows:

1. For the power law region

\[
G(\lambda, F) = \left(\frac{F}{2\Delta}\right)^{1/\alpha} \quad 2\Delta(A)^{\alpha} \leq F \leq 2\Delta \quad (5.17)
\]

and

\[
G'(\lambda, F) = \frac{1}{\alpha} \left(\frac{1}{2\Delta}\right)^{1/\alpha} F^{\frac{1}{\alpha} - 1} \quad (5.18)
\]

and (2) for the linear region.

\[
F(\lambda, r) = 1 - \frac{n_3^2}{n_0^2} \frac{n_1^2 - n_3^2}{n_0^2} \left(\frac{r}{aA}\right) \quad 0 \leq r \leq aA \quad (5.19)
\]

\[
\frac{n_1^2 - n_3^2}{n_0^2} \left(\frac{r}{aA}\right) = 1 - \frac{n_3^2}{n_0^2} - F \quad (5.20)
\]
the inverse profile function for the linear depression in
the index profile at the fibre axis is given by

\[ G(A,F) = \frac{n_o^2 A}{n_1^2 - n_3^2} \left[ 1 - \frac{n_3^2}{n_o^2} - F \right] \quad 2\delta \leq F \leq 2\Delta \]  

(5.24)

where

\[ \delta = \frac{n_o^2 - n_3^2}{2 n_o^2} \]

Substituting equation (5.17) into equation (5.8) gives for the
principal mode number \( m \)

\[ m = \frac{4}{\pi} n_o k \left[ \frac{1}{a} \left( \frac{1}{2\Delta} \right)^{1/\alpha} \right] \frac{2 + a}{2} \sum B(1 + \frac{1}{a}, \frac{1}{2}) \left( I (1 + \frac{1}{a}, \frac{1}{2}) \right) \]

(5.25)

for the power law profile region.
Substituting equation (5.24) into (5.8) gives:

\[ m = \frac{4}{\pi n_0 k} \left[ \frac{1/\alpha}{2\Delta} \right] \frac{2 + \alpha}{2\alpha} \right] P \beta \left( 1 + \frac{1}{\alpha}, \frac{1}{2} \right) \left[ 1 - i_x \left( 1 + \frac{1}{\alpha}, \frac{1}{2} \right) \right]
\]

\[ + \frac{n_0^2 A}{n_1 - n_2^2} \frac{3/2}{(1 - x)^{3/2}} \]

for the depression region.

where

\[ B \left( 1 + \frac{1}{\alpha}, \frac{1}{2} \right) \text{ is the Beta function} \]

\[ X = \frac{2\Delta}{P} (A)^\alpha \] (5.27)

The normalized group delay for the two regions is given by (5.27)

\[ \tau = \frac{1}{\sqrt{1 - P}} \left\{ \left( 1 + D_1 - \frac{D_2}{n_0^2} \right) - (1 + D_1) \frac{2}{\alpha + 2} P \frac{1 - I_x \left( 1 + \frac{1}{\alpha}, \frac{1}{2} \right)}{1 - I_x \left( \frac{1}{\alpha}, \frac{1}{2} \right)} \right\} - 1 \]

for

\[ 2\Delta(A)^\alpha \leq P \leq 2\Delta \] (5.28)

and

\[ \tau = \frac{1}{\sqrt{1 - P}} \left\{ \left( 1 + D_1 - \frac{D_2}{n_0^2} \right) - \frac{2}{3} (1 + D_1)(1 + X)^2P \right\} - 1 \]

for

\[ 2\Delta(A)^\alpha \leq P \leq 2\delta \] (5.29)

where \( I_x \left( \frac{1}{\alpha}, \frac{1}{2} \right) \) is the incomplete Beta function.
If the dispersion parameters $D_1$ and $D_2$ are assumed to be zero then equations (5.28) and (5.29) becomes:

$$
T = \frac{1}{\sqrt{1 - p}} \left[ 1 - p \left\{ \frac{2}{\alpha + 2} \right\} \left\{ 1 - \frac{1}{1 + \frac{1}{\alpha}, \frac{1}{2}} \right\} \right] - 1 \quad (5.30)
$$

for $2\Delta(A)^\alpha \leq p \leq 2\Delta$

and

$$
T = \frac{1}{\sqrt{1 - p}} \left[ 1 - \frac{2}{3} (1 + X)^\alpha_p \right] - 1 \quad (5.31)
$$

for $2\Delta(A)^\alpha \leq p \leq 2\delta$

The impulse response has been calculated using equation (5.13) for a fibre with the following parameters:

$$
n_0 = 1.47028
$$

$$
n_2 = 1.45709
$$

$$
\Delta = 0.00893
$$

$$
\alpha = 2
$$

$$
\alpha = 25 \mu m
$$
Fig. (5.3) Impulse response of power law profile optical fibre with central dip.
Fig. (5.4) Impulse response of power law profile optical fibre with central dip.
Fig. (5.5) Impulse response of power law profile optical fibre with central dip.

\[ A = 10\% a \]
\[ \delta = \Delta \]
Fig. (5.6) Variation of the impulse response with central dip height ($\delta$)
Fig. (5.7) Impulse response of power law profile optical fibre with central dip.
$A = 15\% \, a$

$\delta = \frac{A}{2}$

Fig. (5.8) Impulse response of power law optical fibre with central dip.
A = 15%

Fig. (5.9) Variation of the impulse response with central dip height $\delta$
Fig. (5.10) Variation of impulse response with central dip width (A)
Fig. (5.11) Variation of impulse response with central dip width (A)
5.3 Discussion

Assuming equal excitation of all modes and no mode coupling, it can be seen from the graphs that the impulse response depends primarily on the shape of the refractive index distribution in the core. A dip in the refractive index profile, characterized by its depth $\delta$ and its width $A$, affects the impulse response, and the response to the left of the origin in the graphs is due to the presence of the central dip. Thus the presence of the linear dip increases the pulse dispersion and this represents a reduction in the bandwidth.

It can be seen from the graphs that both the depth and the width of the central dip increase pulse dispersion. Wider depths have drastic effects on the impulse response and hence produce greater pulse dispersion.
CHAPTER 6

EXPERIMENTS

In this chapter some experimental results related to the theoretical content of the thesis will be reported.

Experimenting with fibres is expensive due to the high precision devices required for alignment etc. Additionally, very good resolution is needed to detect the effects of parameters such as non-uniformities in core diameter or the refractive index profile.

The equipment available for this project was rather limited. Hence not all the mathematical derivations could be tested experimentally. Also, the industrial manufacturers of optical fibres were reluctant to provide values for the parameters of their fibres. Specifically no exact values were given of the refractive index exponent $a$, the width and height of the central dip or the ellipticity of the fibre.

The suggestion was that the fibres were circular and had a refractive index variation which was parabolic and the attenuation was 2.5 dB/Km, hence the assumption that $a = 2$ throughout this work.
6.1 Fibre end preparation:

One of the most important criteria when dealing with fibres is the ability to obtain terminations of acceptable quality. Optical fibres have to be externally excited. Therefore, for maximum light to enter the fibre the scattering by the fibre end must be a minimum which means that the fibre end should be as optically flat as possible.

Throughout the recent progress in the field of fibre optics, two types of fibre to fibre connection have come into existence. Fusion arc splicing which may be essentially regarded as the welding of fibres with index matching liquid in between. This is considered as permanent mounting and provides low splicing loss, typically $0.5\,\text{dB}^{(1)}$. Before welding is carried out a clean and flat end is essential for a good seal.

The other type that provides more flexibility is to have the fibre end free and accomplish the coupling via precision alignment. Conventionally 'V' grooves are employed for the task. Whether the fibre is protected by surrounding layers or lies bare in the groove would depend upon the particular requirement.

Much progress has been made in breaking fibres to produce reasonably flat terminations. Gloge\(^{(2)}\) has indicated that to avoid the occurrence of what is called a beak, a known stress must be applied to keep it under
tension while simultaneously bending it over a curved surface.

Another type which was used earlier consisted of curve shaped foam over which the fibre was stretched to the correct tension by a dial gauge. A knife fixed on to a pole would then come down and initiate a crack on the fibre. Another approach is to tape the end of the fibre over any clean curved surface and pull it by hand. Touching it slightly with a knife edge would then effect the cleavage.

The latter method was used here and steps given in reference (3) were followed so that the fibre end was ready for polishing.

This was accomplished as follows:

Two discs were mounted on opposite ends of a small D.C. motor shaft whose speed could be varied using a variac. On one disc the rough polishing pad was fixed and on the other the fine polishing pad. The fibre and the protecting sleeve around it was pushed through a small hole on a mount facing the rotating disc and at right angles to its plane. This arrangement ensured that the fibre was being polished in a plane perpendicular to the fibre axis, since a sloping end is undesirable as it reduces the gathering power or collecting efficiency of the fibre (4).

The rough pad was used to reduce the protruding section and at the same time the process was observed under the microscope. When this was completed, transferring to the fine polishing pad finally yielded the smooth surface desired. The motor provided a faster and easier method for polishing than polishing by hand. The arrangement is shown in Fig. 6.1.
Fig (6.1) Fibre and preparation arrangement
Fig. (6.2) General view of experimental assembly
6.2 Measurements

The equipment supplied for this project was as follows: (see Fig.6.2)

1. Optometer (analogue) plus a separate opto-detector from United Detector Technology.

2. Micro manipulators, three rotational and three translational stages with resolutions of $5 \mu m$ from Unimatic Engineering Ltd.

3. Light sources:
   a - He-Ne laser, $\lambda = 0.632$ nm
   b - Two LEDs, $\lambda_1 = 660$ nm, $\lambda_2 = 890$ nm


5. Various convex and concave lenses and beam splitter

6. Optical bench and cast iron bed on an anti vibration mounting.

The hardware constructed was as follows:

1. Mounting arrangement to fit onto the micro manipulators. These comprised two plates, the lower one was to join the assembly to the micropositioners. The other plate was mounted on the top. Two interchangable blades were also provided to be used on the upper part, one to hold the ferruled terminators and the other for bare fibre ends. 'V' grooves on these plates were cut accordingly. A clamp device with rubber attachment on the lower side was used to hold the fibre down.

2. A housing box and a drive unit for the LEDs. The circuit operated from a $\mu 741$ operational amplifier and two intensity control buttons were provided.
3. A differential amplifier with variable gain. This was used to provide a digital read-out from the output of the photodetector. This also used a μ741 amplifier.

4. Various mounts for examining fibres under the microscope and to couple the fibre into the optometer head. These were cylindrical structures with holes bored through the middle. The mounting device for bare fibres was in two halves each one being individually 'V' grooved. The various items were aligned on an optical bench.

Measurements, using the He-Ne laser as a light source on the variation of the launching efficiency with axial displacement and transverse offsets were explained in (3). Therefore the measurements carried out here were devoted to LED as a light source (incoherent source), and the variation of the launching efficiency with separation, lateral displacement and misalignment was measured. This was carried out using three different lengths of the same fibre. The lengths used were 1 meter, 9 meters and 540 meters. Variation of the launching efficiency with the variation of the light source area using a variable slit could not be verified for the obvious reason that reducing the light source area is decided by the diffraction limit (5) and reducing it further will give inaccurate results. Microscopic objectives were used to collect and focus the light onto fibre. The emitting surface area of the LED was made equal to that of the fibre core diameter. Index matching liquid was used to minimize reflection losses.
Fig. (6.3) Measured variation of the launching efficiency with separation for different optical fibre lengths.
Fig (6.4) Comparison between the variation of calculated and measured launching efficiencies with separation
Fig. (6.5) Measured variation of the launching efficiency with displacement

\[ \frac{b}{a} = 1 \]
\[ \frac{s}{a} = 1 \]
\[ a = 2 \]
Fig (6.6) Comparison between the variation of calculated and measured launching efficiencies with displacement.
Fig (6.7) Measured variation of the launching efficiency with misalignment
Fig. (6.8) Comparison between the variation of the calculated and measured launching efficiencies with misalignment.
6.3 Discussion

Throughout the measurements, care was taken to keep the fibre ends clean and free from dust. 'Inhibisol' solvent and 'Omit plus' sprays were used for this purpose.

Due to diffraction limitations, verification of the variation of the launching efficiency with the variation of the light source surface area, was not attempted.

The slight discrepancy between the theoretical and practical results could either be due to the exact value of the refractive index profile exponent \( a \) not being equal to 2, as stated by the manufacturer, or to the presence of the central dip in the refractive index profile which was not taken into account in the theoretical analysis.
CHAPTER 7

CONCLUSIONS

The major object of this report is to study the effects of various fibre parameters on the source-fibre coupling using the familiar approach whose theoretical basis has been established, namely geometrical ray theory.

Coupling theory, which plays an important role in the study of optical fibres, was presented in Chapter 2. The investigation covered guided rays only and the contribution of leaky rays was taken into consideration later.

Simple analytical expressions were obtained for both $\alpha$-power law profiles and inverse $\alpha$-power low profiles.

It can be seen that for LED-Fibre coupling, when the source area is larger than the fibre core area, direct coupling represents the best coupling condition, and the use of self focussing lenses or micro lenses suggested by many authors appears to have no effect on improving the coupling efficiency. The use of such devices may improve the angular properties of the emitted light so as to decrease the broadening of the temporal pulse along the fibre. When the core area is larger than the source area, the coupling efficiency can be improved up to its maximum value when a lens is placed between the course and the fibre. This improvement will result from collimation
as the source is magnified to fill the fibre core cross-section.

It was found that the launching efficiencies for both \( \alpha \)-power law profiles and inverse \( \alpha \)-power law profiles, approached that of the step index fibre as a limiting case. Unlike \( \alpha \)-power law profile fibres, inverse \( \alpha \)-profile fibres offer minimum launching efficiency at the core centre but this increases gradually towards the core cladding interface. This is due to their acceptance angle and their numerical aperture which is a function of the dip height at the core centre.

In multimode optical fibres, the presence of small ellipticity in the fibre core, was found to improve the launching efficiency of parabolic index (\( \alpha = 2 \)) fibres. This improvement is significant only in parabolic index fibres, and in these fibres an optimum value of ellipticity can be found that improves the launching efficiency without affecting the fibre bandwidth. This is of practical importance since the values of \( \alpha \) in the region \( \alpha = 2 \) are the values of main interest as the optimum profile for minimum pulse dispersion is close to parabolic.

The high temperature collapse of fibres produced by the (CVD) method yields a refractive index dip on the fibre axis. The effect of this dip on the launching efficiency of power law optical fibres has been investigated and analytical expressions, relating the launching efficiency to the dip height have been obtained. It was found that
the dip height affects the launching efficiency adversely and this effect can be minimized by making the refractive index variation in the dip region an \( \alpha \)-power law profile.

Regarding birefringence in graded index, single mode, elliptical fibres, the expression obtained for calculating the phase difference shows that the higher the dip height \( \delta \) the lower the maximum value of the birefringence. Making the refractive index profile in the dip region an \( \alpha \)-profile does not appear to have much effect on birefringence compared with its effect on the launching efficiency as explained in Chapter 3, i.e., the shape of the dip profile is not important but the depth \( \delta \) of the dip is the governing factor. Increasing the dip height increases the position of maximum birefringence and reduces its magnitude.

Previous investigations reported in the literature have shown that any deviation of the refractive index of an optical fibre from its optimum shape dramatically reduces the fibre bandwidth. A deviation from the optimum index profile is caused by the depression in the index distribution on the fibre axis due to the evaporation of the dopant material from the innermost layers during the collapsing process. Linear variation of the refractive index distribution in the dip region was found to affect pulse dispersion, and the deeper the dip, the higher the dispersion.
Suggestions for further studies

1. The presence of small ellipticity in the fibre core was found to improve the launching efficiency of multimode parabolic index optical fibres. An optimum value of ellipticity could be found that improves the launching efficiency without affecting the bandwidth of the fibre. This is of practical importance since the optimum profile for minimum pulse dispersion is near parabolic.

2. The improvement of the launching efficiency in non-circular parabolic index optical fibres, leads to the conclusion that the ratio of leaky to bound rays is reduced. This in turn means that there may be a certain value of ellipticity for which no leaky modes can propagate in the fibres, so that for the near field scanning profile measurement a reduced correction factor may be needed for elliptical fibres with $\alpha = 2$. Experimental evidence indicates that the full circular correction factor is required, and the resulting inconsistency could be associated with the presence of the dip at the core centre. The effect of which has not been taken into consideration.

3. It has been shown that a dip at the core centre affects pulse dispersion and the higher the dip $\delta$, the longer the pulse trail.
In other words, the deeper the dip the higher the dispersion. This has been investigated assuming that the refractive index profile in the dip region is linear.

An investigation could be made to determine how an inverse $\alpha$-profile refractive index in the dip region affects pulse dispersion.
APPENDIX A

The power emitted by the source is

\[ P = \int_{r_0}^{b} r' dr' \int_{\psi_0}^{2\pi} d\psi' \int_{\phi_0}^{2\pi} d\phi' \int_{0}^{\pi/2} R'(r', \psi', \theta', \phi') \sin \theta' \cos \theta' d\theta' \]  

(1A)

and the power accepted and guided by the fibre is

\[ P_o = \int_{r_o}^{a'} r_o dr_o \int_{\psi_o}^{2\pi} d\psi_o \int_{\phi_o}^{2\pi} d\phi_o \int_{0}^{\theta'_m} R(r_o, \psi_o, \theta_o, \phi_o) T(r_o, \theta'_o) \sin \theta'_o \cos \theta'_o d\theta'_o \]

(2A)

\[ \cdot n = \int_{r_o}^{b} r' dr' \int_{\psi_o}^{2\pi} d\psi' \int_{\phi_o}^{2\pi} d\phi' \int_{0}^{\pi/2} R'(r', \psi', \theta', \phi') \sin \theta' \cos \theta' d\theta' \]

where \( \theta'_m \) is given by

\[ n' \sin \theta'_m = n \quad \text{when considering leaky rays} \]

and

\[ n' \sin \theta'_m = n \quad \text{when considering bound rays only} \]

When the source and the fibre are very close and concentric

\[ r' = r_o, \quad \psi' = \psi_o, \quad \phi' = \phi_o, \quad \theta' = \theta'_o \]
\[ n = \frac{\int_0^{\theta'} \int_0^{2\pi} \int_0^{2\pi} R(r',\psi',\theta',\phi') \times \mathcal{T}(r',\theta') \sin\theta' \cos\theta' \, d\theta' \, d\phi' \, d\psi'}{\int_0^{\beta} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} R(r',\psi',\theta',\phi') \sin\theta' \cos\theta' \, d\theta' \, d\phi' \, d\psi'} \]  

(4A)

assuming an LED, in which

\[ R'(r',\psi',\theta',\phi') = R_0 \]

\[ n = \frac{\int_0^{a'} \int_0^{\theta'} \int_0^{2\pi} R_0 \times \mathcal{T}(r',\theta') \sin \theta' \cos\theta' \, d\theta' \, d\phi' \, d\psi'}{\int_0^{b} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} R_0 \sin\theta' \cos\theta' \, d\theta' \, d\phi' \, d\psi'} \]

assuming \( T(r_0,\theta_0) = T_0 \)

\[ n = \frac{2T_0}{b^2} \int_0^{a'} \left[ \sin^2 \theta_0 \right] r_o \, dr_o \]  

(5A)

When considering all rays (bound + skew)

\[ \sin \theta'_m = \frac{1}{n'} \left[ \frac{n^2(r_0) - n_2^2}{1 - \left( \frac{r_o}{a} \right) \cos^2 \phi} \right]^{\frac{1}{2}} \]  

(6A)

and for bound rays only

\[ \sin \theta'_m = \frac{1}{n} \left[ n^2(r_0) - n_2^2 \right]^{\frac{1}{2}} \]  

(7A)

\[ n = \frac{2T_0}{b^2 n'2} \int_0^{a'} \left[ \frac{n^2(r_0) - n_2^2}{1 - \left( \frac{r_0}{a} \right)^2 \cos^2 \phi} \right] r_o \, dr_o \]  

for all rays

(8A)
\[ n' = \frac{2 \tau_o}{b^2 n^2} \int_0^a \left[ n^2(r_o) - n_2^2 \right] r_o \, dr_o \quad \text{for bound rays} \quad (9A) \]
APPENDIX B

It was shown in equation (2.22) that the launching efficiency is given by:

\[ n' = \frac{2 T_o}{b^2} \int_0^{a'} \left[ 1 - \left( 1 - \frac{n^2(r) - n_2^2}{n_1^2} \right) \frac{\xi + 2}{2} \right] r_0 \, dr_0 \]

for \( \xi = 0 \), we have a lambertion source

\[ n' = \frac{2 T_o}{b^2} \int_0^{a'} \left[ \frac{n^2(r) - n_2^2}{n_1^2} \right] r_0 \, dr_0 \quad (1B) \]

i) for \( b > a \)

Taking the refractive index variation as

\[ n^2(r) = n_o^2 \left[ 1 - 2\Delta \left( \frac{r}{a} \right)^\alpha \right] \]

Substituting this into equation (1B)

\[ n' = \frac{2 T_o}{b^2} \int_0^{a'} \left\{ n_o^2 - 2\Delta n_o^2 \left( \frac{r}{a} \right)^\alpha - n_2^2 \right\} r_0 \, dr_0 \quad (2B) \]

\[ = \frac{2 T_o}{b^2} \int_0^{a'} n_o^2 \left[ \left( \frac{n_o^2 - n_2^2}{n_o^2} \right) - 2\Delta \left( \frac{r}{a} \right)^\alpha \right] \, r_0 \, dr_0 \quad (3B) \]

\[ = \frac{2 T_o n_o^2}{b^2} \int_0^{a'} \left[ 2\Delta - 2\Delta \left( \frac{r}{a} \right)^\alpha \right] \, r_0 \, dr_0 \quad (4B) \]

after some manipulation this gives:

\[ n' = 4 T_o n_o^2 \Delta \frac{a^2}{b^2} \left[ \frac{1}{2} - \frac{1}{\alpha + 2} \right] \quad (5B) \]
ii) $b < a$

In this region the acceptance of the fibre is given by (9):

$$\sin \theta_m = \frac{a}{b} (n^2(r) - n_2^2)^{\frac{1}{2}}$$

$$\therefore \eta' = \frac{2 T_0}{a^2} \int_0^b \left[ \frac{a^2}{b^2} (n^2(r) - n_2^2) \right] r_o \, dr_o$$

(6B)

assuming the same refractive index variation as before

$$\eta' = \frac{2 T_0}{b^2} \int_0^b \frac{n_o^2}{n_0^2} \left[ \left( \frac{n_o^2}{n_2^2} - 2 \Delta (\xi_a^2) \right) \right] \, rdr$$

(7B)

$$= \frac{2 T_0 n_0^2}{b^2} \int_0^b \left[ 2 \Delta - 2 \Delta (\xi_a^2) \right] \, rdr$$

(8B)

$$= \frac{4 T_0 n_0^2}{b^2} \left[ \frac{b^2}{2} - \frac{1}{\alpha + 2} \left( \frac{b}{a} \right)^{\alpha} \right]$$

(9B)

$$= 4 T_0 n_o^2 \Delta \left[ \frac{1}{2} - \frac{1}{\alpha + 2} \left( \frac{b}{a} \right)^{\alpha} \right]$$

(10B)
APPENDIX C

The scalar wave equation in elliptical co-ordinates can be represented in the separable form given by equations (2.39) and (2.40) where

\[ G(\epsilon) = e^2 r^2 \left[ (n_o^2 - \beta^2) \sinh^2 \epsilon - g(\epsilon) \right] - (n_o^2 - n_2^2) r^2 \mu \]  \hspace{1cm} (1C)

For parabolic index elliptical fibre \((\alpha = 2)\), \(g(\epsilon)\) is given by equation (2.33) i.e.

\[ g(\epsilon) = (n_o^2 - n_2^2) e^2 \sinh^2 \epsilon \cosh^2 \epsilon \] \hspace{1cm} (2C)

Substituting (2C) in (1C) gives

\[ G(\epsilon) = e^2 r^2 \left[ (n_o^2 - \beta^2) \sinh^2 \epsilon - (n_o^2 - n_2^2) e^2 \sinh^2 \epsilon \cosh^2 \epsilon \right] - (n_o^2 - n_2^2) r^2 \mu \] \hspace{1cm} (3C)

\[ = e^2 r^2 \left[ (n_o^2 - \beta^2) \sinh^2 \epsilon - (n_o^2 - n_2^2) e^2 \sinh^2 \epsilon (1 + \sinh^2 \epsilon) \right] - (n_o^2 - n_2^2) r^2 \mu \] \hspace{1cm} (4C)

\[ = e^2 r^2 (n_o^2 - \beta^2) \sinh^2 \epsilon - e^4 r^2 (n_o^2 - n_2^2) \sinh^2 \epsilon - e^4 r^2 (n_o^2 - n_2^2) \sinh^4 \epsilon \] \hspace{1cm} (5C)

An E-type mode in elliptical fibres is confined to the region (23)

\[ \epsilon_{\text{min}} < \epsilon < \epsilon_{\text{tp}} \]

and

\[ G(\epsilon) = 0 \text{ at } \epsilon = \epsilon_{\text{tp}} \text{ and } \epsilon = \epsilon_{\text{min}} \]

\[ e^4 r^2 \gamma^2 \sinh^4 \epsilon_{\text{tp}} + \left[ e^4 r^2 \gamma^2 - e^2 r^2 (n_o^2 - \beta^2) \right] \sinh^2 \epsilon_{\text{tp}} + \gamma^2 r^2 \mu = 0 \] \hspace{1cm} (6C)

where \( \gamma^2 = (n_o^2 - n_2^2) \)
\[
\sinh^4 \varepsilon_{tp} + \left[1 - \frac{1}{\gamma^2 e^2} (\beta^2 - \beta_o^2)\right] \sinh^2 \varepsilon_{tp} + \frac{\mu}{e^4} = 0
\] (7C)

By completing the square, factorizing and re-arranging:

\[
2 \sinh^2 \varepsilon_{tp} + \left[1 - \frac{1}{\gamma^2 e^2} (\beta^2 - \beta_o^2)\right] = \left[\frac{1}{e^2 \gamma^2} (\beta_o^2 - \beta^2) - 1\right]^2 - \frac{4\mu}{e^4}
\] (8C)

\[
\therefore 2 \sinh^2 \varepsilon_{tp} = \frac{1}{e^2 \gamma^2} (\beta_o^2 - \beta^2) - 1 + \left[\frac{1}{e^2 \gamma^2} (\beta_o^2 - \beta^2) - 1\right]^2 - \frac{4\mu}{e^4}
\] (9C)

Similarly:

\[
2 \sinh^2 \varepsilon_{\text{min}} = \frac{1}{e^2 \gamma^2} (\beta_o^2 - \beta^2) - 1 - \left[\frac{1}{e^2 \gamma^2} (\beta_o^2 - \beta^2) - 1\right]^2 - \frac{4\mu}{e^4}
\] (10C)
APPENDIX D

1. Field Equations:

The well known Maxwell's equations are

\[ \nabla \times E = -j\omega \mu_0 H \]  \hspace{1cm} (1D)  
\[ \nabla \times H = j\varepsilon\varepsilon_0 E \]  \hspace{1cm} (2D)

\( E \) and \( H \) are the electric and magnetic field vectors respectively.

\[ \omega = \text{frequency of radiation} \]
\[ \varepsilon_0 (\mu_0) = \text{permittivity (permeability) of vacuum} \]
\[ \varepsilon = \text{permittivity of propagation medium} \]

Assuming the dependence \( e^{j(\omega t - m\phi - \beta z)} \) where \( m \) is an integer and represents the azimuthal mode number.

\( \beta \) is the propagation constant and \( t \) represents the time variation.

Using cylindrical co-ordinates system where \( r, \phi, z \), denote the radial, angular, and axial co-ordinates and through the use of the curl operator, Maxwell's equations become

\[ j\omega \varepsilon_0 \varepsilon E_z = \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) + \frac{jm}{r} H_r \]  \hspace{1cm} (3D)
\[ j\omega \varepsilon_0 \varepsilon E = -j \beta H_r - \frac{\partial}{\partial r} H_z \]  \hspace{1cm} (4D)
\[ j\omega \varepsilon_0 \varepsilon E_r = j\beta H_\phi - \frac{jm}{r} H_z \]  \hspace{1cm} (5D)
By subsequent substitutions, it is possible to obtain a coupled second order differential equation for $E_z$ and $H_z$.

$$E''_z + \left[ \frac{1}{r} - \frac{\epsilon' \beta^2}{\epsilon (k_0^2 \epsilon - \beta^2)} \right] E'_z + \left( k_0^2 \epsilon - \beta^2 - \frac{m^2}{r^2} \right) E_z = - \frac{j \omega \mu \beta m \epsilon'}{\epsilon r (k_0^2 \epsilon - \beta^2)} H_z \quad (9D)$$

and for $H_z$

$$H''_z + \left[ \frac{1}{r} - \frac{\epsilon' k_0^2}{\epsilon (k_0^2 \epsilon - \beta^2)} \right] H'_z + \left( k_0^2 \epsilon - \beta^2 - \frac{m^2}{r^2} \right) H_z = j \frac{\omega \epsilon_0 \beta m \epsilon'}{r (k_0^2 \epsilon - \beta^2)} E_z \quad (10D)$$

where the prime denotes differentiation with respect to $r$.

The coupling term on the right hand side of equations (9D) and (10D) vanishes if $m = 0$ or if the permittivity is constant.
For $n = 0$, there is no angular variation and both TE and TM modes exist since $E_z = 0$ does not imply that $H_z = 0$. It is possible to eliminate one of the axial field components $E_z$ or $H_z$ to obtain a linear homogeneous fourth order differential equation which is cumbersome to handle. An approximate solution could be obtained to equations (9D) and (10D) for modes whose fields are largely confined to the region of maximum permittivity at the core centre. This approximate solution is described in Reference (5) of Chapter 2.
2. **Bessel Functions**

The differential equation

\[ z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 + \nu^2) = 0 \]  

(11D)

will accept as solutions the linearly independent combination of \( J_\nu(z) \) and \( Y_\nu(z) \)

while the equation

\[ z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2) = 0 \]  

(12D)

have as solutions combinations of \( I_\nu(z) \) and \( K_\nu(z) \)

where

\( J \) = Bessel function (1st kind)

\( Y \) = Bessel function (2nd kind)

\( I \) = modified Bessel function (1st kind) (modified Hankel function)

\( K \) = modified Bessel function (2nd kind) (modified Hankel function)

In the region where \( z \) is real and \( \nu \) is an integer, \( J \) and \( Y \) exhibit
an oscillatory behaviour.

For a fixed order (\( \nu \)), \( K \) decreases with increasing argument while
\( I \) increases with increasing argument.
These functions have a series representation of the form

\[ J_0(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^k}{k! \Gamma(\nu + k + 1)} \]  

\[ Y_\nu(z) = \frac{2}{\pi} \left[ \ln \left(\frac{z}{2}\right) - \psi(m + 1)\right] J_\nu(z) - \frac{m! \left(\frac{z}{2}\right)^{-m}}{\pi} \sum_{k=0}^{m-1} \frac{\left(\frac{z}{2}\right)^k \psi_k(z)}{(m-k)k!} \]

\[ - \left(\frac{z}{2}\right) \sum_{k=1}^{\infty} \frac{(-1)^k (m + 2k)}{k(m + k)} J_m + 2k(z) \]  

(13D)  

(14D)

The subscript \(m\) indicates that the validity of the series is limited to integer orders. \(\psi\) is known as the logarithmic derivative of the Gamma function

\[ \psi(m) = \frac{d}{dz} \left[ \ln \Gamma(m) \right] = \frac{\Gamma'(m)}{\Gamma(m)} \]  

(15D)

\[ \psi(m) = -\text{Eu} + \sum_{k=1}^{m-1} k^{-1} \]  

(16D)

where \(\text{Eu}\) is Euler's constant.

Similarly, \(I\) and \(K\) are represented by the following series:

\[ I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z}{4}\right)^k}{k! \Gamma(\nu + k + 1)} \]  

(17D)
\[
K_m(z) = \frac{1}{2} \left( \frac{z}{2} \right)^{-m} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left( -\frac{z^2}{4} \right)^k
\]

\[
+ (-1)^m + 1 \ \text{Im} \left( \frac{z}{2} \right) I_m(z)
\]

\[
+ (-1)^m \frac{1}{2} \left( \frac{z}{2} \right) \sum_{k=0}^{\infty} \left\{ \psi(k+1) + \psi(m+kH) \right\} \frac{(\frac{z^2}{4})^k}{k!(m+k)!}
\]

(18D)

where \( \psi \) is as defined by equation (15D).
3. **Gamma Function**

This is given by

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad \text{for } \Re(z) > 0 \quad (19D) \]

which is known as Euler's Integral.

It can also be expressed as (2)

\[ \Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{z+k} + \int_0^1 e^{-1/t} t^{-z-1} \, dt \quad (20D) \]

The representation given by (20D) allows for negative values of the argument as well as positive.

4. **Beta Function**

is defined as:

\[ B(z,\omega) = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+\omega}} \, dt \quad (21D) \]

\[ = \int_0^1 t^{z-1} (1-t)^{\omega-1} \, dt \]

and can be expressed in terms of Gamma function as

\[ B(z,\omega) = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)} \quad (22D) \]
5. **Incomplete Beta Function**

\[ B_x(a,b) = \int_0^x t^{a-1} (1-t)^{b-1} \, dt \]  

\[ I_x(a,b) = B_x(a,b) / B(a,b) \]  

or

\[ I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1} (1-t)^{b-1} \, dt \quad 0 \leq X \leq 1 \]  

where \( B(a,b) \) is the Beta function

and in a series form, it can be represented as

\[ I_x(a,b) = \frac{x^a (1-x)^b}{a B(a,b)} \left\{ 1 + \sum_{n=0}^{\infty} \frac{B(a+1, n+1)}{B(a+b, n+1)} \times n+1 \right\} \]
REFERENCES

CHAPTER 1


CHAPTER 3


CHAPTER 4


CHAPTER 5


CHAPTER 6


APPENDIX D


