Digital spectral analysis using unconventional sampling methods

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DIGITAL SPECTRAL ANALYSIS USING UNCONVENTIONAL SAMPLING METHODS

By

ROY JULIAN ORME

Submitted in partial fulfilment of the requirements for the award of
Doctor of Philosophy of the Loughborough University of Technology
February 1974

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Department of Electronic and Electrical Engineering

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The work presented in this thesis covers the design of various sampling/processing schemes which allow the capabilities of a digital spectral analysis system to be extended to cope with specific spectral analysis applications where sampling rates are restricted. It is shown how the sampling/processing methods can be tailored to suit various types of signals so as to minimise the sampling rate.

Uniform sampling of multiple narrowband signals is the first specialised sampling technique to be discussed. Here sampling rate reductions are devised by employing spectral band interlacing which is similar in principle to the classical case of sampling a single narrowband signal but on a much more complicated level. Periodic staggered sampling is also considered with application to the sampling of multiple bandlimited signals. It is shown that such a sampling technique can be constructed with a mean sampling rate equal to the occupied frequency space of the signal, yet be insensitive to the signals frequency distribution of energy from the point of view of reconstruction ambiguity.

To cover the spectral analysis of random signals a general resumé is given to both statistical and random sampling techniques in the estimation of the power spectral density of stationary random waveforms. Statistical sampling is put forward as an unconventional way of digital spectral analysis in an attempt to break the "engineering" approach of uniform time sampling. Random sampling is suggested as a way of reducing the sampling rate for spectral analysis of stochastic waveforms but problems with spectral noise levels were encountered. A new approach to autocorrelation estimation is then given in an attempt to combat the deficiencies of random sampling but to maintain
sampling rates as low as possible. The sampling/processing techniques involved with this new method are again based on a staggered periodic sampling waveform but of a specially designed staggered pattern.

The thesis finishes with a short description of the practical problem of measuring turbine blade vibrations using blade-tip displacement detection. Suggestions are given as to the use of two of the data processing methods which are based on periodic staggered sampling. It is shown possible to use these two processing techniques in parallel while operating on a common source of data samples and giving separate output spectral analyses on different constituent vibrations present on the turbine blading.
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<td>Matrix notation (capital letter)</td>
</tr>
<tr>
<td>$a$</td>
<td>Vector notation (small letter)</td>
</tr>
<tr>
<td>{•}</td>
<td>Number sequence notation</td>
</tr>
<tr>
<td>$A^{-1}$</td>
<td>Inverse matrix</td>
</tr>
<tr>
<td>$A^T$</td>
<td>Transpose matrix</td>
</tr>
<tr>
<td>$A^*$</td>
<td>Transpose conjugate matrix</td>
</tr>
<tr>
<td>$A^*$</td>
<td>Conjugate matrix</td>
</tr>
<tr>
<td>*</td>
<td>Convolution (aperiodic)</td>
</tr>
<tr>
<td>⨀</td>
<td>Circular convolution</td>
</tr>
<tr>
<td>{{•}}</td>
<td>Integer part of function</td>
</tr>
<tr>
<td>D.F.T.</td>
<td>Digital Fourier Transform</td>
</tr>
<tr>
<td>$e(t)$</td>
<td>Sampling impulse time waveform</td>
</tr>
<tr>
<td>${e}$</td>
<td>Sampling sequence (binary)</td>
</tr>
<tr>
<td>$E_p{{\cdot}}$</td>
<td>Statistical expectation</td>
</tr>
<tr>
<td>$f$</td>
<td>Frequency in Hertz (Hz)</td>
</tr>
<tr>
<td>$f_s$</td>
<td>Mean sampling frequency</td>
</tr>
<tr>
<td>$f_{\text{min}}$</td>
<td>Minimum sampling frequency</td>
</tr>
<tr>
<td>F.F.T.</td>
<td>Fast Fourier Transform (computational algorithm)</td>
</tr>
<tr>
<td>$F_{\text{max}}$</td>
<td>Maximum frequency content of $s(t)$</td>
</tr>
<tr>
<td>$F_{\text{occ}}$</td>
<td>Spectral space occupied by $s(t)$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>Time impulse function</td>
</tr>
<tr>
<td>$\delta_{\text{nm}}$</td>
<td>Kronecker delta function</td>
</tr>
<tr>
<td>$\Delta g$</td>
<td>Time interval spacing of the uniform time grid</td>
</tr>
<tr>
<td>$\Delta n$</td>
<td>Time interval spacing when sampling at the Nyquist rate</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>Time interval spacing of a uniform sampling process</td>
</tr>
<tr>
<td>${\gamma}$</td>
<td>Autocorrelation of a number sequence</td>
</tr>
<tr>
<td>I.D.F.T.</td>
<td>Inverse D.F.T.</td>
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<td>$N$</td>
<td>Number of uniformly spaced sampling points in a time interval of $T$ seconds</td>
</tr>
<tr>
<td>$\phi(f)$</td>
<td>Power spectral density function</td>
</tr>
<tr>
<td>$p(t)$</td>
<td>Probability density function</td>
</tr>
<tr>
<td>$\text{Prob}[\cdot]$</td>
<td>Probability designation</td>
</tr>
<tr>
<td>p.s.d.</td>
<td>Power spectral density</td>
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<tr>
<td>$P(f)$</td>
<td>Power spectral density estimate</td>
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<tr>
<td>$Q(f)$</td>
<td>Power spectrum estimate</td>
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<td>$\rho$</td>
<td>Mean sampling density</td>
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<td>$R(t)$</td>
<td>Autocorrelation function estimate</td>
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<tr>
<td>${s}$</td>
<td>Sequence of sampled values from $s(t)$</td>
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<tr>
<td>$s(t)$</td>
<td>Continuous time waveform</td>
</tr>
<tr>
<td>$s_s(t)$</td>
<td>Subscript &quot;s&quot; denotes sampled waveform</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Delay lag in autocorrelation functions (seconds)</td>
</tr>
<tr>
<td>$t$</td>
<td>Time (seconds)</td>
</tr>
<tr>
<td>$T$</td>
<td>Fast Fourier Transform matrix</td>
</tr>
<tr>
<td>$T_r$</td>
<td>Time length of waveform $s(t)$ available for analysis</td>
</tr>
<tr>
<td>$V(f)$</td>
<td>Complex amplitude spectrum</td>
</tr>
<tr>
<td>$U(f)$</td>
<td>Power spectrum</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$2\pi f$</td>
</tr>
<tr>
<td>$\Omega(t)$</td>
<td>Autocorrelation function</td>
</tr>
<tr>
<td>$X(f)$</td>
<td>Complex spectral density estimate</td>
</tr>
<tr>
<td>$Y(f)$</td>
<td>Complex amplitude spectrum estimate</td>
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INTRODUCTION

Digital computer implementation of spectral analysis has now moved from being a laboratory technique and is quickly becoming another convenient tool in the practical world of engineering application. Application in either a remote way, in which analogue information is measured to be later digitally processed on a general purpose computer, or in a direct way with on site sampling and processing which is now possible with the use of the new generation of specialised Fourier Analyser systems based on a small digital computer. Most of the digital spectral analysis computer systems are equipped with analogue to digital converter facilities and the Fourier software to complement the use of uniform sampling techniques. This type of computing system is adequate for analogue signals with lowpass frequency characteristics, which are within the analogue to digital converters capabilities, and with signals which are available for sampling at the required uniformly spaced time instants. It is often the case in engineering problems that these requirements cannot be fulfilled in part or in whole when measuring information in a practical situation. Two typical restrictions often encountered are, firstly, high frequency signals which are beyond the capabilities of the analogue to digital converter. Secondly, that it is physically impossible to sample the information at the required frequency or in a manner compatible with the normal Fourier software.

In this thesis various sampling/processing techniques are presented which have been designed to combat physical restrictions, imposed by practical engineering difficulties, on the digital spectral analysis of various types of signal. In all the techniques discussed, advantage has been taken of the known characteristics of the signals so as to allow flexibility in sampling design. All the processes have been constructed with a view to computer (1)
processing of the sampled time data through the normally available Fourier software which accepts uniformly spaced samples. In this way the capabilities of a normal spectral analysis system have been extended beyond the bounds of lowpass uniformly sampled data. Since these sampling techniques are assumed integral with a computer system, it has been assumed that post-sampling digital processing of the data is easily accomplished. Consequently the sampling methods evolved are backed up by post-sampling processing techniques ranging from simple filtering to complicated harmonic signal reconstruction involving matrix manipulation which necessitates large computing capabilities.

The particular engineering objective which initiated the work presented in this thesis was the need to use digital techniques to measure the frequency spectrum of vibrating turbine blading. This was to be done by measuring the displacement of the turbine blade tips with stationary proximity detection probes placed around the bladed disc's periphery. The physical constraints of the problem made it both necessary to sample the vibrational motion of the blading at a frequency below twice the highest frequency of the vibrational motion, while contending with the restriction of periodicity in the placing of sampling positions in time. Therefore some of the sampling techniques, particularly in chapters three and four, reflect the preoccupation with designing sampling techniques which have periodic characteristics while trying to maintain a minimum number of sampling points.

The aim of the thesis is to show through the use of these various sampling techniques that digital spectral analysis is a very versatile process in its own right rather than being a reflection in the discrete domain of the continuous spectral analysis process. To this end it is hoped to show that the sampling process is an integral part of the total digital spectrum analysis technique requiring sampling pattern design and the use of these properties.
ORGANISATION OF THE WORK

The thesis is based on four major chapters, each presenting a different aspect of sampling in digital spectral analysis processing. Chapter one is based on the well known uniform sampling procedure for narrowband signals, sometimes called first-order sampling, this technique is developed to cater for a generalised form of signal with a multiple narrowband spectral format. The concept behind the use of this sampling method is in its simplicity of application and post-sampling processing techniques and hence forms an ideal introduction to the thesis.

Chapter two presents a review of statistical and random sampling spectral processing methods applied to stationary random waveforms, some of which are fairly well appreciated but others which are possibly new to the engineering application of digital spectral processing. In chapter three a completely new approach to sampling methods is presented for the estimation of the power spectral density function of stationary random stochastic waveforms. The design concept of the sampling process is aimed at the estimation of the autocorrelation function of the stochastic waveform with the lowest most efficient time sampling rate possible.

Chapter four tackles the problem of determining an Ith order periodic staggered sampling process that will allow any signal with an arbitrary multiple bandlimited spectrum to be sampled unambiguously at a mean rate equal to the total occupied frequency space of the signal's spectrum. This interpolation/sampling problem is analysed in a totally discrete manner with the aid of matrix/vector representation of digital spectral analysis processing.
The last chapter, five, stands apart from the other chapters in that it attempts to apply two of the sampling methods described in chapters three and four to a practical problem. The problem is one of measuring the vibrational motion of turbine blading by the detection of the displacement of the blade tips at certain instants. Application wise, chapter five is by no means complete but the object is to lay the foundations for practical implementation of these sampling methods.
1.1 INTRODUCTION

When analogue signals are analysed digitally, an initial stage in the analysis is the conversion of the analogue waveform into a sequence of numerical values. In most applications, the sampling frequency is chosen to exceed twice the highest frequency at which the signal contains significant power. The limited maximum sampling rate of the digitisation equipment thus normally sets a limit to the bandwidth of signals which can be analysed directly without intermediate processing. However, if the signal which is to be analysed has a spectrum which is effectively zero at all frequencies except at a number of isolated frequency bands, Figure 1.1, in principle it is no longer necessary to sample at above twice the highest frequency present. It is, in fact, possible to specify such a signal by a sequence of numbers generated at an average rate not less than twice the occupied bandwidth of the signal. Highly resonant mechanical systems when randomly excited will produce vibrations which have their spectral power isolated in discrete frequency bands. In such cases the spectrum of the vibration signal may have narrow bands of high energy corresponding to the resonances of the system; apart from these bands the spectrum is otherwise practically zero.

As stated earlier it is always possible, in principle, to specify a signal by a sequence of numbers generated at twice the occupied bandwidth but in many cases the equipment and analysis procedures needed to make use of this fact would be too complicated for practical use. This chapter shows how it is possible, in some circumstances, to use simple uniform sampling and straightforward analysis procedures, yet to obtain an unambiguous analysis of the spectrum. While not achieving the ideal of a sampling frequency of
SPECTRUM OF A SIGNAL WITH ISOLATED SPECTRAL POWER BANDS

FIG. 1.1

SPECTRUM OF A NARROWBAND SIGNAL

FIG. 1.2

(Figures 1.1 and 1.2 depict the power distribution for signals with isolated spectral bands and a narrowband signal, respectively.)
twice the occupied bandwidth, a sampling rate can be achieved which is significantly below twice the highest frequency present in the signal.

The method used makes use of the available information on the approximate location of the power in the spectrum in choosing a suitable sampling rate. After the signal is sampled at this rate, the ambiguity or aliasing produced by the low sampling rate is removed by a simple filtering process.

1.2 INTRODUCTION TO THE SAMPLING OF NARROWBAND SIGNALS

This section describes the theory covering the application of uniform sampling to single narrowband signals, which will serve as an introduction to the later sections where the sampling of signals with more complicated spectral structures will be discussed. The theory for this introduction is given in references (1) to (3).

The Nyquist sampling theory although usually interpreted as applying to bandlimited signals is really specified in relation to the total occupied bandwidth of a signal. As an example consider a narrowband signal, \( s_2(t) \), with a power spectrum as shown in Figure 1.2. The signal is assumed real therefore the spectrum is symmetrical about the zero frequency component but the important characteristic is that the spectral power is distributed into two isolated bands of frequencies. The maximum frequency of \( s_2(t) \) is denoted by \( F_{\text{max}} \) but the total occupied frequency space is denoted by \( F_{\text{occ}} \):

\[
F_{\text{occ}} = 2B \quad \text{Hz} \quad 1.1
\]

(Total occupied bandwidth includes both +ve and -ve frequencies)

Where \( F_{\text{occ}} \) is much smaller than \( F_{\text{max}} \). The Nyquist theory states that if a signal such as \( s_2(t) \) is sampled at a mean rate just above \( F_{\text{occ}} \), then the original continuous signal \( s_2(t) \) should be reconstructable from knowledge of (7)
the sampled values, sampling instants and the occupied spectral space. The occupied spectral space in the case of a bandlimited signal would be described by knowing the spectral limit $F_{\text{max}}$ but in the case of a waveform with a narrowband spectral form it is necessary to know the boundaries of the spectral bands

\[ i.e. \quad f_o - \frac{B}{2}, \quad -f_o + \frac{B}{2}, \quad \text{Figure 1.1} \]

So according to the Nyquist theory, provided that all the information mentioned above is available, then $s_2(t)$ can be sampled at well below $2F_{\text{max}}$ and yet still allow unambiguous reconstruction of $s_2(t)$ from the sampled values.

The theoretical minimum uniform sampling rate for any time waveform is $F_{\text{occ}}$, this is optimum but it is of course advantageous to be able to sample at any frequency below $2F_{\text{max}}$. As with all spectrums of uniformly sampled waveforms, aliased versions of the original occupied spectral bands will be placed at uniform intervals across the spectrum plane. In the normal situation where sampling is taking place at a rate above $2F_{\text{max}}$ there is no chance of any of the aliased spectral bands actually coinciding in frequency with one of the original occupied spectral bands. In other words there can be no overlap of the original and aliased spectral bands. In a situation where sampling is taking place at below $2F_{\text{max}}$ there is a likelihood that spectral overlap of the original and aliased spectral bands will occur but to be able to separate the original spectral bands from the aliased versions by a simple linear filtering operation it is necessary to avoid this likelihood by selection of sampling frequency. This type of sampling is termed first-order periodic sampling and is referred to in references (2) and (5).
Spectrum of Sampled Waveform Derived from $s_n(t)$ with Nonoptimum Sampling Rate

Fig. 1.3

Spectrum of Sampled Waveform
Sampling Rate Adjusted for No Spectral Overlap

Fig. 1.4

--- Reconstruction Filter Response

(9)
To approach the optimum sampling rate, $F_{\text{occ}}$, and yet avoid spectral overlap the nature of the spectral aliasing must be controlled. Control is exercised by simply varying the frequency of sampling, keeping it as low as possible while placing the aliased frequency bands where they will not overlap the original spectral bands.

**Technique**

Consider a narrowband signal $s_2(t)$ with spectrum as in figure 1.2, it is sampled at a uniform rate of $f_s$ samples per second ($f_s < 2F_{\text{max}}$). The spectrum of the sampled signal is shown in Figure 1.3. Using this arbitrary sampling rate of $f_s$ Hz it is impossible to apply simple filtering to isolate the original spectrum because of overlap occurring between the original and aliases of the occupied spectrum. The simple solution is to adjust the sampling frequency, $f_s$, so that the aliases do not overlap the original spectrum components, as in Figure 1.4. Selection of the sampling rate, to avoid spectral overlap, is in this case a simple matter of formulating the interlacing effects of the aliases with respect to the sampling frequency. Equation 1.2 does this and expresses the minimum possible sampling rate, $f_{\text{min}}$, for a given bandwidth $B$ Hz and centre frequency $f_o$ Hz.

$$f_{\text{min}} = \frac{f_o}{B} + \frac{1}{2} \cdot 2B$$

(1.2)

The optimum sampling frequencies occur at values of $f_o$ equal to

$$B/2, \ 3B/2, \ 5B/2, \ . \ . \ . \ . \ . \ Hz$$

In these cases $f_{\text{min}} = F_{\text{occ}}$ for $s_2(t)$

" $= 2B$

Where $B$ is the bandwidth of a single narrowband signal.
At other values of $f_0$ the sampling rate derived by equation 1.2 is a minimum for the use of uniform sampling but is not optimum in the sense of the mean sampling rate being equal to $F_{occ}$.

Once the sampling rate has been chosen and the signal sampled, the recovery of the original time waveform or its complex spectrum from the discrete values is simply carried out by applying a linear filter. Figure 1.4 shows an example of a filter applied to the aliased complex spectrum of the sampled values so as to remove all but the original narrowband spectral components. The use of such filters totally depends on the knowledge of the spectrum format and the ability of the sampling rate chosen to avoid spectral overlap. The recovery technique is both theoretically easy to design and practically easy to implement.

The following sections go on to discuss the design of a uniform sampling process to allow signals with spectral structures which are much more complicated than $s_2(t)$ to be sampled at below $2F_{max}$. The work was initially directed at producing the equivalent equation to equation 1.2 for these more complicated signals, it was in fact found impossible to express $f_{min}$ in such a simple single equation. The result was the development of a general computer program to calculate $f_{min}$, the next sections go on to describe the philosophy and design of such a program.

1.3 SIGNALS WITH MULTIPLE NARROWBAND SPECTRAL FORMATS

In the last section the type of signals that were processed are classified as narrowband signals, the characteristic being that the bandwidth $B$ is much less than the frequency $f_0$, figure 1.2. The type of signal that is to be considered in this chapter has a spectrum which is more complicated than the...
Spectrum of Multiple Narrowband Signal $S_3(t)$  

**Figure 1.5**

Example of an Interlaced M.N.B Signal Spectrum  

**Figure 1.6**

Original Spectrum  

[Alias Versions of Spectrum]
single narrowband signal but has some common characteristics, figure 1.5. The spectrum instead of having one isolated occupied spectral band has a number of such bands distributed across the frequency plane from zero frequency to $\pm F_{\text{max}}$. The classification characteristic of this type of spectrum form is based on the ratio of the total occupied spectrum, $F_{\text{occ}}$, to the maximum frequency content, $F_{\text{max}}$. Waveforms with this type of isolated spectral band structure which has a large ratio of $F_{\text{max}}$ to $F_{\text{occ}}$ will be referred to as multiband narrowband signals (M.N.B.) in the rest of the chapter.

1.3.1 SPECTRAL INTERLACING

As with the narrowband signal it is intended that a uniform sampling process will be used to sample the M.N.B. signal. Although non-uniform sampling waveforms could undoubtedly be used to obtain an optimum sampling rate for each M.N.B. signal spectral formation, it is felt that this advantage is outweighed by the ease of computer manipulation and signal recovery associated with uniform sampling. So for most of the M.N.B. signal forms the minimum uniform sampling rate will not be optimum in the sense of occupied frequency bandwidth.

$$(\text{Optimum Sampling rate for a M.N.B. Signal}) f_{\text{opt}} = F_{\text{occ}}$$  \hspace{1cm} 1.3$$

Even so, the actual sampling rate, $f_s$, will be usually well below $2F_{\text{max}}$. The design of the sampling rate will be aimed at causing the sampled waveform's spectrum to exhibit interlacing of the original spectrum and its aliases without overlap, see figure 1.6. This is the same as the narrowband signal case but the complexity is much greater and therefore is only feasible in computer programming form.
1.3.2 DESCRIBING A SPECTRUM FORMAT BY A BINARY SEQUENCE

Calculation of the minimum uniform sampling rate for a M.N.B. signal \( s_3(t) \) will be carried out by a digital computer simulating, in a limited way, the interlacing process of the spectrum and its aliases. The input data to the program must be a digital representation of the spectral bands occupied within \( \pm F_{\text{max}} \). The simplest way to describe whether a particular frequency is occupied by a signal or not is to denote occupancy by 1 and vacancy by 0; this allows a binary representation of the spectrum format to be built up. Only real waveforms have been sampled in practice but the theory treats both the negative and positive frequency components separately, so the work to follow is applicable also to complex waveforms.

Consider a M.N.B. waveform \( s_3(t) \) with a positive half spectrum as shown in figure 1.7. The boundaries of the spectrum are denoted by \( f_0, f_1, f_2, \ldots, f_5 \). The simplest way to convert this information into an integer format is to divide the total spectrum plane into small cells each \( \Delta_c \) Hz wide as in figure 1.7. A sequence \( \{F\} \) of numbers can now be formed with each corresponding to a cell ranging from \(-F_{\text{max}}\) negative frequencies to \(+F_{\text{max}}\) positive frequencies. If any significant portion of the waveform's spectral power lies within a cell then the corresponding sequence element will be given a 1 value. These types of cells are denoted as signal cells. Cells in which no signal power lies are called null cells and consequently the corresponding sequence element is denoted by 0. As an example, the following part of a total sequence \( \{F\} \) would represent the encoding of the positive frequencies of figure 1.7.

\[ \ldots, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, \ldots \]

The size, \( \Delta_c \), of the cells used to encode the spectrum will determine the accuracy of the sequence description of the spectral band boundaries within
A TYPICAL M.N.B. SIGNAL: SPECTRUM FORMAT

Fig. 1.7

Fig. 1.8

WIDE CELL WIDTH

1, 0, 1, 1, 0 — [F] b)

COMPUTER SIMULATION OF SPECTRUM FORMAT

NARROW CELL WIDTH
d)

0, 1, 1, 1, 0, · · · 0, 1, 1, · · · 1, 0, · · · — [F] e)

COMPUTER SIMULATION OF SPECTRUM FORMAT

f)

Frequency Hz

(15)
Computer Simulation of Aliased Spectral Format

Fig. 1.9a

Detailed View of Fig. 1.9a

Fig. 1.9b

(16)
The computer program tries to interlace the input sequences in simulation of the spectrum interlacing at low sampling rates. If the proportion of 0's in the sequence is kept low by the $\Delta_c$ width being too wide then the spectral interlacing will be restricted, figure 1.8a. Reducing $\Delta_c$ in this case will increase the proportion of 0's which will lead to closer spectral interlacing and hence sampling at a lower sampling rate, figure 1.8d. On the other hand the narrower the cells the longer the total sequence will be in order to describe $-F_{\text{max}}$ to $F_{\text{max}}$ frequencies. Longer sequences mean that it will take longer to calculate the $f_{\text{min}}$ sampling frequency. The computer time can be approximated by:

$$\text{Computer time } \propto \sum_{k=1}^{K} (K - k)$$

where $K$ denotes the length of the binary spectrum sequence $\{F\}$. A compromise is sought and possibly the best, which ensures that errors do not become too large, is one in which $\Delta_c$ is chosen to be equal to the width of the narrowest frequency band within the M.N.B spectrum.

1.3.3 ADDITION OF SPECTRAL GUARDBANDS TO THE SPECTRUM

As explained in the last section and shown in figure 1.8, the computer is only aware of the spectral bands as interpreted in a binary sequence form. The computer program, to be described, will calculate a sampling rate which when used will cause spectral interlacing without overlapping occurring but to get the lowest sampling rate the aliases and original spectrum will be packed as closely as possible. In the computer simulation this means that spectrum simulations will be allowed to interlace with adjacent blocks touching, these representing aliased and original frequency bands, figure 1.9. When the actual sampled waveform spectrum is considered, figure 1.9b, where it is superimposed on the binary sequence representation, it is found that adjoining spectral bands, alias and original, leave no interleaving space to
**Fig 1.10a**  
SPECTRAL LEAKAGE OUTSIDE OCCUPIED BAND

**Fig 1.10b**  
SPECTRAL GUARD BANDS

**Sequence**  
\{F\}  
```
0 1 1 1 1 1 1 1 1 1 1 1
```

**Additional Guard 1's**

(18)
allow application of practical filters with finite cutoff limits. Also a less obvious fact is that in spectral processing only finite length waveforms are sampled, possibly with the application of window functions, so there will always be a certain amount of spectral leakage at the boundaries of the spectral bands, figure 1.10. To limit leakage and to give room to apply practical filters it is required to add guardbands to each spectral band within the M.N.B. signals spectrum. This can be thought of in terms of a mythical extension of the continuous band structure or the addition of 1's to the ends of each spectral band representation in sequence form, figure 1.10b. In either case the computer program will now, when supplied with the new spectral representation, provide a sampling rate which when applied will not allow the aliased and original forms of the spectrum to come closer than twice the guardband width. This will quite probably necessitate a higher sampling rate but it is practically impossible to reconstruct without guardbands.

Summarizing at this point, the original multiband narrowband signal spectrum distribution has been simulated by a binary sequence which allows for filtering and leakage effects of the spectral band yet has a high ratio of 0's to 1's. This high ratio being a consequence of the definition of M.N.B. signals that \( F_{\text{max}} \) to the total occupied spectrum \( F_{\text{occ}} \) be a high ratio. The binary sequence is signified by \{F\}, \( F_1, F_2, F_3, \ldots, F_K \).

1.4 DESIGN AND PHILOSOPHY OF A COMPUTER PROGRAM TO DETERMINE THE MINIMUM UNIFORM SAMPLING RATE FOR A M.N.B. WAVEFORM

When a continuous time waveform \( s_3(t) \) is sampled at uniform time instants by a uniform impulse train \( e(t) \) the effect on the spectrum of \( s_3(t) \) is to convolve it with the spectrum of \( e(t) \).
\[ e(t) = \sum_{n=1}^{N} \delta(t - n\Delta t) \]

Sampling in the time domain:

\[ s_s(t) = \sum_{n=1}^{N} s_3(t)\delta(t - n\Delta t) \]

where:

- \( s_s(t) \) denotes the sampled time waveform
- \( \Delta t \) denotes the sampling time interval, \( 1/f_s \)

Equivalently in the frequency domain:

\[ S_s(f) = \int_{-\infty}^{\infty} S_3(f)E(f - f_s)df \]

where:

\[ S_s(f) = \int_{-\infty}^{\infty} s_s(t)e^{-j\omega t}dt \]

Similarly \( S_3(f) \) and \( E(f) \) are Fourier transforms of \( s_3(t) \) and \( e(t) \) respectively. It is the convolution process which folds the aliases of the original spectrum back over the frequency domain. So if the interlacing process is to be studied it is the convolution process between \( E(f) \) and \( S_3(f) \) which controls where the alias spectral bands are placed. Since uniform sampling is to be used the spectrum \( E(f) \) will consist of a series of impulses at harmonics of the sampling frequency.

\[ E(f) = \sum_{k=-\infty}^{\infty} \delta(f - kf_s) \]

Therefore when the convolution is performed between \( E(f) \) and \( S_3(f) \) for every spectral band on the original \( S_3(f) \) signal spectrum there will be aliased bands placed at uniform intervals of \( \pm f_s \) Hz either side, figure 11a. It can therefore be seen that if any original spectrum component is spaced an integral multiple of \( f_s \) Hz away from any other component in the spectrum then the convolution process will cause the aliases of each component to overlap.
One Aliased Spectral Band  

Fig. 1.11a

Two Spectral Bands (overlap)  

Fig. 1.11b

Two Spectral Bands (actual spectrum)  

Fig. 1.11c

(21)
part of the original spectrum, figure 11c. Therefore the conditions for no
spectral overlap can be written down:

\[ j f_s \neq \Delta f \]  \hspace{1cm} 1.7

\[ j = 1, 2, 3, \ldots, \infty \]

where:

\( \Delta f \) signifies the difference between the frequency locations of
two signal occupied frequencies within the spectrum.

So when considering the spectrum of a signal to see whether spectral overlap
will occur between the original and aliases of the spectrum when sampled, it
is imperative that the spectral spacing, \( \Delta f \), of various occupied spectral
bands within \( \pm F_{\text{max}} \) are known. In fact these spacings are of more importance
than the absolute frequencies when considering spectral overlap. If equation
1.7 is initially considered with \( j = 1 \), i.e.

\[ f_s \neq \Delta f \]

Then the range of all possible \( \Delta f \)'s for a complete M.N.B. spectrum format
will represent exactly all the first order sampling frequencies to avoid.
Also information as regarding all submultiple sampling frequencies can be
found by further processing of these \( \Delta f \) values.

1.4.1 FIRST ORDER FORBIDDEN SAMPLING FREQUENCIES

Calculation of the frequency spacings between occupied frequencies in a given
signal spectrum is a time consuming process. Consider the signal spectrum
in figure 1.12, the possible frequency spacings, \( \Delta f \), that can be generated,
form two continuous bands with limits:

\[ 0 \text{ to } (f_4 - f_3) = FB_1 \]

This band includes all differential frequencies which can be generated from
internal differencing within a single band and therefore includes the band
\[ 0 \text{ to } (f_2 - f_1) \]. So in general terms the first band \( FB_1 \) will always reach
from zero frequency to a frequency equal to the width of the widest band

(22)
A TYPICAL M.N.B. SPECTRUM WITH THE ASSOCIATED FORBIDDEN SAMPLING FREQUENCIES IN Fig. 1.13

Fig. 1.12

A GRAPH OF THE FIRST ORDER FORBIDDEN CONTINUOUS SAMPLING FREQUENCIES

Fig. 1.13

WHERE: 1—LEVEL DENOTES A FORBIDDEN SAMPLING FREQUENCY

0—LEVEL DENOTES A ACCEPTABLE SAMPLING FREQUENCY

(23)
within the M.N.B. structure.

\[ \text{FB}_1 = 0 \text{ to } \{ \text{the width in Hz of the largest occupied frequency band within the M.N.B. structure} \} \]

The second difference band calculated from figure 1.12 will be:

\[ (f_3 - f_2) \text{ to } (f_4 - f_1) = \text{FB}_2 \]

If aliased and original forms of the spectrum are not to overlap when the signal is sampled, no sampling frequency equal to any values in FB\(_1\) or FB\(_2\) can be used. Hence any frequencies within bands such as FB\(_1\) and FB\(_2\) are termed first order forbidden sampling frequencies, see figure 1.13. In this example only a signal with two spectral bands was considered but of course if the signal was a real time waveform there would be corresponding negative frequency bands. In a more complex signal it can be envisaged that the number and extent of these forbidden sampling frequencies will expand enormously.

It is the analytical effort of deriving the forbidden sampling band structure for a complicated signal that drives us to calculating them by computer methods. A discrete equivalent of the continuous forbidden frequencies can easily be calculated from the binary simulation of the input spectrum, \(S_3(f)\). Consider figure 1.14, where a binary sequence representation of the spectrum in figure 1.12 is shown. In terms of the sequence the forbidden frequency band would be:

\[
\begin{align*}
0 \text{ to } 4 & \quad = \text{FS}_1 \\
10 \text{ to } 16 & \quad = \text{FS}_2
\end{align*}
\]

where multiplying by \(\Delta c\) gives the actual frequency in Hz. In fact, it would be a simple matter to generate a discrete representation of the forbidden frequencies bands by differencing all the indices of the non zero elements of \(\{F\}\), this would give a sequence \(D\) which would represent the discrete
Computer Binary Simulation of the Spectral Format of Figure 1.12 — \( \{F_n\} \)

**Fig 1.14**

Discrete Frequency \( nA_4 \) Hz

Discrete Representation of the First Order Forbidden Sampling Rates from Fig. 1.14

**Fig 1.15**

Discrete Frequency \( nA_4 \) Hz

(25)
forbidden sampling frequencies, \(\Delta f\), spaced at \(\Delta c\) Hz intervals i.e.

Let 

\[
\{H\} = \text{indices of non zero values in the binary sequence} \ \{F\}
\]

Then 

\[
D_i = |H_n - H_{n-\ell}|
\]

\[n = 1, 2, \ldots, N - \ell\]

\[\ell = 1, 2, \ldots, N - 1\]

where \(\{D\}\) contains only one difference value once.

As an example \(\{D\}\) is found from the discrete spectrum \(\{F\}\) in figure 1.14 and represented in figure 1.15.

\[
\{D\} = (0, 1, 2, 3, 4, 10, 11, 12, 13, 14, 15, 16) \ \Delta c \ \text{Hz}
\]

This sequence of discrete frequencies although it spans the forbidden frequency gap does not totally describe it because a sampling rate with a frequency not equal to any of the above could be chosen yet lie within the forbidden band, i.e. 11.5 \(\Delta c\) Hz. So in order to simplify the problem the sampling frequencies are quantised. Instead of allowing the sampling frequency to vary between 0 and say 2\(F_{\text{max}}\), the Nyquist rate, it is restricted to integer multiples of \(\Delta c\) i.e.

\[
f_s = k \ \Delta c
\]

\[k \ \text{is an arbitrary integer}.
\]

In this way sampling rates cannot be chosen which are not integer multiples of \(\Delta c\). Hence the above sequence of numbers describing the forbidden band of sampling frequencies \(\Delta f\) is complete, in that it bars the new form of quantised sampling frequency from the bands 0 to 4 \(\Delta c\) and 10 \(\Delta c\) to 16 \(\Delta c\) Hz.

### 1.4.2 DEFINING THE TOTAL EXTENT OF THE FORBIDDEN SAMPLING FREQUENCIES

A comprehensive list of the forbidden sampling frequencies, should include the first order frequencies, \(j = 1\) equation 1.7, also higher order forbidden sampling frequency, \(j = 2, 3, 4, \ldots\). Repeating equation 1.7:

\[
j \ f_s \neq \Delta f
\]

\[j = 1, 2, 3, \ldots J_\ell
\]

(26)
where \( J_k \) denotes the first integer number > \( 2F_{\text{max}} \).

The higher order forbidden sampling frequencies \( j = 2, 3, \ldots, J_k \) correspond to the spacings of spectral components, \( \Delta f \), being twice, three or more times the basic interval \( f_s \) Hz. All the higher order forbidden sampling frequencies can be derived from the first order ones because they correspond to the basic \( \Delta f \) values for a particular spectral format. In principle the method of deriving the higher order sampling frequencies is simple, in that the derivation of the submultiples of all the values of \( \Delta f \) will include all the possible forbidden sampling frequencies. This comes directly from the consideration of equation 1.7. As an example, if a first order forbidden sampling frequency was denoted by \( 8 \Delta c \) Hz, then forbidden second and fourth order sampling frequency would be, \( 4 \Delta c \) and \( 2 \Delta c \) Hz respectively. To calculate all forbidden sampling frequencies for a continuous distribution of \( \Delta f \) is practically wise impossible; so for practical implementation one has revert to the discrete equivalent of equation 1.7, i.e.

\[
j f_s \neq \Delta c D_i
\]

for all \( i \) and \( j \) combinations

\[
i = 1, 2, 3, \ldots, D_k
\]

\[
j = 1, 2, 3, \ldots, J_k
\]

where \( D_k \) denotes the length of sequence \( \{D\} \).

Using equation 1.9 it is possible to derive the higher order forbidden sampling frequencies by breaking down each individual element of \( \{D\} \) into its submultiples.

Speaking hypothetically, if it were possible to analyse each element of \( \{D\} \) and derive all possible submultiples then a second sequence could be generated, \( \{G\} \), in which would be included every possible submultiple of all elements in \( \{D\} \). This newly derived sequence \( \{G\} \) would then contain the
sequence of elements corresponding to all the forbidden sampling frequencies. Therefore to choose a sampling frequency which would give no overlap in the spectrum of the sampled waveform, it would be necessary to select a frequency which is not in the sequence \( \{G\} \). Hence a sampling frequency which is not a forbidden one. Reverting to practical matters, it is a time consuming process to generate a sequence such as \( \{G\} \). On consideration it is not really required to know all the factored values of the \( \{D\} \) elements, just the values of frequencies which are not factors of any of the \( \{D\} \) elements. In fact this can be taken one stage further, in that it is really the lowest sampling frequency possible that is required. So the problem is one of finding the lowest number which is not a factor of any of the elements in the sequence \( \{D\} \), this will correspond to the lowest possible sampling frequency, \( f_{\text{min}} \), for a given sequence \( \{F\} \).

1.4.3 SEQUENTIAL TRIAL AND ERROR DETECTION

There are two methods that can be used to find the lowest possible sampling frequency, \( f_{\text{min}} \), they both require the detection of the lowest value which is not a factor of the sequence \( \{D\} \). The first method is the obvious way to approach the problem. Starting with the lowest possible non factor of \( \{D\} \), which would be 2, a test is made to see if this is a factor of \( D_1 \); then \( D_2 \), \( D_3 \), ..., \( D_n \). If the test detects that 2 is a factor of \( \{D\} \) then the next possible non factor is tested, 3. Hence the procedure goes on until a number is found which is not a factor of any of the elements in \( \{D\} \) i.e. \( T_m \).

Sequential test numbers: \( 2, 3, 4, 5, \ldots, T_m \)

Thus the minimum sampling frequency which will avoid spectral overlap is,

\[
f_{\text{min}} = \Delta c \, T_m
\]

(28)
This procedure is very simple to programme but it suffers from the fact that the computer has to test all values from 2 to \( T_m \) to find the required sampling rate. In an attempt to shorten the time necessary to come to the correct sampling rate a second method was devised, based on the use of prime numbers.

1.4.4. PRIME NUMBER TRIAL AND ERROR DETECTION

The second method entails basically the same trial and error procedure as adopted in the first method but the choice of possible non factors of \( \{D\} \) is based on a prime number sequence. So in computing \( T_m \) the first choice would be 2, then 3, 5, 7, 9, \ldots\, to \( T_m \). Therefore \( T_m \) is going to always be a prime. Hence this method will find the lowest possible prime sampling rate, not necessarily the absolute lowest. In practice the results from methods one and two are very close, if not identical. The great benefit in this method is the speed in finding the correct sampling rate. As an example, if \( T_m = 43 \), then using method one, 42 trials would be necessary; using method two, only 14 trials.

To substantiate the use of a prime number sequence it is necessary to consider the method used for searching for a non factor of sequence \( \{D\} \). If the number of factors in \( \{D\} \) is large in comparison to \( T_m \) the existence of factors from 2 to \( T_m \) can be described on a statistical basis. If the \( \{D\} \) elements are considered as a random selection from a field of numbers ranging from 0 to \( N \) the probability of a particular prime number, \( p \), occurring as a factor in an element of \( \{D\} \) is :-

\[
\text{Prob}(p \text{ being a factor}) = \frac{1}{p}
\]

As an example, the number 2 will have a probability of 0.5 of occurring as a factor in a \( \{D\} \) element. This can be seen simply because probability wise
half the choices of \( \{D\} \) would be even and half odd. Consider the search pattern as used in the first method, i.e.

\[
\begin{array}{ccccccc}
\text{Number} & 2 & 3 & 4 & 5 & 6 & \ldots & T_m \\
\text{Prob. of occurring} & 0.5 & 0.33 & ? & 0.20 & ? & \ldots & ?
\end{array}
\]

An efficient search pattern should be designed to test first the numbers with least probability of occurring. On this basis the first number to try would be the largest possible prime, \( P \), followed by \( P - 1 \), etc. The main over-riding factor of this search process is to find the smallest non factor in the least number of steps. Therefore a search method based on ascending numbers must be chosen but to make it more efficient than just a sequential trial and error method only numbers with progressively decreasing probability of occurrence as factors are tested. At first sight, this might signify method one, seeing that each prime element in ascending order has decreasing probability of being found. In fact this is not the case. If numbers 2 and 3 have been found as factors of elements of \( \{D\} \) this will affect the probability of finding 6, in fact it will increase the probability. Every time a factor is found the probabilities of all associated numbers which include this factor will increase. As an example:

\[
\text{Prob}(4 \text{ being a factor}) = 0.25
\]

Given that 2 exists as a factor then:

\[
\text{Prob}(4 \text{ being a factor} | \text{that } 2 \text{ exists}) = 0.5
\]

The only numbers which will not have increased probability of existing due to other numbers being found as factors, are prime numbers. This is simply because the definition of a prime precludes the existence of factors. Therefore searching for non factors using a prime number pattern is statistically optimum in the sense that it selects numbers which have a declining statistical probability of occurring as the numbers ascend in order.
KNOWLEDGE OF M.N.B.
SPECTRAL FORMAT

Choose \( \Delta c \)

Describe spectral format by a binary sequence

\( \{F\} \)

Specify nonzero elements in \( \{F\} \)

\( \{H\} \)

Generate difference values between elements in \( \{H\} \)

\( \{D\} \)

Prime number search

Apply to all elements in \( \{D\} \)

Sequential number search

Apply to all elements in \( \{D\} \)

\( f_{\text{min}} \) (PRIME)

\( f_{\text{min}} \)
1.4.5 DESCRIPTION OF COMPUTER PROGRAMMING

Although the concepts behind the mathematical manipulation may be complicated, the actual computer implementation is very simple. To consolidate the theory a short practical description of the programming required is given.

The first part of the problem is calculating from the binary sequence \( \{F\} \), which describes the spectrum format, the first order forbidden sampling frequencies. This is done by:

1) From \( \{F\} \) a second sequence \( \{H\} \) is generated containing the indices, \( k \), of the elements of \( \{F\} \) which are non zero. Remember that positive and negative frequencies are counted as completely separate components which have both to be taken into account.

2) Next the first order forbidden sampling frequencies or more correctly the sequence equivalent are calculated. This is done by generating a third sequence \( \{D\} \) from \( \{H\} \) in which all the difference values between the sequence elements of \( \{H\} \) are calculated, i.e.

\[
D_k = |H_i - H_j|
\]

For all possible combinations of \( i \) and \( j \), with \( \{D\} \) containing any difference value only once. If there are \( N \) elements in \( \{H_i\} \) the maximum length of \( \{D\} \) will be:

\[
\text{Maximum length of } \{D\} = \sum_{j=1}^{N-1} (N - j)
\]

It is quite likely that the length, \( D_k \), of \( \{D\} \) will be less than this maximum because only differences which are of different values are noted in \( \{D\} \).

The second part of the programming entails the modification of sequence \( \{D\} \) to give either:

(32)
1) The lowest number which is not a factor of \( \{D\} \).

or

2) The lowest prime number which is not a factor of \( \{D\} \).

In both cases the programming is similar. Case 1) requires a sequential test using 2, 3, 4, 5, \( \ldots \), \( T_m \) and the ability to detect when a number is not a factor of any of the \( \{D\} \) elements. When this is detected, \( T_m \), the required sampling frequency, \( f_{\text{min}} \), equals:

\[
f_{\text{min}} = \Delta c T_m
\]

In case 2) exactly the same program is used but a sequential test using 2, 3, 5, 7, 11, \( \ldots \), \( T_m \) is implemented.

1.5 EXAMPLES OF SPECTRAL INTERLACING

To finish this chapter, two examples of spectral interlacing are shown. The two examples are based on the interlacing of ten spectral bands, five negative frequencies and five positive frequencies so that they simulate a real signal source. In both cases the frequency bands are centred at the same frequencies but their widths are varied to show the effect on the sampling rate. The basic spectral format is shown in table 1. In the examples, signals with various amplitudes were placed in each spectral band to help show the effects of interlacing. Simply by inspection of the figures 1.17 and 1.18 the general spectral nature of the interlacing process can be seen. The sampling rate for signal one in figure 1.18 is much closer to the optimum sampling rate than signal two, figure 1.17. This can be seen by the close density packing of the original and aliased occupied bands, in fact in figure 1.18 the frequency scale has to be expanded to show the individual signal bands closely interlaced.

(33)
<table>
<thead>
<tr>
<th>Number of Spectral Band</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central Frequency of Spectral Band (Hz)</td>
<td>-1835</td>
<td>-700</td>
<td>-598</td>
<td>-375</td>
<td>-200</td>
<td>200</td>
<td>375</td>
<td>598</td>
<td>700</td>
<td>1835</td>
</tr>
<tr>
<td>Relative Amplitude of Test Signal</td>
<td>0.45</td>
<td>1.00</td>
<td>0.39</td>
<td>0.68</td>
<td>0.15</td>
<td>0.15</td>
<td>0.62</td>
<td>0.39</td>
<td>1.00</td>
<td>0.45</td>
</tr>
</tbody>
</table>

Arbitrary Spectral Cell Width Chosen: 1.35 Hz
### TABLE 2

<table>
<thead>
<tr>
<th></th>
<th>SIGNAL 1</th>
<th>SIGNAL 2</th>
</tr>
</thead>
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<tr>
<td><strong>Width of Spectral Bands</strong></td>
<td>$5\Delta_c = 6.75 \text{ Hz}$</td>
<td>$20\Delta_c = 27.0 \text{ Hz}$</td>
</tr>
<tr>
<td><strong>Optimum Sampling Rate ($F_{o,\text{cc}}$)</strong></td>
<td>67.5 Hz</td>
<td>270 Hz</td>
</tr>
<tr>
<td><strong>Actual Sampling Rate ($f_{\text{min}}$)</strong></td>
<td>132 Hz</td>
<td>663 Hz</td>
</tr>
<tr>
<td><strong>$F_{\text{max}}$: Original Spectrum</strong></td>
<td>1839 Hz</td>
<td>1848 Hz</td>
</tr>
<tr>
<td><strong>Ratio of $2F_{\text{max}}/F_{o,\text{cc}}$</strong></td>
<td>54.5</td>
<td>13.7</td>
</tr>
<tr>
<td><strong>Ratio of $2F_{\text{max}}/f_{\text{min}}$</strong></td>
<td>27.9</td>
<td>5.57</td>
</tr>
</tbody>
</table>
WAVEFORM 2

INTERLACED ALIASED SPECTRUM

FILTERED SPECTRUM

\[ \text{Frequency (Hz)} \]

\[ \text{Complex Spectrum} \]

\[ \text{Filter Spectrum} \]

Fig. 1.17
INTERLACED ALIASED SPECTRUM

WAVEFORM 1

Fig. 1.18

[Graph showing complex spectrum with frequency axis ranging from 0 to 2650 Hz, and an enlarged section of the spectrum highlighting frequencies.]
Real Time Digital Filtering

\[ \text{Time Sampled Data Rate} = f_s \text{ Hz} \]

\[ \text{Clocking Rate} = n f_s \text{ Hz} \]

\[ \text{Filtered Data Rate} = n f_s \text{ Hz} \]

Where: \( n \) = the first positive integer greater than \( 2 \Delta F_{\text{max}} \)

Fast Discrete Convolution Filtering

\[ \text{Time Sampled Data Rate} = f_s \text{ Hz} \]

\[ \text{Filtered Time Data Rate} = n f_s \text{ Hz} \]

\( \Delta F_{\text{max}} \)
1.6 RECOVERY OF THE ORIGINAL WAVEFORM/COMPLEX SPECTRUM

To recover the original time waveform or its discrete equivalent sampled at above a $2F_{\text{max}}$, it is necessary to apply linear filtering techniques. The filtering process is best illustrated in the frequency domain, figures 1.17, 1.18, where the spectrum of the sampled waveform shows the aliased spectral components which need to be separated from the original spectrum bands. Using knowledge of the original waveform's spectrum the frequency response of the filter can be determined by designing passbands where the original spectral bands existed and stopbands where no spectral components originally existed. The filters transition from a pass to a stopband should be rapid and at least contained within the guardband spaces originally allowed for in the interlacing process. The actual filtering operation can be implemented by analogue or digital means. If the complicated filter characteristic can be formed into an analogue network then direct filtering from digital data to a continuous waveform can be implemented in one step. The conventional analogue filter could be replaced by its digital counterpart, figure 1.19; note in this case that the clock rate of the digital filter must be faster than twice the highest frequency component in the original waveform to avoid ambiguity in the data output sequence. By far the easiest way of filtering the data values is to use an aperiodic fast convolution algorithm, figure 1.20, this is because this method allows easy and flexible design of the filtering characteristic. In this method the sampled data is filtered by processing in blocks of N samples at a time. Each individual block is digitally convoluted with the impulse response of the ideal filter characteristic, the results are then recombined to form the filtered discrete data using the overlap and add method, reference (4).
FIG 1.21

BANDLIMITED SIGNALS

\[ f_1 - f_2 \]
\[ f_3 - f_4 \]
\[ f_5 - f_6 \]

SAMPLE

TRANSMISSION CHANNEL

BANDWIDTH \( 2f_{\text{min}} \)

FILTER

FIG 1.22

M.N.B. SUBGROUP

ONE

TWO

THREE

SELECT COMMON SAMPLING RATE \( f_{\text{min}}(T) \)

\[ f_{\text{min}}(1) = f_{\text{min}}(2) = f_{\text{min}}(3) = f_{\text{min}}(T) \]
1.7 NOTE ON THE SAMPLING RATE REDUCTION ACHIEVED WITH SPECTRAL INTERLACING

The optimum sampling rate for a M.N.B. signal is equal to the total spectral space occupied by the signal and this is denoted by $F_{occ}$. It would be convenient to be able to specify generally the minimum possible sampling rate, $f_{\text{min}}$, derived by the interlacing method in terms of the optimum rate, $F_{occ}$. In practice this is impossible to do because the sampling rate, using the interlacing process, is not only dependent on $F_{occ}$ but also sensitive to the distribution of the occupied spectral space within $\pm F_{\text{max}}$. Therefore the minimum sampling rate can, in practice, only be specified for a particular spectrum format. The only general conclusion that can be drawn in relation to the possible sampling rate achieved by interlacing is that the greater the ratio of $2F_{\text{max}}$ to occupied spectral space, $F_{occ}$, then the greater will be the reduction in sampling rate below $2F_{\text{max}}$.

1.8 COMMENTS

The sampling technique presented in this chapter is applicable not only to spectrum analysis but also to general engineering problems of sampling rate reduction. As an example it can be applied in digital communication/data transmission systems where bandlimited signals which have diversely distributed spectral power are required to be transmitted. The simplicity in operation of this sampling technique make it attractive to systems where circuitry should be a minimum at the transmission terminal, see figure 1.21. As with the spectrum analysis case one has to make sure that the signals at 1, 2, 3, ..., are truly bandlimited and that the frequencies involved do not cause the sampling aperture effect to distort significantly the components filtered at a, b, c, ... .

A variation of this sampling technique, possibly to some advantage, can be
made if a degree of time multiplexing is introduced. The idea being that the total group of signals to be sampled is split into subgroups, see figure 1.22, the sampling rate for each subgroup is optimised for the lowest possible. Then a common sampling rate $f_{\text{min}}(T)$ is selected which will fulfil the sampling conditions for each subsystem, this provides the time multiplexing sampling rate. This technique introduces the possibility of optimising the distribution of the narrowband signals between subgroups so that $f_{\text{min}}(T)$ can be approached as close to $F_{\text{occ}}(T)$ as possible. With this technique one is then balancing information compression between frequency interweaving and time multiplexing so as to create a minimum uniform sampling rate.

1.9 SUMMARY

In this chapter it has been shown possible to reduce the sampling rate of a multiple narrowband signal to below the "safe" limit of $2F_{\text{max}}$ towards the lowest possible rate of $F_{\text{occ}}$ by employing simple uniform sampling and filtering techniques. This has been done by calculation of the lowest possible sampling rate which can be used without causing spectral overlap between the aliased and original spectral bands. Therefore successful use of this sampling technique relies on the original accurate specification of the spectral power bands and ensuring that they are in fact bandlimited.

The emphasis in this chapter has been on the simplification of the actual data handling processes and to this end considerable effort is spent on presampling computation based on a priori knowledge of the signal. In chapter four this type of sampling problem is tackled again this time with a multiple bandlimited signal but emphasis is put on the use of a fixed sampling method with highly complex post-sampling signal processing.
CHAPTER 2
RANDOM SAMPLING TECHNIQUES

2.1 INTRODUCTION

A formidable quantity of theory has been written on the subject of estimation of spectral functions using digital techniques to analyse time waveforms, references (6) to (8). The theoretical analysis of sampled waveforms or discrete time sequences with application to digital spectrum analysis has developed in parallel with the continuous time analogue methods of spectrum analysis. A parallel development because one can list continuous time equations with the corresponding digital equivalent. The main example of this being the Fourier transform.

Continuous time and frequency domains :-

\[ S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt \]

Discrete time and frequency domains :-

\[ X_k = \sum_{n=0}^{N-1} s_n e^{-jnk2\pi/N} \quad k = 0, 1, 2, \ldots, N-1 \]

The adoption of digital techniques of spectrum analysis has brought the universal acceptance of uniform sampling methods at above twice the highest frequency content in the time waveform. This sampling method has become the standard method of transforming between the continuous and discrete time domains simply due to its ability to provide a reversible, unambiguous transformation from one domain to the other. In some signal processing applications where it is easier to carry out the required processing in the discrete domain it is necessary to use such a reversible sampling method. Digital filtering is a prime example of the need for reversibility between the continuous/discrete time domains. In the field of spectral function estimation no such reversible process is required because reconstruction of the original time waveform is never needed.

(43)
The only prerequisite of a sampling process used for spectrum estimation is that it must be possible to estimate the spectral function of the original continuous waveform from the discrete samples taken by the sampling process. This redefinition of the sampling process widens the field of possible sampling methods. Random sampling techniques, in which the sampling points are described on a statistical basis, form a class of irreversible sampling techniques which can be used with advantage to estimate spectral functions. The attribute of random sampling is that it avoids aliasing caused by the interaction of periodicity in the sampling process with periodicities within the time waveform. This means that the sampling rate is not set by the frequency content of the waveform being sampled. Spectral estimates of deterministic and random time functions can be made using random sampling but each application has its own characteristics. This chapter will discuss the use of random sampling techniques in relation to both random and deterministic waveforms with an emphasis on computer compatibility.

2.2 STATISTICAL SAMPLING WITH APPLICATION TO POWER SPECTRAL DENSITY ESTIMATES OF STATIONARY STOCHASTIC RANDOM WAVEFORMS

This first class of random sampling methods to be discussed can be termed statistical because not only is the random sampling process based on statistical distributions but also the processing techniques applied to the samples taken are also statistical. Beutler and Leneman have provided the mathematical background to this sampling scheme in references (9) and (10). This type of sampling has been presented first to highlight the fact that spectrum estimation does not necessarily go hand in hand with Nyquist sampling. Although the theoretical background and possible computer applications in digital estimation are presented here, no actual practical work has been carried out on this sort of sampling process. Other sampling processes, to be considered later, promised greater practical
Randomly Sampled Waveform

Fig. 2.1

Measured Information

\{S\} = s_1, s_2, s_3, s_4, \ldots, s_N
benefits at the time. This small section is therefore presented partly as a resume of a sampling method which is unfamiliar, at least in engineering circles, but mainly as a complementary section to normal random sampling so as to complete the picture of random sampling as applied to spectrum analysis.

No time information is collected with this sampling process, therefore it is only directly applicable to the estimation of the power density spectrum of random waveforms where phase information is irrelevant. The main example of this type of sampling, as given in reference (11), is in the application of Stationary Point Processes to the design of sampling impulse waveforms.

2.2.1 PROCESSING THEORY

The actual statistics of the sampling scheme will be presented later when the aim is more evident, but first consider the initial steps in the sampling process. With reference to figure 2.1 the sampling procedure is simply one of measuring the amplitudes $s_1, s_2, s_3, \ldots, s_N$ of $s(t)$ at time instants $t_1, t_2, t_3, \ldots, t_N$ and stacking the values into a sequence form, a finite section of the sequence is given by:

$$\{s\} = s_1, s_2, s_3, \ldots, s_N$$

The times of sampling are lost. Sequence $\{s\}$ along with the statistical parameters of the sampling process are the only terms of information preserved for the estimation of the power density function of $s(t)$.

Instead of trying to directly form $P(f)$, a power spectral density estimate, the autocorrelation function estimate of $s(t)$, $R(\tau)$, is sought. The first step is to estimate the autocorrelation of the sequence $\{s\}$:

$$\gamma_m = E\left\{ \sum_{n=1}^{N} s_n s_{n+m} \right\}$$

(46)
As the available number of sampled values, \( N \), increases the estimate will converge to the autocorrelation of \( \{s\} \). It is possible, using the statistics of the sampling process, to construct a relationship between the true autocorrelation function \( \Omega(\tau) \), of which \( R(\tau) \) is an estimate, and the autocorrelation function \( \gamma_m \).

\[
\gamma_m = \int_{\tau=0}^{\infty} \Omega(\tau) \cdot p_m(\tau) d\tau \quad \text{2.2}
\]

where:
- \( \Omega(\tau) \) is the autocorrelation function of \( s(t) \)
- \( p_m(\tau) \) is the probability density distribution of the time interval between the \( i \)th and \( i+m \)th sampling instant
- \( \gamma_m \) is the autocorrelation calculated from the sampled sequence \( \{s\} \)

The function \( p_m(\tau) \) defines the statistical nature of the intervals of the sampling process. The first distribution \( p_1(\tau) \) characterises the time intervals between two consecutive sampling points, as an example with reference to figure 2.1

\( t_2 - t_1, t_3 - t_2, \ldots \)

It is useful to note that \( p_1(\tau) \) defines the distribution of the intervals of a sampling process. The \( p_2(\tau) \) distribution again describes a time interval but between two sampling instants with two intervals between, eg.

\( t_3 - t_1, t_4 - t_2, \ldots \)

Hence \( p_n(\tau) \) describes the distribution of the time interval between sampling points \( n \) intervals apart. All these statistical functions are stationary, in that they are independent of absolute time. The mathematical subject of Stationary Point Processes is directly applicable to the design and description of sampling processes which are stationary, reference (9).

Equation 2.2 represents a transformation or a mapping of \( \Omega(\tau) \) on to \( \gamma_m \), if the transformation is a one to one process then in theory it is invertable.

A typical invertable transformation related pair of equations is now (47)
considered. Any function from 0 to $\infty$, $\tau$ can be described by a complete set of linearly independent functions within the interval 0 to $\infty$, $\tau$. Using the autocorrelation $\Omega(\tau)$ as an example

$$\Omega(\tau) = \sum_{n=1}^{\infty} \beta_n A_n(\tau)$$  \hspace{1cm} 2.3

where: \( A_n(\tau) \) denotes a complete set of linearly independent functions within 0 to $\infty$, $\tau$.

\( \beta_n \) denotes a set of real coefficients.

The second equation in the pair can be described by the operation $\phi_n$ on $\Omega(\tau)$ to give the $\beta_n$ coefficients i.e.

$$\beta_n = \phi_n(\Omega(\tau))$$  \hspace{1cm} 2.4

Then the equations 2.3 and 2.4 provide a reversible transformation pair.

The operation $\phi_n$ is difficult to define without knowing the actual independent functions but if now the set $A_n(\tau)$ are replaced by a complete set of orthonormal functions in the region 0 to $\infty$, $\tau$ then the operation $\phi_n$ can easily be defined. The equation 2.4 becomes

$$\beta_n = \int_0^{\infty} \Omega(\tau) q_n(\tau) d\tau$$  \hspace{1cm} 2.5

since

$$\int_0^{\infty} q_n(\tau)q_m(\tau) d\tau = \delta_{mn}$$

where $\delta_{mn}$ denotes the Kronecker Delta function.

Rewriting equation 2.3

$$\Omega(\tau) = \sum_{n=1}^{\infty} \beta_n q_n(\tau)$$  \hspace{1cm} 2.6

where $q_n(\tau)$ defines the complete set of orthonormal functions within 0 to $\infty$, $\tau$.

Equations 2.5 and 2.6 form a basis for the investigation of the transformation denoted by equation 2.2.

(48)
The functions $p_n(T)$ do in fact form a complete set of linearly independent functions in the range $0$ to $\infty$, $\tau$, but are not necessarily orthonormal.

Since $p_n(T)$ do form a complete set it is possible to construct any function within $0$ to $\infty$, $\tau$, by a linear combination of $p_n(T)$ functions. Hence the functions can be used to construct a complete set of orthonormal functions, starting with, $q_1(\tau) = b_{11}p_1(\tau)$, i.e.

\[ q_n(\tau) = \sum_{k=1}^{n} b_{kn}p_k(\tau) \quad n > 1 \tag{2.7} \]

This is all very well if one has access to the $p_n(T)$ functions, in fact only the $\gamma_n$ functions are available for processing. Since it is only the $p_n(T)$ variables within the integral, equation 2.2 that needs conversion, it is possible to overcome the problem by combining equation 2.7 and equation 2.5.

If instead of taking a linear combination of $p_n(T)$ functions a linear combination of $\gamma_n$ functions is taken, i.e.

\[ U_n = b_{1n}\gamma_1 + b_{2n}\gamma_2 + \cdots + b_{nn}\gamma_n \quad n > 1 \]

\[ = b_{1n} \int_0^\infty p_1(\tau)\Omega(\tau)d\tau + b_{2n} \int_0^\infty p_2(\tau)\Omega(\tau)d\tau + \cdots + b_{nn} \int_0^\infty p_n(\tau)\Omega(\tau)d\tau \]

\[ U_n = \int_0^\infty \Omega(\tau)(b_{1n}p_1(\tau) + b_{2n}p_2(\tau) + \cdots + b_{nn}p_n(\tau))d\tau \quad n = 1, 2, 3, \ldots \infty \tag{2.8} \]

In this way the $p_n(T)$ functions are orthogonalised within the integral sign.

By arranging to multiply by suitable coefficients $b_{kn}$ it is possible to arrange the bracketed forms in equation 2.8 to equal $q_n$. Hence comparing equations 2.8 and 2.5 we find $U_n$ is equivalent to $\beta_n$. Therefore by operating on $\gamma_n$ with various sets of $b_{kn}$ coefficients, the coefficients $\beta_n$ for the $q_n(T)$ set of orthonormal functions can be found. Hence invoking equation 2.6 $\Omega(\tau)$ is reconstructed.

\[ \Omega(\tau) = \sum_{n=1}^{n=\infty} \beta_n q_n(\tau) \quad n > 1 \tag{2.9} \]

(49)
2.2.2 THE USE OF LAGUERRE FUNCTIONS

Thus one comes to the first practical difficulty, determining the coefficients $b_{kn}$. The problem is solved in this particular case by selection of the $p_n(\tau)$ forms which can be related easily to some well tabulated orthonormal function in $\tau$. The most obvious and well known sampling scheme which exhibits stationary statistics plus a simple relation to a tabulated orthonormal function is the Poisson sampling process, reference (11).

Poisson Distribution:

$$p(n, \Delta t) = \frac{(\rho \Delta t)^n e^{-\rho \Delta t}}{n!}$$

$n \geq 1$

giving:

$$p_n(\tau) = \frac{\rho(\rho \tau)^{n-1} e^{-\rho \tau}}{(n-1)!}$$

$n \geq 1$

where $\rho$ = the density of sampling points

$p(n, \Delta t) =$ probability density of getting $n$ samples in the time interval $\Delta t$.

The advantage of the Poisson sampling process is in its relation to the Laguerre set of orthogonal polynomials which when weighted by a simple factor are orthonormal in $0$ to $\infty$, $\tau$. The Laguerre functions themselves are described by

$$L_n(\tau) = \frac{n!}{\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\tau}{k!}\right)^{k} (-1)^{k-k}}$$

$n \geq 1$

where $\binom{n}{k}$ signifies the binomial coefficient $\frac{n!}{k!(n-k)!}$.

The function $q_n(\tau)$ denotes the orthonormal weighted Laguerre function:

$$q_n(\tau) = \sqrt{2\rho} L_{n-1}(2\rho \tau) \exp(-\rho \tau)$$

The relation between $p_n(\tau)$ and $q_n(\tau)$ is such that the set of $b_{kn}$ coefficients turn out to be a particularly simple form:

$$b_{kn} = \sqrt{\frac{2}{\rho}} 2^k \binom{n-1}{k} (-1)^{k}$$

(50)
A Typical Autocorrelation Function of a Stochastic Waveform

Fig. 2.2

Weighted Laguerre Functions

Fig. 2.3

\[ L_0 = e^{-\alpha \tau} \]

\[ L_1 = (1 - \tau) e^{-\alpha \tau} \]

\[ L_3 = \left(1 - \frac{10}{3} \tau + \frac{5}{2} \tau^2 - \frac{\tau^3}{6}\right) e^{-\alpha \tau} \]
ILLUSTRATION OF THE BUFFERING EFFECT OF STATISTICAL SAMPLING

**Fig. 2.4**
The Laguerre functions are particularly suited to the estimation of auto-
correlation functions of random processes due to their shape being
somewhat akin to the general oscillating pulse shape of such autocorrelations.
In this way it is reasonable to postulate that a representation of the auto-
correlation function will be possible with a number of Laguerre functions
small compared with the corresponding number of complex exponential
functions of the Fourier expansion. Attention at this stage is directed to
reference (12) where the functions of Legendre are used to increase the
efficiency of description of transients in speech waveforms. Figure 2.3
gives some general shapes of low order weight Laguerre functions.

2.2.3 COMPUTATIONAL COMPATIBILITY

The theoretical basis of digital processing for this type of sampling is
rather involved but the manipulation is compatible with digital techniques.
The generation of the sequence \( s \) is based on a sampling/buffering technique,
figure 2.4, then the actual calculation of the \( \gamma_n \) autocorrelation can be
implemented by direct Digital Fourier Transform (D.F.T.) of the sequence \( s \).

\[
\gamma_n = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} s_k s_n e^{-j\pi nk/N} |^2 e^{j\pi nk/2N} \]

Making allowances for the periodic effects of autocorrelation calculated in
this way. Once the sequence \( \gamma \) has been generated the rest of the
processing, orthornormalisation and Laguerre functions contraction, can be
put into a matrix format.

Vector element of \( a \) : \( a_n = \text{Est} |\gamma_n| \) : Sequence element estimate
\( n = 1, 2, \ldots, N \)

Vector element of \( b \) : \( b_n = \beta_n \) : Sequence element

The orthornormalisation is simply represented by the matrix operation :

\[
b = B a
\]
where \( B \) is a \( N \times N \) matrix containing the coefficients \( b_{kn} \).

\[
B = \begin{bmatrix}
b_{11} & 0 & 0 & \cdots & 0 \\
b_{12} & b_{22} & 0 & \cdots & 0 \\
b_{13} & b_{23} & b_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{1N} & \cdots & \cdots & \cdots & b_{NN}
\end{bmatrix}
+ \begin{bmatrix} b_{k1} \\ b_{k2} \\ \vdots \\ b_{kN} \end{bmatrix} \quad \text{Coefficient Sets}
\]

Each row makes up a complete set of coefficients for the generation of one element of \( \beta \). Generation of matrix \( B \) is simple and it can be stored for standard reference. The final equation, 2.9, can also be put into a matrix format. Instead of generating the time autocorrelation function \( \Omega(\tau) \) in a continuous fashion it is necessary to estimate it at discrete points.

Rewriting equation 2.6:

\[
\Omega(\ell \Delta \tau) = E \left\{ \sum_{n=1}^{N} \beta(n)q_R(\ell \Delta \tau) \right\} \quad \ell = 0, 1, 2, \ldots, N
\]

The interval spacing \( \Delta \tau \) can be as small as required but it must not be greater than \( 1/2F_{\text{max}} \), where \( F_{\text{max}} \) is the highest frequency contained in the power spectrum. Note, the Nyquist criteria is being used here simply because of the discrete representation of a continuous function, this has no direct effect on the sampling rate in the time domain. Converting equation 2.12 to a matrix form:

\[
\mathbf{r} = Q \mathbf{h}
\]

where:

Vector element of \( \mathbf{r} : r_k = R(k \Delta \tau) \) : Sampled value of estimate of \( \Omega(\tau) \)

\[ k = 1, 2, \ldots, N \]
Matrix \( Q \) of order \( N \times N \)

\[
Q = \begin{bmatrix}
q_{11} & q_{12} & \cdots & q_{1N} \\
q_{21} & q_{22} & & \\
& \cdots & & \\
q_{N1} & \cdots & \cdots & q_{NN}
\end{bmatrix}
\]

1st 2nd Nth Weighted Laguerre functions

Matrix element : \( q_{kn} = q_n(k \delta \tau) \) : Sampled value of nth weighted Laguerre function.

The total conversion from vector \( a \) to \( r \) can be written as a combined equation, combining equations 2.11 and 2.13 one gets :

\[
r = Q \beta a
\]

2.14

The matrices \( Q \) and \( \beta \) are a priori knowledge, therefore a combined equation can be written, i.e.

\[
r = V a
\]

2.15

where :

\[
V = Q \beta
\]

Matrix \( V \) can be calculated and stored in a computing system for a fixed order \( N \).

Theoretically the order \( N \) should tend to infinity so as to get total transformation from \( \gamma_n \) to \( \Omega(\tau) \) from 0 to \( \infty \), \( \tau \), because weighted Laguerre functions are orthonormal in the interval 0 to \( \infty \), \( \tau \). In practice since both \( \Omega(\tau) \) and \( \gamma_n \) are autocorrelations of random variables, beyond a certain number of \( \Delta t \) intervals both functions will tend to zero. Hence one can limit the order of
Randomly Sampled Waveform

\[ s(t) \]

\[ \{ s \} = s_1, s_2, s_3, s_4, \ldots, s_N \]

\[ \{ t \} = t_1, t_2, t_3, t_4, \ldots, t_N \]
the equation 2.6 to \( N \), i.e.

\[
\Omega(\tau) + \varepsilon(\tau) = \sum_{n=1}^{N} \beta_n q_n(\tau)
\]

2.16

where :

\( \varepsilon(\tau) \) is an error function which can be made as small as necessary by increasing \( N \).

This approximation is based on two assumptions, firstly that above a lag value \( N_{AT} \) the autocorrelation \( \Omega(\tau) \) does not contribute significantly to the power content of \( P(f) \). Similarly that above element \( N \) in the sequence autocorrelation function \( \{ \gamma_n \} \) there is no significant contribution to the estimation of \( R(\tau) \). The second assumption is that \( \Omega(\tau) \) is lowpass band-limited. This double fold time and frequency limitation in necessary in order to curtail the number of significant Laguerre description coefficients to a finite order. This is because the Laguerre functions are themselves approximations to a time-frequency limited function.

2.3 RANDOM SAMPLING (AMPLITUDE AND TIME MEASUREMENT)

2.3.1 INTRODUCTION

In this section we will deal with the normally encountered type of waveform sampling in which sampled values \( s_1, s_2, \ldots, s_\infty \) are measured along with the timing information \( t_1, t_2, \ldots, t_\infty \), figure 2.5. The spectral function of the sampled waveform, \( s(t) \), is estimated using both pieces of information.

This section discusses two spectral estimation techniques. First, the use of a scheme in which samples are taken at random time instants, described by an impulse train which is in itself a stationary random process. This type of sampling is applied to both stationary random waveforms and to deterministic harmonic waveforms to show the relative spectral effects of random sampling on the power density and complex amplitude spectrums.
respectively. The second type of random sampling is termed random periodic sampling in which samples can only be taken from the time waveform at uniformly spaced time instants. The probability of a sample occurring at one of the uniformly spaced time instants is independent of absolute time, therefore in the discrete sense this type of sampling is also stationary. It is necessary to consider this type of sampling because of all the random, amplitude and time, sampling techniques this type is the only one which is truly computer compatible.

2.3.2 THE EFFECTS OF RANDOM SAMPLING ON THE ESTIMATION OF THE P.S.D. FUNCTION OF THE STATIONARY RANDOM WAVEFORMS

Consider a section \( s(t) \) of the stationary random time waveform \( u(t) \), i.e.

\[
\begin{align*}
s(t) &= u(t) & 0 < t < T \\
s(t) &= 0 & 0 > t > T
\end{align*}
\]

This is sampled by a randomly distributed sampling impulse waveform \( e(t) \):

\[ e(t) = \sum_n \delta(t - t_n) \]

The sampled waveform is therefore given by

\[ s_s(t) = \sum_n s(t) \delta(t - t_n) \]

The complex spectrum and power spectral density estimates are respectively:

\[
X_s(f) = \int_0^T s_s(t) e^{-j2\pi ft} \, dt
\]

\[
P_s(f) = \frac{1}{T} \left| X_s(f) \right|^2 \quad \text{as } T \to \infty
\]

Due to the fact that \( s(t) \) is a random stochastic waveform the periodogram is a defined inconsistent spectral estimate of \( \phi_s(f) \), the power spectral density of \( s_s(t) \). This is overcome in practice by the use of window functions to help the estimate to converge, see references (6) and (8).

(58)
Since the sampled and sampling functions, \( s(t) \) and \( e(t) \), each constitute a random process, between which there is no correlation, the autocorrelation function of the sampled waveform can be expressed as follows:

\[
\Omega_s(\tau) = \Omega_e(\tau) \Omega(\tau) \tag{2.17}
\]

where the autocorrelation of \( u(t) \) is expressed by:

\[
\Omega(\tau) = \int_{-\infty}^{\infty} u(t) u(t+\tau) \, dt
\]

where:

\[
\Omega_s(\tau) = \text{autocorrelation of the } u_s(t) \text{ waveform}
\]

\[
\Omega(\tau) = \text{autocorrelation of the } u(t) \text{ waveform}
\]

\[
\Omega_e(\tau) = \text{autocorrelation of the } e(t) \text{ waveform}
\]

The relationship between the power spectral density and the autocorrelation function is given by the usual Fourier Transform equation pair.

\[
\Phi(f) = \int_{-\infty}^{\infty} \Omega(\tau) e^{-j2\pi ft} \, dt
\]

From equation 2.17 it follows:

\[
\Phi_s(f) = \Phi(f) \Phi_e(f) \tag{2.18}
\]

where:

\[
\Phi_s(f) = \text{Power spectral density of } u_s(t)
\]

\[
\Phi(f) = \text{Power spectral density of } u(t)
\]

\[
\Phi_e(f) = \text{Power spectral density of } e(t)
\]

Equation 2.18 transforms the power spectral density (p.s.d.), \( \Phi(f) \), to the sampled p.s.d., \( \Phi_s(f) \). The form of \( \Phi_e(f) \) must therefore be considered.

The sampling waveform \( e(t) \) although loosely defined as random must be more closely specified. To be consistent in the examples of a random process, the sampling scheme is assumed to be Poisson as in the previous section. This type of statistic generates a sampling process that is easily analysed and
POWER SPECTRAL DENSITY OF A POISSON DISTRIBUTED IMPULSE WAVEFORM $e(t)$ Fig. 2.6

![Power Spectral Density Diagram]

BIASED SPECTRAL ESTIMATE OF FIG. 2.6

![Biased Spectral Estimate Diagram]

(60)
Power Spectral Density of a Lowpass Waveform

Power Spectral Density of a Randomly Sampled Lowpass Waveform

P.S.D. of a Signal with a Wide Spectral Power Distribution

P.S.D. of a Signal with a Narrow Spectral Power Distribution
constructed. The statistics of a Poisson distributed \( e(t) \) function state that the probability of getting a sampling point between \( t_1 \) and \( t_1 + \Delta t \) is constant and independent of absolute time and of any other occurrence of a sampling point at \( t_2, t_1 \neq t_2 \). In this way the probability of getting \( n \) sampling points in a time interval is specified by:

\[
\text{prob}(n, \Delta t) = \frac{(\rho \Delta t)^n e^{-\rho \Delta t}}{n!}
\]

where: \( \rho \) denotes the sampling density of the process \( e(t) \) giving the mean number of samples per unit time interval.

A perfect Poisson sampling process will have an autocorrelation function, which in the limit can be approximated to:

\[
\Omega_e(t) = \rho(\delta(t) + \rho)
\]

The corresponding p.s.d., figure 2.6, can be written down:

\[
\Phi_e(f) = \rho(1 + \rho \delta(f))
\]

Now the form of \( \Phi_e(f) \) can be substituted in the convolution equation 2.18.

\[
\Phi_s(f) = \Phi(f) * (\rho(1 + \rho \delta(f)))
\]

As an example of the general form of \( \Phi_s(f) \) consider figure 2.8, which represents the spectrum of a bandlimited signal, \( u(t) \). Applying the convolution equation 2.20 the corresponding \( \Phi_s(f) \) spectrum of the sampled signal \( u_s(t) \) can be seen in figure 2.9. The original spectrum is visible but a "d.c." level has been added to the spectrum. Consider the spectrum \( \Phi_s(f_1) \) at a given frequency \( f_1 \), expanding 2.20.

\[
\Phi_s(f_1) = \rho \int_{-\infty}^{\infty} \Phi(f)(1 + \rho \delta(f_1 - f)) \, df
\]

\[
\Phi_s(f_1) = \rho \left[ \int_{-\infty}^{\infty} \Phi(f) \, df + \rho \int_{-\infty}^{\infty} \Phi(f) \delta(f_1 - f) \, df \right]
\]

\[
\Phi_s(f_1) = \rho \int_{-\infty}^{\infty} \Phi(f) \, df + \rho^2 \Phi(f_1)
\]

(62)
The first term represents the original total signal power scaled by a factor \( \rho \), this is presented as a uniform d.c. level, independent of \( f_1 \), spread across the spectrum.

In practice since the sampled waveform \( s(t) \) is a random variable it follows that as an estimate of \( \phi_s(f) \), \( P_s(f) \) will have a mean value equal to \( \phi_s(f) \) but will be an inconsistent estimate. This is solved in the usual manner with window functions to form a consistent but biased estimate of \( \phi_s(f) \).

\[
\phi_s(f) \bigg|_{\text{biased}} = E_p \{ P_s(f) \ast W(f) \} \tag{2.22}
\]

The effect of window biasing can be seen by replacing \( \phi_e(f) \) by \( \phi_e(f)|_{\text{biased}} \), figure 2.7, and performing the convolution 2.20. According to the equation 2.21 the addition of a d.c. term across the spectrum can easily be removed to recover the original spectrum \( \phi(f) \), but in practice convergence of the spectrum with \( W(f) \) is hampered by this d.c. level. If the raw spectral estimate \( P_s(f) \) is considered, its mean is equal to \( \phi_s(f) \) and its variance \( |\phi_s(f)|^2 \), therefore the signal to noise power ratio at any point on the raw spectrum will be equal to unity. If the d.c. level is subtracted from the spectrum and the signal to noise power ratio recalculated then one gets:

(consider unity signal power)

\[
\frac{(\rho^2 \phi(f_1))^2}{(\rho^2 \phi(f_1) + \rho)^2} = \frac{(\text{signal})^2}{\text{variance}} \quad \text{at } f_1
\]

Two main points can be drawn from this statement; first that the overall signal to noise ratio is less than unity and that it is dependent on the spectral signal level, in these two ways the raw spectral estimate differs from an estimate derived from a uniformly sampled stochastic waveform. Spectral smoothing can be used on the raw spectral estimates to improve the accuracy but uneven signal to noise ratios will be present on different
levels of the smoothed spectrum. In figures 2.12 and 2.13 four spectral plots of lowpass waveforms are given showing the effects of various sampling rates and lengths of sampled waveform. In figure 2.12 the signal to noise ratio within the bandlimited signal power seems to improve with increase in waveform length, this is because a degree of smoothing has been applied to these particular spectral estimates.

Summary

It has been found possible to use a randomly timed impulse process to sample a random stochastic waveform and to estimate from the samples a power density spectrum, $P_s(f)$. The difference between the spectrum estimate using the random sampling method and that derived from a uniform sampling scheme is two fold.

a) For a given sampling density the variance of the spectral estimate derived from random sampling is higher than that derived from uniform sampling. This is due to an addition of a "d.c." component to the spectrum.

b) Removal of the "d.c." level renders the S/N ratio of the spectrum incomparable with the normally derived spectral estimates because $[P_s(f) - \rho]$ is no longer distributed as an $\chi^2$ function. With heavy smoothing as $P_s(f)_{smoothed}$ becomes Gaussian distributed one can attribute confidence limits again on this basis.

Random Poisson sampling was used in this chapter but any other random sampling distribution could have been used as long as it is statistically stationary. If that is the case the autocorrelation function may be different from that found for the Poisson case, this would add bias distortion to the spectrum estimate analogous to the convergent window effects.
Power Density Spectrums of Randomly Sampled Bandlimited Waveforms

A spectral average over a section of 2 seconds sampled waveform, mean sampling rate 235 samples/sec.

250 Hz Bandlimited Signal Power

Fig 2.12
Power Density Spectrums of Randomly Sampled Bandlimited Waveforms

A spectral average over a section of 2 seconds sampled waveform. Mean sampling rate 470 samples/sec.

2 seconds
1880 samples/sec.

Frequency (kHz)

0 1.0 2.0 3.0 4.0 5.0
As in the previous section the sampling waveform $e(t)$ has a random form:

$$e(t) = \sum_{n} \delta(t - t_n)$$

where $t_n$ = random time instants

The waveform that is to be sampled is classified as a deterministic one, in that it can be constructed from a finite sum of sinusoids, i.e.

$$s(t) = \sum_{k=1}^{N} a_k \sin(\omega_k t - \phi_k)$$

where:

- $a_k$ = amplitude of sinusoidal components
- $\omega_k$ = frequency of sinusoidal components
- $\phi_k$ = phase angle of sinusoidal components

The amplitude spectrum of the sampled waveform $s(t)$ is:

$$V_s(f) = \frac{1}{T} \int_{-\infty}^{\infty} s(t) e^{-j\omega t} dt$$

as $t \to \infty$

The power spectral function is:

$$U_s(f) = |V_s(f)|^2$$

as $t \to \infty$

No trouble will be found with spectrum convergence since $s(t)$ is deterministic, the spectral estimates are therefore consistent. The relationship between the spectrum functions $V(f)$ and $V_s(f)$ is given by the convolution relation, i.e.

$$V_s(t) = V(f) \star e(f)$$

similarly:

$$U_s(f) = |V(f) \star e(f)|^2$$

(67)
but since \( s(t) \) is a random variable with no correlation with \( e(t) \), the equation can be rewritten as:

\[
U_s(f) = \left| V(f) \right|^2 \ast \left| V_e(f) \right|^2
\]

Hence the complex amplitude spectrum \( V_e(f) \) must be determined.

\[
V_e(f) = \frac{1}{T} \int_{t=0}^{T} e(t) e^{-j\omega t} dt
\]

as \( T \to \infty \)

At \( f = 0 \)

\[
V_e(0) = \frac{1}{T} \int_{t=0}^{T} \sum_{n} \delta(t - t_n) dt
\]

as \( t \to \infty \)

\[
V_e(0) = \rho = \overline{\sigma(t)}
\]

At \( f \neq 0 \), the function becomes:

\[
V_e(f) = \frac{1}{T} \int_{t=0}^{T} \sum_{n} \delta(t - t_n) e^{-j\omega t} dt
\]

as \( t \to \infty \)

This is equivalent to the summation of two sets of uniform samples taken from random phase sinusoids. This can be seen by rewriting the summation in the above formula.

\[
\sum_{n} \delta(t - t_n) e^{-j\omega t} = \sum_{n} \delta(t - t_n)(\cos\omega t_n + jsin\omega t_n)
\]

Which is equivalent to

\[
\sum_{n} \delta(t - n\Delta t)(\cos(\omega t + \phi_{1n}) + jsin(\omega t + \phi_{2n}))
\]

where: \( \Delta t \) denotes an arbitrary time interval

\( \phi_{1n} \) and \( \phi_{2n} \) are random variable phase components

(68)
THE COMPLEX SPECTRUM OF THE SAMPLING WAVEFORM $e(t)$

FIG. 2.14

Amplitude

$e(t)$

DISTRIBUTED AS THE PROBABILITY DENSITY OF A RANDOM PHASE SINUSOID

CONVERGES TO

(69)
Since the mean of uniformly sampled random phase sinusoids is always zero then:

\[
0 = \frac{1}{T} \int_0^T \sum_n \delta(t - n\Delta t) (\cos(\omega t + \phi_n) + j\sin(\omega t + \phi_n))
\]

as \( T \to \infty \) at \( f \neq 0 \)

As shown in figure 2.14 the \( V_e(f) \) complex spectrum is equal to \( e(t) \) at \( f = 0 \) and zero everywhere else, this being the expectation of the function in the limit. Hence \( V_s(f) \), from 2.26, is equal to \( V(f) \) scaled by a factor \( e(t) \).

If now the estimate of the complex spectrum, \( Y_s(f) \), is considered. Then as the available waveform length \( T \) increases the estimate \( Y_s(f) \) will tend to the complex spectrum \( V_s(f) \).

\[
E_p\{Y_s(f)\} = \int_{-\infty}^{\infty} E_p\{Y(f)\}E_p\{Y_e(f - f)\}df
\]

The above equation is based on the assumption that \( Y_e(f - f) \) and \( Y(f) \) are random variables with zero correlation.

The general result of this analysis is that when one is estimating amplitude spectrums of randomly sampled continuous deterministic waveforms the signal to noise ratio is only dependent on the data length, \( T \). No trouble is experienced with d.c. levels on the spectrum estimate.

### 2.5 DIGITAL COMPUTER IMPLEMENTATION OF RANDOM SAMPLING

#### Random Periodic Sampling

All the types of random sampling processes described up to now sample waveforms in a continuous time domain and are describable in a continuous frequency domain. This means that the time sampling can be chosen to occur anywhere in time, and the spectrum is continuous and need not be bandlimited.
Practical implementation of sampling and associated spectral processing is almost entirely, today, carried out on a digital computer, this has immediate repercussions on the time and frequency domain representations since now they must both be represented discretely.

In a purely random sampling process the sampled time data information must be contained in two sequences \( \{s\} \) and \( \{t\} \), the sampled values and the relative times of sampling respectively. To calculate a discrete frequency spectrum from this form of data would require implementation of:

\[
X_k = \sum_{n=1}^{N} s_n e^{-j2\pi f_k t_n}
\]

where: \( X_k \) denotes the complex spectrum at frequency \( f_k \), both \( X_k \) and \( f_k \) are sequence elements of sequences \( \{X\} \) and \( \{f\} \) respectively. \( N \) denotes the number of discrete samples.

The continuous spectrum is estimated at discrete frequencies tabulated by the values in sequence \( \{f\} \). Great care has to be exercised in the estimation of the sequence elements of \( \{f\} \) to ensure that the frequency function is adequately sampled. The implementation of equation 2.27 is by no means simple and can be lengthy in computational time since new complex exponential functions will require evaluation for every different combination of \( f_k \) and \( t_n \). Also the computer storage requirements are two fold for each domain representation. A similar situation could be envisaged if estimation of the autocorrelation or p.s.d. functions were required.

Efficient computation of the transform can be provided by the use of the Fast Fourier Transform algorithm (F.F.T.) but the pure random sampling process is incompatible with its use. This incompatibility exists because the input time sequence to the F.F.T. algorithm must be derived from a
PERIODIC RANDOM SAMPLING

Fig 2.15

TIME (SECONDS)

AMPLITUDE

\( S(t) \)

\( \Delta_{\alpha} \)

RANDOM SAMPLES ON THE TIME

UNIFORM SAMPLING TIME GRID

(72)
uniformly spaced time samples. The use of periodic random sampling can alleviate this problem while retaining some of the beneficial properties of pure random sampling. Here the sampling instants are restricted to lie on a uniform time grid of $\Delta$ seconds spacing but each occurrence on the grid has a constant probability, see figure 2.15. The probability distribution which describes the occurrence of $n$ sampling points on a uniform grid can be looked on as a discrete Binomial distribution. This discrete distribution corresponds to the Poisson distribution in the continuous time domain, and is denoted simply by:

$$\text{Prob}(n, k) = \binom{n}{k} q^n (1 - q)^k$$  \hspace{1cm} (2.28)

where:

- $\text{Prob}(n, k)$ denotes the probability of getting $n$ sampling points spread over $k$ points of the uniform time grid.
- $q$ = Probability of getting a sample at any sampling point on the uniform sampling grid.

Using this type of sampling the input time waveform can be represented as a single sequence $\{s\}$ in which the elements $s_1, s_2, \ldots, s_N$ represent samples taken at every time instant on the uniform time grid, i.e.

$$s_n = \delta(t - n\Delta t) s(t) e_n$$

where: $e_n$ = a binary random variable distributed according to equation 2.28

This type of sampling is compatible with the use of the Fast Fourier Transform (F.F.T.) algorithms which can be used to calculate the Digital Fourier Transform (D.F.T.) i.e.

$$X_k = \sum_{n=0}^{N-1} s_n e^{-jnk2\pi/N}$$  \hspace{1cm} j = \sqrt{-1}$$

$k = 0, 1, 2, \ldots, N - 1$
Due to the basic underlying periodicity of this sampling process, the spectral functions estimated will have aliased versions of the original spectra spaced at uniform intervals of $\pm 1/\Delta g$ Hz across the spectral plane. Therefore it is essential that the grid spacing should be so chosen as to avoid spectral overlap of the original and aliased spectra. Hence:

$$\Delta g \leq \frac{1}{2F_{\text{max}}}$$

The mean random sampling rate on the uniform grid is not affected by this condition.

2.6 SUMMARY

Three basic types of waveform/sampling/processing have been considered; they can be classified in the following table:

<table>
<thead>
<tr>
<th>Type of Waveform</th>
<th>Type of Sampling</th>
<th>Type of Processing</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) Random</td>
<td>Random</td>
<td>Statistical</td>
</tr>
<tr>
<td>2) Random</td>
<td>Random</td>
<td>Deterministic</td>
</tr>
<tr>
<td>3) Deterministic</td>
<td>Random</td>
<td>Deterministic</td>
</tr>
</tbody>
</table>

In all three methods, if enough sampled data is available, the autocorrelation estimates can be formed:

1) $\gamma_m$
2) $R(t), R_e(t)$
3) $\langle e(t) \rangle^2, R(t)$

Provided that a priori statistical knowledge is available about the sampling process, these autocorrelation estimates can be formed into the true autocorrelation, $R(t)$.

Random sampling, although providing freedom in the placement of sampling points, does not gain over uniform sampling from the point of view of total samples necessary for a given accuracy of the spectral estimate. In fact,
in method 3), considerable distortion of the S.N.R. of the spectrum is caused by random sampling. The next chapter goes on to discuss the reason for the addition of a "d.c." level to the spectrum in method 3). An attempt is then made to design an efficient deterministic sampling technique with a sampling rate below $2F_{\text{max}}$ which will avoid the addition of a "d.c." level to the spectrum estimate.
CHAPTER 3
NON-UNIFORM PERIODIC SAMPLING TECHNIQUES

3.1 INTRODUCTION

In the previous chapter it was shown that random stationary waveforms can, in some circumstances, be sampled randomly at below twice the highest frequency content of the waveform, \( s(t) \), without ambiguity occurring in the estimate of the power spectral density function. The disadvantage of random sampling is that a "d.c." level is superimposed on the power spectral density estimate, and in order to resolve the true spectrum this level must be removed. The result is a spectrum with S.N.R. dependent on the level of the spectrum signal.

In this chapter the aim is to design a deterministic sampling pattern which will allow estimation of the power spectral density of a stationary random waveform, \( s(t) \), yet have a mean sampling rate below twice the highest frequency in \( s(t) \). By using a known sampling method with special characteristics it is hoped to avoid the "d.c." level associated with pure random sampling techniques.

3.2 ORIGINS OF THE d.c. LEVEL ON THE POWER DENSITY ESTIMATE DERIVED FROM RANDOM SAMPLING

In this section the intention is to show that the d.c. level associated with the spectrum of randomly sampled stochastic data is mainly due to the way in which the autocorrelation is estimated. It is in fact suggested in this section that the cause of the "d.c." component was twofold. Firstly that a raw autocorrelation function generated from a randomly sampled waveform suffers from lack of convergence; secondly that a smoothed autocorrelation function generated from a random waveform sampled by a pure

(76)
random or a random periodic sampling process may converge, but will be
distorted by bias errors which vary in a random manner as a function of
autocorrelation lag.

3.2.1 RANDOM SAMPLING

It has been shown that the information from a randomly sampled waveform is
contained in the two sequences \{s\} and \{t\}, the autocorrelation estimate
is formed by:

\[
R(\tau_m) = \sum_{n=1}^{N} s_n s_{n+m}
\]

where:

\[
\tau_m = t_n - t_{n+m}
\]

If the sampling scheme is truly random with no periodicity, then it can be
expected that the lag intervals, \(\tau_m\), will cause the lag-product estimates
to form a randomly distributed impulse waveform, as shown in figure 3.1.
Although there are a large number of lag-products randomly distributed over
0 to \(T\), at any particular \(\tau\) value the probability of many lag-products
occurring is low. Hence the statistical convergence at any point on the
unsmoothed or raw autocorrelation estimate would be low. If the auto-
correlation is plotted as the average of lags between say \(\tau\) and \(\tau + \Delta \tau\) this
would help to converge \(R(\tau)\) and is essentially the same as convoluting
with a window function.

\[
R_s(\tau) = \int_{-\infty}^{\infty} R(\tau_m) \cdot W_l(\tau - \tau_m) d\tau_m
\]

where:

\(W_l(\tau_m)\) denotes the convoluting window

The number of lag-products averaged to form each estimate on \(R_s(\tau)\) is a
random variable depending on the number of lag-products lying between \(\tau\)
Autocorrelation function of a randomly sampled random waveform

---

The number of lag-products per autocorrelation estimate

---

The number of lag-products per discrete autocorrelation estimate

---

Discrete lag values \( n \)  
\[ \tau = \nu \Delta q \]
and $\tau + \Delta \tau$. This unknown random quantity will make $R_s(\tau)$ biased. The bias will take the form of a random amplitude modulation of the mean of the estimate $R_s(\tau)$ as a function of $\tau$, figure 3.2. This can in fact be termed a random bias error.

3.2.2 RANDOM PERIODIC SAMPLING

The second type of sampling, random periodic, estimates the autocorrelation function directly into a sequence $\{r\}$ which represents estimates at uniform spacing along the $\tau$ axis, i.e.

$$r_m = \sum_{n=1}^{N} s_n s_{n+m}$$

where:

$$R(m\Delta g) = r_m$$

$$N = T/\Delta g$$

Due to the fact that all the time sampling points lie on a uniformly spaced time grid, $\Delta g$ spacing, the autocorrelation lag-products will also lie on a uniform grid, figure 3.3. Since all the lag-products are concentrated at lag values spaced at $\Delta g$ intervals the autocorrelation estimate will not suffer from lack of convergence. The estimate will in fact suffer from random bias error due to the random variation of the number of lag-products at each interval.

In figures 3.4 and 3.5 an example is shown of an autocorrelation of random periodically sampled sinusoid. Figure 3.5 shows the number of lag-products per lag and correspondingly figure 3.4 shows the actual calculated autocorrelation. The random variations in figure 3.4 can be compared with figure 3.5 which indicates that a great proportion of the noise associated
A section of an autocorrelation function calculated from a randomly sampled sinusoid.

The autocorrelation function of the random sampling waveform used to determine the above autocorrelation.

LAG INTERVAL: 1 INTERVAL = 0.2 ms
with the sinusoid is due to the random nature of the number of lag-products per lag interval rather than the statistical convergences of the data.

The general conclusion that can be drawn in both the cases of purely random and random periodic sampling is that additional random variation of $R(\tau)$ is caused by random biasing. This additional noise is then transferred in to the d.c. level imposed on the spectral estimate. Two conditions which must be fulfilled to avoid the d.c. level can now be stated:

a) $R(\tau)$ should be adequately estimated along the $\tau$ axis.

b) The number of estimates at every lag interval should be equal if possible or at least known so that compensation can be made by applying a window function so as to make $R_5(\tau)$ unbiased.

In the next sections sampling patterns are designed which have sampling rates below $2F_{max}$ but possess ideal sampling properties for autocorrelation estimation.

3.3 A NEW APPROACH TO AUTOCORRELATION ESTIMATION OF STATIONARY RANDOM WAVEFORMS

3.3.1 INTRODUCTION

A random periodic sampling process based on a uniformly spaced time grid, spacing $\Delta g$, with $N$ sampling points can at maximum generate $M$ lag-products.

$$M = \sum_{I=1}^{N} (N-I) = \frac{(N-1)(N-2)}{2}$$  \hspace{1cm} 3.1

If it were possible to ensure deterministically that each of these $M$ lag-products were estimated at a different lag value and that the lag values ranged through $\Delta g$, $2\Delta g$, $3\Delta g$, ... $M\Delta g$, then an autocorrelation function could be estimated at uniformly spaced lag values from 0 to $M\Delta g$ using this
sampling process. To fulfil this criterion only \( N \) sampling points per time interval \( M \) need be taken. Since \( M \) is necessarily greater than \( N \), the autocorrelation can be adequately estimated at below the Nyquist rate. It is necessary to design such a sampling waveform to exhibit these properties, so really the sampling waveform cannot be classified as random, therefore the sampling waveforms to be discussed will be termed non-uniform periodic.

3.3.2 SAMPLING PATTERN DESIGN PHILOSOPHY

The general expression for the estimation of an unbiased autocorrelation function at discrete points \( \tau_n \) from an arbitrarily sampled time waveform is:

\[
R(\tau_n, \tau_m) = \frac{1}{N_u(\tau_m)} \sum_{n=1}^{N} s(t_n) s(t_n + \tau_m)
\] 3.2

where:

\[ N_u(\tau_m) \] denotes the number of lag-product estimates at lag \( \tau_m \)

If the waveform \( s(t) \) is stationary then \( \Omega(t, \tau_m) \); the true autocorrelation of \( s(t) \), is no longer a function of \( t_n \) in that it is irrelevant from which absolute time point, \( t_n \), the lags, \( \tau_m \), are measured.

\[
\Omega(t, \tau_m) = E_p\{R(t_n, \tau_m)\}
\]

Stationarity implies that:

\[
\Omega(\tau_m) = \Omega(t, \tau_m)
\]

It is also possible to extend this stationarity to the estimate function \( R(t_n, \tau_m) \) with some reservations.

\[
\Omega(\tau_m) = E_p\{R(\tau_m)\} \quad 3.3
\]

[Subject to conditions]

Due to the stationarity the position of the lag values on the time axis is arbitrary, it is not necessary that each lag-product be contiguous as in the uniform sampling method, figure 3.6. Consider figure 3.7a), here lag-
Fig. 3.6

Uniformly Sampled Time Waveform —
— Contiguous Lag-Product Estimates

Sampling Functions

Fig. 3.7

a)

b)

c)

d)

Time (Seconds)

(83)
products for the estimation of lag $\tau_1$ are distributed randomly, this has
the effect of reducing the mean sampling rate below that in figure 3.6.
The conditions behind the statement of equation 3.3 are in fact restrictions
on the placing of the lag-products in order that the estimate $R(\tau_m)$
converges to $\Omega(\tau_m)$. The conditions are:

1) Even though the waveform $s(t)$ is stationary the lag-products should
take representative samples of lag-products from the available informa-
tion. As an example, figure 3.8 shows how it is possible to compress
all the lag-product values into a small portion of the available wave-
form. In statistical terms in order that 3.3 is true, the probability
of the components of a lag-product, $\tau_1$, lying anywhere on the time axis
should be uniform, this ensures a truly representative estimate of $\Omega(\tau_m)$.

The condition of stationarity of the sampling process is to be some-
what relaxed in this chapter, but uniformity of distribution of lag-
products across $T$ is to be upheld.

2) If the data samples are closer than $1/2F_{max}$, then they will be
statistically dependent. Consequently the convergence of the estimate
$R(\tau)$ per sample taken will be limited if they are closer than $1/2F_{max}$
seconds.

Figure 3.7a shows a possible sampling scheme for the estimation $R(\tau)$ at lag
$\tau_1$, it is also necessary to generate other lag-products at spacings $\tau_2$, $\tau_3$,
$\tau_4$, ... $\tau_m$ to completely specify the autocorrelation estimate. It could be
argued that other sampling schemes should be generated with $\tau_2$, $\tau_3$, ... specifically designed lags. This would mean applying sampling processes to
the time waveform $s(t)$ M times as in figure 3.7. This might be
possible if the data is a repeatable process such as a magnetic
tape playback for instance. A much more efficient sampling method.
NONREPRESENTATIVE SAMPLING OF A WAVEFORM SECTION

Fig. 3.8

AN EXAMPLE OF AN IDEAL SAMPLING PATTERN

Fig. 3.9

A PERIODIC SAMPLING PATTERN

Fig. 3.10

(85)
would result from the combination of the sampling processes into one sampling pattern which would be capable of allowing construction of \( \tau_1, \tau_2, \ldots, \tau_m \) lag-products simultaneously. Combination of the sampling patterns for various \( \tau_m \) values can be simply an addition process, figure 3.7d. This type of combination results in a very wasteful way of estimating \( R(\tau) \). In the example given in figure 3.9, four lag-products for each of three points on the \( R(\tau) \) function are estimated for the use of twenty one time sampling points. Theoretically from twenty one sampling points one hundred lag-products could be estimated if the lags between every sample point were used.

Consider now figure 3.9, here four sampling points are shown and from this single pattern six different lag values can be generated. In this particular example the lags range from \( \Delta g \) to \( 6\Delta g \). It is simple enough to generate small length sampling patterns, as figure 3.9, which generate different lag values, but as the waveform length becomes longer it becomes increasingly difficult to construct by trial and error. Some theory is needed. Even with the help of a theoretical construction method, to generate patterns greater than \( 100\Delta g \) in length is impractical. The simple solution is to design a small length sampling pattern which exhibits the lag generation property required, then construct an infinite length sampling waveform by repeating the simple pattern. In figure 3.10, the pattern in figure 3.9 is shown as a repetitive one. Designing a periodic sampling process involves designing not only the lags generated within one period but also the lags that can be generated between periods. In figure 3.10 an additional \( \tau_3 \) is shown to be generated by making the pattern in figure 3.9 periodic.
The next topic to be discussed will be the theoretical design of a non-uniform periodic sampling process which exhibits the property of generating lag intervals from zero to a defined limit without omission while having the lowest possible mean sampling rate.

3.3.3 USING CYCLIC DIFFERENCE SETS IN SAMPLING PATTERN DESIGN

The theoretical basis for the design of non-uniform periodic sampling patterns was supplied by a standard mathematical number sequence found in the purely theoretical area of number theory. The practical problem of generating a sampling pattern with a minimum number of sampling points yet which will allow discrete delays of $\Delta g$ to be found between the sampling points, ranging from $\Delta g$ to $M\Delta g$ without omission can be interpreted as constructing an integer number sequence. Let the points on a uniform time grid, of spacing $\Delta g$ seconds, be thought of as elements of a positive integer number field $F, f_1, f_2, \ldots, f_v$ ranging from 1 to $v$. A particular sequence of positive integer numbers, \{a\}$\text{MOD}(v)$, is selected.

\[
\{a\} = \{a_1, a_2, \ldots, a_K\} \text{MOD}(v)
\]

Sequence \{a\} has the property of being able to generate all positive integer numbers of the field $F$ by differencing its constituent elements:

\[
|a_i - a_j| = dij\text{MOD}(v)
\]

\[i > j\]

\[i = 1, 2, \ldots, K\]

\[j = 1, 2, \ldots, K - 1\]
As an example \( \{c\} = 0, 1, 3 \) with \( v = 7 \).

\[
\begin{array}{ccc}
  a_i - a_j & d_{ij} \mod(v) \\
  1 - 0 & 1 \text{ or } 8 \text{ or } 15 \\
  3 - 1 & 2 \text{ or } 9 \text{ or } 16 \\
  3 - 0 & 3 \text{ or } 10 \text{ or } 17 \\
  0 - 3 & 4 \text{ or } 11 \text{ or } 18 \\
  1 - 2 & 5 \text{ or } 12 \text{ or } 19 \\
  0 - 1 & 6 \text{ or } 13 \text{ or } 20 \\
\end{array}
\]

Sequences such as \( \{c\} \) are termed cyclic difference sets or perfect difference sets, see references (29) to (32). One can see here the possible analogy between the differencing process generating \( d_{ij} \) and the formation of lag intervals in the autocorrelation process. The resulting sequence of differences \( \{d\} \) includes all integers in the field \( F \) without any deletion, with every integer being generated in \( \{d\} \lambda \) times. In the example above \( \lambda = 1 \) but it is possible to select \( \{c\} \) to generate each integer element more than once. Not only is it possible to generate \( \{d\} \) up to \( v \) but this can be extended to infinity by simply using the modulo notation as shown in the above example. A cyclic difference set is expressed in a notation form as \( (v, K, \lambda) \) incomplete cyclic block design. Where \( v \) denotes the magnitude of the number field on which the sequence is based, \( K \) represents the number of elements in the \( \{a\} \) sequence and \( \lambda \) is the number of times each difference is generated. A more comprehensive explanation of block/set theory and difference sets is given in appendix 1 along with references of tabulated difference sets.

It is easier to visualise the use of the cyclic difference sets if a periodic infinite sequence is made from the sequence \( \{c\} \) thus removing the
module complexity. Using the previous example, 0, 1, 3 MOD(7), one gets :

\[ \{a\} = 0, 1, 3, 7, 8, 10, 14, 15, 17, \ldots \]

Cyclic difference sets can be used to specify the positions of sampling points on a uniform time grid so that the resultant sampling waveform is ideal for autocorrelation estimation sampling. Given a \((v, K, \lambda)\) cyclic difference set it can be converted to a sampling waveform by specifying

\[ e(t) = \sum_n \delta(t - a_n \Delta g) \]

In this way a periodic impulse sampling waveform is built up on a uniform time grid, the period of the waveform will be \(T_r\).

\[ T_r = v \Delta g \]

The spacing of the time grid, \(\Delta g\), is specified as fulfilling the Nyquist criterion.

\[ \frac{1}{\Delta g} \geq 2F_{\text{max}} \]

As an example consider a planar sequence \((13, 4, 1)\), defined planar because \(\lambda = 1\)

\[ \{c\} = 0, 1, 3, 9 \text{ MOD(13)} \]

This gives a sampling sequence \(\{e\}\) of periodicity 13 :

\[ \{e\} = \{\ldots 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, \ldots \} \]

Which when translated to a sampling waveform, \(e(t)\), is shown in figure 3.11. If a periodic autocorrelation of \(\{e\}\) is calculated

\[ R_e(\tau_m) = \sum_{n=1}^{v} e_n e_{n+m} \]

Then the resultant \(R_e(\tau_m)\) function, figure 3.12, shows that every lag-product...
**Fig. 3.11**

**Periodic Nonuniform Sampling Waveform** \( e(t) \)

**Planar Set** \((13, 4, 1)\)

**Fig. 3.12**

**Autocorrelation of** \( e(t) \)
Difference Sets

Fig. 3.13

\[
\begin{align*}
150 & \quad x(156,31,6) \\
100 & \quad x(133,12,1) \\
75 & \quad x(91,10,1) \\
50 & \quad x(73,9,1) \\
25 & \quad x(57,8,1) \\
0 & \quad x(31,6,1) \\
& \quad x(21,5,1) \\
& \quad x(13,4,1) \\
& \quad x(7,3,1) \\
\end{align*}
\]
is estimated from 0 to \( v - 1 \) for one period, hence 0 to infinity for an infinitely long periodic non-uniform waveform, constructed by repeating \( \{e\} \).

In addition completely uniform estimation is given between 0 and \( v \) of height \( \lambda \) where \( \lambda = 1 \). Therefore if \( e(t) \) was used to sample a stationary random waveform \( s(t) \) then it would be possible to construct the autocorrelation estimate \( R(\tau) \) at every interval of \( \Delta g \) along the \( \tau \) axis. This would give \( K \) estimate log-products at \( 0, T_\tau, 2T_\tau, \ldots \) lag spacings and elsewhere \( \lambda \) lag-products, all relative to a length \( T_\tau \) seconds of sampled waveform.

A planar periodic non-uniform sampling scheme based on a Nyquist spaced grid, \( \Delta g = \Delta n \), represents the lowest possible mean sampling rate which can be used without causing ambiguity in the power spectrum estimate. The reasoning for this is that only one estimate per period \( T_\tau \) is made at every lag value on the autocorrelation function when using a planar sampling pattern, any reduction in the number of sampling points per period \( T_\tau \) will result in a lack of estimation at specific lag values on \( R(\tau) \). If \( R(\tau) \) is not estimated at a mean density spacing of \( \Delta n \) then aliasing will occur in the power spectrum function \( P(f) \).

Other cyclic difference sets exist other than planar with \( \lambda \) values of 2, 3, 4, \ldots all these will require higher time sampling rates than planar but give a quicker estimation of \( R(\tau) \) in terms of waveform length sampled. See figure 3.13 for available cyclic difference sets.

Estimation of the Power Spectral Density Function by Use of Cyclic Difference Sets

Periodic non-uniform sampling is ideal for use with digital computational algorithms since it is based on a uniform time grid. The sampled data can
be placed into an array form as an example for the (13, 4, 1) planar set:
0, 1, 3, 9 MOD(13)

... 0, s_1, s_2, 0, s_4, 0, 0, 0, 0, s_10, 0, 0, 0, s_14, s_15, ...

Where s_n denotes the sampled data values as elements of sequence \{s\}.

To derive the p.s.d. estimate for s(t) the calculation of the lag-products for the autocorrelation estimate could be attempted but it is much quicker to go direct to the p.s.d. estimate via the Digital Fourier Transform.

\[
P(f_k) = \frac{\Delta R}{N} \left| \sum_{n=0}^{N-1} s_n e^{-j2\pi kn/N} \right|^2, f_k = \frac{k}{T} \quad k = 0, 1, 2, \ldots, N-1
\]

The estimate \(P(f_k)\) will be statistically inconsistent, since \(s(t)\) is a random process, yet before the application of window functions it is necessary to consider the form of \(R(\tau)\).

\[
R(\tau_m) = \frac{1}{N} \sum_{n=1}^{N} s_n \cdot s_{n+m}
\]

where:

\[
\tau_m = m \Delta g \quad m = 0, 1, 2, \ldots, N-1
\]

Use of computational routines to calculate first equation 3.4 then equation 3.5 will mean that the estimate \(R(\tau_m)\) is a periodic autocorrelation. In the normal case of sampling equation 3.5 would represent an unbiased estimate for a periodic autocorrelation but since non-uniform sampling has been used to construct \(R(\tau_m)\) the function is in fact biased. To rectify this, division of \(R(\tau_m)\) by \(Nu(m)\) is necessary.

Unbiased estimate:

\[
R(\tau_m) = \frac{1}{Nu(m)} \sum_{n=1}^{N} s_n \cdot s_{n+m}
\]

where:

\(Nu(m)\) defines the number of estimates per lag (m)
P.S.D. Estimates using Periodic Nonuniform Sampling

Fig 3.14

Sampled waveform length

1 second

4 seconds

8 seconds

Relative Amplitude

Spectral Power

Frequency KHz
P.S.D. Estimates using Periodic Nonuniform Sampling

Sampled waveform length
- 2 seconds
- 8 seconds
- 20 seconds

Relative Amplitude
- 0.0
- 1.0
- 2.0
- 4.0
- 5.0

Frequency KHz
- 0.0
- 2.5
- 5.0
The function \( N_u(m) \) can be quickly calculated.

\[
N_u(m) = \sum_{n=1}^{N} e_n \cdot e_{n+m}
\]

where \( e_n \) are binary sequence elements of \( \{e\} \) which represent the sampling waveform in discrete form, e.g. \((13, 4, 1)\) planar set.

\[
\{e\} = (1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, \ldots)
\]

Effectively \( \{N_u\} \) represents the autocorrelation of the sampling waveform which is shown in figure 3.12. Once the autocorrelation function has been modified by the division by \( \{N_m\} \), additional window functions can be applied to help statistical convergence in the power spectrum domain, as with usual spectrum techniques. The very fact that additional processing of \( R(\tau_m) \) is performed after its estimation, complicates the calculation of the confidence limits for the final power density estimate. The final sections of this chapter go on to consider the statistical structure of the p.s.d. estimate calculated from non-uniform periodic sampling. In figures 3.14 and 3.15 two examples of p.s.d. estimations are given, each showing the effects of increased data on the accuracy of the estimate. The two spectral functions are based on lowpass and bandpass filtered white noise, 3.14 and 3.15 respectively. In each case the non-uniform periodic sampling waveform was based on the \((37, 9, 2)\) difference set, giving a mean sampling rate approximately four times below the Nyquist rate.

3.4 CONSEQUENCES OF THE USE OF NON-UNIFORM PERIODIC SAMPLING ON THE STATISTICAL ACCURACY OF THE P.S.D. ESTIMATE

Even though power spectral density estimates derived from non-uniform periodic and uniform sampling processes will both converge to the true p.s.d., there are statistical differences between the two processes. This section outlines these fundamental differences.
3.4.1 MEAN LEVEL ESTIMATES

The mean value of the p.s.d. derived from non-uniform sampling is subject to a constant scaling factor as compared to the p.s.d. estimate derived from a uniform sampling method. All the associated mathematics for this section are presented in detail in appendix (2). If as a general case the spectrum of white noise is calculated at the harmonics of the waveform length $T$, we get the usual formula:

$$P(f_k) = \frac{\Delta g}{N} [A^2(f_k) + B^2(f_k)]$$  \hspace{1cm} (3.8)

where:

- $P(f_k)$ is the p.s.d. estimate at harmonic frequencies $0, 1/T, 2/T, \ldots, k/T, \ldots, 1/\Delta g \text{ Hz}$
- $A(f_k)$ and $B(f_k)$ are the usually defined Fourier harmonic coefficients:
  $$A(f_k) = \sum_{n=1}^{N} s(t_n) \cos \frac{2\pi f_k n}{N}$$
  $$B(f_k) = \sum_{n=1}^{N} s(t_n) \sin \frac{2\pi f_k n}{N}$$

Uniform sampling where $s(t)$ is defined at every point on the uniform time grid gives the p.s.d. estimate of white noise as being, in the limit, equal to the variance of the waveform:

$$\phi(f_k) = \Delta g \sigma_s^2$$

for all $k$ as $N \to \infty$

where:

- $\sigma_s^2$ is the variance of the white noise waveform

If this is compared with the result when non-uniform periodic sampling is used:

$(97)$
\[
\phi(f_k) = \frac{N}{N} \Delta g \sigma_s^2
\]

as \(N \to \infty\) for all \(k\)

where:

\(N\) denotes the number of actual samples taken on the \(N\) point uniform time grid.

\[N = \binom{K}{V} N\]

Hence:

\[
\phi(f_k) = \frac{K}{V} \Delta g \sigma_s^2
\]

### 3.4.2 VARIANCE OF ESTIMATES

The variance of the spectral estimates of uniform and non-uniform sampling methods must next be compared. In the uniformly sampled case each p.s.d. harmonic component is distributed as \(\chi^2_2\), the two separate random variable components \(A^2(f_k)\) and \(B^2(f_k)\) each being \(\chi^2_1\) distributions with mean values of:

\[
A^2(f_k) = \sigma_s^2 \frac{N}{2} \quad \text{at harmonics:} \quad k = \pm 1, \pm 2, \ldots, \pm \frac{N-1}{2}
\]

\[
B^2(f_k) = \sigma_s^2 \frac{N}{2}
\]

The corresponding variance of \(P(f_k)\) is:

\[
\text{Var}[P(f_k)] = \sigma^4 \Delta g^2
\]

which is independent of the number of samples taken, hence inconsistent.

In the case of the non-uniformly sampled waveform, in appendix (2) it is shown that due to the variation of the mean values of \(A^2(f_k)\) and \(B^2(f_k)\) with frequency, the variance of \(P(f_k)\) will also be frequency dependent.
\[
\text{Var}[P_s(f_k)] = \Delta g^2 \sigma_s^4 \left( \frac{K}{v} \right)^2 (1 + v^2(f_k))
\]

3.11

at harmonics: \( f_k = \pm \frac{j}{2T_r} \)

\( j = 1, 2, 3, \ldots \)

\[
\text{Var}[P_s(f_k)] = \Delta g^2 \sigma_s^4 \left( \frac{K}{v} \right)^2
\]

3.12

at harmonics: \( f_k \neq \pm \frac{j}{2T_r} \)

The function \( v(f_k) \) can be calculated for a particular non-uniform sampling waveform \( e(t) \), equation 3.13:

\[
v(f_k) = \frac{1}{2} \sum_{n=1}^{N} e(t_n) \cos 2\omega t_n
\]

3.13

It is shown in the appendix (2) that it is possible to eliminate the dependence of p.s.d.'s variance on frequency, simply by the modification of the real and imaginary squared components of \( P_s(f) \), i.e.

\[
P_{sm}(f_k) = \left( \frac{\Delta g}{N} \right) \left[ \frac{A^2_s(f_k)}{(1+v)} + \frac{B^2_s(f_k)}{(1-v)} \right]
\]

which gives:

\[
\text{Var}[P_{sm}(f_k)] = \Delta g^2 \sigma_s^4 \left( \frac{K}{v} \right)^2 \text{ for all } f_k
\]

3.4.3 COVARIANCE BETWEEN SPECTRAL ESTIMATES

In the case of uniform sampling, the spectrum estimates at harmonic frequencies \( f_k \) are always linearly independent, i.e.

\[
\text{Cov}(P(f_i), P(f_j)) = \delta_{ij}
\]

\( i, j = 0, \pm 1, \pm 2, \ldots \)
where:

\[ \text{Cov}(A, B) \] indicates the statistical covariance between two estimates \( A \) and \( B \).

It has already been mentioned that non-uniform sampling produces a frequency dependent variance in the p.s.d. estimate but in addition this type of sampling produces non-zero covariance between spectral estimates. The mathematical analysis of this is given in the appendix (2) as applied to a white noise spectrum. The covariance between spectral harmonic components calculated from the non-uniform sampling method is expressed by:

\[
\text{Cov} \left[ P_s(\omega_i), P_s(\omega_j) \right] = \sum_{k=0}^{N-1} \text{Var}[P(\omega_k)] P_e(\omega_i - \omega_k) P_e(\omega_j - \omega_k) \quad 3.14
\]

where

\[ \text{Var}[P(\omega_k)] \] denotes the variance of the raw p.s.d. estimate derived from a uniformly sampled time waveform.

\[ P_e(\omega_k) \] denotes the p.s.d. of the sampling waveform \( e(t) \).

In this particular application where periodic sampling impulse functions are to be used, then \( P_e(\omega_k) \) takes on the form:

\[
P_e(\omega_k) = D_k \delta(\omega - k/T_r) \quad 3.15
\]

where:

\[ D_k \] denotes a real constant

The periodic impulse form is due to the periodicity of \( e(t) \) in the time domain with period of \( T_r \) seconds. It follows from equation 3.14 and 3.15 that the covariance will only be non-zero at spacings between the spectral harmonics of \( \omega_i \) and \( \omega_j \) at integer multiples of \( 1/T_r \) Hz. The covariance between spectral harmonics can be expressed in the usual covariance matrix form, i.e.
\begin{equation}
\begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1n} \\
C_{21} & C_{22} & & \\
& \ddots & C_{33} & \\
& & \ddots & \\
C_{n1} & \cdots & \cdots & C_{nn}
\end{pmatrix}
\end{equation}

where:

- element \( C_{ij} \) signifies the covariance between the \( i \)th and the \( j \)th harmonic

- when \( i = j \), \( C_{ii} \) signifies the variance of the estimate

As an example, if a time waveform length of \( 3T \) had been analysed the corresponding covariance matrix for spectral harmonics spaced at \( 1/3T \) Hz intervals would be:

\begin{equation}
\begin{pmatrix}
C_{00} & 0 & 0 & C_{03} & 0 & 0 & C_{06} & \cdots & C_{0,N-1} \\
0 & C_{11} & 0 & 0 & & & & & \\
0 & 0 & C_{22} & 0 & & & & & \\
C_{30} & 0 & 0 & C_{33} & & & & & \\
0 & \cdots & \cdots & C_{44} & & & & & \\
0 & \cdots & \cdots & \cdots & \cdots & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
C_{N-1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & C_{N-1,N-1}
\end{pmatrix}
\end{equation}

(101)
3.4.4 POWER SPECTRUM CONFIDENCE LIMITS

Biased, Pre-processed Spectrum Estimate

The confidence limits of a power spectrum derived from uniform sampling are easily determined from standard tables, reference (8). The difficulties which arise with non-uniform sampling are due firstly to the correlation between harmonic spectral estimates, secondly after estimation of the spectral function additional autocorrelation modification is necessary to produce the unbiased power density estimate. These two factors tend to complicate the construction of confidence limits for the final spectrum.

In the usual approach each estimate of say a N point spectrum is assumed statistically independent. This allows the reduction of the variance of each of the N point raw spectral estimates to be totally described in terms of the type of smoothing window used on the spectrum and also the available data length, T. Thus if the degrees of freedom of a spectral estimate plus its distribution form, $\chi^2$, using uniform sampling methods, are known then confidence limits can be prescribed for the spectral estimate.

The situation could be somewhat different for periodic non-uniform sampling in which case interdependence between spectral estimates exist as expressed by the equation 3.14. Consequently the number of degrees of freedom of spectral estimates are not only dependent on the shape of the smoothing function used, but also on the proportion of covariance within the raw spectrum. Luckily the situation is somewhat simplified if the following simple assumption is made. If the number of periods, $T_r$, of the periodic non-uniform sampling scheme included in the total length T is large, then the number of degrees of freedom of the spectrum can be calculated as in normal spectrum analysis. This can be assumed because in the last section

(102)
Fig. 3.16

A TYPICAL POWER DENSITY ESTIMATE $P(x)$

SPECTRAL CONVERGENCE WINDOW

INTERDEPENDENCE INTERVAL

$0 \leq \frac{1}{T} \ll \frac{1}{T_f}$

FREQUENCY $Hz$

POWER DENSITY

(103)
CONVOLUTION OF SPECTRAL FORMS $P_3(f_a)$ AND THE DIGITAL FOURIER TRANSFORM OF THE WINDOW FUNCTION $\left[ \frac{1}{N_{u(m)}} \right]$

Fig. 3.17
it was shown that covariance is only non-zero between harmonics spaced at \(1/T_R\) Hz, therefore if the effective spectral bandwidth of the smoothing window is less than \(1/T_R\) Hz then no summation of inter-related spectral estimates will occur, see figure 3.16. Hence the number of degrees of freedom per smoothed harmonic component in the spectrum estimate is only related to the window shape as in normal spectral estimation.

**Unbiased, Post-processed Spectral Estimates**

It is now required to calculate the confidence limits for the power spectrum estimate after the autocorrelation function \(R_s(\tau_m)\) has been multiplied by the window function \(1/\text{Nu}(m)\) so as to make it an unbiased estimate, as previously shown:

\[
R_s(\tau_m) = \frac{1}{\text{Nu}(m)} \sum_{n=1}^{N} s_n \cdot s_{n+m} \quad 3.6 \text{ Repeat}
\]

Since the window function \(1/\text{Nu}(m)\) is periodic with period \(T_R\) seconds, its spectral function will be a line structure type with harmonics at \(1/T_R\) Hz intervals \(A, B, C, D, \ldots\). So if the effects of the autocorrelation multiplication is viewed as a convolution of \(S_s(f_k)\) with the spectrum of \(1/\text{Nu}(m)\) then only harmonics of \(P_s(f_k)\) at \(1/T_R\) Hz intervals are summated together by the convolution process. To help clarify the problem, consider the effect of the convolution process at any arbitrary points in the convolution shift, see figure 3.17. Let four components from \(P_s(f_k)\) spaced at \(1/T_R\) Hz intervals be:

\[a, b, c, d\]

Each component is distributed as \(\chi^2_2\). In addition there is a covariance between each component, this being described by a covariance matrix \(C\):
The spectral processing as applied to spectrum components \( a, b, c, d \) can be viewed as a convolution process, the resultant spectrum with components \( w, x, y, z \) is formed.

\[
\begin{array}{cccc}
C_{aa} & C_{ab} & C_{ac} & C_{ad} \\
C_{ba} & C_{bb} & C_{bc} & C_{bd} \\
C_{ca} & C_{cb} & C_{cc} & C_{cd} \\
C_{da} & C_{db} & C_{dc} & C_{dd}
\end{array}
\]

\[ w = Aa + Eb + Cc + Dd \]
\[ x = Ea + Ab + Bc + Cd \]
\[ y = Fa + Eb + Ac + Dd \]
\[ z = Ga + Fb + Ec + Ad \]

where

\( A, B, C, D, E, \ldots \) denote the window function which is convolved with the power spectrum (D.F.T. of \( 1/\text{Nu(m)} \)).

To calculate the confidence limits for the spectral components \( w, x, y, z \), it is necessary to know their statistical parameters. The summation for each component \( w, x, y, z \) is not just an addition of \( \chi_2^2 \) random variables but a summation of statistically related variables. Due to this interrelation it is difficult to specify the confidence limits as precisely as in normal spectrum analysis because the statistical distributions of \( w, x, y, z \) are not known.

There are two possible ways this can be resolved:

a) Considering the Chebyshev inequality theorem based on the variance of the spectrum estimates, \( w, x, y, z \).

b) Modification of the confidence limits calculated for the spectrum
estimates $a$, $b$, $c$, $d$ so as to conform to the new spectrum $w$, $x$, $y$, $z$.

The first method is difficult to implement in practice since the variance of the individual components has to be estimated experimentally

$$\text{Est} \left[ \text{Var} \left[ P(f_k) \right] \right] = \frac{1}{N-1} \sum_{n=1}^{N} \left( P_n(f_k) - \bar{P}(f_k) \right)^2$$

where:

$P_n(f_k)$ denotes the power spectrum of the $n$th ensemble.

This requires $N$ ensemble waveform lengths and associated power spectrum estimates. Once an estimate of the variance of the spectrum is obtained the standard deviation can be found, $\sigma$. Evoking the Chebyshev theorem:

$$\text{Prob} \left( |P(f_k) - \bar{P}(f_k)| < m \right) > 1 - \frac{1}{m^2}$$

Hence for a 90% confidence level the limits will be $\pm \sqrt{10} \sigma$

for a 99% confidence level the limits will be $\pm 10 \sigma$

These limits in general will be much wider than the corresponding $\chi^2$ spectral estimates, reference (1).

The second method of determining confidence limits relies on their calculation from $\chi^2$ distributions for the smoothed but as yet unconvoluted spectrum components $a$, $b$, $c$, $d$. Considering one element, say $y$, this is made up of a summation, in the convolution, of the components:

$$y = Fa + Eb + Ac + Bd$$

If the confidence limits for $a$, $b$, $c$, $d$ are known to be $\text{CL}_a$, $\text{CL}_b$, $\text{CL}_c$, and $\text{CL}_d$, then the confidence limit for $y$ can at worst, when $a$, $b$, $c$, ... are totally correlated, be:

$$\text{CL}_y = |F| \cdot \text{CL}_a + |E| \cdot \text{CL}_b + |A| \cdot \text{CL}_c + |B| \cdot \text{CL}_d$$

(107)
Consequently if CL_a, CL_b, ..., are all equal to CL then:

$$CL_y = CL(|F| + |E| + |A| + |B|)$$

A general equation can be given for a periodic spectrum with K components:

$$CL_M = CL_S \sum_{k=0}^{N-1} |\beta_k|$$

where:

- CL_M = confidence limit of convoluted smooth spectrum
- CL_S = confidence limit of unconvoluted smooth spectrum
- $\beta_k$ = N component convoluting periodic spectrum

**Example**

As an example the relative increase in confidence limits is considered for a periodic, non-uniform sampling set of (37, 9, 2).

$$\beta_0 = (1 - \frac{2}{37 \times 9}) \quad k = 0$$

$$\beta_k = -\frac{2}{37 \times 9} \quad k = 1 \text{ to } 36$$

Since $\beta$ is zero everywhere except at intervals of $1/T_x$ Hz, no matter how many components N there are in the spectrum only 37 elements exist in the summation of the circular convolution, hence the confidence limit.

$$CL_M = CL_S \sum_{k=0}^{36} |\beta_k|$$

Substituting:

$$CL_M = CL_S 1.102$$

Hence the confidence limits have increased by processing.
3.4.5 SUMMARY

The resultant picture which emerges from this chapter is that non-uniform periodic sampling in the form of a cyclic difference set design will allow a true unbiased estimate to be made of the autocorrelation function of the sampled waveform. In the case of a planar type set being used, this represented the minimum sampling rate possible for a given $F_{\text{max}}$ value.

If one compares the estimated p.s.d. function derived from a uniform and non-uniform sampling method, after adjustments by a simple scaling factor of $\frac{\nu}{K}$ the expected value of estimates from both methods are identical. The difference in the estimates derived by the two methods lies in the statistical nature of the functions. On a general basis the variance from the lower sampling rate method is $(\frac{\nu}{K})^2$ higher than the uniform method, this is to be expected. In addition, due to the fact that the non-uniform sampling method is periodic the variance of the raw p.s.d. is modulated by a factor of $(1 + \nu^2)$ at frequencies of $\pm j/2T_r$, $j = 1, 2, \ldots$, $T_r/\Delta g$, it is shown that this effect can be avoided by judicious modification of the component elements $A_2^2(f_k)$ and $B_2^2(f_k)$. It has also been found that there exists a degree of covariance between the raw spectral estimates, this will affect the smoothing properties of the window functions as applied to the raw power density spectrum. If the precautions, as mentioned in section 3.4.3, about the waveform length are heeded, then no trouble need arise with the calculation of confidence limits for the p.s.d. estimate.
4.1 INTRODUCTION

Interpolation of sampled signals to recreate the original continuous form of signal, is a well documented area of research reaching back to before the quantitative statements of the Nyquist theory as applicable to lowpass signals. Nyquist stated that it was possible to reconstruct unambiguously a lowpass signal limited to \( F_{\text{max}} \) from samples taken at a mean rate of \( 2F_{\text{max}} \) or above.

It has been accepted for some time that in fact the statement of the interpolation/sampling theory for lowpass waveforms is but a particular specialised section of a much wider theory of sampling. It is, in fact, only necessary to acquire samples of an information bearing signal at a minimum mean frequency equal to the spectral width which contains signal power in order to reconstruct without ambiguity. In the case of a lowpass signal limited to \( F_{\text{max}} \) the frequency space occupied by the signal power is \( 2F_{\text{max}} \) counting positive and negative frequencies separately. Similarly, with a signal bandlimited, \( f_0 \) to \( f_0 + F \), Hz then the minimum sampling rate is \( 2F \) Hz. One can generalise this with a signal which has a spectrum with many isolated spectral bands; this will be termed a Multiple Bandlimited signal. The minimum theoretical sampling rate for this type of signal is equal to \( F_{\text{occ}} \), the total occupied frequency space.

Interpretation of these idealised sampling theories by implementing reconstruction from a waveform sampled at the \( F_{\text{occ}} \) rate has been shown in many references for an idealised lowpass waveform \( (2, 5) \). It is possible to reconstruct any lowpass waveform without ambiguity as long as the mean
sampling rate is above \(2F_{\text{max}}\). If bandlimited signals are considered, interpolation processes have been shown to be able to reconstruct from these sort of signals when sampled at a frequency of their occupied bandwidth, reference (15). With bandlimited signals the interpolation process is not totally independent of the nature of the sampling method in that ambiguity in the interpolated waveform can occur if the position of the sampling points are not chosen with a view to the spectral format of the bandlimited signals.

Kohlenberg in his paper on "Exact Interpolation of Bandlimited Functions", reference (15), shows that unambiguous sampling can be achieved with a sampling impulse train constructed out of two uniform impulse trains, time staggered relative to each other, the mean rate of this sampling method being equal to \(F_{\text{occ}}\) of the bandlimited signal. The method used by Kohlenberg was to design an interpolation kernel so that within the occupied spectral band, \(f_o\) to \(f_o + F\) Hz, the overlapping aliases cancelled each other out leaving the original spectral information.

In general, the sampling processes are restricted, even Kohlenberg's specialised staggered periodic sampling is further restricted in the intervals of stagger in order to avoid reconstruction ambiguity. If now the general case is considered of reconstructing by interpolation a multiple bandlimited sampled signal the problems tackled by Kohlenberg are multiplied many-fold. In fact, if one tries to apply the technique of Kohlenberg, the array of possible overlapping spectral forms becomes overwhelming. In addition, the amount of restrictions on the type of sampling process that could be used without ambiguity occurring at the sampling rate of \(F_{\text{occ}}\) would undoubtedly increase as a function of the complexity of the signals spectral format.

In this chapter the problem of interpolating a generalised waveform with a multiple bandlimited spectrum will be considered. The sampling process will
be based on multiple staggered periodic impulse trains with a mean sampling rate equal to $F_{\text{occ}}$ similar to Kohlenberg's paper. The aim of the analysis is to try and form a general sampling stagger pattern which when applied to an arbitrary multiple bandlimited signal will always avoid ambiguity. The analysis is going to be viewed from the aspect of interpolation, but the actual process of interpolation is to be carried out in a totally digital manner with the object of computer compatibility in mind.

4.2 DISCRETE INTERPOLATION

The aim of this chapter, as was mentioned in the introduction, is to consider the general case of interpolating waveforms which have been sampled at or above their occupied bandwidth, $F_{\text{occ}}$, but not necessarily above twice their upper frequency limit. The way in which this problem is going to be tackled is totally different from the normally encountered methods of interpolation. The problem is going to be looked at as a digital analysis process rather than a continuous function analysis procedure. Interpolation as a method for reconstructing a continuous waveform from a sequence of discrete values is to be replaced by a totally discrete process which is to be termed discrete interpolation. Discrete interpolation can be defined as a process which operates on a discrete time sequence of values which have been sampled from the continuous function, $s(t)$, at a mean rate above twice the occupied bandwidth. The resultant output of the interpolation process is a sequence of numbers which are effectively sampled from $s(t)$ at uniform intervals at or above twice the highest frequency content of $s(t)$. Notation wise, the input sequence to the interpolation process is denoted by a time sequence $\{s_s\}$, this being derived from $s(t)$ by sampling at a rate below $2F_{\text{max}}$ but above $F_{\text{occ}}$.

$$\{s_s\} = s_{s,1}, s_{s,2}, \ldots, s_{s,M}$$  \hspace{1cm} (112)
The output from the interpolation process is a sequence \( \{s\} \) which is
effectively derived from \( s(t) \) at a uniform sampling rate above \( 2F_{\text{max}} \).

\[
\{s\} = s_1, s_2, \ldots, s_N
\]

It is acceptable in many digital processing techniques to have the digital
 equivalent of a continuous process as an interpolated answer or output data
 form. In practice this sort of digital output format is only acceptable if
 the ambiguity presented by the discrete nature of the data can be removed by
 simple lowpass filtering. Thus the process of discrete interpolation can be
 viewed as a technique for processing signals so that the ambiguity can be
 removed by simple lowpass filtering. The discrete and normal interpolation
 processes can be considered as equivalent except for the act of lowpass
 filtering at \( F_{\text{max}} \).

4.3 STAGGERED PERIODIC SAMPLING FUNCTIONS

As explained in the previous section, the interpolation process is going to
 be studied with respect to a certain class of sampling functions. As in the
 paper by Kohlenberg, (15), the sampling process is going to be a staggered
 periodic impulse function with basic period \( T_r \), i.e.

\[
e(t) = \sum_{i=1}^{I} \sum_{l=0}^{\infty} \delta(t - \frac{lT_r - p_i \Delta_g}{N})
\]

\( \Delta_g = T_r/N \)

The actual stagger interval \( p_i \Delta_g \) is totally dependent on \( p_i \) which is an
 arbitrary integer ranging from 0 to \( N - 1 \). If all possible staggered periodic
 sampling processes are present, that is, \( p_i \) ranging from 0 to \( N - 1 \), \( I = N \).

(113)
then \( e(t) \) will effectively be a uniform periodic impulse function with period \( T_r \). The interval \( \Delta \) is chosen to be equal to the Nyquist interval for \( s(t) \), i.e.

\[
\Delta_g = \frac{1}{2F_{\text{max}}} = \Delta_n
\]

This is merely for convenience in the mathematical notation, in practice \( F_{\text{max}} \) can be set to any convenient frequency above all the signal power in \( s(t) \). Due to the fact that the time domain is going to be represented on a discrete basis it is necessary to define a sampling sequence as opposed to a waveform. This can be done because \( e(t) \) is based on a uniform time grid of \( \Delta_g \). Defining a sampling sequence \( \{c\} \)

\[
e_v = \sum_{i=1}^{I} \sum_{z=0}^{\infty} \delta \nu_i u \quad u = (zn - p_i)
\]

where:

\[
\delta_{mn} \text{ denotes the Kronecker delta function}
\]

\[
\delta_{mn} = 1 \quad \text{at } m = n
\]

\[
\delta_{mn} = 0 \quad \text{otherwise}
\]

Now that the sampling sequence has been specified it is possible to relate the sampled waveform sequence \( \{s_s\} \) to the uniform sampled waveform sequence \( \{s\} \). Defining sequence \( \{s\} :-\)

\[
s_n = \sum_{n=0}^{N-1} s(t)\delta(t - n\Delta_g)
\]

Relating \( \{s_s\} \) to \( \{s\} \)

\[
s_{s,n} = s_{e,n} \quad n = 0, 1, 2, \ldots, N - 1
\]

Notice that the sequences only contain \( N \) elements which cover the basic time interval \( T_r \). Since the sampling process is basically periodic in \( T_r \) only
Fig 4.1  INTERPOLATION ON A FINITE TIME DOMAIN

TYPICAL K_n^k INTERPOLATION KERNEL

Fig 4.2 a)  INTERPOLATION ON A PERIODIC TIME DOMAIN

T = T_r

Fig 4.2 b)  T = 2T_r

(115)
\( T = T_r \) will be considered initially for simplicity but in section 4.14 the general case allowing \( T \) to tend to infinity will be discussed.

4.4 INTERPOLATION USING PERIODIC TIME KERNELS

In this chapter the discrete interpolation process is going to be implemented from the frequency domain, in that a process is to be devised that will construct the estimate of the original complex spectrum of \( s(t) \) from the estimate of the complex spectrum of \( s_s(t) \). The term "estimate" is applicable because the process is operating on finite waveform lengths as is the case with normal practical interpolation processes. The transformation of interpolation from one spectral form into another will be carried out by computer so the spectral functions both of the original and sampled waveforms must be represented discretely. Discrete frequency domain functions require correspondingly periodic time domain functions. This is done by making a modification to the usual procedure for time waveform interpolation.

The reconstruction of the continuous time waveform \( s(t) \) is practically implemented by interpolating between a finite number of samples, to increase the resultant accuracy the number of samples is then allowed to tend to infinity. Hence the interpolation error can be made as small as necessary, i.e.

\[
s(t) = \sum_{n=1}^{N} K_n(t) s_{s,n}
\]

as \( N \) tends to infinity

where

\( K_n(t) \) denotes the \( n \)th interpolation kernel.

The method of interpolation to be used in this chapter will work on a slightly different concept. In figure 4.2 a finite length of sampled waveform to be interpolated is shown, before allowing the interval 0 to \( T \) to

(116)
expand to infinity it is made periodic in $T$, from minus to plus infinity.

The effect this has on the interpolation kernel $K_n$ is to make it periodic,

$$
K_{pn} :=
$$

$$
s_p(t) = \sum_{n=1}^{N} K_{pn}(t) \ s_{s,n}
$$

Now as the waveform length, $T$, is increased to infinity the periodic time
function $s_p(t)$ will tend to the original time function $s(t)$. Thus with this
periodic form of interpolation, all the time waveforms and the interpolation
kernels can be considered as constructed of harmonic components of $1/T$ hence
each having discrete spectral functions.

4.5 SPECTRAL REPRESENTATION OF INTERPOLATION

The equation relating the original and sampled sequence, $(s)$ and $(s_s)$
respectively is given by equation 4.3. This provides the basis for inter-
polation from $(s_s)$ back to $(s)$. The interpolation process itself is to be
carried out in the frequency domain therefore equation 4.3 can be best
expressed in the frequency domain, i.e.

$$
D.F.T.(s_{s,n}) = D.F.T.(s_n) \circ D.F.T.(e_n)
$$

where

$\circ$ denotes circular discrete convolution

Let sequence $\{X_s\}$ denote the Digital Fourier Transform of the sequence $\{s_s\}$,
i.e.

$$
X_{s,k} = \sum_{n=0}^{N-1} s_{s,n} e^{-jnk2\pi/N}
$$

$k = 0, 1, 2, \ldots N - 1$

Similarly

$\{X\} = D.F.T.(s)$

and

$\{X_e\} = D.F.T.(e)$

(117)
Rewriting equation 4.4

\[ \{X_s\} = \{X_e\} \circ \{X\} \]  \hspace{1cm} 4.5

If, at this stage, we stop and consider the physical significance of the sequences in equation 4.5; they are \( N \) element sequences derived from a uniformly sampled portion of \( s(t) \) of length \( T \) which has been assumed periodic in \( T \), section 4.4. Therefore, each element of \( \{X\} \) on the spectrum will represent the area of a frequency impulse at harmonics of \( 1/T \) Hz.

Positive frequencies:

\[ X_n = \sum_{n=0}^{(N-1)/2} U(f) \delta(f - n/T) \]

where the function \( U(f) \) is defined in relation to the spectrum \( S(f) \) of the uniformly sampled \( s(t) \) function from \( \pm \infty \) in time

\[ U(f) = \int_{-\infty}^{\infty} S(f) W(n/T - f) \, df \]

\( n = 0, 1, 2, \ldots, N-1 \)

where

\( W(f) \) signifies a window function

In the case of unwindowed time functions \( W(f) \) would be the familiar sinc function:

\[ W(n/T - f) = \frac{\sin (n/T - f)2\pi}{(n/T - f)2\pi} \]

Therefore the sequence \( \{X\} \) does not represent points on the true spectrum \( S(f) \) but weighted approximations. Only in the limit, as \( T \) tends to infinity, does the approximation become exact. On the other hand, if the waveform \( s(t) \) is made up of harmonic sinusoidals of \( T \), \( \{X\} \) is always an exact specification of \( S(f) \). This fact is used in chapter 5. In the theory of interpolation of bandlimited waveforms, it is necessary to be able to specify the
Fig. 4.3  

**DESCRIPTION OF SPECTRAL INFORMATION IN SEQUENCE FORM**

![Diagram showing frequency space and spectral occupancy]

\[
\{X\} = u_1 u_2 \ldots u_5 \ldots u_6 u_7 \ldots \ldots u_{t+1} 0 0 0 \ldots
\]

- Original occupied frequency space
- \( U(t) \) — time limited spectral occupancy
- Leakage
the parts of the frequency spectrum, $S(f)$, which contain power, this will specify the minimum sampling rate, $F_{\text{occ}}$. The occupied frequency space of $s(t)$ may be ideally bandlimited, figure 4.3, but the time limited waveform may not have a so well defined bandlimited spectrum, $U(f)$, see figure 4.3, where $U(f)$ represents the spectral structure of the sampled waveform $s(t)$ of length $T$. It is assumed that $T$ can be made large enough so that $U(f)$ can be a close approximation to $S(f)$ from the point of view of the occupied frequency space. In this chapter we initially only consider a waveform length of $T_x$ but this is for convenience only and is a partial analysis based on harmonics of $T_x$. So one can write

Positive frequencies:

$$U(n/T_x) = X_n \quad n = 0, 1, 2, \ldots (N-1)/2$$

As a close approximation

$$S(n/T_x) \approx X_n$$

So the distribution of the signal power in frequency can be accurately defined at the harmonics of $1/T_x$, figure 4.3.

4.5.1 MATRIX/VECTOR NOTATION

Most of the work in this chapter is carried out in the frequency domain and a matrix/vector representation of the spectra in this domain not only presents a convenient way of manipulating the quantities but provides valuable insight into the digital spectral analysis problems. The complex sequences $\{X_s\}$, $\{X\}$ and $\{X_e\}$ are replaced by vectors, i.e.

Vector element: $c_k = X_{sk}$ : Sequence element

Vector element: $b_k = X_k$ : Sequence element

Vector element: $d_k = X_{ek}$ : Sequence element

$k = 0, 1, 2, \ldots N - 1$

where: $b_k$, $c_k$ and $d_k$ are elements of vectors $b$, $c$ and $d$ respectively (120)
The sequence equation 4.5 expressing circular convolution between \( \{X_s\} \) and \( \{X_e\} \) can be replaced by a vector circular convolution.

\[
\mathbf{c} = \mathbf{d} \circ \mathbf{b}
\]  

Equation 4.6

A circular convolution process can be represented by a matrix/vector multiplication, reference (16). The technique requires that \( \mathbf{d} \) is used to generate a cyclic matrix \( \mathbf{D} \), this is shown in the appendix 3.

Rewriting equation 4.6

\[
\mathbf{c} = \mathbf{D} \mathbf{b}
\]  

Equation 4.7

where

\( \mathbf{c} \) denotes the vector representation of the spectrum of the waveform sampled at a rate of \( F_{\text{occ}} \).

\( \mathbf{b} \) denotes the vector of the spectrum of the waveform sampled at or above \( 2F_{\text{max}} \).

Equation 4.7 represents the starting point for the work in this chapter, so it is useful to know the physical meaning of the vector structure of \( \mathbf{c} \) and \( \mathbf{b} \).

\[
\begin{bmatrix}
\mathbf{c}_0 \\
\mathbf{c}_1 \\
\mathbf{c}_2 \\
\vdots \\
\mathbf{c}_{N/2} \\
\mathbf{c}_{N/2+1} \\
\vdots \\
\mathbf{c}_{N-1}
\end{bmatrix}
\]

\( \mathbf{c} \) or \( \mathbf{b} \)

(121)
From basic matrix theory, if the vector $c$ and matrix $D$ are known, then vector $b$ can be found by the inversion of the matrix $D$. Hence the original spectrum structure derived from uniform sampling can be found from a spectrum structure derived from a signal sampled at $F_{occ}$. This is the foundation of the discrete to discrete interpolation process on which this chapter is based, most of the work is directed at the analysis of the matrix $D$.

It is possible through the analysis of matrix $D$ purely from the viewpoint of matrix algebra to find the conditions of sampling which will allow unambiguous interpolation of the original sequence $\{s\}$. There are two main forms of ambiguity in the reconstruction of the vector $b$ from $c$.

1) The first condition for ambiguity occurs when the mean sampling rate of $s(t)$ is less than $F_{occ}$, this condition corresponds to matrix $D$ being singular, but, more important, the rank of the matrix is less than the number of non-zero elements in the vector $b$ or its corresponding sequence $\{s\}$. In other words, the number of independent bits of spectral information (amplitude plus phase) contained in vector $b$ exceeds the capacity of the matrix $D$ to transform independent bits of information to vector $c$. It is pointless to attempt reconstruction of $b$ under these conditions because this would contradict the basic sampling theorems.

2) The second possible condition of ambiguity is when the mean sampling rate of $s(t)$ is above $F_{occ}$ but below $2F_{max}$. In this situation the matrix $D$ is singular but its rank is above the number of non-zero elements in the vector $b$. This is the condition that is to be investigated in this chapter with respect to a generalised multiple bandlimited signal where $F_{occ} < 2F_{max}$.
The section that now follows attempts to break down the basic matrix $D$ into a simpler form. A feature of all $D$ type matrices is that they are, by necessity, of a circulant form, which gives us the initial starting point in the analysis, the circulant matrix.

4.6 THE CIRCULANT MATRIX

Since $D$ is a circulant matrix, certain properties are already known because this type of matrix lends itself to simple analysis, reference (17).

Firstly, the eigenvectors are based on the $N$th roots of unity where $N$ is the dimension of the matrix, i.e.

N roots of unity:

$$y_i = e^{j2\pi i/N} \quad j = \sqrt{-1}$$

$$i = 0, 1, 2, \ldots N-1$$

Let $x_i$ denote the $i$th normalised eigenvector

$$x_i = e^{ji}$$

$$i = 0, 1, 2, \ldots N-1$$

$$j = 0, 1, 2, \ldots N-1$$

This is developed from the standard matrix equation:

$$D x_i = \lambda_i x_i$$

(123)
where $\lambda_i$ denotes the $i$th eigenvalue which is expressible in terms of the elements of $D$ and is always positive for a Hermitian circulant such as $D$. It is sufficient to realise that the set of normalised eigenvectors are orthonormal and that none are zero. To show that matrix $D$ is in fact singular one needs to look at the eigenvalues of $D$, the diagonalised form of $D$ will give these values. Using the non-zero eigenvectors as columns in a matrix $T$, of order $N \times N$, $D$ is postmultiplied, i.e.

$$D \quad T = \quad U \quad 4.9$$

Referring back to equation 4.8, one can see that $U$ will be made up of columns which are:

$$\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_N x_N$$

Finally, if $U$ is premultiplied by $T^T$:

$$T^T D \quad T = \quad T U = \quad G \quad 4.10$$

The $x_i$ vectors are orthonormal, i.e.

$$x_i^T x_j = \delta_{ij}$$

Therefore $G$ will be a diagonal matrix with elements $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N$ on the diagonal. This is a congruent transform, $T$, supplying the unitary matrix.

Pre and post multiplying by $T$ and $T^T$ gives:

$$T \quad T^T D \quad T \quad T^T = \quad T \quad G \quad T^T$$

$$G = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3 \\
0 & 0 & 0 & \lambda_4
\end{bmatrix} \quad (124)$$
Now the structure of the eigenvalues are considered with reference to (17).

Consider a matrix of dimensions 5 x 5, \( N = 5 \).

\[
\begin{align*}
\lambda_1 &= a + b + c + \overline{c} + \overline{b} \\
\lambda_2 &= a + b\omega + c\omega^2 + \overline{c}\omega^3 + \overline{b}\omega^4 \\
\lambda_3 &= a + b\omega^2 + c\omega^4 + \overline{c}\omega + \overline{b}\omega^3 \\
\lambda_4 &= a + b\omega^3 + c\omega + \overline{c}\omega^4 + \overline{b}\omega^2 \\
\lambda_5 &= a + b\omega^4 + c\omega^3 + \overline{c}\omega^2 + \overline{b}\omega \\
\end{align*}
\]

where \( \omega = e^{j2\pi/N} \)  \( j = \sqrt{-1} \) \( \overline{c} = c^* \)

The eigenvalues can be put in a vector form, \( \vec{\lambda} \), where the \( i \)th element equals the \( i \)th eigenvalue \( \lambda_1 \).

So rewriting the above set of equations in matrix form

\[
\vec{\lambda} = T \vec{d}
\]

Again matrix \( T \) occurs and from the above example one can see that it is in fact the well known Inverse Digital Fourier Transform matrix, (I.D.F.T.). It is now obvious that if \( \vec{d} \) represents the spectrum of the sampling sequence then \( \vec{\lambda} \) must represent the actual sampling sequence itself if the eigenvalues are normalised, i.e.

\[
\frac{\lambda_i}{|\lambda_i|} \quad i = 0, 1, 2, \ldots N-1
\]

The vector \( \vec{\lambda} \) must represent the 1, 0 time sampling pattern of \( \{e\} \). Hence the matrix \( G \) can be expressed as:-
Summarising what has been learnt about the circulant matrix $D$

a) Matrix $D$ can be expressed as a product of three matrices, two of which are standard form only dependent on their dimension, these being I.D.F.T. matrix forms.

b) The third matrix in the product is a diagonal but its elements along the main diagonal are either 1 or 0 and are totally dependent on the sampling pattern $\{e\}$.

It has been shown in the last section that the existence of sampling points in the sequence $\{e\}$ is a direct indication of the eigenvalues of the matrix $D$. If any zero elements exist in the sequence $\{e\}$ then this will give the matrix $D$ a zero eigenvalue and so make $D$ singular. Singularity in $D$ means that the solution of equation 4.7 for $b$ is ambiguous. This condition is equivalent to missing out one or more of the possible, $N$, uniformly staggered periodic sampling patterns which make up $\{e\}$. To give a practical example of singularity in $D$ consider a sampling process where only one periodic sampling pattern is present, $p_i = 0$, this giving a sampling point at the beginning of the interval $T_R$, see figure 4.4. The corresponding matrix $D$ would be:

\[
\begin{bmatrix}
  e_0 & e_1 & 0 \\
  & & \\
  0 & & e_N-1
\end{bmatrix}
\]

(126)
ONE PERIODIC SAMPLING WAVEFORM — ZERO STAGGER

Fig. 4.4

N PERIODIC SAMPLING WAVEFORMS STAGGERED AT Δg INTERVALS

Fig. 4.5

(127)
All the elements are 1; it is obvious that the matrix is singular, also that its rank is equal to one. At the other end of the scale if the time waveform $s(t)$ is sampled by $N$ periodic sampling impulse trains staggered uniformly at $\Delta g$ intervals, the corresponding $D$ matrix would be:

$$D = 
\begin{bmatrix}
    1 & 1 & 1 & \cdots \\
    1 & 1 & \cdots & \\
    1 & \cdots & \\
    \vdots & \ddots & \\
    1 & \cdots & \\
\end{bmatrix}
$$

This matrix is invertable and therefore has rank equal to the matrix order, this corresponds to uniform sampling at a rate of $2F_{\text{max}}$.

Whatever the order of matrix $D$ the general form will be a complex circulant, i.e.
where \( a \) is a real number, represents the zero frequency of \( d \).  
\( b, c, \ldots \) -complex numbers, represent the +ve frequencies of \( d \).  
\( b^*, c^*, \ldots \) -complex conjugate numbers, represent the -ve frequencies of \( d \).

Due to the fact that here one is considering the case where the sampling rate is below \( 2F_{\text{max}} \) but above \( F_{\text{occ}} \), it will always mean that some of the possible \( N \) staggered sampling impulse trains are missing. This means that one or more of the eigenvalues in matrix \( D \) must be zero, hence in all the conditions where \( D \neq I \) the matrix \( D \) is singular.

Therefore, now in all cases of sampling at below twice the highest harmonic in \( s(t) \), the equation 4.7 must have ambiguous solutions due to the singularity of \( D \). The equation is not insoluble, but there are a multitude of possible forms of vector \( b \) which could have generated vector \( c \).

Repeating equation 4.7:

\[
\mathbf{c} = \mathbf{D} \mathbf{d}
\]

This is the typical case of ambiguity associated with spectral aliasing. In principle, if the rank of matrix \( D \) is say \( R \), then it is theoretically possible
to get an unambiguous solution to the equation 4.7 if only R non-zero elements occur in vector $b$. This statement implies that if the rank of $D$ is R then a soluble equation can be written around a nonsingular matrix of order R derived by deletion of $(N - R)$ rows and columns from matrix $D$.

Writing a reduced equation:

$$\begin{align*}
R \times 1 & \quad R \times R & \quad R \times 1 \\
\mathbf{c}_r & = D_r \cdot \mathbf{b}_r
\end{align*}$$

where the $R$ elements of $\mathbf{b}_r$ are the $R$ non-zero elements of $\mathbf{b}$, the $R$ elements of vector $\mathbf{c}_r$ are a particular selection of $R$ elements of $\mathbf{c}$ vector.

A "particular selection" because this has to be determined and may not be an arbitrary choice. The reduced matrix $D_r$ is constructed out of elements of the original $D$ matrix by deletion of columns in $D$ corresponding to the non-zero elements of vector $b$ and deletion of rows of $D$ corresponding to the selected elements of $c$. It is now necessary to know the relationship between the structure of matrix $D$ and its rank so that the minimum row/column deletion can be specified.

4.7 THE RELATION BETWEEN THE RANK AND NON-ZERO EIGENVALUES OF MATRIX $D$

Even though the matrix $D$ may be singular its rank does not necessarily equal zero, in fact, under certain conditions the number of non-zero eigenvalues can be related directly to the rank of the matrix. Matrix $D$ is a semisimple matrix of order $N \times N$, in that it has $N$ linearly independent eigenvectors. Hence the eigenvectors can form a basis for a $N$ dimensional space. The difference between the dimensionality, $N$, of a matrix and its rank is defined as equal to the order of the null space generated by the matrix when it operates on vectors in a space of dimension $N$. Since any vector within a
vector space can be expressed by a linear combination of the base vectors of that space the general condition for a null space can be expressed by allowing the matrix to operate on the base vectors themselves. The dimensions of the null space for a semisimple matrix can therefore be related to the number of linearly independent eigenvectors with zero eigenvalues, i.e.

\[ D \mathbf{x}_i = \lambda_i \mathbf{x}_i \]

where

- \( \mathbf{x}_i \) denotes the \( i \)th eigenvector
- \( \lambda_i \) denotes the \( i \)th eigenvalue

Matrix \( D \) is said to annihilate the eigenvectors \( \mathbf{x}_i \) with associated \( \lambda_i \) eigenvalues equal to zero, see reference (19). In section 4.6, it was shown that the normalised eigenvectors correspond to the sequence elements \( \{e_i\} \), i.e.

\[ \lambda_n = e_n \quad n = 0, 1, 2, \ldots, N-1 \]

The elements \( e_n \) are zero or one depending on the presence of a periodic sampling pattern at that particular stagger, i.e.

\[ e_n = 1 \quad \text{at} \ n = p_i \]
\[ i = 1, 2, \ldots, I \]
\[ e_n = 0 \quad \text{at} \ n \neq p_i \]

Hence, it follows that the number of non-zero eigenvalues is equal to \( I \) and consequently the rank \( R \) of matrix \( D \) is also equal to \( I \).

\[ (\text{Rank of } D) \ R = I \]

So one can relate the number of periodic staggered impulse trains used in the sampling process \( e(t) \) to the rank, \( R \), of matrix \( D \).

(131)
Fig 4.6
\[
\begin{bmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_N
\end{bmatrix}
\begin{bmatrix}
    D
\end{bmatrix}
\begin{bmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_N
\end{bmatrix}
\]

DELETE ALL BUT THESE COLUMNS

Fig 4.7
\[
\begin{bmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_N
\end{bmatrix}
\begin{bmatrix}
    d_{15} & d_{16} & d_{17} \\
    d_{25} & d_{26} & d_{27} \\
    \vdots & \vdots & \vdots \\
    d_{n5} & d_{n6} & d_{n7}
\end{bmatrix}
\begin{bmatrix}
    v_1 \\
    v_2 \\
    \vdots \\
    v_N
\end{bmatrix}
\]

SELECT 4 ROWS SO AS TO MAKE \( U \) NONSINGULAR
4.8 REDUCTION OF THE MATRIX D

It is required to solve the equation 4.7 for

\[ \mathbf{c} = \mathbf{D} \mathbf{b} \]  

4.7 Repeat

certain elements in the vector \( \mathbf{b} \), given the complete \( \mathbf{c} \) vector and the structure of the matrix \( \mathbf{D} \). It was shown in earlier sections that under the conditions where one or more of the staggered periodic pulse trains is missing, then the matrix \( \mathbf{D} \) is always singular. Although \( \mathbf{D} \) is singular its rank is not zero and therefore the matrix in some reduced non-singular form should be capable of relating some elements of \( \mathbf{b} \) unambiguously to elements of \( \mathbf{c} \). The reduction process can be looked on as determining the elements of \( \mathbf{c} \) and \( \mathbf{b} \) that can be related by a non-singular matrix operation. This of course requires a complete re-appraisal of the above equation 4.7, since singularity defines only the non-invertability of \( \mathbf{D} \) and its rank only gives the dimension of the largest non-singular matrix. None of these parameters describes the ability of the singular matrix to relate limited quantities of information unambiguously between \( \mathbf{c} \) and \( \mathbf{b} \) vectors.

Consider figure 4.6 in the vector \( \mathbf{b} \) there are only four non-zero elements, if all but these elements are deleted the corresponding columns in matrix \( \mathbf{D} \) can also be deleted because they take no part in the actual matrix equation. The partially reduced equation can be rewritten:

\[ \begin{align*}
\text{Dimensions: } & N \times 1 & N \times 4 & 4 \times 1 \\
\mathbf{c} & = H \cdot \mathbf{p} \\
\end{align*} \]  

4.14

To solve this equation for \( \mathbf{p} \) it is necessary to select four rows of the matrix \( H \) so as to create a non-singular matrix. This might not be possible, due to the preselection of the four columns in which only less than four rows of \( H \) are linearly independent. The selection of linearly independent
rows and columns is a topic to be covered next. Assuming that four rows of \( H \) can be chosen to generate a non-singular matrix, the resultant matrix equation becomes:

\[
\begin{align*}
\text{Dimensions: } &\quad 4 \times 1 \quad 4 \times 4 \quad 4 \times 1 \\
q &= U \cdot p
\end{align*}
\]

in which matrix \( U \) is invertable. Hence the equation can be solved for \( p \), this requires two important pieces of a priori information.

1) The bandlimited structure of the signal \( s(t) \) must be known so that the position of the non-zero elements in \( b \) which indicate the presence of power at that particular frequency in the spectrum are known. This allows the correct columns of \( D \) to be deleted without loss of signal description in \( c \).

2) Secondly, it must be known that irrespective of which columns are kept to form \( H \), there must be \( N \) linearly independent rows in this matrix. It is also advantageous to know which rows these are if possible.

Thus the reduction of the \( D \) matrix is basically dependent on selecting a non-singular matrix \( U \) from the columns and rows of \( D \) in such a way as to include all the non-zero elements of \( b \) in vector \( p \). To select the required matrix \( U \) for various combinations of power distribution within vector \( \{ b \} \) necessitate a knowledge of the linear interdependencies of rows/columns of the singular matrix \( D \). This would allow instant selection of independent rows given matrix \( H \) and also indicate the feasibility of such a selection.

The next section goes on to consider the internal structure of the matrix \( D \) in order to throw some light on reducing a singular to a non-singular matrix given a particular number of column deletions.
4.9 ANALYSIS OF MATRIX D

Direct analysis of the matrix $H$ to determine the independent rows for all possible forms of $H$ for a given matrix $D$ is a very involved computer process. Also one mapping of the column/row dependencies will only be applicable to one particular power distribution pattern within $b$ with relation to a set sampling pattern $\{e\}$. This situation is not very satisfactory since the possible $N \choose 1$ combinations of power distribution within $b$ will be large, therefore it will be a lengthy process to evaluate the dependence of rows/column for every possible signal $s(t)$. An analytic approach to the problem seemed initially the best way to look for a solution.

Starting with the matrix $D$, it can be reduced to its diagonal form, this was shown in section 4.6.

$$D = T S T^T$$  \hspace{1cm} \text{(4.11 Repeat)}

Considering that the normalised eigenvalues are either 1 or 0 the diagonalised form can be simplified into two matrix forms, i.e.

$$D = T_a T_b$$ \hspace{1cm} \text{(4.16)}

where $T_a$ is a rectangular matrix formed from the columns of matrix $T$ which does not have zero eigenvalues in the corresponding row of $G$. Similarly $T_b$ is a rectangular matrix made from the rows of $T^t$ which do not contain zero eigenvalues in the corresponding columns of $G$. Now consider the deletion of columns and rows in matrix $D$ expressed in the new expanded form. Deletion of the $j$th column in $D$ corresponds to a deletion of the $j$th column in matrix $T_b$, i.e.
Similarly, if the \( i \)th row is deleted in \( D \), then the \( i \)th row in \( T_a \) is deleted. The minimum number of columns and rows which must be deleted are \((N - R)\) where \( R \) denotes the rank of \( D \). Let it be assumed that no more than this minimum number of columns and rows are removed, hence, \( I = R \). This will mean that the two matrices \( T_a \) and \( T_b \) will be reduced to two square matrices of order \( I \), by deletion of their rows and columns respectively. Giving finally:

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
U
\end{bmatrix}
&=
\begin{bmatrix}
T_{ar} & T_{br}
\end{bmatrix}
\end{align*}
\]

If \( U \) is non-singular then both \( T_{ar} \) and \( T_{br} \) must also be invertable in their own right, this can be seen from direct observation or by considering Steinmetz's replacement theorem, reference (18). Therefore, in order to study the reduction of \( D \) to an invertable form it is equally instructive to study the reduction of \( T_a \) and \( T_b \) to an invertable form. This is indeed easier to study since the initial \( T \) matrix is a standard form irrespective of the sampling scheme.

Of the two matrices \( T_{ar} \) and \( T_{br} \), the study of \( T_{br} \) is more important since the deletion of the columns in this matrix is controlled by the distribution of power within the spectrum of \( s(t) \) and cannot be arranged by post-processing techniques or sampling design. Since both matrices are based on
the same order of D.F.T. matrix, the study of a I.D.F.T. matrix for $T_{br}$ is
directly applicable to the study of a D.F.T. matrix for $T_{ar}$. The reduction
technique is considered from the viewpoint of arbitrary column deletions
and then row deletions to form the non-singular matrix $T_{br}$, the latter
determining the pattern {$e$} of the periodic staggered sampling pulses. The
ideal situation would exist where no matter which or how many columns are
deleted from $T$ that an arbitrary choice of rows from the $T_{br}$ matrix could be
chosen to generate $T_{br}$.

The next section goes on to consider the inter-relation between the columns
and rows that must be deleted from a I.D.F.T. matrix in order to produce a
reduced order non-singular matrix.

4.10 PRIME ORDER I.D.F.T. MATRICES

The general form of a Nth order I.D.F.T. matrix is :-

$$
T = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & w & w^2 & w^3 & \ldots \\
1 & w^2 & w^4 & \ldots \\
1 & w^3 & \ldots \\
1 & \ldots & \ldots & \ldots & w^{N-1}
\end{bmatrix}
$$

where :

$$
w = e^{j2\pi/N} \quad \quad j = \sqrt{-1}
$$

The elements are defined by :

$$
T_{ij} = w^{ik \mod (N)}
$$

$\ell = (i - 1) \quad \quad k = (j - 1)$

$T$ is a complex matrix with very symmetrical properties as an example $N = 4$. 

(137)
It is always non-singular, whatever its order \( N \) and it follows that the rows and columns are linearly independent but in this case they are also orthogonal.

Initial investigations into the viability of reducing a \( N \)th order non-singular I.D.F.T. matrix down into an arbitrary order non-singular matrix were more random than constructive in nature. Small orders of D.F.T. matrices were constructed and arbitrary column and row deletions were carried out to see whether singular matrices could be generated. The idea was that a general pattern might emerge. It soon became obvious that when using even ordered dimensioned I.D.F.T. matrices that it was possible to delete to a singular form. As a simple example a matrix of order four can be reduced to a singular form by deletion of columns (2 and 4) and rows (2 and 4). Knowing that even orders of I.D.F.T. matrices might cause singularity in \( T_{br} \), it was decided to look at odd orders. Odd ordered I.D.F.T. matrices also exhibit this properly. Considering the matrix of order nine one can immediately see that at least a \( 3 \times 3 \) matrix made up of 1's could be extracted by deletion.

The obvious causes of singularity that can be seen by inspection in matrices of odd and even order seemed to be caused by the occurrence of a given power of \( w \) on a particular row or column more than once. With this obvious cause of singularity in mind, matrices with orders equal to prime numbers were constructed and it was apparent that singularity could not be detected by

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & w & w^2 & w^3 \\
1 & w^2 & 1 & w^2 \\
1 & w^3 & w^2 & w
\end{bmatrix}
\]
simple inspection in these types of matrices. If a matrix of order five is considered the characteristic structure of a prime order I.D.F.T. matrix can be seen.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & w & w^2 & w^3 & w^4 \\
1 & w^2 & w^4 & w^3 \\
1 & w^3 & w^2 & w \\
1 & w^4 & w & w^2 \\
\end{bmatrix}
\]

\[ w = e^{i\frac{2\pi}{5}} \]

\[ j = \sqrt{-1} \]

Every row and column contains every power of \( w \) from zero to \((N-1)\), this means no repetition occurs, it is shown in appendix (4) that this is true for all prime order D.F.T. matrices. So no matter which or how many columns/rows are deleted from the matrix \( T_b \), the resultant matrix \( T_{br} \) may not, by inspection at least, be singular. Although this is not a proof of non-singularity, this type of matrix at least seemed to hold the promise of arbitrary reduction to a \( N \)th order non-singular matrix.

4.11 USE OF THE VANDERMONTE MATRIX

While searching for an analytical proof of the non-singularity of the prime order I.D.F.T. matrix in reduced form, the Vandermonde matrix was found to supply proof of non-singularity in some cases. The basic Vandermonde matrix form is:

\[
\begin{bmatrix}
1 & 1 & \cdots & \cdots & 1 \\
x_1 & x_2 & \cdots & \cdots & x_N \\
x_1^2 & x_2^2 & \cdots & \cdots & x_N^2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
x_1^{N-1} & x_2^{N-1} & \cdots & \cdots & x_N^{N-1} \\
\end{bmatrix}
\]

(139)
where \( x_i \neq x_j \) for \( i \neq j \)

This is a standard matrix form and has been proven non-singular, reference (17). Substituting:

\[
x_n = w^{n-1}
\]

The Vandermonde matrix becomes equivalent to the I.D.F.T. matrix. Since \( w^0 \neq w^1 \neq w^2 \neq w^3 \neq w^4 \) \ldots it can be said that the I.D.F.T. matrix is non-singular by the Vandermonde theory. This is trivial since it is well known that an I.D.F.T. matrix is non-singular. The great benefit of this theory is in the application to the reduced I.D.F.T. matrix \( T_{br} \). If \( T_{br} \) can in some way be reduced to a Vandermonde form with all the conditions fulfilled then it can be classified as invertable.

The conditions which must be fulfilled by a matrix derived from a I.D.F.T. matrix to be classified as a Vandermonde are two fold:

a) A power law relationship must be found between all elements in a row or column, eg.

\[
\begin{array}{c}
1, w, w^2 \\
1, w^3, w^6
\end{array}
\]

Due to the symmetrical structure of the I.D.F.T. matrix only one column or row need be found to adhere to this power law relation, then automatically all the columns or rows follow this form.

b) The mantissa in each row/column power law relation must be different. In the case of the I.D.F.T. this simply means that the rows/columns must be different.

It is next considered what form the sampling pattern \( \{e\} \) must take in order that a Vandermonde matrix is formed from \( T_{rb} \).

(140)
Fig 4.8

![Graph showing periodic staggered sampling waveform with time (seconds) on the x-axis and amplitude on the y-axis.]

Fig 4.9

\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & w & w^2 & \cdots & \cdots \\
1 & w^2 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\]
4.12 SELECTING A PERIODIC STAGGERED SAMPLING PATTERN

Initially a simple well understood sampling method will be tried. Let \( T \) periodic sampling waveforms be placed contiguously on the uniform time grid \( \Delta_g \), see figure 4.8. If the form of \( D \) is considered:

\[
D = T G T^T
\]

The diagonal matrix \( G \) equals:

\[
G = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

If now the form of the \( T_b \) matrix is written down it is equivalent to a \( N \times N \) I.D.F.T. matrix with all but the first \( I \) rows deleted, figure 4.9. Due to the operation of the power law relationship of elements in every column of \( T_b \) it is irrelevant which columns are deleted from \( T_b \) to form \( T_{br} \) because the latter will always be a Vandermonte matrix. Similarly the matrix \( T_{ar} \) will also be a Vandermonte matrix due to the contiguous sampling patterns.

It therefore follows that irrespective of where the spectral power lies in the harmonics of the vector \( b \) in the fundamental equation 4.7 as long as there are only, at maximum, \( I \) harmonics or elements in \( b \) which are non zero.
It can be seen that even if the block of contiguous samples were moved to any position within $T$ the reduced matrices $T_{br}$ and $T_{ar}$ would still be of a Vandermonte form. In fact the I.D.F.T. matrix form, possesses the property that all the rows/columns can be shifted cyclically by multiplication of columns and rows by powers of $w$. As an example a $3 \times 3$ matrix constructed by selection of the $(2, 3, 4)$ rows and the $(3, 4, 5)$ columns of a large I.D.F.T. matrix equals:

$$
\begin{bmatrix}
w^2 & w^3 & w^4 \\
w^4 & w^6 & w^8 \\
w^6 & w^9 & w^{12}
\end{bmatrix}
$$

By multiplication of rows 1 to 3 by $w^{-2}$, $w^{-4}$, $w^{-6}$ respectively and columns 1 to 3 by $1$, $w^{-2}$, $w^{-2}$ respectively, the matrix can be converted to the top left hand corner of a I.D.F.T. matrix which is always of a Vandermonte form, i.e.

$$
\begin{bmatrix}
1 & 1 & 1 \\
1 & w & w^2 \\
1 & w^2 & w^4
\end{bmatrix}
$$

This cyclic rotation of rows and columns is very useful when the "raw" reduced matrices $T_{br}$ and $T_{ar}$ are required to be formed into a Vandermonte matrix.

Sampling in a restricted contiguous pattern of periodic stagger impulse trains is effectively similar to sampling at a rate above twice the highest frequency in $s(t)$ but for a short period of time, I.A. In practical situations it is desirable that not only the mean sampling rate should be low but due to restrictions on the data acquisition equipment the time
Closely Staggered Sampling

Difficult to interpolate in large time intervals

Widely Staggered Sampling

A typical interpolation kernel

Time

(144)
spacing between sampling instants must be made larger than $\Delta g$ seconds. Therefore it is advantageous to be able to distribute the impulse trains with staggers greater than $\Delta g$. Also one can look at the problem from the point of view of pure interpolation accuracy in that it is less accurate to interpolate a length of waveform $T_f$ from a group of closely spaced samples than it is to estimate the waveform from samples evenly distributed across $T_f$, see figure 4.10.

Staggering the periodic sampling impulse trains in a non-contiguous pattern brings about difficulty in proving non-singularity with use of the Vandermonde matrix. A class of sampling methods to be described next is designed so that there always exists a power law relationship in the columns of the partially reduced $T_b$ matrix. So irrespective of the method of reduction to $T_{br}$ this type of matrix can always be expressed in a Vandermonde form. A power law relationship within the column elements is controlled by the positioning of the staggered periodic impulse functions. The positions of these impulse functions on the uniform time grid, as explained earlier, is signified by the sampling pattern $\{e\}$, the elements of which can take on the values of $e_n$:

\[ e_n = \sum_{n=0}^{N-1} \delta_{n,p_i} \]

where:

- $p_i$ is an arbitrary integer constant ranging from 0 to $N-1$
- $\delta_{n,p_i}$ is the Kronecker delta function.

The relationship between the positioning of the periodic pulse trains and the exponential values of $w$ in a particular column $k$ of the partially reduced matrix $T_b$ can be expressed by:

\[(145)\]
column \( k \) of \( T_b \) = \( w^{kp_i} \mod(N) \)

\[ i = 1, 2, \ldots I \]

In order that the relationship 0, \((\emptyset), 2 (\emptyset), 3 (\emptyset), \ldots\) is maintained in the exponential powers of a column in \( T_b \) it is necessary that the values \( p_i \) have a constant period relationship, i.e.

\[ \{p\} = 0, 2, 4, 6, \quad \text{(period } 2\Delta g) \]

or \[ 0, 5, 10, 15 \quad \text{(period } 5\Delta g) \]

The actual exponential values of the elements in \( T_b \) are subject to a {	extsc{modulus}} operator, even though the power law relationship is not. As an example, \( N = 14 \) with a uniform periodic stagger of \( 4\Delta g \) and six staggered impulse trains, \( I = 6 \), this gives values of \( p_i \) as :-

\[ \{p_i\} = 0, 4, 8, 12, 16, 20 \]

This gives the structure of \( T_b \) as :-

\[
\begin{bmatrix}
1 & 1 & 1 & \ldots & \ldots & 1 \\
1 & w^4 & w^8 & \ldots & \ldots & w^{4(N-1)\mod(N)} \\
1 & w^8 & w^2 & \ldots & \ldots & \ldots \\
1 & w^{12} & w^8 & \ldots & \ldots & \ldots \\
1 & w^2 & w^4 & \ldots & \ldots & \ldots \\
1 & w^6 & w^{12} & \ldots & \ldots & w^{6(N-1)\mod(N)} \\
\end{bmatrix} = T_b
\]

The great problem with row deletion based on the necessity of only generating the power law relation is that the rows may become dependent making \( T_b \) singular in consequence. As discussed earlier, in order to make certain of obtaining a Vandermonte matrix form, given that a row or column power law relation holds for the matrix, the basic stagger interval \( \Delta g \) must be a prime integer multiple of \( T_r \). In other words the order of \( I, N, \) is a prime number.
4.13 AN UNPROVEN CLASS OF SAMPLING PATTERNS

To conclude this investigation into the distribution of periodic staggered impulse trains an attempt was made to find a more general rule for distribution other than a power law relation within the columns of $T_{br}$. This idea was fostered by the use of the sampling patterns as discussed in chapter three. It will be recalled that the sampling patterns there were based on cyclic difference sets, in particular sets based on prime numbers. These number sequences were used as a basis for the selection of the rows in the matrix $T_b$. Examples of the sequences use are:

$$(v, K, \lambda) : (37, 9, 2), (57, 8, 1) \text{ and } (7, 3, 1)$$

When the reduced matrix $T_b$ is based on these sampling patterns it is found impossible in every case to convert it by column deletion into a Vandermonde form. Consider the matrix $T_b$ formed from the sampling pattern $(7, 3, 1)$ by deletion of all but the 1st, 2nd and 4th rows of a 7th order I.D.F.T. matrix, $T$.

$$T_b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 \\ 1 & w^3 & w^6 & w^2 & w^5 & w & w^4 \end{bmatrix}$$

No matter how one tries it is impossible to get any column to form a power law relation. This state of affairs is not really surprising, since the cyclic difference sets are designed to have no periodicity, hence the column elements they generate in $T_b$ will exhibit no power law relation. Thus there will be a whole class of sampling patterns which do not allow the Vandermonde matrix to be formed out of the $T_b$ matrix. Even though no Vandermonde matrix could be formed from $T_b$ it proved impossible, both theoretically and practically, to show that $T_{br}$, reduced from a prime order $T$ was also singular.

(147)
BANDLIMITED SPECTRAL OVERLAP — KOHLENBERG'S METHOD

2nd. ORDER PERIODIC STAGGERED SAMPLING

Fig. 4.11

SPECTRAL OVERLAPS

ORIGINAL SPECTRUM

FREQUENCY $H_2$
It soon became obvious that it was likewise impossible to show singularity in any prime order I.D.F.T. matrix reduced to a square order by any arbitrary row/column deletion.

At this point it was thought that the Kohlenberg paper, reference (15), might throw some light on the situation by introducing an heuristic approach to the multiple bandlimited case based on the results he found for a signal with a single bandlimited spectrum. Kohlenberg in his paper visualises the spectral plane split into sections of width \(1/T_r\) Hz. Any of these sections can be aliased or caused to overlap onto any one of the other spectral sections in the spectrum of the sampled signal. The only restrictions he places on the stagger of the two periodic pulse trains so as to allow unambiguous interpolation, is that the reciprocal of the time stagger must not be a submultiple of any possible spectral spacing between sections of the original spectrum power of the signal. In a single bandlimited example the spacing between the two spectral bands, shown in figure 4.11, would be, \(r = 6, r = 7\), that is \(6/T_r\) and \(7/T_r\) Hz.

\[
\begin{align*}
\text{STAGGER} & \quad \frac{1}{n T_r} \neq \frac{6}{n T_r} \quad \text{or} \quad \frac{7}{n T_r} \\
\end{align*}
\]

\[n = 1, 2, \ldots \infty\]

This can be expanded in an heuristic way to cover signals with multiple bandlimited spectra in which \(r\) could take on the value from 1 to \(N-1\) with spectral power distributed within the limits \(\pm F_{\text{max}}\).

where:

\[
N = \{2F_{\text{max}} T_r\}
\]

\[
\begin{align*}
\text{STAGGER} & \quad \frac{1}{n T_r} \neq \frac{r}{n T_r} \\
\end{align*}
\]

\[n = 1, 2, \ldots \infty\]

\[r = 1, 2, \ldots N-1\]

(149)
In our particular case, the stagger is based on a uniform grid of spacing \( \Delta g \) seconds, hence

\[
\text{STAGGER} = T_r \left( \frac{\gamma}{N} \right)
\]

where \( \gamma \) is an integer with values \( \gamma = 1, 2, \ldots, N-1 \)

Rewriting equation 4.20:

\[
\frac{N}{\gamma} \neq \frac{r}{n}
\]

To make sure of this inequality \( N \) is set equal to a prime number

\[ N n \neq \gamma r \]

Since \( \gamma \) and \( r \) cannot take on values greater than \( N-1 \) the inequality is fixed for all \( n \), \( \gamma \) and \( r \) if \( N \) is a prime.

The unproven postulate is now made that any sampling pattern based on a uniform grid spacing \( \Delta_g \) where \( T_r/\Delta_g \) is a prime integer will always produce a non-singular reduced matrix \( T_{\text{br}} \). This suggestion is totally based on the reasoning given above. This conclusion effectively gives an arbitrary sampling pattern within \( T_r \) as long as it is based on the uniform prime length sampling grid. Although this assumption has never proved incorrect, it might be wise in the use of these type of arbitrary sampling process to include a detector of \( T_{\text{br}} \) singularity in the programming. The general programming structure for the use of these sampling schemes is shown in section 4.15.

4.14 WAVEFORM LENGTH EXTENSION BEYOND \( T_r \)

As explained at the beginning of this chapter a section of length \( T \) is taken from the waveform \( s(t) \) and the waveform within this interval is
**Extension of the Sampled Waveform Length**

![Graph showing extension of waveform length](image)

**Fig. 4.12**

**Fig. 4.13**

Harmonics from original waveform length $T_r$

Additional harmonics due to waveform extension to $3T_r$

(151)
Fig. 4.14

$D_x$
approximated by harmonic sinusoids based on the period T. In all of the work described in this chapter the waveform length T has been restricted to \( T_r \), the period of the staggered sampling impulse trains. In this final section, it is shown how the waveform length T can be extended so that the waveform \( s(t) \) is more accurately formed by the interpolation process. Extension of the sampled section of \( s(t) \) is simply done by repeating the sampling pattern \( \{ e \} \) as designed in \( T_r \) so as to form the periodic staggered impulse waveform, figure 4.12.

The effect of increasing the sampled waveform length by a factor of \( x \), \( x \cdot T_r = T \), is to increase the order of the matrix equation 4.7 to \( xN \). Comparing the expanded matrix \( \mathbf{D}_x \) of order \( xN \times xN \), figure 4.14, with the initial \( N \times N \) order matrix \( \mathbf{D} \), one can see the similarities. In fact, if the elements enclosed within the squares are extracted the original \( \mathbf{D} \) matrix is formed. Therefore instead of writing a \( xN \) order matrix equation such as 4.23,

\[
\mathbf{c}_x = \mathbf{D}_x \mathbf{b}_x
\]

\( 4.23 \)

\( x \) separate \( N \) order matrix equations can be written:

\[
\begin{align*}
\mathbf{c}_{x1} &= \mathbf{D} \mathbf{b}_{x1} \\
\mathbf{c}_{x2} &= \mathbf{D} \mathbf{b}_{x2} \\
&\quad \vdots \\
\mathbf{c}_{xx} &= \mathbf{D} \mathbf{b}_{xx}
\end{align*}
\]

where vectors \( \mathbf{c}_{xn} \) and \( \mathbf{b}_{xn} \) are constructed from the elements \( n, n + x, n + 2x, \ldots, n + (N-1)x \) of vectors \( \mathbf{c} \) and \( \mathbf{b} \) respectively. Thus as the sampled waveform length is increased the additional interweaving harmonics that are
A Breakdown of a Harmonic Spectrum

Fig. 4.15

$T = 5T_r$

Harmonic Power Concentration:

- $C_{x1}$
  - Frequency: $3, 7, 10$

- $C_{x2}$
  - Frequency: $3, 7, 10$

- $C_{x3}$
  - Frequency: $3, 7, 9, 10$

- $C_{x4}$
  - Frequency: $3, 10$

- $C_{x5}$
  - Frequency: $3, 6, 9$

Frequency $Hz$

(154)
created can be treated as separate vectors $c_{xn}$ in their own right, see figure 4.13. Using this method of harmonic separation it is possible to extend the interpolation technique, involving the inversion of $D$, to larger sequences of data without the need for inversion of a matrix larger than $N \times N$. Figure 4.15 shows how a multiple bandlimited spectrum can be split up into its basic harmonic components spaced at $1/T_0$ Hz.

4.15 COMPUTER PROGRAMMING STRUCTURE

The work presented in this chapter has described the theoretical principles of discrete interpolation and the design of Nth order periodic staggered sampling waveforms which can be used to sample multiple bandlimited signals at optimum rates without fear of ambiguity occurring. The practical implementation of these sampling schemes requires considerable post sampling signal processing in order to de-alias the spectrum vector $c$ into its original harmonic component form $b$. A theoretical background has been given for the computational de-aliasing procedures leaving only the practical implementation of the program. It is meaningless to attempt to describe the details of the programming procedures since each operation is formed out of well known matrix or spectral analysis subroutine programs. Instead, in figure 4.16, an information flow/operation process diagram is given for the total computing requirements from sampled data acquisition to continuous waveform reconstruction. The program was implemented on a I.C.L. 1904A series computer system with all the standard complex matrix operations being supplied by the I.C.L. scientific software library. The Fourier analysis parts of the of the programming were performed by the CHIRPZ as opposed to F.F.T. Digital Fourier Transform algorithms due to the fact that data sequence lengths were not necessarily multiples of two, see references (20) and (21).
The programming itself can be split into three sections:

1) Transformation of the time sampled data into the frequency domain and retransforming the de-aliased frequency domain data back into the time domain. The data handling/storage capacity required by this section of the program is equal to \( xN \), in that sequences \( \{s_n\}, \{s\} \) and \( \{X_n\} \) are of length \( xN \). For one section of waveform \( s(t) \) of length \( T \) this section of the program is traversed only twice, once into and once out of the frequency domain.

2) The second part of the program is totally involved with the type of sampling process used, this requires the construction of matrix \( D \) and its reduction to \( U \). The required data storage capacity of this section of the program is initially \( N^2 \) reducing to \( I^2 \) corresponding to the dimensions of \( D \) and \( U \) respectively. The reduction process of \( D \) to \( U \) depends on the power distribution within the subspectral vectors \( b_{x1}, b_{x2}, b_{x3}, \ldots b_{xx} \) and this must be supplied as a priori information. If the power distribution varies from one subspectral vector to another then this section of the program might, at most, need to regenerate a new reduced form of \( U \), \( x \) times.

3) The third part of the program is the actual data processing unit in which the input complex spectrum vector \( c \) is decimated into \( x \) separate \( I \) element subspectral vectors \( c_{x1}, c_{x2}, \ldots c_{xx} \). These are then processed by matrix multiplication with \( U^{-1} \) and finally recombined to form the output complex spectrum vector \( b \). This section of the programming requires a data capacity of \( I^2 + I \) but is traversed \( x \) times in order to completely de-alias the complex spectrum vector of \( c \) into \( b \).
BLOCK FUNCTION DIAGRAM OF THE DEALIASING PROGRAM

Fig. 4.16

SAMPLE \( s(n) \) \( \rightarrow \) D.F.T. \((\text{CHIRP-Z})\) \( \rightarrow \) ALIASED SPECTRUM

SPECTRAL FORMAT \( \rightarrow \) POWER DISTRIBUTION IN \( b \)

D.F.T. \( \rightarrow \) CONSTRUCT \( D \) \( \rightarrow \) REDUCE MATRIX \( D \rightarrow U \) \( \rightarrow \) INVERT MATRIX \( U \) \( \rightarrow \) DETECT SINGULARITY

APRIORI KNOWLEDGE \( \rightarrow \) TYPE OF SAMPLING PATTERN \( \rightarrow \) DECONSTRUCTION OF HARMONIC COMPONENTS AT INTERVALS OF \( V/T, HZ \) \( \rightarrow \) MODIFY HARMONIC SELECTION

\( \rightarrow \) MATRIX MULTIPLY \( U^{-1} \xi \) \( \rightarrow \) RECONSTRUCTION OF SPECTRUM FROM DEALIASED HARMONIC COMP.

DETECT SINGULARITY \( \rightarrow \) UNIFORMLY SAMPLED INTERPOLATED DATA \( \rightarrow \) I.D.F.T. \((\text{CHIRP-Z})\) \( \rightarrow \) DEALIASED SPECTRUM

LOWPASS FILTER \( \rightarrow \) \( s(n) \)

STOP
4.16 SUMMARY

In this chapter the distribution of uniform impulse trains into a staggered periodic impulse sampling waveform has been considered with a view to finding a distribution pattern which will allow a general multiple band-limited signal to be sampled without ambiguity. This problem has been tackled by manipulation of the harmonic spectral structures of the periodic sampling patterns by a discrete matrix/vector representation. The key to the solution of the problem has been shown to be the partial inversion of the singular matrix $D$. This was modified so that the solution could be interpreted as the reduction of a general I.D.F.T. matrix $T$ to a non-singular lower dimensioned matrix $T_r$, by column and row deletion. It was possible to show that an ideal sampling pattern should have a fixed stagger interval between all the uniform sampling impulse trains and this interval should be based on a time grid, $\Delta_g$, where $\Delta_g$ was a prime submultiple of $T_r$. This type of sampling pattern is convenient because one can start with a basic stagger interval, say $n\Delta_g$ between two impulse trains, this will allow one sinusoid to be monitored. Then, if additional sinusoids need resolving, more impulse trains can be put at $2n\Delta_g$, $3n\Delta_g$, ..., $In\Delta_g$, ..., $Nn\Delta_g$ intervals. In this way they will overlap the $T_r$ periodic if $In\Delta_g$ is greater than $N\Delta_g$, but since $T_r/\Delta_g$ is a prime no coincidence will occur, see appendix 4. It was also shown that, heuristically that the very fact that $T_r/\Delta_g$ was equal to a prime integer was in itself sufficient for unambiguous sampling, but this was left unproven.
CHAPTER 5

TURBINE BLADE VIBRATION MEASUREMENT

5.1 INTRODUCTION

The purpose of this chapter is to show how two of the spectrum analysis techniques described in chapters three and four can be used as a partial solution to a practical engineering application. The application is to measure the vibrational motion of turbine blades by monitoring the vibrational displacement of the turbine blade tip with stationary proximity detection probes placed at the bladed disc’s periphery. The aim of this measurement is to give an indication to the turbine design engineer as to the degree and nature of the vibrational stress in a turbine blade configuration. This chapter is arranged in two distinct sections, firstly a short description is given of the physical engineering side of the problem and how it is related to a time sampling process. Secondly, a description of the way in which sampling processes, as designed in previous chapters, can be used in this particular situation. A final comment is given on the possible compatibility between sampling methods from chapters three and four.

5.2 PROBLEM DESCRIPTION

5.2.1 PHYSICAL DESCRIPTION OF VIBRANT ROTATING TURBINE BLADES

Turbine blading is used as a collective term to include all forms of blading in a turbine engine, ranging from the massive fan blades, which can be found on the compressor input of the new generation of turbofan jet engines, to the small driven blades. In this chapter we are interested in the vibrational motion set up in the rotor blading rather than the stator blading. The idealised steady state condition for all turbine blading should be a steady driving force, this being loaded against a steady aero-
VIBRATIONAL MODES OF A SIMPLE CANTILEVER

Fig. 5.1

FLAP MODES OF VIBRATION

1st

2nd

TORSIONAL VIBRATION MODE

EDGWISE VIBRATION MODE

(160)
dynamic fluid stream. In the practical situation conditions are far from ideal, in that all the blading is subject to an alternating load superimposed on a steady load. The causes of this unsteady state fall into two classes, firstly the interaction of the blading with the air stream as it rotates through the non-linear aerodynamic flow stream around the annular channel through the engine, reference (22). Secondly imperfections in the mechanical system can cause inbalanced forces which will provide periodic forcing functions causing such things as, gear box vibration and shaft whirling. These alternating forcing functions set up vibrations in the blading as it rotates so that in addition to designing for the steady stressing conditions the designer has to take into account the vibrational stresses and associated fatigue life reductions.

If the vibrational problem is simplified somewhat, and a single blade considered only, there are three possible types of vibration the blade can take on, figure 5.1.

1) Flap vibration
2) Torsional vibration
3) Edge vibration

In all these types it is possible to get natural frequencies of vibration, in this particular instance it is intended to concentrate on the measurement of the flap mode of vibration. Through the measurement of the amplitude and frequency of the flap vibrational motion it is possible to relate this, under certain conditions, to the stress level in the blade. The cyclic stresses can in turn be related to an estimate of the fatigue life of the blade, see references (23) to (25). In this way the critical vibrations can be isolated in the design stage of the turbine blading.
The maximum stress in a turbine blade vibrating at one of its natural frequencies can be related simply to the frequency and amplitude of vibration at the blade tip, references (22) and (26).

\[ \sigma_{\text{max}} = f a \left( \frac{2\pi y \sqrt{Ey}}{K} \right) \]  

where:

- \( K \) = Radius of gyration of root section
- \( y \) = Distance from neutral axis to point of maximum stress
- \( E \) = Young's Modulus of the blade material
- \( \rho \) = Density of blade material
- \( a \) = Amplitude of blade tip vibration
- \( f \) = Frequency of blade tip vibration
- \( \sigma_{\text{max}} \) = Maximum stress at root of blade

This relation forms the basis for blade stress level estimation using extrinsic blade tip displacement measurement. It must be noted that 5.1 is only true when the blade is vibrating at one of its natural frequencies, although at these frequencies usually the highest stress levels occur. The natural frequencies of an undamped cantilever are non-linearly distributed in frequency. From consideration of the mode of flexure it would be reasonable to assume the distribution of the natural frequencies in the ratio of 1 : 3 : 5 : ..., in fact, due to one end of the cantilever being clamped rather than suspended, the theory covering this case, references (34) and (35), state that the ratios are 1 : 6.27 : 17.65 : ... . These ratios will be modified in the practical case where the blade will be subject to damping and therefore the natural frequencies must be located by experimentation.

The Lycombing blade tip displacement measurement technique, to be described next, was originally devised to take advantage of the relation 5.1 by
The Lycoming Blade Tip Displacement Measuring System

Fig. 5.2

Turbine Blading

Blade Referencing Impulse Waveform

Detecting Probes

Blade 1 2 3 4 5 6

Pulse Position Modulated Waveform

Synchronising Impulse Waveform

(163)
monitoring the actual vibrational displacement of a blade, reference (27). From a "frozen" scope time trace of the vibrational displacement of the blading it was a simple matter to measure the amplitude and frequency of the waveform and so evaluate 5.1. It is equally informative to the stress engineer to be presented not with the vibrational waveform but with a spectral representation of the blade motion. In this chapter the Lycombing measurement system is used in a modified form so that it can cope with higher frequency vibrations. The information obtained from the system is then used to estimate a spectrum of the blade vibrations.

5.2.2 THE LYCOMBING MEASURING SYSTEM

The Lycombing system was devised to measure the blade tip displacement of turbine blades without physical contact, a diagrammatic illustration is given in figure 5.2. A detailed description of the system is given in reference (27). Basically the system has positioned on the periphery of the bladed disc a proximity detector which gives out a pulse when a blade passes. The proximity detector could be a capacitive, magnetic light guide or microwave type. If initially, pulses generated by one blade only are considered, the output from the detector will then consist of a uniform train of pulses with the engine running at a constant speed and no blade vibration. When the blade starts to vibrate the pulse train will be modified by pulse position modulation, figure 5.2. Demodulation of the resultant pulse train will retrieve the original vibration motion of the blade tip. Since each blade in the turbine disc will generate a pulse in the detector, additional referencing pulses are required during each revolution so that the resulting interwoven pulse trains can be indexed to the relevant blade.
The very fact that the Lycombing system has only one sensing probe per revolution limits the upper frequency of vibration on the blades that can be resolved without ambiguity. Improvement of this system with respect to frequency response means effectively increasing the sampling rate, necessitating the use of more sampling probes positioned around the bladed disc's periphery. The sampling scheme can be looked upon as a periodic sampling process, with a period equal to the time for one revolution of the bladed disc. Hence addition of other sampling probes, above the single one in the Lycombing system, is essentially constructing a staggered periodic sampling process. The practical difficulty with increasing the number of detection probes is that to sample at a rate above $2F_{\text{max}}$ of the vibrational waveform would require a prohibitive number of probes. As explained earlier, instead of trying to resolve the actual time vibration, as in the Lycombing system, it is more enlightening to measure or estimate the spectrum functions of the vibration. Power or complex spectrum functions can be used to estimate the peak stresses in the blading at various frequencies. Although phase is not a factor in this calculation, knowledge of relative phase between blade-blade vibrations is sometimes useful in recognising the type of forcing function. Therefore the problem is one of the measurement of the spectral function of the vibration of the blades with a sampling rate below $2F_{\text{max}}$.

5.3 A MATHEMATICAL DESCRIPTION OF THE SAMPLING PROCESS

The sampling process for measuring the vibration of the turbine blade is both a function of time and the spatial distribution of the probes around the bladed disc. The spatial dependence can be removed from the sampling process if the velocity of the engine is known and the acquisition of data can be viewed as a totally time sampling process. In figure 5.3 the sampling process is shown. The actual sampled values of $s(t_i + \Delta t_i)$, when the p.p.m. (165)
Sampling Error \( \Delta \varepsilon = f_n(s(t), V, V_v) \)
is demodulated, are given by:

\[ s(t_i + \Delta t_i) = V_b \Delta t_i \]

where \( V_b \) denotes the velocity of the blade tip when not vibrating.

The actual moment the blade passes the probe is therefore \( t_i + \Delta t_i \). In this particular sampling technique the actual measurements are deemed to occur at the \( t_i \) probe positions hence there is an error, \( \Delta \varepsilon \), between \( s(t_i + \Delta t_i) \) sampled points and the ideal sampled values taken instantaneously at \( t_i \), \( s(t_i) \). The error is a function of the amplitude of vibration at \( t_i \), \( s(t_i) \), the velocity of the blade tip, \( V_b \), and the additional velocity vector due to the blade tip vibration, \( V_v \).

\[ \Delta \varepsilon = f_n(s(t_i), V_b, V_v) \]

In the Lycoming system used to measure blade tip displacement on high speed turbines the effect of the error \( \Delta \varepsilon \) is treated as a frequency dependent error on the estimated spectrum, see appendix (6). If the blade tip velocity, \( V_b \), is at a minimum, much greater than the highest velocity \( V_v \) encountered in the blade vibration the error is kept within acceptable limits. Also the maximum deviation \( \Delta t_i \) of the blade tip sampling point is very small in comparison to the period of the highest permitted sampling frequency for \( s(t) \), see Note 1 at the end of this section. Hence by neglecting \( \Delta t_i \) the sampling time waveform is denoted by:

\[ e(t) = \sum_{i=1}^{I} \delta(t - t_i) \]

where:

\( t_i \) denotes the time sampling instants

(167)
Effective Time Sampling Pattern as the Turbine Accelerates

Fig 5.4a)

Uniform Sampling Pattern with the Turbine Running at a Constant Velocity

(168)
This corresponds to the spatial sampling of:

\[ e(\theta) = \sum_{i=1}^{I} \delta(\theta - \theta_i) \]

where:

- \( \theta \) denotes the position of a blade, in radians, relative to datum
- \( \theta_i \) denotes the arbitrary positioning of the probes
- \( I \) denotes the number of sampling probes

In both space and time the sampling processes are of course periodic in \( 2\pi \) and \( T_r \) respectively. Using the general law of motion these two forms of sampling can be related.

\[ \theta = \omega_1 t + \frac{1}{2} \alpha t^2 \]

where:

- \( \omega_1 \) denotes the initial angular velocity, radians/second
- \( \alpha \) denotes the angular acceleration, radians/second^2

The space sampling points \( \theta_i \) can therefore be related to the time sampling points \( t_i \), i.e.

\[ t_i = -\frac{\omega_1}{\alpha} + \sqrt{\left(\frac{\omega_1}{\alpha}\right)^2 + \frac{2\theta_i}{\alpha}} \]

\[ i = 1, 2, \ldots, I \]

So with knowledge of the velocity and acceleration of the turbine blade at the moment of vibration measurement it is possible to relate the actual space samples taken, to a time dependent measurement. In other words the problem is reduced to a time waveform sampling problem. This time sampling process is basically periodic in \( T_r \) but the value of \( T_r \) can vary if the engine is accelerating. So one gets sampling patterns as shown in figure 5.4a. It can be assumed that under test conditions when the turbine is accelerated that the rate of change of \( T_r \) with time can be made as small as necessary simply by accelerating the engine slowly. Therefore to a degree of approximation
Power Spectral Density Estimate of Typical Turbine Blade Vibration

Fig. 5.5

NATURAL FREQUENCY
FUNDAMENTAL
OF BLADE

1ST OVERTONE
OF BLADE

FORCED
HARMONIC BLADE VIBRATIONS
LOCKED TO THE ROTATIONAL
SPEED OF THE TURBINE

NATURAL FREQUENCIES
EXCITED BY RANDOM NOISE

Frequency

Power Density (NORMALISED)

0

0

0.51

1.0

1.68

2.0

2.5

0.0

0.0

1.0

2.0

5.0
the truly time periodic sampling process as shown in figure 5.4b can be assumed.

It was the very fact that the sampling process is periodic that initiated research into periodic sampling processes. Hence some of the techniques described in the earlier chapters are applicable to this sampling problem.

**NOTE 1**

In reference (36) a turbine blade tip vibration is modelled as a stationary random Gaussian process and the theoretical measurement of this process is shown by analysing the sampled values $s(t_i + \Delta t_i)$ as actually occurring at the $t_i + \Delta t_i$ instants. In this way the Gaussian waveform is sampled by a periodic impulse process with random perturbations, $\Delta t_i$, on the periodic sampling points. This sampling technique relies on the fact that the random perturbations are a significant proportion of the period of the sampling process. In this way the aliasing effects of the sidelobes of a low rate periodic sampling process are somewhat reduced.

5.4 THE SPECTRAL STRUCTURE OF THE TURBINE BLADE VIBRATIONS

In practice one cannot make the assumption that the vibrations present on the rotating blades are purely random or deterministic, they are in fact a mixture of both. Figure 5.5 shows a typical p.s.d. estimate plot of a 0.2 second section of vibration motion of a typical turbine blade. In this spectrum one can recognise various components. First of all at A and B the natural frequencies of the blade itself occur, the ratio of fundamental to first overtone being 1 : 3.26, which is in fact closer to the estimate based on the simply supported mode than to the estimate based on the clamped cantilever configuration. The higher overtones are ignored because
P.S.D. Estimates as the Turbine Accelerates

Fig. 5.6

Forcing Function Loci

Accelerations

Constant Speed

Frequency kHz

(172)
they contain very little power, and so the spectrum is treated as being bandlimited at 2.5 kHz. These natural frequencies, in the conditions of figure 5.5, are being excited by random aerodynamic buffeting so the part of the spectrum outlined in red constitutes a stochastic random process. Superimposed on the random spectral background are deterministic spikes which are interspaced at uniform intervals of \(1/T_r, 2/T_r, \ldots\) Hz across the spectrum plane. The spikes are at harmonics of the rotational frequency of the engine and are caused by fixed disturbances in the annular air flow stream in the engine which exert a periodic forcing function on the blading, see "wake vibrations" reference (19). In figure 5.6 a spectral history of the vibration blade is shown, giving the spectrum as the turbine accelerates to a constant speed. All the graphs in 5.6 are individually normalised to highlight the forced vibrant modes. In conditions f) and j) in 5.6 one can see that the fundamental and first overtone are excited. Using spectral history plots such as figure 5.6 allows one to determine the different causes of the vibration excitation, flutter, stall cell, wake or random forces, all these sources are explained in references (22, 27 and 28).

In order to attempt to measure this type of vibration, a simplified signal model was considered, this consisted of two parts, first a stochastic signal \(s(t)\) and secondly a deterministic signal \(d(t)\). The resulting summation of these two signals was then taken to be the model of the vibrational motion of a typical turbine blade. Additional restraints were placed on \(s(t)\) and \(d(t)\). Signal \(s(t)\) was deemed to be a stationary random signal, this is reasonable to assume under the conditions of constant engine velocity. The deterministic signal \(d(t)\) was restricted to being constructed from sinusoidal harmonics of the rotational speed of the engine.

\[
d(t) = \sum_{i=1}^{M} D_i \sin(2\pi n_i t/T_r + \phi_i)
\]

(173)
FIG 5.7

Sampled Data
Random + Deterministic
(Stochastic) (Sinusoidal)

Alternative Methods

Spectral Averaging Process
Random vibration tends to zero
Deterministic part is resolved

Filtering Process
Filtering out the spectral components at harmonics of the rotational speed of the turbine

Aliased Sinusoidal Components

Dealiasing Process
$D^{-1}c$

Subtraction of original time sinusoidal components

Complex Spectrum of Deterministic Harmonics

Random Stochastic Time Data

Estimation of the power spectral density of the random portion of the vibration

(174)
where:

\[ M = \{(T_r/\Delta g)\} \]

This takes into account the forcing function harmonics whether they coincide with the natural frequencies of the blade or not.

The spectral analysis of this type of vibration must be done with two types or processing, as is normally the case with this mixture of vibrations, see reference (7). First the deterministic harmonics must be removed from the combined waveform to leave the stochastic random data to be analysed separately, figure 5.7. The problem which is posed here is not one of simple spectral analysis but one of estimating the spectral functions from sampled data supplied at a mean rate below \( 2F_{\text{max}} \) for the combined signal, \( s(t) + d(t) \). The permitted practical sampling scheme must have a mean rate as low as possible, to minimise the number of proximity detectors around the bladed disc's periphery, and also be periodic with period \( T_r \) seconds. It is difficult to design a sampling scheme for \( s(t) \) or \( d(t) \) alone, but it is required to design a scheme suitable to sample both types of signal, \( s(t) + d(t) \), simultaneously. Before attempting to suggest a compatible sampling scheme, the two types of waveform must be analysed on their own merits.

5.5 THE RANDOM COMPONENT OF THE TURBINE BLADE VIBRATION

Looking first at the random vibrational motion of the turbine blading rotating at a constant speed with steady state air stream conditions, it is noted that the statistics of the vibration can be assumed stationary, at least to the second order. Given the assumption that the vibration is a stationary random process then the sampling method as described in chapter three can be applied to the estimation of the autocorrelation function. The
sampling probe distribution around the periphery of the bladed disc can be based on a period, $T_r$, of a cyclic difference set design. This will allow the autocorrelation function of the random stochastic portion of the blade vibration to be estimated while maintaining a sampling rate below $2F_{\text{max}}$.

In practical terms a random vibration with spectral power up to 2 kHz will require the autocorrelation function to be estimated at least at every 0.25 millisecond interval. If the turbine is rotating at say 6,000 r.p.m., then:

$$T_r = 0.01 \text{ seconds}$$

So to estimate the autocorrelation function adequately the spacing $\theta_g$ of the uniform angular grid on the periphery of the bladed disc will have to be less than $2\pi/40$ radians. This corresponds to a time grid spacing, $\Delta_t$, equal to a maximum of 0.25 milliseconds. The most ideally suited difference set sampling pattern would be the $(57, 8, 1)$ planar set with sampling points as in figure 5.8, according to:

$$(57, 8, 1) \quad 1, 6, 7, 9, 19, 38, 42, 49 \pmod{57}$$

This will give the lowest number of sampling probes, 8, yet allow the estimation of the p.s.d. up to 2.65 kHz at a turbine speed of 6,000 r.p.m. without spectral ambiguity occurring. The speed of estimating the autocorrelation function can be increased by adoption of nonplanar sets, i.e.

$$(37, 9, 2) \quad 1, 7, 9, 10, 12, 16, 33, 34 \pmod{37}$$

Giving measurement up to 1.85 kHz at 6,000 r.p.m.

or

$$(40, 13, 3) \quad 1, 2, 3, 5, 6, 9, 14, 15, 18, 20, 25, 27, 35 \pmod{40}$$

Giving measurement up to 2.00 kHz at 6,000 r.p.m.
THE POSITIONING OF DETECTION PROBES AROUND THE
PERIPHERY OF THE TURBINE'S BLADED DISC

DESIGN PATTERN — (37, 9, 2)

Fig. 5.8
The deterministic vibrations at harmonic frequencies of the turbines rotational speed, $1/T_r$, are difficult to sample at below $2F_{\text{max}}$ without ambiguity occurring because the periodicity in the sampling pattern is also $1/T_r$ Hz. Hence spectral overlap must always occur in the aliased spectrum of the sampled waveform between the harmonic components. This can be visualised by spectral convolution between the spectrum of the waveform $d(t)$ and the spectrum of the sampling waveform $e(t)$ where spectral power spaced at $1/T_r$ Hz intervals will be overlapped. In this sort of situation, of all the sampling/processing techniques discussed in previous chapters, only the one in chapter four allows aliased spectral harmonics to be de-aliased into the original harmonic structure.

As explained in chapter four, the principle of de-aliasing is based on the ability to represent the spectrum of the original and sampled signals in harmonics of the sampled waveform length $T$ seconds. It was also shown that due to the periodic nature of the sampling pattern it was possible to split the original and aliased spectral forms into subspectra each containing a selection of the total spectrum harmonics, i.e.

<table>
<thead>
<tr>
<th>Subspectrum</th>
<th>Harmonic Components (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1/T_r$ $2/T_r$ $3/T_r$</td>
</tr>
<tr>
<td>2</td>
<td>$1/T_r + 1/T$ $2/T_r + 1/T$</td>
</tr>
<tr>
<td>3</td>
<td>$1/T_r + 2/T$</td>
</tr>
<tr>
<td>4</td>
<td>... ... ...</td>
</tr>
</tbody>
</table>

In this particular problem of turbine blade vibration the only deterministic spectral elements that need resolving are harmonics at $1/T_r$, $2/T_r$, ..., Hz so only the subspectrum one need be considered in this application.
Therefore the procedure is one of resolving the aliased harmonic components out of the random background spectral noise then applying the de-aliasing process as described in chapter four so as to reconstitute the original harmonic spectral components. The positioning of the sampling or detecting probes around the engine periphery will determine the stagger intervals in the periodic staggered sampling process and as shown in chapter four they must be so positioned as to allow the de-aliasing process to operate correctly. It follows from the theory that with I sampling or detecting probes around the periphery of the bladed disc it is possible to detect unambiguously a signal with at maximum \( \left\lfloor I/2 \right\rfloor \) harmonics.

The main shortcoming with this sort of technique for resolving harmonic components is that the frequency of the harmonic components present in the vibration must be known a priori. This is simply because the de-aliasing process of chapter four is essentially a multiple bandlimited interpolation process in which one must know the extent of the bandlimited spectral bands. Therefore summarising, the use of periodic staggered sampling as envisaged in chapter four will allow monitoring of the phase and amplitude of any number or combination of harmonics of \( 1/T_f \) Hz within the frequency space \( \pm F_{\text{max}} \) but the frequency of the harmonics must be known a priori.

5.7 COMPATIBILITY BETWEEN RANDOM AND DETERMINISTIC SAMPLING PROCESSES

It was suggested at the end of chapter four that if periodic staggered sampling processes were based on a time grid, \( \Delta_g \), so that \( T_r/\Delta_g \) is a prime number integral, then de-aliasing of any combination of harmonics, \( n/T_r \), \( n = 1, 2, \ldots \), up to \( 1/2\Delta_g \) Hz irrespective of the positioning of the staggered impulse trains, would be possible. Sampling methods designed from difference sets, \( (v, K, \lambda) \), which are based on prime values of \( v \), will form (179)
a periodic stagger sampling pattern with ratio $T_r/\Delta_g$ equal to a prime number. Therefore sampling with a $(v, K, \lambda)$ formed pattern will not only produce samples which adequately estimate the p.s.d. of the stochastic part, $s(t)$, but will also allow de-aliasing of the deterministic harmonics, $d(t)$. Consequently a staggered periodic sampling pattern based on a prime order difference set design will act as a compatible sampling scheme for the vibrational motion of a turbine blade. Samples taken from such a sampling process will allow $\{K/2\}$ harmonic components to be monitored up to a frequency of $F_{max}$ without ambiguity and from the random portion of the vibration a p.s.d. function can be estimated up to $F_{max}$.

5.8 SUMMARY

It has been shown in this chapter how the capabilities of the simple Lycombing turbine blade tip displacement measuring system can be extended by the addition of more sampling probes in a specially designed sampling pattern and the application of spectral analysis techniques. The restriction on the minimum number of sampling probes is dictated by the number of elements, $K$, in the difference planar set design used to sample the random portion of the vibrational motion. If the number of deterministic harmonic components present is above $K/2$ then the minimum number of sampling probes must be increased by employing a nonplanar set so that $K$ is equal to or greater than twice the number of deterministic harmonics. Thus an effective saving in sampling rate is only possible when some of the harmonics of $1/T_r$ Hz are missing below $1/\Delta_g$ Hz. If all harmonics from $1/T_r$ Hz to $1/\Delta_g$ Hz are present then all periodic staggered impulse trains would be required, hence uniform sampling at the Nyquist rate would be taking place. Therefore the distribution of the additional probes around the bladed disc's periphery is still subject to the Nyquist relation for sampling rate versus bandwidth when
analysing deterministic waveforms. What is gained by the use of the sampling distributions as described in chapter four is that advantage can be taken of the null regions in a sparsely distributed power spectrum to reduce the sampling rate without causing spectral ambiguity wherever the null occurs.
GENERAL CONCLUSION

In this thesis, four types of sampling and processing techniques have been presented. This is but a small percentage of the methods that can be designed to suit particular spectral analysis applications. The common factor in all these sampling methods is the way in which a priori information about the signal has been used to design a specific sampling scheme. In chapter one and four, signal frequency band location information was required, in chapter two and three the signal was known a priori to be a bandlimited stationary random process. In all these cases the information has been used to adopt a restricted sampling method to the spectral estimation of the signal.

A general conclusion on the sampling methods for continuous processes can be split into two sections depending on the reasons for requiring discrete data.

(a) To have a discrete equivalent description of the continuous process.

(b) To acquire discrete data for the estimation of various parameters of the original continuous process.

In case (a) it is required that the total information contained in the original continuous process is transferred unambiguously to the discrete samples, hence the continuous process can be reconstructed if required. If a finite section of a continuous process contains $H$ pieces of information, be it a deterministic or a random process, then for a total description of the process by the least number of samples it is necessary that $H$ linearly independent samples be taken. In this way the information originally
contained in the continuous function will have been transferred unambiguously to a discrete representation. As shown in chapter four one of the ways of describing the discrete information contained within a finite section of a continuous process can be thought of as the Fourier harmonics which can form the finite section of the process. Hence two linearly independent pieces of information are required for each harmonic present in the continuous process. An indication of the linear independence between samples taken is given in chapter one where no spectral overlap or aliasing in the spectrum of the samples taken indicates independence. Conversely spectral overlap indicates the possibility of dependence but in chapter four it is shown that in fact dependence is a function of the sampling method and continuous process and independence can be assured even with spectral overlap.

If the purpose of sampling is to gather information to estimate a statistical function of the continuous process, then the criterion for sampling is slightly different to that of case (a). Information has to be gathered in such a manner that the expectation of the estimate derived from the samples is equal to the same estimate derived from the continuous process. An estimate requires convergence and therefore the sampling process should supply samples with as small a covariance as possible to facilitate efficient convergence for a given number of samples. The actual physical placement of the sampling points must be considered in context with the type of continuous process to be sampled. As shown in chapters two and three, strictly fixed sampling rates are not required for estimation of second order statistical functions from stationary waveforms.

Individual mention must be given to the cyclic difference set sampling patterns as shown in chapter three, for they possess some very unusual autocorrelation and spectral properties. A great deal of thought has been
given to the use of these patterns in other than sampling techniques but they still elude application.

Data acquisition, not just simply for spectral analysis application, is becoming an increasingly important aspect of signal processing. Advanced methods of computer implemented digital signal processing are now beginning to appear on the industrial scene. Data acquisition can be easily placed under the control of the computing system so in future a degree of flexibility in the sampling to suit the continuous process could be envisaged.

In this thesis the application of sampling methods has been highlighted as a flexible link in the process chain of digital spectrum analysis and if future readers recognise sampling as other than a forgotten physical operation then all has not been in vain.
APPENDIX 1

BLOCK/SET DESIGN (Introduction and Definitions)

A set design in practical terms is a sequence of $K$ positive integer numbers, i.e.

$$B_1 = b_1, b_2, b_3, \ldots, b_K$$

The values of $\{B_1\}$ are chosen from a total field, $F$, of positive integers. The order or magnitude of the field is denoted by $v$.

$$F = a_1, a_2, a_3, \ldots, a_v$$

The sequence elements of $\{B_1\}$ are chosen in a particular ordered fashion, the resultant sequence is called a set or block, $B$, which is a subset of $F$, i.e.

$$B_1 \subseteq F$$

Similarly, other sets $B_j$ can be generated which are also subsets of $F$. A simple way of describing the complete numbers of all possible blocks is to use an incidence matrix, $Q$. This is simply a matrix where the column numbers represent the elements of the field $F$ and the row numbers the particular block. If an element is in a particular set one is placed in the relative position in the matrix, if not a zero, i.e.

If $a_1$ is in block $B_j$ then matrix element:

$$q_{ij} = 1$$

If $a_1$ is not in the block $B_j$, then:

$$q_{ij} = 0$$

It is simple to show that given a particular field $F$ of $v$ integers, then there are possibly $2^v$ different block designs.

In this way there are no restrictions placed on the block designs and if any $b$ blocks are grouped together to form an incidence matrix, the number of possible different matrices is $\binom{2^v}{b}$.
BALANCED INCOMPLETE BLOCK DESIGNS

Restrictions are now placed on the form of the blocks that can be included in a group described by an incidence matrix. This in effect restricts the dimension $b$ of the incidence matrix so the resultant incidence matrix is an incomplete descriptor of the total block design available. The type of restrictions which are to be placed on the incidence matrix are balanced in the sense that each block is subject to the same constraints. Listing three constraints:

1) Each block $B_j$ must contain the same number of elements, $K$.
2) Each element, $a_1$, is in the same number of blocks, $r$.
3) For each unordered pair, $a_i, a_j$, of elements the number of blocks containing both of them is the same number, $\lambda$.

So the general notation associated with an incomplete block design is $(b, v, r, K, \lambda)$. From the restrictions 1), 2), 3) the general equations connecting the various parameters are:

\[ bK = vr \quad \text{and} \quad r(K - 1) = \lambda(v - 1) \]

CYCLIC BLOCK DESIGN

To relate the incomplete block design to a cyclic block design one puts a final restriction on the form of the incidence matrix.

4) The incidence matrix must be cyclic, that is $Q$ must be circulant.

This automatically means that $b = v$, because $Q$ becomes square ($v \times v$). From the above equations it can be said that $r = K$. So the description of a cyclic block design can be noted in terms of the $(v, K, \lambda)$ parameters only.
CYCLIC DIFFERENCE SETS

It is now a consequence of restrictions 1) to 4) that a \((v, K, \lambda)\) cyclic block design is also a perfect difference set design. To explain this, consider the case where \(\lambda = 1\). Choose any arbitrary element combination \((a_j, a_i)\), it should be possible to find these two within the same block design once in the incidence matrix \(Q\), due to 3). If the block containing \(a_j\) and \(a_i\) is called \(B\):

\[
B = b_1, b_2, ..., b_K
\]

Then because \(Q\) is cyclic a new sequence generated by shifting the elements cyclically is also a block design contained within \(Q\), i.e.

\[
B_e = (b_1 + e, b_2 + e, ..., b_K + e) \mod (v)
\]

where \(e\) denotes an arbitrary positive integer.

So each row of matrix \(Q\) is an automorphism of any other row, this means that the pattern of each row is identical but shifted relative to each other. This means that if any combination of \(a_i, a_j\) can be found in \(Q\) once, then the difference value generated by differencing \(a_i\) and \(a_j\)

\[
d_k = |a_j - a_i|
\]

must be present in every block represented by \(Q\). Hence in any block represented by \(Q\), any difference value from 0 to \(v\) can be generated once by differencing the elements in any one block:

\[
d_k = |a_j - a_i|, \quad d_k = 1, 2, ..., v \quad j = 1, 2, ..., K
i = 1, 2, ..., K
\]

Each row of \(Q\) represents basically the same difference set since each is a shifted version of the basic pattern. Thus a basic difference block pattern

(187)
design is denoted by \((v, K, \lambda)\) in which the difference sequence has \(K\) elements in a field of maximum dimension \(v\) and it is possible to generate \(\lambda\) differences from 1 to \((v - 1)\) without omission.

All the theory and description given in this section helps define a difference set and link it with the rest of the block design theory but construction is another matter.

**CONSTRUCTION OF DIFFERENCE BLOCK DESIGNS**

**The Multiplier Theorem**

A specific property of all known difference sequences is that they possess a nontrivial integer multiplier which when applied to the sequence simply regenerates its own elements (isomorphism). Using the above sequence \(B :=\)

\[
B' = (b_1 g, b_2 g, b_3 g, \ldots, b_K g) \mod (v)
\]

where \(g\) is a positive integer

If \(B'\) is calculated, then the sequence re-ordered, it is equivalent to \(B\) sequence. As an example a sequence \((13, 4, 1)\) with \(g = 3\).

\[
B = 0, 1, 3, 9 \pmod{13}
\]

\[
B' = 0, 3, 9, 1
\]

Many of the construction theorems are based on finding the multiplier for a given \((v, K, \lambda)\) system, if this exists. There is a great mass of theory connected with the subject of proving that difference sequences based on various combinations of \((v, K, \lambda)\) exist. Equally there is a class of theories devoted to the construction of the proven difference sequences. References (29) to (32) cover a great amount of information on the subject of difference set construction. As an example in the use of one of the many basic theorems the construction of the difference set \((37, 9, 2)\) will be shown.
Theorem:

Let $D$ be a difference set with parameters $(v, K, \lambda)$. Let $p$ be a prime of $K - \lambda$ and suppose that $p$ is not a submultiple of $v$ and $p > \lambda$. Then $p$ is a multiplier of the difference set.

The difference set $(37, 9, 2)$ has initially been proven to exist.

$$K - \lambda = 7$$

Hence since $K - \lambda$ is a prime in itself, set $p$ equal to 7, $v$ is also a prime, $v = 37$, which consequently has no submultiples. So the selection of $p = 7$ fulfils all the requirements of the above theorem. To actually construct the difference set, initially assume that one of the elements is equal to one. Then by successive multiplier operations the whole set can be calculated.

Using

$$b_i g^n \pmod{v} = b_j$$

$$n = 1, 2, \ldots, K$$

where $b_i$ and $b_j$ are elements of the sequence connected by multiplier $g$.

Step 1

$$1 \times 7 \pmod{37} = 7$$

Step 2

$$7 \times 7 \pmod{37} = 12$$

Step 3

$$12 \times 7 \pmod{37} = 10$$

In this way the total set of 9 numbers can be generated.

$$1, 7, 9, 10, 12, 16, 26, 33, 34 \pmod{37}$$

Most of the construction theorems are very involved in number theory so the best practical way to use these $(v, K, \lambda)$ sets is to refer to a table of listed sets which have been constructed, references (30) and (32). In many combinations of $(v, K, \lambda)$ no difference set exist. In others they do exist but have not yet been constructed. To elaborate, a survey has been carried out for sets with $K$ in the interval:

$$\quad (189)$$
\[ 3 \leq K \leq 50 \]

with

\[ K < \frac{\nu}{2} \]

Remembering that \( K(K - 1) = \lambda(\nu - 1) \), exactly 268 choices of \((\nu, K, \lambda)\) fulfil these requirements. In 101 cases the nonexistence has been proven by theory. Of the missing 167, 46 difference sets have been constructed and the nonexistence of a further 121 have been found, see reference (29).
APPENDIX 2

THE VARIANCE OF THE POWER DENSITY SPECTRUM OF WHITE NOISE AS DERIVED FROM NON-UNIFORM PERIODIC SAMPLING

Power spectral density estimate:

\[ P_s(f_k) = \frac{\Delta \bar{r}}{N} \left[ A_s^2(f_k) + B_s^2(f_k) \right] \]

where:

\[ A_s(f_k) = \sum_{n=1}^{N} s(t_n) \cos 2\pi f_k t_n \]

\[ B_s(f_k) = \sum_{n=1}^{N} s(t_n) \sin 2\pi f_k t_n \]

Variables \( A_s(f_k) \) and \( B_s(f_k) \) are zero mean Gaussian distributions (central limit theorem).

Let

\[ \frac{A_s^2(f_k)}{\text{Var}(A_s(f_k))} = X \quad \text{and} \quad \frac{B_s^2(f_k)}{\text{Var}(B_s(f_k))} = Y \]

Then \( X \) and \( Y \) are distributed according to \( \chi_1^2 \).

Hence \( E_p(X + Y) = 2 \)

Since the mean of a normalised \( \chi_2^2 \) function is equal to 2. The variances of \( A_s \) and \( B_s \) depend on the nature of the non-uniform sampling pattern \( e(t) \), i.e.

\[ \text{Var}[A_s] = E_p\left(\sum_{n=1}^{N} s(t_n)e(t_n) \cos \omega_k t_n \right)^2 \]

Since \( s(t_n) \) is a stationary random variable with zero mean, the expectation and square can be taken inside the summation, i.e.

\[ \text{Var}[A_s] = \sigma^2 \sum_{n=1}^{N} e^2(t_n) \cos \omega_k t_n \]

where:

\[ E_p\left(s(t_n)^2\right) = \text{Var}[s(t_n)] = \sigma^2 \]

(191)
Therefore the variance depends on the underlined summation terms, A2.5, and in general this will vary with $f_k$ for non-uniform sampling. To calculate $\text{Var}[A_s]$ is a simple matter, rewriting the summation in terms of a double angle formula one gets:

$$\text{Var}[A_s] = \sigma^2 \left[ \frac{N_1}{2} - \frac{1}{2} \sum_{n=1}^{N} e(t_n) \cos 2\omega_k t_n \right]$$

The summation in this formula can be calculated by a modified Fast Fourier Transform algorithm if necessary.

Denoting

$$V = \sum_{n=1}^{N} e(t_n) \cos 2\omega_k t_n$$

The general results are:

$$\text{Var}[A_s(f_k)] \text{ and } \text{Var}[B_s(f_k)]$$

$$= \sigma^2 \frac{N_1}{2} \quad f_k \neq j/2T_r \text{ Hz}$$

$$j = 0, \pm 1, \pm 2, \ldots$$

$$\text{Var}[A_s(f_k)] = \sigma^2 \frac{N_1}{2} (1 + V) \quad f_k = j/2T_r \text{ Hz}$$

$$\text{Var}[B_s(f_k)] = \sigma^2 \frac{N_1}{2} (1 - V) \quad f_k = j/2T_r \text{ Hz}$$

where:

$$N_1 = N(\frac{K}{v}) \text{ for a } (v, K, \lambda) \text{ non-uniform periodic sampling pattern}$$

If these values of variance are now substituted into equation A2.4, it can be shown that:

$$E_p \{ A_s^2(f_k) + B_s^2(f_k) \} = N_1 \sigma^2$$

for all values of k.
Consequently, the p.s.d. estimate in the limit, as $N$ tends to infinity can be expressed as:

$$\phi(f_k) = \frac{N_1}{N} \mathbb{E}_P \{P_s(f_k)\} = \Delta_{\alpha}^2 \frac{K}{\nu}$$

for all $f_k$

The Variance Of $P_s(f_k)$

$$\text{Var}[P_s(f_k)] = \left( \frac{\Delta_{\alpha}}{N} \right)^2 \mathbb{E}_P \{ [(A_s^2(f_k) - \overline{A_s^2}) + (B_s^2(f_k) - \overline{B_s^2})]^2 \}$$

Since $(A_s^2 - \overline{A_s^2})$ and $(B_s^2 - \overline{B_s^2})$ are random variables, one can rewrite the above equation.

This can be rewritten as:

$$\text{Var}[P_s(f_k)] = \left( \frac{\Delta_{\alpha}}{N} \right)^2 \left[ \mathbb{E}_P \{ (A_s^2 - \overline{A_s^2})^2 \} + \mathbb{E}_P \{ (B_s^2 - \overline{B_s^2})^2 \} \right]$$

Now using $X$ as previously defined:

$$\mathbb{E}_P \{(X - \overline{X})^2\} = \mathbb{E}_P \{X^2\} - \overline{X^2}$$

Since $X$ is distributed as a normalised $\chi^2_1$ function, its variances must be equal to 2.

$$\mathbb{E}_P \{X^2\} - \overline{X^2} = 2$$

Putting in the actual value for $X$ and $\overline{X}$ at $f_k = j/2T_r$, $j = 0, \pm 1, \pm 2, \ldots$ one gets:

$$\mathbb{E}_P \{ (A_s^2 - \overline{A_s^2})^2 \} = (a^2(1 + V)N_1)^2 / 2$$

(193)
Therefore:

\[ \text{Var} |A_s^2| = (\sigma^2 (1 + \nu) N_1)^2 / 2 \]

Similarly:

\[ \text{Var} |B_s^2| = (\sigma^2 (1 - \nu) N_1)^2 / 2 \]

Combining into equation A2.7

\[ \text{Var}[P_s(f_k)] = \sigma^4 \left( \frac{K}{\nu} \right)^2 \Delta g^2 (1 + \nu^2) \left| f_k = j/2T_r \right. \]

\[ j = 0, \pm 1, \pm 2, \ldots \]

Similarly at \( f_k \neq j/2T_r \)

\[ \text{Var}[P_s(f_k)] = \sigma^4 \left( \frac{K}{\nu} \right)^2 \Delta g^2 \left| f_k \neq j/2T_r \right. \]

\[ j = 0, \pm 1, \pm 2, \ldots \]

where:

\[ \left( \frac{N_1}{N} \right) = \left( \frac{K}{\nu} \right) \]

This situation can be somewhat alleviated by modifying the spectral harmonic components \( A_s(f_k) \) and \( B_s(f_k) \) at \( f_k = j/2T_r \), \( j = 0, \pm 1, \pm 2, \ldots \)

Since the frequency variation of variance of \( P_s(f_k) \) is caused by the mean values of \( \text{Var}[A_s(f_k)] \) and \( \text{Var}[B_s(f_k)] \) not being equal, it is logical to modify these inequalities in the transform stage, i.e.

\[ (A_{sm}(f_k))^2 = \frac{(A_s(f_k))^2}{(1 + \nu)} \]

similarly

\[ (B_{sm}(f_k))^2 = \frac{(B_s(f_k))^2}{(1 - \nu)} \]

(194)
Hence:

\[ P_{sm}(f_k) = \frac{\Delta g}{N} \left[ A_{sm}^2(f_k) + B_{sm}^2(f_k) \right] \]

where

\[ \text{Var}[P_{sm}(f_k)] = \sigma^4 \left( \frac{k}{v} \right)^2 \Delta g^2 \mid \text{all } f_k \]
APPENDIX 3

COVARIANCE BETWEEN SPECTRAL ESTIMATES DERIVED FROM A NON-UNIFORM SAMPLING PROCESS

Given a power spectrum estimate \( P_s(f_k) \) at harmonic frequencies \( f_k = 0, \pm 1/T, \pm 2/T, \pm 3/T, \ldots, \pm (N-1)/2T \). It is required to calculate:

\[
\text{Cov}[P_s(f_{i}), P_s(f_{j})]
\]

This is equivalent to:

\[
E_p\{[P_s(f_{i}) - \Phi(f_{i})][P_s(f_{j}) - \Phi(f_{j})]^*]\}
\]

where:

* denotes complex conjugate

Trouble arises here because since \( P_s(f_k) \) is distributed as a \( \chi^2 \) distribution its mean is not zero. Equation A3.1 is replaced by:

\[
\text{Cov}[P_s(f_{i}), P_s(f_{j})] = E_p\{P_s(f_{i})P_s(f_{j})\} - \Phi(s_{i})\Phi(s_{j}) \quad A3.2
\]

Circular convolution between the complex spectra of the time waveform, \( s(t) \), and sampling process, \( e(t) \), replaces \( P_s(f_k) \).

\[
\text{Cov}[P_s(f_{i}), P_s(f_{j})] = \]

\[
E_p\left(\sum_{k=0}^{K} S(f_{k})E(f_{i} - f_{k})|^2 \cdot \sum_{k=0}^{K} S(f_{k})E(f_{j} - f_{k})|^2\right) - \sum_{k=0}^{K} \Phi(f_{k})|E(f_{i} - f_{k})|^2 \cdot \sum_{k=0}^{K} \Phi(f_{k})|E(f_{j} - f_{k})|^2 \quad A3.3
\]

To simplify this expression the squared modulus terms within the expectation bracket can be expanded out into a summation of the terms \( \gamma \) and \( \beta \).

(196)
where

\[ A3.4 \]

\[
\frac{1}{K} \sum_{k=0}^{K} |S(f_k)|E(f_i - f_k)|^2 = \gamma_i + \beta_i
\]

Since \( S(f_k) \) is a random variable with zero mean the term \( \gamma_i \) in equation A3.4 represents the estimate of the mean of the left hand side of the equation, while \( \beta_i \) is a zero mean random variable.

Rewriting the expectation part of equation A3.3:

\[ A3.6 \]

\[
E_p\{(\gamma_i + \beta_i)\cdot(\gamma_j + \beta_j)\}
\]

where \( \beta_i \) and \( \beta_j \) are zero mean random variables. Taking expectations inside brackets.

\[
E_p\{\gamma_i\gamma_j\} + E_p\{\beta_i\beta_j\} + E_p\{\beta_i\gamma_j\} + E_p\{\beta_j\gamma_i\}
\]

Resubstituting in equation A3.3

\[ A3.7 \]

\[
\text{Cov}_p(p_s(f_i), p_s(f_j)) = E_p\{\gamma_i\gamma_j\} + E_p\{\beta_i\beta_j\} - \phi_s(f_i)\cdot\phi_s(f_j)
\]

This could be simplified one stage more by expanding the \( \gamma \) terms. Since

\[
|S(f_k)|^2 \text{ is distributed by } \chi^2_f \text{ function for } k \neq 0, N/2 + 1 \text{ it can be defined}
\]

as:

\[ A3.8 \]

\[
|S(f_k)|^2 = \psi(f_k) + \xi(f_k)
\]

where

\( \psi(f_k) \) denotes the true power spectrum density of \( s(t) \)

\( \xi(f_k) \) denotes a random variable with zero mean

(197)
Then writing $E_p \{ \gamma_i \gamma_j \}$ in these terms:

$$E_p \{ \gamma_i \gamma_j \} = E_p \{ \sum_{k=0}^{K} \phi (f_k) + \xi (f_k) \} \cdot |E(f_i - f_k)|^2. \quad \gamma_j$$

Since $\xi (f_k)$ is a zero mean random variable if one multiplies out the above terms within the estimation bracket and applies the expectation condition the only terms which are non-zero are:

$$\sum_{k=0}^{K} \phi (f_k) |E(f_i - f_k)|^2 \cdot \sum_{k=0}^{K} \phi (f_k) |E(f_j - f_k)|^2$$

and

$$E_p \{ \sum_{k=0}^{K} \xi (f_k) |E(f_i - f_k)|^2 \cdot \sum_{k=0}^{K} \xi (f_k) |E(f_j - f_k)|^2 \}$$

Since the first set of terms is equivalent to $\phi_s (f_i) \phi_s (f_j)$, $E_p \{ \gamma_i \gamma_j \}$ can be resubstituted in equation A3.7 to give:

$$\text{Cov} \left[ P_s (f_i), P_s (f_j) \right] =$$

$$E_p \{ \sum_{k=0}^{K} \xi (f_k) |E(f_i - f_k)|^2 \cdot \sum_{k=0}^{K} \xi (f_k) |E(f_j - f_k)|^2 \} + E_p \{ \beta_i \beta_j \} \quad \text{A3.9}$$

Since $\xi (f_k)$ is defined as a random variable with zero mean

$$\text{Cov} \left[ P_s (f_i), P (f_j) \right] =$$

$$\sum_{k=0}^{K} E_p \{ (\xi (f_k))^2 \} |E(f_i - f_k)|^2 \cdot |E(f_j - f_k)|^2 + E_p \{ \beta_i \beta_j \} \quad \text{A3.10}$$

where

$$E_p \{ (\xi (f_k))^2 \} = \text{Var} P(f_k)$$

$$|E(f_i - f_k)|^2 = P_e (f_i - f_k)$$

(198)
Rewriting equation A3.10:

\[
\text{Cov}\left[ P_s(f_i), P_s(f_j) \right] = \sum_{k=0}^{K} \text{Var} P(f_i) \cdot P_e(f_i - f_k) \cdot P_e(f_j - f_k) + E_p(\beta_i \beta_j)
\]

A3.11

The physical significance of this equation is that the degree of covariance between any two points on the spectrum \( P_s(f) \) at \( f_i \) and \( f_j \) is defined by the correlation between two spectral forms of \( E(f) \) centred at \( f_i \) and \( f_j \). In figure A.1 two examples of possible covariance are given. The term \( E_p(\beta_i \beta_j) \) takes no part in the covariance function if \( i \neq j \) since \( \beta \) is defined as a zero mean random variable.
NONPERIODIC SAMPLING WAVEFORM

Fig. A.1

OVERLAP OF WINDOW FUNCTIONS, \( P_e(f) \),
CENTRED AT \( f_i \) AND \( f_j \) Hz — RESULTING IN NONZERO COVARIANCE

PERIODIC SAMPLING WAVEFORM

Fig. A.2

SPECTRAL INTERWEAVING BUT NO OVERLAP — ZERO COVARIANCE
CONSTRUCTION OF CYCLIC CONVOLUTION MATRICES (SPECTRAL CASE)

Convolution between two spectral vectors $\tilde{d}$ and $\tilde{b}$ can be described by a matrix multiplication formulation. A convolution matrix is constructed out of one of the vectors, consider constructing $\tilde{D}$ out of vector $\tilde{d}$.

$$ d^T = a, b, c, \ldots, c^*, b^* $$

T denotes transpose.

Then matrix $\tilde{D}$ is of the form:

$$
\begin{bmatrix}
  a & b^* & c^* & \ldots & d & c & b \\
  b & a & b^* & c^* & \ldots & d & c \\
  c & b & a & b^* & \ldots & d \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  c^* & \ldots & \ldots & \ldots & a & b^* \\
  b^* & c^* & d^* & \ldots & \ldots & b & a
\end{bmatrix} = \tilde{D}
$$

The matrix is cyclic in nature (known as a circulant) with dimensions $N \times N$, where the first column is equal to $\tilde{d}$ and subsequent columns are cyclically shifted versions of the previous column. The cyclic convolution is expressed by

$$ \tilde{c} = \tilde{D} \tilde{b} $$

where

$\tilde{c}$ is the discrete result of the convolution between the original vectors $\tilde{d}$ and $\tilde{b}$.
APPENDIX 5

PRIME ORDER D.F.T. MATRICES

The general form of an I.D.F.T. matrix is:

\[
I = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & w^1 & w^2 & \cdots & \\
1 & w^2 & w^4 & \cdots & \\
& \ddots & \ddots & \ddots & \\
1 & \cdots & \cdots & \cdots & 1
\end{bmatrix}
\]

where

\[w = e^{j \frac{2\pi}{N}}\]
\[j = \sqrt{-1}\]

The elements are defined by

\[T_{ij} = w^{k \text{MOD}(N)}\]  \hspace{1cm} A5.1

where

\[a = (j - 1) \hspace{1cm} k = (i - 1)\]
\[a < N \hspace{1cm} k < N\]

\[N\] is a prime number

A condition for two elements in a row to be equal is

\[\ell_1 k \text{MOD}(N) = \ell_2 k \text{MOD}(N)\]  \hspace{1cm} A5.2

\[\therefore (\ell_1 - \ell_2)k = mN\]  \hspace{1cm} A5.3

where

\[m\] is an arbitrary integer ranging from 0 to \(N - 1\).

Using :-

\[k < N\]
\[(\ell_1 - \ell_2) < N\]

also \(N\) is prime,

one can expand equation A5.3 into prime factors :
\[ p_1 \times p_2 \times \ldots = p_a \times p_b \times \ldots \times N \quad \text{A5.4} \]
\[ p_1 < N \quad i = 1, 2, \ldots \infty \]

This condition cannot exist because no N prime is present on the left hand side of equation A5.4. Hence the condition of inequality is always true, i.e.:

\[ \lambda_1 \text{ MOD}(N) \neq \lambda_2 \text{ MOD}(N) \quad \text{A5.5} \]

for all \( \lambda \).

Similarly the condition for any column can be proven, i.e.

\[ \xi_1 \text{ MOD}(N) \neq \xi_2 \text{ MOD}(N) \quad \text{A5.6} \]

So in a prime order I.D.F.T. matrix no exponent power exists more than once in any row or column.
APPENDIX 6

TURBINE BLADE DISPLACEMENT MEASUREMENT ERROR

With reference to figure A.3, the sampling instants for \( s(t) \) are timed at \( t_i \) while the actual measurements are taken at \( t_i + \Delta t_i \). The worst condition for error, \( \Delta \varepsilon \), is first approximated by taking the maximum rate of change of the system to be equal to the product of twice the dynamic bandwidth of the system, \( F_{\text{max}} \), and the maximum peak amplitude of the vibration, \( A \), giving \( 2F_{\text{max}}A \), see figure A.4. Then under worst conditions anywhere on the signal path the equations describing the measurement error are:

\[
s(t_i) + \Delta \varepsilon_{\text{max}} = V_b \Delta t_i
\]

\[
2F_{\text{max}}A \Delta t_i = \Delta \varepsilon_{\text{max}}
\]

giving:

\[
\Delta \varepsilon_{\text{max}} = s(t_i)\left(\frac{V_b}{2F_{\text{max}}A} - 1\right)^{-1}
\]

This equation specifies the limits for an error given \( s(t_i) \) and \( V_b \), the blade-tip velocity when not vibrating. It is now assumed that the actual error \( \Delta \varepsilon \) is uniformly distributed within these limits as specified by \( \pm \Delta \varepsilon_{\text{max}} \). The measurement errors \( \Delta \varepsilon \) are assumed to be independent of each other, so the mean squared value of the errors for a given \( s(t) \) can be specified by the summation of the separate variances of each measurement error.

\[
\overline{\Delta^2 \varepsilon} = E_p \left\{ \frac{1}{N} \sum_{n=1}^{N} \Delta^2 \varepsilon_n \right\}
\]

\[
\overline{\Delta^2 \varepsilon} = \frac{1}{N} \sum_{n=1}^{N} E_p \left\{ \Delta^2 \varepsilon_n \right\}
\]

where:

\[
E_p \left\{ \Delta^2 \varepsilon \right\} = \sigma^2_n
\]

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The variance of a random variable uniformly distributed between ±Δε\text{max} can be shown to be equal to \( \frac{\Delta^2\epsilon}{3} \).

From A6.3 \( \sigma_n^2 \) becomes:

\[
\sigma_n^2 = \frac{s^2(t)}{3} \left( \frac{V_b}{2F_{\text{max}} A} - 1 \right)^{-2} \tag{A6.6}
\]

The vibration \( s(t) \) is assumed a random variable with zero mean. Hence \( \sigma_n^2 \) is a random variable and to calculate its mean, as \( N \) tends to infinity, A6.5 is replaced by:

\[
E_p \{ \sigma_n^2 \} = \int_{-\infty}^{\infty} \sigma_n^2(x) p_s(x) \, dx \tag{A6.7}
\]

where:

- \( p_s(x) \) denotes the probability density of the amplitude of \( s(t) \)
- \( x \) denotes the amplitude of \( s(t) \)

Writing in full:

\[
\Delta^2\varepsilon = E_p \{ \sigma_n^2 \} = \frac{1}{3} \left( \frac{V_b}{2F_{\text{max}} A} - 1 \right)^{-2} \int_{-\infty}^{\infty} s^2(t) p_s(x) \, dx \tag{A6.8}
\]

The integral of A6.8 must equal the variance of \( s(t) \), \( \sigma_s^2 \). Hence the mean squared error can be defined in terms of the variance of \( s(t) \), i.e.

\[
\Delta^2\varepsilon = \frac{1}{3} \left( \frac{V_b}{2F_{\text{max}} A} - 1 \right)^{-2} \sigma_s^2 \tag{A6.9}
\]

The distribution of this error, on say the power spectrum of \( s(t) \), will be frequency dependent, zero at zero frequency and peaking at \( F_{\text{max}} \).

As a practical example consider a turbine system with the following parameters:

- Diameter of bladed disc - 24 inches
- Speed of rotation - 6,000 r.p.m.
Vibrational amplitude (maximum) = 0.3 inches, peak to peak

\( F_{\text{max}} = 2,000 \text{ Hz} \)

From the above the blade tip velocity can be calculated.

\[ V_b = 0.12 \times 2\pi \times 100 \text{ inches/second} \]

Using equation A6.9:

\[ \frac{\Delta^2 \epsilon}{\delta^2 s} = 0.0025 \] 

Hence 0.25% of the signal power will be error noise power, which is a value comparable with the general noisy background vibration present on the blading.
Fig A.3

Fig A.4

TIME

(A.3)

TIME

(A.4)
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