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Unitary Double Products as Implementors of Bogolubov Transformations

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A Doctoral Thesis
Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University
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Abstract

This thesis is about double product integrals with pseudo rotational generator, and aims to exhibit them as unitary implementors of Bogolubov transformations. We further introduce these concepts in this abstract and describe their roles in the thesis’s chapters. The notion of product integral, (simple product integral, not double) is not a new one, but is unfamiliar to many a mathematician. Product integrals were first investigated by Volterra in the nineteenth century. Though often regarded as merely a notation for solutions of differential equations, they provide a priori a multiplicative analogue of the additive integration theories of Riemann, Stieltjes and Lebesgue. See Slavík [2007] for a historical overview of the subject.

Extensions of the theory of product integrals to multiplicative versions of Itô and especially quantum Itô calculus were first studied by Hudson, Ion and Parthasarathy in the 1980’s, Hudson et al. [1982]. The first developments of double product integrals was a theory of an algebraic kind developed by Hudson and Pulmannova motivated by the study of the solution of the quantum Yang-Baxter equation by the construction of quantum groups, see Hudson and Pulmaanova [2005]. This was a purely algebraic theory based on formal power series in a formal parameter. However, there also exists a developing analytic theory of double product integral. This thesis contributes to this analytic theory. The first papers in that direction are Hudson [2005b] and Hudson and Jones [2012]. Other motivations include quantum extension of Girsanov’s theorem and hence a quantum version of the Black-Scholes model in finance. They may also provide a general model for causal interactions in noisy environments in quantum physics. From a different direction “causal” double products, (see Hudson [2005b]), have become of interest in connection with quantum versions of the Levy area, and in particular quantum Levy area formula (Hudson [2011] and Chen and Hudson [2013]) for its characteristic function. There is a close association of causal double products with the double products of rectangular type (Hudson and Jones [2012] pp 3). For this reason it is of interest to study “forward-forward” rectangular double products.

In the first chapter we give our notation which will be used in the following chapters and we introduce some simple double products and show heuristically that they are the solution of two different quantum stochastic differential equations. For each example the order in which the products are taken is shown to be unimportant;
either calculation gives the same answer. This is in fact a consequence of the so-called multiplicative Fubini Theorem Hudson and Pulmaanova [2005].

In Chapter two we formally introduce the notion of product integral as a solution of two particular quantum stochastic differential equations.

In Chapter three we introduce the Fock representation of the canonical commutation relations, and discuss the Stone-von Neumann uniqueness theorem. We define the notion of Bogolubov transformation (often called a symplectic automorphism, see Parthasarathy [1992] for example), implementation of these transformations by an implementor (a unitary operator) and introduce Shale’s theorem which will be relevant to the following chapters. For an alternative coverage of Shale’s Theorem, symplectic automorphism and their implementors see Derezinski [2003].

In Chapter four we study double product integrals of the pseudo rotational type. This is in contrast to double product integrals of the rotational type that have been studied in (Hudson and Jones [2012] and Hudson [2005b]). The notation of the product integral is suggestive of a natural discretisation scheme where the infinitesimals are replaced by discrete increments i.e. discretised creation and annihilation operators of quantum mechanics. Because of a weak commutativity condition, between the discretised creation and annihilation operators corresponding on different subintervals of $\mathbb{R}$, the order of the factors of the product are unimportant (Hudson [2005a]), and hence the discrete product is well defined; we call this result the discrete multiplicative Fubini Theorem. It is also the case that the order in which the products are taken in the continuous (non-discretised case) does not matter (Hudson [2005a], Hudson and Jones [2012]). The resulting discrete double product is shown to be the implementor (a unitary operator) of a Bogolubov transformation acting on discretised creation and annihilation operators (Bogolubov transformations are invertible real linear operators on a Hilbert space that preserve the imaginary part of the inner product, but here we may regard them equivalently as liner transformations acting directly on creation and annihilations operators but preserving adjointness and commutation relations). Unitary operators on the same Hilbert space are a subgroup of the group of Bogolubov transformations. Essentially Bogolubov transformations are used to construct new canonical pairs from old ones (In the literature Bogolubov transformations are often called symplectic automorphisms).

The aforementioned Bogolubov transformation (acting on the discretised creation and annihilation operators) can be embedded into the space $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ and limits can be taken resulting in a limiting Bogolubov transformation in the space $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$. It has also been shown that the resulting family of Bogolubov transformation has three important properties, namely bi-evolution, shift covariance and time-reversal covariance, see (Hudson [2007]) for a detailed description of these properties.
Subsequently we show rigorously that this transformation really is a Bogolubov transformation. We remark that these transformations are Hilbert-Schmidt perturbations of the identity map and satisfy a criterion specified by Shale’s theorem. By Shale’s theorem we then know that each Bogolubov transformation is implemented in the Fock representation of the CCR. We also compute the constituent kernels of the integral operators making up the Hilbert-Schmidt operators involved in the Bogolubov transformations, and show that the order in which the approximating discrete products are taken has no bearing on the final Bogolubov transformation got by the limiting procedure, as would be expected from the multiplicative Fubini Theorem.

In Chapter five we generalise the canonical form of the double product studied in Chapter four by the use of gauge transformations. We show that all the theory of Chapter four carries over to these generalised double product integrals. This is because there is unitary equivalence between the Bogolubov transformation got from the generalised canonical form of the double product and the corresponding original one.

In Chapter six we make progress towards showing that a system of implementors of this family of Bogolubov transformations can be found which inherits properties of the original family such as being a bi-evolution and being covariant under shifts. We make use of Shales theorem (Parthasarathy [1992] and Derezinski [2003]). More specifically, Shale’s theorem ensures that each Bogolubov transformation of our system is implemented by a unitary operator which is unique to with multiplication by a scalar of modulus 1. We expect that there is a unique system of implementors, which is a bi-evolution, shift covariant, and time reversal covariant (i.e. which inherits the properties of the corresponding system of Bogolubov transformation). This is partly on-going research. We also expect the implementor of the Bogolubov transformation to be the original double product. In Evans [1988], Evan’s showed that the the implementor of a Bogolubov transformation in the simple product case is indeed the simple product. If given more time it might be possible to adapt Evan’s result to the double product case.

In Hudson et al. [1984] the three properties of bi-evolution, shift covariant, and time reversal covariant (in only one degree of freedom) were used to show uniqueness of the so called “time-orthogonal unitary dilation” in the case of non-Fock quantum stochastic calculus. The “double” case in this thesis is much harder. I show using the bi-evolutionarity and shift-covariance that there exists an unique system of implementors up to multiplication by a family of scalars \(\left(e^{i\mu(b-a)(t-s)}\right)_{0\leq a\leq b, 0\leq s\leq t}\) for some real parameter \(\mu\). It is expected that the parameter can be uniquely specified using the time reversal property. It is not clear, as of yet, how this should be done, even though it is quite simple in the similar case of Hudson et al. [1984].
Keywords: quantum stochastic differential equation, quantum probability, Bogolubov transformation, double product integrals.
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Chapter 1

Introduction

In this chapter, the notations of product integral and double product integral are introduced, which will be elaborated in the following chapter. Explicit computations of some simple double products from first principles are performed, which are the solution of basic quantum stochastic differential equations. These computations, due to there being no non-commuting variables, are easily performed. The later chapters are principally dedicated to the construction of double product integrals involving more complicated quantum stochastic differential equations in two non-commuting variables; namely the differentials of the creation and annihilation operators in Boson Fock space. The solutions of these quantum stochastic differential equations are the so-called double product integrals introduced by Robin Hudson. First, we introduce some important concepts and notation that will be used throughout the thesis.

1.1 Fock space notation

1.1.1 The kernel construction

In order to describe the kernel construction we need to define the notion of a non-negative definite kernel.

**Definition 1.** If $S$ is a set and $\mathcal{R} : S \times S \to \mathbb{C}$ is some function then $\mathcal{R}$ is a non-negative definite kernel over $S$ if, for each integer $n$ with $n \geq 1$, arbitrary $x_1, \ldots, x_n \in S$ and arbitrary $z_1, \ldots, z_n \in \mathbb{C}$ we have

$$\sum_{j,k=1}^{n} z_j z_k \mathcal{R}(x_j, x_k) \geq 0.$$

The following theorem describes the kernel construction from $(\mathcal{R}, S)$ of a Hilbert space $\mathcal{H}$ and map $k : S \to \mathcal{H}$ where $S$ is a set.

**Theorem 1.** Let $\mathcal{R}$ be a non-negative definite kernel over $S$. Then there exists a pair $(\mathcal{H}, k)$ comprising a Hilbert space $\mathcal{H}$ and a map $k : S \to \mathcal{H}$ such that
• \( \{k(x) : x \in S\} \) is total in \( \mathcal{H} \);

• \( \langle k(x), k(y) \rangle_\mathcal{H} = \mathcal{R}(x, y) \) for arbitrary \( x, y \in S \).

If \( (\mathcal{H}', k') \) is a second such pair then there exists a unique Hilbert isomorphism \( U : \mathcal{H} \rightarrow \mathcal{H}' \) such that \( U k = k' \). The pair \( (\mathcal{H}, k) \) is called the Gelfand pair associated with \( \mathcal{R} \).


Consider the separable Hilbert space \( L^2(\mathbb{R}_+, \mathbb{C}) \) and denote this space by \( \mathfrak{h} \). We now define the *Fock space* \( \Gamma(\mathfrak{h}) \) over \( \mathfrak{h} \) as the Hilbert space of the Gelfand pair (see Parthasarathy [1992] pp 93) corresponding to the non-negative definite kernel \( \mathfrak{h} \times \mathfrak{h} \ni (f, g) \rightarrow \exp(f, g) \) over \( \mathfrak{h} \). The pair consists of a Hilbert space \( \Gamma(\mathfrak{h}) \) which is equipped with a generating family of *exponential vectors* \( \psi(f) \) labelled by \( f \in \mathfrak{h} \), such that for all \( f, g \in \mathfrak{h} \), \( \langle \psi(f), \psi(g) \rangle = \exp(f, g) \). An alternative definition of Fock space is as follows (see Reed and Simon [1980]). Denote the \( n \)-fold tensor product of the Hilbert space \( \mathfrak{h} \) by \( \mathfrak{h}^n \) and set \( \mathfrak{h}^0 = \mathbb{C} \). See Parthasarathy [1992] for in depth discussion of tensor products of Hilbert spaces. Let \( \mathcal{P}_n \) be the permutation group on \( n \) elements and let \( \{\phi_k\} \) be a bases for \( \mathfrak{h} \). For each \( \sigma \in \mathcal{P}_n \), we define the operator

\[
\sigma(\phi_{k_1} \otimes \phi_{k_2} \otimes \ldots \otimes \phi_{k_n}) = \phi_{k_{\sigma(1)}} \otimes \phi_{k_{\sigma(2)}} \otimes \ldots \otimes \phi_{k_{\sigma(n)}}.
\]

We extend \( \sigma \) by linearity to a bounded operator on \( \mathfrak{h}^n \) so we can define \( S_n = (1/n!) \sum_{\sigma \in \mathcal{P}_n} \sigma \).

It so happens that \( S_n \) is an orthonormal projection (see Parthasarathy [1992] pp 106-107.) whose range is called the *\( n \)-fold symmetric tensor product of \( \mathfrak{h} \)*. We now define \( \Gamma(\mathfrak{h}) = \bigoplus_{n=0}^\infty S_n \mathfrak{h}^n \) and here the exponential vectors are given by \( \psi(f) = (1, f, \frac{f^2}{2!}, \frac{f^3}{3!}, \ldots) \).

In Chapter 4 we will need to discuss the Fock space over an arbitrary Hilbert space when discussing the family of Weyl operators. In such an occasion we denote this arbitrary Hilbert space by \( \mathfrak{g} \) and its corresponding Fock space by \( \Gamma(\mathfrak{g}) \).

Shorten the notation for \( \Gamma(\mathfrak{h}) \) to \( \mathcal{H} \). For any arbitrary \( t \in \mathbb{R}_+ \), use the notation \( \mathfrak{h}_t \) and \( \mathfrak{h}^t \) for the Hilbert spaces \( L^2(0, t) \) and \( L^2(t, \infty) \) respectively, and \( \mathcal{H}_t \) and \( \mathcal{H}^t \) for the corresponding Fock spaces. For any \( f \in \mathfrak{h} \) the notation \( f_t \) and \( f^t \) denotes the vectors \( f \chi(0,t) \) and \( f \chi(t,\infty) \) in Hilbert subspaces \( \mathfrak{h}_t \) and \( \mathfrak{h}^t \) respectively. The Hilbert space \( \mathcal{H} \) has the decomposition \( \mathcal{H}_t \otimes \mathcal{H}^t \) corresponding to the natural direct sum decomposition \( \mathfrak{h} = \mathfrak{h}_t \oplus \mathfrak{h}^t \), in so far, as for each \( \psi(f) = \psi(f_t) \otimes \psi(f^t) \) for \( f = f_t \oplus f^t \). The notation \( \psi_t(0) \) and \( \psi^t(0) \) is used for the vacuum vectors of \( \mathcal{H}_t \) and \( \mathcal{H}^t \) respectively. The Hilbert spaces \( \mathcal{H}_t \) and \( \mathcal{H}^t \) can be identified with the subspaces of \( \mathcal{H}_t \otimes \psi^t(0) \) and \( \psi_t(0) \otimes \mathcal{H}^t \) of \( \mathcal{H} \) by the inclusion maps \( \mathcal{H}_t \ni \psi \rightarrow \psi \otimes \psi^t(0) \) and \( \mathcal{H}^t \ni \psi \rightarrow \psi^t(0) \otimes \psi \) respectively. The notation \( \mathcal{E}(S) \) denotes the linear span generated by exponential vectors \( \{\psi(g) : g \in S\} \). Let \( \mathcal{E}_t \) and \( \mathcal{E}^t \) denote the linear span of the vectors \( \mathfrak{h}_t \) and \( \mathfrak{h}^t \) respectively. We have the factorisation \( \mathcal{E} = \mathcal{E}_t \otimes \mathcal{E}^t \) where this is the algebraic tensor product. A *unital algebra* \( \mathcal{A} \) is a real or complex vector space equipped with a bilinear product \( \mathcal{A} \times \mathcal{A} \ni (a, b) \rightarrow ab \in \mathcal{A} \) and an element \( 1 \) with the property that \( 1a = a1 = a \) for all \( a \in \mathcal{A} \). We also assume the algebra is associative i.e. \((ab)c = a(bc)\).
1.2 Quantum stochastic processes

Definition 2. A quantum stochastic process $X = (X(t))_{t \geq 0}$ is a family of operators on $\mathcal{H}$ such that for all $t$, the domain of $X(t)$ and its adjoint $X^*(t)$ contains $\mathcal{E}$. The restrictions to $\mathcal{E}$ of the adjoints then form another quantum stochastic process denoted by $X^\dagger = (X^\dagger(t))_{t \geq 0}$.

Definition 3. A quantum stochastic process $X$ is adapted if for all $t \in \mathbb{R}^+$, as an operator on $\mathcal{E}$

$$X(t) = X_t \otimes I,$$

where $X_t$ maps $\mathcal{E}_t$ into $\mathcal{H}_t$ and $\otimes$ denotes the algebraic tensor product on $\mathcal{E} = \mathcal{E}_t \otimes \mathcal{E}_t$. The set of all adapted processes on $\mathcal{H}$ is denoted by $\mathcal{U}(\mathcal{H})$.

Definition 4. An adapted quantum stochastic process $X \in \mathcal{U}(\mathcal{H})$ is called a martingale if for all arbitrary times $s \leq t$ and $f, g \in \mathcal{H}$ both vanishing on $(s, \infty)$

$$\langle \psi(f), X(t)\psi(g) \rangle = \langle \psi(f), X(s)\psi(g) \rangle.$$

We now define new operators by their action on the exponential vectors as follows. For $f, g \in \mathcal{H}$

$$a^\dagger(f)\psi(g) = \frac{d}{dz}\psi(f + zg)|_{z=0} \quad (1.2.1)$$

and

$$a(f)\psi(g) = \langle f, g \rangle\psi(g). \quad (1.2.2)$$

We now introduce two fundamental martingales of quantum stochastic calculus. They are the creation and annihilation processes defined respectively by $A^\dagger(t) = a^\dagger(\chi_{[0,t]})$ and $A(t) = a(\chi_{[0,t]})$ respectively, where $\chi_{[0,t]}$ is the indicator function of the interval $[0, t]$.

The construction of quantum stochastic integrals mirrors that of the classical Itô integral (see Oksendal [2000] for an introduction to the classical theory). We first construct an integral with respect to “simple” processes and extend to more interesting processes using “fundamental estimates” which are akin to the Itô isometry (see Oksendal [2000]).

In order to define the quantum stochastic integral we wish to define the products of an adapted process $E$ with increments of $Q = A^\dagger$ or $A$, the adapted martingale processes defined above. This on first consideration may seem impossible since $Q(s)$, need not map the exponential domain to itself. However, relative to a given splitting time $s$ we set $E(s) = E_s \otimes I$, we define the product to be the algebraic tensor product operator

$$E(s)(Q(t) - Q(s)) = E_s \otimes (Q(t) - Q(s)).$$

We use this definition later in the section.

Definition 5. An adapted process $E(t)$ is elementary if there are times $t_1 < t_2$ such that, for
all times $t$,

$$E(t) = \begin{cases} 
0, & 0 \leq t < t_1 \\
E(t_1), & t_1 \leq t < t_2 \\
0, & t_2 \leq t < \infty.
\end{cases}$$

**Definition 6.** An adapted process $E(t)$ is simple if it is a finite sum of elementary processes.

**Definition 7.** Given an elementary process $E$, and $Q = A^\dagger$ or $A$, we define the stochastic integral process $M$ by

$$M(t) = \begin{cases} 
0, & 0 \leq t < t_1 \\
E(t_1)(Q(t) - Q(t_1)), & t_1 \leq t < t_2 \\
E(t_1)(Q(t_2) - Q(t_1)), & t_2 \leq t < \infty.
\end{cases}$$ (1.2.3)

The stochastic integral can then be extended to simple processes by defining the stochastic integral of a sum of elementary processes to be the sum of the stochastic integrals of the elementary processes summands.

Using the fundamental estimate for simple process $E$ and fundamental martingale $Q$ (Hudson [2003] Theorem 6.3), namely

$$\left\| \int_{0<s<t} E(s)dQ(s)e(f) \right\| \leq 2 \exp \int_{0<s<t} |f(s)|^2 ds \int_{0<s<t} \|E(s)e(f)\|^2 ds,$$

we can extend the integral to processes $E$ such that for all $g, f$ in $\mathfrak{h}$ the map $s \mapsto \langle e(f), E(s)e(g) \rangle$ is Lebesgue measurable, and there exist a sequence of simple processes $E_n$ such that for all $f$ in $\mathfrak{h}$ and all $t$ in $\mathbb{R}_+$, the limits $\|E_n - E\|_{f,t}$ and $\|E^\dagger_n - E^\dagger\|_{f,t}$ exist and are zero, where

$$\|F\|_{f,t} = \sqrt{\int_{0<s<t} \|F(s)e(f)\|^2 ds}.$$ 

We call such processes integrable.

**1.2.1 The first and second fundamental formula of quantum stochastic calculus**

The first fundamental formula of quantum stochastic calculus expresses, in the weak sense (see Parthasarathy [1992] pp 3.) quantum stochastic integrals, as Lebesgue integrals. In order to have a stochastic calculus for quantum stochastic processes we need to be able to express the product of integrals as sums of iterated integrals. This is done by the so-called second fundamental formula of quantum stochastic calculus. Since we are dealing with the operators $A^\dagger$ and $A$ that do not conserve the exponential domain $\mathcal{E}$ we must avoid the explicit products of integrals of these operators, and hence the need to express products in the weak sense (see
Remark 3). We avoid this by utilising the two sides of the Hilbert space inner product to separate the factors in the product. We also define a new process on the domain $E$ called the time process $T(s)$ which is defined as $T(s) = sI$, where $s \in \mathbb{R}_+$ and $I$ is the identity operator. Essentially this map multiples vectors in the exponential domain by the scalar $s$. Quantum stochastic integrals with respect to the time process, by using of Definition 7 and the inequalities that follow the definition, are simply Lebesgue integrals.

**Theorem 2. The first fundamental formula.** Given integrable stochastic processes $E$ and $F$, define the stochastic integral process $M$ by

$$M(t) = \int_{0<s<t} (E(s)dA^\dagger(s) + F(s)dA(s) + GdT(s)).$$

Then for arbitrary $f, g \in \mathfrak{h}$ and $t \in \mathbb{R}_+$

$$\langle \psi(f), M(t)\psi(g) \rangle = \int_{0<s<t} \langle \psi(f), \bar{f}(s)E_2(s) + g(s)F_2(s) + G(s)\psi(g) \rangle ds \quad (1.2.4)$$


**Theorem 3. The second fundamental formula.** Given integrable stochastic processes $E_1$, $F_1$, $E_2$ and $F_2$, the stochastic integral processes $M_1$ and $M_2$ by

$$M_i(t) = \int_{0<s<t} (E_j(s)dA^\dagger(s) + F_j(s)dA(s) + G_jdT(s)$$

for $i = 1$ and $2$. Then for arbitrary $f, g \in \mathfrak{h}$ and $t \in \mathbb{R}_+$

$$\langle M_1(t)\psi(f), M_2(t)\psi(g) \rangle = \int_0^t \langle M_1(s)\psi(f), \bar{f}(s)E_2(s) + g(s)F_2(s) + G(s)\psi(g) \rangle ds$$

$$+ \langle g(s)E_1(s) + f(s)E_1(s) + G(s)\psi(f), M_2(s)\psi(g) \rangle + \langle E_1(s)\psi(f), E_2(s)\psi(g) \rangle ds \quad (1.2.5)$$


1.2.2 Heuristics of the Quantum Itô formula

The second fundamental theorem of stochastic calculus can be informally expressed as an integration by parts formulae by taking illegal adjoints. If we take all operators on the left of the inner product over to the right we obtain

$$\langle \psi(f), M^\dagger M\psi(g) \rangle$$

$$= \int_{0<s<t} \left\langle \psi(f), (1, \bar{f}) \begin{bmatrix} M^\dagger G + G^\dagger M + E^\dagger E & M^\dagger F + F^\dagger M \\ M^\dagger E + E^\dagger M & 0 \end{bmatrix} \begin{bmatrix} 1 \\ g \end{bmatrix} \psi(g) \right\rangle ds \quad (1.2.6)$$
Using the first fundamental theorem of quantum stochastic calculus one can write

\[ M^\dagger M = \int_{0 \leq s \leq t} (M^\dagger E + E^\dagger M)dA^\dagger + (M^\dagger F + F^\dagger M)dA + (M^\dagger G + G^\dagger M + E^\dagger E)dT. \]

We can see clearly the correction term \( E^\dagger E dT \) and hence we have the following Itô algebra multiplication table

<table>
<thead>
<tr>
<th></th>
<th>( dA^\dagger )</th>
<th>( dA )</th>
<th>( dT )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( dA^\dagger )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( dA )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( dT )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Definition 8.** An adapted quantum stochastic process \( X \in \mathcal{U}(\mathfrak{h}) \) is continuous if for all \( f \in \mathfrak{h} \) the map \( t \mapsto X(t)\psi(f) \) from \([0, \infty)\) to \( \mathfrak{h} \) is (strongly) continuous.

A class of quantum stochastic process is now introduced which will be pivotal to the rest of the exposition. This class is a subset of the continuous processes of the previous definition, the class of all iterated quantum stochastic integrals which is defined as the set of linear combinations of iterated integrals. A shorthand for the ease of notation is used to succinctly express integrals against successive creation, annihilation and time processes in iterated quantum stochastic integrals: A lower suffix of \( i \) equal to \(-, + \) or \( 0 \) in \( A_i \) is used to denote either the annihilation, the creation or the time process respectively.

**Definition 9.** For fixed \( s, t \in \mathbb{R}_+ \) with \( s \leq t \) the class of all iterated quantum stochastic integrals are

\[
I^s_t(i_1, i_2, \ldots, i_n) = \int_{s < t_1 < t_2 < \ldots < t_n < t} dA_{i_1}(t_1)dA_{i_2}(t_2) \ldots dA_{i_n}(t_n),
\]

\[
= \int_{s}^{t} \{ \ldots \int_{s}^{t_{j-1}} \{ \int_{s}^{t_2} dA_{i_j}(t_1) \} dA_{i_2}(t_2) \ldots \} dA_{i_n}(t_n), \quad (1.2.7)
\]

where \( i_j \) is either \(-, 0, \) or \(+ \) for \( j = 1 \ldots n \). The notation \( \mathcal{P}_s^t \) is used for the set of all linear combinations of the set of iterated quantum stochastic integrals between \( s \) and \( t \) and multiples of the identity process.

**Remark 1.** As an operator on the exponential domain \( \mathcal{E} \), \( I^s_t(i_1, i_2, \ldots, i_n) \) has an adjoint \( I^s_t(i_1, i_2, \ldots, i_n)^\dagger \) defined on the same domain, in fact given by \( I^s_t(-i_1, -i_2, \ldots, -i_n) \). In fact \( \mathcal{P}_s^t \) is closed under adjoints.

**Remark 2.** Stochastic iterated integrals exist because a stochastic integral of an integrable process with respect to creation, annihilation and time is a continuous process and a continuous process is integrable (as defined and proved in Hudson [2003], Theorem 8.1 and Theorem 8.2).
Remark 3. As \( I^1_s(i_1,i_2,...,i_n) \) does not map \( \mathcal{E} \) to itself, products are not defined directly, but the set \( \mathcal{P}^d_s \) is a unital \( * \)-algebra. The weak product of iterated integrals is a linear combination of iterated integrals i.e. with \( P_1^t \) and \( P_2^t \) in \( \mathcal{P}^d_s \), \( \langle P_1^t \psi(f), P_2^t \psi(g) \rangle = \langle \psi(f), Q \psi(g) \rangle \) for some \( Q \) in \( \mathcal{P}^d_s \) given by the so called sticky shuffle product, which is defined in Section 8 of the thesis.

1.3 Explicit construction of some simple double product integrals

This section discusses the notation for simple and double product integrals. Some of these integrals have explicit solutions which are easy to solve analytically for example using the calculus and Picard iteration. This is in contrast to the double product integral solved in Hudson and Jones [2012] and the work below, in subsequent chapters on the pseudo-rotational case.

1.3.1 The notion of product integral with respect to creation, annihilation and time

We introduce the notion of double product integral in two stages. The notation

\[
\prod_{s}^{t} (1 + F \otimes dA^\dagger + G \otimes dA + H \otimes dT), \ s < t, \tag{1.3.1}
\]

represents the solution \( Y(t) \) at \( t \) of the quantum stochastic differential equation

\[
(I_A \otimes d)Y = Y \left( F \otimes dA^\dagger + G \otimes dA + H \otimes dT \right), \ Y(s) = 1,
\]

where \( Y \) is in \( A \otimes \mathcal{P} \), a suitable topological completion of the algebraic tensor product of a unital system algebra \( A \) and an algebra of processes in Fock space denoted by \( \mathcal{P} \). The \( F, G \) and \( H \) are in \( A \) and \( dA, dA^\dagger \) and \( dT \) are generating elements in the Itô algebra \( \mathcal{I} \) defined as \( \mathbb{C}(dA, dA^\dagger, dT) \) (linear combinations of the differentials of the creation and annihilation martingales \( A^\dagger \), \( A \) and of the time differential \( dT \) equipped with the quantum Itô product multiplication). The right arrow \( \rightarrow \) indicates that the product is taken in the forward direction. Similarly

\[
\prod_{a}^{b} (1 + dA^\dagger \otimes F + dA \otimes G + dT \otimes H), \ a < b
\]

is used for the solution \( X(b) \) at \( b \) of

\[
(d \otimes I_A)X = (dA^\dagger \otimes F + dA \otimes G + dT \otimes H)X, \ X(a) = 1.
\]
The left arrow $\leftarrow$ indicates that the product is taken in the backward direction. When the system algebra is non-unital as is $I$, then the above definitions need to be modified. This leads to the following definition of what is called the decapitated product integral (It is a different object even if $A$ is unital). The notation

$$\overleftarrow{b}^a \prod \left(1 + dA^\dagger \otimes F + dA \otimes G + dT \otimes H \right)$$

is used for the solution of the quantum stochastic differential equation

$$(d \otimes I_A)X = (1_P + X) \left(dA^\dagger \otimes F + dA \otimes G + dT \otimes H \right), \ X(a) = 0.$$

Notice here that the initial condition is not $X(a) = 1$ (the tensor product of unit in the system algebra and identity operator in Fock space), but the tensor product of the zero elements of both spaces. The term $1_P$ is notation for the identity operator of processes in Fock space and naturally $X$, in this case, is in the space formed by tensoring operators from Fock space to itself and the Itô algebra. The notion of a decapitated product integral is vitally important to define what is called a double product integral. Essentially a double product integral is an operator on the tensor product of two Fock spaces, which is the solution of a quantum stochastic differential equation whose driving coefficient is itself the solution of a stochastic differential equation. This is made precise in the next subsection.

### 1.3.2 Double product integrals

Let $dr \in J \otimes J$ and let $[a, b], [s, t] \subset \mathbb{R}_+$ define the double product as

$$\overleftarrow{b}^a \prod \limits_s^t (1 + dr) \tag{1.3.2}$$

as either 1.

$$:= \prod \limits_s^t (1 + \overleftarrow{b}^a \prod \limits_s^t (1 + dr)) \tag{1.3.3}$$

or in an alternative way as 2.

$$:= \overleftarrow{b}^a \prod \limits_s^t (1 + \prod \limits_s^t (1 + dr)) \tag{1.3.4}$$

The terms in version 1. are explained in what follows: Here $\overleftarrow{b}^a \prod \limits_s^t (1 + dr)$ denotes the solution $X(b)$ at $b$ of

$$dX^{1,2,3} = (X^{1,3} + 1^1) dr^{2,3}, \quad X^{1,3} = 0 \otimes 0,$$
where the superscripts 1, 2 and 3 denote places in the tensor product \( \mathcal{P} \otimes \mathcal{I} \otimes \mathcal{I} \). Thus it is of the form \( \alpha \otimes dA^\dagger + \beta \otimes dA + \gamma \otimes dT \), where \( \alpha, \beta \) and \( \gamma \) are members of \( \mathcal{P} \). Now \( \mathcal{P} \prod_t^a (1 + \lambda \prod_s^b (1 + dr)) \) is defined as the solution \( Y(t) \) at \( t \) of

\[
dY^{1,2,3} = Y^{1,2} \left( \alpha^1 \otimes dA^3 + \beta^1 \otimes dA^3 + \gamma^1 \otimes dT^3 \right), \quad Y^{1,2}(a) = 1.
\]

Version 2, can be defined analogously.

### 1.3.3 Examples of explicitly constructed solutions of some simple double product integrals

In this subsection we consider these simple double product integrals:

\[
\prod_t^a (1 + \lambda dT \otimes dT),
\]

\[
\prod_t^a (1 + \lambda dA^\dagger \otimes dT),
\]

and

\[
\prod_t^a (1 + \lambda dA^\dagger \otimes dA^\dagger).
\]

#### 1.3.3.1 Example 1: \( \prod_t^a (1 + \lambda dT \otimes dT) \)

By the first of the two definitions

\[
\prod_t^a (1 + \lambda dT \otimes dT) = \prod_t^s (1 + \lambda \prod_s^b (1 + dT \otimes dT)).
\] (1.3.5)

By Picard iteration and \((dT)^2 = 0\)

\[
\prod_t^s (1 + \lambda(b - a)dT),
\] (1.3.6)

which is by definition the solution \( X(t) \) at \( t \) of

\[
dX = X(b - a)dT, \quad X(s) = 1.
\] (1.3.7)

Solving the above

\[
\prod_t^a (1 + \lambda dT \otimes dT) = e^{\lambda(b-a)(t-s)}.
\] (1.3.8)
A similar argument shows that the second definition yields the same result i.e.

\[
\begin{align*}
\mathbf{b} - \mathbf{a} \prod_t (1 + \lambda dT \otimes dT) &= \mathbf{b} - \mathbf{a} \prod_t (1 + \lambda dT \otimes dT) \\
&= \mathbf{b} - \mathbf{a} \prod_t (1 + \lambda dT \otimes (t - s)).
\end{align*}
\]

### 1.3.3.2 Example 2: \( \mathbf{b} - \mathbf{a} \prod_t (1 + \lambda dA^\dagger \otimes dT) \)

By the first definition

\[
\begin{align*}
\mathbf{b} - \mathbf{a} \prod_t (1 + \lambda dA^\dagger \otimes dT) &= \prod_t (1 + \lambda dA^\dagger \otimes dA^\dagger),
\end{align*}
\]

which equals

\[
\prod_t (1 + \lambda a^\dagger \chi_b \otimes dA^\dagger),
\]

by Picard iteration and \((dT)^2 = 0\). The above product integral is by definition the solution at \( t \) of

\[
dX = \lambda a^\dagger \chi_b X dt, \quad X(s) = 1.
\]

Solving the above heuristically

\[
X(t) = \exp (\lambda a^\dagger \chi_b (t - s))
\]

Using the first definition gives the same answer, by Picard and \((dA^\dagger)^2 = 0\).

### 1.3.3.3 Example 3: \( \mathbf{b} - \mathbf{a} \prod_t (1 + \lambda dA^\dagger \otimes dA^\dagger) \)

By the first of the two definitions

\[
\begin{align*}
\mathbf{b} - \mathbf{a} \prod_t (1 + \lambda dA^\dagger \otimes dA^\dagger) &= \prod_t (1 + \lambda dA^\dagger \otimes dA^\dagger),
\end{align*}
\]

by Picard iteration and \((dA^\dagger)^2 = 0\) equals

\[
\prod_t (1 + \lambda a^\dagger \chi_b \otimes dA^\dagger),
\]

is by definition the solution at \( t \) of

\[
(I \otimes d)X = X a^\dagger \chi_b \otimes dA^\dagger(t), \quad \text{with} \quad X(s) = 1.
\]
Solving the above heuristically for $t$

$$\prod_{a}^{b} (1 + \lambda dA^\dagger \otimes dA) = \exp (\lambda a^\dagger (\chi_b^\dagger) \otimes a^\dagger (\chi_s^\dagger)).$$ (1.3.15)

The symmetry of this solution, makes it clear that the two definitions of the double product integral agree.

### 1.4 Conclusion

The three examples that have been studied can be solved, at least heuristically, by straightforward calculation. The results, for the three examples is the same if either definition 1. or 2. of the double product integral is used. This is a consequence of a so called “multiplicative Fubini” theorem, see Hudson [2005a], or Section 31 for a proof. Here it is shown that the two definitions of double product integrals always agree.
Chapter 2

Double products

2.0.1 Product integrals

The product integrals studied in this exposition arise as solutions of certain quantum stochastic differential equations. The solutions are constructed by the Picard iterative method, and are possibly infinite sums of iterated integrals.

Definition 10. Given a unital associative algebra $\mathcal{A}$, and elements $\eta_-, \eta_+$ and $\eta_0$ in $\mathcal{A}$. The forward-right (resp. left) product integral, denoted by

$$X(t) = \prod_{s}^{t} (1 + \eta_- \otimes dA + \eta_+ \otimes dA^\dagger + \eta_0 \otimes dT)$$  \hspace{1cm} (2.0.1)$$

(resp.

$$X(t) = \prod_{s}^{t} (1 + dA \otimes \eta_- + dA^\dagger \otimes \eta_+ + dT \otimes \eta_0)).$$  \hspace{1cm} (2.0.2)$$

is defined as the solution at $t$ of the following quantum stochastic differential equation

$$dX^{1,2,3} = X^{1,2} \left( \eta_- \otimes dA^3 + \eta_+ \otimes dA^\dagger + \eta_0 \otimes dT^3 \right), \quad X^{1,2}(s) = 1 \otimes 1,$$ \hspace{1cm} (2.0.3)$$

where the superscripts 1, 2 and 3 denote places in the tensor product $\mathcal{A} \otimes \mathcal{P}_s \otimes \mathcal{I}$. (resp. $\mathcal{P}_s \otimes \mathcal{I} \otimes \mathcal{A}$). Here $X$ belongs to a suitable closure of $\mathcal{A} \otimes \mathcal{P}_s$ (resp. $\mathcal{P}_s \otimes \mathcal{A}$). The solutions will always be constructed as limits of sequences of approximations which belong to $\mathcal{A} \otimes \mathcal{P}_s$ (resp. $\mathcal{P}_s \otimes \mathcal{A}$).

Remark 4. Formally the solutions can always be constructed as limits of sequences of approximations which belong to $\mathcal{A} \otimes \mathcal{P}_s$ (resp. $\mathcal{P}_s \otimes \mathcal{A}$). The infinite sum will need a notion of convergence to be meaningful.
**Definition 11.** Given a unital associative algebra \( \mathcal{A} \), and elements \( \eta_- , \eta_+ \) and \( \eta_0 \) in \( \mathcal{A} \). The backward-right (resp. left) product integral, is denoted by

\[
X(t) = \prod_{s}^{t} (1 + \eta_- \otimes dA + \eta_+ \otimes dA^\dagger + \eta_0 \otimes dT)
\]

(resp,

\[
X(t) = \prod_{s}^{t} (1 + dA \otimes \eta_- + dA^\dagger \otimes \eta_+ + dT \otimes \eta_0).
\]

is defined as the solution at \( t \) of the following quantum stochastic differential equation

\[
dX_{1,2,3}(t) = (\eta_- \otimes dA^3 + \eta_+ \otimes dA^3 + \eta_0 \otimes dT^3) X_{1,2}(t), \quad X_{1,2}(s) = 1 \otimes 1,
\]

where the superscripts 1, 2 and 3 denote places in the tensor product \( \mathcal{A} \otimes \mathcal{P}_t^\dagger \otimes I \) (resp.

\[
dX_{1,2,3}(t) = (dA^2 \otimes \eta_-^2 + dA^2 \otimes \eta_+^2 + dT^2 \otimes \eta_0^2) X_{1,3}(t), \quad X_{1,3}(s) = 1 \otimes 1,
\]

where the superscripts 1, 2 and 3 denote places in the tensor product \( \mathcal{P} \otimes \mathcal{P}_s^\dagger \otimes \mathcal{A} \). Here \( X \) belongs to a suitable closure of \( \mathcal{A} \otimes \mathcal{P}_t \) (resp. \( \mathcal{P}_s \otimes \mathcal{A} \)). The solutions will always be constructed as limits of sequences of approximations which belong to \( \mathcal{A} \otimes \mathcal{P}_t \) (resp. \( \mathcal{P}_s \otimes \mathcal{A} \)).

**Remark 5.** The forward-right product integral as a solution of \((2.0.3)\) is obtained by Picard iteration as

\[
X(t) = 1 + \sum_{N=1}^{\infty} \sum_{j_1, j_2, \ldots, j_N \in \{-, 0, +\}} \eta_{j_1} \eta_{j_2} \ldots \eta_{j_N} \otimes I_j^{1, j_2, \ldots, j_N}.
\]

The solution calculated via Picard iteration with initial approximation \( X^{(0)}(t) = X(s) \) and the iterative scheme

\[
dX^{(n)}(t) = X^{(n-1)}(t) (\eta_- \otimes dA(t) + \eta_+ \otimes dA^\dagger(t) + \eta_0 \otimes dT(t)), \quad X^{(n)}(s) = 1,
\]

\( X^{(n-1)}(t) \) is called the preceding approximation to \( X^{(n)}(t) \) for each \( n \in \mathbb{N} \). To illustrate, the approximation for \( n = 1 \) is given by integrating the following quantum stochastic differential equation

\[
dX^{(1)}(t) = 1, (\eta_- \otimes dA(t) + \eta_+ \otimes dA^\dagger(t) + \eta_0 \otimes dT(t)), \quad X^{(1)}(s) = 1.
\]

The solution is

\[
X^{(1)}(t) = 1 + \eta_- \otimes \int_{s}^{t} dA(t_1) + \eta_+ \otimes \int_{s}^{t} dA^\dagger(t_1) + \eta_0 \otimes \int_{s}^{t} dT(t_1).
\]

The approximation \( X^{(2)}(t) \) is obtained using the scheme \((2.0.9)\) with the preceding approxima-
tion \(X^{(1)}(t)\). This amounts to integrating
\[
dX^{(2)}(t) = X^{(1)}(t). (\eta_0 \otimes dA(t) + \eta_1 \otimes dA^1(t) + \eta_0 \otimes dT(t)) \), \(X^{(2)}(s) = 1.\)

This has solution
\[
X^{(2)}(t) = 1 + \eta_0 \otimes I_s^1(0) \]
\[
+ \eta_0 \eta_0 \otimes I_s^1(0, -) + \eta_1 \eta_0 \otimes I_s^1(0, +) + \eta_0 \eta_0 \otimes I_s^1(0, +)
\]
\[
+ \eta_0 \eta_0 \otimes I_s^1(-, 0) + \eta_1 \eta_0 \otimes I_s^1(+, 0) + \eta_0 \eta_0 \otimes I_s^1(0, 0). \tag{2.0.10}
\]

The approximation \(X^{(3)}(t)\) is obtained using the scheme (2.0.9) with the preceding approximation (2.0.10). This amounts to integrating
\[
dX^{(3)}(t) = X^{(2)}(t). (\eta_0 \otimes dA(t) + \eta_1 \otimes dA^1(t) + \eta_0 \otimes dT(t)) \), \(X^{(3)}(s) = 1.\)

This has solution
\[
X^{(3)}(t) = 1 + \sum_{j \in \{-, +\}} \eta_0 \otimes I_s^1(j) + \sum_{j_1, j_2 \in \{-, +\}} \eta_{j_1} \eta_{j_2} \otimes I_s^1(j_1, j_2)
\]
\[
+ \sum_{j_1, j_2, j_3 \in \{-, +\}} \eta_{j_1} \eta_{j_2} \eta_{j_3} \otimes I_s^1(j_1, j_2, j_3). \tag{2.0.11}
\]

The iterative procedure is continued indefinitely. The solution can be formally seem to be (2.0.8) as the iterative procedure extends to infinitely. Similarly the forward-left product integral, backward-right product integral and backward-left product integral as a solutions of (2.0.4), (2.0.6) and (2.0.7) can formally be written as:

\[
X(t) = 1 + \sum_{N=1}^{\infty} \sum_{j_1, j_2, \ldots, j_N \in \{-, +\}} I_s^1(j_1, j_2, \ldots, j_N) \otimes \eta_{j_1} \eta_{j_2} \otimes \eta_{j_N}, \tag{2.0.11}
\]

\[
X(t) = 1 + \sum_{N=1}^{\infty} \sum_{j_1, j_2, \ldots, j_N \in \{-, +\}} \eta_{j_1} \eta_{j_2} \otimes I_s^1(j_1, j_2, \ldots, j_N), \tag{2.0.12}
\]

\[
X(t) = 1 + \sum_{N=1}^{\infty} \sum_{j_1, j_2, \ldots, j_N \in \{-, +\}} I_s^1(j_1, j_2, \ldots, j_N) \otimes \eta_{j_1} \eta_{j_2} \otimes \eta_{j_N} \ldots \eta_{j_N}, \tag{2.0.13}
\]

respectively. The solutions are obtained by “formal” Picard iteration.

**Remark 6.** The algebra \(\mathcal{A}\) used in the definition of the product integral, in this exposition, will either be the so-called Ito algebra (defined in the Section 1.3.1) or the algebra \(\mathcal{P}_a^b\) (an example of the sticky shuffle product defined in Chapter 8), where \(a\) and \(b\) are in \(\mathbb{R}\), \(a \leq b\).
Definition 12. Given an associative non-unital algebra $A$ and from that algebra elements $\eta_-$, $\eta_+$ and $\eta_0$, a decapitated forward-right (resp. left) product integral, denoted by

$$X(t) = \prod_s^t (1 + \eta_- \otimes dA + \eta_+ \otimes dA^\dagger + \eta_0 \otimes d\tau)$$

is defined as the solution at $t$ of the following quantum stochastic differential equation

$$dX^{1,2,3} = \left( X^{1,2} + 1^{2,\eta_+} \right) \left( \eta_1 \otimes dA^3 + \eta_1^1 \otimes dA^3 + \eta_0^1 \otimes d\tau^3 \right), \quad X^{1,2}(s) = 0 \otimes 0,$$

where the superscripts 1, 2 and 3 denote places in the tensor product $A \otimes \mathcal{P}_s^k \otimes \mathcal{I}$ (resp.

$$dX^{1,2,3} = \left( X^{1,3,1} + 1^{1,\eta_+} \right) \left( dA^2 \otimes \eta_3^1 + dA^1 \otimes \eta_3^1 + d\tau^2 \otimes \eta_0^3 \right), \quad X^{1,3}(s) = 0 \otimes 0,$$

where the superscripts 1, 2 and 3 denote places in the tensor product $\mathcal{P}_s^k \otimes \mathcal{I} \otimes A \otimes \mathcal{P}_s \otimes A$. Here $X$ belongs to a suitable closure of $A \otimes \mathcal{P}_s^k$ (resp. $\mathcal{P}_s^k \otimes A$). Thus, the solution will always be limits of sequences of approximations which belong to $A \otimes \mathcal{P}_s^k$ (resp. $\mathcal{P}_s^k \otimes A$).

Definition 13. Given a finite dimensional associative algebra $A$ and the algebra elements $\eta_-$, $\eta_+$ and $\eta_0$. A decapitated backward-right (resp. left) product integral, denoted by

$$X(t) = \prod_s^t (1 + \eta_- \otimes dA + \eta_+ \otimes dA^\dagger + \eta_0 \otimes d\tau)$$

is defined as the solution at $t$ of the following quantum stochastic differential equation

$$dX^{1,2,3} = \left( \eta_1 \otimes dA^3 + \eta_1^1 \otimes dA^3 + \eta_0^1 \otimes d\tau^3 \right) \left( X^{1,2} + 1^{2,\eta_+} \right), \quad X^{1,2}(s) = 0 \otimes 0$$

where the superscripts 1, 2 and 3 denote places in the tensor product $A \otimes \mathcal{P}_s^k \otimes \mathcal{I}$ (resp.

$$dX^{1,2,3} = \left( dA^2 \otimes \eta_3^1 + dA^1 \otimes \eta_3^1 + d\tau^2 \otimes \eta_0^3 \right) \left( X^{1,3,1} + 1^{1,\eta_+} \right), \quad X^{1,3}(s) = 0 \otimes 0,$$

where the superscripts 1, 2 and 3 denote places in the tensor product $\mathcal{P}_s^k \otimes \mathcal{I} \otimes A$. Here $X$ belongs to a closure of $A \otimes \mathcal{P}_s^k$ (resp. $\mathcal{P}_s^k \otimes A$). Thus, the solution will always be limits of sequences of approximations which belong to $A \otimes \mathcal{P}_s^k$ (resp. $\mathcal{P}_s^k \otimes A$).
Remark 7. The decapitated-forward-left product integral, decapitated-forward-right product integral, decapitated-backward-right product integral and decapitated-backward-left product integral as solutions of (2.0.16), (2.0.17), (2.0.20) and (2.0.21) can be formally written as:

\[ X(t) = \sum_{N=1}^{\infty} \sum_{j_1, j_2, \ldots, j_N \in \{-, +\}} \eta_{j_1, j_2, \ldots, j_N} \otimes \mathcal{I}_s^I(j_1, j_2, \ldots, j_N). \]  \hspace{1cm} (2.0.22)

\[ X(t) = \sum_{N=1}^{\infty} \sum_{j_N, j_{N-1}, \ldots, j_1 \in \{-, +\}} \mathcal{I}_s^I(j_1, j_2, \ldots, j_N) \otimes \eta_{j_1, j_2, \ldots, j_N}. \]  \hspace{1cm} (2.0.23)

\[ X(t) = \sum_{N=1}^{\infty} \sum_{j_N, j_{N-1}, \ldots, j_1 \in \{-, +\}} \eta_{j_N, j_{N-1}, \ldots, j_1} \otimes \mathcal{I}_s^I(j_1, j_2, \ldots, j_N). \]  \hspace{1cm} (2.0.24)

\[ X(t) = \sum_{N=1}^{\infty} \sum_{j_N, j_{N-1}, \ldots, j_1 \in \{-, +\}} \mathcal{I}_s^I(j_1, j_2, \ldots, j_N) \otimes \eta_{j_N, j_{N-1}, \ldots, j_1}. \]  \hspace{1cm} (2.0.25)

Examples of all the four different decapitated product integrals defined above will now be given. In fact, these are important examples that will be needed in the exposition of the soon to be defined double product integrals. The algebra \( \mathcal{A} \) will be specialised to the Ito algebra \( \mathcal{I} \). The algebra is non-unital and nilpotent and therefore all sums will be finite. When the sums are finite no topological closure is needed.

Example 1. For each real \( \lambda \) there exist a decapitated-forward-right product integral which is the solution at \( t \) of

\[ dX^{1,2,3} = (X^{1,2} + 1)^2 \lambda (dA^1 \otimes dA^3 - dA^1 \otimes dA^1), \]  \hspace{1cm} (2.0.26)

Using equation (2.0.22), with \( \eta_- = \lambda dA^1, \eta_+ = -\lambda dA \) and \( \gamma_0 = 0 \), and taking note that the product of any three elements of the algebra \( \mathcal{I} \) is zero, the solution is, dropping tensor product notation,

\[ X(t) = \eta_- \mathcal{I}_s^I(-) + \eta_+ \mathcal{I}_s^I(+), \quad \eta_- \mathcal{I}_s^I(-), \quad \eta_+ \mathcal{I}_s^I(+). \]  \hspace{1cm} (2.0.27)

Substituting the values of \( \eta_- \) and \( \eta_+ \) we have

\[ X(t) = \lambda dA^1 \otimes \mathcal{I}_s^I(-) + (-\lambda dA) \otimes \mathcal{I}_s^I(+), \]  \hspace{1cm} (2.0.28)

Of the last four terms, \( dA^1 \lambda dA, \lambda dA^1 dA, \lambda dA \lambda dA \) and \( dA dA \) only \( dA^1 dA \) is non-zero. Hence the
solution is
\[ X(t) = \lambda dA^\dagger \otimes I_3^s(-) - \lambda dA \otimes I_3^s(+) - \lambda^2 dT \otimes I_3^s(+,-). \] (2.0.29)

Similarly the solution at \( b \) of
\[ dX_{1,2,3} = (X_{1,3} + 1)\lambda (dA^3 \otimes dA - dA^2 \otimes dA^1), \quad X_{1,3}(a) = 0 \otimes 0, \] (2.0.30)

the solution at \( t \) of
\[ dX_{1,2,3} = \lambda (dA^1 \otimes dA^3 - dA^2 \otimes dA^3)(X_{1,2} + 1), \quad X_{1,2}(s) = 0 \otimes 0, \] (2.0.31)

and the solution at \( b \) of
\[ dX_{1,2,3} = \lambda (dA^2 \otimes dA^3 - dA^2 \otimes dA^3)(X_{1,3} + 1), \quad X_{1,3}(a) = 0 \otimes 0, \] (2.0.32)

are the decapitated-forward-left, decapitated-backward-right and decapitated-backward-left product integrals which are explicitly:
\[ X(b) = -\lambda I_3^b(\cdot) \otimes dA^3 + \lambda I_3^b(\cdot) \otimes dA - \lambda^2 I_3^b(\cdot,+) \otimes dT, \] (2.0.33)
\[ X(t) = \lambda dA^\dagger \otimes I_3^s(-) - \lambda dA \otimes I_3^s(+) - \lambda^2 dT \otimes I_3^s(+,-), \] (2.0.34)
and
\[ X(b) = -\lambda I_3^b(\cdot) \otimes dA^3 + \lambda I_3^b(\cdot) \otimes dA - \lambda^2 I_3^b(\cdot,+) \otimes dT, \] (2.0.35)
respectively.

**2.0.2 Double product integrals**

Double product integrals are solutions to certain quantum stochastic differential equations, where the driving process is itself a solution of a quantum stochastic equation. The driving process is a decapitated product integral, where \( \eta_- \), \( \eta_0 \) and \( \eta_+ \) are members of the Ito algebra \( \mathcal{I} \). When this is the case we denote both
\[ \eta_- \otimes dA + \eta_+ \otimes dA^\dagger + \eta_0 \otimes dT \]
and
\[ dA \otimes \eta_- + dA^\dagger \otimes \eta_+ + dT \otimes \eta_0 \]
by $dr$, which is a member of the algebra $\mathcal{I} \otimes \mathcal{J}$. So the right-forward and left-backwards product integrals, for example would be denoted by
\[
\prod_a^t (1 + dr) \in \mathcal{J} \otimes \mathcal{P}_a^t \quad \text{and} \quad \prod_b^s (1 + dr) \in \mathcal{P}_b^s \otimes \mathcal{I}
\]
respectively. The first product integral can be used as the driving process of a quantum stochastic differential equation whose solution is either a left-forwards or left-backward product integral i.e.
\[
\prod_a^t (1 + dr) = \prod_a^t (1 + \prod_a^t (1 + dr))
\]
or
\[
\prod_b^s (1 + dr) = \prod_b^s (1 + \prod_b^s (1 + dr))
\]
respectively. For a given $dr$ one can define eight product integrals. We define and classify the four different product integrals in the following subsections.

2.0.2.1 The forward-forward double product integral

The forward-forward product integral is either the right-forward product integral of the decapitated-forward-left product integral denoted as
\[
\prod_a^t (1 + dr) = \prod_a^t (1 + \prod_a^t (1 + dr))
\]
or the left-forward product integral of the decapitated-forward-right product integral denoted as
\[
\prod_a^t (1 + dr) = \prod_a^t (1 + \prod_a^t (1 + dr))
\]

Example 2. Let $dr = \lambda (dA^\dagger \otimes dA - dA \otimes dA\dagger)$ as in Example 1. From that example recall that (2.0.29) is the decapitated-forward-right product integral
\[
X(t) = \prod_a^t (1 + dr),
\]
i.e. the solution of $dX^{1,2,3} = (X^{1,2} + 1^2)dt^{1,3}$, $X^{1,2}(s) = 0 \otimes 0$. Since $X(t) \in \mathcal{J} \otimes \mathcal{P}_a^t$ we can be used as the driving process of a left product integral which by equation (2.0.4) of Definition 10 will be the solution at $b$ to the following quantum stochastic differential equation
\[
dX = X\left(\lambda dA^\dagger \otimes I_s^1(-) - \lambda dA \otimes I_s^1(+) - \lambda^2 dT \otimes I_s^1(+)\right),
\]
\[
X(a) = 1 \otimes 1. \quad (2.0.38)
\]
Note equation (2.0.36), with \(dr = \lambda(dA^\dagger \otimes dA - dA \otimes dA^\dagger)\) is the solution to (2.0.38). Alternatively, the decapitated-forward-left product integral

\[
X(b) = \overleftarrow{b} \prod_a (1 + dr)
\]

of Example 1, equation (2.0.33), can be used, since \(X(b) \in \mathcal{P}^b_a \otimes \mathcal{I}\), as the driving process of a right product integral, which is the solution at \(t\), by equation (2.0.3) of Definition 10, of the following quantum stochastic differential equation

\[
dX = X \left( -\lambda I_{b}^a(-) \otimes dA^\dagger + \lambda I_{b}^a(+) \otimes dA - \lambda^2 I_{b}^a (+,-) \otimes dT \right),
\]

\(X(s) = 1 \otimes 1.\) (2.0.39)

Note equation (2.0.37), with \(dr = \lambda(dA^\dagger \otimes dA - dA \otimes dA^\dagger)\) is the solution to (2.0.39). Note the quantum stochastic differential equation (2.0.38) the dependence of \(X\) on \(s\) and \(t\) has been dropped for notational ease. The driving process is a linear combination of terms of the form Itô differentials tensor integrated quantum stochastic integrals in \(\mathcal{P}^t_s\). In the quantum stochastic differential equation (2.0.39) the dependence of \(X\) on \(a\) and \(b\) has been dropped for notational ease. The driving process is a linear combination of terms of the form integrated quantum stochastic integrals in \(\mathcal{P}^b_a\) tensor Itô differentials.

2.0.2.2 The backward-backward double product integral

The backward-backward product integral is either the right-backward product integral of the decapitated-backward-left product integral denoted as

\[
\overleftarrow{b} \prod_s^t (1 + dr) = \overleftarrow{b} \prod_s^t (1 + \overleftarrow{\prod_s^t} (1 + dr)), \tag{2.0.40}
\]

or the left-backward product integral of the decapitated-backward-right product integral denoted as

\[
\overleftarrow{b} \prod_s^t (1 + dr) = \overleftarrow{\prod_s^t} (1 + \overleftarrow{\prod_s^t} (1 + dr)). \tag{2.0.41}
\]

Example 3. Let \(dr = \lambda(dA^\dagger \otimes dA - dA \otimes dA^\dagger)\) as in Example 1. From that example recall that (2.0.34) is the decapitated-backward-right product integral

\[
X(t) = \prod_s^d (1 + dr),
\]

i.e. the solution of \(dX^{1,2,3} = dr^{2,3}(X^{1,3} + 1^1)\), \(X^{1,3}(s) = 0 \otimes 0\). Since \(X(t) \in \mathcal{I} \otimes \mathcal{P}^t_s\) it can be used as the driving process of a left product integral which by equation (2.0.7) of Definition 10
will be the solution at $b$ to the following quantum stochastic differential equation
\[ dX = \left( \lambda dA^\dagger \otimes I_s(+) - \lambda dA \otimes I_s(-,+) - \lambda^2 dT \otimes I_s(-,,+) \right) X, \]
\[ X(a) = 1 \otimes 1. \quad (2.0.42) \]

Note equation (2.0.40), with \( dr = \lambda (dA^\dagger \otimes dA - dA \otimes dA^\dagger) \), is the solution to (2.0.42). Alternatively, the decapitated-backward-left product integral
\[ X(b) = \left\langle \stackrel{\leftarrow}{a} \right| \prod_{t} (1 + dr) \right\rangle \]

of Example 1, equation (2.0.35), can be used, since \( X(b) \in \mathcal{P}_{a}^b \otimes I \), as the driving process of a right product integral which is the solution, by equation (2.0.6) of Definition 10, as the solution at $t$ of the following quantum stochastic differential equation
\[ dX = \left( - \lambda I_{a}^b (-) \otimes dA^\dagger + \lambda I_{a}^b (+) \otimes dA - \lambda^2 I_{a}^b (-,,+) \otimes dT \right) X, \]
\[ X(s) = 1 \otimes 1. \quad (2.0.43) \]

Note equation (2.0.41), with \( dr = \lambda (dA^\dagger \otimes dA - dA \otimes dA^\dagger) \), is the solution to (2.0.43). Note the in quantum stochastic differential equation (2.0.42) the dependence of $X$ on $s$ and $t$ has been dropped for notational ease. The driving process is a linear combination of terms of the form Ito differentials tensor integrated quantum stochastic integrals in $\mathcal{P}_{a}^b$. In the quantum stochastic differential equation (2.0.43) the dependence of $X$ on $a$ and $b$ has been dropped for notational ease. The driving process is a linear combination of terms of the form integrated quantum stochastic integrals in $\mathcal{P}_{a}^b$ tensor Ito differentials.

### 2.0.2.3 The forward-backward double product integral

The forward-backward product integral is either the left-forward product integral of the decapitated-backward-right integral denoted as
\[ \left\langle \stackrel{\rightarrow}{b} \right| \prod_{s}^t (1 + dr) = \left\langle \stackrel{\rightarrow}{b} \right| \prod_{s}^t (1 + \prod_{s}^{\leftarrow} (1 + dr)). \quad (2.0.44) \]

or the right-backward product integral of the decapitated-forward-left integral product denoted as
\[ \left\langle \stackrel{\leftarrow}{b} \right| \prod_{s}^t (1 + dr) = \prod_{s}^{\leftarrow} (1 + \left\langle \stackrel{\rightarrow}{b} \right| \prod_{s}^{\leftarrow} (1 + dr)). \quad (2.0.45) \]

Example 4. In a similar vein to Examples 2 and 3, when \( dr = \lambda (dA^\dagger \otimes dA - dA \otimes dA^\dagger) \) double
product integrals (2.0.44) and (2.0.45) are the solutions to the following differential equations

\[ dX(b) = \left( \lambda dA^\dagger(b) \otimes I_s(b) - \lambda dA(b) \otimes I_s^+(b) - \lambda^2 dT(b) \otimes I_s^-(b) \right) X(b), \]

\[ X(a) = 1 \otimes 1. \quad (2.0.46) \]

\[ dX(t) = \left( -\lambda I_a^b(-) \otimes dA^\dagger(t) + \lambda I_a^b(+) \otimes dA(t) - \lambda^2 I_a^b(+) \otimes dT(t) \right) X(t), \]

\[ X(s) = 1 \otimes 1, \quad (2.0.47) \]

respectively. Note the driving processes of (2.0.46) and (2.0.47) are the product integrals (2.0.33) and (2.0.34) of Example 1 respectively.

2.0.2.4 The backward-forward double product integral

The backward-forward product integral is either the left-backward product integral of the decapitated-forward-right integral denoted as

\[ \overset{\leftrightarrow}{b \int_s^t} (1 + dr) = \overset{\leftarrow}{b \int_s^t} (1 + \overset{\rightarrow}{a \int_s^t} (1 + dr)), \quad (2.0.48) \]

or the right-forward product integral of the decapitated-backward-left integral denoted as

\[ \overset{\leftrightarrow}{b \int_s^t} (1 + dr) = \overset{\rightarrow}{a \int_s^t} (1 + \overset{\leftarrow}{b \int_s^t} (1 + dr)). \quad (2.0.49) \]

Example 5. In a similar vein to Examples 2 and 3, when \( dr = \lambda (dA^\dagger \otimes dA(t) - dA \otimes dA^\dagger(t)) \) double product integrals (2.0.48) and (2.0.49) are the solutions to the following differential equations

\[ dX(b) = \left( \lambda dA^\dagger(b) \otimes I_s(b) - \lambda dA(b) \otimes I_s^+(b) - \lambda^2 dT(b) \otimes I_s^-(b) \right) X(b), \]

\[ X(a) = 1 \otimes 1. \quad (2.0.50) \]

\[ dX(t) = \left( -\lambda I_a^b(-) \otimes dA^\dagger(t) + \lambda I_a^b(+) \otimes dA(t) - \lambda^2 I_a^b(+) \otimes dT(t) \right) X(t), \]

\[ X(s) = 1 \otimes 1, \quad (2.0.51) \]

respectively. Note the driving processes of (2.0.50) and (2.0.51) are the product integrals (2.0.29) and (2.0.35) of Example 1 respectively.
Remark 8. It can be shown Hudson [2005a] that the two definitions of

\[ b \prod_a^b (1 + dr) \]

are equal for every choice of \( dr \in I \otimes J \). We saw in Section 1.3.2 that this was so for some simple examples. In what follows we shall see that it is the case by explicit evaluations in the more interesting case when

\[ dr = \lambda (dA^\dagger \otimes dA^\dagger - dA \otimes dA) - \frac{1}{2} \lambda^2 dT \otimes dT. \]

Remark 9. In Hudson [2005b] a solution is exhibited (constructed in Hudson [2007] by discretisation of the double product and the taking of heuristic limits) and shown to satisfy both (2.0.46) and (2.0.47) of Example 4. In Hudson and Jones [2012] a solution to (2.0.38) and (2.0.39) of Example 2 was exhibited.
Chapter 3

The Fock representation of the CCR and the von Neumann uniqueness theorem

3.0.3 The Fock representation of the CCR

The Fock Weyl operators over a Hilbert space $g$ are the family of operators $W(f), f \in g$ defined by their action on the exponential vectors in the Fock space $\Gamma(g)$ over $g$ (using the probabilistic normalisation i.e. $\hbar = 2$),

$$W(f)\psi(g) = \exp\left\{-\frac{1}{2}\|f\|^2 - \langle f, g \rangle\right\}\psi(f + g).$$ (3.0.1)

The operator is well defined since the exponential vectors are linearly independent and total in $\Gamma(g)$, see Proposition 1. They satisfy the Weyl form of the CCR (canonical commutation relations), so they comprise the Fock RCCR (see Theorem 6 below for a precise statement and proof),

$$W(f)W(g) = \exp\{-i\text{Im}\langle f, g \rangle\}W(f + g)$$ (3.0.2)

and the Fock vacuum vector $\psi(0)$ is cyclic for them. By putting $g = 0$ in the defining formula (3.0.1) it can be seen that they evidently generate the exponential domain by action on $\psi(0)$.

Some of the basic results regarding the Weyl operators are now stated and proved. Before we do this we recall three well know propositions.

Proposition 1. The set $\{\psi(g) : g \in g\}$ of exponential vectors is linearly independent and total in $\Gamma(g)$.

Proof. See Parthasarathy [1992], pp 126.
Proposition 2. Let \( S_i \) be a total (dense) subset of the Hilbert space \( \mathcal{H}_i \) for \( i = 1, 2 \). Suppose \( U_0: S_1 \to S_2 \) is a scalar product preserving map, so that for all \( u, v \in S_1 \)

\[
\langle U_0u, U_0v \rangle = \langle u, v \rangle.
\]

Then there exist a unique linear isometry \( U: \mathcal{H}_1 \to \mathcal{H}_2 \) which extends \( U_0 \) i.e. \( Uu = U_0u \) for all \( u \in S_1 \). If, in addition \( U_0 \) is onto, then \( U \) is a unitary isomorphism from \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \).


Proposition 3. Let \( \mathcal{H}_1, \mathcal{H}_2 \) be Hilbert spaces and \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \). Then there exists a unique unitary isomorphism \( U: \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2) \) satisfying the relation

\[
U\psi(u_1 + u_2) = \psi(u_1) \otimes \psi(u_2) \text{ for all } u_1 \in \mathcal{H}_1, \ u_2 \in \mathcal{H}_2.
\]


Remark 10. We use \( U \) to identify \( \Gamma(\mathcal{g}_1 \oplus \mathcal{g}_2) \) with \( \Gamma(\mathcal{g}_1) \otimes \Gamma(\mathcal{g}_2) \).

Theorem 4. \( W(\mathcal{g}) : \mathcal{E}(\mathcal{g}) \to \mathcal{E}(\mathcal{g}) \) is isometric for any \( g \in \mathcal{g} \).

Proof. For all \( f, g_1, g_2 \in \mathcal{g} \) it will be shown that

\[
\langle W(f)\psi(g_1), W(f)\psi(g_2) \rangle = \langle \psi(g_1), \psi(g_2) \rangle.
\]

i.e. \( W(f) \) is an isometry from \( \mathcal{E}(\mathcal{g}) \) onto itself.

\[
\langle W(f)\psi(g_1), W(f)\psi(g_2) \rangle = \langle \exp \left( -\langle f, g_1 \rangle - \frac{1}{2}||f||^2 \right) \psi(f + g_1), \exp \left( -\langle f, g_2 \rangle - \frac{1}{2}||f||^2 \right) \psi(f + g_2) \rangle
\]

\[
= \langle \exp \left( -\langle f, g_1 \rangle - ||f||^2 - \langle f, g_2 \rangle \right) \psi(f + g_1), \psi(f + g_2) \rangle
\]

\[
= \exp \left( -\langle f + g_1, f + g_2 \rangle + \langle g_1, g_2 \rangle \right) \exp \langle f + g_1, f + g_2 \rangle
\]

\[
= \langle \psi(g_1), \psi(g_2) \rangle,
\]

where the definition of the Weyl operators (equation (3.0.1)) has been used in the first line and the identity \( \langle f + g_1, f + g_2 \rangle = \langle f, f \rangle + \langle f, g_1 \rangle + \langle f, g_2 \rangle + \langle g_1, g_2 \rangle \) has been used to go from the second to third line.

The following corollary states that the Weyl family extends to the whole Fock space and are unitary operators on the Fock space.

Corollary 1. \( W(\mathcal{g}) \) extends uniquely to a unitary operator on \( \Gamma(\mathcal{g}) \) for any \( g \in \mathcal{g} \).
Proof. Using the result of Proposition 2 and the fact that the map \( W(g) : \mathcal{E}(g) \to \mathcal{E}(g) \) is onto and that \( \mathcal{E}(g) \) is dense in \( \Gamma(g) \) it follows that \( W(g) \) has a unique isometric extension to \( \Gamma(g) \).

From the RCCR (equation (3.0.2)) it can be seen that each \( W(g) \) has an inverse \( W(-g) \). So it is unitary.

\[ \Box \]

**Theorem 5.** The family of Weyl operators \( \{ W(f) : f \in g \} \) is irreducible.

Before we give a proof of the theorem, we discuss a certain unitary isomorphism \( U : \Gamma(l^2) \to L^2(P) \), where \( l^2 \) is the Hilbert space of all absolutely square summable sequences i.e. \( \{(a_1, a_2, \ldots) : a_i \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \} \) and \( P \) is the probability measure of an independent and identically distributed sequence of standard Gaussian random variables \( \xi = (\xi_1, \xi_2, \ldots) \).

We define \( U \) on \( \mathcal{E}(l^2) \) as follows for \( z \in l^2 \)

\[
[U\psi(z)](\xi) = \exp \left( \sum_{j=1}^{\infty} (z_j \xi_j - \frac{1}{2} z_j^2) \right).
\]

(3.0.4)

and \( z_1 \) and \( z_2 \in l^2 \)

\[
\int_{\mathbb{R}^\infty} e^{\sum_{j=1}^{\infty} (\tau_1 x_j - \frac{1}{2} x_j^2)} e^{\sum_{j=1}^{\infty} (\tau_2 x_j - \frac{1}{2} x_j^2)} dP(x) = \langle \psi(z_1), \psi(z_2) \rangle.
\]

(3.0.5)

Since the set of functions \( \{ \exp(\langle z, \xi \rangle) : z \in l^2, \ \xi \in \mathbb{R} \} \) is dense in \( L^2(P) \), by Propositions 1 and 2, \( U \) extends to all of \( \Gamma(l^2) \).

**Proof.** We follow the proof given in Parthasarathy [1992] pp142-143. Let \( T \) be any bounded operator on \( \Gamma(g) \) such that \( TW(f) = W(f)T \) for all \( f \) in \( g \). For any square summable real sequence \( \omega \) we have

\[
\{ UW(ia)U^{-1} f\}(\xi) = e^{-||a||^2 - \frac{1}{2} \sum_{j=1}^{\infty} a_j \xi_j} f(\xi)
\]

(3.0.6)

\[
\{ UW(-\frac{1}{2}a)U^{-1} f\}(\xi) = e^{-\frac{1}{2} ||a||^2 - \frac{1}{2} \sum_{j=1}^{\infty} a_j \xi_j} f(\xi + a)
\]

(3.0.7)

for all \( f \in L^2(P) \). Let \( S := UTU^{-1} \), then \( S \) commutes with \( UW(ia)U^{-1} \) and \( UW(-\frac{1}{2}a)U^{-1} \).

Equation (3.0.6) implies that \( S \) commutes with the operator of multiplication by any bounded random variable \( \phi(\xi) \), and in particular with the characteristic function of a measurable set \( E \).

Then we have

\[ SI_E = SI_E 1 = I_E S1. \]

If \( S1 = \vartheta \) we can conclude that \( (SF)(\xi) = \vartheta(\xi) f(\xi) \) for every \( f \) in \( L^2(P) \). Equation (3.0.7) implies

\[ \vartheta(\xi + a) f(\xi + a) = \vartheta(\xi) f(\xi + a) \ \text{a.e.} \ \xi. \]

for each \( a \) of the form \((a_1, a_2, \ldots, a_n, 0, 0, \ldots)\). Thus \( \vartheta(\xi) \) is independent of \( \xi_1, \xi_2, \ldots, \xi_n \) for each \( n \). Applying Kolmogorov’s 0 – 1 law shows that \( \vartheta \) is constant.

\[ \Box \]
The next theorem states that the family of Weyl operators form a representation of the CCR. They satisfy the Fock RCCR.

**Theorem 6.** For all \( f \in \mathfrak{g} \), \( f \mapsto W(f) \) satisfies the Weyl relation,

\[
W(f_1)W(f_2) = e^{-i\text{Im}(f_1, f_2)} W(f_1 + f_2), \text{ for any } f_1, f_2 \text{ in } \mathfrak{g}.
\]

**Proof.** For arbitrary \( f_1, f_2 \) in \( \mathfrak{g} \),

\[
W(f_1)W(f_2)\psi(g) = W(f_1)e^{-\langle f_2, g \rangle - \frac{1}{2}\|f_2\|^2} \psi(f_2 + g)
= e^{-\langle f_2, g \rangle} e^{-\frac{1}{2}\|f_2\|^2} e^{-\langle f_1, f_2 + g \rangle - \frac{1}{2}\|f_1\|^2} \psi(f_1 + f_2 + g)
= e^{-\text{Im}(f_1, f_2)} W(f_1 + f_2)\psi(g).
\]

**Theorem 7.** The map from \( \mathfrak{g} \to \Gamma(\mathfrak{g}) \), \( f \mapsto W(f)\phi \) is continuous for all \( \phi \in \Gamma(\mathfrak{g}) \) i.e. the map \( f \mapsto W(f) \) is strongly continuous from \( \mathfrak{g} \) into the unitary group on \( \Gamma(\mathfrak{g}) \).

**Proof.** The map \( f \mapsto W(f)\phi \) is continuous if \( \phi = \psi(g) \) for some \( g \) in \( \mathfrak{g} \) since the action of \( W(f) \) on \( \psi(g) \) is the product of two factors which are continuous from \( \mathfrak{g} \) to \( \mathbb{C} \) and to \( \Gamma(\mathfrak{g}) \), namely \( e^{-\langle f, g \rangle - \frac{1}{2}\|f\|^2} \) and \( \psi(f + g) \) respectively. The last factor is continuous since

\[
\|\psi(f_1) - \psi(f_2)\|^2 = e^{\langle f_1, f_1 \rangle - \langle f_2, f_2 \rangle - \langle f_1, f_2 \rangle - \langle f_2, f_1 \rangle} \to 0 \text{ as } f_1 \to f_2
\]

by continuity of \( \langle \cdot, \cdot \rangle \) and of the exponential function. Since the map \( f \mapsto W(f)\psi(g) \) is continuous for all \( g \) in \( \mathfrak{g} \), \( \mathcal{D}(\mathbb{g}) \) is dense in \( \Gamma(\mathfrak{g}) \) and the uniform boundedness of \( W(f) \) due to its unitarity for all \( f \in \mathfrak{g} \) the continuity property can be extended to the all of \( \Gamma(\mathfrak{g}) \) by a similar argument.

We introduce the well known avatar of an observable in quantum probability which we call a **continuous unitary representation** (CUR).

**Definition 14.** A **continuous unitary representation (CUR)** of \((\mathbb{R}, +)\) on a Hilbert space \( \mathcal{H} \) is a group homomorphism from \((\mathbb{R}, +)\) into the unitary group of \( \mathcal{H} \)

\[
\pi : \mathbb{R} \to U(\mathcal{H}),
\]

such that \( \mathbb{R} \ni x \mapsto \pi(x)\eta \) is norm continuous for every \( \eta \in \mathcal{H} \).

The next theorem (**Bochner’s theorem**.) is a standard theorem in probability. It allows CURs to be given probability distributions. The detail of how this is done is given in Corollary 2 below. We are particularly interested in finding CURs which have a Gaussian distribution. For a proof of Bochner’s theorem see Stone [1932].

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Theorem 8. **Bochner’s theorem** A function \( F : \mathbb{R} \to \mathbb{C} \) is the characteristic function, i.e.

\[
F(x) = \int_{\mathbb{R}} e^{ix\lambda} dP(\lambda) \quad \text{for all } x \in \mathbb{R},
\]

of a probability distribution \( P \) on \( \mathbb{R} \) if and only if it satisfies

1. \( F(0) = 1 \).
2. \( F \) is continuous.
3. \( F \) is nonnegative definite i.e. for all \( N \in \mathbb{N}, t_1, t_2, \ldots, t_N \in \mathbb{R} \) and \( z_1, \ldots, z_N \in \mathbb{C} \),

\[
\sum_{j,k=1}^{N} \bar{z}_j z_k F(-t_j + t_k) \geq 0.
\]

\( P \) is then unique.

The following lemma links CURs with Bochner’s theorem which in turn will link a given CUR to a probability distribution.

**Lemma 1.** Let \( \{U_x\}_{x \in \mathbb{R}} \) be a CUR on a Hilbert space \( \mathcal{H} \) and let \( \phi \) be a unit vector from \( \mathcal{H} \). If \( F(x) = \langle \phi, U_x \phi \rangle \), then \( F \) satisfies the conditions (1), (2) and (3) of Bochner’s theorem.

**Proof.**

1. \( F(0) = \langle \phi, U_0 \phi \rangle = \langle \phi, I \phi \rangle = \langle \phi, \phi \rangle = 1 \).

2. \( |F(x) - F(y)| = \| \langle \phi, U_x \phi \rangle - \langle \phi, U_y \phi \rangle \|
\leq \| \langle \phi, (U_x - U_y) \phi \rangle \| \quad \text{by the Cauchy-Schwartz inequality,}
\]

\[
= \| (U_x - U_y) \phi \|.\]

3. \( \sum_{j,k=1}^{N} \bar{z}_j z_k F(-t_j + t_k) = \sum_{j,k=1}^{N} \bar{z}_j z_k \langle \phi, U_{-t_j} U_t \phi \rangle
\]

\[
= \sum_{j,k=1}^{N} \bar{z}_j z_k \langle U_{t_j} \phi, U_{t_k} \phi \rangle
\]

\[
= \left\| \sum_{k=1}^{N} z_k U_{t_k} \phi \right\|^2 \geq 0.\]

\( \square \)

**Remark 11.** By Corollary 1 and Theorem 7 for a given \( f \) in \( g \), \( \{W(xf)\}_{x \in \mathbb{R}^+} \) is a CUR on \( \Gamma(g) \). We now illustrate some facts about the distribution of this CUR in the vacuum state \( \psi(0) \).

By Lemma 1 and Remark 11 we immediately have the following corollary to Bochner’s theorem (Theorem 8).

**Corollary 2.** By Bochner’s theorem \( \{W(xf)\}_{x \in \mathbb{R}^+} \) determines a unique probability distribution \( P \) on \( \mathbb{R} \) such that

\[
\langle \psi(0), W(xf) \psi(0) \rangle = \int_{\mathbb{R}} e^{ixy} P(dy), \quad (3.0.8)
\]

for all \( x \) in \( \mathbb{R} \), where the unit vector \( \phi \) of Lemma 1 is the vacuum vector \( \psi(0) \).
The corresponding probability distribution associated to the CUR

\[ \{W(xf)\}_{x \in \mathbb{R}^+} \]

is well known. We discuss the associated distribution in what follows. The corresponding probability distribution of \( \{W(xf)\} \) for fixed \( f \in \mathfrak{g} \) follows from the following calculation

\[
F(x) = \langle \psi(0), W(xf)\psi(0) \rangle \\
= \langle \psi(0), \exp \left( -\langle xf, 0 \rangle - \frac{1}{2} \|xf\|^2 \right) \psi(xf) \rangle \\
since \langle \psi(0), \psi(xf) \rangle \rangle = 1 \\
= e^{-\frac{1}{2}x^2\|f\|^2}.
\]

This is the characteristic function of a Gaussian random variable with mean 0 and variance \( \|f\|^2 \). We denote the distribution of such a random variable as \( N(0, \|f\|^2) \). Therefore \( (W(xf))_{x \in \mathbb{R}} \) may be thought of as a Gaussian random variable living in the Fock space over \( \mathfrak{g} \).

### 3.0.4 Construction of Brownian motion over the Fock space \( \mathcal{H} \)

It was seen that by taking an arbitrary function \( f \) from \( \mathfrak{g} \) a corresponding quantum random variable could be constructed which had a Gaussian distribution with parameters mean 0 and variance \( \|f\|^2 \). We now specialise our discussion to the Hilbert space \( \mathfrak{h} \) in order to define quantum Brownian motion. Taking \( g \) indexed by \( t \in \mathbb{R}^+ \) i.e. \( g_t \in \mathfrak{h}_t \), we would like to obtain the variance and covariance of Brownian motion i.e.

\[
\|g_t\|^2 = t \text{ and } \langle g_s, g_t \rangle = s \wedge t, \quad \text{for all } s, t \in \mathbb{R}^+. \tag{3.0.9}
\]

The property of independent increments will be achieved if, for \( s < t \), \( g_t - g_s \in L^2(s, t) \). Both properties can be achieved by either taking \( g_t = \chi_{[0,t)} \) for all \( t \in \mathbb{R}^+ \) or by taking it to be \( i\chi_{[0,t)} \) for all \( t \in \mathbb{R}^+ \).

In the following subsection we derive the self adjoint avatars of the CURs

\[
(W(x\chi_{[0,t)}))_{x \in \mathbb{R}} \text{ and } (W(ix\chi_{[0,t)}))_{x \in \mathbb{R}} \text{ for } t \in \mathbb{R}^+.
\]

These avatars are well known in quantum probability as the self-adjoint extensions (with core \( \mathfrak{c} \)) of certain operators processes.
The self adjoint operator avatars of the CURs

\{W(x\chi[0,t])\}_{x \in \mathbb{R}} \text{ and } \{W(xi\chi[0,t])\}_{x \in \mathbb{R}}

**Theorem 9. Stone’s Theorem.** There is a one to one correspondence between CURs \((U_x)_{x \in \mathbb{R}}\) on a Hilbert space \(\mathcal{H}\) and self-adjoint operators \(H\) given by

\[ U_x = \int_{\mathbb{R}} e^{-ix\lambda} dE(\lambda), \]

where \(H = \int_{\mathbb{R}} \lambda dE(\lambda)\) is the spectral resolution of \(H\), and

\[ H = i \frac{d}{dx} U_x \bigg|_{x=0}, \]

where the domain of \(H\) is the set of vectors on which the limits exist.

**Proof.** see Parthasarathy [1992] pp73. □

Formally differentiating with respect to \(x\) the action on an arbitrary exponential vector \(W(x\chi[0,t])\psi(f)\) and setting \(x = 0\) gives the corresponding self-adjoint operator \(P(t)\) as the sum \(i(A^\dagger(t) - A(t))\) of mutually adjoint operators

\[ A^\dagger(t)\psi(f) = \frac{d}{dx}\psi(x\chi[0,t] + f) \bigg|_{x=0} \]

and

\[ A(t)\psi(f) = (\chi[0,t], f)\psi(f). \]

Indeed:

\[ i \frac{d}{dx} W(x\chi[0,t])\psi(f) \bigg|_{x=0} = i \frac{d}{dx} \left( e^{-i(x\chi[0,t], f)} - \frac{i}{2}\|x\chi[0,t]\|^2 \right) \psi(x\chi[0,t] + f) \bigg|_{x=0} \]

\[ = -i(\chi[0,t], f)\psi(f) + i \frac{d}{dx} \psi(x\chi[0,t] + f) \bigg|_{x=0} \]

\[ = -iA(t)\psi(f) + iA^\dagger(t)\psi(f) \]

\[ = i(A^\dagger(t) - A(t))\psi(f). \]

The resulting operator \(i(A^\dagger(t) - A(t))\) has the standard notation as \(P(t)\). A similar calculation on the CUR \(\{W(xi\chi[0,t])\}_{x \in \mathbb{R}}\) yields the operator \(Q(t) = A(t) + A^\dagger(t)\).

The following lemmas, specialised from Parthasarathy [1992] pp136-140, are used to prove Theorem 10 which states in what sense the operators \(P(t)\) and \(Q(s)\) are non-commuting.

**Lemma 2.** Given a finite set of vectors \(\{g_1, g_2, \ldots, g_n\}\) from an arbitrary Hilbert space \(g\), the map \((z_1, z_2, \ldots, z_n) \mapsto e(z_1g_1 + z_2g_2 + \ldots + z_ng_n)\) from \(\mathbb{C}^n\) into \(\Gamma(g)\) is analytic.

**Proof.** The lemma holds for \(n = 1\) since for \(g \in g\), \(\sum_{n=0}^{\infty} \|z^n\frac{\partial^n}{\partial z^n}\| < \infty\) and the map \(z \mapsto \sum_{n=0}^{\infty} z^n\frac{\partial^n}{\partial z^n}\) is analytic. For the case when \(n > 1\), choose an orthonormal basis \(\{v_1, v_2, \ldots, v_m\}\).
for the vector space spanned by \( \{ z_1, z_2, \ldots, z_n \} \). Then \( \sum_{i=0}^{n} z_i g_i = \sum_{j=1}^{m} L_j(z_1, z_2, \ldots, z_n) v_j \), where \( L_j(z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} g_i(z_i, v_j) \). Let \( g_j \), for \( 1 \leq j \leq m \), denote the one-dimensional subspace spanned by \( v_j \) and let \( g_{m+1} = \{ v_1, v_2, \ldots, v_m \} \perp \). Then \( g = \bigoplus_{j=1}^{m+1} g_j \) and we can (using Proposition 3) identify \( \Gamma(g) \) with the tensor product \( \bigotimes_{j=1}^{m+1} \Gamma(g_j) \) so that

\[
e(\sum_{i=1}^{n} z_i g_i) = \left\{ \bigotimes_{j=1}^{m} e \left( L_j(z_1, z_2, \ldots, z_n) v_j \right) \right\} \otimes e(0). \quad (3.0.13)
\]

Since for each \( j \), \( L_j(z_1, z_2, \ldots, z_n) \) is linear in \( z \) the analytically for the general case follows from one-dimensional case.

**Lemma 3.** For any \( g_1, g_2, \ldots, g_m, g'_1, g'_2, \ldots, g'_n, g \) in \( g \) the map

\[(x, y) \mapsto W(x_1 g_1) \cdots W(x_m g_m) e(\sum_{j=1}^{n} y_j g'_j + g) \quad (3.0.14)\]

from \( \mathbb{R}^{m+n} \) into \( \Gamma(g) \) is real-analytic.

**Proof.** Using the definition of the Weyl operators (equation (3.0.1)) and the Fock CCR (equation (3.0.2)) we have

\[W(x_1 g_1) \cdots W(x_m g_m) e(\sum_{j=1}^{n} y_j g'_j + g) = \phi(x, y) e(\sum_{i=1}^{m} x_i g_i + \sum_{j=1}^{n} y_j g'_j + g), \quad (3.0.15)\]

where \( \phi \) is a second degree polynomial in the variables \( x_i \) and \( y_j \), \( 1 \leq i \leq m, 1 \leq j \leq n \). The result follows from Lemma 2.

**Lemma 4.** \( E \subset D(P(t_1) P(t_2) \cdots P(t_n)) \) for all \( n \).

**Proof.** The lemma quickly follows from Lemma 3 by setting, for \( j \) from 1 to \( n \), the \( y_j \)'s to zero, setting the \( g_i \)'s all equal to \( \chi_{[0, t_i]} \) for some \( t_i \). Then using the calculation (3.0.12) successively, where the differentiation is with respect to the \( x_i \)'s for \( i \) from \( m \) to 1 sequentially.

**Lemma 5.** The operator processes \( P(t) \) and \( Q(t) \) on the exponential domain \( E \) have self adjoint extensions to \( \mathcal{H} \).

**Proof.** Clear from Stone’s theorem.

**Remark 12.** In the light of Lemma 5, by Stone’s theorem, the Stone generators of

\[W(x \chi_{[0, t]}), \text{ and } \{ W(x \chi_{[0, t]} \} \}_{x \in \mathbb{R}}\]

are the unique self adjoint extensions of

\[P(t) = i \{ A^\dagger(t) - A(t) \} \text{ and } Q(t) = A(t) + A^\dagger(t)\]
respectively, with core $\mathcal{E}$.

**Lemma 6.**

\[
\langle i \frac{d}{dx} W(xf) \psi(h_1), i \frac{d}{dy} W(yg) \psi(h_2) \rangle \bigg|_{x=y=0} = -\left( \langle f, g \rangle + \langle -h_1, g \rangle + \langle f, h_2 \rangle - \langle g, h_2 \rangle + \langle h_1, g \rangle \right) e^{(h_1, h_2)} (3.0.16)
\]

**Proof.** The left hand side of (3.0.16) is equal to

\[
-\frac{\partial^2}{\partial x\partial y} e^{-\langle (xf, h_1) + \frac{1}{2} x^2 f^2 + (yg, h_2) + \frac{1}{2} y^2 g^2 \rangle, e^{\langle (xf, h_1) + (yg, h_2) \rangle}} \bigg|_{x=y=0} = -\frac{d}{dy} \left( \langle h_1, f \rangle + \frac{1}{2} y^2 g^2 \right) e^{-\langle (yg, h_2) - \frac{1}{2} y^2 g^2 \rangle, e^{\langle (xf, h_1) + (yg, h_2) \rangle}} \bigg|_{y=0}
\]

\[= -\left( \langle f, g \rangle + \langle -h_1, f \rangle + \langle f, h_2 \rangle - \langle g, h_2 \rangle + \langle h_1, g \rangle \right) e^{(h_1, h_2)} (3.0.17)\]

**Theorem 10.** The two operators $P(t)$ and $Q(s)$ are non-commuting (in the weak sense) and in particular

\[
[P(t), Q(s)] = -2i (s \wedge t) I (3.0.18)
\]

in the weak sense i.e. for all $f,g$ in $\mathfrak{g},$

\[
\langle P(t)\psi(f), Q(s)\psi(g) \rangle - \langle Q(s)\psi(f), P(t)\psi(g) \rangle = -2i (s \wedge t) \langle \psi(f), \psi(g) \rangle.
\]

**Proof.** Using Lemma 6 we have

\[
\langle i \frac{d}{dx} W(x\chi(0,t)) \psi(f), i \frac{d}{dy} W(iy\chi(0,s)) \psi(g) \rangle \bigg|_{x=y=0} = -\left( \langle \chi(0,t), i\chi(0,s) \rangle + \langle -f, \chi(0,t) \rangle + \langle \chi(0,s), g \rangle \left( i\langle \chi(0,s), g \rangle + i(f, i\chi(0,s)) \right) \right) e^{(f,g)} (3.0.19)
\]

Swapping the roles of $\chi(0,t)$ and $i\chi(0,s)$ in the left hand side of (3.0.19) and using Lemma 6 a second time we have

\[
\langle i \frac{d}{dx} W(iy\chi(0,s)) \psi(f), i \frac{d}{dy} W(x\chi(0,t)) \psi(g) \rangle \bigg|_{x=y=0} = -\left( \langle i\chi(0,s), \chi(0,t) \rangle + \langle -i(f, i\chi(0,s)) - i(\chi(0,s), g) \rangle \left( -\langle \chi(0,t), g \rangle + \langle f, \chi(0,t) \rangle \right) \right) e^{(f,g)} (3.0.20)
\]

Subtracting the result of the calculation (3.0.20) from that of (3.0.19), using Proposition 1,
Lemmas 4 and 5 we have that $P(t)$ and $Q(s)$ satisfy

$$\langle P(t)\psi(f), Q(s)\psi(g) \rangle - \langle Q(s)\psi(f), P(t)\psi(g) \rangle = -\langle i\langle \chi[0,t], \chi[0,s] \rangle + i\langle \chi[0,s], \chi[0,t] \rangle \rangle(\psi(f), \psi(g))$$

$$= -2i (s \wedge t)\langle \psi(f), \psi(g) \rangle,$$

thus (3.0.18) holds in the weak sense.

**Corollary 3.**

$$[P(t), P(s)] = [Q(t), Q(s)] = 0 \quad (3.0.21)$$

in the weak sense i.e. $P(t)$ commutes with $P(s)$, and $Q(t)$ with $Q(s)$ in the weak sense.

**Proof.** This can be by a similar proof to that of Theorem 10.

**Corollary 4.** The two operators $A(t)$ and $A^\dagger(s)$ are non-commuting (in the weak sense) and in particular

$$[A(t), A^\dagger(s)] = (s \wedge t)I \quad (3.0.22)$$

in the weak sense.

**Proof.**

$$[A(t), A^\dagger(s)] = \frac{1}{2} (Q(t) + iP(t)) \cdot \frac{1}{2} (Q(s) - iP(s))$$

$$= \frac{1}{4} [Q(t), Q(t)] - \frac{1}{4} [Q(t), P(s)] + \frac{1}{4} [P(t), Q(s)] + \frac{1}{4} [P(s), P(s)]$$

$$= \frac{1}{2} (s \wedge t)I + \frac{1}{2} (s \wedge t)I,$$

where the definition of $P$ and $Q$ is used in the first line, the linearity of the bracket in the second and in the third line Corollary 3 and Theorem 10.

**Remark 13.** The two Brownian motions corresponding to the two families of CURs $\{W(x\chi[0,t])\}_{x \in \mathbb{R}}$ and $\{W(x\chi[0,t])\}_{x \in \mathbb{R}}$ are respectively called Momentum-Brownian motion (or $P$-Brownian Motion) and Position-Brownian motion (or $Q$-Brownian motion).

### 3.1 The von Neumann uniqueness theorem and its consequences

In this section the von Neumann uniqueness theorem is introduced without proof. For the time being we discuss the theorem in the contexts of a general Hilbert space $\mathfrak{g}$ and specify later onto the Hilbert space $\mathfrak{h}$. Let $W : f \mapsto W(f)$ denote the Fock RCCR over $\mathfrak{g}$, acting in the Fock space $\Gamma(\mathfrak{g})$ over $\mathfrak{g}$. Let there be given another Hilbert space $\mathcal{K}$. Then $W_{\mathcal{K}} : f \mapsto W(f) \otimes I_{\mathcal{K}}$
is another RCCR called the ampliation of $W$ to $\Gamma(g) \otimes \mathcal{H}$. Given a Hilbert space $\mathcal{L}$ and a unitary transformation $V : \mathcal{L} \to \Gamma(g) \otimes \mathcal{H}$ then $W^V_{\mathcal{H}} : f \to V^{-1} W_{\mathcal{H}}(f)V$ is another RCCR in $\mathcal{L}$.

**Theorem 11.** The von Neumann Uniqueness theorem: Every representation of the Weyl relations over a finite dimensional Hilbert space is unitarily equivalent to an ampliation of the Fock representation.

For a proof of the von Neumann uniqueness theorem see von Neumann [1937] or Derezinski [2003].

**Example 6.** For any real-linear $\phi : g \mapsto \mathbb{R}$, define $W_\phi(f) := e^{i\phi(f)} W(f)$ for $f \in g$. Then $W_\phi$ is a representation of the CCR, since for any $f$ and $g$ in $g$,

$$W_\phi(f)W_\phi(g) = e^{i\phi(f)}W(f)e^{i\phi(g)}W(g) = e^{i(\phi(f)+\phi(g))}W(f)W(g) = e^{i\phi(f+g)}e^{-i\text{Im}(f,g)}W(f+g) = e^{-i\text{Im}(f,g)}W_\phi(f+g)$$

and hence $W_\phi$ satisfies the Weyl relations. However, by Theorem 7, if $W_\phi$ is discontinuous it will not be unitary equivalent to $W$. For example we could construct $\phi$ which is unbounded by considering a sequence $(e_n)_{n \geq 1}$ of linearly independent vectors in $g$. Define

$$\phi(e_n) = n\|e_n\|$$

for each $n = 1, 2, \ldots$. $\phi$ can now be extended uniquely to a linear map on $g$.

### 3.1.1 Bogolubov transformations

Let there be given a complex Hilbert space $g$. We define, on the underlying real Hilbert space of $g$, a Bogolubov transformation $B : g \to g$ to be a bounded real-linear invertible transformation such that for arbitrary $f, g$ in $g$

$$\text{Im} \langle Bf, Bg \rangle = \text{Im} \langle f, g \rangle.$$

We say that $B$ preserves the imaginary part of the inner product.

**Remark 14.** Let $W : f \to W(f)$ be the Fock RCCR over $g$ and define $W_B(f) := W(Bf)$ for any $f \in g$. Then $W_B$ is also a representation of the CCR (RCCR) (3.0.2), since given $f$ and $g$ in $g$

$$W_B(f)W_B(g) = W(Bf)W(Bg) = e^{-i\text{Im}(Bf,Bg)}W(B(f+g)) = e^{-i\text{Im}(f,g)}W_B(f+g),$$

(3.1.2)
where Theorem 6 and then the definition of a Bogolubov transformation are used in the last two steps.

**Remark 15.** Every unitary operator $U$ is a Bogolubov transformation since $U$ preserves inner products, and in particular their imaginary parts. In fact, the following conditions on a Bogolubov transformation $B$ are equivalent

1. $B$ is complex-linear.
2. $B$ is unitary.

To prove that (1) is sufficient for (2),

\[
\begin{align*}
\text{Re} \langle Bf, Bg \rangle &= \text{Im} \langle Bf, -iBg \rangle \\
&= \text{Im} \langle Bf, -Bi g \rangle \\
&= \text{Im} \langle f, -ig \rangle \\
&= \text{Re} \langle f, g \rangle,
\end{align*}
\]

where the fact $\text{Re} \langle \psi_1, \psi_2 \rangle = \text{Im} \langle \psi_1, -i\psi_2 \rangle$ has been used at lines (3.1.3) and (3.1.6), that $B$ is complex linear at line (3.1.4) and $B$ is Bogolubov at line (3.1.5). So $B$ preserves real as well as imaginary parts and, since it is invertible, it is unitary.

### 3.1.2 Shale’s theorem

**Definition 15.** The second quantisation of a unitary operator $U : \mathfrak{g} \rightarrow \mathfrak{g}$, $\Gamma(U)$ is given by the following formula

\[
\Gamma(U) \psi(f) = \psi(Uf),
\]

for all $f$ in $\mathfrak{g}$.

**Question 1.** Given a Hilbert space $\mathfrak{g}$ and a Bogolubov transformation $B$ on $\mathfrak{g}$. Does there exist a unitary operator $U_B$ in the representation Hilbert space $\Gamma(\mathfrak{g})$ such that for all $f$ in $\mathfrak{g}$

\[
W_B(f) = U_B W(f) U_B^{-1}?
\]

i.e. $W_B$ is a unitarily equivalent representation of the CCR.

In general, when $\mathfrak{g}$ is infinite dimensional there are both positive and negative examples. For a positive example: If $B : \mathfrak{g} \rightarrow \mathfrak{g}$ is also complex linear then it is a unitary operator and $W_B$ is implemented by the second quantisation of $B$ i.e. $\Gamma(B)$. This is now shown, for any $f$ and $g$.
\[ \Gamma(B)W(f)\Gamma(B)^{-1}\psi(g) = \Gamma(B)W(f)\psi(B^{-1}g) \]
\[ = \Gamma(B)e^{-\langle f, B^{-1}g \rangle - \frac{1}{2}\|f\|^2}\psi(f + B^{-1}g) \]
\[ = e^{-\langle Bf, g \rangle - \frac{1}{2}\|Bf\|^2}\psi(Bf + g) \]
\[ = W(Bf)\psi(g). \]

For a negative example: Define a transformation \( B : \mathfrak{h} \to \mathfrak{h} \) by \( B(f) = c\text{Re}\{f\} + i\frac{1}{c}\text{Im}\{f\} \) where \( c \) is a non-zero real number with \( c^2 \neq 1 \); \( B \) is a real-linear map. Then \( W_B \) satisfies the Weyl relations
\[ W_B(f)W_B(g) = W(Bf)W(Bg) \]
\[ = e^{-i\text{Im}\langle Bf, Bg \rangle}W_\phi(B(f + g)) \]
\[ = e^{-i\text{Im}\langle f, g \rangle}W_B((f + g)), \]
but is not unitarily equivalent to \( W \), as it fails to satisfy Shale’s criterion (see Theorem 12 below) since
\[ B^*B - I = (c^2 - 1)I_{\text{Re}\{\mathfrak{h}\}} \oplus (c^{-2} - 1)I_{\text{Im}\{\mathfrak{h}\}}, \]
and \( (c^2 - 1)I \) is not Hilbert-Schmidt.

**Theorem 12. Shale’s Theorem.** Consider the Hilbert space \( \mathfrak{h} \). Let \( B : \mathfrak{h} \to \mathfrak{h} \) be a Bogolubov transformation and let \( W \) denote the Fock representation of the CCR over \( \mathfrak{h} \). Then the RCCR \( W_B \) is unitarily equivalent to \( W \) if and only if the real-linear operator \( B^*B - I \) is Hilbert-Schmidt class. This condition is called Shale’s criterion. Here \( B^* \) denotes the adjoint of \( B \) in the sense of the canonical real Hilbert space \( (\mathfrak{h}, \text{Re}\{\}) \) i.e. \( \text{Re}(B^*f, g) = \text{Re}(f, Bg) \).

**Remark 16.** The unitary operator \( U_B \) which implements the equivalence is unique to within multiplication by a scalar of modulus one (This is because the Fock representation is irreducible).
Chapter 4

Towards a Bogolubov transformation which is implemented by

\[ \prod_{s}^{b} (1 + \lambda (dA^\dagger \otimes dA^\dagger - dA \otimes dA) - \frac{1}{2} \lambda^2 dT \otimes dT) \]

4.1 Introduction

The double product integral

\[ \prod_{s}^{b} (1 + \lambda (dA^\dagger \otimes dA^\dagger - dA \otimes dA) - \frac{1}{2} \lambda^2 dT \otimes dT) \]

is of central importance in the thesis. We have seen that by definition double product integrals are solutions of quantum stochastic differential equations. The product integral notation is suggestive of a new way of approximating such solutions, for example by discrete approximations which are finite double products. The integral is approximated by a discrete double product by replacing infinitesimals in (4.1.1) by discrete increments. The approximation is shown to be the implementor of a Bogolubov transformation acting on “discretised creation and annihilation operators”. This transformation can then be considered to act on the direct sums of \((m + n)\) copies of \(\mathbb{C}\). Embedding \(\mathbb{C}^{n+m}\) into \(\mathfrak{h} \oplus \mathfrak{h}\) and taking heuristic limits gives a Bogolubov transformation \(W_{a,s}^{b,t}\) on the new space. Using Shale’s theorem there is, to within a scalar of modulus 1, a unique unitary operator \(U_{a,s}^{b,t}\) which implements \(W_{a,s}^{b,t}\). We conjecture \(U_{a,s}^{b,t}\) to be the original double product integral. We might expect (4.1.1) to be unitary since an argument in Hudson and Jones [2012] shows that

\[ \prod_{s}^{b} (1 + dr) \]
with \( dr \in \mathcal{I} \otimes \mathcal{I} \) is a coisometry if and only if

\[
dr + dr^\dagger + dr dr^\dagger = 0 \tag{4.1.2}
\]
i.e. \( dr^\dagger \) is the quasi-inverse to \( dr \). Isometry, and hence unitarity, is more difficult to characterise. When \( dr = \lambda \left(dA^\dagger \otimes dA^\dagger - dA \otimes dA - \frac{1}{2}\lambda^2dT \otimes dT\right) \) this particular \( dr \) satisfies equation (4.1.2).

By using the first and second fundamental formulae of quantum stochastic calculus one can show that the solution of the quantum stochastic differential equation satisfied by \( b \overset{\rightarrow}{\prod}_a^b (1 + dr) \) must be a isometry.

### 4.2 Approximation of a double product integral

We devote the rest of the chapter to finding a Bogolubov transformation which is plausibly implemented by

\[
b \overset{\rightarrow}{\prod}_a^b (1 + \lambda(dA^\dagger \otimes dA^\dagger - dA \otimes dA - \frac{1}{2}\lambda^2dT \otimes dT)). \tag{4.2.1}
\]

We again specialise our working to the Fock space \( \mathcal{H} \) of \( \mathfrak{h} \). The double product (4.2.1) can be approximated as follows. Partition the intervals \([a,b)\) and \([s,t)\) into \( m \) and \( n \) subintervals of equal lengths \( \frac{b-a}{m} \) and \( \frac{t-s}{n} \) respectively. Define creation and annihilation operators on each of these subintervals as follows

\[
a_j^\# = \sqrt{\frac{m}{b-a}} \left(A^\#(a + \frac{j}{m}(b-a)) - A^\#(a + \frac{j-1}{m}(b-a))\right) \otimes 1 \tag{4.2.2}
\]
and

\[
b_k^\# = 1 \otimes \sqrt{\frac{n}{b-a}} \left(A^\#(s + \frac{k}{n}(t-s)) - A^\#(s + \frac{k-1}{n}(t-s))\right), \tag{4.2.3}
\]
where \(# \in \{-1,+1\} \) and \( A^\# = A \) if \(# = -1\) and \( A^\# = A^\dagger \) if \(# = +1\). Then \((a_j, a_j^\dagger)\), \(j = 1, \ldots, m\) and \((b_k, b_k^\dagger)\), \(k = 1, \ldots, n\) are mutually commuting standard creation-annihilation pairs satisfying \([a_j, a_j^\dagger] = \delta_{j_1,j_2} \) and \([b_k, b_k^\dagger] = \delta_{k_1,k_2} \). Replacing differentials with finite differences it could be expected that

\[
b \overset{\rightarrow}{\prod}_a^b (1 + \lambda(dA^\dagger \otimes dA^\dagger - dA \otimes dA - \frac{1}{2}\lambda^2dT \otimes dT)) \approx \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \exp \left(\lambda \sqrt{\frac{(b-a)(t-s)}{mn}} (a_j^\dagger b_k^\# - a_j b_k)\right) \tag{4.2.4}
\]
for large \( m \) and \( n \), where \( \mathbb{N}_n = \{1, \ldots, n\} \) and \( \mathbb{N}_m \) is defined similarly. The discrete approximations are of the form \( \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} x_{jk} \), where \([x_{jk}, x_{jk'}] = 0\) if both \( j \neq j' \) and \( k \neq k' \), and
are therefore well defined as
\[ m \prod_{j=1}^{m} \left( \prod_{k=1}^{n} x_{jk} \right) \text{ or } n \prod_{k=1}^{n} \left( \prod_{j=1}^{m} x_{jk} \right), \]
The equality of which is the discrete “Fubini” Theorem, as described in Hudson [2005a].

4.2.1 Infinitesimal pseudo rotation matrices

The right hand side of (4.2.4) above is formally a unitary operator. Let \((a, a^\dagger), (b, b^\dagger)\) be mutually commuting standard annihilation-creation pairs and let \(\theta\) be a real number. The formally unitary operator \(\exp \theta (a^\dagger b^\dagger - a b)\) can be constructed rigorously as the unitary implementor of a Bogolubov transformation as follows in Theorem 13. Before we prove Theorem 13 we need the following lemma and we make the following definition ad

**Lemma 7.** Let \((a, a^\dagger), (b, b^\dagger)\) be mutually commuting creation and annihilation pairs i.e. \([a, a^\dagger] = 1, [b, b^\dagger] = 1, [a, b] = 0, [a^\dagger, b] = 0). Then

\[ \text{ad}_{\{a^\dagger b^\dagger - a b\}}(a) = -b^\dagger, \]
\[ \text{ad}_{\{a^\dagger b^\dagger - a b\}}(b^\dagger) = -a, \]
\[ \text{ad}_{\{a^\dagger b^\dagger - a b\}}(b) = -a^\dagger, \]
and
\[ \text{ad}_{\{a^\dagger b^\dagger - a b\}}(a^\dagger) = -b. \]

**Proof.** We use the commutation relations and the relation \(\text{ad}_{X Y}(Z) = X \text{ad}_Y(Z) + \text{ad}_X(Z)Y\). First consider the evaluation of \(\text{ad}_{\{a^\dagger b^\dagger - a b\}}(a)\),

\[ \text{ad}_{\{a^\dagger b^\dagger - a b\}}(a) = [a^\dagger b^\dagger - a b, a] \]
\[ = a^\dagger [b^\dagger, a] + [a^\dagger, a] b^\dagger - a[b, a] - [a, a]b \]
\[ = -b^\dagger, \]
the result is \(-b^\dagger\) as required. The evaluation of \(\text{ad}_{\{a^\dagger b^\dagger - a b\}}(b^\dagger)\) is as follows

\[ [a^\dagger b^\dagger - a b, b^\dagger] = a^\dagger [b^\dagger, b^\dagger] + [a^\dagger, b^\dagger] b^\dagger - a[b, b^\dagger] - [a, b^\dagger] b \]
\[ = -a. \]
The evaluation of \( \text{ad}_{\{a^\dagger b^\dagger - ab\}} (b) \) is as follows

\[
\text{ad}_{\{a^\dagger b^\dagger - ab\}} (b) = [a^\dagger b^\dagger - ab, b],
\]

\[
= a^\dagger [b^\dagger, b] + [a^\dagger, b]b^\dagger - a[b, b] - [a, b]b
\]

\[
= -a^\dagger, \tag{4.2.11}
\]

and \( \text{ad}_{\{a^\dagger b^\dagger - ab\}} (a^\dagger) \) as

\[
[a^\dagger b^\dagger - ab, a^\dagger] = a^\dagger [b^\dagger, a^\dagger] + [a^\dagger, a^\dagger]b^\dagger - a[b, a^\dagger] - [a, a^\dagger]b
\]

\[
= -b^\dagger. \tag{4.2.12}
\]

**Theorem 13.** Let \((a, a^\dagger), (b, b^\dagger)\) be mutually commuting creation and annihilation pairs i.e. \([a, a^\dagger] = 1, [b, b^\dagger] = 1, [a, b] = 0 \text{ and } [a^\dagger, b] = 0\). Then

\[
\left( \exp (\text{ad}_{\{a^\dagger b^\dagger - ab\}}) \right) a = (\cosh \theta)a - (\sinh \theta)b^\dagger
\]

and

\[
\left( \exp (\text{ad}_{\{a^\dagger b^\dagger - ab\}}) \right) b = - (\sinh \theta)a^\dagger + (\cosh \theta)b.
\]

**Proof.** Denoting the operator \( a^\dagger b^\dagger - ab \) by \( H \). Then the operator \( e^{\theta \text{ad}_H} \) acting on \( a \) can be easily evaluated by expanding the exponential as a series and repeatedly using Lemma 7. This is done below

\[
a + \frac{1}{1!} [\theta H, a] + \frac{1}{2!} [\theta H, [\theta H, a]] + \frac{1}{3!} [\theta H, [\theta H, [\theta H, a]]] + \ldots
\]

\[
= a - \frac{\theta}{1!} b^\dagger + \frac{\theta^2}{2!} a - \frac{\theta^3}{3!} b^\dagger + \frac{\theta^4}{4!} a - \frac{\theta^5}{5!} b^\dagger + \frac{\theta^6}{6!} a
\]

\[
- \frac{\theta^7}{7!} b^\dagger + \frac{\theta^8}{8!} a - \frac{\theta^9}{9!} b^\dagger + \ldots
\]

\[
= (1 + \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 + \frac{1}{6!} \theta^6 + \frac{1}{8!} \theta^8 + \ldots)a
\]

\[
- \frac{\theta}{1!} b^\dagger = \cosh (\theta)a - \sinh (\theta)b^\dagger.
\]

Similarly we have

\[
e^{\text{ad}_{\{a^\dagger b^\dagger - ab\}}} (b) = - (\sinh \theta)a^\dagger + \cosh (\theta)b.
\]
Then we can formally write

\[
\exp \left( \text{ad}_{\{a^\dagger b^\dagger - ab\}} \right) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cosh \theta & -\sinh \theta \iota \\ -\iota \sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},
\]

where \( \iota \) maps a creation operator to its conjugate annihilation operator and vice-versa.

**Remark 17.** Instead of Bogolubov transformations acting as linear symmetries of creation and annihilation operators, we may regard them as real linear transformations in the underlying complex Hilbert space, in this case \( \mathbb{C}^2 \), as explained in Section 3.1.1. Then the above one-parameter group assumes the form

\[
\begin{pmatrix} \cosh \theta & -\sinh \theta \kappa \\ -\kappa \sinh \theta & \cosh \theta \end{pmatrix},
\]

where \( \kappa \) is the real linear map \( z \mapsto \bar{z} \) in \( \mathbb{C} \), so that (4.2.14) acts on \( \mathbb{C}^2 \) as the real-linear map

\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \cosh \theta z_1 - \sinh \theta \bar{z}_2 \\ -\sinh \theta z_1 + \cosh \theta \bar{z}_2 \end{pmatrix}
\]

which leads to the next theorem.

**Theorem 14.** As a real-linear map on \( \mathbb{C}^2 \), the matrix operator

\[
B_\theta = \begin{pmatrix} \cosh \theta & -\sinh \theta \kappa \\ -\kappa \sinh \theta & \cosh \theta \end{pmatrix},
\]

preserves the imaginary part of the inner product \( \langle , \rangle_{\mathbb{C}^2} \) and is invertible with inverse

\[
B_{-\theta} = \begin{pmatrix} \cosh \theta & \sinh \theta \kappa \\ \kappa \sinh \theta & \cosh \theta \end{pmatrix},
\]

and is therefore a Bogolubov transformation.
Proof. Take any \((u, v)^\dagger\) and \((u', v')^\dagger\) in \(\mathbb{C}^2\). Then

\[
\begin{align*}
\left( \begin{array}{cc}
\cosh \theta & -\sinh \theta \kappa \\
-\kappa \sinh \theta & \cosh \theta \\
\end{array} \right) \left( \begin{array}{c}
u \\
v' \\
\end{array} \right) & = \left( \begin{array}{cc}
\cosh \theta & -\sinh \theta \kappa \\
-\kappa \sinh \theta & \cosh \theta \\
\end{array} \right) \left( \begin{array}{c}
u' \\
v'' \\
\end{array} \right) \\
= \left( \begin{array}{cc}
\cosh \theta u - \sinh \theta \kappa v \\
-\kappa \sinh \theta u + \cosh \theta v \\
\end{array} \right) \left( \begin{array}{cc}
\cosh \theta u' - \sinh \theta \kappa v' \\
-\kappa \sinh \theta u' + \cosh \theta v' \\
\end{array} \right),
\end{align*}
\]

(4.2.18)

by collecting terms in \(\sinh \theta \cosh \theta\) in line (4.2.19), using

\[
\begin{align*}
\Re(\bar{u}u') & = \Re(\bar{u}u'), \\
(\bar{u}v' + \bar{v}u') & = 2\Re(\bar{u}u'), \\
\Im(\bar{u}u') & = -\Im(\bar{u}u'), \\
\text{and } \bar{u}'v' + v'u' & = 2\Re(\bar{v}u'),
\end{align*}
\]

on line (4.2.20) and finally expressing the left hand side of line (4.2.18) as the sum of its real and imaginary parts (line (4.2.21)).

Hence the imaginary part of the left hand side of (4.2.18) is equal to the imaginary part of the inner product of the vectors \(\left( \begin{array}{c}u \\
v \end{array} \right)\) and \(\left( \begin{array}{c}u' \\
v' \end{array} \right)\). The matrix operator (4.2.16) is real linear and bounded and hence is a Bogolubov transformation. This completes the proof of the theorem. \(\square\)

4.2.2 Embedding the two-dimensional Bogolubov transformation into a \((m+n)\) by \((m+n)\) matrix

The transformation (equation (4.2.16)) of Theorem 14 which was shown to be a Bogolubov transformation can be embedded into a \((n+m)\) by \((n+m)\) matrix. The embedding is done in such a way that the resulting matrix is a Bogolubov transformation over \(\mathbb{C}^{(n+m)}\) (the notation for this embedding is given in equation (4.2.22) below). The embedding is now specifically described as follows. Arbitrary natural numbers \(m\) and \(n\) are fixed and for each \((j, k) \in \mathbb{N}_m \times \mathbb{N}_n\),
let
\[ r_{(m,n)}^{(j,k)}(\theta) = \begin{bmatrix} \cosh \theta & -\kappa \sinh \theta \\ -\kappa \sinh \theta & \cosh \theta \end{bmatrix} \] (4.2.22)
denote the \((m+n)\) by \((m+n)\) matrix obtained by embedding the transform (equation (4.2.16)) in the intersections of the \(j\)th and \((m+k)\)th rows and columns. Vacant diagonal positions are completed with 1’s and off diagonal positions are filled with 0’s.

We now give an example,

Example 7.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cosh \theta & 0 & 0 & 0 & -\sinh \theta \kappa & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\kappa \sinh \theta & 0 & 0 & \cosh \theta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\] (4.2.23)

As a transformation on \(\mathbb{C}^{(m+n)}\), \(r_{(m,n)}^{(j,k)}(\theta)\) maps

\[
\begin{pmatrix} \mathbf{z} \\ \mathbf{w} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} z_1 \\ \vdots \\ z_{j-1} \\ \cosh \theta z_j - \sinh \theta \bar{w}_k \\ z_{j+1} \\ \vdots \\ w_{k-1} \\ -\sinh \theta \bar{z}_j + \cosh \theta w_k \\ \vdots \\ w_n \end{pmatrix}
\] (4.2.24)

where \(\mathbf{z}\) and \(\mathbf{w}\) are elements of \(\mathbb{C}^m\) and \(\mathbb{C}^n\) respectively.

We denote by the vector \(\begin{pmatrix} z^{(j)} \\ w^{(k)} \end{pmatrix} \in \mathbb{C}^{(m+n)}\) made by replacing all components of \(\begin{pmatrix} \mathbf{z} \\ \mathbf{w} \end{pmatrix}\) by zero other than the \(j\)th and the \((m+k)\)th components. The transformation is Bogolubov as it acts (by Theorem 14) on the element of \(\begin{pmatrix} z^{(j)} \\ w^{(k)} \end{pmatrix} \in \mathbb{C}^{(m+n)}\) as a transform that preserves the imaginary part, and acts as the identity on the orthogonal complement of the set of all
vectors of this form.

There are four distinct ordered double products of the matrices (4.2.22). The resulting ordered double products are Bogolubov transformations of $C^{(n+m)}$, since any product of Bogolubov transformations is itself a Bogolubov transformation. We introduce the discrete forward-forward double product with the notation

$$
\overrightarrow{R}_{(m,n)}(\theta) = \prod_{(j,k) \in N_m \times N_n} r^{(j,k)}_{(m,n)}(\theta),
$$

where the double arrows on top of the product symbol $\prod$ indicate the direction of increasing $j$ and $k$. The other three ordered double products are:

$$
\overleftarrow{\overrightarrow{R}}_{(m,n)}(\theta) = \prod_{(j,k) \in N_m \times N_n} r^{(j,k)}_{(m,n)}(\theta) \tag{4.2.26}
$$

which we describe as the discrete backward-forward double product.

$$
\overrightarrow{\overleftarrow{R}}_{(m,n)}(\theta) = \prod_{(j,k) \in N_m \times N_n} r^{(j,k)}_{(m,n)}(\theta) \tag{4.2.27}
$$

which we describe as the discrete forward-backward double product.

$$
\overleftarrow{\overleftarrow{R}}_{(m,n)}(\theta) = \prod_{(j,k) \in N_m \times N_n} r^{(j,k)}_{(m,n)}(\theta) \tag{4.2.28}
$$

which we describe as the discrete backward-backward double product.

As yet there is no explicit general formula for

$$
R_{(m,n)}(\theta) = \prod_{(j,k) \in N_m \times N_n} r^{(j,k)}_{(m,n)}(\theta),
$$

where the absence of arrows indicates that the direction of the product is not important to the discussion. However, we can evaluate certain simpler discrete double products, when either $m$ or $n = 1$, which leads to the next theorem.

**Theorem 15.** The products $\overrightarrow{R}_{(m,1)}(\theta)$, $\overrightarrow{R}_{(1,m)}(\theta)$, $\overrightarrow{R}_{(1,n)}(\theta)$ and $\overrightarrow{R}_{(1,n)}(\theta)$ can be evaluated.
Setting $\alpha = \cosh(\theta)$, $\beta = -\sinh(\theta)$, $\delta = \cosh(\theta)$ and $\gamma = -\sinh(\theta)$ the products are:

\[
\overleftarrow{R}_{\{m, 1\}}(\theta) = \begin{pmatrix}
\alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-3} \gamma & \beta \delta^{m-2} \gamma & \beta \delta^{m-1} \gamma & \kappa \beta \delta^m \\
0 & \alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-4} \gamma & \beta \delta^{m-3} \gamma & \kappa \beta \delta^m \\
0 & 0 & \alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-5} \gamma & \beta \delta^{m-4} \gamma & \kappa \beta \delta^m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \ldots & \alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \kappa \beta \delta^2 \\
0 & 0 & 0 & \ldots & 0 & \alpha & \beta \gamma & \kappa \beta \delta \\
0 & 0 & 0 & \ldots & 0 & 0 & \alpha & \kappa \beta
\end{pmatrix}, \quad (4.2.29)
\]

\[
\overrightarrow{R}_{\{m, 1\}}(\theta) = \begin{pmatrix}
\alpha & 0 & 0 & 0 & \ldots & 0 & 0 & \kappa \beta \\
\beta \gamma & \alpha & 0 & 0 & \ldots & 0 & 0 & \kappa \beta \delta \\
\beta \delta \gamma & \beta \gamma & \alpha & 0 & \ldots & 0 & 0 & \kappa \beta \delta^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\beta \delta^{m-3} \gamma & \beta \delta^{m-2} \gamma & \beta \delta^{m-1} \gamma & \beta \delta^m & \ldots & \beta \delta \gamma & \beta \gamma & \alpha & \kappa \beta \delta^{m-1} \\
\beta \delta^{m-2} \gamma & \beta \delta^{m-3} \gamma & \beta \delta^{m-4} \gamma & \beta \delta^{m-5} \gamma & \beta \delta^m & \ldots & \beta \delta \gamma & \beta \gamma & \alpha & \kappa \beta \delta^{m-2} \\
\delta^{m-1} \gamma \kappa & \delta^{m-2} \gamma \kappa & \delta^{m-3} \gamma \kappa & \delta^{m-4} \gamma \kappa & \delta^{m-5} \gamma \kappa & \ldots & \delta^{m} \gamma \kappa & \gamma \kappa & \delta & \delta^m
\end{pmatrix}, \quad (4.2.30)
\]

\[
\overleftarrow{R}_{\{1, n\}}(\theta) = \begin{pmatrix}
\alpha^n & \kappa \beta & \kappa \alpha \beta & \kappa \alpha^2 \beta & \kappa \alpha^3 \beta & \ldots & \kappa \alpha^{n-2} \beta & \kappa \alpha^{n-1} \beta \\
\gamma \alpha^{n-1} \kappa & \delta & \gamma \beta & \gamma \alpha \beta & \gamma \alpha^2 \beta & \ldots & \gamma \alpha^{n-3} \beta & \gamma \alpha^{n-2} \beta \\
\gamma \alpha^{n-2} \kappa & 0 & \delta & \gamma \beta & \gamma \alpha \beta & \gamma \alpha^2 \beta & \ldots & \gamma \alpha^{n-4} \beta & \gamma \alpha^{n-3} \beta \\
\gamma \alpha^{n-3} \kappa & 0 & 0 & \delta & \gamma \beta & \gamma \alpha \beta & \gamma \alpha^2 \beta & \ldots & \gamma \alpha^{n-5} \beta & \gamma \alpha^{n-4} \beta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\gamma \alpha^2 \kappa & 0 & 0 & 0 & \ldots & \delta & \gamma \beta & \gamma \alpha \beta \\
\gamma \alpha \kappa & 0 & 0 & 0 & \ldots & 0 & \delta & \gamma \beta \\
\gamma \kappa & 0 & 0 & 0 & \ldots & 0 & 0 & \delta
\end{pmatrix}, \quad (4.2.31)
\]
$$\begin{pmatrix}
\alpha^n & \kappa\alpha^{n-1}\beta & \kappa\alpha^{n-2}\beta & \ldots & \kappa^3\beta & \kappa^2\beta & \kappa\beta & \kappa\beta \\
\gamma\kappa & \delta & 0 & \ldots & 0 & 0 & 0 & 0 \\
\gamma\alpha^1\kappa & \gamma\beta & \delta & \ldots & 0 & 0 & 0 & 0 \\
\gamma\alpha^2\kappa & \gamma\alpha\beta & \gamma\beta & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\gamma\alpha^{n-3}\kappa & \gamma\alpha^{n-4}\beta & \gamma\alpha^{n-5}\beta & \ldots & \gamma\beta & \delta & 0 & 0 \\
\gamma\alpha^{n-2}\kappa & \gamma\alpha^{n-3}\beta & \gamma\alpha^{n-4}\beta & \ldots & \gamma\alpha\beta & \gamma\beta & \delta & 0 \\
\gamma\alpha^{n-1}\kappa & \gamma\alpha^{n-2}\beta & \gamma\alpha^{n-3}\beta & \ldots & \gamma\alpha^2\beta & \gamma\alpha\beta & \gamma\beta & \delta
\end{pmatrix}.$$  \hspace{1cm} (4.2.32)

**Remark 18.** Note that, because $\kappa^2 = \text{id}$, in the 2 by 2 block-matrices (4.2.29), (4.2.30), (4.2.31) and (4.2.32) the off-diagonal sub-matrices involve complex conjugation. For example the bottom left block of (4.2.29) is

$$\begin{pmatrix}
\gamma\kappa & \delta\gamma\kappa & \delta^2\gamma\kappa & \ldots & \delta^{m-3}\gamma\kappa & \delta^{m-2}\gamma\kappa & \delta^{m-1}\gamma\kappa \\
-\sinh(\theta)\kappa & -\cosh(\theta)\sinh(\theta)\kappa & -\cosh(\theta)^2\sinh(\theta)\kappa & \ldots \\
-\cosh(\theta)^{m-3}\sinh(\theta)\kappa & -\cosh(\theta)^{m-2}\sinh(\theta)\kappa \\
-\cosh(\theta)^{m-1}\sinh(\theta)\kappa
\end{pmatrix}.$$  \hspace{1cm} (4.2.33)

**Proof.** First we consider the proof of equation (4.2.29). The result is true for $m = 1$ as can be seen by inspection. We will assume the result is true for $m - 1$ and use induction to show that
it is then true for \( m \). This amounts to showing

\[
\begin{align*}
1 & \\
\mathbf{0}^{(m)} & \\
\rightarrow & \\
R_{(m-1,1)}(\theta) & \\
\rightarrow & \\
\mathbb{R} & (m,1) \quad \text{for} \quad \alpha \gamma \kappa \\
\delta \\
\end{align*}
\]

where \( \mathbf{0}^{(m)} \) is the vector of dimension \( m \) with zero in every entry. We express the matrix \( \rightarrow R_{(m-1,1)}(\theta) \) as

\[
\left( \begin{array}{cccccc}
B_{22}^{(m-1)} & B_{23}^{(m-1)} \\
B_{32}^{(m-1)} & B_{33}^{(m-1)}
\end{array} \right),
\]

where

\[
B_{22}^{(m-1)} = \\
\left( \begin{array}{cccccc}
\alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-3} \gamma \\
0 & \alpha & \beta \gamma & \beta \delta \gamma & \ldots & \beta \delta^{m-4} \gamma \\
0 & 0 & \alpha & \beta \gamma & \ldots & \beta \delta^{m-5} \gamma \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \alpha & \beta \gamma \\
0 & 0 & 0 & 0 & 0 & \alpha
\end{array} \right),
\]

\[
B_{23}^{(m-1)} = \left( \begin{array}{ccccccc}
\not{\kappa \beta \delta^{m-2}} & \not{\kappa \beta \delta^{m-3}} & \not{\kappa \beta \delta^{m-4}} & \ldots & \not{\kappa \beta \delta} & \kappa \beta \\
\not{\gamma \kappa} & \not{\delta \gamma \kappa} & \not{\delta^2 \gamma \kappa} & \ldots & \not{\delta^{m-3} \gamma \kappa} & \not{\delta^{m-2} \gamma \kappa} & \delta^{m-1}
\end{array} \right),
\]

\[
B_{32}^{(m-1)} = \left( \begin{array}{ccccccc}
\not{\kappa \beta \delta^{m-2}} & \not{\kappa \beta \delta^{m-3}} & \not{\kappa \beta \delta^{m-4}} & \ldots & \not{\kappa \beta \delta} & \kappa \beta \\
\gamma \kappa & \not{\delta \gamma \kappa} & \not{\delta^2 \gamma \kappa} & \ldots & \not{\delta^{m-3} \gamma \kappa} & \not{\delta^{m-2} \gamma \kappa} & \delta^{m-1}
\end{array} \right),
\]

\[
B_{33}^{(m-1)} = \delta^{m-1}.
\]
Then
\[
\begin{align*}
& \mathbf{r}_{m,1}^{1,1}(\theta) \left( \begin{array}{c}
1 \\
0^{(m)}
\end{array} \right) \rightarrow_{(m-1)}^R (\theta) = \left( \begin{array}{ccc}
\alpha & 0^{(m-1)^T} & \kappa \\
0 & I^{(m-1)} & \beta \\
\gamma & 0^{(m-1)^T} & \delta
\end{array} \right) \\
& \cdot \left( \begin{array}{ccc}
\alpha & 0^{(m-1)^T} & 0 \\
0 & B^{(m-1)}_{22} & B^{(m-1)}_{23} \\
0 & B^{(m-1)}_{32} & \kappa B^{(m-1)}_{33}
\end{array} \right) \\
& = \left( \begin{array}{ccc}
\alpha & \beta B^{(m-1)}_{22} & \kappa \beta B^{(m-1)}_{33} \\
0 & B^{(m-1)}_{22} & B^{(m-1)}_{33} \\
\gamma \kappa & \delta B^{(m-1)}_{32} & \delta B^{(m-1)}_{33}
\end{array} \right), \quad (4.2.40)
\end{align*}
\]

where \( I^{(m-1)} \) and \( 0^{(m-1)} \) are the \((m-1)\) by \((m-1)\) identity matrix and the zero vector with \((m-1)\) rows respectively. The left right hand side of \((4.2.40)\) is clearly \( \rightarrow R_{(m-1)} \) and hence the result is true for all \( m \). The proofs of \((4.2.30)\), \((4.2.31)\) and \((4.2.32)\) are similar.

\[\square\]

**Remark 19.** Consider the matrix \( R_{(m,1)}(\theta) \) (equation \((4.2.29)\)) written as
\[
\left( \begin{array}{ccc}
B^{(m)}_{11} & B^{(m)}_{12} \\
B^{(m)}_{21} & B^{(m)}_{22}
\end{array} \right), \quad (4.2.41)
\]
where

\[
B^{(m)}_{11} = \begin{pmatrix}
\alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-3} \gamma & \beta \delta^{m-2} \gamma \\
0 & \alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-4} \gamma & \beta \delta^{m-3} \gamma \\
0 & 0 & \alpha & \beta \gamma & \beta \delta \gamma & \ldots & \beta \delta^{m-5} \gamma & \beta \delta^{m-4} \gamma \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \alpha & \beta \gamma & \beta \delta \gamma \\
0 & 0 & 0 & \ldots & 0 & \alpha & \beta \gamma \\
0 & 0 & 0 & \ldots & 0 & 0 & \alpha
\end{pmatrix},
\quad (4.2.42)
\]

\[
B^{(m)}_{12}^T = \begin{pmatrix}
\kappa \beta \delta^{m-1} & \kappa \beta \delta^{m-2} & \kappa \beta \delta^{m-3} & \kappa \beta \delta^2 & \kappa \beta \delta & \kappa \beta
\end{pmatrix},
\quad (4.2.43)
\]

\[
B^{(m)}_{21} = \begin{pmatrix}
\gamma \kappa & \delta \gamma \kappa & \delta^2 \gamma \kappa & \ldots & \delta^{m-3} \gamma \kappa & \delta^{m-2} \gamma \kappa & \delta^{m-1} \gamma \kappa
\end{pmatrix},
\quad (4.2.44)
\]

and \(B^{(m)}_{22} = \delta^m\). (4.2.45)

Notice that \(B^{(m)}_{11}\) (equation (4.2.42)) is a linear operator on \(\mathbb{C}^{m-1}\), \(B^{(m)}_{12}\) (equation (4.2.43)) can be considered a vector in \(\mathbb{C}^{m-1}\) multiplied by \(\kappa\), \(B^{(m)}_{21}\) (equation (4.2.44)) can be considered as \(\kappa\) multiplied by a covector on \(\mathbb{C}^{m-1}\) and \(B^{(m)}_{22}\) (equation (4.2.45)) is a scalar. Notice that \(B^{(m)}_{11}\) is not conjugate linear as \(\kappa^2 = \text{id}\). Also notice that the matrices \(\overrightarrow{R}_{(m,1)}(\theta)\) (equation (4.2.30)), \(\overrightarrow{R}_{(1,n)}(\theta)\) (equation (4.2.31)), and \(\overleftarrow{R}_{(1,n)}(\theta)\) (equation (4.2.32)) can also be interpreted as a matrix of: a linear operator of \(\mathbb{C}^{m-1}\), a vector in \(\mathbb{C}^{m-1}\), a covector on \(\mathbb{C}^{m-1}\) and a scalar. The vertical and horizontal line in each of (4.2.29) (4.2.30), (4.2.31) and (4.2.32) shows their division into such a matrix containing these elements (linear operator, \(\kappa\times\text{vector, \text{covector}}\times\kappa\) and scalar).

**Remark 20.** The matrix \(\overrightarrow{R}_{(m,1)}(\theta)\) (equation (4.2.29)) also makes sense if

- \(\alpha\) is a scalar
- \(\beta\) is covector
- \(\gamma\) is a vector and
- \(\delta\) is an operator.

The same hold for \(\overleftarrow{R}_{(m,1)}(\theta)\) (equation (4.2.30)). The matrices \(\overrightarrow{R}_{(1,n)}(\theta)\) and \(\overleftarrow{R}_{(1,n)}(\theta)\) (equations (4.2.31) and (4.2.32) respectively) make sense when

- \(\alpha\) is an operator
- \(\beta\) is vector
- \(\gamma\) is a covector and
- \(\delta\) is a scalar.
These facts will be used in Section 4.5.

4.3 The first limits

For the purpose of taking limits the parameter $\theta$ is made to depend on parameters $m$ and $n$ such that

$$
\theta_{m,n} = \lambda \sqrt{\frac{(b-a)(t-s)}{mn}} = \lambda_m \sqrt{\frac{t-s}{n}} = \lambda^{(n)} \sqrt{\frac{b-a}{m}},
$$

where $\lambda_m = \lambda \sqrt{\frac{b-a}{m}}$ and $\lambda^{(n)} = \lambda \sqrt{\frac{t-s}{n}}$, where $\lambda \in \mathbb{R}$. This choice of parameterisation is so that discretised increments, equations (4.2.2) and (4.2.3), used in the approximation (4.2.4) are standardised creation and annihilation operators.

Heuristic limits of the above four matrices (4.2.29) (4.2.30), (4.2.31) and (4.2.32) can be taken by embedding either $\mathbb{C}^n$ or $\mathbb{C}^m$ into $\mathfrak{h}$. The embedding is done in the natural way by mapping the elements of their canonical orthonormal bases to the normalised indicator functions of the subintervals of $[s, t)$ and $[a, b)$ obtained by equipartitioning them into, respectively $m$ and $n$ subintervals. The limit of each product produces an operator; each of the four cases (4.2.29), (4.2.30), (4.2.31) and (4.2.32) are distinct from each other. The limits

$$
\lim_{m \to \infty} R_{(m,1)}(\theta_{(m,n)}) \quad \text{and} \quad \lim_{m \to \infty} \tilde{R}_{(m,1)}(\theta_{(m,n)})
$$

are operators acting in $\mathfrak{h} \oplus \mathbb{C}$. The limits

$$
\lim_{n \to \infty} \tilde{R}_{(1,n)}(\theta_{(m,n)}) \quad \text{and} \quad \lim_{n \to \infty} \tilde{R}_{(1,n)}(\theta_{(m,n)})
$$

are operators acting in $\mathbb{C} \oplus \mathfrak{h}$. We introduce some notation for ease of reading the limits of (4.2.26), (4.2.27), (4.2.27) and (4.2.28). The integral operator on $\mathfrak{h}$ whose kernel is $K$ (a $L^2$ map from $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$) is denoted by $\text{Op}\{K\}$; thus for a vector $f$ in $\mathfrak{h}$

$$
(\text{Op}\{K\}f)(x) = \int_{\mathbb{R}_+} K(x,y)f(y)dy.
$$

We use the standard notation $\ker(\Xi)$ to denote the kernel of a linear integral operator $\Xi$. The indicator function $\chi_a^b(x)$ takes value one in the interval $[a, b)$ and zero everywhere else. We denote the following product of indicator functions $\chi_a^b(x)\chi_a^x(y)$ as $>_a^b(x, y)$ i.e.

$$
>_a^b(x, y) = \begin{cases} 
1 & b \geq x \geq y > a \\
0 & \text{otherwise.}
\end{cases}
$$

(4.3.2)
The product of indicator functions $\chi^b_a(x)\chi^b_a(y)$ is denoted as $<^b_a(x,y)$ i.e.

$$<^b_a(x,y) = \begin{cases} 
1 & a \leq x \leq y < b \\
0 & \text{otherwise.}
\end{cases} \quad (4.3.3)$$

The limits:

$$\begin{align*}
\lim_{m \to \infty} \vec{R}_{(m,1)}(\theta_{(m,1)}), & \quad \lim_{m \to \infty} \vec{R}_{(m,1)}(\theta_{(m,1)}), \\
\lim_{n \to \infty} \vec{R}_{(1,n)}(\theta_{(1,n)}), & \quad \lim_{n \to \infty} \vec{R}_{(1,n)}(\theta_{(1,n)}),
\end{align*} \tag{4.3.4}$$

of the four operators, (4.2.26), (4.2.27), (4.2.27) and (4.2.28), can be explicitly written, using the new notation, respectively as stated in the following theorem.

**Theorem 16.** The limits of the matrix products $\vec{R}_{(m,1)}(\theta)$, $\vec{R}_{(m,1)}(\theta)$, $\vec{R}_{(1,n)}(\theta)$ and $\vec{R}_{(1,n)}(\theta)$ (equations (4.2.29), (4.2.30), (4.2.31) and (4.2.32) respectively) as either $m$ or $n$ tend to infinity are:

$$\begin{align*}
\lim_{m \to \infty} \vec{R}_{(m,1)}(\theta_{(m,1)}) &= \begin{pmatrix}
\id + D^b_a(\lambda^{(n)}) & f^b_a(\lambda^{(n)}, \cdot) \\
\kappa \left< g^b_a(\lambda^{(n)}, \cdot) \right> & h^b_a(\lambda^{(n)})
\end{pmatrix}, \\
\lim_{m \to \infty} \vec{R}_{(m,1)}(\theta_{(m,1)}) &= \begin{pmatrix}
\id + D^b_a(\lambda^{(n)}) & g^b_a(\lambda^{(n)}, \cdot) \\
\kappa \left< f^b_a(\lambda^{(n)}, \cdot) \right> & h^b_a(\lambda^{(n)})
\end{pmatrix}, \\
\lim_{n \to \infty} \vec{R}_{(1,n)}(\theta_{(1,n)}) &= \begin{pmatrix}
h^b_a(\lambda_m) & \kappa \left< f^b_a(\lambda_m, \cdot) \right> \\
f^b_a(\lambda_m, \cdot) & \id + D^b_a(\lambda_m)
\end{pmatrix}, \\
\lim_{n \to \infty} \vec{R}_{(1,n)}(\theta_{(1,n)}) &= \begin{pmatrix}
h^b_a(\lambda_m) & \kappa \left< g^b_a(\lambda_m, \cdot) \right> \\
g^b_a(\lambda_m, \cdot) & \id + D^b_a(\lambda_m)
\end{pmatrix},
\end{align*} \tag{4.3.5}$$

where

$$h^b_a(\lambda) = \exp \left( \frac{\lambda^2}{2} (b - a) \right), \quad (4.3.10)$$

$$D^b_a(\lambda) = \op \left\{ (x,y) \to \lambda^2 <^b_a(x,y) h^b_a(\lambda) \right\}, \quad (4.3.11)$$

$$D^b_a(\lambda) = \op \left\{ (x,y) \to \lambda^2 <^b_a(x,y) h^b_a(\lambda) \right\}, \quad (4.3.12)$$

$$f^b_a(\lambda, x) = -\lambda \chi^b_a(x) h^b_a(\lambda), \quad (4.3.13)$$

$$g^b_a(\lambda, x) = -\lambda \chi^b_a(x) h^b_a(\lambda), \quad (4.3.14)$$

and $a \leq b$.

**Remark 21.** It can be shown that the limiting operators (4.3.6), (4.3.7), (4.3.8) and (4.3.9) are Bogolubov transformations by similar calculations to those that will be done in Sec-
tions 4.7, 4.8 and 4.9.

4.4 An example of the calculation of a limit

We derive the limit (4.3.6).

4.4.0.1 Limit of the bottom right of (4.2.29)

From the bottom right of (4.2.29) we have

\[ \delta_m = (\cosh \theta_{m,n})^m. \]  \hspace{1cm} (4.4.1)

Using the approximation, valid for large \( m \)

\[ \cosh \left( \frac{\lambda(n)}{m} \sqrt{\frac{b-a}{m}} \right) \approx 1 + \frac{\lambda(n)^2}{2} \left( \frac{b-a}{m} \right), \]  \hspace{1cm} (4.4.2)

formally taking the scalar limit of (4.4.1) as \( m \) tends to \( \infty \) gives

\[ \lim_{m \to \infty} \left( 1 + \frac{\lambda(n)^2}{2} \left( \frac{b-a}{m} \right) \right)^m = \exp \left( + \frac{\lambda(n)^2}{2} \left( \frac{b-a}{m} \right) \right) = h_b^b(\lambda(n)) \]  \hspace{1cm} (4.4.3)

which is the bottom right of the matrix (4.3.6).

**Remark 22.** In the paper Hudson [2007] pp. 175 the sign of \( \frac{\lambda(n)^2}{2} \frac{(b-a)}{m} \) is negative. This is because in that paper the \( r_{(m,n)}^{(j,k)}(\theta_{m,n}) \) are rotations and not pseudo-rotations.

4.4.0.2 Limit of the bottom left of (4.2.29)

In order to find limits of the remaining three matrix elements of (4.2.29) we embed the finite-dimensional Hilbert space \( \mathbb{C}^m \) into the Hilbert space \( L^2(\mathbb{R}^+) \) by mapping each element \( \varepsilon_j = (0,0,...,0, 1,0,...,0)^T \) of the canonical orthonormal basis of \( \mathbb{C}^m \) to the normalised indicator function

\[ \chi_{[a,b]}^j(x) = \left\{ \begin{array}{ll} \sqrt{\frac{m}{(b-a)}} & \text{if } a + \frac{(j-1)}{m}(b-a) \leq x < a + \frac{j}{m}(b-a) \\ 0 & \text{otherwise} \end{array} \right. \]

for \( j = 1, 2, ..., m \). We note that the sequence of projection operators

\[ P_m = \sum_{j=1}^{m} |\chi_{[a,b]}^j\rangle \langle \chi_{[a,b]}^j| \]

converges strongly to the projection onto the subspace \( L^2([a,b]) \) of \( L^2(\mathbb{R}^+) \), as can be seen by considering first the increasing subsequence of projections got by restricting \( m \) to powers of 2.
Using the approximations

\[
\sinh \theta \simeq \theta, \cosh \theta \simeq 1 + \frac{\theta^2}{2},
\]

valid for small \( \theta \), we find approximations for \( \sinh \theta_{m,n} \) and \(( \cosh \theta_{m,n})^i \) for integer \( i, 1 \leq i \leq m \) as

\[
\sinh \theta_{m,n} \simeq \lambda^{(n)} \sqrt{\frac{b-a}{m}} \tag{4.4.4}
\]

and

\[
(\cosh \theta_{m,n})^i = \left( 1 + \frac{\lambda^{(n)}}{2} \left( \frac{y_i - a}{m} \right) \right)^i \\
\approx \left( 1 + \frac{\lambda^{(n)}^2 (y_i - a)}{2} \sqrt{\frac{b-a}{m}} \right)^i \\
\simeq \left( 1 + \frac{\lambda^{(n)}^2 (y_i - a)}{2} \right)^i m, \tag{4.4.5}
\]

where \( y_i = a + i \frac{(b-a)}{m} \) was used on the second line, and on the third line the fact that the first two terms of the binomial expansion of \( \left( 1 + \frac{\lambda^{(n)}^2 (y_i - a)}{2} \sqrt{\frac{b-a}{m}} \right)^i \) are \( 1 + \frac{\lambda^{(n)}^2 (y_i - a)}{2} \sqrt{\frac{b-a}{m}} \).

The bottom left matrix element of (4.2.29), which is the composition of \( \kappa \) with the covector in \( \mathbb{C}^m \)

\[
\sum_{j=1}^{m} \delta^{j-1} \gamma \langle \varepsilon_j \rangle = - \sum_{j=1}^{m} (\cosh \theta_{m,n})^{j-1} \sinh \theta_{m,n} \langle \varepsilon_j \rangle,
\]

is embedded as the composition of \( \kappa \) with

\[
- \sum_{j=1}^{m} (\cosh \theta_{m,n})^{j-1} \sinh \theta_{m,n} \langle \chi^{j,m}_{[a,b]} \rangle \\
\approx - \sum_{j=1}^{m} \left( 1 + \left( \frac{\lambda^{(n)}}{2} \right) \left( \frac{y_{j-1} - a}{m} \right) \right)^m \lambda^{(n)} \sqrt{\frac{b-a}{m}} \langle \chi^{j,m}_{[a,b]} \rangle.
\]

Applying this covector to an arbitrary element \( f \) of \( L^2(\mathbb{R}_+) \) we obtain

\[
- \sum_{j=1}^{m} \left( 1 + \left( \frac{\lambda^{(n)}}{2} \right) \left( \frac{y_{j-1} - a}{m} \right) \right)^m \lambda^{(n)} \sqrt{\frac{b-a}{m}} \langle \chi^{j,m}_{[a,b]} \rangle f \\
= - \sum_{j=1}^{m} \left( 1 + \left( \frac{\lambda^{(n)}}{2} \right) \left( \frac{y_{j-1} - a}{m} \right) \right)^m \lambda^{(n)} \int_{a + \frac{(b-a)}{m}(y_{j-1} - a)}^{a + \frac{(b-a)}{m}(y_{j-1} - a)} f(y) \, dy
\]

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which converges as $m \to \infty$ to

$$-\lambda^{(n)} \int_{a}^{b} e^{\lambda^{(n)}(y-a)} f(y) \, dy = \left\langle g_{a}^{b} (\lambda^{(n)}, \cdot), f \right\rangle.$$

Thus, since $f$ was arbitrary we find the limit of the bottom left of (4.2.29) to be the composition with $\kappa$

$$\kappa \left\langle g_{a}^{b} (\lambda^{(n)}, \cdot) \right\rangle$$

as claimed.

4.4.0.3 Limit of the top right of (4.2.29)

A similar argument to that just given shows that the limit of the top right of (4.2.29) is the composition with the conjugation map $\kappa$ of the vector $f_{a}^{b} (\lambda^{(n)}, \cdot)$, identified with the linear map $z \mapsto f_{a}^{b} (\lambda^{(n)}, \cdot)$ from $\mathbb{C}$ to $L^{2} (\mathbb{R}_{+})$.

4.4.0.4 Limit of the top left of (4.2.29)

Using the embedding above of $\mathbb{C}^{m}$ into $L^{2} (\mathbb{R}_{+})$ the matrix

$$
\begin{pmatrix}
\alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^{2} \gamma & \cdots & \beta \delta^{m-4} \gamma & \beta \delta^{m-3} \gamma & \beta \delta^{m-2} \gamma \\
0 & \alpha & \beta \gamma & \beta \delta \gamma & \cdots & \beta \delta^{m-5} \gamma & \beta \delta^{m-4} \gamma & \beta \delta^{m-3} \gamma \\
0 & 0 & \alpha & \beta \gamma & \cdots & \beta \delta^{m-6} \gamma & \beta \delta^{m-5} \gamma & \beta \delta^{m-4} \gamma \\
0 & 0 & 0 & \alpha & \cdots & \beta \delta^{m-7} \gamma & \beta \delta^{m-6} \gamma & \beta \delta^{m-5} \gamma \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \alpha & \beta \gamma & \beta \delta \gamma \\
0 & 0 & 0 & 0 & \cdots & 0 & \alpha & \beta \gamma \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha \\
\end{pmatrix}
$$

regarded as a linear operator on $\mathbb{C}^{m}$ acting by left multiplication of column vectors, becomes the operator

$$W_{m} = \alpha \sum_{j=1}^{m} \left\langle \chi_{j,a,b}^{j,m} \right| \chi_{j,a,b}^{j,m} \right\rangle + \beta \sum_{1 \leq j < k \leq m} \delta_{k-j}^{j} \gamma \left| \chi_{j,a,b}^{j,m} \right\rangle \left\langle \chi_{j,a,b}^{k,m} \right|$$

$$+ I \{ \chi_{j,a,b}^{j,m}, j=1,2,\ldots,m \}$$

where, in order to obtain a Bogolubov transformation, we have defined the action of $W_{m}$ on the orthogonal complement of the embedding of $\mathbb{C}^{m}$ as the identity operator. Thus, for arbitrary
Consider $f, g \in L^2(\mathbb{R}_+)$, denoting by $f^\perp_m, g^\perp_m$ their components in \( \{ \chi_{a,b}^j, j = 1, 2, \ldots, m \}^\perp \)

\[
\langle f, W_m g \rangle - \langle f^\perp_m, g^\perp_m \rangle
\]

\[
= \alpha \sum_{j=1}^m \langle f, \chi_{a,b}^j \rangle \langle \chi_{a,b}^j, g \rangle + \beta \sum_{1 \leq j < k \leq m} \delta^{k-j-1} \langle f, \chi_{a,b}^j \rangle \langle \chi_{a,b}^k, g \rangle
\]

We take the large $m$ limit of both sides of this equation, noting first that, since $P_m$ converges strongly to the projection onto the subspace $L^2([a,b])$ of $L^2(\mathbb{R}_+)$, \( \langle f^\perp_m, g^\perp_m \rangle \to m \to \infty \)

\[
\langle f, g \rangle - \langle f \chi_{[a,b]}, g \chi_{[a,b]} \rangle
\]

Using the approximations

\[
\beta = \gamma = -\sinh \theta_{m,n} \simeq -\theta_{m,n} = -\lambda(n) \sqrt{\frac{(b-a)}{m}},
\]

the second term

\[
\beta \gamma \sum_{1 < j < k \leq m} \delta^{k-j-1} \langle f, \chi_{[a,b]}^j \rangle \langle \chi_{[a,b]}^k, g \rangle
\]

\[
\simeq \left( \lambda(n) \right)^2 \frac{(b-a)}{m} \sum_{1 < j < k \leq m} \delta^{k-j-1} \langle f, \chi_{[a,b]}^j \rangle \langle \chi_{[a,b]}^k, g \rangle
\]

and finally using the approximation

\[
\delta^{k-j-1} = (\cosh \theta_{m,n})^{k-j-1} \simeq \left( 1 + \frac{\lambda(n)^2}{2} \frac{(y_{k-1} - x_j)}{m} \right)^m
\]

\[
\simeq e^{\frac{\lambda(n)^2}{2} \left( y_{k-1} - x_j \right)}
\]

where, $y_{k-1} = a + (k-1)\frac{b-a}{m}$ and $x_j = a + j\frac{b-a}{m}$, the second term on the right hand side converges as $m \to \infty$ to

\[
\lambda(n)^2 \int_{a \leq x \leq y < b} \tilde{f}(x)e^{\frac{\lambda(n)^2}{2}(y-x)} g(y) \; dy \; dx.
\]

We deduce that the operator $W_m$ converges weakly to the sum of the identity $I$ and the integral operator

\[
\text{Op}\left\{(x,y) \longrightarrow \lambda(n)^2 \mathbb{1}_a^b (x,y) \exp \left( \frac{\lambda(n)^2}{2} (y-x) \right) \right\} = D_\Omega^n(\lambda(n))
\]

as claimed.

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4.5 The second limits

Following the Remark 20 made at the end of Section 4.2.2 we describe in this section the process of putting the entries of the matrices that make up the first limits

\[ \lim_{n \to \infty} R_{(1,n)}(\theta_{(m,n)}) \text{ and } \lim_{n \to \infty} \tilde{R}_{(1,n)}(\theta_{(m,n)}) \]

of Theorem 16 into the product matrices \( \tilde{R}_{(m,1)}(\theta) \) and \( \tilde{R}_{(m,1)}(\theta) \) of Theorem 15 and taking the limit as \( m \) tends to infinity (the second limit after letting \( n \) tend to infinity). We use the notation

\[ \lim_{m \to \infty} \left( \lim_{n \to \infty} \tilde{R}_{(m,n)}(\theta_{m,n}) \right) \text{ and } \lim_{m \to \infty} \left( \lim_{n \to \infty} \tilde{R}_{(m,n)}(\theta_{m,n}) \right) \]  

(4.5.1)

for the results of putting the entries of \( \lim_{n \to \infty} \tilde{R}_{(1,n)}(\theta_{(m,n)}) \) into \( \tilde{R}_{(m,1)}(\theta) \) and \( \tilde{R}_{(m,1)}(\theta) \), and taking the limit, respectively. We use the notation

\[ \lim_{m \to \infty} \left( \lim_{n \to \infty} \tilde{R}_{(m,n)}(\theta_{m,n}) \right) \text{ and } \lim_{m \to \infty} \left( \lim_{n \to \infty} \tilde{R}_{(m,n)}(\theta_{m,n}) \right) \]  

(4.5.2)

for the results of putting the entries of the matrices that make up the first limits

\[ \lim_{n \to \infty} \tilde{R}_{(1,n)}(\theta_{(m,n)}) \]  

into \( \tilde{R}_{(m,1)}(\theta) \) and \( \tilde{R}_{(m,1)}(\theta) \),

(4.5.3)

and taking the limit, respectively.

In this section we also describe the process of putting the entries of the first limits

\[ \lim_{m \to \infty} \tilde{R}_{(m,1)}(\theta_{(m,n)}) \text{ and } \lim_{m \to \infty} \tilde{R}_{(m,1)}(\theta_{(m,n)}) \]  

(4.5.4)

of Theorem 16 into the product matrices \( \tilde{R}_{(1,n)}(\theta) \) and \( \tilde{R}_{(1,n)}(\theta) \) of Theorem 15 and taking limit as \( n \) tends to infinity (the second limit). In a similar manner we denote the four limits resulting from putting in the entries of the matrices that make up the first limits (equations (4.5.4)) into \( \tilde{R}_{(1,n)}(\theta) \) and \( \tilde{R}_{(1,n)}(\theta) \) and taking limits as

\[ \lim_{n \to \infty} \left( \lim_{m \to \infty} \tilde{R}_{(m,n)}(\theta_{m,n}) \right), \lim_{n \to \infty} \left( \lim_{m \to \infty} \tilde{R}_{(m,n)}(\theta_{m,n}) \right) \]

\[ \text{and } \lim_{n \to \infty} \left( \lim_{m \to \infty} \tilde{R}_{(m,n)}(\theta_{m,n}) \right), \lim_{n \to \infty} \left( \lim_{m \to \infty} \tilde{R}_{(m,n)}(\theta_{m,n}) \right) \]  

(4.5.5)

Notice that the notation for the limits of equations (4.5.5) indicates that the order of limits of \( n \) and \( m \) has been reversed to those of equations (4.5.1) and (4.5.2). In Section 4.6 we will show that each of the limits from equations (4.5.1) and (4.5.2) are identical to one and only one from equations (4.5.5).
4.5.1 Second limit notation

In the previous section we introduced eight different second limits. In this section we shorten the notation and terminology to make referring to them easier. We use the notation

\[ W_{b,t}^{a,s} := \lim_{m \to \infty} \left( \lim_{n \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right). \]  

(4.5.6)

We use a tilde over the \( W \) to denote that the limit as \( m \) tend to infinity is taken first and then that of \( n \) to infinity.

\[ \tilde{W}_{b,t}^{a,s} := \lim_{n \to \infty} \left( \lim_{m \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right). \]  

(4.5.7)

We describe the two double products (equations (4.5.6) and (4.5.7)) as both begin continuous forward-forward double products or just forward-forward double products when it is obvious it is a continuous product being discussed.

A reversal of the positions of \( a \) and \( b \) in \( W_{b,t}^{a,s} \) indicates that the product is taken from \( b \) to \( a \) i.e. we have the notation

\[ W_{b,s}^{a,t} := \lim_{m \to \infty} \left( \lim_{n \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right) \]

and

\[ \tilde{W}_{b,s}^{a,t} := \lim_{n \to \infty} \left( \lim_{m \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right) \]

for the continuous backward-forward double product. Similar a reversal of the positions of \( s \) and \( t \) in \( W_{b,t}^{a,s} \) indicates that the product is taken from \( t \) to \( s \) and we have the notation

\[ W_{a,t}^{b,s} := \lim_{m \to \infty} \left( \lim_{n \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right) \]

and

\[ \tilde{W}_{a,t}^{b,s} := \lim_{n \to \infty} \left( \lim_{m \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right) \]

for the continuous forward-backward double product. The continuous backward-backward double product is denoted by \( W_{a,s}^{b,t} \) or \( \tilde{W}_{a,s}^{b,t} \) depending on wether the first limit taken is either \( n \to \infty \) or \( m \to \infty \) respectively.

4.5.2 The forward-forward case

We use the notation for the calculation

\[ W_{a,s}^{b,t} := \lim_{m \to \infty} \left( \lim_{n \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right), \]

which is done as follows. First substitute the elements of \( \lim_{n \to \infty} \rightarrow R_{(1,n)} \) into the matrix (4.2.31), where the integral operator on \( h \) is substituted for \( \alpha \), the vector in \( h \) for \( \beta \), covector in \( h \) for \( \gamma \) and the scalar for \( \delta \). Next, the limit as \( m \) goes to infinity is taken heuristically to give equation

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Similarly to calculate the operator
\[ \tilde{W}^{b,t}_{a,s} = \lim_{n \to \infty} \left( \lim_{m \to \infty} \overrightarrow{R}_{(m,n)}(\theta_{m,n}) \right) \]
the elements of \( \lim_{m \to \infty} \overrightarrow{R}_{(m,1)} \) are substituted into matrix (4.2.29) and the limit as \( m \) goes to infinity is taken in a heuristic manner. Again the resulting operator acts on \( \mathfrak{h} \oplus \mathfrak{h} \). As may be hoped, this operator is the same operator as the previous one. This can be shown by calculating the kernels of the four constituting operators which act on \( \mathfrak{h} \), this is done in the next section (Section 4.6). We do this for both forms of the operators and show the the kernels are indeed the same.

The notation for the integral operator of the indicator functions equations (4.3.2) and (4.3.3) respectively are: \( \Delta^b_a := \text{Op}\{ (x,y) \mapsto >^b_a (x,y) \} \) and \( \Delta^a_b := \text{Op}\{ (x,y) \mapsto <^a_b (x,y) \} \). Notice how the positions of \( a \) and \( b \) on the \( \Delta \) indicate which indicator function is the kernel. We use the notation \( \kappa \) for the map \( \kappa : f \mapsto \overrightarrow{f} \) such that \( \overrightarrow{f}(x) = f(x) \).

We now state the form of the operators \( W^{b,t}_{a,s} \) and \( \tilde{W}^{b,t}_{a,s} \) resulting from the second limiting processes. We also, as part of the subsection, give an example of how the heuristic limits are taken (Section 4.5.2.3).

### 4.5.2.1 The limit \( W^{b,t}_{a,s} \)

\[
W^{b,t}_{a,s} = \lim_{n \to \infty} \left( \lim_{m \to \infty} \overrightarrow{R}_{(m,n)}(\theta_{m,n}) \right) = \begin{pmatrix}
\text{id} + \text{Op}\{ (x,y) \mapsto \lambda^2 <^b_a (x,y) \chi^t_b, \exp \left( \lambda^2 (y-x) \Delta^t_b \right) \chi^t_b \} \\
\kappa \text{Op}\{ (x,y) \mapsto -\lambda \chi^b_a(y) \left( \exp \left( \lambda^2 (y-a) \Delta^t_a \right) \chi^t_a \right)(x) \} \\
\kappa \text{Op}\{ (x,y) \mapsto -\lambda \chi^b_a(x) \left( \exp \left( \lambda^2 (b-x) \Delta^t_b \right) \chi^t_b \right)(y) \} \\
\exp \left( \lambda^2 (b-a) \Delta^t_b \right) \end{pmatrix}.
\] (4.5.8)

### 4.5.2.2 The limit \( \tilde{W}^{b,t}_{a,s} \)

\[
\tilde{W}^{b,t}_{a,s} = \lim_{n \to \infty} \left( \lim_{m \to \infty} \overrightarrow{R}_{(m,n)}(\theta_{m,n}) \right) = \begin{pmatrix}
\exp \left( \lambda^2 (t-s) \Delta^t_a \right) \\
\kappa \text{Op}\{ (x,y) \mapsto -\lambda \chi^t_a(x) \left( \exp \left( \lambda^2 (t-x) \Delta^t_a \right) \chi^t_a \right)(y) \} \\
\kappa \text{Op}\{ (x,y) \mapsto -\lambda \chi^t_a(y) \left( \exp \left( \lambda^2 (y-s) \Delta^t_a \right) \chi^t_a \right)(x) \} \\
\text{id} + \text{Op}\{ (x,y) \mapsto \lambda^2 <^b_a (x,y) \chi^b_a, \exp \left( \lambda^2 (y-x) \Delta^t_a \right) \chi^b_a \} \end{pmatrix}.
\] (4.5.9)
4.5.2.3 An example: the calculation of $\tilde{W}^{k,t}_{a,s}$

This calculation is very much like that of the operator (4.3.6) done in Section 4.4. The resulting
operators of the limit $\lim_{m \to \infty} R_{(m,1)}$ (i.e. equation (4.3.6)) are put into the product matrix
(4.2.31), where

$$\alpha = \text{id} + \text{Op}\left\{ (x,y) \mapsto \lambda^{(n)^2} <_{a}^{b} (x,y) \exp \left( \frac{\lambda^{(n)^2}}{2} (y-x) \right) \right\},$$

$$\beta = \left\lvert - \lambda^{(n)} \chi_{a}^{b} (\cdot) \exp \left( \frac{\lambda^{(n)^2}}{2} (b-\cdot) \right) \right\rvert \kappa,$$

$$\gamma = \kappa \left\langle - \lambda^{(n)} \chi_{a}^{b} (\cdot) \exp \left( \frac{\lambda^{(n)^2}}{2} (\cdot-a) \right) \right\rangle \kappa,$$

$$\delta = \exp \left( \frac{\lambda^{(n)^2}}{2} (b-a) \right).$$

Then the heuristic limit as $m$ tends to $\infty$ is taken.

4.5.2.4 Limit of the top left of (4.2.31)

We first obtain an approximation of the operator

$$\alpha^n = (\text{id} + D_a^b (\lambda_n))^{n},$$

for large $n$. Observe that the kernel of the integral operator $D_a^b (\lambda_n)$ is of the form

$$\lambda^{(n)^2} <_{a}^{b} (x,y) + o(\lambda^{3}).$$

Since $\lambda^{(n)} = \lambda \sqrt{\frac{t-s}{n}}$ (to order $\frac{1}{n}$), $D_a^b (\lambda_n)$ is the operator $\frac{t-s}{n} \Delta_a^b$. Hence for large $n$

$$\alpha^n \approx \left( \text{id}_{L^2 (\mathbb{R}_+)} + \lambda \frac{t-s}{n} \Delta_a^b \right)^n,$$

$$\approx \exp \left( \lambda \frac{t-s}{n} \Delta_a^b \right),$$

which is the top left of matrix (4.5.9).

4.5.2.5 Limit of the top right of (4.2.31)

In order to find the limits of the remaining three matrix element of (4.5.9) we embed the
finite-dimensional Hilbert space $\mathbb{C}^n$ into the Hilbert space $L^2 (\mathbb{R}_+)$ by mapping each element

$$\varepsilon_j = (0,0,...,0,(i),0,...,0)^T$$

of the canonical orthonormal basis of $\mathbb{C}^n$ to the normalised indicator function

$$\chi_{s,t}^{n} \left( x \right) = \begin{cases} \sqrt{\frac{n}{t-s}} & \text{if } s + \frac{(i-1)}{n}(t-s) \leq x < s + \frac{i}{n}(t-s) \\ 0 & \text{otherwise} \end{cases}$$

(4.5.18)
for $j = 1, 2, ..., n$.

The top right matrix element of (4.2.31) is the composition of $\kappa$ with a ket in $\mathbb{C}^n$

$$
\sum_{j=1}^{n} \alpha^{j-1} \beta |\varepsilon_j\rangle \kappa.
$$

(4.5.19)

Observe that the vector $f^b_a(\lambda^{(n)}, \cdots) = -\lambda^{(n)} \chi^b_a(\cdot) \exp \left( \frac{\lambda^{(n)} t}{2} (b - \cdot) \right)$ of $L^2[a,b]$ is of the form $-\lambda^{(n)} \chi^b_a(\cdot) + o(\lambda^{(n)} \beta)$. Since $\lambda^{(n)} = \lambda \sqrt{\frac{t-s}{n}}$, the composition of the ket

$$
-\lambda^{(n)} \chi^b_a(\cdot) \exp \left( \frac{\lambda^{(n)} t}{2} (b - \cdot) \right)
$$

with $\kappa$ (to order $\frac{1}{n}$) is

$$
\beta \simeq -\lambda \sqrt{\frac{t-s}{n}} \chi^b_a(\cdot) \kappa.
$$

(4.5.20)

Embedding (4.5.19) into $L^2(\mathbb{R}_+)$ using (4.5.18) we have

$$
\sum_{j=1}^{n} \alpha^{j-1} \beta |\varepsilon_j\rangle \kappa \simeq \sum_{j=1}^{n} \alpha^{j-1} \beta |\chi_j^{i,n}(\cdot)\rangle \kappa.
$$

(4.5.21)

Using approximation (4.5.16) and (4.5.20) we have for large $n$ and arbitrary $f$ and $g$ in $L^2(\mathbb{R}_+)$ the following approximation

$$
\langle f, \sum_{j=1}^{n} \alpha^{j-1} \beta |\chi_j^{i,n}(\cdot)\rangle \kappa g 
\simeq \left\langle f, -\lambda \sum_{j=1}^{n} \left( \frac{\lambda^2 t-s}{n} \Delta^a_b \right)^{j-1} \sqrt{\frac{t-s}{n}} \chi^b_a(\cdot) |\chi_j^{i,n}(\cdot)\rangle \kappa g(y) \right\rangle 
\simeq \left\langle f, -\lambda \sum_{j=1}^{n} \left( \frac{\lambda^2 (y-s)}{n} \Delta^a_b \right)^{n} \sqrt{\frac{t-s}{n}} \chi^b_a(\cdot) |\chi_j^{i,n}(\cdot)\rangle \kappa g(y) \right\rangle,
$$

(4.5.22)

which converges to the limit

$$
\langle f, -\lambda \int_{s}^{t} \chi^b_a(y) \exp \left( \frac{\lambda^2 (y-s) \Delta^b_a}{\gamma} \right) \kappa g(y) dy \rangle.
$$

(4.5.23)

as $n$ tends to $\infty$. Therefore the limit is

$$
\text{Op} \left\{ (x, y) \rightarrow -\lambda \chi^b_a(y) \left( \exp \left( \frac{\lambda^2 (y-s) \Delta^b_a}{\gamma} \right) \right) \chi^b_a \right\} (x) \kappa
$$

(4.5.24)

which is the desired operator.
4.5.2.6 Limit of the bottom left of (4.2.31)

A similar argument to that just given shows that the limit of the bottom left of the matrix (4.2.31) is the composition of the conjugation map $\kappa$ with the linear operator

$$\text{Op}\left\{(x,y) \rightarrow -\lambda \chi_s^f(x)\left(\exp \left(\lambda^2(t-x)\Delta_\psi^b\right)\chi_s^b(y)\right)\right\}$$

(4.5.25)

from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}_+)$.

4.5.2.7 Limit of the bottom right of (4.2.31)

Using the embedding of $\mathbb{C}^n$ into $L^2(\mathbb{R}_+)$ introduced in Section 4.5.2.5 the matrix

$$
\begin{pmatrix}
\delta & \gamma & \gamma & \gamma^2 & \beta & \ldots & \gamma & \alpha^{n-3} & \beta & \gamma & \alpha^{n-2} & \beta \\
0 & \delta & \gamma & \gamma & \beta & \ldots & \gamma & \alpha^{n-4} & \beta & \gamma & \alpha^{n-3} & \beta \\
0 & 0 & \delta & \gamma & \beta & \ldots & \gamma & \alpha^{n-5} & \beta & \gamma & \alpha^{n-4} & \beta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \gamma & \beta & \gamma & \alpha & \beta \\
0 & 0 & 0 & 0 & \ldots & \delta & \gamma & \beta & \gamma & \alpha & \beta \\
0 & 0 & 0 & 0 & \ldots & 0 & \delta & \gamma & \beta & \gamma & \alpha & \beta \\
\end{pmatrix}
$$

(4.5.26)

regarded as a linear operator on $\mathbb{C}^n$ acting by left multiplication of column vectors becomes the operator

$$W_n = \delta \sum_{j=1}^n \left| \chi_{s,t}^{j,n} \right| \left| \chi_{s,t}^{j,n} \right| + \gamma \sum_{1 \leq j < k \leq n} \alpha^{k-j-1} \beta \left| \chi_{s,t}^{j,n} \right| \left| \chi_{s,t}^{k,n} \right|$$

$$+ \sum_{j=1}^n \left| \chi_{s,t}^{j,n} \right| \left| \chi_{s,t}^{j,n} \right|$$


where, in order to obtain a Bogolubov transformation on $L^2(\mathbb{R}_+)^n \boxplus L^2(\mathbb{R}_+)^n$, we have defined the action of $W_n$ on the orthogonal complement of the embedding of $\mathbb{C}^n$ as the identity operator. Thus, for arbitrary $f,g \in L^2(\mathbb{R}_+)$, denoting by $f_n^l, g_n^l$ their components in $\left\{ \chi_{s,t}^{j,n}, j = 1, 2, \ldots, n \right\}$

$$\left\langle f, W_n g \right\rangle - \left\langle f_n^l, g_n^l \right\rangle$$

$$= \delta \sum_{j=1}^n \left\langle f, \chi_{s,t}^{j,n} \right\rangle \left\langle \chi_{s,t}^{j,n} \right\rangle + \gamma \sum_{1 \leq j < k \leq n} \alpha^{k-j-1} \beta \left\langle \chi_{s,t}^{j,n} \right\rangle \left\langle \chi_{s,t}^{k,n} \right\rangle$$

$$= \sum_{j=1}^n \left\langle f, \chi_{s,t}^{j,n} \right\rangle \left\langle \chi_{s,t}^{j,n} \right\rangle + \gamma \sum_{1 \leq j < k \leq n} \alpha^{k-j-1} \beta \left\langle f, \chi_{s,t}^{j,n} \right\rangle \left\langle \chi_{s,t}^{k,n} \right\rangle$$

Observe that the vector $g_n^l(\lambda^{(n)}, \cdot) = -\lambda^{(n)} \chi_a^b(\cdot) \exp\left(\frac{\lambda^{(n)}^2}{2} (\cdot - a)\right)$ of $L^2[a, b]$ is of the form
\(-\lambda^{(n)} \chi^b_a(\cdot) + o(\lambda^{(n)^3})\). Since \(\lambda^{(n)} = \lambda \sqrt{\frac{t-s}{n}}\), the composition of the ket \(\langle g^b_a(\lambda^{(n)}, \cdot) \rangle\) with \(\kappa\) (to order \(\frac{1}{n}\)) is
\[
\gamma \simeq -\kappa \lambda^b_a(x) \sqrt{\frac{t-s}{n}} \text{Op}\{\chi^b_a\}. \quad (4.5.27)
\]
In a similar fashion the approximation of \(\delta\) to order \(\frac{1}{n}\) is
\[
\delta \approx \left(1 + \lambda \left(\frac{t-s}{n}\right)\right). \quad (4.5.28)
\]
Using the approximation (4.5.28) the first term on the left hand side converges to \(\langle f \chi^b_a [s,t], g \chi^b_a [s,t] \rangle\).

Using the approximations (4.5.20), (4.5.16) and (4.5.27) the second term
\[
\gamma \sum_{1 \leq j < k \leq m} \alpha^{k-j-1} \beta \langle f, \chi^{j,n}_a \rangle \langle \chi^{k,n}_a, g \rangle \approx \lambda^2 \chi^b_a (y_{k-1}) \frac{t-s}{n} \text{Op}\{\chi^b_a\} \sum_{1 \leq j < k \leq n} \left(\text{id}_{L^2(\mathbb{R})} + \lambda^2 \frac{y_{k-1} - x_j}{n} \Delta^a_b\right)^n \chi^b_a (y_{k-1}) \langle f, \chi^{j,n}_a \rangle \langle \chi^{k,n}_a, g \rangle, \quad (4.5.29)
\]
where \(y_{k-1} = a + (k-1) \frac{b-a}{m}\) and \(x_j = a + j \frac{b-a}{n}\), the second term on the right hand side converges as \(m \to \infty\) to
\[
\lambda^2 \int_{s \leq x < y \leq b} \chi^b_a (y) \exp \left(\lambda^2 (y-x) \Delta^a_b\right) \chi^b_a (y) dy g(x) dx.
\]
We deduce that the operator \(W_m\) converges weakly to the sum of the identity \(I\) and the integral operator
\[
\text{Op} \left\{ (x, y) \mapsto \lambda^2 \chi^b_a (x, y) \exp \left(\lambda^2 (y-x) \Delta^a_b\right) \chi^b_a \right\}
\]
as claimed.

### 4.5.3 The backward-backward case

For the backward-backward case the positions of the \(a\) and \(b\) in \(\Delta^b_a\) are exchanged to that of the forward-forward case and similarly the positions of the \(s\) and \(t\) are exchanged too. This ensures that the order of both the products of the double product are in the reverse direction to that of the forward forward case.
4.5.3.1 The limit \( W^{a,s}_{b,t} \)

\[
W^{a,s}_{b,t} = \lim_{m \to \infty} \left( \lim_{n \to \infty} R_{(m,n)}(\theta_{m,n}) \right) = \begin{pmatrix}
\text{id} + \text{Op} \{ (x, y) \to \lambda^2 <_a (x, y) \langle \chi^b_t, \exp (\lambda^2(x-y)\Delta^b_t) \chi^b_t \rangle \\
\kappa \text{ Op} \{ (x, y) \to -\lambda \chi^b_a(y) \langle \exp (\lambda(b-y)\Delta^b_t) \chi^b_t \rangle(x) \\
\kappa \text{ Op} \{ (x, y) \to -\lambda \chi^b_a(x) \langle \exp (\lambda^2(x-a)\Delta^b_t) \chi^b_t \rangle(y) \\
\exp (\lambda^2(b-a)\Delta^b_t)
\end{pmatrix}
\tag{4.5.30}
\]

4.5.3.2 The limit \( \tilde{W}^{a,s}_{b,t} \)

\[
\tilde{W}^{a,s}_{b,t} = \lim_{m \to \infty} \left( \lim_{n \to \infty} R_{(m,n)}(\theta_{m,n}) \right) = \begin{pmatrix}
\text{id} + \text{Op} \{ (x, y) \to \lambda^2 >_a (x, y) \langle \chi^b_t, \exp (\lambda^2(x-y)\Delta^b_t) \chi^b_t \rangle \\
\kappa \text{ Op} \{ (x, y) \to -\lambda \chi^b_a(y) \langle \exp (\lambda(b-y)\Delta^b_t) \chi^b_t \rangle(x) \\
\kappa \text{ Op} \{ (x, y) \to -\lambda \chi^b_a(x) \langle \exp (\lambda^2(x-a)\Delta^b_t) \chi^b_t \rangle(y) \\
\exp (\lambda^2(b-a)\Delta^b_t)
\end{pmatrix}
\tag{4.5.31}
\]

4.5.4 The forward-backward case

For the remaining four second limits (\( W^{b,s}_{a,t}, \tilde{W}^{b,s}_{a,t}, W^{a,t}_{b,s} \) and \( \tilde{W}^{a,t}_{b,s} \)) we simply state the form of the resulting operators from taking the second limit (Sections 4.5.4.1, 4.5.4.2, 4.5.5.1 and 4.5.5.2).

4.5.4.1 The limit \( W^{b,s}_{a,t} \)

\[
W^{b,s}_{a,t} = \lim_{m \to \infty} \left( \lim_{n \to \infty} R_{(m,n)}(\theta_{m,n}) \right) = \begin{pmatrix}
\text{id} + \text{Op} \{ (x, y) \to \lambda^2 <_a (x, y) \langle \chi^b_t, \exp (\lambda^2(y-x)\Delta^b_t) \chi^b_t \rangle \\
\kappa \text{ Op} \{ (x, y) \to -\lambda \chi^b_a(y) \langle \exp (\lambda^2(y-a)\Delta^b_t) \chi^b_t \rangle(x) \\
\kappa \text{ Op} \{ (x, y) \to -\lambda \chi^b_a(x) \langle \exp (\lambda^2(b-x)\Delta^b_t) \chi^b_t \rangle(y) \\
\exp (\lambda^2(b-a)\Delta^b_t)
\end{pmatrix}
\tag{4.5.32}
\]
4.5.4.2 The limit $\hat{W}_{a,t}^{b,s}$

\[
\hat{W}_{a,t}^{b,s} = \lim_{n \to \infty} \left( \lim_{m \to \infty} \hat{R}_{(m,n)}(\theta_{m,n}) \right) = \left( \begin{array}{c}
\exp \left( \lambda^2 (t-s) \Delta_{ba}^a \right) \\
\kappa \text{ Op} \left\{ (x,y) \mapsto -\lambda \chi^a_\ast(x) \langle \exp \left( \lambda^2 (x-s) \Delta_{ba}^a \right) \chi^b_\ast(y) \rangle \right\} \\
\kappa \text{ Op} \left\{ (x,y) \mapsto -\lambda \chi^a_\ast(y) \langle \exp \left( \lambda^2 (y-t) \Delta_{ba}^a \right) \chi^b_\ast(x) \rangle \right\} \\
\text{id} + \text{ Op} \left\{ (x,y) \mapsto \lambda^2 \langle \chi^b_\ast \exp \left( \lambda^2 (x-y) \Delta_{ba}^a \right) \chi^b_\ast \rangle \right\}
\end{array} \right)
\]

(4.5.33)

Above is the operator $\hat{W}_{a,t}^{b,s}$ obtained by reversing the order in which the limits of $n$ and $m$ taken.

4.5.5 The backward-forward case

4.5.5.1 The limit $W_{b,s}^{a,t}$

\[
W_{b,s}^{a,t} = \lim_{m \to \infty} \left( \lim_{n \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right) = \left( \begin{array}{c}
\exp \left( \lambda^2 (t-s) \Delta_{ba}^a \right) \\
\kappa \text{ Op} \left\{ (x,y) \mapsto -\lambda \chi^a_\ast(x) \langle \exp \left( \lambda^2 (x-s) \Delta_{ba}^a \right) \chi^b_\ast(y) \rangle \right\} \\
\kappa \text{ Op} \left\{ (x,y) \mapsto -\lambda \chi^a_\ast(y) \langle \exp \left( \lambda^2 (y-t) \Delta_{ba}^a \right) \chi^b_\ast(x) \rangle \right\} \\
\text{id} + \text{ Op} \left\{ (x,y) \mapsto \lambda^2 \langle \chi^b_\ast \exp \left( \lambda^2 (x-y) \Delta_{ba}^a \right) \chi^b_\ast \rangle \right\}
\end{array} \right)
\]

(4.5.34)

4.5.5.2 The limit $\hat{W}_{b,s}^{a,t}$

\[
\hat{W}_{b,s}^{a,t} = \lim_{n \to \infty} \left( \lim_{m \to \infty} \rightarrow R_{(m,n)}(\theta_{m,n}) \right) = \left( \begin{array}{c}
\exp \left( \lambda^2 (t-s) \Delta_{ba}^a \right) \\
\kappa \text{ Op} \left\{ (x,y) \mapsto -\lambda \chi^a_\ast(x) \langle \exp \left( \lambda^2 (x-s) \Delta_{ba}^a \right) \chi^b_\ast(y) \rangle \right\} \\
\kappa \text{ Op} \left\{ (x,y) \mapsto -\lambda \chi^a_\ast(y) \langle \exp \left( \lambda^2 (y-t) \Delta_{ba}^a \right) \chi^b_\ast(x) \rangle \right\} \\
\text{id} + \text{ Op} \left\{ (x,y) \mapsto \lambda^2 \langle \chi^b_\ast \exp \left( \lambda^2 (x-y) \Delta_{ba}^a \right) \chi^b_\ast \rangle \right\}
\end{array} \right)
\]

(4.5.35)

4.5.6 Conclusion

It can see that there are two operators for each of the four cases; the forward-forward, backward-backward, forward-backward and backward-forward. In turns out that in each case the corresponding $W$’s and $\hat{W}$’s are the same operator. This can be seen by computing the kernels of
the constituent operators of the $W$’s and $\tilde{W}$’s. This process is explained in detail in the next section (Section 4.6).

### 4.6 Kernels

The four double product (4.5.8), (4.5.30), (4.5.32) and (4.5.34) were seen in Section 4.5 to be the result of first taking the limit as $n$ goes to infinity and then taking the limit as $m$ goes infinity; these double product correspond to the $W$’s without the tilde. These four double products have the form

$$\left( \begin{array}{cc} \text{id} + \text{Op} \{ A \} & \kappa \text{Op} \{ B \} \\ \kappa \text{Op} \{ C \} & \Xi \end{array} \right),$$

(4.6.1)

where $A$, $B$ and $C$ are the kernels of the integral operators $\text{Op} \{ A \}$, $\text{Op} \{ B \}$ and $\text{Op} \{ C \}$ and $\Xi$ is an operator acting on $\mathfrak{h}$. We call the operators $\text{Op} \{ A \}$, $\text{Op} \{ B \}$ and $\text{Op} \{ C \}$ and $\Xi$ the constituent operators of the operator $W$. It will be seen later that we can express the kernel $A$ as a convergent power series in $x$ and $y$ (and fixed parameters $s$ and $t$), the kernels $B$ and $C$ as convergent power series in $x$ and $y$ (and fixed parameters $a$, $b$, $s$ and $t$). The integral operator $\Xi$ can be expressed as $\text{id} + \text{Op} \{ D \}$, $D$ being a kernel which can also be expressed as a convergent power series in $a$, $b$, $x$ and $y$. We do not express the operator $\Xi$ as the identity plus integral operator, as $\Xi$ has the form of an exponential of a linear operator (either $\Delta^t_s$ or $\Delta^s_t$). The integral operator is dependant on fixed parameters $s$, $t$. Having $\Xi$ in this form makes easier the calculation to prove (in Sections 4.8 and 4.9) that operators (4.5.8), (4.5.30), (4.5.32) and (4.5.34) are Bogolubov transformations. Note that (4.6.1) is a Hilbert Schmidt operator plus the identity.

The form of the operators (4.5.9), (4.5.31), (4.5.33) and (4.5.35) (the $\tilde{W}$’s) are obtained by taking the limit as $m$ goes to infinity and then taking the limit as $n$ goes to infinity is

$$\left( \begin{array}{cc} \Xi & \kappa \text{Op} \{ B \} \\ \kappa \text{Op} \{ C \} & \text{id} + \text{Op} \{ D \} \end{array} \right),$$

where $B$ and $C$ are kernels expressible as polynomials in $a,b$, $s$, $t$, $x$ and $y$, $D$ as a kernel expressible as a polynomial in $a$, $b$, $x$ and $y$, and $\Xi$ is an operator acting on $\mathfrak{h}$. The operator $\Xi$ can be express as $\text{id} + \text{Op} \{ A \}$, where $A$ is a kernel in the form of a convergent power series in $s$, $t$, $x$ and $y$.

In this section the constituent kernels of the eight operators (the double products (4.5.8), (4.5.9), (4.5.30), (4.5.31), (4.5.32), (4.5.33), (4.5.34) and (4.5.35)), are calculated; all these operators being the result of the second limiting process. It will be shown that the constituent kernels of the two forms of forward-forward operators are equal and hence are the same operators. This process is then repeated for the backward-backward case, the forward-backward case and the finally the backward-forward case.

Before we calculate the kernels $A$, $B$, $C$ and $D$ we state and prove some useful lemmas.
Lemma 8. For $a, b \in \mathbb{R}$, $a < b$ and any $n \in \mathbb{N}$ the following equalities hold

\[
\langle \chi^{b}_{a}, (\Delta^{b}_{a})^{n} \chi^{b}_{a} \rangle = \int_{a}^{b} \int_{a}^{x_{1}} \int_{a}^{x_{2}} \ldots \int_{a}^{x_{n-2}} \int_{a}^{x_{n-1}} dx_{n-1} \ldots dx_{2} dx_{1} \quad (4.6.2)
\]

\[
= \frac{(b-a)^{n+1}}{(n+1)!}.
\]

Proof. The first equality from the left is true by definition of $\Delta^{b}_{a}$ and that of the inner product of the Hilbert space $L^{2}(\mathbb{R})$. The second equality is true for $n = 1$. If we assume result is true for some $n$, then replacing $b$ with a variable $z$ in equation (4.6.2) the following calculation shows that it is also true for $n + 1$,

\[
\int_{a}^{b} \frac{(z-a)^{n+1}}{(n+1)!} dz = \frac{(b-a)^{n+2}}{(n+2)!}. \quad (4.6.3)
\]

The left hand side of equation (4.6.3) is an iterated integral of order $n + 1$ and the right hand side is the result for $n + 1$.

Lemma 9. For $a, b \in \mathbb{R}$, $a < b$ and any $n \in \mathbb{N}$ the following equalities hold

\[
\langle \chi^{b}_{a}, (\Delta^{a}_{b})^{n} \chi^{b}_{a} \rangle = \int_{a}^{b} \int_{a}^{x_{1}} \int_{a}^{x_{2}} \ldots \int_{a}^{x_{n-2}} \int_{a}^{x_{n-1}} dx_{n-1} \ldots dx_{2} dx_{1} \quad (4.6.4)
\]

\[
= \frac{(b-a)^{n+1}}{(n+1)!}.
\]

Proof. The proof is similar to that of Lemma 8.

4.6.1 The constituent kernels of $W^{b,t}_{a,s}$

In this subsection the kernels $A, B, C$ and $D$ of the constituent operators of $W^{b,t}_{a,s}$ are expressed as polynomials in the variable $x$ and $y$, and the fixed parameters $a, b, s$ and $t$.

\[
A = \lambda^{2} \langle \chi^{t}_{a}, \exp \left( \lambda^{2}(y-x)\Delta^{s}_{a}\right) \chi^{t}_{a} \rangle
\]

\[
= \lambda^{2} \langle \chi^{t}_{a}, \sum_{n=0}^{\infty} \frac{\lambda^{2n}(y-x)^{n}}{n!} (\Delta^{s}_{a})^{n} \chi^{t}_{a} \rangle
\]

\[
= \lambda^{2} \langle \chi^{t}_{a}, \sum_{n=0}^{\infty} \frac{\lambda^{2n}(y-x)^{n}}{n!} (t-s)^{n+1} \rangle,
\]

where the exponential has been expanded from the first to second line, and Lemma 9 has been used in the last line.
\[ B = -\lambda \chi_b^y(x) \left( \exp \left( \lambda^2 (b - x) \Delta_t^a \right) \chi_t^y(y) \right) = -\lambda \chi_b^y(x) \chi_t^y(y) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(b - x)^n}{n!} \frac{(y - s)^n}{n!}, \]

where Lemma 8 has been used in the last line.

\[ C = -\lambda \chi_b^y(y) \left( \exp \left( \lambda^2 (y - a) \Delta_t^a \right) \chi_t^x(x) \right) = -\lambda \chi_b^y(y) \chi_t^x(x) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(y - a)^n}{n!} \frac{(t - x)^n}{n!}, \]

where Lemma 9 has been used in the last line.

\[ D = \text{Ker} \left( \exp \left( \lambda^2 (b - a) \Delta_t^a \right) - \text{id} \right) = \lambda^2 <_a^t (x, y) \sum_{n=1}^{\infty} \lambda^{2n} \frac{(b - a)^n}{n!} \frac{((\Delta_y^a)^{n-1})^h \chi_t^x(x)}{(n-1)!} = \lambda^2 <_a^t (x, y) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(b - a)^{n+1}}{(n+1)!} \frac{(y - x)^n}{n!}, \]

where the exponential has been expanded in the second line from the top, Lemma 9 has been used in the third line, and the summation has been re-indexed to start from 0 rather than 1 in the last line.

**4.6.2 The constituent kernels of \( \tilde{W}^{b, t}_{a,s} \)**

In this subsection the kernels for the forward forward case (where first the limit as \( m \) goes to infinity is taken and then secondly when \( n \) goes to infinity) are calculated. The explanation of the calculations is similar those done in the previous subsection (Section 4.6.1).

\[ A = \text{Ker}(\exp \left( \lambda^2 (t - s) \Delta_y^a \right) - \text{id}) = \lambda^2 <_a^h (x, y) \sum_{n=1}^{\infty} \lambda^{2n} \frac{(t - s)^n}{n!} \frac{((\Delta_y^a)^{n-1})^h \chi_y^a}{(n-1)!} = \lambda^2 <_a^h (x, y) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(t - s)^{n+1}}{(n+1)!} \frac{(y - x)^n}{n!} \]
\[ B = -\lambda \chi_t^b (y) \left( \exp (\lambda(y - s)\Delta_a^b) \chi_a^b \right)(x) \]
\[ = -\lambda \chi_t^b (x) \chi_s^t (y) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(y - s)^n (b - x)^n}{n!} \]

\[ C = -\lambda \chi_t^b (x) \left( \exp (\lambda(t - x)\Delta_b^a) \chi_a^b \right)(y) \]
\[ = -\lambda \chi_t^b (y) \chi_s^t (x) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(t - x)^n (y - a)^n}{n!} \]

\[ D = \lambda^2 \chi_t^a (x,y) \left( \chi_a^b, \exp (\lambda^2(y - x)\Delta_a^b) \chi_a^b \right) \]
\[ = \lambda^2 \chi_t^a (x,y) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(y - x)^n (b - a)^n}{n! (n+1)!} \]

**Theorem 17.** The kernels \( A, B, C \) and \( D \) of the constitute operators of
\[ W_{a,s}^{b,t} = \lim_{m \to \infty} \lim_{n \to \infty} \rightarrow \left( m,n \right)(\theta_{m,n}) \]
and
\[ \tilde{W}_{a,s}^{b,t} = \lim_{n \to \infty} \lim_{m \to \infty} \rightarrow \left( m,n \right)(\theta_{m,n}) \]
are the same. As are those of
\[ W_{b,s}^{a,t} = \lim_{m \to \infty} \lim_{n \to \infty} \leftarrow \left( m,n \right)(\theta_{m,n}) \]
and
\[ \tilde{W}_{b,s}^{a,t} = \lim_{n \to \infty} \lim_{m \to \infty} \leftarrow \left( m,n \right)(\theta_{m,n}), \]
those of
\[ W_{a,t}^{b,s} = \lim_{m \to \infty} \lim_{n \to \infty} \leftarrow \left( m,n \right)(\theta_{m,n}) \]
and
\[ \tilde{W}_{a,t}^{b,s} = \lim_{n \to \infty} \lim_{m \to \infty} \leftarrow \left( m,n \right)(\theta_{m,n}), \]
and those of
\[ W_{b,t}^{a,s} = \lim_{m \to \infty} \lim_{n \to \infty} \rightarrow \left( m,n \right)(\theta_{m,n}) \]
and
\[ \tilde{W}_{b,t}^{a,s} = \lim_{n \to \infty} \lim_{m \to \infty} \rightarrow \left( m,n \right)(\theta_{m,n}). \]
Proof. The first part of the theorem was proved by the calculations in this subsection and those of the previous (Sections 4.6.1 and 4.6.2). The rest of the proof follows a similar vein; a brief summary of the calculations follows in the subsequent six subsections (Sections 4.6.3, 4.6.4, 4.6.5, 4.6.6, 4.6.7 and 4.6.8).

4.6.3 The constituent kernels of $W_{a,s}^{a,s}$

\[
A = \lambda^2 >_a (x, y) \langle \chi_s, \exp (\lambda^2 (x - y) \Delta_s^t) \chi_s \rangle \\
= \lambda^2 >_a (x, y) \sum_{n=0}^{\infty} \lambda^{2n} (x - y)^n \frac{(t - s)^{n+1}}{n! (n+1)!} \\

B = -\lambda \chi_a^b (x) \langle \exp (\lambda (x - a) \Delta_s^t \chi_s) \rangle (y) \\
= -\lambda \chi_a^b (x) \chi_s^b (y) \sum_{n=0}^{\infty} \lambda^{2n} (x - a)^n \frac{(t - y)^n}{n!} \\

C = -\lambda \chi_a^b (y) \langle \exp (\lambda^2 (b - y) \Delta_s^t \chi_s) \rangle (x) \\
= -\lambda \chi_a^b (y) \chi_s^b (x) \sum_{n=0}^{\infty} \lambda^{2n} (b - y)^n \frac{(x - s)^n}{n!} \\

D = \text{Ker} (\exp (\lambda^2 (b - a) \Delta_s^t) - \text{id}) \\
= \lambda^2 >_s (x, y) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(b - a)^{n+1}}{(n+1)!} \frac{(x - y)^n}{n!} \\

4.6.4 The constituent kernels of $\tilde{W}_{b,t}^{a,s}$

\[
A = \text{Ker} (\exp (\lambda^2 (t - s) \Delta_a^b) - \text{id}) \\
= \lambda^2 >_a (x, y) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(t - s)^{n+1}}{(n+1)!} \frac{(x - y)^n}{n!} \\

B = -\lambda \chi_a^b (y) \exp (\lambda^2 (t - y) \Delta_a^b) \chi_a^b (x) \\
= -\lambda \chi_a^b (x) \chi_s^b (y) \sum_{n=0}^{\infty} \lambda^{2n} \frac{(t - y)^n}{n!} \frac{(x - a)^n}{n!}
\[ C = -\lambda \chi_a^t(x)(\exp(\lambda(x-s)\Delta_a^s)\chi_b^t)(y) \]
\[ = -\lambda \chi_a^b(y)\chi_a^t(x) \sum_{n=0}^{\infty} \frac{\lambda^{2n}(x-s)^n(b-y)^n}{n!} \]
\[ D = \lambda^2 >_t^s (x,y)(\chi_a^b,\exp(\lambda^2(x-y)\Delta_a^s)\chi_a^b) \]
\[ = \lambda^2 >_t^s (x,y) \sum_{n=0}^{\infty} \frac{\lambda^{2n}(x-y)^n(b-a)^{n+1}}{(n+1)!} \]

4.6.5 The constituent kernels of \( W^{b,s}_{a,t} \)

\[ A = \lambda^2 <_t^s (x,y)(\chi_a^t,\exp(\lambda^2(y-x)\Delta_s^t)\chi_a^t) \]
\[ = \lambda^2 <_t^s (x,y) \sum_{n=0}^{\infty} \frac{\lambda^{2n}(y-x)^n(t-s)^{n+1}}{(n+1)!} \]
\[ B = -\lambda \chi_a^b(x)(\exp(\lambda^2(b-x)\Delta_s^t)\chi_a^b)(y) \]
\[ = \lambda \chi_a^b(x)\chi_a^t(y) \sum_{n=0}^{\infty} \frac{\lambda^{2n}(b-x)^n(t-y)^n}{n!} \]
\[ C = -\lambda \chi_a^b(y)(\exp(\lambda^2(y-a)\Delta_s^t)\chi_a^t)(x) \]
\[ = -\lambda \chi_a^t(x)\chi_a^b(y) \sum_{n=0}^{\infty} \frac{\lambda^{2n}(y-a)^n(x-s)^n}{n!} \]
\[ D = \text{Ker}(\exp(\lambda^2(b-a)\Delta_s^t) - \text{id}) \]
\[ = \lambda^2 >_t^s (x,y) \sum_{n=0}^{\infty} \frac{\lambda^{2n}(b-a)^{n+1}(x-y)^n}{(n+1)!} \]

4.6.6 The constituent kernels of \( \tilde{W}^{b,s}_{a,t} \)

\[ A = \text{Ker}(\exp(\lambda^2(t-s)\Delta_s^t) - \text{id}) \]
\[ = \lambda^2 <_t^s (x,y) \sum_{n=0}^{\infty} \frac{\lambda^{2n}(y-x)^n(t-s)^{n+1}}{(n+1)!} \]
\[ B = -\lambda \chi^t_s(y) \left( \exp \left( \lambda^2 (t - y) \Delta^t_s \right) \chi^b_a(x) \right) \]
\[ = -\lambda \chi^b_a(x) \chi^t_s(y) \sum_{n=0}^{\infty} \frac{\lambda^n (t - y)^n (b - x)^n}{n!} \]

\[ C = -\lambda \chi^b_a(x) \left( \exp \left( \lambda^2 (x - s) \Delta^b_a \right) \chi^t_s(y) \right) \]
\[ = -\lambda \chi^b_a(x) \chi^t_s(y) \sum_{n=0}^{\infty} \frac{\lambda^n (x - s)^n (y - a)^n}{n!} \]

\[ D = \lambda^2 \left< \chi^b_a, \exp \left( \lambda^2 (x - y) \Delta^b_a \right) \chi^t_s \right> \]
\[ = \lambda^2 \left< \chi^b_a, (x - y)^n \frac{(b - a)^n}{(n+1)!} \right> \]

### 4.6.7 The constituent kernels of \( W_{b,s}^{a,t} \)

\[ A = \lambda^2 \left< \chi^b_a, \chi^t_s, \exp \left( \lambda^2 (x - y) \Delta^t_s \right) \chi^b_a \right> \]
\[ = \lambda^2 \left< \chi^b_a, \chi^t_s, (x - y)^n \frac{(b - a)^n}{(n+1)!} \right> \]

\[ B = -\lambda \chi^b_a(x) \left( \exp \left( \lambda^2 (x - a) \Delta^t_s \right) \chi^s_t \right) \]
\[ = -\lambda \chi^b_a(x) \chi^t_s(y) \sum_{n=0}^{\infty} \frac{\lambda^n (x - a)^n (y - s)^n}{n!} \]

\[ C = -\lambda \chi^b_a(y) \left( \exp \left( \lambda^2 (b - y) \Delta^t_s \right) \chi^t_s \right) \]
\[ = -\lambda \chi^b_a(y) \chi^t_s(y) \sum_{n=0}^{\infty} \frac{(b - y)^n (t - x)^n}{n!} \]

\[ D = \text{Ker} \left( \exp \left( \lambda^2 (b - a) \Delta^t_s \right) - \text{id} \right) \]
\[ = \lambda^2 \left< \chi^b_a, (x - y)^n \frac{(b - a)^n}{(n+1)!} \right> \]
4.6.8 The constituent kernels of $\tilde{W}_{b,t}^{a,s}$

$$A = \text{Ker}(\exp(\lambda^2(t - s)\Delta_b^a) - \text{id})$$

$$= \lambda^2 >_b^a(x, y) \sum_{n=0}^{\infty} \frac{\lambda^2(t - s)^{n+1}}{(n+1)!} \frac{(x - y)^n}{n!}$$

$$B = -\lambda \chi_b^a(y)(\exp(\lambda^2(y - s)\Delta_b^a)\chi_b^a)(x)$$

$$= -\lambda \chi_b^a(x)\chi_b^a(y) \sum_{n=0}^{\infty} \frac{\lambda^2(y - s)^{n+1}}{(n+1)!} \frac{(x - a)^n}{n!}$$

$$C = -\lambda \chi_b^a(x)(\exp(\lambda^2(t - x)\Delta_b^a)\chi_b^a)(y)$$

$$= -\lambda \chi_b^a(y)\chi_b^a(x) \sum_{n=0}^{\infty} \frac{\lambda^2(t - x)^{n+1}}{(n+1)!} \frac{(b - y)^n}{n!}$$

$$D = \lambda^2 <^t_s(\chi_b^a, \exp(\lambda^2(y - x)\Delta_b^a)\chi_b^a)$$

$$= \lambda^2 <^t_s(x, y) \sum_{n=0}^{\infty} \frac{\lambda^2(y - x)^{n+1}}{(n+1)!} \frac{(b - a)^n}{n!}$$

4.6.9 Conclusion

Clearly the operators $W_{b,t}^{a,s}$ and $\tilde{W}_{b,t}^{a,s}$ are one for the same, and similarly so are: $W_{a,s}^{b,t}$ and $\tilde{W}_{a,s}^{b,t}$, $W_{b,s}^{a,t}$ and $\tilde{W}_{b,s}^{a,t}$, and $W_{a,t}^{b,s}$ and $\tilde{W}_{a,t}^{b,s}$. We also notice that the operators $W_{b,t}^{a,s}$, $W_{b,s}^{a,t}$, $W_{b,s}^{b,t}$ and $W_{b,t}^{b,s}$ all have Hilbert Schmidt operators for their off-diagonal and identity plus Hilbert Schmidt operators on their diagonal. Since all integral operators with complex valued measurable kernels belonging to $L^2(\mathbb{R}^2)$ are Hilbert Schmidt Yosida [1980] pp 197-198, and by the well know fact that the product of integral operator is a integral operator of the convolution of the two kernels we can conclude that $W_{b,t}^{a,s}W_{b,t}^{a,s} - 1$ is a Hilbert Schmidt operator and hence $W_{b,t}^{a,s}$ satisfies the Shale’s criterion of Theorem 12. By the same reasoning we can say that $W_{b,t}^{a,s}$, $W_{b,s}^{a,t}$, and $W_{b,s}^{b,t}$ also satisfy the Shale’s criterion.

4.7 The semi-Bogolubov conditions for the second limit

In this section we derive three necessary and sufficient conditions that the double products (equations (4.5.8), (4.5.30), (4.5.32), and (4.5.34)) of Section 4.5 have to satisfy in order for them to preserve the imaginary part of the inner product. We call the property that a transformation
preserves the imaginary part the *semi-Bogolubov property* and if an operator has this property we say it a *semi-Bogolubov transformation*. In a later section (Section 4.8) we show that the double products of Section 4.5 are also invertible and hence they are Bogolubov transformations.

Operators of the form \( W = \lim_{m \to \infty} \lim_{n \to \infty} R_{(m,n)}(\theta_{m,n}) \) resulting from the second limit, where the limit of \( n \) goes to infinity taken first and then as \( m \) goes to infinity second, also satisfy the semi-Bogolubov condition. These double products are of the form

\[
W = \begin{pmatrix}
1 + \text{Op}\{A\} & \kappa \text{Op}\{B\} \\
\kappa \text{Op}\{C\} & \Xi
\end{pmatrix},
\]

where Op\{A\}, Op\{B\}, Op\{C\} are the integral operators with kernels \( A, B \) and \( C \) respectively.

**Remark 23.** The conditions which a semi-Bogolubov transformation of the form

\[ \tilde{W} = \lim_{n \to \infty} \lim_{m \to \infty} R_{(m,n)}(\theta_{m,n}) \]

would be derived in the same way. However, we do not derive the conditions as each \( \tilde{W} \) transformation was shown to be also a \( W \) transformation, and verse versa.

Taking an arbitrary vector \( \begin{pmatrix} f \\ g \end{pmatrix} \) in \( \mathfrak{h} \oplus \mathfrak{h} \) and calculating the action of \( \lim_{n \to \infty} \lim_{m \to \infty} R_{(m,n)}(\theta_{m,n}) \) on the vector gives

\[
\begin{pmatrix}
1 + \text{Op}\{A\} & \kappa \text{Op}\{B\} \\
\kappa \text{Op}\{C\} & \Xi
\end{pmatrix}
\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} (1 + \text{Op}\{A\})f + \kappa \text{Op}\{B\}g \\ \kappa \text{Op}\{C\}f + \Xi g \end{pmatrix}.
\]

Similarly any other arbitrary vector \( \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix} \) in \( \mathfrak{h} \oplus \mathfrak{h} \) we have

\[
\begin{pmatrix}
1 + \text{Op}\{A\} & \kappa \text{Op}\{B\} \\
\kappa \text{Op}\{C\} & \Xi
\end{pmatrix}
\begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix} = \begin{pmatrix} (1 + \text{Op}\{A\})\tilde{f} + \kappa \text{Op}\{B\}\tilde{g} \\ \kappa \text{Op}\{C\}\tilde{f} + \Xi \tilde{g} \end{pmatrix}.
\]
Now taking the inner-product of the two resulting vectors we have

\[
\left\langle \left( \begin{array}{c}
(1 + \text{Op}(A))f \\
\kappa \text{Op}(C)f \\
\Xi g
\end{array} \right), \left( \begin{array}{c}
(1 + \text{Op}(A))\tilde{f} \\
\kappa \text{Op}(C)\tilde{f} \\
\Xi \tilde{g}
\end{array} \right) \right\rangle
= \left\langle (1 + \text{Op}(A))f + \kappa \text{Op}(B)\tilde{g}, (1 + \text{Op}(A))\tilde{f} + \kappa \text{Op}(B)\tilde{g} \right\rangle_{L^2}
+ \left\langle \kappa \text{Op}(C)f + \Xi g, \kappa \text{Op}(C)\tilde{f} + \Xi \tilde{g} \right\rangle_{L^2}
= \langle f, \tilde{f} \rangle + \langle f, \text{Op}(A)\tilde{f} \rangle + \langle f, \kappa \text{Op}(B)\tilde{g} \rangle + \langle \text{Op}(A)f, \tilde{f} \rangle + \langle \text{Op}(A)f, \text{Op}(A)\tilde{f} \rangle
+ \langle \kappa \text{Op}(B)\tilde{g}, \text{Op}(A)\tilde{f} \rangle + \langle \kappa \text{Op}(B)g, \text{Op}(A)\tilde{f} \rangle
+ \langle \text{Op}(B)g, \text{Op}(B)\tilde{g} \rangle + \langle \text{Op}(C)f, \text{Op}(C)\tilde{f} \rangle
+ \langle \kappa \text{Op}(C)f, \Xi \tilde{g} \rangle + \langle \Xi g, \kappa \text{Op}(C)\tilde{f} \rangle + \langle \Xi g, \Xi \tilde{g} \rangle.
\]

This calculation results in a rather complicated expression of which the imaginary part of the right hand side is equal to \(\text{Im}\{ \langle f, \tilde{f} \rangle + \langle g, \tilde{g} \rangle \}\) if the semi-Bogolubov conditions hold. Explicitly if the operator preserves the imaginary part then,

\[
\text{Im}\{ \langle f, \tilde{f} \rangle + \langle f, \text{Op}(A)\tilde{f} \rangle + \langle f, \kappa \text{Op}(B)\tilde{g} \rangle + \langle \text{Op}(A)f, \tilde{f} \rangle + \langle \text{Op}(A)f, \text{Op}(A)\tilde{f} \rangle
+ \langle \kappa \text{Op}(B)\tilde{g}, \text{Op}(A)\tilde{f} \rangle + \langle \kappa \text{Op}(B)g, \text{Op}(A)\tilde{f} \rangle
+ \langle \text{Op}(B)g, \text{Op}(B)\tilde{g} \rangle + \langle \text{Op}(C)f, \text{Op}(C)\tilde{f} \rangle
+ \langle \kappa \text{Op}(C)f, \Xi \tilde{g} \rangle + \langle \Xi g, \kappa \text{Op}(C)\tilde{f} \rangle + \langle \Xi g, \Xi \tilde{g} \rangle \}
= \text{Im}\{ \langle f, \tilde{f} \rangle + \langle g, \tilde{g} \rangle \}
\]

which simplifies to, by taking \(\text{Im}\{f, \tilde{f}\}\) away from both sides,

\[
\text{Im}\{ \langle f, \text{Op}(A)\tilde{f} \rangle + \langle f, \kappa \text{Op}(B)\tilde{g} \rangle + \langle \text{Op}(A)f, \tilde{f} \rangle + \langle \text{Op}(A)f, \text{Op}(A)\tilde{f} \rangle
+ \langle \kappa \text{Op}(B)\tilde{g}, \text{Op}(A)\tilde{f} \rangle + \langle \kappa \text{Op}(B)g, \text{Op}(A)\tilde{f} \rangle
+ \langle \text{Op}(B)g, \text{Op}(B)\tilde{g} \rangle + \langle \text{Op}(C)f, \text{Op}(C)\tilde{f} \rangle
+ \langle \kappa \text{Op}(C)f, \Xi \tilde{g} \rangle + \langle \Xi g, \kappa \text{Op}(C)\tilde{f} \rangle + \langle \Xi g, \Xi \tilde{g} \rangle \}
= \text{Im}\{g, \tilde{g}\}. \quad (4.7.1)
\]

The first necessary condition for the semi-Bogolubov property is now derived. Setting \(f = \tilde{f} = 0\) in equation (4.7.1) gives

\[
\text{Im}\{ -\langle \text{Op}(B)g, \text{Op}(B)\tilde{g} \rangle + \{ \langle \Xi g, \Xi \tilde{g} \} \} = \text{Im}\{g, \tilde{g}\}. \quad (4.7.2)
\]
Now set $\tilde{g} = ig$ in the above equation (4.7.2) gives

$$\text{Im} \left\{ -i \langle \text{Op}\{B\} g, \text{Op}\{B\} g \rangle + i \langle \Xi, \Xi \rangle \right\} = \text{Im} \left\{ i \langle g, g \rangle \right\},$$

$$\text{Im} \left\{ -i \langle g, \text{Op}\{B\}^* \text{Op}\{B\} g \rangle + i \langle g, \Xi^* \Xi \rangle \right\} = \text{Im} \left\{ i \langle g, g \rangle \right\},$$

$$\text{Re} \left\{ -\langle g, \text{Op}\{B\}^* \text{Op}\{B\} g \rangle + \langle g, \Xi^* \Xi \rangle \right\} = \langle g, g \rangle.$$ 

However, setting $\tilde{g} = g$ in (4.7.2) gives

$$\text{Im} \left\{ -\langle g, \text{Op}\{B\}^* \text{Op}\{B\} g \rangle + \langle g, \Xi^* \Xi \rangle \right\} = 0$$

and hence the imaginary part of the inner product is zero. Adding real and imaginary parts and using the fact that the vectors $g$ in $\mathfrak{h}$ are chosen arbitrary gives the first necessary condition

$$- \text{Op}\{B\}^* \text{Op}\{B\} + \Xi^* \Xi = I \quad (4.7.3)$$

The second condition is now derived. Setting $g = \tilde{g} = 0$ in (4.7.1) yields

$$\text{Im} \left\{ \langle \text{Op}\{A\} f, \text{Op}\{A\} \tilde{f} \rangle + \langle \text{Op}\{A\} f, \tilde{f} \rangle + \langle f, \text{Op}\{A\} \tilde{f} \rangle 
+ \langle \text{Op}\{C\} f, \text{Op}\{C\} \tilde{f} \rangle \right\} = 0. \quad (4.7.4)$$

Then setting $\tilde{f} = f$ in (4.7.4) gives

$$\text{Im} \left\{ \langle f, \text{Op}\{A\}^* \text{Op}\{A\} f \rangle + \langle f, \text{Op}\{A\} f \rangle + \langle f, \text{Op}\{A\}^* \text{Op}\{A\} f \rangle + \langle f, \text{Op}\{C\}^* \text{Op}\{C\} f \rangle = 0 \right\},$$

and setting $\tilde{f} = if$ in (4.7.4) gives

$$\text{Im} \left\{ i \langle f, \text{Op}\{A\}^* \text{Op}\{A\} f \rangle + i \langle f, \text{Op}\{A\} f \rangle 
+ i \langle f, \text{Op}\{A\}^* \text{Op}\{A\} f \rangle - i \langle f, \text{Op}\{C\}^* \text{Op}\{C\} f \rangle \right\} = \text{Re} \left\{ \langle f, \text{Op}\{A\}^* \text{Op}\{A\} f \rangle + \langle f, \text{Op}\{A\} f \rangle 
+ i \langle f, \text{Op}\{A\}^* \text{Op}\{A\} f \rangle - \langle f, \text{Op}\{C\}^* \text{Op}\{C\} f \rangle = 0 \right\}.$$ 

Therefore the second condition is

$$\text{Op}\{A\}^* \text{Op}\{A\} + \text{Op}\{A\}^* + \text{Op}\{A\} - \text{Op}\{C\}^* \text{Op}\{C\} = 0. \quad (4.7.5)$$
The third and final necessary condition is derived by setting \( f = \tilde{g} = 0 \) in (4.7.1)

\[
\text{Im}\left\{ \langle \kappa \text{Op}(B)g, \tilde{f} \rangle + \langle \kappa \text{Op}(B)g, \text{Op}(A)\tilde{f} \rangle + \langle \Xi g, \kappa \text{Op}(C)\tilde{f} \rangle \right\} = 0 \tag{4.7.6}
\]

and then taking \( \tilde{f} = g \) in (4.7.6) gives

\[
\text{Im}\left\{ \langle g, \kappa \text{Op}(B)^*g \rangle + \langle g, \kappa \text{Op}(B)^*\text{Op}(A)g \rangle + \langle g, \Xi^*\kappa \text{Op}(C)g \rangle \right\} = 0,
\]

and then taking \( \tilde{f} = ig \) in (4.7.6) we have

\[
\text{Re}\left\{ -\langle g, \kappa \text{Op}(B)^*g \rangle - \langle g, \kappa \text{Op}(B)^*\text{Op}(A)g \rangle + \langle g, \Xi^*\kappa \text{Op}(C)g \rangle \right\} = 0.
\]

Since \( g \) is an arbitrary vector the condition is

\[
\text{Op}(B)^* + \text{Op}(B)^*\text{Op}(A) - \Xi^*\text{Op}(C) = 0. \tag{4.7.7}
\]

**Theorem 18.** A necessary and sufficient condition that

\[
\begin{pmatrix}
1 + \text{Op}(A) & \kappa \text{Op}(B) \\
\kappa \text{Op}(C) & \Xi
\end{pmatrix}
\]

is a semi-Bogolubov transformation on \( \mathfrak{h} \oplus \mathfrak{h} \) is that (4.7.3), (4.7.5) and (4.7.7) hold.

**Proof.** The three conditions are:

\[
\begin{align*}
-\text{Op}(B)^*\text{Op}(B) + \Xi^*\Xi &= I, \quad (4.7.8) \\
\text{Op}(A)^*\text{Op}(A) + \text{Op}(A)^* + \text{Op}(A) - \text{Op}(C)^*\text{Op}(C) &= 0, \quad (4.7.9) \\
\text{Op}(B)^* + \text{Op}(B)^*\text{Op}(A) - \Xi^*\text{Op}(C) &= 0. \quad (4.7.10)
\end{align*}
\]

In order to prove sufficiency the right hand side of

\[
\text{Im}\left\{ \left( \begin{array}{cc}
(1 + \text{Op}(A))f & \kappa \text{Op}(B)g \\
\kappa \text{Op}(C)f & \Xi g
\end{array} \right) \cdot \left( \begin{array}{c}
(1 + \text{Op}(A))\tilde{f}

\kappa \text{Op}(C)\tilde{f} \\
\Xi \tilde{g}
\end{array} \right) \right\}
\]

\[
= \text{Im}\{ \langle f, \tilde{f} \rangle + \langle f, \text{Op}(A)\tilde{f} \rangle + \langle f, \kappa \text{Op}(B)\tilde{g} \rangle + \langle \text{Op}(A)f, \tilde{f} \rangle + \langle \text{Op}(A)f, \text{Op}(A)\tilde{f} \rangle

+ \langle \text{Op}(A)f, \kappa \text{Op}(B)\tilde{g} \rangle + \langle \kappa \text{Op}(B)g, \tilde{f} \rangle + \langle \kappa \text{Op}(B)g, \text{Op}(A)\tilde{f} \rangle

+ \langle \text{Op}(B)g, \text{Op}(B)\tilde{g} \rangle + \langle \text{Op}(C)f, \text{Op}(C)\tilde{f} \rangle + \langle \kappa \text{Op}(C)f, \Xi \tilde{g} \rangle

+ \langle \kappa \text{Op}(C)f, \Xi \tilde{g} \rangle \}
\tag{4.7.11}
\]

must be shown equal to \( \text{Im}\{ \langle f, \tilde{f} \rangle + \langle g, \tilde{g} \rangle \} \). Using (4.7.8) the sum of the ninth and thirteenth terms of (4.7.11) are equal to \( \text{Im}\{ \langle g, \tilde{g} \rangle \} \). Using (4.7.9) the second, fourth, fifth and tenth terms sum to zero. Using (4.7.10) the third, sixth and eleventh terms sum to zero, as do the seventh,
eighth and twelfth by the same condition. Hence the right hand side sums to \( \text{Im}\{(f, \tilde{f}) + (g, \tilde{g})\} \) as expected.

\[ \square \]

**Remark 24.** With some work the conditions of Theorem 18 can be derived from those on pp26 of Derezinski [2003]

### 4.8 The second limits are semi-Bogolubov transformations

**Theorem 19.** The operators (4.5.8), (4.5.30), (4.5.32) and (4.5.34) are all semi-Bogolubov transformations.

Before we prove the theorem we will need the following lemmas:

**Lemma 10.** Given a kernel \( K \in L^2(\mathbb{R} \times \mathbb{R}) \) expressible in the form \( K(x,y) = \phi_1(x)\phi_2(y) \), where \( \phi_1, \phi_2 \in L^2(\mathbb{R}) \), and a linear operator \( \mathcal{L} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \). Then

\[
\text{Op}\{(x,y) \mapsto \phi_1(x)\phi_2(y)\}\mathcal{L} = \text{Op}\{(x,y) \mapsto \phi_1(x)\mathcal{L}^*\phi_2(y)\}.
\]

**Proof.**

\[
\text{Op}\{(x,y) \mapsto \phi_1(x)\phi_2(y)\}\mathcal{L} f = \int_\mathbb{R} \phi_1(x)\phi_2(y)\mathcal{L} f(y) \, dy \\
= \int_\mathbb{R} \phi_1(x)\phi_2(y)\mathcal{L} f(y) \, dy = \int_\mathbb{R} \phi_1(x)\mathcal{L}^*\phi_2(y)f(y) \, dy \\
= \text{Op}\{(x,y) \mapsto \phi_1(x)\mathcal{L}^*\phi_2(y)\}f
\]

(4.8.1)

**Lemma 11.** Given kernels \( K_1, K_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) in \( L^2(\mathbb{R}^2) \) we have

\[
\text{Op}\{(x,y) \mapsto K_1(x,y)\}\text{Op}\{(x,y) \mapsto K_2(x,y)\} = \text{Op}\{(x,y) \mapsto \int_\mathbb{R} K_1(x,z)K_2(z,y)dz\}.
\]

**Proof.**

\[
\text{Op}\{(x,y) \mapsto K_1(x,y)\}\text{Op}\{(x,y) \mapsto K_2(x,y)\} f \\
= \int_\mathbb{R} K_1(x,t)\int_\mathbb{R} K_2(t,y)f(y)dy \, dt \\
= \int_\mathbb{R} \int_\mathbb{R} K_1(x,t)K_2(t,y)f(y)dt \, dy = \text{Op}\{(x,y) \mapsto \int_\mathbb{R} K_1(x,t)K_2(t,y)dt\} f
\]

(4.8.3)
Lemma 12. The adjoint of the integral operator with kernel $K \in L^2(\mathbb{R} \times \mathbb{R})$ is given by the formula
\[
\text{Op}\{ (x, y) \mapsto K(x, y) \}^* = \text{Op}\{ (x, y) \mapsto \overline{K(y, x)} \}.
\] (4.8.5)

Proof.
\[
\langle f, \text{Op}\{ (x, y) \mapsto K(x, y) \}^* g \rangle = \int_\mathbb{R} \int_\mathbb{R} f(y) \overline{K(x, y)} g(x) \, dy \, dx
= \int_\mathbb{R} \int_\mathbb{R} \overline{K(x, y)} g(x) \, dx \, dy = \langle f, \text{Op}\{ (x, y) \mapsto \overline{K(y, x)} \} g \rangle
\] (4.8.6)

Lemma 13.
\[
\Delta_t^+ + \Delta_t^- = \text{Op}\{ (x, y) \mapsto \chi_t^+(x) \chi_t^-(y) \}
\] (4.8.7)

Proof.
\[
\Delta_t^+ + \Delta_t^- = \text{Op}\{ (x, y) \mapsto >_{\Delta_t^+}^1 (x, y) + <_{\Delta_t^-}^1 (x, y) \}
= \text{Op}\{ (x, y) \mapsto \chi_t^+(x) \chi_t^-(y) \}
\]

Lemma 14.
\[
(\Delta_t^+)^* = \Delta_t^-
\] (4.8.8)

Proof.
\[
\langle \Delta_t^+ f, g \rangle = \int_s^t \int_x f(y) \, dy \, g(x) \, dx = \int_s^t \int_y \overline{f(y)} \, g(x) \, dx \, dy
\] (4.8.9)

Lemma 15.
\[
>_{\Delta_t^+}^b (x, y) = <_{\Delta_t^+}^a (y, x)
\] (4.8.10)

Proof.
\[
\chi_{\Delta_t^+}^b(x) \chi_{\Delta_t^+}^a(y) = \chi_{\Delta_t^-}^a(x) \chi_{\Delta_t^-}^b(y)
\]

Lemma 16. For $f \in L^2(\mathbb{R})$
\[
\int_a^b >_{\Delta_t^+}^b (x, z) >_{\Delta_t^-}^b (y, z) f(z) \, dz = \left( \int_a^x \chi_{<x<y<b} + \int_y^b \chi_{b>y>a} \right) f(z) \, dz
= \left( \int_a^x \chi_{\Delta_t^+}^b(x) \chi_{\Delta_t^-}^b(y) + \int_a^b \chi_{\Delta_t^+}^b(x) \chi_{\Delta_t^-}^b(y) \right) f(z) \, dz.
\] (4.8.11)
Proof. 
\[
\int_a^b \chi_a^b(x,z) \chi_a^b(y,z) f(z) \, dz = \int_a^\min\{x,y\} \chi_a^b(z) f(z) \, dz = \left( \int_a^x \chi_a<x<y<b + \int_a^y \chi_a>b>y>a \right) f(z) \, dz. \tag{4.8.12}
\]

We now go on to prove the theorem.

Proof. We show that the forward forward second limit 
\[
W_{b,t}^{a,s} = \lim_{m \to \infty} \lim_{n \to \infty} \mathcal{R}_{(m,n)}(\theta_{m,n})
\]
is a semi-Bogolubov transformation. The other cases: \(W_{a,t}^{b,s}\), \(W_{b,s}^{a,t}\) and \(W_{a,s}^{b,t}\) are similar. The constituent integral operators of \(W_{b,t}^{a,s}\) are:
\[
\begin{align*}
\text{Op}\{A\} &= \text{Op}\{(x,y) \to \lambda^2 \chi_a^b(x,y) \exp(\lambda^2 (y-x) \Delta^b_a) \chi_a^b)\}, \\
\text{Op}\{B\} &= \text{Op}\{(x,y) \to -\lambda \chi_a^b(x) \exp(\lambda^2 (b-x) \Delta^b_a) \chi_a^b(y)\}, \\
\text{Op}\{C\} &= \text{Op}\{(x,y) \to -\lambda \chi_a^b(y) \exp(\lambda^2 (y-a) \Delta^b_a) \chi_a^b(x)\},
\end{align*}
\]
and \(\Xi = \exp(\lambda^2 (b-a) \Delta^b_a)\).

The first semi-Bogolubov condition that we show to be satisfied is
\[-\text{Op}\{B\}^*\{B\} + \Xi^* \Xi = I.\]

We now compute the corresponding terms of the conditions. We first consider the product of integral operators \(\text{Op}\{B\}^* \text{Op}\{B\}\) and express the product as an integral operator as follows
\[
\begin{align*}
\text{Op}\{B\}^* \text{Op}\{B\} &= \text{Op}\{(x,y) \to -\lambda \chi_a^b(y) \exp(\lambda^2 (b-y) \Delta^b_a) \chi_a^b(x)\} \\
&\quad \times \text{Op}\{(x,y) \to -\lambda \chi_a^b(x) \exp(\lambda^2 (b-x) \Delta^b_a) \chi_a^b(y)\} \\
&= \text{Op}\{(x,y) \to \lambda \int_a^b \chi_a^b(z) \exp(\lambda^2 (b-z) \Delta^b_a) \chi_a^b(x) \exp(\lambda^2 (b-z) \Delta^b_a) \chi_a^b(y) \, dz\},
\end{align*}
\]
where Lemma 12 has been used on the first line and Lemma 11 on the second. Using Lemma 14 we have
\[
\Xi^* \Xi = \exp(\lambda^2 (b-a) \Delta^b_a) \exp(\lambda^2 (b-a) \Delta^b_a). \tag{4.8.13}
\]
Replacing all the $a$’s in equation (4.8.13) with $z$’s and differentiating $\Xi^*\Xi|_{z=a}$ with respect to $z$ yields
\[
\frac{\partial \Xi^*\Xi}{\partial z} \bigg|_{z=a} = -\lambda^2 \exp\left(\lambda^2(b-z)\Delta_t^s + \Delta_t^s\right) \exp\left(\lambda^2(b-z)\Delta_t^s\right) 
\]
(4.8.14)
\[
= -\lambda^2 \exp\left(\lambda^2(b-z)\Delta_t^s\right) \text{Op}\left\{(x, y) \to \chi^t_s(x, y) \exp\left(\lambda^2(b-z)\Delta_t^s\right)\right\} 
\]
\[
= \text{Op}\left\{(x, y) \to -\lambda^2 \left( \exp\left(\lambda^2(b-z)\Delta_t^s\right) \chi^t_s(x) \right) \right\} 
\]
\[
= \text{Op}\left\{(x, y) \to -\lambda^2 \left( \exp\left(\lambda^2(b-z)\Delta_t^s\right) \chi^t_s(x) \right) \right\}. 
\]
where Lemma 13 has been used on the second line and Lemma 10 on the fourth. Since $\Xi\Xi|_{a=b} = I$, we have
\[
\int_a^b \frac{\partial \Xi^*\Xi|_{a=z}}{\partial z} = I - \Xi^*\Xi 
\]
and integrating equation (4.8.14) between limits $a$ and $b$ gives
\[
\int_a^b \frac{\partial \Xi^*\Xi|_{a=z}}{\partial z} = -\text{Op}\{B\}^*\text{Op}\{B\} 
\]
and hence the condition is satisfied.

The second condition to be satisfied is
\[
\text{Op}\{A\}^*\text{Op}\{A\} + \text{Op}\{A\}^* + \text{Op}\{A\} - \text{Op}\{C\}^*\text{Op}\{C\} = 0. 
\]
(4.8.15)
First we compute the left most term of equation (4.8.15) as an integral operator using Lemmas
11, 12 and 15 as follows

\[
\text{Op}\{A\}^*\text{Op}\{A\} = \text{Op}\left\{(x, y) \to \lambda^2 \chi_a^b(x, y) \langle \chi_s^t, \exp(\lambda^2(x-y)\Delta_t^s)\chi_s^t\rangle \right\}
\]

\[
\text{Op}\left\{(x, y) \to \lambda^2 \chi_a^b(x, y) \langle \chi_s^t, \exp(\lambda^2(y-x)\Delta_t^s)\chi_s^t\rangle \right\}
\]

\[
= \text{Op}\left\{(x, y) \to \lambda^2 \int_a^b (x, z) \langle \chi_s^t, \exp(\lambda^2(x-z)\Delta_t^s)\chi_s^t\rangle \right\}
\]

\[
. \langle \chi_s^t, \exp(\lambda^2(y-z)\Delta_t^s)\chi_s^t\rangle dz \right\}
\]

\[
= \text{Op}\left\{(x, y) \to \lambda^2 \int_a^b (x, z) \langle \chi_s^t, \exp(\lambda^2(x-z)\Delta_t^s)\chi_s^t\rangle \right\}
\]

\[
. \langle \chi_s^t, \exp(\lambda^2(y-z)\Delta_t^s)\chi_s^t\rangle dz \right\}
\]

\[
= \text{Op}\left\{(x, y) \to \lambda^2 \left( \int_a^x \chi_s^t + \int_x^b \chi_s^t \right) \langle \chi_s^t, \exp(\lambda^2(x-z)\Delta_t^s)\chi_s^t\rangle \right\}
\]

\[
(\chi_s^t, \exp(\lambda^2(y-z)\Delta_t^s)\chi_s^t) dz \right\}. \quad (4.8.16)
\]

where Lemma 16 has been used on the last line. We use Lemmas 11 and 12 to express the product of the operator \text{Op}\{C\} against its adjoint (from the left) as follows

\[
\text{Op}\{C\}^*\text{Op}\{C\} = \text{Op}\left\{(x, y) \to -\lambda^2 \chi_a^b(x) \langle \exp(\lambda^2(x-a)\Delta_t^s)\chi_s^t\rangle(y) \right\}
\]

\[
. \langle \chi_s^t, \exp(\lambda^2(y-a)\Delta_t^s)\chi_s^t\rangle(x) \right\}
\]

\[
= \text{Op}\left\{(x, y) \to -\lambda^2 \chi_a^b(y) \langle \exp(\lambda^2(y-a)\Delta_t^s)\chi_s^t\rangle(x) \right\}
\]

\[
. \langle \exp(\lambda^2(x-a)\Delta_t^s)\chi_s^t\rangle(z) \right\}
\]

\[
= \text{Op}\left\{(x, y) \to \lambda^2 \chi_a^b(x) \chi_a^b(y) \int_a^y \langle \exp(\lambda^2(x-a)\Delta_t^s)\chi_s^t\rangle(z) \right\}
\]

\[
. \langle \exp(\lambda^2(x-a)\Delta_t^s)\chi_s^t, \exp(\lambda^2(y-a)\Delta_t^s)\chi_s^t\rangle \right\}. \quad (4.8.17)
\]

where Lemma 10 has been used on the last line. Removing the factor \chi_a^b(x)\chi_a^b(y) from equation (4.8.17), replacing the variable \alpha with \beta in the resulting operator and differentiating with respect to \beta we have
\[ \frac{\partial}{\partial z} \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \langle \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_x, \exp(\lambda^2(y-z)\Delta_t^y)\chi^t_y \rangle \right\} \]
\[ = \text{Op}\left\{ (x, y) \rightarrow -\lambda^4 \langle \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_x, (\Delta_t^x + \Delta_t^y) \exp(\lambda^2(y-z)\Delta_t^y)\chi^t_y \rangle \right\} \]
\[ = \text{Op}\left\{ (x, y) \rightarrow -\lambda^4 \langle \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_x, \chi^t_x \rangle, \exp(\lambda^2(y-z)\Delta_t^y)\chi^t_y \rangle \right\} \]
\[ = \text{Op}\left\{ (x, y) \rightarrow -\lambda^4 \langle \chi^t_x, \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_y \rangle \right\} \]
\[ \langle \chi^t_x, \exp(\lambda^2(y-z)\Delta_t^y)\chi^t_y \rangle \right\}. \] (4.8.18)

where Lemma 14 has been used on the second line and Lemma 13 has been used on the third line. Notice that the kernel of (4.8.16) is an integral of the resulting operator's kernel (equation (4.8.18)) multiplied by negative one. We now integrate equation (4.8.18) between the limits \( a \) to \( \min \{ x, y \} \) as follows

\[ - \text{Op}\{A\}^* \text{Op}\{A\} = \left( \int_a^x \chi_{a<x<y<b} + \int_{y>b>x>y>a} \right) \frac{\partial}{\partial z} \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \langle \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_x, \exp(\lambda^2(y-z)\Delta_t^y)\chi^t_y \rangle \right\} dz \]
\[ = \left[ \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{a<x<y<b} \langle \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_x, \exp(\lambda^2(y-z)\Delta_t^y)\chi^t_y \rangle \right\} \right]_{z=a}^{z=x} \]
\[ + \left[ \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{b>x>y>a} \langle \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_x, \exp(\lambda^2(y-z)\Delta_t^y)\chi^t_y \rangle \right\} \right]_{z=a}^{z=y} \]
\[ = \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{a<x<y<b} \langle \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_x \rangle \right\} \]
\[ + \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{b>x>y>a} \langle \exp(\lambda^2(x-z)\Delta_t^x)\chi^t_x \rangle \right\} \]
\[ - \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{a<x<y<b} \langle \exp(\lambda^2(x-a)\Delta_t^x)\chi^t_x, \exp(\lambda^2(y-a)\Delta_t^y)\chi^t_y \rangle \right\} \]
\[ + \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{b>x>y>a} \langle \exp(\lambda^2(x-a)\Delta_t^x)\chi^t_x, \exp(\lambda^2(y-a)\Delta_t^y)\chi^t_y \rangle \right\} \]
\[ = - \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{a<x<y<b} \langle \exp(\lambda^2(x-a)\Delta_t^x)\chi^t_x, \exp(\lambda^2(y-a)\Delta_t^y)\chi^t_y \rangle \right\} \]
\[ + \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{a<x<y<b} \langle \exp(\lambda^2(y-a)\Delta_t^y)\chi^t_y \rangle \right\} \]
\[ + \text{Op}\left\{ (x, y) \rightarrow \lambda^2 \chi_{b>x>y>a} \langle \exp(\lambda^2(x-a)\Delta_t^x)\chi^t_x, \exp(\lambda^2(y-a)\Delta_t^y)\chi^t_y \rangle \right\} \]
\[ = - \text{Op}\{C\}^* \text{Op}\{C\} + \text{Op}\{A\} + \text{Op}\{A\}^*. \] (4.8.19)

Since the second and fourth term of the third equality sum to \( \text{Op}\{C\}^* \text{Op}\{C\} \). Hence the second condition is shown to hold.
The third condition $\text{Op}\{B\}^*\text{Op}\{A\} + \text{Op}\{C\} + \Xi^*\text{Op}\{C\} = 0$. We express the product of operators, the operator $\text{Op}\{B\}^*\text{Op}\{A\}$, as an integral operator using Lemmas 11 and 12 as follows

$$\text{Op}\{B\}^*\text{Op}\{A\} = \text{Op}\left\{ (x, y) \mapsto -\lambda \int_a^b \chi^b_a(z) \langle \exp(\lambda^2(b-z)\Delta^*_x) \chi^*_x \rangle(x) \right\} \cdot \lambda^2(\chi^*_x, \exp(\lambda^2(y-z)\Delta^*_x) \chi^*_x)dz \right\} = \text{Op}\left\{ (x, y) \mapsto -\lambda^3 \int_a^y \chi^b_a(y) \langle \exp(\lambda^2(b-z)\Delta^*_x) \chi^*_x \rangle(x) \right\} \cdot \langle \chi^*_x, \exp(\lambda^2(y-z)\Delta^*_x) \chi^*_x \rangle dz \right\}. \quad (4.8.20)$$

Using Lemma 14 we have

$$\Xi^*\text{Op}\{C\} = \exp(\lambda^2(b-a)\Delta^*_x)\text{Op}\left\{ (x, y) \mapsto -\lambda \left( \exp(\lambda^2(y-z)\Delta^*_x) \chi^*_x \right)(x) \right\}. \quad (4.8.21)$$

Removing the factor $\chi^b_a(y)$ from the kernel of (4.8.21), setting the parameter $a$ to $z$ in the resulting kernel and taking the derivative with respect to $z$ gives

$$\frac{\partial}{\partial z} \exp(\lambda^2(b-z)\Delta^*_x)\text{Op}\left\{ (x, y) \mapsto -\lambda \left( \exp(\lambda^2(y-z)\Delta^*_x) \chi^*_x \right)(x) \right\} = \lambda^2 \exp(\lambda^2(b-z)\Delta^*_x)\text{Op}\left\{ (x, y) \mapsto \lambda \left( \Delta^*_x + \Delta^*_x \right) \left( \exp(\lambda^2(y-z)\Delta^*_x) \chi^*_x \right)(x) \right\} = \exp(\lambda^2(b-z)\Delta^*_x)\text{Op}\left\{ (x, y) \mapsto \lambda^3 \langle \chi^*_x, \exp(\lambda^2(y-z)\Delta^*_x) \chi^*_x \rangle \right\}, \quad (4.8.22)$$

where Lemma 13 has been used on the last line. Integrating (4.8.22) between $a$ and $y$ with respect to $z$ the following can be seen to be true

$$\int_a^y \frac{\partial}{\partial z} \exp(\lambda^2(b-z)\Delta^*_x)\text{Op}\left\{ (x, y) \mapsto -\lambda \left( \exp(\lambda^2(y-z)\Delta^*_x) \chi^*_x \right)(x) \right\} = -\text{Op}\{B\}^*\text{Op}\{A\} = \exp(\lambda^2(b-y)\Delta^*_x)\text{Op}\left\{ (x, y) \mapsto -\lambda \chi^*_x(\chi^*_x)(y) \right\} + \Xi^*\text{Op}\{C\} = -\text{Op}\{B\}^* + \Xi^*\text{Op}\{C\}. \quad (4.8.23)$$

As the integral of (4.8.22) between $a$ and $y$ is the operator (4.8.20) and hence it has been shown that the operator $W^h_{a,\ast}^{b,\ast}$ has the semi-Bogolubov property.

In the next section it is shown that $W^h_{a,\ast}^{b,\ast}$ also has both a right and left inverse.
4.9 The second limits are Bogolubov transformations

**Theorem 20.** The operators (4.5.8), (4.5.30), (4.5.32) and (4.5.34) are all Bogolubov transformations.

**Proof.** We show that the second limit $W_{b,t}^{a,s}$ has an inverse. The proof for the other cases $W_{b,s}^{a,t}$, $W_{a,t}^{b,s}$ and $W_{a,s}^{b,t}$ are similar. Using obvious notation, the second limit is expressed as

\[
\left( 1 + \text{Op}\{A\} \kappa \text{Op}\{B\} \right) \left( \kappa \text{Op}\{C\} \Xi \right)
\]

Looking for the inverse of the form

\[
\left( 1 + \text{Op}\{\tilde{A}\} \kappa \text{Op}\{\tilde{B}\} \right) \left( \kappa \text{Op}\{\tilde{C}\} \Xi \right),
\]

where

\[
\begin{align*}
\text{Op}\{\tilde{A}\} &= \text{Op}\{ (x, y) \rightarrow \lambda^2 b_{a}^b (x, y) \chi_{a}^b, \exp (\lambda^2 (x - y) \Delta^b_{a}) \chi_{a}^b \} , \\
\text{Op}\{\tilde{B}\} &= \text{Op}\{ (x, y) \rightarrow \lambda \chi_{a}^b (x) \exp (\lambda^2 (x - a) \Delta^b_{s}) \chi_{a}^b (y) \} , \\
\text{Op}\{\tilde{C}\} &= \text{Op}\{ (x, y) \rightarrow \lambda \chi_{b}^b (y) \exp (\lambda^2 (b - y) \Delta^b_{t}) \chi_{a}^b (x) \}, \\
\text{and} \quad \Xi &= \exp (\lambda^2 (b - a) \Delta^b_{s}).
\end{align*}
\]

Multiplying $W_{b,t}^{a,s}$ by its proposed inverse

\[
\left( 1 + \text{Op}\{A\} \kappa \text{Op}\{B\} \right) \left( \kappa \text{Op}\{C\} \Xi \right) = \left( 1 + \text{Op}\{\tilde{A}\} \kappa \text{Op}\{\tilde{B}\} \right) \left( \kappa \text{Op}\{\tilde{C}\} \Xi \right)
\]

(4.9.1)

\[
\begin{align*}
\text{Op}\{A\} + \text{Op}\{\tilde{A}\} + \text{Op}\{A\} \text{Op}\{\tilde{A}\} + \text{Op}\{B\} \text{Op}\{\tilde{C}\} &= 0, \\
\text{Op}\{\tilde{B}\} + \text{Op}\{A\} \text{Op}\{\tilde{B}\} + \text{Op}\{B\} \Xi &= 0, \\
\text{Op}\{\tilde{C}\} + \text{Op}\{C\} \text{Op}\{\tilde{A}\} + \text{Op}\{\tilde{C}\} &= 0, \\
\text{and} \quad \text{Op}\{C\} \text{Op}\{\tilde{B}\} + \Xi \Xi &= 1.
\end{align*}
\]
We begin by showing the first condition,

\[ \text{Op}\{A\} + \text{Op}\{\tilde{A}\} + \text{Op}\{A\}\text{Op}\{\tilde{A}\} + \text{Op}\{B\}\text{Op}\{\tilde{C}\} = 0. \]

Using Lemma 11 to express the product of operators as an integral operator we have

\[ \text{Op}\{\tilde{A}\}\text{Op}\{A\} \]

\[ = \text{Op}\{\langle x, y \rangle \rightarrow \lambda^2 \chi_a^{b} \langle \chi^t_s, \exp(\lambda^2(y - x)\Delta^t_s)\chi_s^t \rangle\} \]

\[ \cdot \text{Op}\{\langle x, y \rangle \rightarrow \lambda^2 \chi_a^{b} \langle \chi^t_s, \exp(\lambda^2(x - y)\Delta^t_s)\chi_s^t \rangle\} \]

\[ = \text{Op}\{\langle x, y \rangle \rightarrow \lambda^2 \int_a^b \chi_a^{b} \langle \chi^t_s, \exp(\lambda^2(z - x)\Delta^t_s)\chi_s^t \rangle dz \} \]}

\[ = \text{Op}\{\langle x, y \rangle \rightarrow \lambda^2 \left( \int_0^y \chi_{(y>x)} + \int_x^0 \chi_{(x>y)} \right) \]

\[ \cdot \langle \chi^t_s, \exp(\lambda^2(x - z)\Delta^t_s)\chi_s^t \rangle \langle \chi^t_s, \exp(\lambda^2(y - z)\Delta^t_s)\chi_s^t \rangle dz \} \]

where Lemma 16 was used on the last line. Next we express \( \text{Op}\{\tilde{B}\}\text{Op}\{C\} \) as an integral operator to find its kernel using Lemma 11 as follows

\[ \text{Op}\{\tilde{B}\}\text{Op}\{C\} \]

\[ = \text{Op}\{\langle x, y \rangle \rightarrow -\lambda\chi_a^{b}(x)\left( \exp(\lambda^2(x - a)\Delta^t_s)\chi_s^t \right)\}(y) \}

\[ \cdot \text{Op}\{\langle x, y \rangle \rightarrow \lambda\chi_a^{b}(y)\left( \exp(\lambda^2(y - a)\Delta^t_s)\chi_s^t \right)\}(x) \}

\[ = \text{Op}\{\langle x, y \rangle \rightarrow -\int_0^x \lambda\chi_a^{b}(x)\left( \exp(\lambda^2(x - a)\Delta^t_s)\chi_s^t \right)\}(z) \]

\[ \cdot \lambda\chi_a^{b}(y)\left( \exp(\lambda^2(y - a)\Delta^t_s)\chi_s^t \right)\}(z)dz \}

\[ = \text{Op}\{\langle x, y \rangle \rightarrow -\lambda^2 \int_a^b \chi_a^{b}(x)\chi_a^{b}(y)\left( \exp(\lambda^2(x - a)\Delta^t_s)\chi_s^t \right)\}(z) \]

\[ \cdot \left( \exp(\lambda^2(y - a)\Delta^t_s)\chi_s^t \right)\}(z)dz \}

\[ = \text{Op}\{\langle x, y \rangle \rightarrow -\lambda^2 \int_a^b \chi_a^{b}(x)\chi_a^{b}(y)\chi_s^t(z) \exp(\lambda^2(x - a)\Delta^t_s) \]

\[ \cdot \left( \exp(\lambda^2(y - a)\Delta^t_s)\chi_s^t \right)\}(z)dz \} \]

where Lemma 10 has been used on the second to last line.

Removing the factor \( \chi_a^{b}(x)\chi_a^{b}(y) \) from (4.9.3), replacing \( a \) with \( z_1 \), and differentiating with
respect to $z$ we have

$$
\frac{\partial}{\partial z_i} \text{Op} \left\{ (x, y) \to -\lambda^2 \int_s^t \chi_s^i (z) \exp (\lambda^2 (x - z_i) \Delta_s^i) \right\} = \\
\text{Op} \left\{ (x, y) \to \lambda^2 \int_s^t \chi_s^0 (x) \chi_s^0 (y) \chi_s^i (z) \exp (\lambda^2 (x - z_i) \Delta_s^i) \right\}.
$$

where we have used Lemma 13 once and Lemma 10 twice. Notice that the kernel of (4.9.4) is an integral of that of (4.9.2). This observation leads to the next calculation

$$
\left( \int_a^x \chi_{a < x < y < b} + \int_a^y \chi_{a < y < x < b} \right) \frac{\partial}{\partial z} \text{Op} \left\{ (x, y) \to -\lambda^2 \int_s^t \chi_s^i (z) \right\} = \text{Op} \left\{ \chi_{a < x < y < b \Delta s} \chi^i_s \right\} + \text{Op} \left\{ \chi_{a < y < x < b \Delta s} \chi^i_s \right\}.
$$

where $\Delta s$ is an integral of that of (4.9.2).

The next condition to be shown to hold is $\text{Op} \{ \tilde{B} \} + \text{Op} \{ A \} \text{Op} \{ \tilde{B} \} + \text{Op} \{ B \} \Xi = 0$. First we express the product of integral operators $\text{Op} \{ A \} \text{Op} \{ \tilde{B} \}$ as an integral operator, using Lemma
11, as follows

\[
\text{Op\{A\} Op\{B\} = Op\left\{ (x, y) \to \lambda^2 <_a (x, y) \langle \chi^+_a, \exp(\lambda^2(y - x)\Delta^+_a) \rangle \chi^+_a \right\}
\]

\[
\text{Op\left\{ (x, y) \to \lambda \chi^+_a \left( \exp(\lambda^2(x - a)\Delta^+_a) \right) \chi^+_a \right\} (y) \right\}
\]

\[
= \text{Op}\left\{ (x, y) \to \int_a^b \left( \lambda^2 <_a (x, z) \langle \chi^+_a, \exp(\lambda^2(z - x)\Delta^+_a) \rangle \chi^+_a \right) \right\}
\]

\[
\cdot \left( \lambda \chi^+_a (z) \left( \exp(\lambda^2(z - a)\Delta^+_a) \right) \chi^+_a \right) (y) \right\} dz
\]

\[
= \text{Op}\left\{ (x, y) \to \lambda^3 \int_a^b <_a (x, z) \langle \chi^+_a, \exp(\lambda^2(z - x)\Delta^+_a) \rangle \chi^+_a \right\}
\]

\[
\cdot \left( \exp(\lambda^2(z - a)\Delta^+_a) \right) \chi^+_a (y) \right\}.
\]

(4.9.6)

Using Lemma 10 and 14 we express the product of operators Op\{B\} \tilde{\Xi} as an integral operator as follows

\[
\text{Op\{B\} \tilde{\Xi} = Op\left\{ (x, y) \to -\lambda \chi^+_a (x) \left( \exp(\lambda^2(b - x)\Delta^+_a) \right) \langle \chi^+_a, \exp(\lambda^2(b - a)\Delta^+_a) \rangle \right\}
\]

(4.9.7)

\[
= \text{Op}\left\{ (x, y) \to -\lambda \chi^+_a (x) \left( \exp(\lambda^2(b - x)\Delta^+_a) \right) \right\}
\]

\[
\cdot \left( \exp(\lambda^2(b - a)\Delta^+_a) \right) \chi^+_a (y) \right\}.
\]

Removing the factor \chi^+_a(x) in Op\{B\} \tilde{\Xi}, replacing the a’s with z’s and differentiating the resulting operator with respect the z we have

\[
\frac{\partial}{\partial z} \text{Op}\left\{ (x, y) \to -\lambda \left( \exp(\lambda^2(z - x)\Delta^+_a) \right) \right\}
\]

\[
\cdot \left( \chi^+_a (y) \exp(\lambda^2(z - a)\Delta^+_a) \chi^+_a (y) \right) \right\}
\]

(4.9.8)

\[
= \text{Op}\left\{ (x, y) \to -\lambda^3 \chi^+_a (x) \left( \exp(\lambda^2(z - x)\Delta^+_a) \right) \right\}
\]

\[
\cdot \left( \exp(\lambda^2(z - a)\Delta^+_a) \right) \chi^+_a (y) \right\}.
\]

\[
= \text{Op}\left\{ (x, y) \to -\lambda^3 \chi^+_a (x) \langle \chi^+_a, \exp(\lambda^2(z - a)\Delta^+_a) \rangle \right\}
\]

\[
\cdot \left( \exp(\lambda^2(z - a)\Delta^+_a) \right) \chi^+_a (y) \right\}.
\]

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Integrating between $b$ and $x$ we have

$$ - \text{Op}\{A\}\{\tilde{B}\} = \chi^b_a(x) \int_x^b \frac{\partial}{\partial z} \text{Op}\{(x, y) \to -\lambda \left( \exp \left( \lambda^2(z - x)\Delta_x^i\right) \right) \left( \chi^y_z(y) \exp \left( \lambda^2(z - a)\Delta_y^i\right) \right) dz = \text{Op}\{B\}\tilde{\Xi} + \text{Op}\{B\} \tag{4.9.9} $$

which shows the result.

The third condition which needs to be shown for the candidate to be a right inverse is $\text{Op}\{C\} + \text{Op}\{C\}\text{Op}\{\tilde{A}\} + \Xi\text{Op}\{\tilde{C}\} = 0$. We express the product of operators $\text{Op}\{C\}\text{Op}\{\tilde{A}\}$ as an integral operator, using Lemma 11, as follows

$$ \text{Op}\{C\}\text{Op}\{\tilde{A}\} = \text{Op}\{(x, y) \to -\lambda \chi^b_a(y) \left( \exp \left( \lambda^2(y - a)\Delta^i_x\right) \right) \chi^x_y(x)\}\tag{4.9.10} $$

$$ = \text{Op}\{(x, y) \to \lambda^2 \int_a^z \exp \left( \lambda^2(z - a)\Delta^i_x\right) \chi^x_y(x)\} $$

$$ = \text{Op}\{(x, y) \to -\lambda^3 \int_a^b \exp \left( \lambda^2(z - a)\Delta^i_x\right) \chi^x_y(x)\} $$

We express the product of operators $\Xi\text{Op}\{\tilde{C}\}$ as an integral operator, using Lemma 10, as follows

$$ \Xi\text{Op}\{\tilde{C}\} = \exp \left( \lambda^2(b - a)\Delta^i_x\right) \text{Op}\{(x, y) \to \lambda \chi^b_a(y) \left( \exp \left( \lambda^2(b - y)\Delta^i_y\right) \chi^y_z(x)\right)\} \tag{4.9.11} $$

Removing the factor $\chi^b_a(y)$ from (4.9.11), replacing the parameter $b$ by $z$ and differentiating
with respect to \( z \) we have

\[
\frac{\partial}{\partial z} \text{Op} \left\{ (x, y) \rightarrow \lambda \exp \left( \lambda^2(z - a)\Delta_s^i \right) \chi^i_{a} \chi^i_{a} \right\}(x) \]

\[
= \lambda^2 \exp \left( \lambda^2(z - a)\Delta_s^i \right) \Delta_i^s \text{Op} \left\{ (x, y) \rightarrow \lambda \chi^b_{a}(y) \right\}(x)
+ \lambda^2 \exp \left( \lambda^2(z - a)\Delta_i^s \right) \Delta^s \text{Op} \left\{ (x, y) \rightarrow \lambda \chi^b_{a}(y) \right\}(x)
= \text{Op} \left\{ (x, y) \rightarrow \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_s^i \right) \Delta_i^s \chi^i_{a} \right\}(x)
+ \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_i^s \right) \Delta^s \chi^i_{a} \right\}(x)
= \text{Op} \left\{ (x, y) \rightarrow \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_s^i + \Delta_i^s \right) \right\}
+ \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_i^s \right) \Delta^s \chi^i_{a} \right\}(x)
= \text{Op} \left\{ (x, y) \rightarrow \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_s^i \right) \chi^i_{a} \right\}(x)
+ \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_i^s \right) \chi^i_{a} \right\}(x)
= \text{Op} \left\{ (x, y) \rightarrow \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_s^i \right) \chi^i_{a} \right\}(x)
+ \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_i^s \right) \chi^i_{a} \right\}(x)
\]

and then integrating (4.9.12) between \( y \) and \( b \) and using equation (4.9.10) we have the result as follows

\[
- \chi^b_{a}(y) \text{Op} \{ C \} \text{Op} \{ \tilde{A} \}
= \int_{y}^{b} \text{Op} \left\{ (x, y) \rightarrow \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_s^i \right) \chi^i_{a} \right\}(x)
+ \lambda^3 \chi^b_{a}(y) \exp \left( \lambda^2(z - a)\Delta_i^s \right) \chi^i_{a} \right\}(x) \right\}(x)
= \Xi \text{Op} \{ \tilde{C} \} - \text{Op} \left\{ (x, y) \rightarrow \lambda^2 \chi^b_{a}(y) \exp \left( \lambda^2(y - a)\Delta_s^i \right) \chi^i_{a} \right\}(x)
= \Xi \text{Op} \{ \tilde{C} \} + \text{Op} \{ C \}.
\]

The condition is satisfied.

To show that \( \text{Op} \{ C \} \text{Op} \{ \tilde{B} \} + \Xi \Xi = 1 \) we first express the product of operators \( \text{Op} \{ C \} \text{Op} \{ \tilde{B} \} \)
as an integral operator, using Lemma 11, as follows

\[ \text{Op}\{C\}\text{Op}\{\tilde{B}\} \]

\[ = \text{Op}\{(x, y) \rightarrow -\lambda \chi_{a}^{b}(y)\left( \exp\left(\lambda^{2}(y-a)\Delta_{t}^{a}\right)\chi_{s}^{t}(x)\right)\} \]

\[ \cdot \text{Op}\{(x, y) \rightarrow \lambda \chi_{a}^{b}(x)\left( \exp\left(\lambda^{2}(x-a)\Delta_{t}^{a}\right)\chi_{s}^{t}(y)\right)\} \]

\[ = \text{Op}\{(x, y) \rightarrow \int_{a}^{b} \left( (-\lambda \chi_{a}^{b}(z))\left( \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right)\chi_{s}^{t}(x)\right) \right. \]

\[ \left. \cdot \left( \lambda \chi_{a}^{b}(z)\left( \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right)\chi_{s}^{t}(y)\right) \right) dz \} \]

\[ = \text{Op}\{(x, y) \rightarrow -\lambda^{2} \int_{a}^{b} \left( \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right)\chi_{s}^{t}(x) \right. \]

\[ \left. \cdot \left( \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right)\chi_{s}^{t}(y)\right) dz \}, \quad (4.9.13) \]

Replacing all occurrences of the variable \( b \) with \( z \) in the product

\[ \tilde{\Xi} \Xi = \exp\left(\lambda^{2}(b-a)\Delta_{t}^{a}\right) \exp\left(\lambda^{2}(b-a)\Delta_{t}^{a}\right) \]

and differentiating with respect to \( z \) we have

\[ \frac{\partial}{\partial z} \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right) \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right) \]

\[ = \lambda^{2} \left( \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right)\Delta_{t}^{a} + \Delta_{t}^{a} \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right) \right) \]

\[ = \lambda^{2} \exp\left(\lambda^{2}(b-a)\Delta_{t}^{a}\right) \text{Op}\{(x, y) \mapsto \chi_{s}^{t}(x)\chi_{s}^{t}(y)\} \exp\left(\lambda^{2}(b-a)\Delta_{t}^{a}\right) \]

\[ = \text{Op}\{(x, y) \mapsto \lambda^{2} \left( \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right)\chi_{s}^{t}(x) \right. \]

\[ \left. \cdot \left( \exp\left(\lambda^{2}(z-a)\Delta_{t}^{a}\right)\chi_{s}^{t}(y)\right) dz \}, \quad (4.9.14) \]

where the Lemma 13 has been used in the third line and Lemma 10 has been used twice on the second line. The kernel of equation (4.9.13) is seen to be the integral between \( a \) and \( b \) of the kernel of (4.9.14) with respect to \( z \). Therefore we have

\[ - \int_{a}^{b} \frac{\partial}{\partial z} \tilde{\Xi} \Xi \bigg|_{z} = -\tilde{\Xi} \Xi + 1 = \text{Op}\{C\}\text{Op}\{\tilde{B}\}, \quad (4.9.15) \]

which is the expected result.

It has been shown that \( W_{a,b}^{b,a} \) has a right inverse. Next it must be shown that the right inverse is also a left inverse.
### 4.9.1 The left inverse

To show that the right inverse is also the left, the following has to be shown

\[
\begin{pmatrix}
1 + \text{Op}\{\tilde{A}\} & \kappa \text{Op}\{\tilde{B}\} \\
\kappa \text{Op}\{\tilde{C}\} & \Xi
\end{pmatrix}
\begin{pmatrix}
1 + \text{Op}\{A\} & \kappa \text{Op}\{B\} \\
\kappa \text{Op}\{C\} & \Xi
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

The left hand side is

\[
\begin{pmatrix}
1 + \text{Op}\{\tilde{A}\} + \text{Op}\{A\} + \text{Op}\{\tilde{A}\}\text{Op}\{A\} + \text{Op}\{\tilde{B}\}\text{Op}\{C\} \\
\kappa \text{Op}\{B\} + \text{Op}\{\tilde{A}\}\kappa \text{Op}\{B\} + \kappa \text{Op}\{\tilde{B}\}\Xi \\
\kappa \text{Op}\{\tilde{C}\} + \kappa \text{Op}\{\tilde{C}\}\text{Op}\{A\} + \tilde{\Xi}\kappa \text{Op}\{C\} \\
\text{Op}\{\tilde{C}\}\text{Op}\{B\} + \tilde{\Xi}\Xi
\end{pmatrix}
\]

which gives us the four equalities that we need to show. They are:

\[
\begin{align*}
\text{Op}\{\tilde{A}\} + \text{Op}\{A\} + \text{Op}\{\tilde{A}\}\text{Op}\{A\} + \text{Op}\{\tilde{B}\}\text{Op}\{C\} &= 0, \\
\text{Op}\{B\} + \text{Op}\{\tilde{A}\}\text{Op}\{B\} + \kappa \text{Op}\{\tilde{B}\}\Xi &= 0, \\
\text{Op}\{\tilde{C}\} + \text{Op}\{\tilde{C}\}\text{Op}\{A\} + \tilde{\Xi}\kappa \text{Op}\{C\} &= 0, \\
\text{Op}\{\tilde{C}\}\text{Op}\{B\} + \tilde{\Xi}\Xi &= 1.
\end{align*}
\]

To show that the first condition

\[
\text{Op}\{\tilde{A}\} + \text{Op}\{A\} + \text{Op}\{\tilde{A}\}\text{Op}\{A\} + \text{Op}\{\tilde{B}\}\text{Op}\{C\} = 0
\]

is satisfied, we express the following two products of operators \(\text{Op}\{\tilde{A}\}\text{Op}\{A\}\) and \(\text{Op}\{\tilde{B}\}\text{Op}\{C\}\) as integral operators. Using Lemma 11 we have

\[
\text{Op}\{\tilde{A}\}\text{Op}\{A\}\]

\[
= \text{Op}\left\{(x, y) \rightarrow \lambda^2 \int_{a}^{b} (x, y) \left\langle \chi_s^t, \exp \left(\lambda^2(x - y)\Delta^s_t\right)\chi_s^t\right\rangle\right\}
\]

\[
= \text{Op}\left\{(x, y) \rightarrow \lambda^2 \int_{a}^{b} \left\langle \chi_s^t, \exp \left(\lambda^2(x - y)\Delta^s_t\right)\chi_s^t\right\rangle\right\}
\]

\[
= \text{Op}\left\{(x, y) \rightarrow \lambda^2 \int_{a}^{b} \left\langle \chi_s^t, \exp \left(\lambda^2(x - z)\Delta^s_t\right)\chi_s^t\right\rangle dz\right\}
\]

\[
= \text{Op}\left\{(x, y) \rightarrow \lambda^2 \left(\int_{a}^{y} \chi_{y>y} + \int_{a}^{x} \chi_{x>y}\right)\right\}
\]

\[
\left\langle \chi_s^t, \exp \left(\lambda^2(x - z)\Delta^s_t\right)\chi_s^t\right\rangle, \left\langle \chi_s^t, \exp \left(\lambda^2(y - z)\Delta^s_t\right)\chi_s^t\right\rangle dz\right\}.
\]
where Lemmas 15 and 16 have been used on the last line. Using Lemma 11 we have

\[ \text{Op}\{\hat{B}\}\text{Op}\{C\} = \text{Op}\{(x, y) \rightarrow \lambda \chi_a^t(x)(\exp(\lambda^2(x-a)\Delta_s^t)\chi_a^t(y)) \}
\]  
\[ \text{.Op}\{(x, y) \rightarrow -\lambda \chi_a^t(y)(\exp(\lambda^2(y-a)\Delta_s^t)\chi_a^t(x)) \}
\]  
\[ = \text{Op}\{(x, y) \rightarrow \int_s^t \lambda \chi_a^t(x)(\exp(\lambda^2(x-a)\Delta_s^t)\chi_a^t(z)) \)
\[ \text{.(-\lambda)\chi_a^t(y)(\exp(\lambda^2(y-a)\Delta_s^t)\chi_a^t(z))dz} \}
\]  
\[ = \text{Op}\{(x, y) \rightarrow -\lambda^2 \int_s^t \chi_a^t(x)(\exp(\lambda^2(x-a)\Delta_s^t)\chi_a^t(z)) \)
\[ \text{.\exp(\lambda^2(y-a)\Delta_s^t)\chi_a^t(z))dz} \}
\]  
\[ = \text{Op}\{(x, y) \rightarrow -\lambda^2 \int_s^t \chi_a^t(x)\chi_a^t(y)\chi_a^t(z)(\exp(\lambda^2(x-a)\Delta_s^t) \)
\[ \text{.\exp(\lambda^2(y-a)\Delta_s^t)\chi_a^t(z))dz} \}, \quad (4.9.17)
\]  
where Lemma 10 have been used on the last line. Removing the factor \(\chi_a^t(x)\chi_a^t(y)\) from (4.9.17), replacing \(a\) with \(z_1\), and differentiating with respect to \(z_1\) we have

\[ \frac{\partial}{\partial z_1} \text{Op}\{(x, y) \rightarrow -\lambda^2 \int_s^t \chi_a^t(z)(\exp(\lambda^2(x-z_1)\Delta_s^t) \)
\[ \text{.\exp(\lambda^2(y-z_1)\Delta_s^t)\chi_a^t(z))dz} \} = 
\]  
\[ \text{Op}\{(x, y) \rightarrow \lambda^4 \int_s^t \chi_a^t(z)(\exp(\lambda^2(x-z_1)\Delta_s^t) \)
\[ \text{.\exp(\lambda^2(y-z_1)\Delta_s^t)\chi_a^t(z))dz} \}
\]  
\[ = \text{Op}\{(x, y) \rightarrow \lambda^4 \langle \chi_a^t, \exp(\lambda^2(x-z_1)\Delta_s^t)\chi_a^t \)
\[ \text{.\langle \chi_a^t, \exp(\lambda^2(y-z_1)\Delta_s^t)\chi_a^t \rangle} \}, \quad (4.9.18)
\]
where Lemma 13 has been used on the last line. Notice that the kernel in equation (4.9.18) is an integral of that of (4.9.16). We use this fact in the next calculation,

\[ \chi^b_a(x)\chi^b_a(y) \left( \int_a^x \chi_{x<y} + \int^y_a \chi_{y<x} \right) \text{Op}\{(x, y) \rightarrow \lambda^4 \langle \chi^{'}, \exp \lambda^2(x - z)\Delta^s_x \rangle \chi^{'}, \exp \lambda^2(y - z)\Delta^s_y \} \langle \chi^{'}, \exp \lambda^2(x - z)\Delta^s_x \rangle \} \rangle \langle z \rangle = \text{Op}\{A\} \text{Op}\{A\}

= \text{Op}\{(x, y) \rightarrow -\chi_{x<y}\chi^b_a(x)\chi^b_a(y)\lambda^2 \langle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \}

- \text{Op}\{(x, y) \rightarrow -\chi_{x<y}\chi^b_a(x)\chi^b_a(y)\lambda^2 \langle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \}

\langle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \}\}

+ \text{Op}\{(x, y) \rightarrow -\chi_{x<y}\chi^b_a(x)\chi^b_a(y)\lambda^2 \langle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \}

- \text{Op}\{(x, y) \rightarrow -\chi_{x<y}\chi^b_a(x)\chi^b_a(y)\lambda^2 \langle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \}

\langle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \}\}

= -\text{Op}\{A\} - \text{Op}\{\bar{A}\} - \text{Op}\{\bar{B}\} \text{Op}\{C\} \quad (4.9.19)

which shows that the condition is satisfied.

To show that the condition \text{Op}\{B\} + \text{Op}\{\bar{A}\} \text{Op}\{B\} + \text{Op}\{\bar{B}\} \Xi = 0 holds we express the product of operators \text{Op}\{\bar{A}\} \text{Op}\{B\} as an integral operator as follows, using Lemma 11, we have

\[ \text{Op}\{\bar{A}\} \text{Op}\{B\} = \text{Op}\{(x, y) \rightarrow \lambda^b_a (x, y) \langle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \} \quad (4.9.20) \]

\[ \rightarrow -\lambda^b_a (x) \langle \exp \lambda^2(b - x)\Delta^s_x \rangle \chi^{'}, \exp \lambda^2(x - y)\Delta^s_x \rangle \} \]

\[ = \text{Op}\{(x, y) \rightarrow -\lambda^b_a (x, z) \chi^b_a (z) \langle \chi^{'}, \exp \lambda^2(x - z)\Delta^s_x \rangle \chi^{'}, \exp \lambda^2(x - z)\Delta^s_x \rangle \} \}

\[ \times \chi^b_a (z) \langle \exp \lambda^2(b - z)\Delta^s_x \rangle \chi^{'}, \exp \lambda^2(b - z)\Delta^s_x \rangle \} \} \}

\]

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We next express the product of operators $\text{Op}\{\tilde{B}\}\Xi$ as an integral operator, we have

$$\text{Op}\{\tilde{B}\}\Xi = \text{Op}\{(x, y) \to \lambda \chi_a^b(x) \left( \exp \left( \lambda^2 (x - a) \Delta_t^a \right) \chi_s^t \right)(y) \}$$

$$\quad \cdot \exp \left( \lambda^2 (b - a) \Delta_t^a \right)$$

$$\quad = \text{Op}\{(x, y) \to \lambda \chi_a^b(x) \exp \left( \lambda^2 (x - a) \Delta_t^a \right) \chi_s^t \}$$

$$\quad \cdot \left( \exp \left( \lambda^2 (b - a) \Delta_t^a \right) \right)(y). \quad (4.9.21)$$

Removing the factor $\chi_a^b(x)$ from (4.9.21), replacing the $a$’s with $z$’s, and differentiating with respect to $z$ gives

$$\frac{\partial}{\partial z} \text{Op}\{(x, y) \to \lambda \exp \left( \lambda^2 (x - z) \Delta_t^a \right) \left( \exp \left( \lambda^2 (b - z) \Delta_t^a \right) \chi_s^t \right)(y) \}$$

$$\quad = \text{Op}\{(x, y) \to -\lambda^3 \exp \left( \lambda^2 (x - z) \Delta_t^a \right) \}$$

$$\quad \cdot (\Delta_t^a + \Delta_t^a) \left( \exp \left( \lambda^2 (b - z) \Delta_t^a \right) \chi_s^t \right)(y) \}$$

$$\quad = \text{Op}\{(x, y) \to -\lambda^3 \left( \exp \left( \lambda^2 (x - z) \Delta_t^a \right) \chi_s^t \right)(x)$$

$$\quad \cdot \left( \chi_s^t, \exp \left( \lambda^2 (b - z) \Delta_t^a \right) \chi_s^t \right) \} \quad (4.9.22)$$

Integrating (4.9.22) between $a$ and $x$ and using equation (4.9.20) gives

$$\chi_a^b(x) \int_a^x \text{Op}\{(x, y) \to -\lambda^3 \left( \exp \left( \lambda^2 (x - z) \Delta_t^a \right) \chi_s^t \right)(x)$$

$$\quad \cdot \left( \chi_s^t, \exp \left( \lambda^2 (b - z) \Delta_t^a \right) \chi_s^t \right) \} \, dz = \text{Op}\{(\tilde{A})\text{Op}\{B\} \}$$

$$\quad = \text{Op}\{(x, y) \to \lambda \chi_a^b(x) \left( \exp \left( \lambda^2 (b - x) \Delta_t^a \right) \chi_s^t \right)(y) \} - \text{Op}\{\tilde{B}\}\Xi$$

$$\quad = -\text{Op}\{B\} - \text{Op}\{\tilde{B}\}\Xi. \quad (4.9.23)$$

and the condition is clearly seen to be satisfied.

The second to last condition is $\text{Op}\{\tilde{C}\} + \text{Op}\{\tilde{C}\}\text{Op}\{A\} + \tilde{z}\text{Op}\{C\} = 0$. We express the product of operators $\text{Op}\{\tilde{C}\}\text{Op}\{A\}$ as an integral operator, using Lemma 11, as follows

$$\text{Op}\{\tilde{C}\}\text{Op}\{A\} = \text{Op}\{(x, y) \to \lambda \chi_a^b(y) \left( \exp \left( \lambda^2 (b - y) \Delta_t^a \right) \chi_s^t \right)(x) \}$$

$$\quad \cdot \text{Op}\{(x, y) \to \lambda^3 \int_a^b \chi_a^b(z) \chi_a^h(z, y) \left( \exp \left( \lambda^2 (b - z) \Delta_t^a \right) \chi_s^t \right)(z)$$

$$\quad \cdot \left( \chi_s^t, \exp \left( \lambda^2 (y - z) \Delta_t^a \right) \chi_s^t \right) \} \quad (4.9.24)$$
We express the product of operators ˜ΞOp\{C\} as an integral operator as follows
\[
\tilde{\Xi}*Op\{C\} = \exp(\lambda^2(b - a)\Delta_s^t).Op\{(x, y) \to -\lambda\chi^b_a(y)\left(\exp(\lambda^2(y - a)\Delta_s^t)\chi^s_y\right)(x)\}.
\]
\[
= Op\{(x, y) \to -\lambda\chi^b_a(y)\exp(\lambda^2(y - a)\Delta_s^t)\left(\exp(\lambda^2(y - a)\Delta_s^t)\chi^s_y\right)(x)\}.
\]
Removing the factor \chi^b_a(y) from (4.9.24), replacing the variable \(a\) with \(z\), and differentiating with respect to \(z\) we have
\[
\frac{\partial}{\partial z} Op\{(x, y) \to -\lambda\exp(\lambda^2(b - z)\Delta_s^t)\left(\exp(\lambda^2(y - z)\Delta_s^t)\chi^s_y\right)(x)\}
= Op\{(x, y) \to \lambda^3\exp(\lambda^2(b - z)\Delta_s^t)
\cdot \left(\Delta_s^t + \Delta_s^t\right)\left(\exp(\lambda^2(y - z)\Delta_s^t)\chi^s_y\right)(x)\}
= Op\{(x, y) \to \lambda^3\exp(\lambda^2(b - z)\Delta_s^t)\chi^s_y(y)
\cdot \left(\chi^s_y, \exp(\lambda^2(y - z)\Delta_s^t)\chi^s_y\right)\}.
\]
where Lemma 13 has been used on the last line. Notice that the kernel of (4.9.24) is an integral of the kernel in equation (4.9.26). We use this fact in the next calculation
\[
\chi^b_a(x)\chi^b_a(y) Op\{(x, y) \to \lambda^3 \int_a^b \exp(\lambda^2(b - z)\Delta_s^t)\chi^s_y(z) d\lambda\}
= Op\{\tilde{C}\} Op\{A\}
= Op\{(x, y) \to -\lambda\chi^b_a(y)\left(\exp(\lambda^2(b - y)\Delta_s^t)\chi^s_y\right)(x)\} - \tilde{\Xi}*Op\{C\}
= -Op\{\tilde{C}\} - \tilde{\Xi}*Op\{C\}.
\]
The condition is clearly satisfied. The final condition to be shown is \(Op\{\tilde{C}\} Op\{B\} + \tilde{\Xi}* = 1.\) To that end, the product of operators \(Op\{\tilde{C}\} Op\{B\}\) is expressed as an integral operator using Lemma 11 as follows
\[
Op\{\tilde{C}\} Op\{B\} = Op\{(x, y) \to \lambda\chi^b_a(y)\left(\exp(\lambda^2(b - y)\Delta_s^t)\chi^s_y\right)(x)\},
\]
\[
\cdot Op\{(x, y) \to -\lambda\chi^b_a(x)\left(\exp(\lambda^2(b - x)\Delta_s^t)\chi^s_x\right)(y)\}.
\]
\[
= Op\{(x, y) \to -\lambda^2 \int_a^b \chi^b_a(z)\left(\exp(\lambda^2(b - z)\Delta_s^t)\chi^s_y\right)(x)
\cdot \chi^b_a(z)\left(\exp(\lambda^2(b - z)\Delta_s^t)\chi^s_y\right)(y) d\lambda\}.
\]
We have the product
\[ \tilde{\Xi} = \exp(\lambda^2(b - a)\Delta_t^s) \exp(\lambda^2(b - a)\Delta_t^s), \]
replacing the \( a \)'s with \( z \)'s and differentiating with respect to \( z \) we have
\[
\frac{\partial \tilde{\Xi}}{\partial z} \bigg|_{a=z} = -\lambda^2 \exp(\lambda^2(b - z)\Delta_t^s)(\Delta_t^s + \Delta_t^s) \exp(\lambda^2(b - z)\Delta_t^s)
= \text{Op}\{ (x, y) \rightarrow -\lambda^2 \left( \exp(\lambda^2(b - z)\Delta_t^s)\chi_t^s(x) \right) \}
\cdot \left( \exp(\lambda^2(b - z)\Delta_t^s)\chi_t^s(y) \right),
\]
where Lemma 13 has been used on the last line. Integrating (4.9.28) with appropriate limits and using (4.9.27) it can been seen that
\[
\int_a^b \frac{\partial \tilde{\Xi}}{\partial z} \bigg|_{a=z} dz = +\text{Op}\{ \tilde{C} \} K \{ B \}
= 1 - \tilde{\Xi}
\]
which clearly shows the condition to be satisfied.
\[ \square \]
Chapter 5

Towards the classification of double product integrals

5.1 Introduction

In this chapter we generalise the canonical form of the double product studied in Chapter four by the use of gauge transformations (see Hudson and Parthasarathy [1984] for a description of the three canonical forms). We show that all the theory of Chapter four carries over to these generalised double product integrals. This is because there is unitary equivalence between the Bogolubov transformation got from the generalised canonical form of the double product and the corresponding original one. In papers of Robin Hudson and the author (for example Hudson [2005b] and this thesis) two canonical forms of double products have been studied:

\[ dr = \lambda (dA^\dagger \otimes dA - dA \otimes dA^\dagger), \]

\[ dr = \lambda (dA^\dagger \otimes dA^\dagger - dA \otimes dA), \]

where \( \lambda \) is a real number. However, the above can be modified to give a new, but related, double product by the use of gauge transformations as follows,

\[ dr = \lambda dA^\dagger \otimes dA - \overline{\lambda} dA \otimes dA^\dagger, \]

\[ dr = \lambda dA^\dagger \otimes dA^\dagger - \overline{\lambda} dA \otimes dA, \]

where now \( \lambda \) is a complex number.

Lemma 17. Writing \( \chi = |\chi|e^{i\omega}, \) where \( |\chi| \) is the modulus of \( \chi \) and \( e^{i\omega} \) its phase, and denoting the complex conjugation transform as \( \kappa, \)

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Proof. Using the commutation relations and the relation \( \text{ad}_{XY}(Z) = X \text{ad}_{Y}(Z) + \text{ad}_{X}(Z)Y \) we have, dropping tensor product notation,

\[
\text{ad}_{e^{i\omega}a^{\dagger}b^{\dagger} - e^{i\omega}ab}(a) = [e^{i\omega}a^{\dagger}b^{\dagger} - e^{i\omega}ab, a] \\
= e^{i\omega}(a^{\dagger}[b^{\dagger}, a] + [a^{\dagger}, a]b^{\dagger}) - e^{i\omega}(a[b, a] + [a, a]b) = -e^{i\omega}b^{\dagger},
\]

\[
\text{ad}_{e^{i\omega}a^{\dagger}b^{\dagger} - e^{i\omega}ab}(-e^{i\omega}b^{\dagger}) = [e^{i\omega}a^{\dagger}b^{\dagger} - e^{i\omega}ab, -e^{i\omega}b^{\dagger}] \\
= e^{i\omega}(a^{\dagger}[b^{\dagger}, e^{i\omega}b^{\dagger}] + [a^{\dagger}, e^{i\omega}b^{\dagger}]b^{\dagger}) + e^{i\omega}(a[b, e^{i\omega}b^{\dagger}] + [a, e^{i\omega}b^{\dagger}]b) = e^{i\omega}e^{i\omega}a = a.
\]

Therefore we can write \( e^{\text{ad}_{|\chi|e^{i\omega}a^{\dagger}b^{\dagger} - |\chi|e^{i\omega}ab}}(a) \) as

\[
a + \frac{1}{1!}|\chi|^2 e^{i\omega}a^{\dagger}b^{\dagger} - |\chi|e^{i\omega}ab, a] + \frac{1}{2!}[[|\chi|^2 e^{i\omega}a^{\dagger}b^{\dagger} - |\chi|e^{i\omega}ab, a], a] \\
+ \frac{1}{3!}[[|\chi|^2 e^{i\omega}a^{\dagger}b^{\dagger} - |\chi|e^{i\omega}ab, a], a], a] + \ldots
\]

\[
= a - e^{i\omega}b^{\dagger}|\chi|^2 - \frac{|\chi|^4}{3!}a + a \frac{|\chi|^6}{6!} - e^{i\omega}b^{\dagger}|\chi|^4 - \frac{|\chi|^8}{8!}a + \ldots
\]

\[
= a(1 + \frac{|\chi|^2}{2!} + \frac{|\chi|^4}{4!} + \frac{|\chi|^6}{6!} + \frac{|\chi|^8}{8!} + \ldots) - e^{i\omega}b^{\dagger}(\frac{|\chi|^2}{3!} + \frac{|\chi|^4}{5!} + \frac{|\chi|^6}{7!} + \ldots) \\
= \cosh(|\chi|)a - e^{i\omega}\kappa \sinh(|\chi|)b.
\]

This justifies the first row of the pseudo-rotational matrix of the lemma. For the second row, the proof continues along the same lines,

\[
[e^{i\omega}a^{\dagger}b^{\dagger} - e^{i\omega}ab, b] = e^{i\omega}(a^{\dagger}[b^{\dagger}, b] + [a^{\dagger}, b]b^{\dagger}) - e^{i\omega}(a[b, b] + [a, b]b)
\]

\[
= -e^{i\omega}a^{\dagger},
\]

\[
[e^{i\omega}a^{\dagger}b^{\dagger} - e^{i\omega}ab, -e^{i\omega}a^{\dagger}] = -e^{i\omega}(a^{\dagger}[b^{\dagger}, e^{i\omega}a^{\dagger}] + [a^{\dagger}, e^{i\omega}a^{\dagger}]b^{\dagger}) \\
+ e^{i\omega}(a[b, e^{i\omega}a^{\dagger}] + [a, e^{i\omega}a^{\dagger}]b)
\]

\[
= e^{i\omega}e^{i\omega}b = b.
\]

As in Chapter 4 these relation can be used to write

\[
e^{\text{ad}_{|\chi|e^{i\omega}a^{\dagger}b^{\dagger} - |\chi|e^{i\omega}ab}}(b) = -e^{i\omega}\kappa \sinh(|\chi|)a + \cosh(|\chi|)b
\]

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and so
\[
\exp \left( \text{ad}_{\chi a^\dagger b^\dagger - \tau_{ab}} \right) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cosh |\chi| & -e^{i\omega} (\sinh |\chi|) \kappa \\ -e^{i\omega} \kappa \sinh |\chi| & \cosh |\chi| \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.
\]

**Lemma 18.**
\[
B = \begin{pmatrix} \cosh |\chi| & -e^{i\omega} (\sinh |\chi|) \kappa \\ -e^{i\omega} \kappa \sinh |\chi| & \cosh |\chi| \end{pmatrix}
\]
is a Bogolubov transformation.

**Proof.** A direct proof of the preservation of the imaginary part is given. Writing \(c = \cosh \mu\) and \(s = \sinh \mu\), we have, for any real \(\mu\),
\[
\left\langle \begin{pmatrix} c & -e^{i\omega} s \kappa \\ -e^{i\omega} \kappa s & c \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} c & -e^{i\omega} s \kappa \\ -e^{i\omega} \kappa s & c \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} ca_1 - e^{i\omega} s \bar{b}_1 \\ -e^{i\omega} s \bar{a}_1 + cb_1 \end{pmatrix}, \begin{pmatrix} ca_2 - e^{i\omega} s \bar{b}_2 \\ -e^{i\omega} s \bar{a}_2 + cb_2 \end{pmatrix} \right\rangle = \langle ca_1 - e^{i\omega} s \bar{b}_1, ca_2 - e^{i\omega} s \bar{b}_2 \rangle + \langle -e^{i\omega} s \bar{a}_1 + cb_1, -e^{i\omega} s \bar{a}_2 + cb_2 \rangle = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle + c^2 \langle b_1, b_2 \rangle + s^2 \langle b_1, b_2 \rangle - 2 s c e^{i\omega} ((\langle b_1, \bar{a}_2 \rangle + \langle a_1, \bar{b}_2 \rangle) + \langle a_1, a_2 \rangle) + \text{Im} \left\{ \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle \right\}.
\]

The imaginary part of the above right hand side is indeed \(\text{Im} \left\{ \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle \right\}\) and so \(B\) goes preserve the imaginary part of the inner product. Obviously \(B\) is a continuous real-linear operator with bounded inverse, and therefore a Bogolubov transformation.

**Proposition 4.** \( \left( \begin{pmatrix} \cosh |\chi| & -e^{i\omega} (\sinh |\chi|) \kappa \\ -e^{i\omega} \kappa \sinh |\chi| & \cosh |\chi| \end{pmatrix} \right) \) is a Bogolubov transformation acting in \(\mathbb{C}^2\) which is implemented by \(\exp (\chi a^\dagger b^\dagger - \overline{\tau}_{ab})\).

**Proof.** By Lemma (17) and Lemma (18) \(\square\)

**Lemma 19.** \(\exp \left( \text{ad}_{\chi a^\dagger b^\dagger - \tau_{ab}} \right)\) is unitarily equivalent to \(\exp \left( \text{ad}_{|\chi| a^\dagger b^\dagger - |\chi| \tau_{ab}} \right)\). The equivalence is implemented by the unitary matrix
\[
\begin{pmatrix} 1 & 0 \\ 0 & e^{i\omega} \end{pmatrix}.
\]

**Proof.** Writing \(U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\omega} \end{pmatrix}\), by definition \(U \overline{U}^t = I\), so for any \(n \in \mathbb{N}\),
\[
\text{ad}_{\chi a^\dagger b^\dagger - \tau_{ab}} = U (\text{ad}_{|\chi| a^\dagger b^\dagger - |\chi| \tau_{ab}})^n U^t.
\]

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and the result follows by the definition of the exponential function.

5.1.1 Embedding the two-dimensional Bogolubov operators into a 
\((m + n)\) by \((m + n)\) matrix

As in the Chapter 4, a Bogolubov transformation, in this case \(B\), is embedded into a matrix, while still preserving the property of being a Bogolubov transformation. Let us fix natural numbers \(m\) and \(n\) and for each \((j, k) \in \mathbb{N}_m \times \mathbb{N}_n\). Let

\[
t^{(i, j)}_{(m, n)}(\chi) = \begin{pmatrix}
\cosh |\chi| & -e^{i\omega} \sinh |\chi| \\
-e^{i\omega} \kappa \sinh |\chi| & \cosh |\chi|
\end{pmatrix}_{(m, n)}^{(j, k)}
\]

(5.1.1)

dozen the \(m + n\) by \(m + n\) matrix obtained by embedding in the intersections of the \(j\)th and \((m + k)\)th rows and columns and completing by placing 1’s in vacant diagonal positions and 0’s in the off diagonal positions. Hence we may form the ordered double products which are also Bogolubov by the same reasoning as in Section 4.2.2.

\[
\rightarrow R_{(m, n)}^{(\omega)}(|\chi|) = \prod_{(j, k) \in \mathbb{N}_m \times \mathbb{N}_n} t^{(i, j)}_{(m, n)}(\chi),
\]

(5.1.2)

where the double arrows on top of the product symbol \(\prod\) indicate the direction of increasing \(k\) and \(j\) as in Section 4.2.2. Three more ordered double products, namely \(\leftarrow R_{(m, n)}^{(\omega)}(\chi)\), \(\rightarrow R_{(m, n)}^{(\omega)}(\chi)\) and \(\leftarrow R_{(m, n)}^{(\omega)}(\chi)\) can be similar defined as in Section 4.2.2. Writing \(\alpha = \cosh |\chi|\), \(\beta = \sinh |\chi|\), \(\delta = \cosh |\chi|\) and \(\gamma = \sinh |\chi|\), a lemma is now stated which relates \(\rightarrow R_{(m, 1)}^{(\omega)}(|\chi|)\) to \(\rightarrow R_{(m, 1)}^{(\omega)}(|\chi|)\) from a previous section (Section 4.2.2).

Theorem 21.

\[
\rightarrow R_{(m, 1)}^{(\omega)} = \begin{pmatrix}
\alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-2} \gamma & e^{i\omega} \beta \kappa \delta^{m-1} \\
0 & \alpha & \beta \gamma & \beta \delta \gamma & \ldots & \beta \delta^{m-3} \gamma & e^{i\omega} \beta \kappa \delta^{m-2} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ddots & \beta \delta \gamma & e^{i\omega} \beta \kappa \delta^2 \\
0 & 0 & 0 & 0 & \ldots & \beta \gamma & e^{i\omega} \beta \kappa \delta \\
0 & 0 & 0 & 0 & \ldots & 0 & \alpha & e^{i\omega} \beta \kappa \\
e^{i\omega} \kappa \gamma & e^{i\omega} \delta \kappa \gamma & e^{i\omega} \delta^2 \kappa \gamma & e^{i\omega} \delta^3 \kappa \gamma & \ldots & e^{i\omega} \delta^{m-1} \kappa \gamma & \delta^m
\end{pmatrix}
\]
and \( R^{(\omega)}_{(m, 1)}(|\chi|) \) is unitarily equivalent to \( R^{(\omega)}_{(m, 1)}(|\chi|) \) defined in the Section 4.2.2, with implementor

\[ \begin{pmatrix}
  1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
\end{pmatrix} e^{i\omega} \]

Proof. The first part of the proof is similar to that of Theorem 15. The last part is a simple calculation.

The heuristic limit of \( \lim_{m \to \infty} \overrightarrow{R}^{(\omega)}_{(m, 1)}(|\chi_{m,n}|) \) is taken by embedding \( \mathbb{C}^m \) into \( \mathfrak{h} \), as before, but where

\[ \chi_{m,n} = \lambda \sqrt{\frac{(b-a)(t-s)}{mn}} = \lambda_m \sqrt{\frac{t-s}{n}} = \lambda_n \sqrt{\frac{b-a}{m}} \]

and \( \lambda, \lambda_m \) and \( \lambda_n \) are also complex numbers with the same phase \( \omega \) as \( \chi_{m,n} \). This is done by mapping their canonical orthonormal bases to the normalised indicator functions of the subintervals of \( [s, t) \) and \( [a, b) \) obtained by equipartitioning them into, respectively \( m \) and \( n \) subintervals. The resulting operator is summarised in the following theorem.

Theorem 22.

\[
\lim_{m \to \infty} \overrightarrow{R}^{(\omega)}_{(m, 1)}(|\chi_{m,n}|) = \begin{pmatrix}
  \text{id} + \text{Op} \left\{ (x, y) \mapsto \lambda_n |\chi|^2 \right\} \\
  e^{i\omega} \left\langle -\kappa |\lambda_n| |\chi|^2 \exp \left( \frac{|\lambda_n|^2}{2} (y-x) \right) \right\rangle \\
  e^{i\omega} \left\langle -\kappa |\lambda_n| |\chi|^2 \exp \left( \frac{|\lambda_n|^2}{2} (b-y) \right) \right\rangle \\
  e^{i\omega} \left\langle -\kappa |\lambda_n| |\chi|^2 \exp \left( \frac{|\lambda_n|^2}{2} (b-a) \right) \right\rangle \\
\end{pmatrix}
\]

Remark 25. The theorem could naturally be extended to include the cases

\[
\lim_{m \to \infty} \overrightarrow{R}^{(\omega)}_{(m, 1)}(|\chi_{m,n}|),
\]

\[
\lim_{m \to \infty} \overrightarrow{R}^{(\omega)}_{(1, m)}(|\chi_{m,n}|)
\]

and

\[
\lim_{n \to \infty} \overrightarrow{R}^{(\omega)}_{(1, n)}(|\chi_{m,n}|)
\]

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5.2 The second limit

The limit \( \lim_{m \to \infty} \lim_{n \to \infty} R_{(m,n)}^{(\omega)(\chi_{m,n})} \) is calculated as in a similar fashion as

\[
\lim_{m \to \infty} \lim_{n \to \infty} R_{(m,n)}(\chi_{m,n}),
\]

which was done in Section 4.5. The resulting operator acts from and to \( \mathfrak{h} \oplus \mathfrak{h} \). The second limits are below which we denote by \( W_{b,t}^{(\omega)}_{a,s} \).

Theorem 23.

\[
W_{b,t}^{(\omega)}_{a,s} = \lim_{m \to \infty} \lim_{n \to \infty} R_{(m,n)}^{(\omega)(\chi_{m,n})} = \left( \begin{array}{c}
\text{id} + \text{Op} \left\{ (x, y) \rightarrow \chi^2 <_b^a \langle x, y \rangle (\chi^a_t, \exp (\lambda^2 (y - x) \Delta_t^a) \chi^a_t) \right\} \\
\text{Op} \left\{ (x, y) \rightarrow -e^{i \omega \kappa \lambda} \chi^b_s (y) (\exp (\lambda^2 (y - a) \Delta_t^a) \chi^b_s) (x) \right\} \\
\text{Op} \left\{ (x, y) \rightarrow -e^{i \omega \kappa \lambda} \chi^b_s (x) (\exp (\lambda^2 (b - x) \Delta_t^a) \chi^b_s) (y) \right\} \\
\exp (\lambda^2 (b - a) \Delta_t^a) \end{array} \right).
\]

is equal to

\[
\tilde{W}_{b,t}^{(\omega)}_{a,s} = \lim_{n \to \infty} \lim_{m \to \infty} R_{(m,n)}^{(\omega)(\chi_{m,n})} = \left( \begin{array}{c}
\exp (\lambda^2 (t - s) \Delta_t^a) \\
\text{Op} \left\{ (x, y) \rightarrow -e^{i \omega \kappa \lambda} \chi^b_s (x) (\exp (\lambda^2 (t - x) \Delta_t^a) \chi^b_s) (y) \right\} \\
\text{Op} \left\{ (x, y) \rightarrow -e^{i \omega \kappa \lambda} \chi^b_s (y) (\exp (\lambda^2 (y - s) \Delta_t^a) \chi^b_s) (x) \right\} \\
\text{id} + \text{Op} \left\{ (x, y) \rightarrow \lambda^2 <_s^t (x, y) \langle \chi^b_s, \exp (\lambda^2 (y - x) \Delta_t^a) \chi^b_s \rangle \right\} \\
\end{array} \right).
\]

Remark 26. Naturally the theorem could be extended to include similar results for

\[
\lim_{m \to \infty} \lim_{n \to \infty} R_{(m,n)}^{(\omega)(\chi_{m,n})},
\]

\[
\lim_{m \to \infty} \lim_{n \to \infty} R_{(m,n)}^{(\omega)(\chi_{m,n})},
\]

and

\[
\lim_{m \to \infty} \lim_{n \to \infty} R_{(m,n)}^{(\omega)(\chi_{m,n})}.
\]

Theorem 24. The operators \( \lim_{m \to \infty} R_{(m,1)}^{(\omega)(\chi_{m,n})} \) and \( W_{b,t}^{(\omega)}_{a,s} \) are unitarily equivalent to \( \lim_{m \to \infty} R_{(m,1)} \) and \( \tilde{W}_{b,t}^{(\omega)}_{a,s} \) respectively.

Proof. The proof is a simple calculation. \( \square \)
Chapter 6

The uniqueness of implementors

In this chapter we show that the system of Bogolubov transformations

\[(W_{b,t}^{a,s})_{0 \leq a < b, 0 \leq s < t}\]

satisfies three important properties: One is bi-evolution and the other two are bi-covariance properties, namely shift bi-covariance and time reversal bi-covariance. We then show that there exists a unique system of implementors of

\[(W_{a,t}^{b,s})_{0 \leq a < b, 0 \leq s < t}\]

that also has these properties.

6.1 The bi-evolutional property

**Theorem 25.** The forward-forward limit \(\lim_{m \to \infty} \lim_{n \to \infty} (R_{(m,n)}(\theta_{m,n})) = W_{a,t}^{b,s}\) has the bi-evolution property i.e. for all \(a < b < c\) and \(s < t < u\) we have

\[W_{a,s}^{b,t}W_{c,s}^{b,t} = W_{a,s}^{c,t}\] and \[W_{a,s}^{b,t}W_{a,t}^{b,u} = W_{a,s}^{b,u}\]. \hspace{1cm} (6.1.1)

**Remark 27.** The theorem could be extended to the backward-backward, forward-backward and backward-forward limits.

**Proof.** We show the evolution property in the variable \(a\) and \(b\) in \(W_{a,t}^{b,s}\). The proof that
We perform the following calculation

\[ W_{a,s}^{b,t} = \lim_{m \to \infty} \lim_{n \to \infty} \overrightarrow{R}_{(m,n)}(\theta_{m,n}) \]

\[ = \begin{pmatrix}
\text{id} + \text{Op} \{(x, y) \mapsto \lambda^2 \chi_b^a(x, y)(\chi_s^b \exp(\lambda^2(y-x)\Delta_t^s)\chi_s^b)\}
\kappa \text{Op} \{(x, y) \mapsto -\lambda^2 \chi_b^a(y)(\exp(\lambda^2(y-a)\Delta_t^s)\chi_s^b)(x)\}
\kappa \text{Op} \{(x, y) \mapsto -\lambda^2 \chi_b^a(x)(\exp(\lambda^2(b-x)\Delta_t^s)\chi_s^b)(y)\}
\exp(\lambda^2(b-a)\Delta_t^s) & .
\end{pmatrix} \] (6.1.2)

We perform the following calculation

\[ W_{a,s}^{b,t} W_{b,s}^{c,t} = \begin{pmatrix}
\text{id} + \text{Op} \{A_{a,s}^{b,t}\} & \kappa \text{Op} \{B_{a,s}^{b,t}\}
\kappa \text{Op} \{C_{a,s}^{b,t}\} & \Xi_{a,s}^{b,t}
\end{pmatrix} \begin{pmatrix}
\text{id} + \text{Op} \{A_{a,s}^{c,t}\} & \kappa \text{Op} \{B_{a,s}^{c,t}\}
\kappa \text{Op} \{C_{a,s}^{c,t}\} & \Xi_{a,s}^{c,t}
\end{pmatrix} \]

\[ = \begin{pmatrix}
\text{id} + \text{Op} \{A_{a,s}^{b,t}\} + \text{Op} \{A_{a,s}^{b,t}\} \text{Op} \{A_{b,s}^{c,t}\} + \text{Op} \{B_{a,s}^{b,t}\} \text{Op} \{C_{b,s}^{c,t}\}
\kappa \text{Op} \{C_{a,s}^{b,t}\} + \kappa \text{Op} \{C_{a,s}^{b,t}\} \text{Op} \{A_{b,s}^{c,t}\} + \kappa \Xi_{a,s}^{b,t} \text{Op} \{C_{b,s}^{c,t}\}
\kappa \text{Op} \{B_{b,s}^{c,t}\} + \kappa \text{Op} \{A_{a,s}^{b,t}\} \text{Op} \{B_{b,s}^{c,t}\} + \kappa \text{Op} \{B_{a,s}^{b,t}\} \Xi_{b,s}^{c,t}
\text{Op} \{C_{a,s}^{b,t}\} \text{Op} \{B_{b,s}^{c,t}\} + \Xi_{a,s}^{b,t} \Xi_{b,s}^{c,t}
\end{pmatrix} \] (6.1.3)

Equating the righthand side of equation (6.1.3) to

\[ W_{a,s}^{c,t} = \begin{pmatrix}
\text{id} + \text{Op} \{A_{a,s}^{c,t}\} & \kappa \text{Op} \{B_{a,s}^{c,t}\}
\kappa \text{Op} \{C_{a,s}^{c,t}\} & \Xi_{a,s}^{c,t}
\end{pmatrix} \]

we have to prove the following four equalities:

\[ \text{Op} \{A_{a,s}^{b,t}\} + \text{Op} \{A_{b,s}^{c,t}\} + \text{Op} \{A_{a,s}^{b,t}\} \text{Op} \{A_{b,s}^{c,t}\} + \text{Op} \{C_{a,s}^{b,t}\} \text{Op} \{B_{b,s}^{c,t}\} = \text{Op} \{A_{a,s}^{c,t}\}, \] (6.1.5)

\[ \text{Op} \{B_{a,s}^{c,t}\} + \text{Op} \{A_{b,s}^{b,t}\} \text{Op} \{B_{b,s}^{c,t}\} + \text{Op} \{B_{a,s}^{b,t}\} \Xi_{b,s}^{c,t} = \text{Op} \{B_{a,s}^{c,t}\}, \] (6.1.6)

\[ \text{Op} \{C_{a,s}^{b,t}\} + \text{Op} \{C_{a,s}^{b,t}\} \text{Op} \{A_{b,s}^{c,t}\} + \Xi_{b,s}^{c,t} \text{Op} \{C_{b,s}^{c,t}\} = \text{Op} \{C_{a,s}^{c,t}\}, \] (6.1.7)

\[ \text{Op} \{C_{a,s}^{b,t}\} \text{Op} \{B_{b,s}^{c,t}\} + \Xi_{b,s}^{c,t} \Xi_{b,s}^{c,t} = \Xi_{a,s}^{c,t}. \] (6.1.8)

To prove the first equality (equation (6.1.5)) holds for \( W_{a,s}^{b,t} \) we only need show that

\[ \text{Op} \{C_{a,s}^{b,t}\} \text{Op} \{B_{b,s}^{c,t}\} = \text{Op} \{(x, y) \mapsto \lambda^2 \chi_b^a(x)\chi_s^b(y)(\chi_s^b \exp(\lambda^2(y-x)\Delta_t^s)\chi_s^b)\} \] (6.1.9)

since \( \text{Op} \{A_{a,s}^{b,t}\} \text{Op} \{A_{b,s}^{c,t}\} = 0 \). Since \( \text{Op} \{A_{a,s}^{c,t}\} \) is only non-zero on \( L^2[b, c] \) and \( \text{Op} \{A_{a,s}^{b,t}\} \) is
non-zero on $L^2([a,b])$.

$$\text{Op}(C_{a,s}^{b,t})\text{Op}(B_{b,s}^{c,t}) = \text{Op}\{(x,y) \mapsto -\lambda \chi_a^b(x)(\exp(\lambda^2(b-x)\Delta_s^b)\chi_a^c(y))\}$$

$$= \text{Op}\{(x,y) \mapsto -\lambda \chi_a^b(y)(\exp(\lambda^2(y-b)\Delta_s^b)\chi_a^c(x))\}$$

$$= \text{Op}\{(x,y) \mapsto \lambda^2 \chi_a^b(x)\chi_a^c(y) \int_s^t (\exp(\lambda^2(b-x)\Delta_s^b)\chi_a^c(z))(\exp(\lambda^2(y-x)\Delta_s^b)\chi_a^c(z))dz\}$$

which is the desired result.

To prove the second equality (equation (6.1.6)) we only need show that

$$\text{Op}(B_{b,s}^{c,t})\Xi_{b,s}^{c,t} = \text{Op}\{(x,y) \mapsto -\lambda^2 \chi_a^b(x)\exp(\lambda^2(c-x)\Delta_s^b(y))\}$$

as $\text{Op}(A_{a,s}^{b,t})\text{Op}(B_{b,s}^{c,t}) = 0$, since $\text{Op}(B_{b,s}^{c,t})$ is only non-zero on $L^2[b,c]$ and $\text{Op}(A_{a,s}^{b,t})$ is non-zero on $L^2[a,b]$. The calculation is as follows

$$\text{Op}\{(x,y) \mapsto -\lambda \chi_a^b(x)(\exp(\lambda^2(b-x)\Delta_s^b)\chi_a^c(y))\exp(\lambda^2(c-b)\Delta_s^b)\}$$

$$= \text{Op}\{(x,y) \mapsto -\lambda \chi_a^b(x)\exp(\lambda^2(c-b)\Delta_s^b)(\exp(\lambda^2(b-x)\Delta_s^b)\chi_a^c(y))\}$$

which is the desired result.

To prove the third equality (equation (6.1.7)) we only need show that

$$\Xi_{a,s}^{b,t}\text{Op}(C_{a,s}^{c,t}) = \text{Op}\{(x,y) \mapsto -\lambda^2 \chi_a^b(y)\exp(\lambda^2(y-a)\Delta_s^b(x))\}$$

as $\text{Op}(C_{a,s}^{b,t})\text{Op}(A_{a,s}^{c,t}) = 0$, since $\text{Op}(A_{a,s}^{c,t})$ is only non-zero on $L^2([b,c])$ and $\text{Op}(C_{a,s}^{b,t})$ is non-zero on $L^2([a,b])$. To that end we have

$$\Xi_{a,s}^{b,t}\text{Op}(C_{a,s}^{c,t}) = \exp(\lambda^2(b-a)\Delta_s^b)\text{Op}\{(x,y) \mapsto -\lambda^2 \chi_a^b(y)\exp(\lambda^2(y-b)\Delta_s^b)(x)\}$$

$$= \text{Op}\{(x,y) \mapsto -\lambda^2 \chi_a^b(y)\exp(\lambda^2(y-a)\Delta_s^b)(x)\}. \quad (6.1.14)$$

To prove the fourth equality (equation (6.1.8)) we only need show that $\Xi_{a,s}^{b,t}\Xi_{b,s}^{c,t} = \Xi_{a,s}^{c,t}$, since the product of operators $\text{Op}(C_{a,s}^{b,t})\text{Op}(B_{b,s}^{c,t})$ is zero. However, this calculation is clear by inspection.
Remark 28. Similar calculations to that of Theorem 25 were performed in Hudson [2007] pp 186-187 for the rotational case.

6.2 Covariance properties

The operator $W_{a,s}^b$ satisfy two further properties: shift covariance and time-reversal covariance. We describe these properties in the following two subsections.

6.2.1 Shift bi-covariance

We introduce the forward shift isometry on $L^2(\mathbb{R}^+)$ given by

$$S_r f(x) = \begin{cases} 
0 & \text{if } x \leq r, \\
 f(x-r) & \text{if } x > r.
\end{cases} \quad (6.2.1)$$

Using this isometry we state the shift covariance property of $W_{a,s}^b$ as a theorem.

Theorem 26. For arbitrary times $a < b$, $s < t$ and $p, q \in \mathbb{R}^+$

$$(S_p \oplus S_q)^* W_{a+p,s+q}^{b+p,t+q} (S_p \oplus S_q) = W_{a,s}^{b,t}. \quad (6.2.2)$$

Proof. The argument is exactly the same as in Hudson [2007] pp 188-189. For any $f \in L^2(\mathbb{R}^+)$ and kernel $Z$ we have

$$S^*_p \text{Op}(Z) S_q f(x) = \text{Op}(Z) S_q f(x + p) \quad (6.2.3)$$

$$= \int_{\mathbb{R}} Z(x+p,y) S_q f(y) dy \quad (6.2.4)$$

$$= \int_{\mathbb{R}} Z(x+p, y + q) f(y) dy \quad (6.2.5)$$

showing that $S^*_p \text{Op}(Z) S_q$ is the integral operator with kernel $(x, y) \mapsto Z(x+p, y+q)$. It follows that the operator $(S_p \oplus S_q)^* W_{a+p,s+q}^{b+p,t+q} (S_p \oplus S_q)$ has upper diagonal element differing from $S^*_p S_p = \text{id}_{\mathbb{R}^+}$ by the integral operator whose kernel takes value at $(x, y)$

$$\lambda^2 <_{a+p}^b (x + p, y + p) \langle \chi_s^t, \exp (\lambda^2 (y+p-(x+p)) \Delta^t_s) \chi_s^t \rangle \quad (6.2.6)$$

$$\lambda^2 <_{a}^b (x, y) \langle \chi_s^t, \exp (\lambda^2 (y-x) \Delta^t_s) \chi_s^t \rangle \quad (6.2.7)$$
and the upper off diagonal element whose kernel takes value
\[
- \lambda \chi_{a+p}(x+p) \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} \frac{(b+p-(x+p))^n}{n!} \frac{(y+q-(s+q))^n}{n!}
\]

which are the corresponding kernels of \(W_{b,t,a,s}^{b,t,a,s}\). The argument for the remaining two kernels is similar.

\[6.2.8\]

6.2.2 Time-reversal bi-covariance

We introduce the time-reversal map from \(L^2(\mathbb{R}^+\times \mathbb{R}^+)\) to \(L^2(\mathbb{R}^+\times \mathbb{R}^+)\) given by

\[
R^*_s f(x) = \begin{cases} 
 f(x) & \text{if } x \leq s, \\
 f(s + t - x) & \text{if } s < x \leq t, \\
 f(x) & \text{if } t < x.
\end{cases} \quad (6.2.9)
\]

The operator \(W_{b,t,a,s}^{b,t,a,s}\) has the following time-reversal property which we now state as a theorem.

**Theorem 27.** For arbitrary times \(a < b\), \(s < t\)

\[(R^b_a \oplus R^t_s)^* W_{b,t,a,s}^{b,t,a,s} (R^b_a \oplus R^t_s) = W_{a,s}^{a,s} . \quad (6.2.10)\]

**Proof.** The argument is exactly the same as in Hudson [2007] pp 188-189. For arbitrary \(f \in L^2(\mathbb{R}^+)\) and kernel \(Z\) we have

\[
(R^b_a)^* \text{Op}[Z] R^t_s(x)f
\]

\[
= \chi^b_a(x) \int_{\mathbb{R}^+} Z(a + b - x, y)(R^*_s f(y))dy \quad (6.2.11)
\]

\[
= \chi^b_a(x) \int_{\mathbb{R}^+} Z(a + b - x, y)\chi^t_s(y) f(s + t - y)dy \quad (6.2.12)
\]

\[
= \chi^b_a(x) \int_{\mathbb{R}^+} Z(a + b - x, s + t - y)\chi^t_s(y) f(y)dy \quad (6.2.13)
\]

\[
= \int_{\mathbb{R}^+} \chi^b_a(a + b - x)Z(a + b - x, s + t - y)\chi^t_s(s + t - y) f(y)dy \quad (6.2.14)
\]

\[
= \int_{\mathbb{R}^+} Z(a + b - x, s + t - y) f(y)dy, \quad (6.2.15)
\]

where we have used the identities on the second line from the bottom

\[
\chi^b_a(a + b - x) = \chi^b_a(x), \quad \chi^t_s(s + t - y) = \chi^t_s(y).
\]

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are easily carried over to the transformations $W$, the unitary implementor $U$, the subspace $U$, and each Bogolubov transformation $6.3.1$ Choosing bi-evolutionary implementors. We only consider the Bogolubov transformation is itself bi-evolutionary and second-quantised shift and time reversal covariant. In this chapter depending on a real parameter $\lambda$, which the bi-evolutionary, shift-covariant system of unitary implementors is covariant under. This system is unique to within multiplication by a particular scalar-valued bi-evolution. Next we show that this scalar-valued bi-evolution can be chosen so as to make the bi-evolutionary system of unitary implementors bi-covariant under second-quantised shifts. This system is unique to within multiplication by a particular scalar-valued bi-evolution. The map which sends each element of the group of unitarity implementable Bogolubov operators to the whole Fock space $\Gamma(U)$ is non-unique only to within multiplication by a scalar $a,s$ (6.2.17) which is the value of the corresponding kernel for $W_{b,t}^{a,s}$. The argument for the other entries are similar and is omitted. 

6.3 Uniqueness of the system of implementors of $W_{a,s}^{b,t}$.

In this section we show that there exist a system of unitary implementors of the bi-evolutionary shift and time reversal covariant system of Bogolubov transformations $(W_{a,s}^{b,t})_{0 \leq a < b, 0 \leq s \leq t}$, which is itself bi-evolutionary. This system is unique to within multiplication by an arbitrary scalar-valued bi-evolution. Next we show that this scalar valued bi-evolution can be chosen so as to make the bi-evolutionary system of unitary implementors bi-covariant under second-quantised time reversals. There is thus a unique system of unitary implementors which is itself bi-evolutionary and second-quantised shift and time reversal covariant. In this chapter we only consider the Bogolubov transformation $(W_{a,s}^{b,t})_{0 \leq a \leq b, 0 \leq s \leq t}$. However, the arguments are easily carried over to the transformations $W_{b,s}^{a,t}$, $W_{a,t}^{b,s}$ and $W_{b,t}^{a,s}$.

6.3.1 Choosing bi-evolutionary implementors.

Each Bogolubov transformation $W_{a,s}^{b,t}$ satisfies the Shale criterion for unitary implementability. The unitary implementor $U_{a,s}^{b,t}$ is non-unique only to within multiplication by a scalar $\omega$. Thus the unitary ray $U_{a,s}^{b,t} = \mathcal{T} U_{a,s}^{b,t} = \{ \omega U_{a,s}^{b,t} | \omega \in \mathbb{T} \}$ is unique. Since $W_{a,s}^{b,t}$ acts nontrivially only on the subspace $L^2([a,b]) \oplus L^2([s,t])$ of $\mathfrak{h} \oplus \mathfrak{h}$ and as the identity on the orthogonal complement of this subspace, $U_{a,s}^{b,t}$ is the ampliation to the whole Fock space $\Gamma(\mathfrak{h} \oplus \mathfrak{h})$ of a unitary operator on the Fock space $\Gamma(L^2([a,b]) \oplus L^2([s,t])) = \Gamma(L^2([a,b]) \otimes \Gamma(L^2([s,t])))$. It follows that, for disjoint pairs of intervals $[[a,b],[c,d]]$ and $[[s,t],[u,v]]$, $U_{a,s}^{b,t}$ commutes with any choice $U_{c,u}^{d,v}$ of unitary implementor of $W_{c,u}^{d,v}$ since they are ampliations of operators in different tensor factors. Equivalently $U_{a,s}^{b,t}$ commutes with $U_{c,u}^{d,v}$, where the product of unitary rays $U_1 = \mathcal{T} U_1$ and $U_2 = \mathcal{T} U_2$ is defined to be $U_1 U_2 = \mathcal{T} U_1 U_2$ and is independent of the choice of representatives $U_1$ and $U_2$.

The map which sends each element of the group of unitarity implementable Bogolubov
transformations to the unitary ray generated by its unitary implementor is a homomorphism into the group of all unitary rays equipped with this product. It follows that, for arbitrary non-negative $a \leq b \leq c$ and $s \leq t$.

$$U_{a,s}^{b,t}U_{b,s}^{c,t} = \xi(a, b, c|s, t)U_{a,s}^{c,t},$$  
(6.3.1)

where $\xi(a, b, c|s, t)$ is in $T$. Similarly, for fixed $a$ and $b$ and for any $0 < s < t < u$ we have

$$U_{a,s}^{b,t}U_{a,t}^{b,u} = \eta(a, b|s, t, u)U_{a,s}^{b,u},$$  
(6.3.2)

where $\eta(s, t, u|a, b)$ is in $T$. Therefore, we can further conclude that given a system of bi-evolutionary Bogolubov transformations $\{W_{a,s}^{b,t}\}$ the corresponding family of implementors $\{U_{a,s}^{b,t}\}$ are representatives of a system of unitary rays

$$\{TU_{a,s}^{b,t}\}$$

which is also a bi-evolution. The question may be asked: Is there a system of representatives which is a bi-evolution? The answer is affirmative and in fact we can say more which leads us to the next theorem (see Hudson and Jones [2013] for a similar theorem in the context of non-Fock quantum stochastic calculus).

**Theorem 28.** There exists a system of representatives $\{U_{a,s}^{b,t}\}$ which forms a bi-evolution. Moreover, any other system which forms a bi-evolution is got by multiplying this system by a $T$-valued bi-evolution.

**Proof.** Using (6.3.1) and (6.3.2), for an arbitrary system of representatives and arbitrary non-negative $a \leq b \leq c$ and $s \leq t \leq u$ we can expand $U_{a,s}^{c,u}$ in two different ways to obtain the relation

$$\xi(a, b, c|s, t)\xi(a, b, c|t, u)\eta(a, b, c|s, t, u)\xi(a, b, c|s, u).$$  
(6.3.3)

The justifying calculations are as follows

$$U_{a,s}^{b,t}U_{b,s}^{c,t}U_{a,t}^{b,u} = \xi(a, b, c|s, t)U_{a,s}^{c,t}\xi(a, b, c|t, u)U_{a,s}^{c,u}$$  
(6.3.4)

$$= \xi(a, b, c|s, t)\xi(a, b, c|t, u)\eta(a, b, c|s, t, u)U_{a,t}^{c,u}$$  
(6.3.5)

and since $U_{b,s}^{c,t}$ and $U_{a,t}^{b,u}$ commute we have

$$U_{a,s}^{b,t}U_{b,s}^{c,t}U_{a,t}^{b,u}U_{b,t}^{c,u} = \xi(a, b, c|s, t, u)U_{a,s}^{b,u}\eta(b, c|s, t, u)U_{a,s}^{c,u}$$  
(6.3.6)

$$= \eta(a, b|s, t, u)\eta(b, c|s, t, u)\xi(a, b, c|s, t, u)U_{a,s}^{c,u}.$$  
(6.3.7)

$$= \eta(a, b|s, t, u)\eta(b, c|s, t, u)\xi(a, b, c|s, u)U_{a,s}^{c,u}.$$  
(6.3.8)
By equating (6.3.5) and (6.3.8) we have equation (6.3.3). Using equation (6.3.3) we are able to show the existence of functions \( \xi' \) and \( \eta' \) such that

\[
\begin{align*}
\xi(0, a, c|s, u) - \xi'(a, c, s, u) &= \eta(a, c|0, s, u) - \eta'(a, c, s, u), \\
\eta'(a, c, s, t)\eta'(a, c, t, u) &= \eta'(a, c, s, u),
\end{align*}
\]

(6.3.9) (6.3.10)

so that the family of representatives \( \tilde{U}_{a,s}^{c,u} \) defined by

\[
\tilde{U}_{a,s}^{c,u} = \xi(a, c, s, u)
\]

(6.3.12)
is a bi-evolution. We observe that by the associative of multiplication we have \( (U_{a,0}^{b,u_0} U_{b,s}^{c,u}) U_{c,t}^{a,u} = U_{a,0}^{b,u_0} (U_{b,0}^{c,u} U_{c,t}^{a,u}) \) and \( (U_{a,0}^{c,u} U_{c,t}^{a,u}) U_{a,t}^{c,u} = U_{a,0}^{c,u} (U_{a,t}^{c,u} U_{a,s}^{c,u}) \). From which we have the relations

\[
\begin{align*}
\xi(a, b, c|s, u) &= \frac{\xi(0, a, b|s, u)\xi(0, b, c|s, u)}{\xi(0, a, c|s, u)}, \\
\eta(a, c|s, t, u) &= \frac{\eta(a, c|0, s, u)\eta(a, c|0, u, t)}{\eta(a, c|0, s, t)}.
\end{align*}
\]

(6.3.13) (6.3.14)

Using (6.3.9), (6.3.10), (6.3.11), (6.3.13), and (6.3.14) the bi-evolution property of \( \tilde{U}_{a,s}^{c,u} \) can be shown.

We now show the existence of solutions \( \xi' \) and \( \eta' \) of equations (6.3.9), (6.3.10) and (6.3.11). Putting \( a = s = 0 \) into (6.3.3) we have

\[
\frac{\xi(0, a, c|0, u)}{\xi(0, b, c|0, t, u)} = \frac{\eta(0, a, c|0, u)}{\eta(0, b|0, t, u)}
\]

(6.3.15)

setting \( b = a \) and \( t = s \) into (6.3.15)

\[
\xi(0, a, c|s, u)\frac{\eta(0, a|0, s, u)}{\eta(0, a|0, s, u)} = \eta(a, c|0, s, u)\frac{\xi(0, a, c|0, u)}{\xi(0, a, c|0, s)}
\]

(6.3.16)

Defining

\[
f(a|s, u) = \eta(0, a|0, s, u) \quad \text{and} \quad g(a, c|s) = \xi(0, a, c|0, s)
\]

(6.3.17)

and using these definitions in (6.3.16) we have

\[
\xi(0, a, c|s, u)\frac{f(c|s, u)}{f(a|s, u)} = \eta(a, c|0, s, u)\frac{g(a, c|u)}{g(a, c|s)}
\]

(6.3.18)

Therefore setting

\[
\xi'(a, c, s, u) = \frac{f(a|s, u)}{f(c|s, u)} \quad \text{and} \quad \eta'(a, c, s, u) = \frac{g(a, c|s)}{g(a, c|u)}
\]

(6.3.19)

we have that \( \xi' \) and \( \eta' \) are evolutions i.e. satisfy equations (6.3.10) and (6.3.11) respectively,
and also satisfy equation (6.3.9). This proves the existence of the bi-evolutionary system of representatives $\tilde{U}^{b,t}_{a,s}$. If $\hat{U}^{b,t}_{a,s}$ is another such system then writing $\hat{U}^{b,t}_{a,s} = \zeta(a,b,s,t)\tilde{U}^{b,t}_{a,s}$ we see that the function $\zeta$ is a $T$-valued bi-evolution by comparing the bi-evolutionary identities for the two systems of representatives.

### 6.3.2 The shift bi-covariance property of the implementors of $W^{b,t}_{a,s}$

In this subsection we show that there exists a system of implementors that is shift bi-covariant as well as being a bi-evolution and is in fact, unique up to multiplication by a scalar with a particular functional form which is parameterised by a real number.

In the proof of Theorem 30 below, we use the following theorem, whose proof is taken from Hudson and Jones [2013].

**Theorem 29.** Every $T$-valued bi-evolution $\zeta$ is of the form

$$\zeta(a,b,s,t) = F(a,s)\overline{F(b,s)}\overline{F(a,t)}F(b,t)$$  \hspace{1cm} (6.3.20)

for some function $F$ on $\mathbb{R}^2$. $F$ is non unique to the extent that we may replace $F$ by $G$ where

$$G(a,s) = \theta f(a)g(s)F(a,s),$$  \hspace{1cm} (6.3.21)

where $\theta$ is a fixed element of $T$ and $f$ and $g$ are arbitrary $T$-valued functions. In particular there is a unique choice of $F$ satisfying

$$F(a,0) = F(0,s) = 1$$  \hspace{1cm} (6.3.22)

for arbitrary $a, s \in \mathbb{R}$.

**Proof.** Using the bi-evolution property of $\zeta$ we have that, by evolution in the first two parameters of $\zeta$

$$\zeta(0,b,s,t) = \zeta(0,a,s,t)\zeta(a,b,s,t).$$  \hspace{1cm} (6.3.23)

Rearranging the above we have

$$\zeta(a,b,s,t) = \frac{\zeta(0,b,s,t)}{\zeta(0,a,s,t)}.$$  \hspace{1cm} (6.3.24)

Again using the bi-evolution property, we have, by evolution in the last two parameters of $\zeta$

$$\zeta(a,b,s,t) = \frac{\zeta(0,b,0,t)\zeta(0,a,0,s)}{\zeta(0,b,0,s)\zeta(0,a,0,t)}.$$  \hspace{1cm} (6.3.25)

Setting $F(a,s) = \zeta(0,a,0,s)$, using (6.3.25) we can write $\zeta$ as

$$\zeta(a,b,s,t) = F(b,t)F(a,s)\overline{F(b,s)}\overline{F(a,t)}.$$  \hspace{1cm} (6.3.26)
Now suppose that as well as (6.3.26) we have
\[ \zeta(a, b, s, t) = G(b, t)G(a, s)G(b, s)G(a, t) \] (6.3.27)
for some function \( G \). Equating the right hand sides of (6.3.26) and (6.3.27) we have that
\[ F(b, t)F(a, s)F(b, s)F(a, t) = G(b, t)G(a, s)G(b, s)G(a, t). \] (6.3.28)
Writing \( H = G^F \) we have
\[ H(a, s)H(b, t) = H(a, t)H(b, s). \] (6.3.29)
Collecting all terms involving \( a \) on the left and all terms involving \( b \) on the right in (6.3.29) we have
\[ H(a, s)H(a, t) = H(b, s)H(b, t). \] (6.3.30)
Since the left hand side is independent of \( b \) and the right hand side is independent of \( a \) we have,
\[ H(a, s)H(a, t) = H(b, s)H(b, t) = \alpha(s, t), \] (6.3.31)
for some function \( \alpha : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{T} \). The function \( \alpha \) is an evolution since for example
\[ \alpha(s, t)\alpha(t, u) = H(a, s)H(a, t)H(a, u) \]
\[ = H(a, s)H(a, u) \]
\[ = \alpha(s, u). \]
We write
\[ \alpha(s, t) = f(s)\overline{f(t)} \] (6.3.32)
for unique \( f : \mathbb{R}^+ \to \mathbb{T} \) satisfying \( f(0) = 1 \). Thus from (6.3.31) we have
\[ H(a, s)H(a, t) = H(b, s)H(b, t) = f(s)\overline{f(t)}. \] (6.3.33)
Similarly, collecting all terms involving \( s \) and the left and all terms involving \( t \) on the right in (6.3.29) we have
\[ H(a, s)H(b, s) = H(a, t)H(b, t) = g(a)\overline{g(b)}, \] (6.3.34)
for unique \( g : \mathbb{R}^+ \to \mathbb{T} \) satisfying \( g(0) = 1 \). Dividing equation (6.3.33) by (6.3.34) we have
\[ \frac{H(a, t)H(b, s)}{H(a, s)H(b, s)} = f(s)\overline{f(t)}g(a)\overline{g(b)}, \] (6.3.35)
and thus we have
\[ H(b, s)\overline{f(s)g(b)} = H(a, t)\overline{f(t)g(a)}. \] (6.3.36)
The left hand side is independent of \( a \) and \( t \), and the right hand side is independent of \( b \) and \( s \).
so both sides are equal to a constant \( \theta \). Therefore from (6.3.36) we have

\[
H(b, s) = \theta f(s)g(b) \quad \text{and} \quad H(a, t) = \theta f(t)g(a). \tag{6.3.37}
\]

Therefore we have shown that \( F \) is non-unique to the extent that we may replace \( F \) by \( G \) where \( G(a, s) = \theta f(s)g(a)F(a, s) \).

Assuming \( H(b, s) = \theta f(s)g(b) \) with arbitrary choice of \( f \) and \( g \) with \( f(0) = g(0) = 1 \) then (6.3.29) is satisfied. If we choose

\[
f(s) = F(0, 0)F(0, s), \quad g(a) = F(0, 0)F(a, 0) \quad \text{and} \quad \theta = F(0, 0) \tag{6.3.38}
\]

we find that we can replace \( F \) by \( G = HF \) which satisfies \( G(a, 0) = G(0, s) = 1 \) and this choice is unique.

**Theorem 30.** There exists a system of representatives \( \{U^{b,t}_{a,s}\}_{0 \leq a \leq b, 0 \leq s \leq t} \) which forms a shift bi-covariant bi-evolution. This system of representatives is unique up to multiplication by

\[
e^{i\lambda(b-a)(t-s)}
\]

for some \( \lambda \in \mathbb{R} \).

**Proof.** Choosing a bi-evolutionary system of representatives \( \{U^{b,t}_{a,s}\}_{0 \leq a < b, 0 \leq s < t} \) which implements the system of Bogolubov tranformations \( \{W^{b,t}_{a,s}\}_{0 \leq a < b, 0 \leq s < t} \) we have

\[
W \left( \left( (S_p \oplus S_q)^* W_{a+p,s+q}^{b+p,t+q}(S_p \oplus S_q) \right) (f_1 \oplus f_2) \right) = U^{-1}_{b+p,t+q}(S_p \oplus S_q)(f_1 \oplus f_2) W_{b,t}^{a+p,s+q}(S_p \oplus S_q)
\]

which is equal to, by the shift bi-covariance property of \( W^{b,t}_{a,s} \),

\[
U^{a,b}_{b,t} W(f_1 \oplus f_2) U^{b,t}_{a,s} = W \left( W^{b,t}_{a,s}(f_1 \oplus f_2) \right) \tag{6.3.40}
\]

Thus, we have, using (6.3.40) and the fact that \( U_{b+p,t+q}(S_p \oplus S_q)U_{a+p,s+q}(S_p \oplus S_q) \) can be shown to be also a bi-evolution. Therefore, there exists a \( \mathbb{T} \)-valued bi-evolution \( \zeta \) by Theorem 28 such that

\[
\zeta_{b,t}(a, b|s, t)U^{b,t}_{a,s} = U_{b+p,t+q}(S_p \oplus S_q)U_{a+p,s+q}(S_p \oplus S_q) \tag{6.3.41}
\]

for some \( \mathbb{T} \)-valued bi-evolution \( \zeta_{p,q} \).
Evaluating $U_{a+p_1+p_2,t+q_1+q_2}^{b+p_1+p_2,t+q_1+q_2}$ in two different ways using the fact that

$$U_{s_1} S_{p_1} \oplus U_{s_2} = (U_{s_1} \oplus U_{s_1}) \left(U_{s_2} \oplus U_{s_2}\right)$$

we have on one hand

$$U_{a+p_1+p_2,s+q_1+q_2}^{b+p_1+p_2,t+q_1+q_2} = U_{s_2}^{(s_2 \oplus s_2)} \zeta_{p_2,q_2} (a + p_1, b + p_1 | s + q_1, t + q_1) U_{s_2}^{(s_2 \oplus s_2)}$$

$$= U_{s_2}^{(s_2 \oplus s_2)} \zeta_{p_2,q_2} (a + p_1, b + p_1 | s + q_1, t + q_1) U_{s_2}^{(s_2 \oplus s_2)} \zeta_{p_1,q_1} (a, b | s, t) U_{s_2}^{b,t} U_{s_2}^{(s_2 \oplus s_2)} U_{s_2}^{(s_2 \oplus s_2)}.$$  

(6.3.43)

On the other hand

$$U_{a+p_1+p_2,s+q_1+q_2}^{b+p_1+p_2,t+q_1+q_2} = U_{s_2}^{(s_2 \oplus s_2)} \zeta_{p_1+p_2,q_1+q_2} (a, b | s, t) U_{s_2}^{b,t} U_{s_2}^{(s_2 \oplus s_2)}.$$  

(6.3.44)

Thus

$$\zeta_{p_1+p_2,q_1+q_2} (a, b | s, t) = \zeta_{p_2,q_2} (a + p_1, b + p_1 | s + q_1, t + q_1) \zeta_{p_1,q_1} (a, b | s, t).$$  

(6.3.45)

Setting $q_1 = q_2 = 0$ and denoting $\zeta_{p,0}$ by $\zeta_p$ we have,

$$\zeta_{p_1+p_2} (a, b | s, t) = \zeta_{p_2} (a + p_1, b + p_1 | s, t) \zeta_{p_1} (a, b | s, t).$$  

(6.3.46)

Representing each $\zeta_p$ as in Theorem 29 as

$$\zeta_p (a, b | s, t) = F_p (a | s) F_p (a | t) F_p (b | s) F_p (b | t),$$  

(6.3.47)

where $F_p (a | 0) = F_p (0 | s) = 1$. Substituting (6.3.47) in equation (6.3.46) we have

$$F_{p_1+p_2} (a | s) F_{p_1+p_2} (b | s) F_{p_1+p_2} (a | t) F_{p_1+p_2} (b | t)$$

$$= F_{p_2} (a + p_1 | s) F_{p_2} (b + p_1 | s) F_{p_2} (a + p_1 | t) F_{p_2} (b + p_1 | t)$$

$$\cdot F_{p_1} (a | s) F_{p_1} (b | s) F_{p_1} (a | t) F_{p_1} (b | t).$$  

(6.3.48)

Setting $a = 0$ and $s = 0$ we have

$$F_{p_1+p_2} (b | t) = \frac{F_{p_2} (b + p_1 | t)}{F_{p_2} (p_1 | t)} F_{p_1} (b | t).$$  

(6.3.49)

Multiplying both sides of (6.3.49) by $F_{p_2} (p_1 | t)$ we have

$$F_{p_1+p_2} (b | t) F_{p_2} (p_1 | t) = F_{p_2} (b + p_1 | t) F_{p_1} (b | t).$$  

(6.3.50)
Writing \( \omega_t(p, b) = F_p(b|t) \) we have

\[
\omega_t(p_1 + p_2, b) \omega_t(p_2, p_1) = \omega_t(p_2, b + p_1) \omega_t(p_1, b).
\] (6.3.51)

It is known (see Hudson et al. [1984]), that the general continuous solution of the 2-cocycle equation (6.3.51) on \( \mathbb{R}_+ \) is

\[
\omega_t(r, s) = \frac{f_t(r) f_s(s)}{f_t(r + s)}
\] (6.3.52)

for some \( f : \mathbb{R}_+ \to \mathbb{T} \) which is unique up to multiplication by the function \( r \mapsto e^{i \lambda_t r} \) for some \( \lambda_t \in \mathbb{R} \), further more \( f_t(0) = 1 \). Therefore we have the 2-cocycle equation parameterised by \( t \), a positive real number.

Let

\[
\tilde{U}_{a,s,p}^{b,t} := U_{(S_p \oplus \mathrm{id})} U_{a+p,s+t}^{b,t} U_{(S_p \oplus \mathrm{id})}.
\] (6.3.53)

It can be shown that the system of representatives \( (\tilde{U}_{a,s,p}^{b,t})_{0 \leq a < b, 0 \leq s < t} \) is a bi-evolution. Since \( (\tilde{U}_{a,s,p}^{b,t})_{0 \leq a < b, 0 \leq s < t} \) is a bi-evolution by Theorem 28 there exists \( \zeta_p(a, b|s, t) \), a \( \mathbb{T} \)-valued bi-evolution such that

\[
\tilde{U}_{a,s,p}^{b,t} = \zeta_p(a, b|s, t) U_{a,s}^{b,t}
\] (6.3.54)

\[
= F_p(a | s) F_p(b | t) F_p(a | t) F_p(b | t) U_{a,s}^{b,t}
\] (6.3.55)

\[
= \omega_t(p, a) \omega_t(p, b) U_{a,s}^{b,t}
\] (6.3.56)

\[
= \frac{f_s(a)}{f_s(b)} \frac{f_t(b + p)}{f_t(b)} U_{a,s}^{b,t}
\] (6.3.57)

Rearranging the above we have

\[
\frac{f_s(a + p)}{f_s(b + p)} \frac{f_t(b + p)}{f_t(a + p)} \tilde{U}_{a,s,p}^{b,t} = \frac{f_s(a)}{f_s(b)} \frac{f_t(b)}{f_t(a)} U_{a,s}^{b,t}.
\] (6.3.58)

Substituting (6.3.53) into (6.3.58) we have

\[
U_{(S_p \oplus \mathrm{id})} \frac{f_s(a + p)}{f_s(b + p)} \frac{f_t(b + p)}{f_t(a + p)} U_{a+s,t}^{b+2p,t} U_{(S_p \oplus \mathrm{id})} = \frac{f_s(a)}{f_s(b)} \frac{f_t(b)}{f_t(a)} U_{a,s}^{b,t}.
\] (6.3.59)

So the system of representatives

\[
\hat{U}_{a,s,p}^{b,t} := \frac{f_s(a)}{f_s(b)} \frac{f_t(b)}{f_t(a)} U_{a,s}^{b,t}
\] (6.3.60)

satisfies

\[
\hat{U}_{a,s}^{b,t} = U_{(S_p \oplus \mathrm{id})} \hat{U}_{a+p,s+t}^{b+p,t} U_{(S_p \oplus \mathrm{id})}
\] (6.3.61)

Therefore the system of representative \( (\hat{U}_{a,s}^{b,t})_{0 \leq a < b, 0 \leq s < t} \) is shift covariant in \( a \) and \( b \) for any
s and t. Hence any bi-evolution \( U_{a,s}^{b,t} \) that is shift covariant in its parameters \( a \) and \( b \) is unique up to multiplication by the factor
\[
e^{i(\lambda_t - \lambda_s)(b-a)}.
\]

(6.3.62)

From the previous argument we can choose a bi-evolutionary system of representatives \((U_{a,s}^{b,t})_{0 \leq a < b, 0 \leq s < t}\) which is also shift covariant in \( a \) and \( b \). The \( \zeta_{p,q}(a,b,s,t) \) of equation (6.3.41) is, with this system of representatives, of the form
\[
\zeta_{p,q}(a,b,s,t) = e^{i(\lambda_t(q) - \lambda_s(q))(b-a)},
\]
where \( \lambda_t : \mathbb{R}_+ \rightarrow \mathbb{R} \) for all \( t \in \mathbb{R}_+ \), since the system \((U_{a,s}^{b,t})_{0 \leq a < b, 0 \leq s < t}\) is bi-evolutionary system shift covariant in \( a \) and \( b \). Using (6.3.63) in equation (6.3.45) we have
\[
e^{i(\lambda_t(q_1 + q_2) - \lambda_s(q_1 + q_2))(b-a)} = e^{i(\lambda_t(q_2) - \lambda_s(q_2))(b-a)} e^{i(\lambda_t(q_1) - \lambda_s(q_1))(b-a)}.
\]

(6.3.64)

Setting \( \omega_t(q) = e^{i\lambda_t(q)} \) and substituting into (6.3.64) we have
\[
\omega_t(q_1 + q_2)\omega_s(q_1)\omega_t(q_2) = \omega_{t+q_1}(q_2)\omega_{s+q_1}(q_2)\omega_t(q_1)\omega_s(q_1).
\]

(6.3.65)

Since \( \omega_s(q) = 1 \) for \( s = 0 \) and for all \( q \in \mathbb{R}_+ \) (as the \( F_p(b(t))'s \) of (6.3.48) are equal to one whenever \( t = 0 \) for all \( p \) and \( b \) in \( \mathbb{R}_+ \)), setting \( s = 0 \) in (6.3.65) we have
\[
\omega_t(q_1 + q_2) = \omega_{t+q_1}(q_2)\omega_{s+q_1}(q_2)\omega_t(q_1).
\]

(6.3.66)

We write \( \omega(t,q) = \omega_t(q) \). Then \( \omega \) is a 2-cocycle on \( \mathbb{R} \). It is well known that the general continuous solution of the 2-cocycle equation on \( \mathbb{R} \) is
\[
\omega(t,q) = \frac{f(t)f(q)}{f(t+q)}
\]
for some \( f : \mathbb{R} \rightarrow \mathbb{T} \) with \( f(0) = 1 \). The \( f \) is unique up to multiplication by the function \( r \rightarrow e^{i\lambda r} \) with \( \lambda \in \mathbb{R} \). Therefore given any bi-evolutionary system of representatives \((U_{a,s}^{b,t})\) which is also shift covariant in \( a \) and \( b \) we have
\[
e^{i(\lambda_t(q) - \lambda_s(q))(b-a)} U_{a,s}^{b,t} = \left( \frac{f(t)f(q)}{f(t+q)} \right)^{(b-a)} \left( \frac{f(s+q)}{f(s)} \right)^{(b-a)} U_{a,s}^{b,t} = U_{(S_p\oplus S_q)^*}^{\lambda_t+p,t+q} U_{(S_p\oplus S_q)^*}^{b,t+1}. \]

(6.3.68)

Cancelling out common factors and multiplying both sides by \( \left( \frac{f(t+q)}{f(s+q)} \right)^{(b-a)} \) we have
\[
\left( \frac{f(t)}{f(s)} \right)^{(b-a)} U_{a,s}^{b,t} = \left( \frac{f(t+q)}{f(s+q)} \right)^{(b-a)} U_{(S_p\oplus S_q)^*}^{\lambda_t+p,t+q} U_{(S_p\oplus S_q)^*}^{-1}. \]

(6.3.69)
In other words, we have constructed a shift bi-covariant bi-evolutionary system of representatives from a system of representative which is only bi-evolutionary and shift covariant in the $a$ and $b$ parameters. Furthermore, this new system of representatives is unique up to multiplication by

$$e^{i\lambda(t-s)(b-a)},$$

where $\lambda$ is a real number.
Chapter 7

Conclusion

We have constructed the Bogolubov transformations $W_{b,t}^{b,t}$, $W_{a,s}^{a,t}$, $W_{a,t}^{b,s}$ and $W_{b,s}^{a,s}$. Using the bievolutionary and shift bi-covariance of the $W$’s we can show that their implementors are unique up to multiplication by the factor $e^{i\lambda(b-a)(t-s)}$ where $\lambda$ is a real number. Using the time covariance of the $W$’s we expect that the value of $\lambda$ can be fixed. We conjecture that the implementors in the Fock representation CCR got from $W$’s are indeed the corresponding double products

$$b \prod_{s}^{t} \left( 1 + \lambda (dA^\dagger \otimes dA^\dagger - dA \otimes dA - \frac{1}{2} \lambda^2 dT \otimes dT) \right).$$ (7.0.1)

We proved in Hudson and Jones [2012] that the implementors of Bogolubov transformations for the double product integral

$$b \prod_{s}^{t} \left( 1 + \lambda (dA^\dagger \otimes dA - dA \otimes dA^\dagger) \right)$$ (7.0.2)

(which are second quantisations of the $W$’s) are indeed double products. They form the unique bi-evolutionary bi-covariant and time reversal covariant system of implementors of the corresponding Bogolubov transformation, which are unitary in the rotational case.
Chapter 8

Appendix A

8.1 The shuffle Hopf algebra

Assume a complex vector space \( \mathcal{J} \). A unital algebra structure is imposed on the vector space \( \mathcal{T}(\mathcal{J}) = \bigoplus_{i=0}^{\infty} \mathcal{J}^\otimes_{n} \) (equipped with the natural componentwise operations) of all tensors \( \alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots) \), where \( \alpha_n \in \mathcal{J}^\otimes_{n} = \mathcal{J} \otimes \mathcal{J} \otimes \ldots \otimes \mathcal{J} \) for all \( n \) and in particular \( \mathcal{J}^\otimes_{0} = \mathbb{C} \).

If \( \mathcal{J} \) has in addition to being a vector space, the structure of an associative algebra then, by modifying the shuffle product, a new algebra can be created which is called the *sticky shuffle product algebra*. We call this modified multiplication the *sticky shuffle product* and denote it by \( \sigma \). This algebra models the multiplication of both classical and quantum stochastic iterated integrals. For \( dA_1, dA_1, \ldots, dA_m \in \mathcal{J} \) we denote by \( \{dA_1 \otimes dA_2 \otimes \ldots \otimes dA_m\} \) the natural embedding \((0, 0, 0, \ldots, dA_1 \otimes dA_2 \otimes \ldots dA_m, 0, \ldots)\)

8.1.1 The shuffle product algebra

We describe the shuffle product algebra \((\mathcal{T}(\mathcal{J}), \sigma, \eta)\). The details of \( \sigma \), \( \eta \) are given below.

8.1.1.1 The shuffle product \( \sigma \)

The multiplication map \( \sigma : \mathcal{T}(\mathcal{J}) \times \mathcal{T}(\mathcal{J}) \to \mathcal{T}(\mathcal{J}) \) on the vector space \( \mathcal{T}(\mathcal{J}) \) is defined as follows. For any \( \alpha \) and \( \beta \in \mathcal{T}(\mathcal{J}) \), each component of the product \( \alpha \beta = \gamma = (\gamma_0, \gamma_1, \gamma_2 \ldots \gamma_n, \ldots) \) is given by

\[
\gamma_n = \sum_{A \cup B = \{1, 2, \ldots n\}} \sum_{A \cap B = \emptyset} \alpha_A^B \beta_B^A
\]
where the summation is over all $2^n$ ordered pairs $(A, B)$ of disjoint subsets of $\{1, \ldots, n\}$ whose union is $\{1, 2, \ldots, n\}$, $|A|$ denotes the cardinality of the ordered pair $A$, similarly for $B$. Finally, $\alpha^A_{|A|} \beta^B_{|B|}$ denotes the element of the $n$-fold homogeneous tensor product $\otimes^n I$, where the copies of the elements in $I$ of the $|A|$-fold homogenous component of $\alpha$ are put into positions labelled by $A$, while preserving the ordering of the components of the $|A|$-fold homogenous component of $\alpha$. The $|B|$ fold homogeneous component of $\beta$ occupies positions in $\otimes^n I$ in exactly the same way. Since $A \cup B = \{1, 2, \ldots, n\}$ all positions in $\otimes^n I$ are occupied exactly once.

The shuffle product can also be defined by linear extension of the rule

$$\{dA_1 \otimes dA_2 \otimes \ldots \otimes dA_m\}\{dA_{m+1} \otimes dA_{m+2} \otimes \ldots \otimes dA_{m+n}\} = \sum_{\pi \in S(m,n)} \{dA_{\pi(1)} \otimes dA_{\pi(2)} \otimes \ldots dA_{\pi(m+n)}\}, \quad (8.1.1)$$

where $S(m,n)$ is the set of $(m,n)$-shuffles, defined as the subset of the set $S(m+n)$ of all permutations $\pi$ of $(1, 2, \ldots, m+n)$, such that $\pi(1) < \pi(2) < \ldots < \pi(m)$ and $\pi(m+1) < \pi(m+2) < \ldots < \pi(m+n)$. See Hudson [2012] for a proof of the equivalence.

8.1.1.2 The unit element $\eta$

The unit element is denoted for brevity by $\eta = (1, 0, 0 \ldots)$.

Remark 29. The point of defining the unit in this complicated way is so that the “counit” is exactly “dual” to the unit; in particular it is a map $\mathcal{T}(I) \to \mathbb{C}$

8.2 The sticky shuffle product algebra

We describe the sticky shuffle product algebra $(\mathcal{T}(I), M, \eta)$. The details of $M$ and $S$ are given below.

8.2.0.3 The sticky shuffle product $M$

By assuming that $I$ also has the structure of an associative algebra, we can define the following multiplication map. The multiplication map $M : \mathcal{T}(I) \times \mathcal{T}(I) \to \mathcal{T}(I)$ on the vector space $\mathcal{T}(I)$ is defined as follows. For any $\alpha$ and $\beta \in \mathcal{T}(I)$, each component of the product $\alpha \beta = \gamma = (\gamma_0, \gamma_1, \gamma_2 \ldots \gamma_n, \ldots)$ is given by

$$\gamma_n = \sum_{A \cup B = \{1, 2, \ldots, n\}} \alpha^A_{|A|} \beta^B_{|B|},$$

where the summation is over all $3^n$ ordered pairs $(A, B)$ of not necessary disjoint subsets of $\{1, \ldots, n\}$ whose union is now $\{1, 2, \ldots, n\}$, and $\alpha^A_{|A|} \beta^B_{|B|}$ denotes the element of the $n$-fold homogeneous tensor product $\otimes^n I$, where double occupancies are resolved by the multiplication.
rule of \( J \); the element in \( I \) from the \(|A|\)-fold homogenous component of \( \alpha \) on the left, and the element in \( I \) of the \(|B|\)-fold component of \( \beta \) on the right of the multiplication in \( J \). The sticky shuffle product is associative (see Hudson [2012]) and we have for any \( \alpha, \beta \) and \( \gamma \in \mathcal{T}(J) \)

\[
(\alpha \beta \gamma)_n = (\alpha \beta \gamma)_n = \sum_{A \cup B \cup C = \{1, 2, \ldots, n\}} \alpha_A^A \beta_B^B \gamma_C^C \tag{8.2.1}
\]

where now the summation is over all ordered triples \((A, B, C)\) whose union is \(\{1, 2, \ldots, n\}\).

The sticky shuffle product can also be defined by linear extension of the rule

\[
\{dA_1 \otimes dA_2 \otimes \ldots \otimes dA_m\} \{dA_{m+1} \otimes dA_{m+2} \otimes \ldots \otimes dA_{m+n}\} = \sum_{P \in \mathcal{P}(m, n)} \{dA_{P_1} \otimes dA_{P_2} \otimes \ldots dA_{P_k}\}, \tag{8.2.2}
\]

where \(\mathcal{P}(m, n)\) is the set of sticky shuffles. The set of sticky shuffles is the set of all ordered partitions \(P = \{P_1, P_2, \ldots, P_k\}\) of \(\{1, 2, \ldots, m+n\}\) into subsets \(P_j\) which are either singletons \([s]\) or ordered pairs \((s, t)\) with \(s \in \{1, 2, \ldots, m\}\) and \(t \in \{m+1, m+2, \ldots m+n\}\), and such that if the internal brackets of \(P\) are deleted the sets \(s \in \{1, 2, \ldots, m\}\) and \(t \in \{m+1, m+2, \ldots m+n\}\) retain their natural order. The elements of \(P\) are used in the right hand side of (8.2.2) as follows, if \(P_j = \{s\}\) then \(dL_{P_j} = dL_s\), and if \(P_j = (s, t)\) then \(dL_{P_j} = dL_s dL_t\). Hence this resolves all double occupancies.

Using the sticky shuffle product we can define, in the weak sense, the product of two or more stochastic integrals. For any arbitrary \(\alpha, \beta \in \mathcal{T}(J)\) and \(a < b\) then we have

\[
I^b_a(\alpha)I^b_a(\beta) = I^b_a(\alpha \beta) \tag{8.2.3}
\]

in the weak sense, where \(\alpha \beta\) is calculated using the sticky shuffle product.

**Example 8.** Using the notation of Definition 9 we have the following

\[
I^b_a(-, +)I^b_a(-, +) = 4I^b_a(-, -, +, +) + 2I^b_a(-, +, -, +) + I^b_a(-, 0, +) \tag{8.2.4}
\]

### 8.3 The double product “Fubini theorem”

**Theorem 31.** Given any \( dr \in J \otimes J \) the double product integral

\[
\overrightarrow{b} \overrightarrow{a} \prod_s (1 + dr) \tag{8.3.1}
\]

can be defined either as either

\[
\overrightarrow{b} \overrightarrow{a} \prod (1 + \overrightarrow{t} (1 + dr)) \text{ or } \overrightarrow{a} \overrightarrow{b} \prod (1 + \overrightarrow{a} (1 + dr)) \tag{8.3.2}
\]

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and the two definitions are equivalent.

Before we prove the equivalence we state and prove a lemma needed for the proof.

**Lemma 20. The rearrangement lemma.** Given arbitrary \( dr \in J \otimes J \) we have the following equivalence

\[
\sum_{A_1 \cup \ldots \cup A_m \in N_n} \prod_{j=1}^m dr_j^{j,m+A_j} = \sum_{M \in M_{m,n}} \prod_{(j,k) \in N_m \times N_n} (dr_j)^{(j,m+k)}_{M_{j,k}} \tag{8.3.3}
\]

Here summation on the left of (8.3.3) is over all order \( m \)-tuples \((A_1, A_2, \ldots, A_m)\) of non-empty subsets whose union is \( N_n \). The notation \( A_j \) is for the set \( \{a_1, a_2, \ldots, a_{|A_j|}\} \) with \( a_1 < a_2 \ldots < a_{|A_j|} \). Using this notation we define \( dr_{j,A_j} \) as the element

\[
\prod_{l=1}^{|A_j|} dr_{j,m+a_l} \in J^m \times \mathbb{I}_j \times \ldots \times \mathbb{I}_j \otimes \cdots \otimes \mathbb{I}_j \otimes \mathbb{I}_j \times \mathbb{I}_m \times \ldots \times \mathbb{I}_m \times \mathbb{I}_m \times \ldots \times \mathbb{I}_m,
\]

where the superscripts indicate places within the \((m+n)\)-fold tensor product \((\otimes^m J) \otimes (\otimes^n J)\). The factors \( dr_{B_k,m+k} \) are defined similarly.

The notation \( M_{m,n} \) denotes the set of all \( m \times n \) matrices \( M = [M_{j,k}]_{(j,k) \in N_m \times N_n} \) with each \( M_{j,k} \in \{0,1\} \) and such that each row and each column of each \( M \) contains at least one entry 1. The factors \( (dr_j)^{(j,m+k)}_{M_{j,k}} \) are defined as \( dr \) if \( M_{j,k} = 1 \) and void (formally 1) if \( M_{j,k} = 0 \).

We now prove the equivalence of the two expressions for a double product:

**Proof.** The equality of the three sums is established using the correspondence

\[
k \in A_j \iff M_{j,k} = 1 \iff j \in B_k. \tag{8.3.5}
\]

\( \square \)

We now go on to prove Theorem 31.

**Proof.** We consider the definition in the form

\[
k \prod_{a} (1 + \prod_{t} (1 + dr)) \tag{8.3.6}
\]

and expand the inner product integral using Picard iteration to give

\[
\prod_{t} (1 + dr) = \sum_{i=2}^{\infty} (1 \otimes I_i) \cdot dr_{1;2} \cdot \ldots \cdot dr_{1;1}. \tag{8.3.7}
\]
Using (8.3.7) in the chosen definition we have

\[
\int_{a}^{b} \prod \left(1 + \sum_{i=2}^{\infty} (1 \otimes I_i) \, dr^1 \cdot \ldots \cdot dr^i \right)
\]

\[= 1 + \sum_{m=1}^{\infty} (I_a^m \otimes 1) \sum_{n_1=1, \ldots, n_m=1}^{\infty} (1 \otimes I_i^m) \, dr^{1+m+1} \cdot \ldots \cdot dr^{m+n_1+n_2+\ldots (1 \otimes I_s^m) \, dr^{m+m+1}}
\]

\[= 1 + \sum_{m=1}^{\infty} (I_a^m \otimes I_s^m) \sum_{n_1=1, \ldots, n_m \in N}^{\infty} \prod_{k=1}^{m} dr^{m+A_{j_k}} \quad (8.3.8)
\]

where repeated Picard iteration is used to arrive at the first equality and the sticky shuffle product is used repeatedly to arrive at the second equality. A similar argument shows that the other definition of the double product integral leads to

\[
\prod_{s} \left(1 + \int_{a}^{b} (1 + dr) \right) = 1 + \sum_{m=1}^{\infty} (I_a^m \otimes I_s^m) \sum_{n=1}^{\infty} \sum_{B_1 \cup \ldots \cup B_n}^{\infty} \prod_{k=1}^{n} dr^{B_k, m+k} \quad (8.3.9)
\]

Using the rearrangement lemma it follows that the sums (8.3.8) and (8.3.9) are equal.
Nomenclature

Roman Symbols

\( \overrightarrow{\Pi} \), page 15
\( \overrightarrow{\Pi} \), page 12

\((a_j, a_j^\dagger)\) standardised annihilation and creation operators, page 37

\((b_j, b_j^\dagger)\) standardised annihilation and creation operators, page 37

\( <_a \) , page 49
\( >_a \), page 49
\( \lambda_{m,n} \), page 100
\( \Delta_a^<_b \) For \( a < b \), page 57
\( \Delta_a^>_b \) For \( a > b \), page 57

\( \Gamma(g) \) Fock space over a Hilbert space \( \mathfrak{h} \), page 2

\( \iota \) Maps a creation operator to its conjugate annihilation operator and vice-versa, page 40

\( \kappa \), page 40
\( \lambda^{(\alpha)} \), page 49
\( \lambda_m \), page 49

\( \mathcal{J} \) The Itô algebra \( \mathbb{C}(dA, dA^\dagger, dT) \), page 7

\( g \) Arbitrary separable Hilbert space, page 2

\( \mathfrak{h} \) \( L^2(\mathbb{R}_+, \mathbb{C}) \), page 2

\( \mathfrak{h}' \) The Hilbert space \( L^2(t, \infty) \), page 2
The Hilbert space $L^2(0, t)$, page 2

$\mathcal{K}$ non-negative definite kernel, page 1

$\mathcal{E}(S)$ Denotes the linear span generated by exponential vectors $\{\psi(g): g \in S\}$, page 2

$\mathcal{E}^t$ Denote the linear span of the vectors $\mathcal{E}$, page 2

$\mathcal{E}_t$ Denote the linear span of the vectors $\mathcal{E}_t$, page 2

$\mathcal{H}$ Fock space over $\mathcal{E}$, page 2

$\mathcal{H}^t$ Fock space of $L^2(t, \infty)$, page 2

$\mathcal{H}_t$ Fock space of $L^2(0, t)$, page 2

$\mathcal{W}(\mathcal{H})$ The set of all adapted processes on $\mathcal{H}$, page 3

$\mathcal{R}_{(m,n)}$, page 43

$\mathcal{R}_{(m,n)}$, page 43

$\mathcal{R}_{(m,n)}$, page 43

$\mathcal{R}_{(m,n)}$, page 99

$\mathcal{R}_{(m,n)}$, page 43

$\prod^t_s (1 + dr)$, page 19

$\prod^t_s (1 + dr)$, page 21

$\prod^t_s (1 + dr)$, page 20

$\prod^t_s (1 + dr)$, page 18

$\prod^t_s$, page 12

$\prod^t_s$, page 15

$\ker$, page 49

$\mathrm{Op}$, page 49

$\theta_{m,n}$, page 49
\( \tilde{W}_{a,s}^{b,t} \), page 57
\( \tilde{W}_{a,t}^{b,s} \), page 63
\( \tilde{W}_{b,t}^{a,s} \), page 63
\( \tilde{W}_{b,s}^{a,t} \), page 62
\( f^t \) The vector \( f_{X(t,\infty)} \), page 2
\( f_t \) The vector \( f_{X(0,t)} \), page 2
\( P(t) \) Momentum operator at time \( t \), page 29
\( Q(t) \) Position operator at time \( t \), page 29
\( R^t_s \) The forward shift isometry on \( L^2(\mathbb{R}_+) \), page 106
\( r^{(j,k)}_{(m,n)}(\theta) \) Embedded rotation matrix of angle \( \theta \), page 42
\( S_t \) The forward shift isometry on \( L^2(\mathbb{R}_+) \), page 105
\( U_{a,s}^{b,t} \) The unitary implementor of the Bogolubov transformation \( W_{a,s}^{b,t} \), page 107
\( W_B \), page 33
\( X^{1,2,3} \) Superscript notation to denote places in a tensor product, page 13
\( \mathcal{I}_s \) The set of all linear combinations of the set of iterated quantum stochastic integrals between \( s \) and \( t \) and multiples of the identity process, page 6
\( I_s^{i_1,i_2,\ldots,i_n} \) iterated quantum stochastic integral, page 6
\( W_{a,s}^{b,t} \), page 57
\( W_{a,t}^{b,s} \), page 62
\( W_{b,t}^{a,s} \), page 63
\( W_{b,s}^{a,t} \), page 62
\( W(f) \) The Weyl operator of a vector \( f \) of some Hilbert space, page 23
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