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Static and dynamic analysis of multi-cracked beams with local and non-local elasticity

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PhD Student

A Doctoral Thesis Submitted for the Degree of

Doctor of Philosophy

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Supervisor

Dr Mariateresa Lombardo
Supervisor
All physical phenomena are non-local.
Locality is a fiction invented by idealists.
A. Cemal Eringen
The thesis presents a novel computational method for analysing the static and dynamic behaviour of a multi-damaged beam using local and non-local elasticity theories. Most of the lumped damage beam models proposed to date are based on slender beam theory in classical (local) elasticity and are limited by inaccuracies caused by the implicit assumption of the Euler-Bernoulli beam model and by the spring model itself, which simplifies the real beam behaviour around the crack. In addition, size effects and material heterogeneity cannot be taken into account using the classical elasticity theory due to the absence of any microstructural parameter.

The proposed work is based on the inhomogeneous Euler-Bernoulli beam theory in which a Dirac’s delta function is added to the bending flexibility at the position of each crack: that is, the severer the damage, the larger is the resulting impulsive term. The crack is assumed to be always open, resulting in a linear system (i.e. nonlinear phenomena associated with breathing cracks are not considered). In order to provide an accurate representation of the structure’s behaviour, a new multi-cracked beam element including shear effects and rotatory inertia is developed using the flexibility approach for the concentrated damage. The resulting stiffness matrix and load vector terms are evaluated by the unit-displacement method, employing the closed-form solutions for the multi-cracked beam problem. The same deformed shapes are used to derive the consistent mass matrix, also including the rotatory inertia terms. The two-node multi-damaged beam model has been validated through comparison of the results of static and dynamic analyses for two numerical examples against those provided by a commercial
finite element code. The proposed model is shown to improve the computational efficiency as well as the accuracy, thanks to the inclusion of both shear deformations and rotatory inertia.

The inaccuracy of the spring model, where for example for a rotational spring a finite jump appears on the rotations’ profile, has been tackled by the enrichment of the elastic constitutive law with higher order stress and strain gradients. In particular, a new phenomenological approach based upon a convenient form of non-local elasticity beam theory has been presented. This hybrid non-local beam model is able to take into account the distortion on the stress/strain field around the crack as well as to include the microstructure of the material, without introducing any additional crack related parameters. The Laplace’s transform method applied to the differential equation of the problem allowed deriving the static closed-form solution for the multi-cracked Euler-Bernoulli beams with hybrid non-local elasticity. The dynamic analysis has been performed using a new computational meshless method, where the equation of motions are discretised by a Galerkin-type approximation, with convenient shape functions able to ensure the same grade of approximation as the beam element for the classical elasticity. The importance of the inclusion of microstructural parameters is addressed and their effects are quantified also in comparison with those obtained using the classical elasticity theory.

*Keywords:* Cracked beams, Flexibility crack model, Damaged structures, Finite element analysis, Euler-Bernoulli beam theory, Timoshenko beam theory, Rotatory inertia, Dirac’s delta function, Aifantis’ strain gradient, Eringen’s stress gradient, Hybrid gradient elasticity, Non-local elasticity.
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Nomenclature

Main Symbols

\( C_i \) \quad \text{ith Integration constant, } (i = 0, \cdots, 5)

\( \text{sech}(\cdot) \) \quad \text{Hyperbolic secant } (= \cosh^{-1}(\cdot))

\( \text{csch}(\cdot) \) \quad \text{Hyperbolic cosecant } (= \sinh^{-1}(\cdot))

\( \delta(\cdot) \) \quad \text{Dirac’s delta function}

\( \mathcal{L}\{\cdot\} \) \quad \text{Laplace’s transform operator}

\( \mathcal{L}^{-1}\{\cdot\} \) \quad \text{Inverse Laplace’s transform operator}

\( \overset{\wedge}{(\cdot)} \) \quad \text{The over-hat stands for the dimensionless form of this quantity}

\( (\cdot)' \) \quad \text{Derivative with respect to the abscissa along the beam’s axis } (= \frac{d(\cdot)}{dx})

\( (\cdot)\dot{} \) \quad \text{Derivative with respect to time } (= \frac{d(\cdot)}{dt})

\( (\cdot)^{[j]} \) \quad j\text{th anti-derivative of a generic function } (\cdot)

A, A_i \quad \text{Cross sectional area (of the } i\text{th section})

\( A_s \) \quad \text{Shear area}

b \quad \text{Width of the beam}

c_{gi} \quad \text{Centre of gravity of the } i\text{th section}

c_{g\infty} \quad \text{Centre of gravity of the fully composite section}
\( \kappa \) Shear correction factor (or Timoshenko shear coefficient)
\( e \) Core/interlayer thickness
\( E, E_i \) Young’s modulus (of the \( i \)th section)
\( G \) Shear modulus
\( \nu \) Poisson’s ratio
\( \rho \) Mass per unit volume
\( I_0 \) Second moment of the undamaged cross section
\( I_\infty \) Second moment of the fully composite section
\( k_0, k_{G0} \) Axial and shear stiffness of the spring layers
\( E A_0 \) Axial stiffness of the undamaged section
\( E I_0 \) Flexural stiffness of the undamaged section
\( E I_\infty \) Flexural stiffness of the fully composite section
\( G A_{s0} \) Shear stiffness of the undamaged section
\( E A(x) \) Axial stiffness function
\( E I(x) \) Flexural stiffness function
\( G A_s(x) \) Shear stiffness function
\( \Upsilon_A(x) \) Axial flexibility function
\( \Upsilon_I(x) \) Flexural flexibility function
\( \Upsilon_S(x) \) Shear flexibility function
\( \Upsilon_{A0} \) Axial flexibility of the undamaged section
\( \Upsilon_{I0} \) Flexural flexibility of the undamaged section
\( \Upsilon_{S0} \) Shear flexibility of the undamaged section
\( F_\alpha(x) \) Axial damage function
\( F_\beta(x) \) Rotational damage function
\( F_\gamma(x) \) Shear damage function
\( H(x) \) Heaviside’s unit step function
\( K_{\alpha i} \) Elastic stiffness of the \( i \)th axial spring
\( K_{\beta i} \) Elastic stiffness of the \( i \)th rotational spring
\( K_{\gamma i} \) Elastic stiffness of the \( i \)th transversal spring
\( K_T \)  
Slip modulus

\( \ell_\varepsilon \)  
Strain gradient length scale parameter

\( \ell_\sigma \)  
Stress gradient length scale parameter

\( \ell_f, a_f \)  
Constants related with \( k_0 \) and \( k_{G0} \)

\( \rho_\ell \)  
Dimensionless microstructural parameter ratio

\( L \)  
Beam’s length

\( M \)  
Bending moment, positive if sagging

\( N \)  
Axial force

\( N_\phi \)  
Number of shape functions

\( n \)  
Number of cracks

\( p(x) \)  
Foundation reaction

\( q(x), q_T(x) \)  
Transverse load function, positive if downward

\( q_A(x) \)  
Axial load function, positive to right

\( Q \)  
Transformed load function

\( s \)  
Laplace’s coordinate

\( b^{(r)}(x) \)  
r\textsuperscript{th} rotations’ shape function

\( c^{(r)}(x) \)  
r\textsuperscript{th} curvature’s shape function

\( d^{(r)}(x) \)  
r\textsuperscript{th} displacement’s shape function

\( m^{(r)}(x) \)  
r\textsuperscript{th} bending moment shape function

\( u_T \)  
Transverse deflection, positive if downward

\( u_b \)  
Pure bending displacement, positive if downward

\( u_s \)  
Pure shearing displacement, positive if downward

\( V \)  
Shear force

\( w(x) \)  
Foundation deflection

\( x \)  
Abscissa along the beam’s axis, positive to right

\( \xi \)  
Dimensionless abscissa along the beam’s axis

\( \xi_i \)  
Dimensionless position of the \( i \text{th} \) point load along the beam’s axis

\( \bar{x}_i \)  
Position of the \( i \text{th} \) concentrated damage
$\xi_i$ Dimensionless position of the $i$th concentrated damage

$y$ Horizontal coordinate along the beam’s neutral axis

$z$ Vertical coordinate, positive downward

$\alpha_i$ Dimensionless severity of the $i$th axial damage

$\beta_i$ Dimensionless severity of the $i$th rotational damage

$\gamma_i$ Dimensionless severity of the $i$th shear damage

$\omega_i$ $i$th natural circular frequencies of vibration

$\varepsilon$ Normal strain

$\sigma$ Normal stress

$\varphi$ Rotation of the cross section

$\chi$ Local curvature, positive is sagging

$\tilde{\chi}$ Effective non-local curvature

$\tilde{\chi}_0(x)$ Effective non-local curvature of the undamaged beam

$\Delta \tilde{\chi}_j(x)$ Contribution of the $j$th concentrated damage to the non-local curvature

$\chi_0(x)$ Non-local curvature of the undamaged beam

$\Delta \chi_j(x)$ Contribution of the $j$th concentrated damage to the non-local curvature

$\varphi_0(x)$ Rotation of the undamaged beam

$\Delta \varphi_j(x)$ Contribution of the $j$th concentrated damages to the rotation

$u_0(x)$ Transverse displacement of the undamaged beam

$\Delta u_j(x)$ Contribution of the $j$th concentrated damage to the transverse displacement

$\Pi(t)$ Total potential energy

$\mathcal{W}(t)$ Potential energy due to the external forces

$\mathcal{T}(t)$ Kinetic energy

$\mathcal{U}(t)$ Elastic internal strain energy

$\mathbf{u}$, $\mathbf{u}_T$ Vector of the transverse nodal displacements

$\mathbf{u}_A$ Vector of the dimensionless axial nodal displacements
**Nomenclature**

- **K**  Stiffness matrix
- **M**  Mass matrix
- **\( \hat{K}_A \)**  Dimensionless stiffness matrix block related to the axial displacements
- **\( \hat{K}_T \)**  Dimensionless stiffness matrix block related to the transverse displacements
- **\( \theta \)**  Vector of the unknown constant \( \theta^{(r)} \)
- **\( \phi \)**  Vector of the collecting the shape functions \( \phi^{(r)} \)
- **F**  Vector of the dimensionless nodal forces
- **\( \hat{F}_A \)**  Vector of the dimensionless axial nodal forces
- **\( \hat{F}_T \)**  Vector of the dimensionless transverse nodal forces and bending couples

**Acronyms and Abbreviations**

- **2D**  Two dimensional
- **3D**  Three dimensional
- **BCs**  Boundary conditions
- **DoF**  Degree of freedom
- **DS**  Discrete spring
- **EB**  Euler-Bernoulli
- **FE**  Finite element
- **LSR**  Local Stiffness Reduction
- **MDB**  Multi-damaged beam
- **MWR**  Method of weighted residual
- **PL**  Point load
- **UDL**  Uniformly distributed load
Performances of structures, including their static and dynamic behaviour, are highly influenced by the presence of damage, which can lead them to different types and levels of failure, ranging from loss of serviceability to complete loss of functionality. This work explores the mathematical models to represent damaged structures, which play a key role in the health monitoring of structures. In the technical literature, several cracks models have been developed, including “always open” and “breathing” cracks models (see Section 2.3). In this context, one of the key objectives is to reduce the computational costs, especially when the goal is the damage identification [64, 87]. A common procedure is then to represent the cracks through a “discrete spring” (DS) model, where the damage is lumped at a given position and rotational springs simulates the residual stiffness at the position of each crack (e.g. Refs. [88, 116, 161, 204]).

The main shortcoming with the DS model is that, if a conventional beam element is used, two additional FE nodes must be placed at the location of each concentrated damage, i.e. one node on each side. This could be particularly cumbersome if the DS model is used for the purposes of damage identification, as it would also require re-meshing the beam during the identification process. The need then arises for an accurate and computationally efficient beam element able to account for the damages, without increasing the size of the FE assembly for the structural frame under investigation.
One further inadequacy of the DS model in the classical continuum is in the description of the rotations’ profile, where a jump discontinuity appears as a consequence of concentrating the increased curvature at a single abscissa. Moreover, classical continuum is unable to account for size effects and the unavoidable material’s heterogeneity. In order to overcome the aforementioned deficiencies, many higher-order theories of elasticity have been proposed in the past years. These theories enrich the constitutive law by including gradients, in which the stress at a given point depends on the strain in the neighbourhood points, and not at that point only [171, 247]. Focusing on the non-local beam models, many publications are available describing bending, buckling and free transverse vibrations [183, 171, 166, 235]. Almost all the publications deal with undamaged elements and only a few deals with single damage [136, 220], despite the importance of the presence of cracks, which may substantially change the static and dynamic behaviour of the structures.

This thesis addresses the lack of a computationally efficient and accurate DS beam model and the need to account for the distortion on the stress/strain field at the crack location, as well as to include the microstructure of the material.

1.1 Aim and Objectives

The overall aim of this thesis is to “develop efficient computational methods for the static and dynamic analysis of beams with local and non-local elasticity in presence of concentrated damages”. It has been accomplished through achieving six main objectives:

1. To review the existing models for the analysis of damaged structures.
   An extensive review of the existing literatures concerning the damage investigation methods will be carried out, focusing mainly on the different models used to describe damaged beams and columns. This survey will allow to identify fields that have not been covered yet, in particular, it existing methods, in order to get accurate results, discretise the structure with a large number of finite elements, resulting in an high computational effort (especially when they are used for damage identification purposes).

2. To develop a two-node multi-cracked beam element with classical elasticity.
   Given the lack of models able to describe with a single finite element a line beam incorporating multiple cracks, a new two-node multi-damaged beam element will be proposed using the recently developed closed-form solutions for beams with singularities [34, 162]. This model will be based on the classical continuum theory, including shear effects and rotatory
inertia, and it is able to describe damaged short structures without enlarging the size of the Finite Element model.

3. **To review the higher-order gradient theories in particular when applied to the beam model.**

A detailed review on the main application of the higher-order gradient theories will be carried out, highlighting the main advantage as the ability to remove singularities from the stress or strain field, which makes these theories particularly suitable to study discontinuities like those in the multi-cracked beam. The review will focus on a particular type of gradient enrichment, the so-called non-local beam theories.

4. **To develop a theoretical model for the beam with multiple concentrated damage using higher order gradients.**

The closed-form solution of a beam with multiple-cracks will be extended to cope with enriched continuum theories, taking the gradients of both strain and stress, as well as the associated microstructural length scale parameters.

5. **To develop an efficient method to solve the dynamic problem of the multi-cracked beam enriched with higher order gradients.**

A new computational method will be developed, where a Galerkin-type approximation will be implemented to evaluate the stiffness and mass matrices for the non-local multi-cracked beam, which in turn allows evaluation of the natural frequencies and the modal shape of the beam.

6. **To study the effects of the non-local parameters on the static and dynamic structural response.**

A number of numerical investigations will be carried out to assess the effects of the new parameters introduced by using the theory of non-local elasticity. In particular, the effects on the final response of the main factors of the enriched continuum theory, such as the length scale parameters, will be evaluated. Furthermore in order to validate the proposed method, static and dynamic comparisons between the proposed model and others analytical solutions will be performed.

**1.2 Structure of the Thesis**

This thesis is organised into seven chapters, plus Bibliography and the list of Published Works.
Chapter 1 - Introduces the aim and objectives and describes the structure of the thesis.

Chapter 2 - Reviews the literature of damaged structure, in particular the main mathematical representation of cracked beam models so far implemented by researchers, along with a discussion of their analytical and numerical limitations and corresponding solutions, with particular interest on the DS beam model.

Chapter 3 - Provides stiffness matrix and consistent mass matrix, as well as equivalent force vectors, for the multi-damaged beam element using as shape functions the exact closed-form solutions of the multi-cracked Timoshenko beam, where the concentrated damages are represented using the flexibility modelling for the sets of axial, rotational and translational springs. Two numerical examples, a cantilever beam and a portal frame, of the multi-cracked beam element are studied together with validations against alternative less efficient numeric models. The results of this Chapter have been presented at an international conference, and a paper has been accepted for publication in Computers & Structures.

Chapter 4 - Reviews the main higher-order enrichment approaches, their advantages and applications with particular interest on the enrichment of beam theories. Both non-local and gradient elasticity beam theories are investigated, focusing on the application of these models to the static and dynamic behaviour of beams.

Chapter 5 - Presents the closed-form solution for the static deflection of the proposed hybrid non-local multi-damaged Euler-Bernoulli beam, where the microstructural effects are taken into consideration via the stress and strain gradients theory, while cracks are mathematically represented through Dirac’s delta functions centred at each damage position. A cantilever beam and a clamped-clamped beam with multiple cracks are investigated in order to study the effects of the microstructural length scale parameters. The results of this Chapter have been presented at an international conference, and a paper has been published in the International Journal of Solids and Structures.

Chapter 6 - Formulates a new numerical method to solve the dynamic problem for the hybrid non-local multi-damaged beams. The method is based on a Galerkin discretisation using closed-form solutions of the hybrid beam model as the basis for the shape functions. To assess the advantages of the proposed model, three numerical examples are investigated, providing a deeper insight into the effects of the length scale parameters. A third journal paper is planned with the results of this Chapter, and the target agreed with the supervisors is Computer Methods in Applied Mechanics and Engineering.
Chapter 7 - Summarises the main outcomes of the thesis, outlines the achievements of this research project and suggests recommendations for future work.
2.1 Introduction

An unavoidable fact during the service life of engineering structures is the occurrence of damage. In some cases design mistakes or manufacture imperfections can affect the structural behaviour from the construction stage, while natural and/or anthropic actions increase over the years the probability to find damage, which may drastically reduce the structural reliability. For this reason damage effects have been intensively investigated in many engineering fields, as aeronautical, mechanical and civil, and they are still an active area of research, with the aim to devise more efficient and accurate models and analysis/prediction tools.

Indeed cracks are a serious threat to the integrity of engineering structures. For instance, damage in frame structures may substantially change their static and dynamic response, reducing the performance and eventually leading to failure. Typical examples of damage mechanisms include growth of micro cracks under cycling loading and the continuous wear and tear, especially if the yield limit is exceeded. As a result, a concentrated loss of stiffness appears in the structural element.
2.1.1 Damage detection

In order to prevent unexpected collapses, possibly causing catastrophic consequences in terms of economic and human losses, adequate structural damage monitoring and detection is required. The importance of crack detection in structural members motivates the wealth of research papers published in this field. Their main goal is to develop non-destructive techniques to investigate the integrity of operative structures. Indeed a better knowledge of the status of the structure allows to plan more efficiently the maintenance at reduced cost, improving the safety and the system performance.

The non-destructive damage identification techniques can be classified in many categories, e.g. considering the portion of structure being inspected [66]. In this respect, two groups can be defined:

- Local damage identification techniques (acoustic or ultrasonic methods, magnetic field methods, X-ray methods, eddy-current methods or thermal field method);
- Global damage identification techniques (static or vibration-based damage identification methods).

Another common classification of damage detection is based on the effects of the damage on the structure [66], namely:

- The linear damage problem, which refers to cases where the structure is linear-elastic and it continues to be linear-elastic even after the damage;
- The non-linear damage problem, which arises when the damaged structures behaves in a non-linear manner.

Another way to classify damage identification methods is as parametric and non-parametric [105]:

- Parametric identification is when the estimation of system parameters is involved;
- Non-parametric identification uses an analytical representation to determine the transfer function of the system.

In this thesis, the attention will be focused on global, linear, parametric models in beam frame structures.

2.1.1.1 Vibration-based techniques

A very popular class of damage detection techniques uses the dynamic behaviour to monitor the structural health. The basic idea is that dynamic properties like stiffness and/or damping of a structure are affected by the presence of damages; as consequence, the vibration response due to dynamic loads will also be affected. These changes on the structural behaviour can then
be related with the damage position and severity, giving a tool able to identify and quantify the damage.

The vibration-based damage identification technique started in the sixties of the last century, initially in the area of electrical engineering, and has been briskly expanding over the last four decades to the field of mechanical/control engineering. As evidence, several papers have been published on this topic [64]. This section will present an overview of the most popular methods to detect, locate, and characterise damage in structural systems by investigating changes in the dynamic response.

Vibration-based structural identification systems have traditionally focused on estimating some unknown or uncertain parameters in a linearised model of the structure. Algorithms which try to minimise a given norm of a certain error can be used to estimate such parameters. These algorithms fall into two complementary categories, depending on whether they operate on the data in the time domain or on the Fourier transform of the data in the frequency domain. In the time domain approaches, data sampled at equally spaced time instants are used to determine the system parameters; while in the frequency domain method, modal quantities are identified using measurements in the frequency domain [134]. Frequency domain algorithms have been so far the most popular due to their simplicity, however these algorithms involve averaging temporal informations, which inevitably lose most of the details, including any non-linear behaviour, while the parametric identification techniques in the time domain are more relevant to vibration problems related to non-stationary loading and/or non-linear behaviour. A state-of-the-art review on beam damage identification methods can be found in the paper of Dimarogonas [64].

The vibration damage-identification process can be subdivided in four different steps [194]:

**Level 1** - Detection, which is the determination whether damage is present in the structure;
**Level 2** - Localisation, that corresponds with the geometric location of the crack or other type of damage within the structure;
**Level 3** - Quantification of the severity of the damage;
**Level 4** - Prediction remaining service life of the structure.

Vibration damage identifications methods can be classified depending on the availability of a numerical model to analyse the structure [81]. The corresponding classes are:

- Model-based methods (a numerical model of the structure is available)
- No-model techniques or Response-based methods (which do not require models, but just experimental response data)
No-model methods do not suffer from modelling errors while enjoy a relatively low computational cost, which is particularly important in iterative procedures. However, using these methods only Level 1 and Level 2 can be efficiently carried out, while the quantification of the damage (Level 3) requires higher number of sensors, which are often technically complicated to install and lead to heavy signal processing analysis. Instead model-based methods adopt appropriate linear or non-linear models to represent the discontinuities effects under the given load conditions. They can be used to detect, localise and also quantify and predict the damages by using experimental data, often with a few sensors only. Certainly, their main shortcoming is the requirements of a model able to adequately represent the damage phenomena [45]. The need then arises to have accurate mathematical models for the structural elements able to describe the local and/or global effects of concentrated damages.

2.2 Modelling approaches for concentrated damage

Damaged structures can be modelled using several different approaches, ideally a three dimensional representation of the structure able to take into account all the non-linearities as well as the stress and strain concentrations of a damage structures would be the ideal model to adopt for the analyses. In reality, the actual computational limits and the uncertainties on the crack phenomenon do not allow such a detailed model. Instead, depending on the type of problem to be solved, a compromise must be found between computational costs and accuracy. When an accurate description of the stress or strain field around the crack is required, a three dimensional (3D) or a two dimensional (2D) Finite Element (FE) model of the structural element is preferred, while for the damage identification of beam/frame structures, a one dimensional (1D) analytical model of the damaged element can be a more efficient approach.

2.2.1 Two or three dimensional damaged models

Indeed, as consequence of the presence of the crack the stress and strain distribution along the beam are perturbed. In particular, the stresses tend to increase to a maximum value close to the crack position.

2D and 3D FE models (Fig. 2.1) can be used to study the stress and strain fields in a cracked beam and take into account the breathing phenomena by means of imposition of constrain equation or use of non-linear contact elements [124, 119]. Even though the resulting model is precise and refined, the mesh resolution could require very powerful processors and very long computational time. Furthermore more complicated and more time consuming analysis
are required in crack identification procedures where the unknown parameters, as location and depth, are sought using iterative procedures including automatic re-meshing [45].

The use of 2D or 3D FE models may produce detailed and accurate results, but such computationally intensive approaches are more appropriate to tackle problems of crack initiation and/or propagation, while global analysis of framed structures and damage detection in beams and columns can be carried out by less sophisticated FE models.

2.2.2 One dimensional/beam damage model

Beams and columns are three dimensional continua but, because one dimension is much bigger than the other two, they can be represented using a one-dimensional model, which condenses the geometric properties in the short directions into a single point, leaving only one spatial dimension along the axis of the beam [110]. The straight beam is a very important model for structural analysis, widely applied to study the mechanical response of cracked beams and to understand the involved phenomena [172]; it can drastically reduce the computational cost, while still able to represent the basic characteristics of a cracked member.

The simplest one-dimensional beam theory is the Euler-Bernoulli (EB) model [57]. It dates back more than three centuries ago, when Jacob Bernoulli (1654-1705) discovered the proportionality among curvature and bending moment of an elastic beam. Subsequently Jacob’s nephew, Daniel Bernoulli (1700-1782), formulated the differential equation for a beam in bending. Later Leonhardt Euler (1707-1783) improved the theory giving the actual form. Among engineers, this is the most widely used theory, because of its simplicity and reasonable approximation for many engineering problems. The most accurate applications are for slender beams (i.e. slenderness \( \lambda > 4 \), where \( \lambda \) is the dimensionless ratio of the radius of gyration \( r = \sqrt{I/A} \) to the effective length of the beam \( L_{eff} \)) and to evaluate low vibration frequencies. Indeed, because the theory does not take into account rotational and shear effects, the high frequencies are overestimated. Attempts to further improve the theory has been done by Rayleigh [182], who included the rotatory inertia effects of the cross-section (Rayleigh beam model). Then Timoshenko [216, 217] first proposed a beam theory which added both shear effects and rotatory inertia to the EB beam (Timoshenko beam model).
The beam theories have been enriched to include local variations in material properties or geometric characteristics. One of the first attempts to study stiffness discontinuities using beam models dates back to the 40s, when Thompson [215] investigated the dynamic behaviour of a uniform beam with a slot. Later, in the 60s, it was found that the strain energy concentration around the crack tip under load is responsible for a local increase of flexibility [106] and its effects on the behaviour of the cracked structures have been discovered shortly after [132]. The damages effects have then been included as a link [100] or as structural portion with reduced stiffness [249]. The different approaches employed to include the presence of cracks can be grouped in three basic categories [64]:

- Local Stiffness Reduction (LSR), or “smeared” models
- Continuous models
- Discrete spring (DS), or lumped flexibility models

2.2.2.1 Local stiffness reduction (LSR) models

The LSR is conceptually the simplest approach to build a FE model of a damaged beam, as it just requires to mesh the member with a sufficient number of beam elements and to reduce the relevant stiffness component (e.g. the flexural stiffness) of the element at the position where the damage occurs [229, 230, 47]. To be efficient, this approach requires a fine mesh, and the problem arises of quantifying the stiffness reduction in each FE to match the global effects of the actual concentrated damage. This representation consists of a FE model of the structure where the damage is introduced by reducing the stiffness of the elements where the damage is located (see Fig. 2.2). This approach can be used in inverse problem to localise and quantify the damage from static to vibration response. One of the main limitations is that the equivalent stiffness of the cracked element not only depends on the crack severity but also on the model refinement [44]. There are many application of the LSR in the literature [249, 196, 164, 181, 163]. In the paper by Yuen [249], for instance, it is shown that reducing the modulus of elasticity and the second moment of area leads to identical results. Unfortunately this study
did not relate the damage parameters, position and depth, to the stiffness reduction but merely studied the detection of zones with reduced stiffness. This model can be useful to study beams with an abrupt change of the cross section as in the work of Sato [197].

2.2.2.2 Continuous cracked beam models

Differently from the LSR, in the continuous cracked model the stress distribution around the crack is explicitly defined by a decay function and then combined with the kinematic assumptions to derive the governing equilibrium equation [44]. The inclusion of the stress concentration function is motivated by the attempt to explicitly consider the significant parameters (location and depth) in the equation of motion of the damaged structure. This effect has been taken into account by some researchers through the insertion of a local function for the stresses [57]; the assumed function is centred at the crack abscissa and exponentially decreases with the distance from the crack (e.g. Fig. 2.3). It includes parameters which require experimental estimation and vary for different cross sections and crack geometries. The exponential decay has been assumed because of the analogy with many decay rates found in analytical solutions relating to the De Saint Venant’s Principle [57]. Once the stress function is evaluated, then the equation of motion can be obtained from the variational principle after integration over the cross-section of the beam. The strain and velocity field is assumed unaffected by the crack presence (see Fig. 2.3), furthermore, it is assumed that the refined definition of the stress distribution is not required because the aim of this work is not the local effects, like crack propagation or initi-
2. Literature Review

Figure 2.4: Stiffness along the beam for the linear crack model [205] (solid) and the approach by Christides and Barr [57] (dashed)

...ation, but instead the overall static and dynamic behaviour of the beam structures where small discrepancies on the stresses are negligible [57]. The proposed stress function is valid for rectangular cross section with symmetric sided cracks, one on the upper surface and the other at the bottom surface, and depends on the section geometry and on the crack depth. Christides and Barr [57] carried out some experiments, and the natural frequencies obtained match quite well the predicted ones. Shen and Pierre [203] validate the same formulation by comparison with FE models showing good agreement, they also introduced a different estimation of the parameter using 2D FE models and extended the model by applying the same theory to single edge breathing crack [202].

Similar to Christides and Barr [57], Sinha et al. [205] developed a model for multi-cracked beam using EB theory with modified local flexibility at the neighbourhood of the crack. Compared to other formulations this model adopts a simple linear approximation of the stiffness reduction (see Fig. 2.4) while keeps including the location and depth of the crack directly. Starting from an effective distance from the crack, the stiffness $EI$ linearly decreases to a minimum value at the crack position where the stiffness assume the value of the cracked section and is related to the crack depth. The model is not accurate at high frequencies but on the other hand it is relatively simple and is directly related to the crack depth.

2.2.2.3 Discrete spring (DS) models

In the DS model the local change in the cracked beam section is represented by an equivalent spring element with no mass. The beam is assumed as split in two parts at the crack location and these two parts are linked together by the additional equivalent spring element (see Fig. 2.5). One of the main advantages of the DS model is the effective representation of the crack in
2.3 Cracked beam with DS model

In many practical applications the jump discontinuities in deflections, slope, or in the geometric properties as transversal dimensions or flexural stiffness as well as discontinuous loading along a beam member are modelled by partitioning the whole element into sub-segments among any discontinuity points [246]. The DS model groups together all those techniques where the damaged structure is represented by a combination of undamaged portions and crack model usually represented by lumped springs or a compliance matrix obtained from stress intensity factors or strain energy release rate [107, 35, 53]. Depending on the characteristics of the damage, the residual stiffness is simulated by axial, and/or rotational and/or shear springs.

For slender beams in bending, for instance, the presence of cracks affects mainly the flexural stiffness $EI(x)$, and therefore the DS model just consists of joining the adjacent split elements of the beam with a hinge (i.e. axial and shear flexibility are not considered), which are then coupled with a rotational spring, whose stiffness is related to the intensity of the damage: that is, the severer the damage, the lesser stiff is the spring. In the limiting case in which the cross section is fully damaged, the stiffness of the spring becomes zero.

Depending on the behaviour of the crack, it can be modelled using the “always open” or the “breathing” spring models. The simplest one is the so-called “always open” model, which is suitable for static problems and also for dynamic analysis, but in the latter case the static deflection must be bigger than the vibration amplitude. The main advantage is that the model of the damaged structure remains linear, provided that all the parameters involved in the structural analysis are linear. In the “breathing” models, on the contrary, the dynamic input causes the...
cracks to open and close, creating more complicated phenomena that have to be treated with a non-linear analysis. Although the proposed approach can be extended to any type of crack behaviour, in this thesis the crack is assumed as always open to avoid the complexities of a non-linear field due to a breathing crack.

2.3.1 Applications of the DS model

The DS model has been adopted to study the static, dynamic and stability behaviours of many damaged structures, in civil but also mechanical engineering. It made its first appearance in the 50s [106], afterwards it has been applied by many other authors. Dimarogonas et al. [91], in particular, have related the crack depth with the stiffness of the rotational spring used to replace the discontinuity due to the crack [55, 188]; they further studied the effect of the crack position and depth on the natural frequencies and mode shapes of the beam. The DS model has been applied firstly to study beam with a single crack and indeed the majority of the studies consider this case [35, 240, 158, 130]; only later the double-cracked beam [193, 200] and the multi-cracked beam [116, 204, 162, 38] have been investigated. Some applications can be found in the papers cited in the literature review of Dimarogonas [64] as well as in [88, 116, 161, 204, 37].

The equivalent spring model requires the assumption of an appropriate value for the spring stiffness. Different attempts can be found in the literature to provide the values for the rotational spring stiffness using crack parameters as depth or geometry. Among them, Bilello [31] related the crack depth and the section geometry, also different equations for the rotational spring stiffness have been proposed by other authors [249, 197, 138] (more details on the spring stiffness can be found in Section 2.3.4). Only a few formulations are available for the axial spring stiffness [62], in most of the works where the crack was studied by using an axial spring the relation among the crack geometry and the axial stiffness of the spring has not been provided [3]. Indeed it can be useful to have a model able to take into account the presence for axial, rotational or transversal damages, to study for example bolted joint in steel structures or wooden connection where axial or transversal movements can occurs.

2.3.2 Analytical solution for the DS model

Theoretical solutions of damaged beam models are very useful for many engineering problems as they can provide a computationally efficient alternative to the numerical methods, especially for identifications purposes, when iterative analysis are implemented for the localisation and quantification of the damage. Classical analytical approaches are generally available only for structures with a simple geometry or that can be simplified into one dimensional model.
2.3 Cracked beam with DS model

The classical approach to analytically solve a damaged straight beam with cracks modelled as DS consists of splitting the beam at the discontinuity points and analysing each undamaged beam segment so obtained by solving the their differential equation. The final enforcement of the boundary conditions (BCs) and continuity conditions at each crack positions yields the deflection equation for the whole beam. Even though this approach consistently provides the analytical solution for a multi-damage beam, it may result in very cumbersome values due to the enforcement of several continuity conditions at each jump. Among the alternative procedures, Clebsch [59] was the first to give a simplified method to obtain the closed-form solution of problems with discontinuities by writing a single expression in terms of bending moment. Later, Macaulay [140] introduced the singularity function method, which reduces the uncoupled ordinary differential equations system to a single ordinary equation; in his formulation he introduced the so-called Macaulay’s bracket (see Sec. 4.5.1.2). This singularity function method has been extended by Arbabi [18] for a beam with internal hinge and a beam with jump discontinuities in flexural stiffness. Another procedure can be found in the work of Yavari and Sarkani [246], who developed a EB beam formulation with jump discontinuities in the flexural stiffness introduced as generalised functions (in particular the Dirac’s delta function) giving some examples of beam deflection and buckling problems solved using Laplace transform method. This was the first attempt to include generalised functions to develop the closed-form solution of a beam with local discontinuities where, however, the enforcement of continuity conditions at the crack position was still required.

Following the same approach, Biondi and Caddemi proposed the closed-form solution for a single crack EB beam [33] subsequently extended to the case of multi-cracked EB beam [34]. Their work represents a big step forward on the modelling of cracked beam as for the first time a closed-form (exact) solution was available for the EB multi-damaged beam where no continuity conditions were required. In their approach the crack is modelled via negative impulses (i.e. Dirac’s delta functions with negative sign) introduced into the flexural stiffness $EI(x)$ expressions, yielding:

$$EI(x) = EI_0 \left[1 - \sum_{j=1}^{n} \beta_{r,j} \delta(x - \bar{x}_j)\right] ,$$

(2.1)

where $n$ is the number of concentrated cracks, $EI_0$ is the flexural stiffness of the undamaged beam’s section, $\bar{x}_j$ is the position along the abscissa of the $j$th crack and $\beta_{r,j}$ is its coefficient related to the damage intensity. Fig. 2.6 clearly shows how the inhomogeneous flexural stiffness is constant between two consecutive damages and assumes negative impulses at the discontinuities. Even though efficient and able to deliver the exact closed-form solutions for
the static analysis of multi-cracked slender beams in bending, this mathematical representation where the stiffness along the beam assumes negative values is not physically consistent with the definite-positive nature of the flexural rigidity \((EI(x) > 0 \ \forall x \in [0, L])\).

Aimed at overcoming this theoretical flaw, Palmeri and Cicirello [162] have recently presented a physically consistent dual representation of cracks, i.e. “flexibility modelling”, in which Dirac’s delta functions with a positive sign are introduced in the bending flexibility of the beam, i.e. the inverse of its flexural stiffness. They also extended the model to cope with Timoshenko beams, to take into account the contribution of the shear deformations in the un-cracked regions of the member. A rigorous theoretical justification of this flexibility modelling has been subsequently presented by Caddemi and Morassi [42]. Following Palmeri and Cicirello’s approach, the bending flexibility, which does not have any mathematical limitation \((\frac{1}{EI(x)} > 0 \ \forall x \in [0, L])\), is defined as:

\[
\frac{1}{EI(x)} = \frac{1}{EI_0} \left[ 1 + \sum_{j=1}^{n} \beta_{f,j} \delta(x - \bar{x}_j) \right],
\]

where \(\beta_{f,j}\) is the coefficient related to the severity of the \(j\)th damage, and is illustrated with Fig. 2.7.

**2.3.2.1 Analytical solution in dynamics**

One of the more interesting applications of multi-cracked beam is to study the dynamic behaviour of the structure, as this can be particularly useful in damage identification procedures. Given the dynamic differential equation of motion of beam with one or multiple cracks, the resulting vibrational problem can be solved using different approaches: analytical, numerical
or a combination of the two. The problem can be treated analytically by using the Rayleigh-Ritz method [57]; while the numerical alternative is commonly the FE method for the dynamic problem. Chondros et al. [56] have obtained the natural frequencies equation for a beam with lumped crack flexibility from the harmonic vibration modes of the two undamaged segments of the beam combined together using the BC equations at the crack location (i.e. the equilibrium of the forces or the congruence of the displacements at the two sides of the crack). The global governing equation of flexural vibrations of a EB beam is given by:

$$[EI(x) u''(x, t)]'' + \rho A \ddot{u}(x, t) = 0$$

where $EI(x)$ is the flexural stiffness; $u_b(x)$ is the transverse deflection; $\rho$ is the mass per unit volume and $A$ is the cross sectional area. The apex $'$ stands for the derivative with respect to the coordinate along the beam $x$, i.e. $(\cdot)' = d(\cdot)/dx$, while the dot stands for the derivative with respect to the time $t$, i.e. $(\cdot) = d(\cdot)/dt$.

If we now consider a beam with a single crack, both undamaged beam segments obtained by splitting the beam at the crack location can be treated separately, and the transversal displacements are obtained:

$$u_1(x) = A_1 \cos(\lambda x) + B_1 \cosh(\lambda x) + C_1 \sin(\lambda x) + D_1 \sinh(\lambda x)$$

$$u_2(x) = A_2 \cos(\lambda x) + B_2 \cosh(\lambda x) + C_2 \sin(\lambda x) + D_2 \sinh(\lambda x)$$

where $A_i$, $B_i$, $C_i$ and $D_i$ ($i = 1, 2$) are the integration constants and $\lambda$ is a frequency parameter related to the mechanical and geometrical properties of the beam, defined as:

$$\lambda = \sqrt[4]{\frac{\omega^2 \rho A}{EI_0}}$$

where $\omega$ is the natural circular frequency of the beam; $E$ is the Young’s modulus of the material; $I_0$ is the second moment of area of the undamaged section.

This approach has been adopted also by other authors [188, 242, 161], and the solution of the resulting equation can then be solved numerically to provide the natural frequencies $\omega$. One of the first papers was published by Rizos et al. [188], who proposed a $(4n + 4)$ equations system for the vibration analysis of a uniform beam with $n$ cracks. Later Shifrin and Ruotolo [204] gave another improved analytical method for calculating the natural frequencies leading to a $(n + 2)$ linear equation system, drastically reducing the size of the problem. Their
general solution has the form:

\[ u(x) = A \cos(\lambda x) + B \cosh(\lambda x) + C \sin(\lambda x) + D \sinh(\lambda x) + \]
\[ + \frac{\lambda}{4} \sum_{j=1}^{n} \Delta \varphi_j \int_{0}^{x} \{ \sinh[\lambda (x - s)] - \sin[\lambda (x - s)] \} |s - x_j| \, ds + \frac{\Delta \varphi_j}{2} |s - x_j| , \]

(2.6)

where again \( A, B, C \) and \( D \) are constants, while \( \Delta \varphi_j \) is the jump on the rotations at the crack position. Introducing the BCs then leads to a linear system, and its non-trivial solutions \( \lambda \) are extracted by imposing the determinant of the system to vanish, giving the sought eigenfrequencies of the cracked beam. Nandwana and Maiti [158] further extended this approach to a stepped EB cantilever beam, where again each segment with uniform material and geometrical properties has been studied separately and then combined together by imposing the continuity conditions. Li [129] extended Shifrin and Ruotolo approach to any kind of end supports and any finite number of cracks and concentrated masses. He expressed the jump of the slope and the jump of the shear force using the spring crack model and the concentrated masses respectively. The problem can be reduced to a second order determinant and therefore with significant savings in the computational effort.

Further attempt to solve the dynamic problem is the work of Mazanoglu et al. [144] who presented a model for vibration analysis of non-uniform EB multi-cracked beam. This energy-based model considers not only the strain energy at the cracked beam surface but also the effect of stress field, which is a consequence of the angular displacement of the beam due to bending. Another mathematical technique is that one used by Khien [116], who implemented a transfer matrix method for the natural frequency analysis of a multicracked beam reducing the numerical problem to a \( 4 \times 4 \) determinant calculation, with significant saving of the computational time.

The exact closed-form solution for the vibration modes of an EB straight beam with multiple open cracks has been proposed by Caddemi and Caliò [38], where the crack presence is modelled by introducing a sequence of Dirac’s delta functions in the flexural stiffness. In their approach, the fourth order differential equation has been split in two differential ordinary equations by assuming the displacement function as product of two independent functions \( \Phi(x) \) and \( y(t) \), only dependent on the spatial coordinate and on the time respectively. The sought solution can be expressed as:

\[ u(x, t) = \Phi(x) \, y(t) . \]

(2.7)
As consequence the governing equation Eq.(2.3) can be separated into two ordinary differential equations:

\[
\begin{align*}
\ddot{y}(t) + \omega^2 y(t) &= 0, \\
\dddot{\Phi}(x) - \lambda^4 \Phi(x) &= B(x),
\end{align*}
\]

where \(\omega\) and \(\lambda\) are the same as defined in Eq. (2.5) and \(B(x)\) is a function which includes all the Dirac's delta terms and their derivatives. The dot stands for the derivative with respect to the time \(t\) i.e. \(\dot{\cdot} = d(\cdot)/dt\), while the over check \(\check{\cdot}\) indicates the distributional derivative \(^1\). As the undamaged beam the spatial dependent function \(\Phi(x)\) has been assumed as a combination of trigonometric and hyperbolic functions:

\[
\Phi(x) = d_1(x) \sin \left( \lambda \frac{x}{L} \right) + d_2(x) \cos \left( \lambda \frac{x}{L} \right) + d_3(x) \sinh \left( \lambda \frac{x}{L} \right) + d_4(x) \cosh \left( \lambda \frac{x}{L} \right),
\]

where the four functions \(d_i(x)\) \((i = 1, 2, 3, 4)\) are unknown generalised functions obtained by imposing four independent conditions, which are the requirement to fulfil the governing differential equation and the conditions that only the first distributional derivatives of the unknown functions \(d_i(x)\) are involved. In Eq. (2.10) the damage effect and the corresponding generalised functions are included through additional coefficients. The complete procedure with all the formulations can be found in their paper [39], which gives explicit solutions in terms of natural frequencies and vibration modes, obtained by enforcing the standard BCs, for four different classical cases: (i) the simply supported beam (pinned-pinned); (ii) the clamped-clamped beam; (iii) the cantilever beam (clamped-free) and (iv) the free-free Euler-Bernoulli beam. These closed-form solutions have been used in this thesis to validate when possible the frequencies obtained by the proposed multi-damaged FE beam element.

### 2.3.3 Finite Element method for the DS model

Most of the structural problems can not be solved using analytical solutions because of the complexity of the structure or because the involved equations require a big computational effort, that can only be carried by computer methods.

The common numerical method to study the behaviour of almost any structure is the FE method. Born in the 50s of the last century, it has grown exponentially over the years, replace-
2. Literature Review

cing many of the analytical approaches used to study structural problems. Nowadays it is the most widely used numerical method, especially for large and/or irregular structures as it allows modelling different geometry configurations, with virtually no size restrictions [116].

FE methods discretise the whole structure in a finite number of geometrical elements, the finite elements, which are then combined together using the equilibrium, kinematic and constitutive equation to give a global system of structural problem solvable using computational algorithms to give an approximate solution of the structural problem. Using the FE method, the equilibrium equations governing the dynamic response of the structure with no damping can be written as:

\[ M \ddot{u}(t) + K u(t) = F(t), \]  

where \( K \) and \( M \) are the stiffness and mass matrices respectively; while \( u(t), \dot{u}(t) \) and \( \ddot{u}(t) \) are the displacement, velocity and acceleration vectors, respectively. The equilibrium equations for the static response can be obtained from Eq. (2.11) by imposing null acceleration, yielding:

\[ K u = F. \]  

The modal analysis problem assumes the following form:

\[ \omega^2 M \Phi = K \Phi, \]  

where \( \omega \) is the modal circular frequency and \( \Phi \) is the associated modal shape. These equations can then be solved using computational methods to provide approximate solutions for the structural systems in terms of displacements, internal forces, natural frequencies and mode shapes.

Focusing now on the applications of the FE beam method to damaged structures using the DS model, a review of the existing models with single or multi-cracks has been conducted as part of this study. In many of the published contributions [64], the local discontinuities effects due to the damages have been taken into account by altering the local flexibilities. In detail, the crack can be included using two different procedures [195]: (i) the first one, similar to an equivalent spring, considers a stiffness matrix at the crack section obtained by inverting the crack flexibility matrix. Indeed the inversion procedure can lead to numerical problems when the cracks are small, and consequently the stiffness matrix coefficient can be very large; (ii) the second procedure assembles to the undamaged element an additional element stiffness matrix due to the crack with the advantages of avoiding any numeric uncertainty in the stiffness computation.
Historically one of the first examples of crack flexibility matrix is the one introduced by Dimaragonas and Paipetis [65], where the torsion phenomena is neglected. Interestingly the components of the direct compliance matrix, related to tension, bending and their coupling terms, were previously computed by other authors as Okamura [160], Liebowitz [132, 131], Rice and Levy [186]. Papadopoulos and Dimarogonas [165] computed a full $6 \times 6$ matrix for a shaft with a transverse surface crack. The elements of this matrix are obtained from the stress intensity factors and the associated strain energy density function. A beam FE including cracked sections has been developed by Gounaris and Dimarogonas [91] in the late 80s of the last century, using the strain energy concentration arguments form to obtain the compliance matrix.

Papadopulos et al. [165], Sekhar et al. [200] and Krawczuk and Ostachowicz [120] expressed the flexibility matrix of the global cracked beam element $C_c$ (with crack at the mid-section) as sum of the undamaged $C_0$ part and an additional damaged part $C_1$, which includes the crack effects:

$$C_c = C_0 + C_1. \quad (2.14)$$

The flexibility (or compliance) matrix for the case of two-dimensional structure is:

$$C_1 = \begin{bmatrix} C_{xx} & 0 & C_{xr} \\ 0 & C_{zz} & 0 \\ C_{rx} & 0 & C_{rr} \end{bmatrix}, \quad (2.15)$$

where $x$ is the direction along the beam; $z$ is the vertical coordinate and $r$ is the rotation of the cross section. $C_{xx}$, $C_{zz}$ and $C_{rr}$ are the coefficient of the compliance matrix obtained by an energy balance among external work and fracture work; the off-diagonal terms $C_{xr}$ and $C_{rx}$ are responsible for the coupling among axial force and bending moments [64, 165]. Once the compliance matrix for the crack is known, it can be added to the flexibility of the un-cracked beam to give the global element flexibility matrix which inverse yields the global stiffness matrix of the damaged beam element.

Differently Saavedra and Cuitino [195] expressed the crack matrix $C_c$ as:

$$C_c = B^T \left( C_0^a + C_0^b \right) B + C_1 \quad (2.16)$$

where the matrix $B$ is obtained from equilibrium conditions while $C_0^a$ and $C_0^b$ are matrices related to the undamaged part and only depend on the crack position. In case the crack is
located in the beam middle section, Eq. (2.16) can be re-rewritten as:

\[ C_c = \mathbf{B}^\top \mathbf{C}_0 \mathbf{B} + \mathbf{C}_1 \]  

(2.17)

where \( \mathbf{B}^\top \mathbf{C}_0 \mathbf{B} \) can be recognised as \( \mathbf{C}_0 \) of Eq. (2.14).

A further FE approach is the work of Bouboulas and Anifantis [36], who developed a cracked FE beam including a non-propagating edge crack which stiffness matrix includes additional flexibility terms evaluated using strain energy density factors and where the whole beam element consists of two sub-beams plus a zero-length spring-like element at the damaged section. Similarly, Qian et al. [177] derived an element stiffness matrix from integration of the stress intensity factors. Viola et al. [231] developed a single lumped crack model using the Timoshenko theory giving the shape functions from a combination of the solution for the two undamaged parts. They also developed the stiffness and consistent matrices for the cracked Timoshenko beam element by using the direct approach where each column is obtained by imposing unit displacement of one DoF while the others are nulls. The equilibrium equations of each component have been combined together using the compatibility conditions at the damaged node and after static condensation, the procedure of elimination of DoF, yields the \( 6 \times 6 \) stiffness matrix for a two dimensional damaged beam element. They employed the same lumped mass matrix as the undamaged beam for the framed element assuming negligible loss of mass due to the crack and a small displacement approach.

Lee [127] proposed a damage identification technique where the damage beam has been modelled using a FE discretisation. Differently from other methods, an additional rotational DoF at the crack node has been adopted, which results to have three DoFs instead of two. The element before and after the crack are connected through the cracked stiffness matrix:

\[ \mathbf{K}_{e_i} = \begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix}, \]

(2.18)

where \( k_i \) is the rotational stiffness per unit width of the crack (its detailed evaluation can be found in Sec. 2.3.4) The stiffness matrix \( \mathbf{K}_i \), mass matrix \( \mathbf{M}_i \) of each segment \( e_i \) and the crack matrix \( \mathbf{K}_{e_i} \) can be assembled to give the global stiffness and mass element matrix. This approach can also be extended to element with multiple cracks by assembling all the \( n - 1 \) crack matrices.

Carniero et al. [45] gave the stiffness \( \mathbf{K} \) and mass \( \mathbf{M} \) matrices starting from a continuous cracked model. The partial differential equation of motion has been solved via Galerkin method
assuming an approximate solution for the displacement as combination of cubic splines shape function \([203]\), opportunely scaled by an unknown constant parameters \(\theta^{(r)}\). The solution is then expanded in the following form:

\[
\mathbf{u}^*(x) = \sum_{r=1}^{N_\phi} \theta^{(r)} \phi^{(r)}(x),
\]

(2.19)

where \(\phi^{(r)}(x)\) is the \(r\)th mode shape of the corresponding uncracked beam and it is defined as \(\phi^{(r)}(x) = \sin \left(\frac{i\pi x}{L}\right)\), while \(N_\phi\) is the number of terms in the Galerkin expansion. By substituting the approximate solution into the differential equation Eq. (2.13) and by applying Galerkin’s method the following discretised eigenvalue problem is obtained:

\[
\mathbf{K} \theta - \omega^2 \mathbf{M} \theta = 0,
\]

(2.20)

where \(\theta\) is the \(N_\phi\)-vector of generalised co-ordinates and the eigenvalues \(\omega\) correspond to the natural frequencies of the system. The structural dynamic behaviour of the damaged structure can then be evaluated using the FE discretisation and the resulting equations system can be solved by implementing convenient mathematical techniques. However, this study has only considered slender Euler-Bernoulli beams in bending and masses lumped at the two nodes of the resulting FE, which may limit its applicability.

Similarly, Skrinar and Pliberšek \([206, 207]\) extended the single lumped crack problem to a multi-crack problem giving the stiffness matrix \(\mathbf{K}\) and assuming a consistent mass matrix \(\mathbf{M}\) for an multi-damaged EB beam element. In their method cubic splines are used to represent the field of transverse displacements in each uncracked region of the beam, while the additional kinematic and static unknowns arising at each crack have been eliminated with the help of compatibility and equilibrium equations combined with the Hooke’s law for the rotational springs simulating the cracks. The displacement solution is assumed as sum of all the distinct displacement functions of the \(n\) undamaged parts \(u_{e_i}(x)\):

\[
u(x) = \sum_{i=1}^{n} u_{e_i}(x).
\]

(2.21)

Figure 2.8 shows the two nodes multicracked beam element adopted in this formulation. The
undamaged displacement function is then defined as:

\[
\begin{align*}
\left\{ \begin{array}{ll}
  u_e(x_i) = N_i^\top u & \text{if } x_{i-1} \leq x \leq x_i; \\
  u_e(x) = 0 & \text{if } x < x_{i-1} \text{ and } x_i < x,
\end{array} \right.
\end{align*}
\]

(2.22)

where \( u_e = \{u(x_{i-1}), \varphi(x_{i-1}), u(x_i), \varphi(x_i)\}^\top \) is the vector of the unknown end nodal displacements. \( N_i = \{\phi_{e_i}^{(1)}, \phi_{e_i}^{(2)}, \phi_{e_i}^{(3)}, \phi_{e_i}^{(4)}\} \) is the vector of the interpolation functions which are assumed as a polynomial of the third order:

\[
\phi_{e_i}^{(j)} = a_{e_i}^{(j)} + b_{e_i}^{(j)} x + c_{e_i}^{(j)} x^2 + d_{e_i}^{(j)} x^3.
\]

(2.23)

The polynomial coefficients \( a_{e_i}^{(j)}, b_{e_i}^{(j)}, c_{e_i}^{(j)} \) and \( d_{e_i}^{(j)} \) can be obtained by imposing the continuity conditions at each crack position:

- Continuity of the displacements:
  \[ u_{e_i}(x_i) = u_{e_{i+1}}(x_i); \]

(2.24a)

- Continuity of the rotations:
  \[ u'_{e_i}(x_i) + \frac{k_i}{EI_0} u''_{e_i}(x_i) = u'_{e_{i+1}}(x_i); \]

(2.24b)

- Continuity of the bending moment:
  \[ u''_{e_i}(x_i) = u''_{e_{i+1}}(x_i); \]

(2.24c)

- Continuity of the shear forces:
  \[ u'''_{e_i}(x_i) = u'''_{e_{i+1}}(x_i); \]

(2.24d)

as well as the BCs at the end of the beam element. In the previous equations \( k_i \) is the stiffness of the \( i \)th rotational spring. Once the interpolation functions \( u_e(x) \) are known the usual FE procedure can be implemented, where the stiffness and mass matrices are obtained from the deformation energy and kinematic energy of the whole element. Interestingly, the deformation energy is proposed as sum of the deformation energy of the undamaged parts and the energy...
2.3 Cracked beam with DS model

due to the rotational springs:

\[ U = \frac{1}{2} \sum_{i=1}^{n} \int_{\bar{x}_{i-1}}^{\bar{x}_{i}} E I_0 \left( \frac{\partial^2 u_{e,i}(x)}{\partial x^2} \right)^2 \, dx + \frac{1}{2} \sum_{i=1}^{n-1} k_i \left( \frac{\partial u_{e,i}(x)}{\partial x} \bigg|_{\bar{x}_i} - \frac{\partial u_{e,i+1}(x)}{\partial x} \bigg|_{\bar{x}_i} \right)^2, \tag{2.25} \]

Stiffness and mass matrices are thus:

\[ K = E I_0 \sum_{i=1}^{n} \int_{\bar{x}_{i-1}}^{\bar{x}_{i}} N'_e_i N_e_i^T \, dx + \sum_{i=1}^{n} k_i (N'_{e,i} - N'_{e,i+1}) (N'_{e,i} - N'_{e,i+1})^T; \tag{2.26a} \]
\[ M = \sum_{i=1}^{n} \int_{\bar{x}_{i-1}}^{\bar{x}_{i}} \rho(x) A(x) N_e_i N_e_i^T \, dx + \sum_{i=1}^{n} \int_{\bar{x}_{i-1}}^{\bar{x}_{i}} \rho(x) I(x) N'_e_i N_e_i^T \, dx, \tag{2.26b} \]

where the mass matrix accounts for both translational and rotatory inertia.

Another FE approach for multi-cracked beam element has been recently proposed by Cademi et al. [41, 40], who have studied the effects of the crack position and depth, showing how these parameters affect the dynamic properties of the structures, in particular the frequencies and modal shapes. Their formulation also includes the shear deformations (i.e. the Timoshenko beam theory has been adopted), and rotational and transverse springs are considered at the position of each crack. They have employed the rigidity modelling of concentrated damage to derive the exact closed-form expressions for the deformed shape of the two-node multi-cracked beam element subjected to unitary nodal displacements, which in turn have been used to derive the stiffness matrix and consistent mass matrix, without accounting though for the rotatory inertia.

2.3.4 Crack functions for the DS model

The presence of a crack in a structural element increases the local flexibility due to the stress concentration at the crack tip. These stresses produce a local increased strain \( \varepsilon(x, y, z) \) and deformation \( u(x, y, z) \), similarly to an equivalent spring. The linear fracture mechanics theory can be used to compute additional displacements [195]. According to Castigliano’s theorem, the equation for the displacement of a force \( p_i \) is defined as:

\[ u_i(x) = \frac{\partial}{\partial p_i} \int_{0}^{a} I(\alpha) \, d\alpha, \tag{2.27} \]
where $I(\alpha)$ is the release strain energy density function, defined as:

$$I(\alpha) = \frac{1 - \nu^2}{E} \left[ \left( \sum_{i=1}^{N} k_{I,i} \right)^2 + \left( \sum_{i=1}^{N} k_{II,i} \right)^2 + \frac{1}{1 - \nu} \left( \sum_{i=1}^{N} k_{III,i} \right)^2 \right], \quad (2.28)$$

while $k_{I,j}$, $k_{II,j}$ and $k_{III,j}$ are the stress intensity factors, corresponding to three independent cracking modes (opening, sliding and tearing) [189], and their values are available in literature [65, 161]. The tip plastic zone is assumed to be small compared to the crack dimensions and therefore linear fracture mechanics can then be applied. The additional flexibility $C_{ij}$ can be obtained from the combination of the crack displacement of Eq. (2.27) and the compliance of Eq. (2.28)

$$C_{ij} = \frac{\partial u_i}{\partial p_j} = \frac{\partial^2}{\partial p_i \partial p_j} \int_a I(\alpha) d\alpha, \quad (2.29)$$

Other approaches adopt the Irwin’s relationship for the stress at crack tip $\sigma_{ij}$, which can be written as:

$$\sigma_{ij} = \frac{k}{\sqrt{r}} f_{ij}(\theta), \quad (2.30)$$

where $k$ is the stress intensity factor; $r$ is the distance from the crack tip and $f_{ij}(\theta)$ is a function of the orientation angle $\theta$. This equation illustrates local deformations at close neighbourhood of the crack tip due to the stress concentration. Consequently loss of local stiffness is a result of the local deformations. However the dynamics of the structure would be influenced due to enhanced flexibility. This approach has been used by many authors e.g. Rice and Levy [186], Freud and Herrmann [86], Levy and Herrmann [128] or Gudmundson [94]. One limitation of this approach is that an accurate definition of the stress intensity factor is required, and this is easily achievable only for simple structures.

Many approaches in the literature relate the rotational spring stiffness to the crack depth $d$ (see Fig. 2.9). All these models can be unified with the following general expression for the stiffness $k_{eq}$:

$$k_{eq} = E I_0 \frac{1}{h C(\beta)}, \quad (2.31)$$

where $E$ is the Young’s modulus of the material; $I_0$ is the second moment of inertia of the

![Figure 2.9: Sketch of crack geometry](image-url)
2.3 Cracked beam with DS model

undamaged cross section; \( \beta = d/h \) is the ratio of the crack depth and the cross section height \( h \). Additionally, \( C(\beta) \) is the dimensionless local compliance, which depends on the crack orientation and magnitude as well as on the applied loadings and the deformation shape. Different approaches have been adopted to evaluate the local compliance. Ostachowicz and Krawczuk [161] used the expression for the elastic energy to relate the equivalent stiffness at the crack location with the stress intensity factor, which can be expressed in terms of the crack depth using a correction factor. For one-sided crack the equation for the compliance yields:

\[
C(\beta) = 6\pi \beta^2 (0.6384 - 1.035 \beta + 3.7201 \beta^2 - 5.1773 \beta^3 + 7.553 \beta^4 - 7.332 \beta^5 + 2.4909 \beta^6).
\]

Rizos et al. [188] also computed the compliance starting from the strain energy density function yielding:

\[
C(\beta) = 5.346(1.86 \beta^2 - 3.95 \beta^3 + 16.375 \beta^4 - 37.226 \beta^5 + 76.81 \beta^6 - 126.9 \beta^7 + 172 \beta^8 - 143.97 \beta^9 + 66.56 \beta^{10}).
\]

Bilello in his PhD thesis [31] proposed an expression for the stiffness of the equivalent rotational spring \( k_{eq} \), which is valid only for beams with uniform rectangular cross section and with the crack perpendicular to the beam longitudinal axis, which is realistic for cracks caused by service loads [32]. The model directly relates spring stiffness with the crack depth \( d \), as well as the section geometry and material properties, through the following expression of compliance function:

\[
C(\beta) = \frac{\beta(2 - \beta)}{0.9(\beta - 1)^2}.
\]

Chondros et al. [56] developed a model where the crack has been represented using a displacement field in the vicinity of the crack which, is consistent with fracture mechanics methods:

\[
C(\beta) = 6\pi(1 - \nu^2)(0.6272\beta^2 - 1.04533 \beta^3 + 4.5948\beta^4 - 9.9736\beta^5 + 20.2948 \beta^6 - 33.0351 \beta^7 + 47.1063 \beta^8 - 40.7556\beta^9 + 19.6\beta^{10}).
\]

Parmeter and Reid [168] expressed the same crack function as:

\[
C(\beta) = 0.5\beta^2 (12 - 19.5 \beta + 70.1 \beta^2 - 97.6 \beta^3 + 142 \beta^4 - 138 \beta^5 + 128 \beta^6 - 132 \beta^7 + 379 \beta^8 - 417 \beta^9 + 131 \beta^{10} + 313 \beta^{12} - 357 \beta^{13} + 102 \beta^{14}), \text{ if } 0 \leq \beta < 0.5,
\]

(2.36)
\[ C(\beta) = \frac{0.66}{(1 - \beta)^2} - 0.89, \text{ if } 0.5 \leq \beta \leq 1, \] (2.37)

Applications of this formulation can be found in the paper of Viola and Pascale [232]. These crack functions are valid for specific crack configurations and limited cross section type only, because of that any different cross section requires to identify suitable expressions able to give the stiffness value of the equivalent spring.

Importantly, once the crack function is obtained for a given material and cross section, it enables the application of the DS model.

### 2.3.5 Limitations of the discrete spring beam model

As a matter of fact in the DS model the damage is represented using an axial, rotational and/or transverse spring which represents a simplification of the real crack phenomena. As a consequence this model has limitations and therefore it is necessary to specify its applicability.

#### 2.3.5.1 Beam models

A limitation is given by the type of beam model used in the analysis. For example, the classical EB beam, tends to overestimate the natural frequencies and proves to be inadequate to identify the higher modes of vibration or to analyse short, stubby beams, where the cross-sectional dimensions are comparable to the length of the beam [241, 242, 142]. The main shortcoming of the EB theory is the absence of shear deformations and rotatory inertia, which leads to higher value of the natural frequencies of the beam if compared with the experimental results, especially for materials with low shear modulus, as wood or composite structure. Therefore, if such effects are neglected, this can result in wrong estimations of the structural parameters, as for example in the quantification of the elastic constants, Young’s modulus and damping, of a wooden structure using a vibration method. For instance, Hearmon [101] has showed how neglecting the shear and the rotatory inertia effects can lead to less than half of the expected frequencies. These effects have been studied by Wang et al [242], showing that different sections can considerably increase, about 50\%, the frequencies. In the same work, these effects have been proven to be small and negligible for slender beam, where the vibration parameters converge to the classical EB theory.

In order to overcome these inefficiencies, there have been many attempts to improve the EB theory. The more noticeable are the inclusion of the shear flexibility effect and of the rotatory inertia. The effects of rotatory inertia was first included in a beam theory by Rayleigh
2.3. Cracked beam with DS model

et al. [182], subsequently Timoshenko [216, 217] extended this theory by adding the transverse shear effect. Over the years several structural configurations been investigated by using the rotatory inertia and shear effects. The whole Timoshenko theory has been applied to beam problems by Anderson [17] and Dolph [67]. Huang [103, 102] applied Ritz and Galerkin methods to solve a pinned-pinned beam problem; they gave the frequencies and mode shapes of six common types of beam with different boundary conditions (BCs). Another approach has been adopted by Kapur [112] who applied the FE method to present solutions at similar problems. Wang [241] included the two effects to study multiple span beams using the general dynamic three-moment equation. Grant [92] published a paper regarding the Timoshenko beam carrying a concentrated mass. Gupta [95], Rouch [190] and To [218] developed a linearly tapered Timoshenko beam element and studied its dynamic characteristics. Shastry [201] investigated the buckling problem and studied the phenomena for different slenderness ratio varying from 10 to 500.

As previously mentioned, composite beams (or laminated beams), and in particular those with high normal to shear modulus ratio, represent one of those examples where shear deformation and rotatory inertia can not be neglected, leading to erroneous predictions of the natural frequencies [111]. Considering the increasing use of the laminated beam concept in several fields like mechanical and engineering, transportation vehicles, marine, aviation and aerospace, the effort to include these effects is fully justified. Exact solution for the free vibrations of composite structures has been studied by many different authors [248, 51, 142, 111, 54, 170]. Among all Abramovich [2] highlighted the importance of considering shear deformation and rotatory inertia especially for fibrous composite materials. In the late nineties Al-Ansary [13] included the rotatory inertia to study the flexural vibrations of rotating beams showing an increase up to 10% on the extensional tensile force; the problem has also been studied by Rao and Gupta [180] few years after. An estimation of the shear effect and rotatory inertia effect has been presented through an approximate equation by Traill-Nash [222]; the comparison the two contributions shows how the shear flexibility effect more relevant, three to four times bigger, compared with the rotatory inertia.

2.3.5.2 DS model

In most of the DS model the spring is located along the longitudinal axis of the beam, in doing so it is not able to account for any eccentricity of the crack e.g. if the damage is on one side only of the section or it is not symmetric with respect of the longitudinal axes. In practice, the eccentricity effects are usually neglected especially if the beam has a low or null
axial load or if the eccentricity is relatively small as well as if the cracks are symmetrical to the longitudinal axis (null eccentricity of the cracked section). Indeed, the effect of the out of axis at the crack sections can be relevant in stubby beams where it can be relatively important bringing to coupling effects among axial and transversal displacements. In order to overcome these limitations a DS model able to include the eccentricity and coupling effects need to be assumed; few are the authors who took into account the coupling phenomena, among them Adams mentioned the double resonance phenomenon resulting in highly eccentric damages [3]; while Papadopoulos and Dimarogonas [165] in their research enriched the stiffness and mass matrices by including coupling terms.

Further limitation is due to the localisation of all the damage effects in a single point which brings to inaccuracies on the description of the displacement and rotational field around the crack, especially at the high frequencies. Fan and Qiao, in their review [81], judged the rotational spring methods to be less efficient to investigate deep crack configurations where higher frequencies are required; they believed that these methods can be efficiently applied only to structures as slender beams with small cracks and they can correctly provide only the first few modes.

This is the reason why a more accurate DS model able to represent the structural behaviour around the crack and in particular able to avoid the abrupt changes in the rotation profile at the crack position can then be valuable approach. This will be the aim of Chapter 5 and Chapter 6 where this limitation will be overcome thanks to the inclusion of material parameters related to the microstructure.

2.3.6 Reasons of the proposed DS model

In summary, the DS model represents the best trade-off among accuracy and computational effort thanks to its capability to well represent the crack in terms of position and severity. It has been extensively applied within different schemes of structural health monitoring, aimed at identifying presence, location and severity of concentrated damage in frame structures. In this context, the size of the FE assembly for the structural frame under investigation plays an important role, as ideally it should be as small as possible. Indeed, the vast majority of the identification algorithms proceed iteratively until convergence, and therefore any little saving in the computational cost for a single analysis may result in a significant advantage on the whole process. Moreover traditional damage detection approaches may require FE re-meshing throughout the identification process, which inevitably causes an additional increase in the computational effort.
The main shortcoming with the DS model is that, if a conventional beam element is used, two additional FE nodes must be placed at the location of each concentrated damage, i.e. one node on each side. This could be particularly cumbersome if the DS model is used for the purposes of damage identification, as it would also require re-meshing the beam during the identification process.

Aimed at addressing these issues, an analytical and numerical study has been carried out as part of this thesis to develop an efficient two-node multi-damaged Timoshenko beam (MDB) element for the FE analysis of frame structures, which is able to account for any number and location of concentrated damages without increasing the size of the problem in comparison with the corresponding undamaged structure. In particular the representation of cracks proposed by Palmeri and Cicirello [162] has been exploited, and the resulting MDB FE formulation is presented and numerically validated in Chapter 3. It is worth mentioning here that within the present study, any localised increase in the flexibility of the beam is considered as a concentrated damage, provided that the portion of the beam affected by such increase is less than its cross sectional dimensions. Similar approaches have been recently pursued by other authors [40], whose studies differ in the analytical formulations adopted to get the closed-form expressions for stiffness matrix, load vector and mass matrix. The main difference of the proposed work with these formulations are:

i) the inclusion of the axial damage in the MDB element, so that it is possible to consider for each crack a set of axial, rotational and shear springs, i.e. the beam is fully articulated at the position of the cracks, allowing relative longitudinal, rotational and transverse movements (this is particularly important in the dynamic analysis of structures in which both axial and transversal deformations affects the modes of vibrations, as for instance in camshaft [3, 165]);

ii) the rotatory inertia is also taken into account, which results in lowering the vibrational frequencies, particularly the higher ones, which can be critical in the accurate identification of position and severity of cracks [123, 133, 178, 179, 192]. Furthermore, when the material has a relatively high ratio of bending to shear modulus, e.g. wood, the effects of shear deformations and rotatory inertia may not become negligible even for slender beam [58];
Aimed at defining the shape functions for the proposed Multi-damaged beam (MDB) element, the flexibility modelling recently proposed by Palmeri and Cicirello [162] for a concentrated flexural damage (i.e. a crack-induced lumped rotation) has been extended to include axial and shear lumped deformations at the position of the concentrated damage, and it has been then used to derive the exact closed-form solutions for MDBs subjected to both axial and transverse loads. Such solutions will then allow the definition of stiffness matrix and consistent mass matrix.

### 3.1 Exact closed-form solutions

For derivation purposes, let us consider an elastic straight beam with abscissa-dependent axial and flexural stiffness, $EA(x)$ and $EI(x)$, where $E$ is the Young’s modulus, $A(x)$ is the cross-sectional area and $I(x)$ the second moment of area respectively; while $x$ is the axial spatial coordinate spanning from 0 to the length $L$ of the beam. The beam is subjected to a generic axial and transverse load, $q_A(x)$ and $q_T(x)$, distributed along the longitudinal axis $x$ (see Fig. 3.5). The expressions for the generic axial (longitudinal) and transverse displacements, $u_A(x)$ and $u_T(x)$, can be derived combining together the equilibrium, kinematic (or compatibility) and constitutive equations.
3. Multi-damaged beam element

3.1 Equilibrium equations

For the problem in hand, the equilibrium equations are obtained from the balance of forces and moments acting on a small (infinitesimal) section of beam (see Fig. 3.1). In detail the longitudinal, transversal and rotational equilibrium equations read:

\[ N'(x) + q_A(x) = 0 ; \]
\[ V'(x) + q_T(x) = 0 ; \]
\[ M'(x) - V(x) = 0 , \]

where the prime \( ' \) stands for the spatial derivative with respect to the abscissa \( x \), i.e. \( (\cdot)' = \frac{d (\cdot)}{dx} \); \( N(x) \) is the axial force, \( M(x) \) is the bending moment (positive if sagging) and \( V(x) \) is the shear force.

3.1.1 Kinematic equations

To take into account the shear effects, the transverse deflection \( u_T(x) \) is assumed as superposition of two contributions, the pure bending displacement \( u_b(x) \) and the pure shearing displacement \( u_s(x) \):

\[ u_T(x) = u_b(x) + u_s(x) . \]

It follows that, the slope of the beam \( u_T'(x) \) with respect to the longitudinal axes is the superposition of the pure rotation \( \varphi(x) \) of the cross section and the average shearing strain \( \Gamma(x) \). This relation can be mathematically defined as:

\[ u_T'(x) = \Gamma(x) - \varphi(x) . \]

Figure 3.2 shows the kinematic approximation involved in the EB beam model for the bending deflection \( u_b(x) \) only, while Fig. 3.3 shows the combination of bending and shear. The kin-
3.1 Exact closed-form solutions

Figure 3.2: Kinematic approximation involved in the Euler-Bernoulli beam

Kinematic equations for the assumed Timoshenko beam model including the axial displacements are:

\[
\varepsilon_A(x) = u_A'(x) ; \\
\chi(x) = \frac{1}{\rho(x)} = \varphi'(x) = -u''_b(x) ; \\
\varphi(x) = -u'_b(x) ; \\
\Gamma(x) = u'_s(x) ,
\]

where \( \varepsilon_A(x) \) is the axial deformation of the centroid position and \( \chi(x) \) is the curvature of the beam, equal to the inverse of the radius of curvature \( \rho(x) \).

3.1.1.2 Constitutive equations

Within the classical elasticity theory, the constitutive equations which link together the deformations and forces can be expressed as:

\[
EA(x) \varepsilon(x) = N(x) ; \\
EI(x) \chi(x) = M(x) ; \\
GA_s(x) \Gamma(x) = V(x) ,
\]
where \( G = E/(2(1 + \nu)) \) is the shear modulus (which can be computed once the Young’s modulus \( E \) and the Poisson’s ratio \( \nu \) are known); \( A_s \) is the shear area, which can be expressed as \( A_s(x) = A(x)/\kappa \), \( \kappa \) being the shear correction factor (or Timoshenko shear coefficient), which takes into account the non uniformity of the stress distribution over the cross-section [216] and is defined as the ratio of the average shear strain on a section to the shear strain at the centroid:
\[
\kappa = \frac{\int A \gamma_{xz} \, dA}{\Gamma}.
\]
(3.6)

Combining Eqs. (3.1), (3.4) and (3.5) yields to the three well known differential equations linking together displacements and external loads:
\[
\begin{align*}
[EA(x)u_A'(x)]' &= -q_A(x); \quad (3.7a) \\
[EI(x)u_b''(x)]'' &= q_T(x); \quad (3.7b) \\
[GA_S(x)u_s'(x)]' &= -q_T(x). \quad (3.7c)
\end{align*}
\]

### 3.1.1.3 Flexibility functions

Fig. 3.4 displays a \( n \)-damaged prismatic beam element, with length \( L \) in the dimensional system of reference, with discontinuities shown at abscissa \( x = \bar{x}_1, x = \bar{x}_j \) and \( x = \bar{x}_n \), where the first, the \( j \)th and the \( n \)th damage are located. Each point of the beam possesses three
3.1 Exact closed-form solutions

Figure 3.4: Sketch of the multi-damaged beam

degrees of freedom, i.e. the two axial and transverse translations, \( u_A(x) \) and \( u_T(x) \), and the in-plane rotation, \( \varphi(x) \), whose positive directions are shown in Fig. 3.4.

According to the flexibility modelling [162], the presence of a concentrated damage at the generic abscissa \( x = \bar{x}_j \) results in a positive impulse in the flexibility functions, which in turn can be mathematically represented with a Dirac’s delta function centred at the position of the damage. It follows that, if \( n \) concentrated damages occur in the beam, the axial flexibility \( \Upsilon_A(x) \), bending flexibility \( \Upsilon_I(x) \) and shear flexibility \( \Upsilon_S(x) \) take the expressions:

\[
\Upsilon_A(x) = \frac{1 + F_\alpha(x)}{E \ A_0};
\]

\[
\Upsilon_I(x) = \frac{1 + F_\beta(x)}{E' I_0};
\]

\[
\Upsilon_S = \frac{1 + F_\gamma(x)}{G A_{s0}},
\]

where \( E \) is the Young’s module of the material; \( A_0, I_0 \) and \( A_{s0} \) are the geometric properties of the undamaged cross section of the beam; the functions \( F_\alpha(x) \), \( F_\beta(x) \) and \( F_\gamma(x) \) describes mathematically the localised effect of the \( n \) damages:

\[
F_\alpha(x) = \sum_{j=1}^{n} \alpha_j \delta(x - \bar{x}_j);
\]

\[
F_\beta(x) = \sum_{j=1}^{n} \beta_j \delta(x - \bar{x}_j);
\]

\[
F_\gamma(x) = \sum_{j=1}^{n} \gamma_j \delta(x - \bar{x}_j),
\]
where the impulsive function $\delta(x - \bar{x}_j)$ is the Dirac’s delta function, centred at the position of the $j$th damage, $x = \bar{x}_j$, and is formally defined as the derivative of the Heaviside’s unit step function, i.e. $\delta(x - \bar{x}_j) = H'(x - \bar{x}_j)$, with $H(x) = 0$ for $x < 0$; $H(x) = 1$ for $x > 0$; $H(x) = 1/2$ for $x = 0$. Moreover, the dimensionless parameters $\alpha_j$, $\beta_j$ and $\gamma_j$ define the intensity of the $j$th impulses in the three flexibility functions, which in turn are related to severity of the corresponding longitudinal, rotational and transverse damages, i.e. the more severe the damage, the higher are the impulses (while the condition $\alpha_j = \beta_j = \gamma_j = 0$ corresponds to the absence of damage at $x = \bar{x}_j$). The proposed model can be used in both direct and inverse problems; in the first case, the damage parameters are assumed either experimentally or via a detailed FE analysis of the damaged zone (e.g. multi-scale analysis), and then introduced to model the global response of the structure; while typically in inverse problems, the axial, rotational and transversal damage parameters are unknown variables, which can be quantified using the experimental data and any standard technique of model updating.

Indeed it can be mathematically proven that the three damage parameters of a generic damage $j$ are related to the axial, rotational and transversal spring stiffness coefficients $k_{\alpha j}$, $k_{\beta j}$ and $k_{\gamma j}$ by the following relationships:

\begin{align}
  k_{\alpha j} &= \frac{EA}{\alpha_j L}, \quad (3.10a) \\
  k_{\beta j} &= \frac{EI}{\beta_j L}, \quad (3.10b) \\
  k_{\gamma j} &= \frac{GA_s}{\gamma_j L}, \quad (3.10c)
\end{align}

and for instance, the rotational spring stiffness $k_{\beta j}$ may be evaluated as a function of the crack depth (e.g. Refs. [161, 31]).

### 3.1.2 Axial displacement function of the multi-damaged beam

Let us consider first the axial displacement of a multicracked beam with $n$-damages and subjected to a generic axial load $q_A(x)$ distributed along the beam. The exact closed-form solution in terms of the resulting field of axial displacements can be obtained by solving the differential equation (Eq. (3.7a)), which by integrating with respect to the abscissa $x$ gives:

\begin{equation}
  EA(x) u_\alpha'(x) = - \left( q_A^{[1]}(x) + C_1 \right), \quad (3.11)
\end{equation}
3.1 Exact closed-form solutions

where the superscripted expression \([i]\) denotes the primitive (or anti-derivative \(^1\)), that is: \(d^nf^{[n]}(x)/dx^n = f(x)\), while \(C_1\) is an integration constant evaluated once the boundary conditions (BCs) are enforced. Combining Eq. (3.11) and Eq. (3.8a) yields:

\[
 u_A'(x) = -\mathcal{Y}_{A0} \left[ 1 + \sum_{j=1}^{n} \alpha_j \delta(x - \bar{x}_j) \right] \left( q_A^{[1]}(x) + C_1 \right) , \tag{3.13}
\]

where \(\mathcal{Y}_{A0}\) is the axial flexibility of the undamaged section. In order to derive the mathematical solution for the problem in hand, the Laplace’s transform operator \(L\langle \cdot \rangle\) can be applied to both sides of Eq. (3.13):

\[
 s \mathcal{L}\langle u_a(x) \rangle - C_2 = -\mathcal{Y}_{A0} \left[ \frac{C_1}{s} + \mathcal{L}\langle q_A^{[1]}(x) \rangle + \sum_{j=1}^{n} \alpha_j \left( q_A^{[1]}(\bar{x}_j) + C_1 \right) e^{-\bar{x}_j s} \right], \tag{3.14}
\]

where \(s\) is the Laplace’s variable associated with the abscissa \(x\), while \(C_2\) is the second additional integration constant. Applying now the inverse Laplace’s transform operator \(L^{-1}\langle \cdot \rangle\), we obtain the mathematical expression of the axial displacements:

\[
 u_A(x) = C_2 L^{-1}\langle \frac{1}{s} \rangle - \mathcal{Y}_{A0} \left[ \frac{C_1}{s} L^{-1}\langle \frac{1}{s} \rangle + \frac{\mathcal{L}\langle q_A^{[1]}(x) \rangle}{s} \right] + \sum_{j=1}^{n} \alpha_j \left( q_A^{[1]}(\bar{x}_j) + C_1 \right) L^{-1}\langle \frac{e^{-\bar{x}_j s}}{s} \rangle = \tag{3.15}
\]

\[
 = C_2 + \mathcal{Y}_{A0} \left[ C_1 x + q_A^{[2]}(x) + \sum_{j=1}^{n} \alpha_j \left( q_A^{[1]}(\bar{x}_j) + C_1 \right) H(x - \bar{x}_j) \right].
\]

The two integration constants (\(C_1\) and \(C_2\)) can be then obtained by imposing the two BCs as in the case of an undamaged EB beam. In Eq. (3.15) one can easily identify one part which is independent of the damage and another one which is the sum of all the damages effects. Based on this consideration, it is possible to rewrite the Eq. (3.15) as sum of the two contributions:

\[
 f(x)^{[1]} = \int_0^x f(t)dt. \tag{3.12}
\]

\(^1\) The anti-derivative \(f(x)^{[1]}\) is given by the following definite integral of \(f\):
the undamaged part $u_{A0}(x)$ and all the damaged effects $\Delta u_{Aj}(x)$.

$$u_A(x) = u_{A0}(x) + \sum_{j=1}^{n} \Delta u_{Aj}(x), \quad (3.16)$$

where the contribution due to the undamaged response is:

$$u_{A0}(x) = C_2 - \mathcal{Y}_{A0} \left[ C_1 x + q_{A}^{[2]}(x) \right], \quad (3.17)$$

while for the generic concentrated damage, the following contribution appears:

$$\Delta u_{Aj}(x) = -\mathcal{Y}_{A0} \sum_{j=1}^{n} \alpha_j \left( q_{A}^{[1]}(\bar{x}_j) + C_1 \right) H(x - \bar{x}_j). \quad (3.18)$$

From Eq. (3.18) it can be shown that the generic finite jump $\Delta u_{Aj}(\bar{x}_j)$ appearing in the axial displacements’ profile at the position of the crack $x = \bar{x}_j$, takes the expression:

$$\Delta u_{Aj}(\bar{x}_j) = u_{Aj}(\bar{x}_j^+) - u_{Aj}(\bar{x}_j^-) = -\mathcal{Y}_{A0} \alpha_j \left( q_{A}^{[1]}(\bar{x}_j) + C_1 \right)$$

$$= \mathcal{Y}_{A0} \alpha_j N(\bar{x}_j), \quad (3.19)$$

i.e. it is proportional to the axial flexibility of the beam $\mathcal{Y}_{A0}$, the damage parameter $\alpha_j$ and the axial force at the position of the damage, $N(\bar{x}_j)$.

The first derivative of the axial displacement gives the axial deformation of the centroidal fibre:

$$\varepsilon_A(x) = u_A'(x) = -\mathcal{Y}_{A0} \left[ C_1 + q_{A}^{[1]}(x) + \sum_{j=1}^{n} \alpha_j \left( q_{A}^{[1]}(\bar{x}_j) + C_1 \right) \delta(x - \bar{x}_j) \right]$$

$$= -\mathcal{Y}_{A0} \left( C_1 + q_{A}^{[1]}(x) \right) \left[ 1 + \sum_{j=1}^{n} \alpha_j \delta(x - \bar{x}_j) \right]. \quad (3.20)$$

It follows that the presence of a damage at the position $x = \bar{x}_j$ results in a finite jump in terms of axial displacements, represented by the Heaviside’s unit step function $H(x - \bar{x}_j)$ in Eq. (3.15), and an impulse in terms of the axial strain, represented by the Dirac’s delta function $\delta(x - \bar{x}_j)$ in Eq. (3.20). These singularities disappear in the normal force:

$$N(x) = EA(x) \varepsilon_A(x) = -q_{A}^{[1]}(x) - C_1 \quad (3.21)$$
3.1 Exact closed-form solutions

3.1.3 Transverse displacement function of the multi-damaged beam

Let us consider now the case in which the transverse load \( q_T(x) \) is applied on the same \( n \)-damaged beam, and the Timoshenko beam theory is adopted for the kinematics of the undamaged portions of the beam. The resulting field of the transverse displacements \( u(x) \) can be decomposed in pure-bending (i.e. Euler-Bernoulli) contribution, \( u_b(x) \), and a pure-shearing contribution, \( u_s(x) \), which are ruled by coupled differential equations. By extending the formulation presented by Palmeri and Cicirello [162] to include the presence of a transverse spring at the position of the generic concentrated damage (see Eq. (3.8c)), the exact closed-form solution for this problem has been derived. The procedure adopted is the same as the axial case where the equilibrium equation has been solved using the Laplace’s transform.

3.1.3.1 Euler-Bernoulli component \( u_b(x) \) of the multi-damaged beam

To avoid the double derivation of the Dirac’s delta, let us first integrate with respect to \( x \) both sides of Eq. (3.7b):

\[
EI(x) u_b''(x) = q_T^{[2]}(x) + C_3 x + C_4, \tag{3.22}
\]

where \( C_3 \) and \( C_4 \) are integration constants, evaluated once the BCs are enforced. Combining Eq. (3.22) and Eq. (3.8b) yields:

\[
u b''(x) = T_{10} \left[ 1 + \sum_{j=1}^{n} \beta_j \delta(x - \bar{x}_j) \right] \left( q_T^{[2]}(x) + C_3 x + C_4 \right) \tag{3.23}
\]
The application of the Laplace’s transform operator $L\langle \cdot \rangle$ to both sides of Eq. (3.23) yields:

$$s^2 L\langle u_b(x) \rangle - C_{6b} s - C_5 = \Upsilon_0 \left[ \frac{C_3}{s^2} + \frac{C_4}{s} + L\langle q^2_T(x) \rangle + \sum_{j=1}^n \beta_j \left( q^2_T(\bar{x}_j) + C_3 \bar{x}_j + C_4 \right) e^{-\bar{x}_j s} \right],$$  

(3.24)

where again $s$ is the Laplace’s variable associated with the abscissa $x$, while $C_5$ and $C_{6b}$ are the two additional integration constants. Applying now the inverse Laplace’s transform operator $L^{-1}\langle \cdot \rangle$, we obtain the final equation for the transverse displacement:

$$u_b(x) = C_{6b} L^{-1}\left( \frac{1}{s} \right) + C_5 L^{-1}\left( \frac{1}{s^2} \right) + \Upsilon_0 \left[ C_4 L^{-1}\left( \frac{1}{s^3} \right) + C_3 L^{-1}\left( \frac{1}{s^4} \right) + L^{-1} \left( \frac{L\langle q^2_T(x) \rangle}{s^2} \right) + \sum_{j=1}^n \beta_j \left( q^2_T(\bar{x}_j) + C_3 \bar{x}_j + C_4 \right) L^{-1}\left( \frac{e^{-\bar{x}_j s}}{s^2} \right) \right],$$  

(3.25)

and then:

$$u_b(x) = C_{6b} + C_5 x + \Upsilon_0 \left[ C_4 \frac{x^2}{2} + C_3 \frac{x^3}{6} + q^4_T(x) + \sum_{j=1}^n \beta_j \left( q^2_T(\bar{x}_j) + C_3 \bar{x}_j + C_4 \right) (x - \bar{x}_j) H(x - \bar{x}_j) \right].$$  

(3.26)

The bending component for the transverse displacement of Eq. (3.26) can also be expressed in the form:

$$u_b(x) = u_{b0}(x) + \sum_{j=1}^n \Delta u_{bj}(x),$$  

(3.27)

where the contribution due to the undamaged response is given by:

$$u_{b0}(x) = C_{6b} - C_5 x - \Upsilon_0 \left[ C_4 \frac{x^2}{2} + C_3 \frac{x^3}{6} + q^4_T(x) \right];$$  

(3.28)

while for the generic concentrated damage, the following term appears:

$$\Delta u_{bj}(x) = -\Upsilon_0 \beta_j (x - \bar{x}_j) H(x - \bar{x}_j) \left[ C_4 + C_3 \bar{x}_j + q^2_T(\bar{x}_j) \right].$$  

(3.29)
3.1 Exact closed-form solutions

3.1.3.2 Shear effect component $u_s(x)$ of the multi-damaged beam

The same approach can be adopted for the shear component $u_s(x)$; in this case the bending differential equation Eq. (3.7c) can be rewritten as:

$$GA_s(x) u_s'(x) = -q^{[1]}_T(x) - C_3,$$

and then introducing Eq. (3.8c) yields:

$$u_s'(x) = -\mathcal{T}_S \left[ 1 + \sum_{j=1}^{n} \gamma_j \delta(x - \bar{x}_j) \right] (q^{[1]}_T(x) + C_3).$$

Similarly to the previous cases, the Laplace’s transform operator applied to both sides of Eq. (3.31) yielding:

$$s \mathcal{L}\{u_s(x)\} - C_{6s} = -\mathcal{T}_S \left[ \frac{C_3}{s} + \mathcal{L}\{q^{[1]}_T(x)\} + \sum_{j=1}^{n} \gamma_j \left( q^{[1]}_T(\bar{x}_j) + C_3 \right) e^{-\bar{x}_j s} \right],$$

where $C_{6s}$ is an additional integration constant. Applying now the inverse Laplace’s transform operator $\mathcal{L}^{-1}\{ \cdot \}$, we obtain the final equation for the shear component of the displacement:

$$u_s(x) = C_{6s} - \mathcal{T}_S \left[ C_3 x + q^{[2]}_T(x) + \sum_{j=1}^{n} \gamma_j \left( q^{[1]}_T(\bar{x}_j) + C_3 \right) H(x - \bar{x}_j) \right].$$

3.1.3.3 Transverse displacement function $u_T(x)$ of the multi-damaged beam

Given the equations for the bending and shearing effect, they can be superimposed by following Eq. (3.2), yielding to the closed-form expression of the total transverse deflection $u_T(x)$ in multi-damaged Timoshenko beams:

$$u_T(x) = C_6 + C_5 x + \mathcal{T}_I \left[ C_4 \frac{x^2}{2} + C_3 \frac{x^3}{6} + q^{[4]}_T(x) + \sum_{j=1}^{n} \beta_j \left( q^{[2]}_T(\bar{x}_j) + C_3 \bar{x}_j + C_4 \right) (x - \bar{x}_j)H(x - \bar{x}_j) \right] +$$

$$- \mathcal{T}_S \left[ C_3 x + q^{[2]}_T(x) + \sum_{j=1}^{n} \gamma_j \left( q^{[1]}_T(\bar{x}_j) + C_3 \right) H(x - \bar{x}_j) \right].$$

(3.34)
The integration constants \((C_3, C_4, C_5\text{ and } C_6)\) can be then obtained by imposing the four BCs as in the case of an undamaged beam. It must be noted here that the integration constant \(C_6\) appearing in Eq. (3.34) can be considered as the sum of the constants \(C_{6b}\) and \(C_{6s}\) due to the bending and shearing contributions, i.e. \(C_6 = C_{6b} + C_{6s}\). The solution for EB beams is recovered when the shearing flexibility \(T_{S0}\) equal zero. In Eq. (3.34) it can be easily identified one part independent of the damage and another one, which is the sum of all the damages effects in terms of transversal displacement. It is then possible to rewrite Eq. (3.34) as sum of the two contributions: the undamaged part \(u_{T0}(x)\) and all the damage effects \(\Delta u_{Tj}(x)\):

\[
u_T(x) = u_{T0}(x) + \sum_{j=1}^{n} \Delta u_{Tj}(x), \tag{3.35}
\]

where the contribution due to the undamaged response is given by:

\[
u_{T0}(x) = C_6 + C_5 x + \eta_{I0} \left( C_4 \frac{x^2}{2} + C_3 \frac{x^3}{6} + q_{T}^{[4]}(x) \right) +
- \eta_{S0} \left( C_3 x + q_{T}^{[2]}(x) \right), \tag{3.36}
\]

while for the generic concentrated damage, the following contribution appears:

\[
\Delta u_{Tj}(x) = \sum_{j=1}^{n} \left[ \eta_{I0} \beta_j(x - \bar{x}_j) \left( q_{T}^{[2]}(\bar{x}_j) + C_3 \bar{x}_j + C_4 \right) +
- \eta_{S0} \gamma_j \left( q_{T}^{[1]}(\bar{x}_j) + C_3 \right) \right] H(x - \bar{x}_j). \tag{3.37}
\]

The spatial derivative of Eq. (3.34) with respect to the abscissa \(x\) gives the slope function of the beam, which takes into account both bending and shearing contributions, while taking the derivative of the sole bending term, Eq. (3.26), delivers the rotation of the cross section, with opposite sign to take into account the adopted system of reference (see Fig. 3.4), that is:

\[
\varphi(x) = -u_b(x) = -C_5 - \eta_{I0} \left[ C_4 x + C_3 \frac{x^2}{2} + q_{T}^{[3]}(x) +
+ \sum_{j=1}^{n} \beta_j \left( q_{T}^{[2]}(\bar{x}_j) + C_3 \bar{x}_j + C_4 \right) H(x - \bar{x}_j) \right]. \tag{3.38}
\]
From Eq. (3.38) it can be shown that the generic finite jump $\Delta \varphi(\bar{x}_j)$ appearing in the rotations’ profile at the position of the crack $x = \bar{x}_j$, takes the expression:

$$
\Delta \varphi(\bar{x}_j) = \varphi(\bar{x}^+_j) - \varphi(\bar{x}_j^-) = \Upsilon_{I0} \beta_j \left( q_T^{[2]}(\bar{x}_j) + C_3 \bar{x}_j + C_4 \right)
$$

$$
= \Upsilon_{I0} \beta_j M(\bar{x}_j).
$$

The rotation of the cross section $\varphi(x)$, similarly to Eq. (3.27), can be expressed as superposition of undamaged contribution, $\varphi_0(x)$, and the individual damage effect, $\Delta \varphi_j(x)$:

$$
\varphi(x) = -u'_b(x) = \varphi_0(x) + \sum_{j=1}^{n} \Delta \varphi_j(x); 
$$

where:

$$
\varphi_0(x) = -u'_{b0}(x) = C_5 + \Upsilon_{I0} \left[ C_4 x + \frac{C_3 x^2}{2} + q^{[3]}(x) \right];
$$

$$
\Delta \varphi_j(x) = -\Delta u'_{bj}(x) = \Upsilon_{I0} \beta_j \left( q_T^{[2]}(\bar{x}_j) + C_4 + C_3 \bar{x}_j + q^{[2]}(\bar{x}_j) \right) \delta(x - \bar{x}_j).
$$

The dimensionless bending curvature of the beam is obtained by taking the spatial derivative of the Eq. (3.38) with respect to the abscissa $x$:

$$
\chi(x) = \varphi'(x) = -\Upsilon_{I0} \left[ C_4 + C_3 x + q_T^{[2]}(x) + \sum_{j=1}^{n} \beta_j \left( q_T^{[2]}(\bar{x}_j) + C_3 \bar{x}_j + q^{[2]}(\bar{x}_j) \right) \delta(x - \bar{x}_j) \right],
$$

while the average shearing strain is given by:

$$
\Gamma(x) = u'_s(x) = -\Upsilon_{S0} \left[ C_3 + q_T^{[1]}(x) + \sum_{j=1}^{n} \gamma_j \left( q_T^{[1]}(\bar{x}_j) + C_3 \right) \delta(x - \bar{x}_j) \right].
$$

Interestingly, like in the case of the axial load, the two measures of the deformation introduced for the transverse load, i.e. the dimensionless bending curvature of Eq. (3.42) and the average shearing strain of Eq. (3.43), show impulsive terms at the position of the generic singularity, which are proportional to the corresponding dimensionless measures $\beta_j$ and $\gamma_j$ of the severity of the damage. These impulses disappear in the expressions of the associated internal forces,
3. Multi-damaged beam element

Figure 3.6: Sketch of the multi-damaged beam element

i.e. bending moment $M(x)$ and shear force $V(x)$, respectively:

$$M(x) = -C_3 x - C_4 - q_T^{[2]}(x); \quad (3.44a)$$

$$V(x) = -C_3 - q_T^{[1]}(x) = M'(x). \quad (3.44b)$$

It can be noticed that the two integration constants $C_3$ and $C_4$ appearing in the expressions above are equal to the shear force and the bending moment at $x = 0$ (i.e. $C_3 = V(0)$ and $C_2 = M(0)$). Similarly, the other integration constants $C_5$ and $C_6$ in Eq. (3.28) and Eq. (3.41a) are the values of deflection and rotation at $x = 0$ (i.e. $C_5 = \varphi(0)$ and $C_6 = u(0)$), while the slope at $x = 0$ is a combination of $C_5$ and $C_3$, i.e. $u_T'(0) = C_5 - \tau_{50} C_3$ (see Eq. (3.34)).

3.2 Multi-damaged beam element

The aim of this section is to provide stiffness matrix and consistent mass matrix including rotatory inertia for the MDB presented in Section 3.1. The flexibility functions (Eqs. (3.8a) to (3.8c)), along with the resulting fields of kinematic quantities (i.e. displacements and rotations, given by Eq. (3.15), (3.34) and (3.38)) and static quantities (i.e. the internal forces, given by Eqs. (3.21), (3.44a), (3.44b)) are rewritten as dimensionless functions (denoted with the over-hat) of the dimensionless spatial coordinate $\xi = x/L$, spanning from 0 to 1, being the Young’s modulus $E$ and the length of the beam $L$ the two dimensional reference variables assumed for the static problem. Figure 3.6 shows a sketch of the MDB element.

3.2.1 Dimensionless flexibility functions

It follows that, if $n$ concentrated damages occur in the beam, dimensionless axial flexibility $\hat{\Upsilon}_A(\xi) = E L^2 / EA(\xi L)$, dimensionless bending flexibility $\hat{\Upsilon}_I(\xi) = E L^4 / EI(\xi L)$ and di-
3.2 Multi-damaged beam element

Dimensionless shear flexibility \( \hat{\Upsilon}_S(\xi) = E L^2 / G A_s(\xi L) \) take the expressions:

\[
\hat{\Upsilon}_A(\xi) = \hat{\Upsilon}_{A0} \left[ 1 + \sum_{j=1}^n \alpha_j \delta(\xi - \bar{\xi}_j) \right] ; \tag{3.45a}
\]

\[
\hat{\Upsilon}_I(\xi) = \hat{\Upsilon}_{I0} \left[ 1 + \sum_{j=1}^n \beta_j \delta(\xi - \bar{\xi}_j) \right] ; \tag{3.45b}
\]

\[
\hat{\Upsilon}_S(\xi) = \hat{\Upsilon}_{S0} \left[ 1 + \sum_{j=1}^n \gamma_j \delta(\xi - \bar{\xi}_j) \right], \tag{3.45c}
\]

where \( \bar{\xi}_j \) is the dimensionless abscissa at the \( j \)th damage position, while \( \hat{\Upsilon}_{A0} = L^2 / A \), \( \hat{\Upsilon}_{I0} = L^4 / I \) and \( \hat{\Upsilon}_{S0} = 2 \kappa (1 + \nu) L^2 / A \).

3.2.2 Dimensionless displacement functions

The exact closed-form solution in terms of the resulting field of dimensionless axial displacements \( \hat{u}_A(\xi) = u_A(x)/L \), the dimensionless transverse displacements \( \hat{u}_T(\xi) = u_T(x)/L \) and rotations \( \hat{\varphi}(\xi) = \varphi(x) \) can be expressed as:

\[
\hat{u}_A(\xi) = \hat{\Upsilon}_{A0} \left[ C_1 \xi + C_2 - \hat{q}_A^{[2]}(\xi) + \sum_{j=1}^n \alpha_j \left( -\hat{q}_A^{[1]}(\bar{\xi}_j) + C_1 \right) H(\xi - \bar{\xi}_j) \right]; \tag{3.46a}
\]

\[
\hat{u}_T(\xi) = \hat{\Upsilon}_{I0} \left[ \frac{1}{6} C_3 \xi^3 + \frac{1}{2} C_4 \xi^2 + C_5 \xi + C_6 + \hat{q}_T^{[4]}(\xi) + \right.
\]

\[
\left. + \sum_{j=1}^n \beta_j \left( \hat{q}_T^{[2]}(\bar{\xi}_j) + C_3 \bar{\xi}_j + C_4 \right) (\xi - \bar{\xi}_j) H(\xi - \bar{\xi}_j) \right] + \hat{q}_T^{[3]}(\xi) + \hat{\Upsilon}_{S0} \left[ C_3 \xi + \hat{q}_T^{[2]}(\xi) + \sum_{j=1}^n \gamma_j \left( C_3 + \hat{q}_T^{[1]}(\bar{\xi}_j) \right) H(\xi - \bar{\xi}_j) \right]; \tag{3.46b}
\]

\[
\hat{\varphi}(\xi) = -\hat{u}_b(\xi) = -\hat{\Upsilon}_{I0} \left[ \frac{1}{2} C_3 \xi^2 + C_4 \xi + C_5 + \hat{q}_T^{[3]}(\xi) + \right.
\]

\[
\left. + \sum_{j=1}^n \beta_j \left( \hat{q}_T^{[2]}(\bar{\xi}_j) + C_3 \bar{\xi}_j + C_4 \right) H(\xi - \bar{\xi}_j) \right], \tag{3.46c}
\]

in which \( \hat{q}_A(\xi) = q_A(\xi L)/E L \) and \( \hat{q}_T(\xi) = q_T(\xi L)/E L \) are the dimensionless axial and transversal dimensionless load respectively, while \( \hat{\varphi}(\xi) = \varphi(\xi L) \).
3.2.3 Dimensionless internal forces

The expressions of the associated dimensionless internal forces, i.e. the dimensionless normal force \( \hat{N}(\xi) = N(\xi L)/(EL^2) \), the dimensionless bending moment \( \hat{M}(\xi) = M(\xi L)/(EL^3) \), and shear force \( \hat{V}(\xi) = V(\xi L)/(EL^2) \), are respectively:

\[
\begin{align*}
\hat{N}(\xi) &= C_1 - \hat{q}_A^{[1]}(\xi) ; \\
\hat{M}(\xi) &= -C_3 \xi - C_4 - \hat{q}_T^{[2]}(\xi) ; \\
\hat{V}(\xi) &= -C_3 - \hat{q}_T^{[1]}(\xi).
\end{align*}
\]

Figure 3.7 illustrates the positive sign convention for the dimensional internal forces \( N(x), \ V(x) \) and \( M(x) \) at the position of the two end nodes 0 (at \( \xi = 0 \)) and 1 (at \( \xi = 1 \)), which will be used to derive the elements of the stiffness matrix and the equivalent load vector.

Figure 3.7: Internal forces at the two end nodes of the MDB element

3.2.4 Stiffness matrix

The exact closed-form solutions for a beam with \( n \) concentrated damages under axial and transverse loads, as presented in the previous section, are employed in the following to determine the \((6 \times 6)\) dimensionless stiffness matrix \( \hat{K} \) of the MDB element. The method adopted has been the unit-displacement method (e.g. Ref.[233]), which evaluates the dimensionless internal forces at the two end nodes due to a unit displacement imposed to a single nodal degree of freedom (DoF) per time, while the other BCs are zeroed and the beam is unloaded. Exploiting the generalised Hooke’s law, \( \hat{F} = \hat{K} \hat{u} \), each column of the stiffness matrix \( \hat{K} \) is given by the vector \( \hat{F} \) computed for a unit displacement of the associated DoF.

For the purposes of mathematical derivation, the vector of the nodal forces \( \hat{F} \) is partitioned as:

\[
\hat{F} = \begin{bmatrix} \hat{F}_A^\top ; \hat{F}_T^\top \end{bmatrix}^\top,
\]

(3.48)
where \( \hat{\mathbf{F}}_A = [-\hat{N}(0); \hat{N}(1)]^\top \) collects the axial nodal forces and \( \hat{\mathbf{F}}_T = [-\hat{V}(0); -\hat{M}(0); \hat{V}(1); \hat{M}(1)]^\top \) is the vector listing the transverse forces and bending couples externally applied at the two end nodes, the superscripted symbol \(^\top \) meaning the transpose operator; analogously, the vector of the DoFs \( \hat{\mathbf{u}} \) is partitioned as:

\[
\hat{\mathbf{u}} = \left[ \begin{array}{c}
\hat{\mathbf{u}}_A^\top ;
\hat{\mathbf{u}}_T^\top
\end{array} \right]^\top,
\]

(3.49)

where \( \hat{\mathbf{u}}_A = [\hat{u}_A(0); \hat{u}_A(1)]^\top \) and \( \hat{\mathbf{u}}_T = [\hat{u}_T(0); \hat{\varphi}(0); \hat{u}_T(1); \hat{\varphi}(1)]^\top \) are the \((2 \times 1)\) and \((4 \times 1)\) sub-vectors of the axial and transverse components. Importantly, the sign of all the static (forces and couples) and kinematic (displacements and rotations) quantities in the vectors \( \hat{\mathbf{F}} \) and \( \hat{\mathbf{u}} \) is evaluated according to the same system of reference displayed in Fig. 3.4.

Following this decomposition, the stiffness matrix can be expressed as:

\[
\hat{\mathbf{K}} = \begin{bmatrix}
\hat{\mathbf{K}}_A & \mathbf{0}_{2 \times 4} \\
\mathbf{0}_{4 \times 2} & \hat{\mathbf{K}}_T
\end{bmatrix};
\]

(3.50)

where \( \hat{\mathbf{K}}_A \) and \( \hat{\mathbf{K}}_T \) are the \((2 \times 2)\) and \((4 \times 4)\) stiffness blocks related to the axial and transversal DoFs, respectively, while the symbol \( \mathbf{0}_{r \times s} \) stands for a zero matrix having \( r \) rows and \( s \) columns. The presence of the zero out-of-diagonal blocks in the stiffness matrix of Eq. (3.50) comes from the assumption that the coupling among axial and transverse displacements is negligible, as typically accepted for small concentrated damages, or cracks appearing on both sides of beam’s cross section, or for relatively low values of the axial force.

For the sake of clarity, the formulation for the axial and transverse components will be presented separately in Section 3.2.5 and Section 3.2.6. Interestingly, it can be shown that for the axial component the dimensionless stiffness coefficients of the MDB element only depend on the constants \( \hat{\Gamma}_{A0} \) (normalised axial flexibility of the undamaged member) and \( a_0 = \sum_{j=1}^{n} \alpha_j \) (which is effectively an overall measure of the additional axial flexibility due to the presence of the \( n \) damages); while the dimensionless stiffness coefficients for transverse component depends on \( \hat{\Gamma}_{10} \) and \( \hat{\Gamma}_{S0} \) (i.e. the dimensionless constants playing the same role as \( \hat{\Gamma}_{A0} \) for the pure-bending and pure-shearing deflections), along with the additional dimensionless quantities \( b_m = \sum_{j=1}^{n} \beta_j \xi^m \), with \( m \in \{0, 1, 2\} \), and \( c_0 = \sum_{j=1}^{n} \gamma_j \).
3.2.5 Axial stiffness matrix

The stiffness matrix blocks $\hat{K}_A$ associated with the axial deformation can be expressed as follows:

$$
\hat{K}_A = \begin{bmatrix}
\hat{K}_{11}^A & \hat{K}_{12}^A \\
\hat{K}_{21}^A & \hat{K}_{22}^A
\end{bmatrix}.
$$

(3.51)

Its components can be calculated from the forces developed in the above beam member when unit displacement is imposed along each DoF holding all other displacements to zero. That means, for the $\hat{u}_A$ displacements, applying the BC $\hat{u}_A = [1, 0]^\top$ while $\hat{q}_A(\xi) = 0$ because no external loads must be considered in deriving the stiffness matrix (see Fig. 3.8).

Figure 3.8: Clamped-clamped MDB with an imposed unit displacement at the first BC: $\hat{u}_A(0)$

One than obtains:

$$
\begin{cases}
\hat{u}_A(0) = 1 \\
\hat{u}_A(1) = 0
\end{cases} \Rightarrow \begin{cases}
\hat{N}_A(0) = 1 \\
\hat{N}_A(1) = \hat{N}_A(1) = -\hat{K}_{11}^A = -\hat{N}_A(1) = (1 + a_0)^{-1}.
\end{cases}
$$

(3.52)

where $a_0 = \sum_{j=1}^n \alpha_j$. Once evaluated the constants $C_1, C_2$ and substituted into Eq. (3.47a), the equivalent nodal axial forces can be computed:

$$
\hat{N}_A(0) = \hat{N}_A(1) = -\hat{K}_{11}^A = -\hat{N}_A(1) = (1 + a_0)^{-1}.
$$

(3.54)

Imposing the second set of BCs is $\hat{u}_A = [0, 1]^\top$, plus the load condition $\hat{q}_A(\xi) = 0$ (see Fig. 3.9), gives the following constants:

$$
\begin{cases}
\hat{u}_A(0) = 0 \\
\hat{u}_A(1) = 1
\end{cases} \Rightarrow \begin{cases}
\hat{N}_A(0) = 0 \\
\hat{N}_A(1) = \hat{N}_A(1) = -\hat{q}_A(1 + a_0)^{-1}.
\end{cases}
$$

(3.55)
3.2 Multi-damaged beam element

Figure 3.9: Clamped-clamped MDB with an imposed unit displacement at the second BC: \( \hat{u}_A(1) \)

\[
\begin{align*}
C_2 &= 0 \\
C_1 &= \hat{T}_{A0}^{-1} (1 + a_0)^{-1} \, ,
\end{align*}
\tag{3.56}
\]

and then the following axial forces:

\[
\hat{N}^{(2)}(0) = \hat{N}^{(2)}(1) = -\hat{K}_A^{21} = \hat{K}_A^{22} = \hat{T}_{A0}^{-1} (1 + a_0)^{-1} \, .
\tag{3.57}
\]

Finally, the axial stiffness matrix can be expressed by assembling the node reactions giving:

\[
\hat{K}_A = \hat{T}_{A0}^{-1} (1 + a_0)^{-1} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \, .
\tag{3.58}
\]

It is worth noting that if \( a_0 = 0 \) (\( \Rightarrow \hat{N}(\xi) = -\hat{T}_{A0}^{-1} \)), the axial stiffness matrix of the undamaged beam is recovered as a particular case:

\[
\hat{K}_{A0} = \frac{A}{L} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \, .
\tag{3.59}
\]

### 3.2.6 Transverse stiffness matrix

The stiffness matrix blocks \( \hat{K}_T \) associated with the transverse deformation can be expressed as follows:

\[
\hat{K}_T = \begin{bmatrix}
\hat{K}_T^{11} & \hat{K}_T^{12} & \hat{K}_T^{13} & \hat{K}_T^{14} \\
\hat{K}_T^{21} & \hat{K}_T^{22} & \hat{K}_T^{23} & \hat{K}_T^{24} \\
\hat{K}_T^{31} & \hat{K}_T^{32} & \hat{K}_T^{33} & \hat{K}_T^{34} \\
\hat{K}_T^{41} & \hat{K}_T^{42} & \hat{K}_T^{43} & \hat{K}_T^{44}
\end{bmatrix}_{\text{sym.}} \, .
\tag{3.60}
\]

All the stiffness components can be obtained from the reaction forces of a clamped-clamped beam with imposed an unit displacement on the DoF related with the correspondent column of the matrix. In the following all the terms of the stiffness matrix are summarised:
• First column (BCs $\hat{\mathbf{u}}_T = [1; 0; 0; 0]^T$ and $\hat{\mathbf{q}}_T (\xi) = 0$, see Fig. 3.10):

\[
\begin{align*}
\hat{K}^{31} & = -\Psi; \\
\hat{K}^{41} & = -\Psi
\end{align*}
\]

where $b_0 = \sum_{j=1}^{n} \beta_j = \beta_0$, $b_1 = \sum_{j=1}^{n} \beta_j \xi_j$, $b_2 = \sum_{j=1}^{n} \beta_j \xi_j^2$ and

\[
\hat{\Psi} = \frac{\rho_{10}^{-1}}{12} - \frac{b_1 - b_0/4 + b_2}{1 + b_0} + b_2 + \frac{\rho_{20} (1 + \gamma_0)}{Y_0}.
\]

(3.61)

• Second column (BCs $\hat{\mathbf{u}}_T = [0; 1; 0; 0]^T$ and $\hat{\mathbf{q}}_T (\xi) = 0$, see Fig. 3.11):

\[
\begin{align*}
\hat{K}^{12} & = -\Psi \frac{1+2b_1}{2(1+b_0)}; \\
\hat{K}^{22} & = \Psi \frac{1+2b_0 + 4b_0 b_1 - 4b_1^2}{4(1+b_0)^2} - \frac{\rho_{10}^{-1}}{1+b_0}; \\
\hat{K}^{32} & = \Psi \frac{1+2b_1}{2(1+b_0)}; \\
\hat{K}^{42} & = \Psi \frac{1+2b_0}{2(1+b_0)}.
\end{align*}
\]

(3.63)

• Third column (BCs $\hat{\mathbf{u}}_T = [0; 0; 1; 0]^T$ and $\hat{\mathbf{q}}_T (\xi) = 0$, see Fig. 3.12):
3.2 Multi-damaged beam element

Figure 3.12: Clamped-clamped MDB with an imposed unit displacement at the fifth BC: $\hat{u}_T(1)$

\[
\begin{align*}
\hat{K}_{T}^{13} &= -\Psi; & \hat{K}_{T}^{23} &= \Psi \frac{1 + 2b_1}{2(1 + b_0)}; \\
\hat{K}_{T}^{33} &= \Psi; & \hat{K}_{T}^{43} &= -\Psi \left[ \frac{1 + 2b_1}{2(1 + b_0)} - 1 \right].
\end{align*}
\] (3.64)

- Fourth column (BCs $\hat{u}_T = [0; 0; 0; 1]^T$ and $\hat{q}_T(\xi) = 0$, see Fig. 3.13):

Figure 3.13: Clamped-clamped MDB with an imposed unit displacement at the sixth BC: $\hat{\phi}(1)$

\[
\begin{align*}
\hat{K}_{T}^{14} &= -\Psi \frac{1 + 2b_0 - 2b_1}{2(1 + b_0)}; & \hat{K}_{T}^{24} &= -\Psi \frac{1 - 4b_1^2 + 2b_0(1 + 2b_1)}{4(1 + b_0)^2} + \frac{\hat{T}_{f_0}^{-1}}{1 + b_0}; \\
\hat{K}_{T}^{34} &= \Psi \frac{1 + 2b_0 - 2b_1}{2(1 + b_0)}; & \hat{K}_{T}^{44} &= \Psi \frac{[1 + 2(b_0 - b_1)]^2}{4(1 + b_0)^2} + \frac{\hat{T}_{f_0}^{-1}}{1 + b_0}.
\end{align*}
\] (3.65)

It can be verified that the derived stiffness matrix $\hat{K}_T$ is symmetric and positive definite; moreover the stiffness matrix for the undamaged Timoshenko beam is recovered if the condition $b_0 = b_1 = b_2 = c_0 = 0$ is met.

\[
\hat{K}_T = \begin{bmatrix}
\Psi_0 & -\Psi_0/2 & -\Psi_0/2 & -\Psi_0/2 \\
\Psi_0/4 + \hat{T}_{f_0}^{-1} & \Psi_0/2 & \Psi_0/4 - \hat{T}_{f_0}^{-1} & \Psi_0/2 \\
sym. & \Psi_0/2 & \Psi_0/2 & \Psi_0/4 + \hat{T}_{f_0}^{-1}
\end{bmatrix},
\] (3.66)
where
\[
\Psi_0 = \frac{12}{\bar{T}_{t0} + 12\bar{T}_{s0}}. 
\] (3.67)

### 3.3 Equivalent nodal forces

In this section, a procedure will be presented to evaluate the vector \( \hat{Q} \) of the equivalent nodal forces for a generic distribution of actual loads \( \hat{q}_A(\xi) \) and \( \hat{q}_T(\xi) \) acting on the MDB element. Similarly to the stiffness matrix, the sought vector \( \hat{Q} \) can be partitioned into axial and transverse element, that is:
\[
\hat{Q} = \begin{bmatrix} \hat{Q}_A^T; \hat{Q}_T^T \end{bmatrix}^T, 
\] (3.68)
where \( \hat{Q}_A = [-\hat{N}(0); \hat{N}(1)]^T \) collects the axial nodal forces and \( \hat{Q}_T = [-\hat{V}(0); -\hat{M}(0); \hat{V}(1); \hat{M}(1)]^T \) collect the end-beam reactions due to the external load applied to the clamped-clamped system. The following subsection illustrates the procedure for some typical load distributions.

#### 3.3.1 Uniform axial external load

The equivalent nodal forces vector for a uniform external axial load \( \hat{q}_A(\xi) = \hat{q}_A \) can be determined with the help of the same exact closed-form solutions reported in Section 3.1, by taking zeroed BCs (\( \hat{u}_A = [0, 0]^T \)) and evaluating the corresponding internal forces at the two end nodes (see Fig. 3.14).

![Figure 3.14: Clamped-clamped MDB with a uniform axial load: \( \hat{q}_A \)](image)

One then obtains:
\[
\begin{cases}
\hat{u}_A(0) = 0 \\
\hat{u}_A(1) = 0
\end{cases} \quad \Rightarrow \quad \hat{T}_{A0} \begin{bmatrix} C_1 + C_2 - \frac{1}{2} \hat{q}_A + \sum_{j=1}^{n} \alpha_j (C_1 - \hat{q}_A \bar{\xi}_j) \end{bmatrix} = 0
\]
3.3 Equivalent nodal forces

\[
\begin{cases}
C_2 = 0 \\
C_1 - \frac{1}{2} \hat{q}_A + C_1 a_0 - \hat{q}_A a_4 = 0
\end{cases} \Rightarrow \begin{cases}
C_2 = 0 \\
C_1 = \hat{q}_A \frac{1+2a_1}{2(1+a_0)}
\end{cases}, \quad (3.69)
\]

where \( a_1 = \sum_{j=1}^{n} \alpha_j \xi_j \) and

\[
\hat{q}^{[1]}_A(\xi) = \hat{q}_A; \quad (3.70a)
\]
\[
\hat{q}^{[2]}_A(\xi) = \frac{1}{2} \hat{q}_A \xi^2; \quad (3.70b)
\]

are first and second primitive of the uniform axial load. Substituting the constants into Eq. (3.21) yields:

\[
\hat{Q}^\top_A = \begin{bmatrix}
\frac{1+2a_1}{2(1+a_0)} & -\frac{1+2(a_0-a_1)}{2(1+a_0)}
\end{bmatrix}^\top. \quad (3.71)
\]

3.3.2 Uniform transverse external load

The same approach can be used for the uniform transverse load \( \hat{q}_T(\xi) = \hat{q}_T \). In this case the closed-form solutions used are the Eq. (3.46b) and Eq. (3.46c) where the primitives of the uniform transverse load are

\[
\hat{q}^{[1]}_T(\xi) = \hat{q}_T \xi; \quad (3.72a)
\]
\[
\hat{q}^{[2]}_T(\xi) = \frac{1}{2} \hat{q}_T \xi^2; \quad (3.72b)
\]
\[
\hat{q}^{[3]}_T(\xi) = \frac{1}{6} \hat{q}_T \xi^3; \quad (3.72c)
\]
\[
\hat{q}^{[4]}_T(\xi) = \frac{1}{24} \hat{q}_T \xi^4. \quad (3.72d)
\]

![Figure 3.15: Clamped-clamped MDB with a uniform transverse load: \( \hat{q}_T \)](image)

By imposing zeroed displacements and rotations at the two end nodes ( BCs \( \hat{u}_T = [0, 0, 0, 0]^\top \), see Fig. 3.15) it is possible to obtain the resulting internal forces which gives the
components of the load vector $\hat{Q}_T$:

$$\hat{Q}_1^T = -\hat{V}(0) = -\hat{q}_T\hat{T}_{10}\Psi \left[ \frac{1}{24} + \frac{b_0 - 2b_1 - 3b_2 - 6b_1b_2}{2(1 + b_0)} + \frac{b_3}{2} + \frac{\hat{T}_{S0}(1 + 2c_1)}{2\hat{T}_{10}} \right] ; \quad (3.73a)$$

$$\hat{Q}_2^T = -\hat{M}(0) = \frac{1 + 2b_1}{2(1 + b_0)} \hat{V}(0) - \frac{1 + 3b_2}{6(1 + b_0)} \hat{q}_T ; \quad (3.73b)$$

$$\hat{Q}_3^T = \hat{V}(1) = \hat{V}(0) - \hat{q}_T ; \quad (3.73c)$$

$$\hat{Q}_4^T = \hat{M}(1) = -\left[ \frac{1 + 2b_1}{2(1 + b_0)} - 1 \right] \hat{V}(0) + \frac{1 + 3b_2}{6(1 + b_0)} \hat{q}_T - \frac{1}{2} \hat{q}_T . \quad (3.73d)$$

The same approach can be used for any type of axial and/or transverse load. Similarly to the axial case, by imposing the following BC $\hat{u}_T = [0, 0, 0, 0]^T$ and $\hat{q}_T(\xi) = \hat{q}_T$, the equivalent nodal loads due to an uniform transverse load are obtained.

### 3.3.3 Concentrated external load

The equivalent nodal forces vector for a concentrated external axial load $\hat{q}_A(\xi) = \hat{q}_A\delta(\xi - \xi_k)$ (see Fig. 3.16) can be determined using the same procedure. For this type of load, the primitives $\hat{q}_A^{[1]}(\xi)$ and $\hat{q}_A^{[2]}(\xi)$ assume the following form:

$$\hat{q}_A^{[1]}(\xi) = \hat{q}_A H(\xi - \xi_k) ; \quad (3.74a)$$

$$\hat{q}_A^{[2]}(\xi) = \hat{q}_A(\xi - \xi_k)H(\xi - \xi_k) . \quad (3.74b)$$

Figure 3.16: Clamped-clamped MDB with a concentrated axial load: $\hat{q}_A(\xi) = \hat{q}_A\delta(\xi - \xi_k)$

By imposing the BCs ($\hat{u}_A = [0, 0]^T$) yields:

$$\hat{N}(0) = \hat{q}_A a_{0k} - a_0 - \xi_k , \quad \hat{N}(1) = \hat{q}_A a_{0k} - a_0 - \xi_k , \quad (3.75)$$

where $a_{0k} = \sum_{j=1}^n \alpha_j H(\xi_j - \xi_k) = \sum_{j=1}^n \alpha_j$ represents the sum of all damages to the right side of the $k$th damage.
3.3.4 Concentrated transverse load

For the transversal point load case, once again the following BCs $\hat{u}_T = [0, 0, 0, 0]^T$ are adopted, and $\hat{q}_T(\xi) = \hat{q}_T \delta(\xi - \xi_k)$, leads to (see Fig. 3.17):

\[
\hat{V}(0) = \hat{q}_T \Psi \left[ \frac{(1 - \xi_k)^2(1 + 2 \xi_k) + 6(b_0 - b_1)(1 - \xi_k)^2 - 2b_0(1 - \xi_k)^3}{12(1 + b_0)} \right] + \frac{6(b_{1k} - \xi_k b_{0k})[1 + 2(b_0 - b_1)]}{12(1 + b_0)} + \frac{-12(1 + b_0)[b_{1k} - b_{2k} - \xi_k(b_{0k} - b_{1k})]}{12(1 + b_0)} + \frac{\hat{f}_{S0}}{T_{f0}}(1 - \xi_k + c_{0k}) \right] \right] \hat{q}_T \Psi \right] \frac{V(0)}{2(1 + b_0)} + \hat{M}(0) = -\frac{1 + 2b_1}{2(1 + b_0)} \hat{V}(0) + \frac{(1 - \xi_k)^2 + 2b_{1k} - 2b_{0k}}{2(1 + b_0)} \hat{q}_T \hat{M}(1) = -\left[ \frac{1 + 2b_1}{2(1 + b_0)} - 1 \right] \hat{V}(0) + \frac{(1 - \xi_k)^2 + 2b_{1k} - 2b_{0k}}{2(1 + b_0)} \hat{q}_T \hat{q}_T(1 - \xi_k) \]

where $b_{0k} = \sum_{j=k}^n \beta_j; b_{1k} = \sum_{j=k}^n \xi_j \beta_j; b_{2k} = \sum_{j=k}^n \xi_j^2 \beta_j; \text{ and } c_{0k} = \sum_{j=k}^n \gamma_j$. In this case the primitives for the assumed load case $\hat{q}_T(\xi) = \hat{q}_T \delta(\xi - \xi_k)$ are:

\[
\hat{q}_T^{[1]}(\xi) = \hat{q}_T H(\xi - \xi_k) \]
\[
\hat{q}_T^{[2]}(\xi) = \hat{q}_T (\xi - \xi_k) H(\xi - \xi_k) \]
\[
\hat{q}_T^{[3]}(\xi) = \hat{q}_T \frac{(\xi - \xi_k)^2}{2} H(\xi - \xi_k) \]
\[
\hat{q}_T^{[4]}(\xi) = \hat{q}_T \frac{(\xi - \xi_k)^3}{6} H(\xi - \xi_k) \]
3.4 Mass matrix

To perform the FE analysis of a MDB subjected to dynamic loading, the dimensionless mass matrix $\hat{M}$, consistent with the dimensionless stiffness matrix $\hat{K}$, is highly desirable. However, little attention has been paid in the past to this issue, and the approximation of lumped masses has been generally adopted (e.g. Ref. [206]), therefore neglecting the effects of the damaged cross sections on the inertial forces. Indeed, when the approximation with lumped masses is resorted to, the mass of the whole FE is concentrated at the two end nodes, like in the undamaged beam, despite the fact that the damage on the member can significantly alter the distribution of the inertial forces (see Refs. [68, 41]). It will be shown in the next Section that the implementation of the lumped mass matrix in conjunction with a MDB element reduces (and potentially nullifies) most of the computational advantage of such formulation for applications of Structural Dynamics, as more FEs are required to capture the eigenproperties (modal shapes and modal frequencies) of the damaged beam.

Similarly to the stiffness matrix considered above, the mass matrix can be decoupled in two blocks related to the axial and the transversal components.

$$\hat{M} = \begin{bmatrix} \hat{M}_A & 0_{2\times4} \\ 0_{4\times2} & \hat{M}_T + \hat{M}_R \end{bmatrix},$$

(3.78)

where $\hat{M}_A$ is the $(2 \times 2)$ axial sub-matrix, while $\hat{M}_T$ and $\hat{M}_R$ are the $(4 \times 4)$ sub-matrices associated with transverse and rotatory inertia, respectively. To the best of our knowledge, the term $\hat{M}_R$ has never been taken into account in the formulation of a MDB element, even though this contribution proves to be very important for the higher modes of vibration (e.g. when the flexural wavelength becomes comparable to the cross sectional depth of the beam).

3.4.1 Axial inertia

In order to evaluate the consistent mass matrix for the problem in hand, the same deformed shapes derived by assigning unit displacements at the two end nodes of the MDB element have been used, meaning that the same set of shape functions discretise both potential and kinetic energy of the FE. For the axial component, the dimensionless $(2 \times 2)$ consistent mass matrix can be evaluated as:

$$\hat{M}_A = \hat{\mathcal{F}}_{A0}^{-1} \int_0^1 \hat{h}_A(\xi) \hat{h}_A(\xi)^\top d\xi,$$

(3.79)
where the mass density of the material $\rho$ has been selected as third dimensional reference variable of the dynamic problem, and therefore it does not appear in the above expression, while $\widehat{h}_A(\xi) = [\widehat{u}^{(1)}_A(\xi); \widehat{u}^{(2)}_A(\xi)]^T$ is the $(2 \times 1)$ vector collecting the dimensionless functions representing the axial displacements in the MDB for unit axial displacements at the end nodes 0 and 1.

If follows from Eq. (3.79) that the generic element of the mass matrix $\widehat{M}_A$ requires the evaluation of the integral:

$$\widehat{M}_A^{(r,s)} = \hat{T}_A^{-1} \int_0^1 \widehat{u}^{(r)}_A(\xi) \widehat{u}^{(s)}_A(\xi) \, d\xi,$$

(3.80)

with $\{r, s\} \subset \{1, 2\}$, and for this purpose each shape function $\widehat{u}_A$ in the r.h.s. of Eq. (3.80) can be decomposed into the superposition of a continuous part, $\widehat{u}_{A,0}$, obtained when all the coefficients $\alpha_j$ are zeroed (and therefore the beam is assumed to be undamaged), and $n$ additional terms, $\Delta \widehat{u}_{A,j}$, each one contributing for $\xi \geq \xi_j$. That is, for the $r$th axial shape function, one can write:

$$\widehat{u}^{(r)}(\xi) = \hat{T}_A \left( \widehat{u}^{(r)}_{A,0}(\xi) + \sum_{j=1}^{n} \Delta \widehat{u}^{(r)}_{A,j}(\xi) \right),$$

(3.81)

where (see Eq. (3.46a)):

$$\widehat{u}^{(r)}_{A,0}(\xi) = C^{(r)}_1 \xi + C^{(r)}_2;$$

(3.82)

$$\Delta \widehat{u}^{(r)}_{A,j}(\xi) = \alpha_j C^{(r)}_1 H(\xi - \xi_j).$$

(3.83)

Substitution of Eq. (3.81) into Eq. (3.80) leads to:

$$\widehat{M}_A^{(r,s)} = \hat{T}_A^{-1} \int_0^1 \left( \widehat{u}^{(r)}_{A,0}(\xi) \widehat{u}^{(s)}_{A,0}(\xi) + \sum_{j=1}^{n} \widehat{u}^{(r)}_{A,0}(\xi) \Delta \widehat{u}^{(s)}_{A,j}(\xi) + \sum_{j=1}^{n} \widehat{u}^{(s)}_{A,0}(\xi) \Delta \widehat{u}^{(r)}_{A,j}(\xi) + \sum_{j=1}^{n} \sum_{k=1}^{n} \Delta \widehat{u}^{(r)}_{A,j}(\xi) \Delta \widehat{u}^{(s)}_{A,k}(\xi) \right) \, d\xi,$$

(3.84)

and each term in the r.h.s. can be evaluated in closed-form using Eq. (3.84) as:

$$\int_0^1 \widehat{u}^{(r)}_{A,0}(\xi) \widehat{u}^{(s)}_{A,0}(\xi) \, d\xi = \frac{1}{3} C^{(r)}_1 C^{(s)}_1 + \frac{1}{2} \left( C^{(r)}_2 C^{(s)}_1 + C^{(r)}_1 C^{(s)}_2 \right) + C^{(r)}_2 C^{(s)}_2;$$

(3.85)
\[
\int_0^1 \hat{u}_{A,0}^{(r)}(\xi) \Delta \hat{u}_{A,j}^{(s)}(\xi) \, d\xi = \alpha_j C_1^{(s)} \left[ C_1^{(r)} \frac{1 - \hat{\xi}_j^2}{2} + C_2^{(r)} (1 - \hat{\xi}_j) \right];
\]

(3.86)

\[
\int_0^1 \Delta \hat{u}_{A,j}^{(r)}(\xi) \Delta \hat{u}_{A,k}^{(s)}(\xi) \, d\xi = \alpha_j \alpha_k C_1^{(s)} \left[ C_1^{(r)} \Delta \hat{u}_{A,k}^{(s)}(\xi) \right],
\]

(3.87)

with \(\hat{\xi}_{j,k} = \max\{\hat{\xi}_j, \hat{\xi}_k\}\), \(C_2^{(1)} = \hat{T}_{A0}^{-1}\) and \(C_2^{(2)} = 0\).

### 3.4.2 Transverse inertia

Similarly to the axial case, the dimensionless \((4 \times 4)\) consistent mass matrix for the transverse inertia can be evaluated as:

\[
\hat{M}_T = \hat{T}_{A0}^{-1} \int_0^1 \hat{h}_T(\xi) \hat{h}_T(\xi)^\top \, d\xi,
\]

(3.88)

where \(\hat{h}_T(\xi) = [\hat{u}_T^{(3)}(\xi) ; \hat{u}_T^{(4)}(\xi) ; \hat{u}_T^{(5)}(\xi) ; \hat{u}_T^{(6)}(\xi)]^\top\) is the \((4 \times 1)\) vector collecting the dimensionless functions representing the transverse displacements in the MDB for unit transversal and rotational displacements at end nodes 0 and 1. The matrix Eq. (3.88) can be rewritten in terms of the generic element within the transverse mass matrix as:

\[
\hat{M}_T^{(r,s)} = \hat{T}_{A0}^{-1} \int_0^1 \hat{u}_T^{(r)}(\xi) \hat{u}_T^{(s)}(\xi) \, d\xi,
\]

(3.89)

with \(\{r, s\} \subset \{3, 4, 5, 6\}\). In order to solve this integral in closed form, each shape function \(\hat{u}_T\) in the r.h.s. of Eq. (3.89) can be decomposed into the superposition of a continuous part, \(\hat{u}_{T,0}\), obtained when all the coefficients \(\beta_j\) and \(\gamma_j\) are zeroed (and therefore the beam is assumed to be undamaged), and \(n\) additional terms, \(\Delta \hat{u}_{T,j}\), each one contributing for \(\xi \geq \hat{\xi}_j\). That is, for the \(r\)th transversal shape function, one can write:

\[
\hat{u}_T^{(r)}(\xi) = \hat{T}_{I0} \left( \hat{u}_{T,0}^{(r)}(\xi) + \sum_{j=1}^n \Delta \hat{u}_{T,j}^{(r)}(\xi) \right),
\]

(3.90)

where (see Eq. (3.46b)):

\[
\hat{u}_{T,0}^{(r)}(\xi) = \frac{1}{6} C_3^{(r)} \xi^3 + \frac{1}{2} C_4^{(r)} \xi^2 + \left( C_5^{(r)} - \frac{\hat{T}_{S0}}{\hat{T}_{I0}} C_3^{(r)} \right) \xi + C_6^{(r)},
\]

(3.91)
and

\[ \Delta \tilde{u}_{r,j}^{(r)}(\xi) = \left[ \beta_j (C_3^{(r)} \xi_j + C_4^{(r)}) (\xi - \xi_j) - \frac{\hat{T}_{s_0}}{T_{j0}} \gamma_j C_3^{(r)} \right] H(\xi - \xi_j). \] (3.92)

Substitution of Eq. (3.90) into Eq. (3.89) leads to:

\[
\begin{aligned}
\int_0^1 & \left( \tilde{u}_{r,0}^{(r)}(\xi) \tilde{u}_{r,0}^{(s)}(\xi) + \sum_{j=1}^n \tilde{u}_{r,0}^{(r)}(\xi) \Delta \tilde{u}_{r,j}^{(s)}(\xi) \\
& + \sum_{j=1}^n \tilde{u}_{r,0}^{(s)}(\xi) \Delta \tilde{u}_{r,j}^{(r)}(\xi) + \sum_{j=1}^n \sum_{k=1}^n \Delta \tilde{u}_{r,j}^{(r)}(\xi) \Delta \tilde{u}_{r,k}^{(s)}(\xi) \right) d\xi,
\end{aligned}
\] (3.93)

and the exact solution for each term in the r.h.s. is offered below:

\[
\begin{aligned}
& \int_0^1 \tilde{u}_{r,0}^{(r)}(\xi) \tilde{u}_{r,0}^{(s)}(\xi) d\xi = \frac{1}{252} C_3^{(r)} C_3^{(s)} + \frac{1}{20} C_4^{(r)} C_4^{(s)} + C_6^{(r)} C_6^{(s)} + \frac{1}{72} C_3^{(r)} C_4^{(s)} \\
& + \frac{1}{30} C_3^{(r)} C_3^{(s)} + \frac{1}{24} C_3^{(r)} C_6^{(s)} + \frac{1}{72} C_4^{(r)} C_3^{(s)} + \frac{1}{8} C_4^{(r)} C_4^{(s)} \left( C_5^{(s)} - \frac{\hat{T}_{s_0}}{T_{J0}} C_3^{(s)} \right) \\
& + \frac{1}{30} \left( C_5^{(r)} - \frac{\hat{T}_{s_0}}{T_{J0}} C_3^{(r)} \right) C_3^{(s)} + \frac{1}{8} \left( C_5^{(r)} - \frac{\hat{T}_{s_0}}{T_{J0}} C_3^{(r)} \right) C_4^{(s)} + \frac{1}{6} C_4^{(r)} C_6^{(s)} \\
& + \frac{1}{3} \left( C_5^{(r)} - \frac{\hat{T}_{s_0}}{T_{J0}} C_3^{(r)} \right) \left( C_5^{(s)} - \frac{\hat{T}_{s_0}}{T_{J0}} C_3^{(s)} \right) + \frac{1}{2} \left( C_5^{(r)} - \frac{\hat{T}_{s_0}}{T_{J0}} C_3^{(r)} \right) C_6^{(s)} \\
& + \frac{1}{24} C_6^{(r)} C_3^{(s)} + \frac{1}{6} C_6^{(r)} C_4^{(s)} + \frac{1}{2} C_6^{(r)} \left( C_5^{(s)} - \frac{\hat{T}_{s_0}}{T_{J0}} C_3^{(s)} \right);
\end{aligned}
\] (3.94)
\[
\int_{0}^{1} \hat{u}_{T,0}^{(r)}(\xi) \Delta \hat{u}_{T,j}(\xi) d\xi = \\
- \beta \mathcal{T}_0 \mathcal{C}_b^{r}(\xi_j + C_4^{r})(C_3^{s} \xi_k + C_4^{s}) \left( \frac{1 - \xi_k^3}{30} - \xi_k \frac{1 - \xi_k^3}{24} \right) \\
+ \beta \mathcal{T}_0 \mathcal{C}_b^{r}(\xi_j + C_4^{r})(C_3^{s} \xi_k + C_4^{s}) \left( \frac{1 - \xi_k^3}{8} - \xi_k \frac{1 - \xi_k^3}{6} \right) \\
- \beta \mathcal{T}_0 \mathcal{C}_b^{r}(\xi_j + C_4^{r})(C_3^{s} \xi_k + C_4^{s}) \left( \frac{1 - \xi_k^3}{3} - \xi_k \frac{1 - \xi_k^3}{2} \right) \\
+ \beta \frac{\mathcal{T}_0}{T_0} \mathcal{C}_b^{r}(\xi_j + C_4^{r})(C_3^{s} \xi_k + C_4^{s}) \left( \frac{1 - \xi_k^2}{2} - \xi_k(1 - \xi_j) \right) ; \\
\]

\[
\int_{0}^{1} \Delta \hat{u}_{T,j}(\xi) \Delta \hat{u}_{T,k}(\xi) d\xi = \\
\beta \gamma \mathcal{T}_0 \mathcal{C}_b^{r}(\xi_j + C_4^{r})(C_3^{s} \xi_k + C_4^{s}) \left( \frac{1 - \xi_j^3}{3} - (\xi_j + \xi_k) \frac{1 - \xi_j^2}{2} + \xi_j \xi_k(1 - \xi_j) \right) \\
- \beta \gamma \mathcal{T}_0 \mathcal{C}_b^{r}(\xi_j + C_4^{r})(C_3^{s} \xi_k + C_4^{s}) \left( \frac{1 - \xi_j^2}{2} - \xi_j(1 - \xi_j) \right) \\
- \gamma \beta \frac{\mathcal{T}_0}{T_0} \mathcal{C}_b^{r}(\xi_j + C_4^{r})(C_3^{s} \xi_k + C_4^{s}) \left( \frac{1 - \xi_j^2}{2} - \xi_k(1 - \xi_j) \right) \\
+ \frac{\mathcal{T}_0^2}{T_0^2} \gamma \beta \mathcal{C}_b^{r}(\xi_j + C_4^{r})(C_3^{s} \xi_k + C_4^{s}) \left( \frac{1 - \xi_j^2}{2} - \xi_k(1 - \xi_j) \right) . \\
\]
The constants \( C_i^{(r)} \) or \( C_i^{(s)} \) are obtained by imposing the four BCs, and take the values:

\[
C_3^{(3)} = -C_3^{(5)} = \Psi ; \\
C_3^{(6)} = \Psi \frac{1 + 2(b_0 - b_1)}{2(1 + b_0)} ; \\
C_4^{(5)} = C_4^{(3)} = -C_3^{(4)} = \frac{\Psi (1 + 2b_1)}{2(1 + b_0)} ; \\
C_4^{(4)} = \frac{\hat{T}_{I_0}^{-1}}{1 + b_0} \frac{1 + 2b_1}{2(1 + b_0)} C_3^{(4)} ; \\
C_4^{(6)} = -\frac{\hat{T}_{I_0}^{-1}}{1 + b_0} \frac{1 + 2b_1}{2(1 + b_0)} C_3^{(6)} ; \\
C_6^{(3)} = -C_5^{(4)} = \hat{T}_{I_0}^{-1} ; \\
C_5^{(3)} = C_5^{(5)} = C_5^{(6)} = C_6^{(4)} = C_6^{(5)} = C_6^{(6)} = 0 ,
\]

where \( \Psi \) is the same quantity as defined in Eq. (3.62).

### 3.4.3 Rotatory inertia

As a further contribution to the consistent mass matrix, the rotatory inertia has been considered. In this case, the dimensionless functions representing the field of rotations \( \hat{\phi} \) in the MDB appear within the integrals. That is, the additional block \( \hat{M}_R \) can be evaluated as:

\[
\hat{M}_R = \hat{T}_{I_0}^{-1} \int_0^1 \hat{h}_R(\xi) \hat{h}_R(\xi)^\top d\xi ,
\]

where \( \hat{h}_R(\xi) = [\hat{\phi}^{(3)}(\xi); \hat{\phi}^{(4)}(\xi); \hat{\phi}^{(5)}(\xi); \hat{\phi}^{(6)}(\xi)]^\top \) is the \((4 \times 1)\) vector collecting the dimensionless functions representing the rotations of the sections in the MDB for unit transversal and rotational displacements at end nodes 0 and 1. It follows from Eq. (3.98) that the generic element of the mass matrix associated with rotatory inertia \( \hat{M}_R \) requires the following integral:

\[
\hat{M}_R^{r,s} = \hat{T}_{I_0}^{-1} \int_0^1 \hat{\phi}^{(r)}(\xi) \hat{\phi}^{(s)}(\xi) d\xi ,
\]

with \( \{r, s\} \subset \{3, 4, 5, 6\} \). As in the case of axial displacement \( \hat{u}_A \) and the transverse displacement \( \hat{u}_T \), the rotation function \( \hat{\phi} \) in the r.h.s. of Eq. (3.99) is conveniently decomposed into the superposition of a continuous part, \( \hat{\phi}_0 \), equivalent to \( \hat{\phi} \) when the beam is undamaged, and \( n \)
additional terms, $\Delta \hat{\varphi}_j$, each one contributing for $\xi \geq \bar{\xi}_j$, leading to:

$$
\hat{\varphi}^{(r)}(\xi) = \hat{Y}_0 \left( \hat{\varphi}_0^{(r)}(\xi) + \sum_{j=1}^{n} \Delta \hat{\varphi}_j^{(r)}(\xi) \right),
$$

(3.100)

where (see Eq. (3.46c)):

$$
\hat{\varphi}_0^{(r)}(\xi) = \frac{1}{2} C_3^{(r)} \xi^2 + C_4^{(r)} \xi + C_5^{(r)} ;
$$

(3.101)

$$
\Delta \hat{\varphi}_j^{(r)}(\xi) = \beta_j (C_3^{(r)} \bar{\xi}_j + C_4^{(r)}) H(\xi - \bar{\xi}_j).
$$

(3.102)

Substitution of Eq. (3.100) into Eq. (3.99) gives:

$$
\tilde{M}_{R^s} = \hat{Y}_0 \int_0^1 \left( \hat{\varphi}_0^{(r)}(\xi) \hat{\varphi}_0^{(s)}(\xi) + \sum_{j=1}^{n} \hat{\varphi}_0^{(r)}(\xi) \Delta \hat{\varphi}_j^{(s)}(\xi) + \sum_{j=1}^{n} \sum_{k=1}^{n} \Delta \hat{\varphi}_j^{(r)}(\xi) g_{R,k}^{(s)}(\xi) \right) d\xi,
$$

(3.103)

and each term in the r.h.s. can be evaluated in closed-form:

$$
\int_0^1 \hat{\varphi}_0^{(r)}(\xi) \hat{\varphi}_0^{(s)}(\xi) d\xi = \frac{1}{20} C_3^{(s)} C_3^{(r)} + \frac{1}{3} C_4^{(s)} C_4^{(r)}
$$

$$
+ C_5^{(s)} C_5^{(r)} + \frac{1}{8} \left( C_3^{(r)} C_4^{(s)} + C_3^{(s)} C_4^{(r)} \right) + \frac{1}{6} \left( C_3^{(r)} C_5^{(s)} + C_3^{(s)} C_5^{(r)} \right) + \frac{1}{2} \left( C_4^{(r)} C_5^{(s)} + C_4^{(s)} C_5^{(r)} \right);
$$

(3.104)

$$
\int_0^1 \hat{\varphi}_0^{(r)}(\xi) \Delta \hat{\varphi}_j^{(s)}(\xi) d\xi = -\frac{1}{6} \beta_k (\bar{\xi}_k - 1) \left( C_3^{(s)} \bar{\xi}_k + C_4^{(s)} \right)
$$

$$
\left\{ C_3^{(r)} (\bar{\xi}_k^2 + \bar{\xi}_k + 1) + 3 \left[ C_4^{(r)}(\bar{\xi}_k + 1) + 2 C_5^{(r)} \right] \right\};
$$

(3.105)

$$
\int_0^1 \Delta \hat{\varphi}_j^{(r)}(\xi) g_{R,k}^{(s)}(\xi) d\xi = \beta_j \beta_k (1 - \bar{\xi}_{j,k})
$$

$$
\left( C_3^{(r)} \bar{\xi}_j + C_4^{(r)} \right) \left( C_3^{(s)} \bar{\xi}_k + C_4^{(s)} \right).
$$

(3.106)
3.5 Numerical applications

In this Section, the performance of the proposed MDB element has been tested by means of two numerical applications, namely: i) a cantilever beam with three damages \( n = 3 \) subjected to concentrated and uniform distributed loads, inducing both axial and transverse displacements (Section 3.5.1); ii) a planar frame with two damages and subjected to both vertical and lateral loads (Section 3.5.2). In both examples, the effects of shear deformations and rotatory inertia have been studied. It is worth mentioning here that, as discussed in Section 2.3, the axial and/or rotational and/or transversal deformation concentrated at a given abscissa could be associated to construction reasons, e.g., a steel girder with inner bolted connections, responsible for a localized partial fixity, rather than to an actual damage.

In each example, the response analysis of the objective structure to the given static loads has been carried out by implementing the proposed MDB element within an in-house MATLAB [141] code, and it has been verified that the results so obtained with a single FE per member coincide with those delivered by the commercial FE code SAP2000 when at the position of each concentrated damage an additional node is introduced and the relevant partial fixity factors are considered for the axial and/or rotational and/or transversal spring at the damaged section. The exact static response of these two DS models have been also compared with the approximate response computed by implementing the LSR model with an increasing number of FEs per member. In both examples, the static analysis has been followed by a dynamic (modal) analysis of the bare objective structure, looking at the convergence of different FE models in terms of the first few modal frequencies of vibration, which has once again demonstrated the improved performance of the proposed MDB element.

3.5.1 Cantilever beam

In the first application, a cantilever beam of length \( L = 1.20 \) m has been studied. The beam is made of steel, with Young’s modulus \( E = 210 \) GPa and Poisson’s ratio \( \nu = 0.30 \), and the cross section is a solid square with dimensions of \( 240 \times 60 \) mm (shear correction factor \( \kappa = 1.2 \)). As shown within Fig. 3.18, three damages are assumed at the abscissas \( x = 0.1L, x = 0.2L \) and \( x = 0.4L \); the first and the second concentrated damage \((j = 1, 2)\) are modelled with a longitudinal spring of stiffness \( K_A = 25,200 \) kN/mm in parallel with a rotational spring \( K_R = 120,960 \) kN×m and with a shear spring \( K_T = 8,076 \) kN/mm, corresponding to the dimensionless damage parameters \( \alpha_j = \beta_j = \gamma_j = 0.1 \), while the third concentrated damage
Figure 3.18: Example one – Multi-cracked cantilever beam

<table>
<thead>
<tr>
<th>Method</th>
<th>$\hat{u}_A(1)/10^{-5}$</th>
<th>$\hat{u}_T(1)/10^{-6}$</th>
<th>$\hat{\varphi}(1)/10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS Proposed (1 FE)</td>
<td>1.0079</td>
<td>1.3431</td>
<td>-1.1310</td>
</tr>
<tr>
<td>DS SAP2000 (4 FEs)</td>
<td>1.0079</td>
<td>1.3431</td>
<td>-1.1310</td>
</tr>
<tr>
<td>LSR Model (4 FEs)</td>
<td>1.0119</td>
<td>3.4492</td>
<td>-1.1260</td>
</tr>
</tbody>
</table>

Table 3.1: Example one – Displacements and rotation at the free end of the cantilever beam

$(j = 3)$ simply consists of a rotational spring, having half of the stiffness of the previous damage $K_R/2$ and therefore $\beta_3 = \beta_2/2 = 0.05$ and $\alpha_3 = \gamma_3 = 0$.

In a first stage, the static analysis of the cantilever beam has been carried out by adopting the Timoshenko beam theory. Two loads have been considered in this case (see Fig. 3.18): a uniformly distributed load $q = 5.0 \sqrt{2}$ kN/m, forming an angle of $45^\circ$ with the longitudinal axis of the beam; and a point force $F_1 = 80.0 \sqrt{2}$ kN, inclined by $45^\circ$ and applied at the position $x = 0.7 L$.

The results in terms of dimensionless displacements and rotation at the free end of the cantilever beam are compared in Table 3.1 for three different methods of analysis, showing that the DS model with the proposed MDB formulation and just 1 FE is in perfect agreement with the conventional DS model built with SAP2000, which however requires 4 FEs; while the LSR model with 4 FEs is still affected by significant inaccuracies (in particular, the transverse displacement at the free end is overestimated by 157%).

This aspect has been further investigated, and Fig. 3.19 shows the convergence of the LSR model (dashed line) to the exact solution (solid line) as the number $N_e$ of FEs used to discretise the MDB increases. Transverse deflection and rotation at the free end of the beam have been considered, and it can be observed that in both cases the convergence is oscillatory, as the
3.5 Numerical applications

Figure 3.19: Example one – Convergence in terms of static transverse deflection (a) and rotation (b) at the free end

Figure 3.20: Example one – Fields of transverse displacements (a) and rotations (b) when 4 (top) and 16 (bottom) FEs are used to discretise the beam
Table 3.2: Example one – Modal frequencies [Hz] of the cantilever beam discretised with 4 elements

<table>
<thead>
<tr>
<th></th>
<th>1&lt;sup&gt;st&lt;/sup&gt; mode</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; mode</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt; mode</th>
<th>4&lt;sup&gt;th&lt;/sup&gt; mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS Consistent</td>
<td>110.16</td>
<td>665.38</td>
<td>1593.47</td>
<td>2915.13</td>
</tr>
<tr>
<td>DS Lumped</td>
<td>107.44</td>
<td>607.73</td>
<td>1323.21</td>
<td>2104.13</td>
</tr>
<tr>
<td>DS SAP2000</td>
<td>107.44</td>
<td>607.73</td>
<td>1323.21</td>
<td>2104.13</td>
</tr>
<tr>
<td>LSR</td>
<td>105.67</td>
<td>590.83</td>
<td>1328.52</td>
<td>2098.37</td>
</tr>
</tbody>
</table>

The exact solution can be either underestimated or overestimated by the LSR model, depending on the adopted mesh. Interestingly, more than 60 FEs are required for the LSR model to deliver accurate results, the reason being that the assumption of concentrated damage means that the increase in the flexibility of the beam is distributed over a very small region, and a conventional FE formulation can achieve this only with a very fine mesh. On the contrary, the proposed MDB element gives the exact results independently of the number of elements used in the mesh.

Figures 3.20a and 3.20b display the deformed shape (left column) and the rotation (right column) of the cantilever beam as computed with 4 FEs (top row) and 16 FEs (bottom row). In these graphs, the DS model (solid lines), implemented with the proposed MDB element, represents the exact solution, as increasing the number $N_e$ of FEs reduces the sampling interval, but does not change the values at a given abscissa, e.g. deflection and rotation at the free end do not vary with the meshing size. On the contrary, the approximate solution computed with the LSR model improves with the number of FEs.

In a second stage, the modal analysis of the MDB has been carried out. Table 3.2 compares the values of the first four modal frequencies computed with 4 FEs for four different approximate models, namely: the DS model with the proposed MDB element and consistent or lumped mass matrix; the DS model implemented in the commercial FE software SAP2000 (lumped mass matrix); the LSR model with lumped masses. The FE analysis carried out with SAP2000 delivers exactly the same results as the proposed MDB element with lumped mass matrix, as the same mesh has been adopted for validation purposes. Interestingly, roughly the same level of inaccuracy is shown by the LSR model with lumped masses, which therefore appears as a very crude approximation for dynamic applications.

This is confirmed by the semi-log convergence diagram of Fig.3.21, showing that the DS model with the proposed MDB element and consistent mass matrix (solid lines) is able to accurately estimate the first $n$ modal frequencies with just $n$ FEs, $\omega_i$ being the $i$th modal circular
frequency. All the other models, adopting lumped masses, are less efficient and require at least \( 2n \) FEs to provide an accurate estimate of the \( n \)th modal frequency.

Aimed at assessing the influence of shear deformations and rotatory inertia, with or without lumping the masses at the nodes, the convergence of eight different FE models with an increasing number \( N_e \) of FEs has been studied for the third mode of vibration. This model has been chosen because for the beam under consideration the flexural wavelength to cross sectional depth ratio is about 4, and then falls within the limits commonly accepted for the Timoshenko beam theory. The results of this investigation are shown in Fig. 3.22, where the solid lines identify the adoption of a consistent mass matrix, while dashed lines are used for the lumped
mass matrix; circles and squares denote models with or without the contribution of the shear deformations, respectively; filled markers mean that the model account for the rotatory inertia, while the unfilled markers mean the opposite. Moreover, the dotted line represents the theoretical value obtained by Caddemi and Calìò [38] for a multi-damaged Euler-Bernoulli beam without rotatory inertia (denoted with 0 in Fig. 3.22). As expected, the MDB element with the same assumptions (model 1, see Table 3.3) converge to this theoretical value as \( N_e \) increases. The modal frequency so obtained, however, largely overestimates (+45% for \( N_e = 10 \)) the more realistic value computed when both shear deformation and rotatory inertia are taken into account. Interestingly, in this case the models with a lumped mass matrix (dashed lines) always underestimate the modal frequency delivered by the corresponding models with a consistent mass matrix (solid lines).

### 3.5.2 Portal frame

Focusing on the modal response, a similar trend of results has been observed for the portal frame shown within Fig. 3.23. The columns AB and CD are \( L = 2 \) m and 1 m tall, respectively, while the horizontal beam has a span of 3 m.
The material properties are $E = 30$ GPa and Poisson’s ratio $\nu = 0.25$; the cross section of columns and beam is a solid square, $b = h = 0.2L$ mm, and the shear correction factor is $\kappa = 1.2$. The frame has three concentrated damages, modelled as a rotational $K_R = 3.2 \times 10^5$ kNm and a shear spring of stiffness $K_T = 8 \times 10^6$ kN/mm (corresponding to $\beta = 0.01$ and $\gamma = 0.01$). They occur on the two columns at the abscissa $x = 0.10L$, and on the top beam on the abscissa $x = 0.10L$ taken from the left end B. The modal analysis of the bare portal frame has been performed, confirming the superior performance of the proposed MDB element with consistent mass matrix. The semi-log Fig. 3.24 shows that with $N_e = 2$ (i.e. two FEs per member and six in total) the proposed model (solid lines) accurately predicts the first three modal frequencies, while with $N_e = 3$ (i.e. three FEs per member and nine in total), the analysis substantially converges also for the fourth and fifth modal frequency. More FEs are needed for the two models with lumped masses, showing also in this circumstance a slower convergence. The attention has been focused on the third vibration mode of the frame, looking at the eight different models listed in Table 3.3. In this case, the lumped mass matrix (dashed lines) can either underestimate or overestimate the modal frequency given by the consistent mass matrix (solid lines). It is also confirmed that neglecting shear deformations and rotatory inertia may result in a significant inaccuracy ($+12\%$ for $N_e = 10$) in the evaluation of the modal frequency.
Figure 3.24: Example two – Modal frequencies convergence diagram

Figure 3.25: Example two – Third modal frequency convergence diagram for different DS models
3.6 Conclusions

In this chapter, a two-node MDB (multi-damaged beam) element for the static and dynamic analysis of planar frame structures with concentrated damages has been presented and numerically tested, and all the elements of both stiffness matrix and mass. The proposed model follows the DS (discrete spring) representation of linear elastic, concentrated damages, in which the beam is fully articulated at the position of each singularity, and a set of axial, rotational and transversal elastic springs takes into account the residual stiffness of the damaged section, while axial, bending and shear deformations are considered in the undamaged regions of the beam among two consecutive singularities (i.e. the end nodes and the damaged sections).

The (physically consistent) flexibility modelling of concentrated damages has been adopted to derive the exact closed-form expressions for the deformed shape and internal forces of a beam with \( n \) damages of arbitrary severity and position, subjected to static axial and transverse loading. By exploiting the generalised Hooke’s law and the action-reaction principle, these results have been used to determine the dimensionless stiffness matrix and vector of equivalent nodal forces. The consistent mass matrix has been also computed, by adopting the same shape functions to represent the inertial forces on the MDB element, including the rotatory inertia.

Unlike the conventional DS models, which require the beam to be split with two FE nodes added to both sides of each singularity, the proposed MDB element embeds the effects of the concentrated damages without enlarging the size of the FE model. It follows that the exact static solution is retrieved independently of the mesh, while a faster convergence is achieved for the dynamic problems. This has been confirmed with two numerical examples, which also demonstrate the improved performance of the proposed MDB element in comparison with the approximate LSR approach, very often used for problems of damage detection. The important effects due to shear deformations and rotatory inertia (which cannot be neglected, particularly in the higher modes of vibration) are also highlighted and quantified in the numerical applications.

This work clearly demonstrates the improved efficiency of the proposed two-node MDB element with respect to the LSR model as well as the effects of the inclusion of both shear deformations and rotatory inertia. To do this, the results of static analyses are validated against those provided by the commercial FE code SAP2000 [61]. It has shown that, independently of the number of concentrated damages, the proposed MDB element is able to deliver, for both Euler-Bernoulli and Timoshenko kinematic models, the same exact solutions with just a single FE for each beam and column in the frame structure, while the LSR model only gives an approximate solution (whose accuracy depends on the size of the mesh) and SAP2000
needs an additional node at the position of each damage. In order to facilitate the practical implementation of the proposed MDB element, the closed-form expressions for stiffness matrix (see Eq. 3.50) and consistent mass matrix (see Eq. 3.78) are also provided and they have been also tested for the modal analyses. It is shown that using lumped masses with the proposed MDB element allows recovering the same eigenproperties given by SAP2000, provided that the same mesh is adopted; besides, using the consistent mass matrix increases the accuracy, as the eigenproperties so obtained converge more rapidly to the exact solution, with the additional advantage that the FE mesh is independent of the position of the concentrated damages.

This work has been published in a conference paper [68] and has been accepted for publication in Computers & Structures [69] with the further extension to include the rotatory inertia, which results in lowering the vibrational frequencies, particularly the higher ones, and therefore can be critical in the accurate identification of position and severity of the damages [123, 133].

The crack presented in this Chapter has successfully addressed a few gaps identified in technical literature, namely:

- the availability of single, closed-form analytical solutions for multi-damaged beams, accounting for the shear deformation and axial, rotational and shear springs at each damage position, under static loads;

- the availability of a two-node finite element for the multi-damaged beam, in which the stiffness matrix $K$ depends on a few damage parameters $(a_0, b_0, b_1, b_2, c_0)$, and the consistent mass matrix $M$ also account for the rotatory inertia;

- a parametric study, which has quantified the relative impact of various assumptions that can be made in the computational dynamic analysis of multi-damaged Euler-Bernoulli and Timoshenko beams.
In the previous Chapter, a novel multicracked beam element has been proposed, which includes shear effects and rotatory inertia, providing a realistic model applicable not just to slender beams, but to short beams as well using a discrete spring (DS) for the axial, transverse and rotational flexibility. Moreover, when the classical elasticity theory is used in conjunction with the DS model, the bending effects of the damage are effectively concentrated at a single position and a finite jump appears in the rotations’ profile along the beam’s axis. This oversimplifies the actual behaviour of the beam, which is expected to experience larger curvature in the neighbourhood of the damage, resulting in a smoother variation of the beam’s rotations.

In order to address this issue, different approaches are available in the literature; for instance, the already mentioned work of Christides and Barr [57] and Sinha et al. [205] who introduced a modified continuous flexibility without Dirac’s delta; or, as alternative, by replacing the Dirac’s delta appearing in the flexibility function with its Fourier’s transform or with an equivalent continuous equation as a probability density function.

A more attractive approach is the introduction of regularising higher-order gradients into the constitutive equation of the material, which allows smoothing the non-uniformity [23]. This approach has been used already in many sub-fields of structural mechanics, including in elasticity, plasticity, and damage mechanics, where the higher-order gradients have been postulated from a phenomenological point of view or derived from micromechanical basis.
4. Non-local elasticity

4.1 Classification of higher-order gradient theories

Higher-order gradient theories refers to all those differential equations, used to study physical problems, which include higher-order spatial derivatives. They are at the basis of many physics principles; they literally started in the nineteenth century, at the same time of the continuum theories. For example the Maxwell’s electromagnetism law is a gradient theory of the first order of the electromagnetic potentials, or Einstein’s theory of gravitation is a second-gradient theory of the metric of curved space-time [143]. These are just two main examples of the several gradient theories.

Focusing on elasticity problems, the inclusion of higher-order gradient elasticity terms into the continuum equations is not a novel idea. Indeed, more than a century and a half has elapsed since Cauchy suggested the use of higher-order spatial derivatives in order to model with more accuracy the behaviour of discrete lattice effects [46]. The general idea behind the inclusion of higher-order gradients into the classical constitutive equations is that, differently from the classical continuum theories where the stress at one point only depends on the strain at that point, now the stress at one point are assumed to depend on the strain in that point and in the neighbourhood of the point.

The two main approaches to include higher-order gradients into the classical constitutive equations are:

- **Phenomenological (or regularisation) approach**, in which the classical continuum theory is enhanced with higher-order gradients to avoid singularities in localised phenomena. Conceptually, such approach moves from the macro-structure, and higher-order terms are added following a phenomenological nature (smoothing of the stress/strain unrealistic peaks).
4.2 Reasons of higher-order gradient theories

Non-local modes, gradient elasticity models and micro-polar model are some examples of this approach.

- **Microstructural (or homogenisation) approach**, is a direct consequence of the homogenisation of the discrete medium, where the influence of the microstructure is considered in an average sense [239]. The homogenisation may bring higher-order terms which are not guaranteed to be free from undesirable destabilising effects [52]. Differently than regularisation, homogenisation procedure starts from the microstructure, achieving a homogenised a continuum (macroscopic) formulation. To do this, a Representative Volume Element (RVE), i.e. a sub-domain large enough to capture the microstructural parameters, is studied, whose macroscopic behaviour can then be derived from the solution of pertinent boundary value problems.

The two procedures are not mutually exclusive, as some continuum theory can be obtained using regularisation techniques and also can be derived through homogenisation procedures e.g. when regularising gradients are achieved via microstructure homogenisation as for the Cosserat’s continuum [20]. Both approaches, and in particular the phenomenological approach, will be investigated in detail in the next Subsections.

### 4.2 Reasons of higher-order gradient theories

Not all the mechanical phenomena can be properly described by the classical or standard continuum theory, e.g. classical elasticity leads to singularities in the stress field at the crack tip as well as dislocation lines. As solution to this inefficiencies, one of the available approaches can be the enrichment of the governing equations with gradients or with weighted spatial averages of an internal variable (e.g. the strain or the stress). In addition to the already mentioned regularising effect and removal of singularities from stress and strain fields, some other reasons to include higher-order theories into the physical problems are here summarised:

- to capture small-scale deviations from local continuum models caused by material heterogeneity [176];
- to capture size effects observed in experiment and discrete simulations [228, 8];
- to achieve objective and convergent numerical solutions for localised damage [26];
- to introduce non-uniformity in the strain field [23].
4.2.1 To capture small-scale deviations from local continuum models caused by material heterogeneity

As shown by experimental observations, materials’ microstructure can affect the macroscopic mechanical behaviour, which can be significantly different from the prediction of the classical continuum theories [113]. These effects are more noticeable when the size of the investigated structure is comparable with the size of the microstructure of the materials. Examples of material with a clear influence of microstructure are porous media, cellular or granular materials, bones, metallic or polymeric foams, polymers, polycristals, granular material, graphite, particle or fibre reinforced composites as well as asphalt and concrete which consist of aggregates and matrix material randomly mixed together. Figs. 4.2(a)-(f) show the microstructure of some of these materials.

In standard continuum mechanics, a solid body is an ideal continuum made of infinitesimal material volumes, where each volume can be described independently from the others; that is, the volumes are not isolated but they interact only on the level of balance equations of mass,
momentum, energy and entropy. This ideal uniform continuum does not exist in the reality. Any kind of material has an internal microstructure, more or less complicated, whose size usually ranges over several orders of magnitude. Microstructure refers of any type of internal material structure, from the micrometers at atomistic level to meters of a lattice structure. One way to include the characteristic of the microstructure is by describing the material properties in a refined model where each crystal or fibre is modelled separately. Indeed, this is possible just within a certain range of scale, otherwise the model would be computationally too expensive for practical applications.

Furthermore the continuum description is often not adequate at small scale and needs to be replaced by a discrete model, which takes into account inter-atomic forces. When the resolution level is not accurate enough to describe relevant components of the deformation field, a more effective method than refinement is to enrich the model, by using the so-called enriched or generalised continuum formulations, which better approximate the real processes keeping the number of additional constitutive parameters to a minimum. Since direct modelling of the microstructure inevitably results in huge computational costs, enriched elastic continua, in which additional parameters are related to microstructure (e.g. the length scale parameters), can be seen as a viable approach. The material is then assumed as macroscopically homogeneous, and therefore the same equilibrium and kinematic conditions as in the classical elasticity theory can be used. Due to the enrichment of the constitutive law, however, the order and the complexity of the equations increase.

Indeed, classical elasticity does not take into account some unavoidable effects of materials heterogeneity (e.g. crystals in metals, fibres in timber or aggregates in concrete), therefore it fails for problems where the microstructural effects are relevant. Different elastic theories can be adopted, to link the microstructure with the macro-structure e.g. the general Cosserat elasticity theory [60], the Cosserat theory with constrained rotator or couple stress [118, 150, 221], the strain gradient theory [221], the multipolar theory of continuum mechanics [93], the elastic theory with microstructure [147, 148], the micromorphic, microstretch and micropolar elastic theories [78], the non-local elasticity [77].

4.2.2 To capture size effects

A broad range of problems and applications in civil, chemical, electrical, geological, mechanical and materials engineering have been solved using the classical continuum solid mechanics theories, where the stress is related to the strain at one point, without any higher-order derivatives of the field variables. These theories, such us linear elasticity, non-linear elasticity,
or plasticity were initially designed to describe phenomena and processes in the macro level (structural) size ranging from millimetres to meters. In the last century they were used to describe problems at different scales, like the elastic theory of dislocation at atomistic scale [253, 151], or faults and earthquakes at earth scale [43, 198], or the relativistic elastic solids at astronomic scale [29]. It is in the description of these size-dependent responses that classical continuum theories results inadequate.

Higher-order gradient elasticity theories can be viewed as an extension of the classical continuum elasticity, where the included gradient terms, associated with characteristic lengths or dimensional parameters, account for the size effects at different scales (i.e. the scale at the macroscopic level and the microstructural level). This enriched theories have gained the attention of the researchers from different disciplines as solid mechanics or material science. The big advantage of these theories is the possibility to model phenomena that otherwise could not be studied with the classical elasticity theory [10]. Such phenomena are for example fracture dynamics, localisation processes as well as the presence of size effects which is dependent on the macroscopic properties or specimen size.

The importance to include size effects is one of the main motivation to the inclusion of non-locality into a physical problem. Using the classical theories of elasticity the stress of a structure is assumed independent of the size of the structure e.g. a large beam and a tiny beam will have the same stress field if they are made of the same material (scale-free character [136]). In the real world, because of the heterogeneity of the material, the stress in small beam may be different. The dependence of the material property on the size of the sample/specimen is called size effect [109]; differently than scaling which is a transformation that enlarges or shrinks objects by a scale factor. Many physical problems contain size effects as observed in experimental tests of concrete specimens [226] as well as at the micrometer scale as bending of metallic films [187, 208], carbon nanotubes [11], micro-indentation [159, 139, 175] or fracture experiments on concrete and limestone [24, 27]. Size effects were also measured on porous materials such as human bones and foams [122].

A deep investigation of size effects started in the seventies, when the big scale difference among the size of laboratory tests and the size of concrete structure as dam, bridges or reactor containments, lit up the interest of many researchers. This scale gap can span from one to many orders of magnitude. Size effects have become a crucial aspect especially in the last years with the development of fibre composites and sandwiches for large ship hulls, decks stack as well as masts and bulkheads [24]. One of the field where size effects are particularly relevant is geotechnical engineering and geomechanics [28, 25] where the whole structure can
be reproduced only significantly reducing its size e.g. the study of mountain slide or the risk evaluation of an excavation of a tunnel.

Even though the problem of size effects, and more in general scaling, can be less important in some engineering field as mechanical and aerospace where a full size sample can be tested, scaling can be considered one of the “fundamental characteristic of any physical theory” and if a theory does not have scaling properties the theory itself is incorrect [12]. In the field of nano and micro technologies, where the length range from few centimetres of small electrical devices or a sheet metal forming down to few nanometres of a molecule chain, the study of size effects plays a key role.

Experiments proved that there is a connection among the range of length scales involved and then the response may be linked to the size of the tested specimen. Aifantis et al. [7, 213] studied the ability of gradient elasticity to embed size effects in small-sized samples with uniform microstructure or large-size specimens with irregular microstructure. Beveridge et al. [30] observed, from experimental tests of different sizes of heterogeneous beam samples, as the characteristic length value is remarkably similar to the length scale parameter appearing in the micropolar theory.

Gradient-dependent constitutive equations can be used to consider the size effect [9] because of the higher-order gradients in the constitutive equation which can be seen as a measure of the heterogeneity of the deformation field which global effect may depends on the specimen size.

### 4.3 Applications of higher-order gradient elasticity

There are several applications of higher-order gradient elasticity models in different engineering fields. For example it has been used in fracture mechanics to smoothen the singularities in the strain field near the crack tips [77, 80, 7] or, combined with the gradient plasticity, to study the strain field of the dislocations [76, 4], as well as to study the softening behaviour, like in localisation problems where the well-position is ensured by the higher-order terms.

In dynamics, waves propagation in discrete lattice materials [73, 145, 135], composites structure [243] or granular media [166] have been studied using higher-order gradient theories, which can reproduce with higher accuracy the experimentally proved dispersive behaviour of waves propagating through the medium [23].

Applications of gradient elasticity are composite materials: fibre pull-out from a matrix; stress transfer in short fibre composites; transverse shearing of a sandwich plate [212]; post-
buckling problems for long elastic beams [125]. Altan et al. [15] studied the dispersion relationship for harmonic shear waves propagating in layered composite infinite medium using the gradient theory; Suiker [209] applied the strain gradient to study the mechanical behaviour of ballasted railway tracks.

The knowledge of the stress/strain distributions in geometric singularities or material discontinuities is very useful e.g for the design of micro/nano devices, where size effects are more noticeable. Thanks to the recent developments on the experimental techniques it is now possible to experimentally capture the stress and deformation fields near crack tips (singularities) or interfaces (discontinuities) giving informations to validate the non-local models. Experimental works which focus on capturing the localised gradient and size effects are for example the work of Sciammarella et al. [199] or Aifantis [7, 8] on micro tensile preforated specimens.

The enrichment with higher-order gradients approach for strains and stress is required to study problems where the structure and microstructure have approximate the same scale (e.g. micro-devices) or when the material has a clear granular or crystal macro-structure (e.g. micro-composite). Furthermore, classical theories do not work when the magnitude of the deformation is comparable to the characteristic length of the microstructure. Focusing on the micro and nano-scales, recent observations in advanced optical and electron microscopes, involving nano-tubes and nano-scales objects, have been interpreted by using the classical continuum mechanics theory, but the experimental evidence and observations have suggested that these theories are not accurate enough to describe the behaviour of corresponding deformation phenomena. Certainly one of the main area of interest is the study of the vibration and buckling of carbon nanotubes, which electronic and mechanical properties let them to become one of the most promising materials for nanotechnology in the last decade [239, 137]. Atomic-force microscope, field emitters, nanofillers for composite materials, nanoscale electronic devices, and frictionless nanoactuators, nanomotors, nanobearings, and nanosprings are just few examples where carbon nanotubes have been deeply applied [239]. Some of the main papers on this area are those by Peddieson et al.[171], Zhang et al. [176, 236] and Wang et al. [237] where the non-local elasticity constitutive equations of beam and shell theories are studied [184]. Further area where the non-local constitutive equations have been applied is to study the fluids surface tension [75].
4.4 Microstructural approaches for higher-order gradient elasticity theories

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The classic continuum is essentially the theory of Cauchy’s continua at the beginning of the nineteenth century. Cauchy theory refers for isotropic homogeneous elastic solids in small strains in an Euclidean (a vectorial space enriched with the notions of distance, length and angles) and connected (made of one piece) physical space and material manifold where the basic assumptions are:

- There are only volume and surface forces applied and no couples;
- There are no additional degrees of freedom for each point, consequently the concept of “microstructure” is not applicable.

Basis of all the theory is the Cauchy’s postulate: “the stress vector does not changes for all surfaces passing through a generic point of the body and it has the same normal vector” [143]. Then the traction on a facet cut depends only on the normal, first order dependence, and it no depends on the geometry of the internal surfaces. Follow the notion of stress tensor, the state of stress in a point of the body is defined by the infinite planes that pass through that point and it is the only “internal force” in the theory. According to the Cauchy’s stress theorem there exists a second-order tensor field, the Cauchy stress tensor, such that it linear relates the stress vectors and the normal function [143].
One of the solutions suggested to take into account microstructural effects has been the addition of DoF to the standard continuum equations at the macro-level to incorporate the inhomogeneous material behaviour. The enrichment is associated with the inclusion of a characteristic material length, which macroscopically represents the microstructural parameters, such as the geometry and the distribution of the micro-elements. One of the first attempts to enhance the continuum is the micro-polar (or Cosserat) continuum, where three rotational DoFs are added to the standard formulation. The continuous media so obtained appears to have 6 DoFs, like any rigid body and, not the only three DoFs typical of a point in the Euclidean space as in the classical elasticity theory. The three additional rotations are created by the so called “couple stress” and correspond to three directors. The appearance of couple stresses in the governing equations represents the novel feature of the Cosserat theory. This formulation includes many characteristic length scales, which are related to the ratio among the normal stiffness and an additional “bending stiffness” linked to the introduced rotational DoFs [210].

4.4.1 Brief history of microstructural theories

The aim to describe the behaviour of materials has been a human ambition, since Galileo Galilei’s pioneering intuition. One of the first mathematical examples is the “Hooke’s Law”, the theory of elasticity conceived by Robert Hooke, in 1678, who expressed a direct proportion among the power of the spring and the tension thereof. It was in the 1821, about two hundreds and an half years later, when Navier gave the equations for the equilibrium and motion of elastic solids, and this can be considered the birth of the “modern” theory of elasticity [219]. The theory has been improved and extended in the following years by many scientists e.g. Cauchy, Poisson, Stokes applied the theory to other materials (e.g. viscous fluids) or at different structural levels (e.g. molecular and atomic level). In the nineteenth and beginning of the twentieth centuries the improvements and extensions continued, from the relativistic continuum mechanics to the visco-elasticity and thermo-elasticity. Indeed, there are still fields that remain active and require further research, like those in granular and porous elastic solids, composite materials and polymeric materials.

Among them, a big improvements in the elasticity theory has been made by the Cosserat brothers with their book, in 1909, entitled Théorie des Corps Déformables [60]; this was the beginning of the generalised continua. The two brothers added new DoFs to the material points of an elastic solid and introduced the concept of couple stress as well as a new conservation law for the momentum, leading to a formulation which can be used to study a wider range of materials than the classical theory e.g. anisotropic fluids like liquid crystals and blood. Even
though it was innovative it did not attract much attention due to the non-symmetric nature of the couple stresses and a lack of specific constitutive relations which make it complex for practical application [221].

Fifty years later the theory was further developed leading to a multitude of new couple stress theories. New Cosserat-like theories were published under a variety of different nomenclature e.g. polar elasticity, strain gradient theories, micropolar elasticity, multipolar theory [93], etc. The subject was also deeply investigated by Mindlin [150, 147, 148, 149], Toupin [221], Kröner [121], Koiter [118] and Eringen [72]. Among these theories, the most explored was one the special case where the three extra rotations are assumed to coincide with the local rotations of classical elasticity, and then the so obtained couple theories include only three components of displacement vector as independent variables. As result eight of eighteen new components of the first gradient of strain were included in the strain energy density function, in addition to the classical six components of strains. A strain gradient theory including all the first gradient of strain has been proposed by Toupin [221]. The linear version of Toupin theory has been developed by Mindlin [147, 148]. Mindlin theory introduced eighteen new constants in the formulation, resulting to be physically and mathematically complicated. Three simplified version, named as Form-I, Form-II and Form-III, were subsequently proposed assuming the same deformation for macro and micro structure; they are valid for big wave-lengths, reducing to seven the number of characteristic parameters. In details:

**Form-I** - the strain density function is assumed to be a quadratic form of the classical strains and the second gradient of displacement;

**Form-II** - the second displacement gradient is replaced by the gradient of strains leading to a symmetric total stress tensor;

**Form-III** - the strain energy function is written in terms of the strain, the gradient of rotation, and the fully symmetric part of the gradient of strain ( = second gradient of macroscopic displacement) [113].

Three further theories (able to include microstructural effects) were also proposed by Eringen [78]:

**Micromorphic theory** - is defined in the classical continuum with additional DoFs, twelve in total, coming from micro deformations of the physical particle, and represented by three deformable directors. Its linear form coincides with the micro-structure theory of Mindlin [147];
Microstretch theory - is defined in a continuum where the deformable directors do not contain micro-shears but stretches only.

Micropolar theory - is defined in a continuum where the three directors are rigid and represent three independent rotations.

All three theories are aimed at describing physical phenomena among atoms, molecules or particles at nano scales. As we can see in Fig. 4.3, the classical theory is a subclass of the micropolar theory, which in turn is a subclass of the microstretch and of the micromorphic.

### 4.5 Phenomenological approaches for higher-order gradient elasticity theories

Looking at the microstructural theories, in most of them a Laplacian-type gradient appears, where the Laplacian operator describes diffusion process; analogously, in the phenomenological approach Laplacian-type gradients are introduced in order to have redistribution effects [19].

In elasticity, there are two main classes of phenomenological higher-order gradient theories:

- **Integral non-local model** - where the stress is obtained from integral operators applied to the local strain [234]. Non-local continuum differs from classical and other enriched continuum theories because the constitutive equation is a functional of the independent variables over all points of the domain; and because the balance laws are formulated to be non-local (global) by introducing non-local residuals, whose global (integral) values are assumed to vanish;
Gradient elastic model - where the stress is defined explicitly from the local strain and its derivatives.

An integral non-local model can be defined as a model where weighted averages of a state variable in the neighbourhood of a generic point are involved in the constitutive law at that point, abandoning the principle of local action in the classical continuum mechanics. A gradient type non-local model indirectly adheres to this principle, in fact as consequence of the inclusion of first or higher gradients of some state variables or thermodynamic forces a field around the point is taken into account. Interestingly both theories contains a characteristic length (or material length) in the constitutive law.

4.5.1 Strong and weak non-local theory

The phenomenological approach leads to the enriched models with higher-order gradients, which are referred to as non-local theories, meaning that they account for the influence of the field variables at the neighbouring points. In details, non-local theories can be classified according to the extension of the neighbourhood area as strong or weak non-local formulations. Bažant and Jirásek [26] gave clear mathematical definition of the two type of non-locality. Generally speaking, weak non-local theories use explicit gradient models, while strong non-local theories are those expressed by implicit gradient models or by integral formulations with weighted spatial averaging [108]. As example the Cosserat and multi-polar media can be considered as a continua having weak non-local formulation represented by the inclusion of higher-order gradients of the stress and strain field and additional material parameters [96] and the so defined gradient model is classified as weak non-local.

4.5.1.1 Mathematical definition of non-local theory

The general equation to express a physical theory assume the form:

\[ Au = f, \quad (4.1) \]

where \( f \) is a given excitation; \( u \) is the unknown response; \( A \) is the operator, which characterises the system. The equation and its functions are defined in a certain spatial domain \( V \). The problem is considered local when the operator \( A \) applied to two functions that are identical, for \( x \) in the neighbourhood of the point \( x_0 \), then gives \( A u(x_0) = A v(x_0) \). This is for the differential operators which does not change inside the neighbourhood of the point where the derivatives are taken. For example the EB differential equation \[ EI(x) u''(x) = q(x) \] is
local and the resulting local constitutive equation is $M(x) = EI(x)\chi(x)$ where $M(x)$ is the bending moment and $\chi(x)$ is the curvature. Instead the non-local constitutive equation may be expressed as:

$$M(x) = \int_{-\infty}^{+\infty} EI(x, \xi) \chi(\xi) d\xi,$$  \hspace{1cm} (4.2)

where $EI(x, \xi)$ is the kernel of the elastic integral operator. The combination of the equilibrium and the kinematic equations yields:

$$\left[ \int_{-\infty}^{+\infty} EI(x, \xi) u''(\xi) d\xi \right]'' = f(x).$$  \hspace{1cm} (4.3)

This equation is strongly non-local due to the presence of the spatial integral [26] which requires the equality $u(x) = v(x)$ not just in the neighbourhood $O$ but in all the domain.

A further aspect to be considered when distinguishing among local and non-local theories is the presence of a characteristic length. Mathematically, when the scaling of the spatial coordinate (the size of the element) does not change the fundamental equation, the theory is considered strictly local as it does not show any characteristic length. Instead if the theory is not invariant with respect to spatial scaling is defined as weakly non-local. Typical these weakly non-local theories are differential equations with derivatives of different orders where the characteristic length can be deduced from the ratio among the coefficient multiplying the different derivatives which have different dimensions. An example of weakly non-local formulation is the EB beam on the elastic (Winkler) foundation theory [26]:

$$u'''_{b}(x) + \frac{4}{\ell_{w}^{4}} u_{b}(x) = \frac{q(x)}{EI},$$  \hspace{1cm} (4.4)

where $u_{b}(x)$ is the transversal displacement, $q(x)$ is the applied external load and the characteristic length $\ell_{w} = \sqrt{\frac{4EI}{k}}$ is proportional to the ratio between flexural stiffness $EI$ and the elastic constant of the foundation $k$. Another example is the Timoshenko beam, which homogeneous beam of constant cross-section can be expressed as:

$$EI u'''_{T}(x) = q(x) - \ell_{T}^{2} q''(x),$$  \hspace{1cm} (4.5)

in this case the characteristic length $\ell_{T}$ is given by the square root of the ratio among bending and shear stiffness $\ell_{T} = \sqrt{\frac{EI}{\kappa GA}}$. When the cross section does not change along the beam, this ratio and then the characteristic length are proportional to the beam depth of the model. In the one-dimensional case the only geometric dimension of the model is the beam length while
the beam cross depth represents a property of the beam. If there is a characteristic length, the solutions for different spans can be obtained by properly scaling a reference solution without scaling the material beam properties. If there is not a characteristic length the simple scaling is not possible without also scaling the material properties, as the beam depth for the Timoshenko beam theory.

The solution of the physical problem depends on the ratio among physical dimensions of the structure and its intrinsic microstructural material length. The reduction of the geometric model dimensions to a one-dimensional excludes some of the material variables, which can be partially reintroduced using the material length.

In conclusion, the local/non-local classification proposed by Bažant and Jirásek [26] can be summarise as follow:

- **strictly local models**: nonpolar simple materials;
- **weakly non-local models**: polar theories and gradient theories (higher-grade materials);
- **strongly non-local models**: models of the integral type or of a gradient type equivalent to integral-type with weight functions.

### 4.5.1.2 Non-local averaging operator ($\alpha$)

Roughly speaking, using a non-local integral approach, the local value of a variable $f(x)$ is replaced by its non-local counterpart $\tilde{f}(x)$ through its weighted average over a spatial neighbourhood $V$ of each point [1]:

$$\tilde{f}(x) = \int_V \alpha_w(x, \hat{x}) f(\hat{x}) d\hat{x},$$

where $\alpha_w(x, \hat{x})$ is the non-local weight function, which is a measure of the dependence of the stresses at the source point $x$ to the stress at the receiver point $\hat{x}$. The spatial neighbourhood around a certain value $x$ can be identified by a circle or a sphere, whose size is related to the weight function by the non-local interaction radius (or influence distance) $R$, defined as the smallest distance $r = ||x - \hat{x}||$ which corresponds to null or negligible values of the weight function. In other words, $R$ is the maximum distance at which $\alpha_w(x, \hat{x})$ is meaningful and, so it represents an internal characteristic length of the material [174]. Fig. 4.4a shows how the strain value at the centre of the representative volume $\bar{\varepsilon}(\hat{x})$ may differ from the strain corresponding to the average strain $\varepsilon(\hat{x})$ over this volume.

The operator $\alpha_w(x, \hat{x})$ is a positive attenuation function, usually assumed as a non-negative bell-shaped function centred at the investigated point $x$ and monotonically decreases for in-
creasing distance parameter \( r(x, \hat{x}) \) which, in the hypothesis of infinite, isotropic and homogeneous medium, represents the only dependent variable for the weight function. Mathematically \( \alpha_w(x, \hat{x}) \) attenuates to zero for large distance:

\[
\lim_{r \to \infty} \alpha_w(r) = 0, \tag{4.7}
\]

and usually satisfies the normalizing condition:

\[
\int_{\Omega} \alpha_w(x, \hat{x}) \, d\hat{x} = 1, \tag{4.8}
\]

which guarantees that a uniform field is not altered [26]. As example, the Gaussian distribution function is:

\[
\alpha_w(r) = \left( \ell \sqrt{2\pi} \right)^{-N_{\text{dim}}} e^{-\frac{r^2}{2\ell^2}}, \tag{4.9}
\]

where \( \ell \) is a parameter with the dimension of length and \( N_{\text{dim}} \) is the number of spatial dimension. This function has infinite support, which means it assumes meaningful values in all the domain; as consequence a non-local interaction take place among any point of the domain (see Fig. 4.4b). An example of finite supported averaging operator is the function:

\[
\alpha_w(r) = c \left( 1 - \frac{r^2}{R^2} \right)^2, \tag{4.10}
\]

which is null outside the radius \( R \): \( \alpha_w(r) = 0 \) for \( r \geq R \) (see Fig. 4.4c). The brackets \( \langle \cdot \rangle \), usually called Macaulay brackets, denote the positive part of the function \( \langle \alpha_w(r) \rangle = \max(0, \alpha_w(r)) \).

### 4.5.2 Brief history of the phenomenological theories

It was back in the eighties when Eringen [76] derived a simple stress-gradient theory from the integral non-local theory. Followed by Aifantis and co-workers [5, 16, 223] which formulated a gradient elasticity theory for finite deformations and infinitesimal deformations with fewer higher-order terms and then a smaller number of constitutive constants [19]. Aifantis article [9] gives some recent developments of gradient theories.

After the aforementioned pioneering works, a considerable amount of literature has been published in the last thirty years dealing with new versions of these enriched theories or giving solutions for the boundary value problems. Other theories that are worth to mention are the
gradient theory of Aifantis [5], the gradient elasticity theory with surface energy of Vardoulakis and Sulem [211] and the theory of Fleck and Hutchinson [84, 85].

The simplest special case of Form-II is the gradient elastic problem proposed by Aifantis [5] and Ru and Aifantis [191] as it requires, in addition to the standard Lamé constant, only one new material parameter. Nevertheless, the absence of a variational formulation in Aifantis model leads to BCs not compatible with the corresponding Mindlin ones. This has been solved by Vardoulakis et al. [227] with their gradient elasticity theory with surface energy. Fleck and Hutchinson [84, 85] gave an alternative to Mindlin’s Form-I gradient elasticity theory, where the second gradient of displacement is decomposed into stretch gradient and rotation gradient tensors.

4.5.3 Strain gradient, stress gradient and hybrid gradient elasticity

In the classical elasticity theory, the constitutive equation is represented by an algebraic relationship among the stress and strain tensors at a generic point, while the non-local elasticity involves spatial integrals, which represent some weighted averages of the contributions of the tensors around a given point [171, 247]. In this context, the two main theories are those due to
Eringen and Aifantis [9], which enrich the classical elastic constitutive law by adding stress or strain gradients, respectively.

### 4.5.3.1 Eringen’s (or stress gradient) theory

The non-local theories proposed by Eringen [74, 76] and Eringen and Edelen [79] use the constitutive assumptions that the stress at one point is a function of strains at all points in the continuum. Originally in stress gradient theory the non-local variable is computed from its local counterpart using an integral formulation representing a volume average of the relevant state variable:

$$\tilde{\sigma}(x) = \int_V \alpha(|x - \hat{x}|, \ell_\sigma) \sigma(\hat{x}) \, dV,$$

where $\tilde{\sigma}(x)$ is the non-local stress; $\sigma(x)$ local stress; $\alpha(|x - \hat{x}|, \ell_\sigma)$ is a kernel function (see Section 4.5.1.2); while $|x - \hat{x}|$ is the Euclidean distance and $\ell_\sigma$ is the length scale parameter (a material property related to the microstructure, see Section 4.5.1.1).

Later, Eringen introduced an equivalent formulation, valid only for certain kernel functions, where differential operators were used instead of the spatial integrals. For the uni-axial state the Eringen’s differential equation assumes the following form:

$$\sigma(x) - \ell_\sigma^2 \sigma''(x) = \tilde{\sigma}(x) = E \varepsilon(x),$$

where $E$ is the elastic Young modulus of the classical isotropic elasticity. In the Eringen’s (or stress gradient) theory the local strain $\varepsilon(x)$ is related not only to the local stress $\sigma(x)$ but also to its second-order derivative.

### 4.5.3.2 Aifantis’ (or strain gradient) theory

Eringen’s gradient elasticity has been revisited by Triantafyllidis and Aifantis [223], resulting in a constitutive equation which incorporates the second order gradients of the strain. The so called Aifantis’ (or strain gradient) theory is a complementary formulation, in which the stress-strain uni-axial constitutive law is expressed as:

$$\sigma(x) = E \left[\varepsilon(x) - \ell_\varepsilon^2 \varepsilon''(x)\right],$$
where now $\ell_\varepsilon$ is the strain length scale parameter, which again can be related to the microstructural arrangement [9]. The theory was successful applied to eliminate the singularity at the crack tip and dislocation line [16, 191, 225, 21].

Differently than Eringen’s model, Aifantis’ gradient elasticity equation is a development of Mindlin’s work, where the linear elastic constitutive relations have been extended with the inclusion of the Laplacian of the strain [5]. Interestingly, in both theories if the microstructural parameter ($\ell_\sigma = 0$ or $\ell_\varepsilon = 0$) is zero, then the classical elastic continuum theory is recovered. Eqs. (4.12) and (4.13) represent the gradient elastic model where the stress/strain gradients are directly introduced to define the stress field; on the other hand in the integral non-local elastic model to obtain the stresses an integration of the local strain is required. For the uniaxial case, combining constitutive equation Eq. (4.13), equilibrium equation ($\sigma'(x) = q(x)$) and kinematic equation ($u'(x) = \varepsilon(x)$), yields:

$$E \left[ \dddot{u}'(x) - \ell_\varepsilon^2 \dddot{u}''(x) \right] + q(x) = 0,$$

(4.14)

where $q(x)$ is the axial body force component and $u(x)$ is the axial displacements component. Eq. (4.14) can be decomposed into an equivalent two equations system:

$$\begin{align*}
E \dddot{u}'(x) + q(x) &= 0; \\
\ddot{u}(x) - \ell_\varepsilon^2 \dddot{u}''(x) &= u(x).
\end{align*}$$

(4.15a)

(4.15b)

Eq. (4.15a) can be solved first and its solution $u(x)$ can be used to solve Eq. (4.15b) and then to obtain the gradient-enriched displacements $\dddot{u}(x)$. Eq. (4.15b) can be also expressed in terms of stresses by differentiating and then multiplying by the elastic Young’s moduli $E$:

$$\dddot{\sigma}(x) - \ell_\varepsilon^2 \dddot{\sigma}''(x) = \sigma(x),$$

(4.16)

where $\sigma(x) = E u'(x)$. Using the stress field formulation there is the big advantage that the BCs are consistent with the results with no singularities at the crack tip, differently from the strain field formulation where the BCs (variational consistent) are not able to remove all singularities [21].

Even though for $\ell_\varepsilon = \ell_\sigma$ Eq. (4.12) and Eq. (4.16) seem identical, they were formulated to solve totally different problems: Eringen’s theory for dynamics problems, while Aifantis’ theory for static cases [21]. In the first case the equation of motion ($\dddot{\sigma}'(x) + q(x) = \rho \dddot{u}(x)$) contains the non-local stress $\dddot{\sigma}$, while in the Aifantis’s model the static equilibrium equation includes the local stress $\sigma(x)$. Both equilibrium equations can be solved if combined with the
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Table 4.1: Stress gradient elasticity theories of Eringen and Aifantis - overview of field equations for the uniaxial case

The BCs can be obtained from the variational form, and can be classified in: i) essential BCs (or kinematic BCs), expressed in terms of kinematic variables (displacements or their normal derivatives); ii) natural BCs, which are the conjugate tractions expressed in terms of stress type variables, both standard and high order [22]. In detail the essential BCs are prescribed displacements $u(x_0) = u_0$ or prescribed normal derivatives $u'(x_0) = \varepsilon_0$ where $\varepsilon_0$ is a prescribed value of the strain at $x_0$. 

constitutive relation of local stresses, the kinematic equation and the non-local stress assumption of Eringen Eq. (4.12) or of Aifantis Eq. (4.16). Table 4.1 gives an overview of the field equations for both Eringen’s and Aifantis’ theories for the uniaxial case. Reduced and irreducible forms are obtained by subtracting to the balance of momentum its Laplacian multiplied by $\ell^2$, where the second derivative of the body forces is neglected. The two sets of equations in the Eringen reduced form are coupled and solved simultaneously, while Aifantis’ sets are uncoupled because $\tilde{\sigma}(x)$ does not appear in Eq. (4.15a).
4.5.3.3 Hybrid gradient elasticity

A more general gradient theory of elasticity was firstly introduced by Aifatis [9] and Ru and Aifantis [191], when they extended the strain gradient elasticity model by adding a stress gradient term. The so obtained hybrid gradient constitutive equation assumes the form:

\[ \sigma(x) - \ell_\sigma^2 \sigma''(x) = E \left[ \varepsilon(x) - \ell_\varepsilon^2 \varepsilon''(x) \right], \]

where \( \ell_\sigma \) and \( \ell_\varepsilon \) are two different gradient coefficients. This theory is an unification of the theories of Eringen, Eq. (4.12), and Aifantis, Eq. (4.13), able to smooth both stress and strain singularities at the crack tip or dislocation core [97, 99]. Using appropriate BCs, Eq. (4.17) can be solved by studying the following three equations system:

\[
\begin{align*}
\ddot{\varepsilon}(x) &= \varepsilon(x) - \ell_\varepsilon^2 \varepsilon''(x) ; \\
\ddot{\sigma}(x) &= \sigma(x) - \ell_\sigma^2 \sigma''(x) ; \\
\ddot{\sigma}(x) &= E \ddot{\varepsilon}(x),
\end{align*}
\]

where the constitutive equation has been reformulated in terms of the classical local strain \( \varepsilon(x) \) and stress \( \sigma(x) \) fields. It can be easily verified that for particular values of such parameters it is possible to retrieve the Eringen’s model (\( \ell_\sigma > 0 \) and \( \ell_\varepsilon = 0 \)) or the Aifantis’ model (\( \ell_\sigma = 0 \) and \( \ell_\varepsilon > 0 \)), as well as the classical model without non-local effects (\( \ell_\sigma = \ell_\varepsilon \geq 0 \)), and this versatility makes the hybrid gradient model particularly attractive.
4.5.3.4 Other non-local theories

It is possible to employ higher-order gradients to remove singularities or discontinuities; they can be used both to smoothen the heterogeneity that exists and, conversely, to introduce heterogeneity in the medium. The homogenisation procedures, where a discrete lattice is used to derive the higher-order terms, maintain a close link with the underlying microstructure and the new material parameters can be directly related to the microstructural properties [145]. In this case the equation that can be used to describe the gradient elasticity can be written as:

$$\sigma(x) = E[\varepsilon(x) + \ell^2 \varepsilon''(x)], \quad (4.19)$$

where $\sigma(x)$ are the stresses and $\varepsilon(x)$ the strains, and $\ell$ is the scale length. In this procedure, where the higher-order gradients are derived from a discrete lattice, the heterogeneity is introduced naturally and, differs from others methods where the higher-order terms are included with the purpose of smoothing heterogeneity. An equation that can be used to describe the second main group of gradient elasticity is:

$$\sigma(x) = E[\varepsilon(x) - \ell^2 \varepsilon''(x)]. \quad (4.20)$$

Eq. (4.19), which differs from Eq. (4.20) only for the negative sign, possesses excellent mathematical properties, but the link with the microstructure is less obvious. It should be noted that, because the inertia terms have been neglected and the higher-order gradients have been added on the constitutive equation, the dynamic equations including these methods are not exhaustive.

In order to study dispersive waves propagation, Metrikine and Askes [145, 146] proposed a dynamically consistent generalisation of the Eringen’s irreducible form obtained by a continualisation method, where the higher-order elasticity term is combined with a higher-order inertia term and therefore a new length scale parameter $\ell_m$ has been introduced to account the interaction among discrete and continuous models. Their proposed model assumes the following form:

$$\rho [\dddot{u}(x) - \ell_m^2 \dddot{u}''(x)] = E [\dddot{u}''(x) - \ell_e^2 \dddot{u}''''(x)], \quad (4.21)$$

where the length scale $\ell_m$ is related to the inertia gradient while $\ell_e$ is related to the strain gradients. The same model has been derived from microstructural considerations when the two length scale parameters are related to the size of the RVE [90].
4.5.4 Length scale parameters

All the gradient or non-local theories include one or more additional parameters in the governing equations, typically some characteristic length scales, here usually indicated with the letter $\ell$. These are related to the heterogeneity and the microstructure [19], as e.g. the inter-particle distance in a regular lattice material or the dimension of the RVE for a heterogeneous materials.

Indeed, the microstructural parameter estimation is still a complicated task and remains one of the main issues of gradient theories. One of the first attempts is the work of Yang and Leak [245], who experimentally studied the torsion and bending of elastic material with microstructure, in particular bones and polymeric foams; while Aifantis [7] and Vardoulakis et al. [228] have investigated the gradient microstructural coefficients for polymeric, biological and metallic materials as well as geomaterials.

The value of the gradient coefficients can be inferred from measurements of phenomena where size effects are relevant, as strain/stress fields of dislocation lines or crack tips. They can also be estimated from properly designed experiments [10]. Aifantis [5] and Murphy et al. [157] gave some results in case of non-local plasticity, valid also for elastic problem. Another estimation of the gradient coefficients can be obtained from the grain size of elasto-plastic polycristals [6]. Kioseoglou et al. [117] used a different approaches involving high-resolution optical methods for strain field in crack problem to relates the experimental results with the dislocation core and cohesive zone estimated using gradient theories. Le Bellégo et al. [126], instead, calibrated the non-local parameters from the size effect three point bend test on notched concrete beams.

In phenomena with relevant size effects the length scale parameters can also be obtained by fitting the experimental data with the gradient theory’s parameters, as in Tsagarakis and Aifantis [224] or Aifantis [7, 8]; as well as by comparison of the wave propagation curves of a gradient elasticity model with the classical dispersion relation of lattice dynamics [14].

Gutkin and Aifantis [97, 98], in order to study the screw and edge dislocations of a dipole, proposed a simple gradient elasticity theory able to eliminate the strain singularities at the crack tip and dislocations cores. In their study they related the lattice constant $\varsigma$ to the characteristic length scale $\ell$ of the gradient model using an empirical equations ($\ell = \sqrt{\varsigma/4}$).

The length scales can also be derived using atomistic or homogenization techniques, some examples can be found in the work in Fish et al. [82, 83] or Triantafyllidis and Aifantis [223]. Instead, Picu [173] have deduced the non-local coefficients from the weight function $\alpha(|x - \hat{x}|, \ell_\sigma)$ (appearing in Eq. (4.11)).
4. Phenomenological approaches

4.6 Beams with non-local elasticity theories

Although the vast majority of studies on non-local theories have been carried out on axially loaded bars and two-dimensional shell elements, an increasing attention has been recently devoted to non-local beams in bending, under static and dynamic loads. One of the main reasons is the development of micro or nano devices (e.g. carbon nanotubes), albeit it should be mentioned that the same governing equations are applied to a larger scale to take into account size effects as well as to study composite beams with interlayer slip (e.g. sandwich beams or multi-layered beams with shear connectors) or the two parameters elastic foundations beam [49].

4.6.1 Non-local (stress gradient) beam model

Peddisen et al. [171] formulated an EB beam theory using the stress gradient elasticity and derived the deflection function of a simply supported beam for four different load configurations. In their work, they assumed the following non-local constitutive equation, linking bending moment $M(x)$ and curvature $\chi(x)$:

$$M(x) - \ell_\sigma^2 M''(x) = EI \chi(x). \quad (4.22)$$

where $EI$ is the bending stiffness of the beam and $\ell_\sigma$ is an internal characteristic length. Eq. (4.22) has been obtained by multiplying both sides of non-local uniaxial constitutive equation (Eq. (4.12)), by the distance $z$ from the neutral axis (positive if downward, as displayed within Fig. 4.6), and integrating the result over the cross sectional area $A$ under the assumption that the non-local behaviour is neglected in the thickness direction. The bending moment $M(x)$ is defined as:

$$M(x) = \int_A \sigma(x, y, z) z \, dA, \quad (4.23)$$

and $\sigma(x, y, z)$ is the normal stress at a generic point $P$ of coordinates $P \equiv \{x, y, z\}$; while, under the kinematic EB assumption that the beam’s cross section remains plane, and normal to the deflection line and transverse normal stresses are negligible, axial strain $\varepsilon(x, y, z)$ and curvature $\chi(x)$ are related by the expression:

$$\varepsilon(x, y, z) = z \chi(x). \quad (4.24)$$
Figure 4.6: Sketch of a non-local EB beam representing a carbon nanotube, showing the Cartesian coordinate system.

Eq. (4.24) is consistent with the Cartesian reference system for the multi-cracked non-local beam depicted within Figure 4.6, in which: $x$ is the longitudinal axis of the member; $y$ is directed along the neutral axis; $z$ is orthogonal to the $xy$ plane and points downwards, forming a right-handed reference system ($xz$ is therefore the bending plane).

Eq. (4.22) can be rewritten in an equivalent differential equations, where the constitutive relation appears in terms of bending moment and a new kinematic quantity, the non-local curvature $\tilde{\chi}_{NL}(x)$, as follows:

\[
\begin{align*}
\chi(x) &= \tilde{\chi}_{NL}(x) - \ell^2 \tilde{\chi}_{NL}''(x); \\
M(x) &= EI \tilde{\chi}_{NL}(x),
\end{align*}
\]

alternatively the constitutive relation can appear in terms of non-local bending moment $\tilde{M}_{NL}(x)$ as follows:

\[
\begin{align*}
\tilde{M}_{NL}(x) &= M(x) - \ell^2 \sigma M''(x); \\
\tilde{M}_{NL}(x) &= EI \chi(x).
\end{align*}
\]

Eq. (4.25a) and Eq. (4.26a) show as the non-local curvature $\tilde{\chi}_{NL}(x)$ and the non-local bending moment $\tilde{M}_{NL}(x)$ are the spatial weighted averages \(^1\) of the local curvature $\chi(x)$ and the bending moment $M(x)$ respectively.

Interestingly, combining Eq. (4.22) with the equilibrium condition Eq. (3.1) leads to:

\[
M(x) = EI \chi(x) + \ell^2 q(x),
\]

\(^1\) The spatial weighted average is defined as the integral of the weighted function on the overall domain

\[
\tilde{f}(x) = \int_0^L G(x,y) f(y) \, dy
\]

where $G(x, y)$ is the Green’s function of the differential system where the usual BC’s are often assumed as $\chi'(0) = \chi'(L) = 0$. 
which clearly shows how the solution is obtained as sum of the classical solution and an additional term \( \ell_2^2 q(x) \) related with the load function \( q(x) \). A troubling feature resulting from Eq. (4.28) is that when \( M''(x) = 0 \), e.g. for beam under a point load, the non-local effect vanishes and the non-local constitutive equation is simplified to the local one.

A double derivation of Eq. (4.28) allows rewriting the equation in terms of transverse displacement \( u_b(x) \) and load function \( q(x) \) as follows:

\[
 u_b''''(x) = \left[q(x) - \ell_2^2 q''(x)\right] \frac{1}{EI}.
\]  

(4.29)

The so obtained differential equation has the same order of the local model and the non-local effect results as a spatial variation of the applied load.

### 4.6.1.1 Internal potential energy of the stress gradient beam

In the non-local elastic beam, the expression of the internal potential energy \( U_{NL} \) is assumed as integral over the domain of the product between local \( \chi(x) \) and non-local curvature \( \tilde{\chi}_{NL}(x) \) or between local curvature and non-local bending moment \( \tilde{M}_{NL}(x) \):

\[
 U_{NL} = \frac{1}{2} \int_0^L EI \chi(x) \tilde{\chi}_{NL}(x) \, dx = \frac{1}{2} \int_0^L \tilde{M}_{NL}(x) \chi(x) \, dx.
\]  

(4.30)

By considering the variation \( \delta \) of the potential energy \( U_{NL} \), one obtains:

\[
 \delta U_{NL} = \frac{1}{2} \int_0^L EI \left[ \tilde{\chi}_{NL}(x,t) \delta \chi(x,t) + \chi(x,t) \delta \tilde{\chi}_{NL}(x,t) \right] \, dx.
\]  

(4.31)

Integrating by parts the second term in the right-hand side of Eq. (4.31), the following expression is obtained:

\[
 \int_0^L \chi(x) \delta \tilde{\chi}_{NL}(x) \, dx = \int_0^L \left[ \tilde{\chi}_{NL}(x) - \ell_2^2 \tilde{\chi}_{NL}''(x) \right] \delta \tilde{\chi}_{NL}(x) \, dx \\
= \int_0^L \tilde{\chi}_{NL}(x) \delta \chi(x) \, dx - \ell_2^2 \left[ \tilde{\chi}_{NL}'(x) \delta \tilde{\chi}_{NL}(x) \right]_0^L - \ell_2^2 \left[ \tilde{\chi}_{NL}(x) \delta \tilde{\chi}_{NL}'(x) \right]_0^L.
\]  

(4.32)
where \( \delta \chi(x) = \delta \tilde{\chi}_{NL}(x) - \ell_0^2 \delta \tilde{\chi}_{NL}''(x) \). The variation of the potential energy then assumes the following form:

\[
\delta \mathcal{U}_{NL} = \int_0^L M(x) \delta \chi(x) \, dx - \frac{EI}{2} \ell_0^2 \left[ \tilde{\chi}'_{NL}(x) \tilde{\delta \chi}_{NL}(x) \right]_0^L + \frac{EI}{2} \ell_0^2 \left[ \tilde{\chi}_{NL}(x) \tilde{\delta \chi}'_{NL}(x) \right]_0^L .
\] (4.33)

The BCs can be obtained by imposing the null variation of the total potential energy:

\[
\delta \left[ \mathcal{U}_{NL} - \mathcal{W} \right] = 0 ,
\] (4.34)

where \( \mathcal{W} \) is the total work done by external forces:

\[
\delta \mathcal{W} = \int_0^L q(x) \delta u_b(x) \, dx - \left[ V(x) \delta u_b(x) \right]_0^L + \left[ M(x) \delta u'_b(x) \right]_0^L .
\] (4.35)

The BCs are usually assumed as null values of the derivative of the non-local curvature at the ends of the beam:

\[
\tilde{\chi}'_{NL}(0) = \tilde{\chi}'_{NL}(L) = 0 .
\] (4.36)

### 4.6.1.2 Others stress gradient beam models

Alternative interpretation of the stress gradient constitutive law has also been investigated, using a formulation similar to the strain gradient, which only differs for the sign on the gradient term [171]. The equation is obtained by expansion of the uniaxial integral constitutive equation of non-local elasticity for \( \ell_0^2 \ll 1 \) interrupted after the second terms [76].

\[
\sigma(x) = E \left[ \varepsilon(x) + \ell_0^2 \varepsilon''(x) \right] .
\] (4.37)

Eq. (4.37) substituted in the equilibrium equation and, after performing the integration over the cross section \( A \), leads to:

\[
M(x) = EI \left[ u''_b(x) + \ell_0^2 u'''_b(x) \right] ;
\] (4.38)

\[
u_b''''(x) + \ell_0^2 u_b''''(x) = \frac{q(x)}{EI} .
\] (4.39)

Differently form Eq. (4.29), Eq. (4.39) is of higher order than the local EB beam model and consequently it requires additional BCs, which are not physically obvious but can obtained...
using a variational approach. Between the proposed example, it is of particular interest the case of cantilever beam with point load at the free end, where the resulting non-local deflection is the same as the local EB beam. In Eq. (4.22) it can be observed that if $M''(x) = 0$, corresponding to concentrated load along the beam, the non-local effects vanish and then the non-local bending curvature reduce to the local one, loosing the capability to capture the small length scale effect [48].

Further attempt to include non-local effects into the beam model is the beam strain gradient formulation introduced by Papagyri-Besku et al. [167]. The year after the same authors extended the formulation to bending and stability analysis of gradient elastic beam including surface effects. The strain gradient formulation requires additional BCs, which despite the absence of a straightforward physical meaning, can be obtained from the variational principles. Their formulation can be viewed as a simplified version of Mindlin theory [147] adopting the same concepts, ideas and structures of surface energy but using only four elastic material constants (two classical $E$, $G$, and two non-classical $\ell$, $g$), instead of eighteen. The reduced number of parameters makes this theory more convenient than other theories as Cosserat brothers [60], which has eighteen constants or the micropolar elasticity of Eringen with six [71]. According to the one dimensional gradient theory with surface energy the internal potential energy $U$ is defined as:

$$U = \frac{1}{2} \int_0^L EI \left[ u''_b^2(x) + g^2 u''_b^2(x) + 2 \ell u''_b(x) u''_b(x) \right] dx$$

Instead, the external work is the sum of the work of the external forces $q(x)$, the work of the boundary shear forces $V(x)$, the boundary bending moment $M(x)$ and the non-local (double) moments $m(x)$ as follows:

$$\delta U = - \int_0^L q(x) \delta u_b(x) dx - \left[ V(x) \delta u_b(x) \right]_0^L + \left[ M(x) \delta u'_b(x) \right]_0^L + \left[ m(x) \delta u''_b(x) \right]_0^L.$$  (4.41)

Their main outcomes were a decreasing deflection for increasing values of the ratio $g/L$, as well as the surface energy effects has been found to only slightly influence the results and the amount depends on the non-classical BCs. The solution of the buckling problem shows increasing values for increasing gradient coefficient and the classical elasticity solution represents a lower bound [167].
4.6.1.3 Stress gradient Timoshenko beam theory

The Timoshenko beam theory relaxes the EB assumption that straight lines normal to the mid-plane remain straight and normal after deformation. Constant shear strain and shear stress are considered in the Timoshenko beam, and to do that a shear force factor has been introduced to obtain a zero shear stress at the beam-free surfaces. Even though the Timoshenko beam can be interpreted as a non-local theory (see Sec. 4.5.1.1), it is inadequate to capture the small-scale effect in the material mechanical properties. As an example, the microstructure has a significant influence on the flexural wave dispersion, and the decrease in phase velocities of wave propagation cannot be predicted by the classical beam theories [238]. Pioneered by Eringen, a considerable number of papers have been published on the non-local beam EB theory, although a consistent variational formulation of the governing equations and boundary conditions was not provided since Wang et al. [235].

The non-local Timoshenko beam has been studied by a number of authors. Studies on the inclusion of stress gradient into the Timoshenko theory can be found in [137, 240, 237, 236, 239, 114]. Reddy [183] provided the static, buckling and vibration analytical solution of a simply supported beam with Eringen’s gradient theory, Eq. (4.11), for four different beam models (EB; Timoshenko; Reddy; and Levinson), where displacements, rotations and loads have been expressed using the following series expansions:

\[
\begin{align*}
    u_b(x) &= \sum_{n=1}^{\infty} U_n \sin \frac{n \pi x}{L} e^{i \omega t}; \\
    \varphi(x) &= \sum_{n=1}^{\infty} \varphi_n \cos \frac{n \pi x}{L} e^{i \omega t}; \\
    q(x) &= \sum_{n=1}^{\infty} Q_n \sin \frac{n \pi x}{L}.
\end{align*}
\]

One of the main outcomes of this work was to find out that the introduction of the non-local elasticity parameters increases the magnitude of deflections and decreases buckling loads and natural frequencies; it also evidenced that non-local effects are different and potentially higher than the introduction of a shear concentration factor. Wang et al. in 2006 [237] studied a Timoshenko beam with stress-gradient, where they assumed the non-local effects only for the normal stress and strain while no non-local effects where introduced in the shear constitutive law because the Eringen’s non-local constitutive model has been assumed not to be valid for the z-direction, as the beam theory is a one-dimensional problem and the cross section repres-
ents a point value in the formulations. Interesting outcomes was on the buckling load of the Timoshenko beam: that is, if \( \ell^2_\sigma \gg \sqrt{\frac{EI}{\kappa_s G_{\chi}}} \), the small scale effect is dominant and therefore the shear deformation can be neglected; the opposite happen if \( \ell^2_\sigma \ll \sqrt{\frac{EI}{\kappa_s G_{\chi}}} \). Wang and Varadan [240] found out as the scale effect is evident only for nano-structures with the size of nano-meter with highest effect at the location of the point force. They have shown as non-local Timoshenko theory compared to the non-local EB beam introduce additional scale effect only for statically indeterminate structure.

Different hypotheses as been taken by Ke et al. [114], who considered the stress gradients for the shear stress \( \sigma_{xz}(x, y, z) \) as:

\[
\sigma_{xz}(x, y, z) - \ell^2_\sigma \sigma''_{xz}(x, y, z) = G \gamma_{xz}(x, y, z),
\]

(4.43)

where \( \gamma_{xz}(x, y, z) \) is the shear strain; while \( G \) is the shear modulus. They derived the governing equations using the Hamilton’s principle:

\[
\int_0^l \left[ \delta U(t) - \delta W(t) \right] dt = 0,
\]

(4.44)

which were then solved using the differential quadrature method, showing the effects of the non-local parameter on the natural frequencies.

In this thesis, for the first time an attempt is made to include hybrid non-local theory into the multicrocked beam model. Due to the mathematical difficulties of combining non-local gradient elasticity and concentrate damage, only the EB theory will be considered in the following, so to keep the equations as simple as possible. The extension to the Timoshenko theory could represent one of the main future developments.

### 4.6.2 Strain gradient elastic beam model

The strain gradient elastic beam model obtained is somehow alternative to the stress-gradient theory. Moving from Aifantis’s strain gradient constitutive equation, Eq. (4.13), the same procedure as in the previous subsection can be followed leading to:

\[
\Upsilon_{I0} M(x) = \chi(x) - \ell^2_\epsilon \chi''(x),
\]

(4.45)
where $\Upsilon_0 = \frac{1}{EI}$ is the bending flexibility of the undamaged beam. Eq. (4.45) can be rewritten as:

\[
\begin{align*}
\tilde{\chi}_{GR}(x) & = \chi(x) - \ell_\varepsilon^2 \chi''(x); \\
M(x) & = EI \tilde{\chi}_{GR}(x),
\end{align*}
\]

where $\tilde{\chi}_{GR}(x)$ is the non-local curvature for the strain gradient beam model. It can be noticed that in Eq. (4.46a) the non-local curvature is defined as difference among the local curvature $\chi(x)$ and a quantity proportional to its gradient $\chi''(x)$, unlike Eq. (4.25a) where the local curvature is defined as the difference of the non-local terms.

### 4.6.2.1 Internal potential energy of the strain gradient beam

The internal potential energy for the strain gradient beam model assumes the following form:

\[
U_{GR} = \frac{1}{2} \int_0^L EI \left[ \chi^2(x) + \ell_\varepsilon^2 \chi''^2(x) \right] \, dx.
\]

(4.47)

By considering the variation of the potential energy $U$ and then integrating by parts the following expression is obtained:

\[
\delta U_{GR} = \int_0^L M(x) \delta \chi(x) \, dx + EI \ell_\varepsilon^2 [V(x) \delta \chi(x)]_0^L,
\]

with $M(x) = EI \chi(x)$; while $V(x) = EI \chi'(x)$. Indeed, two terms valued at the end of the beam $[\chi'(x) \delta \chi(x)]_0^L$ have to be null, in so defining the two additional BCs $\chi'(0) = \chi'(L) = 0$.

### 4.6.3 Hybrid non-local beam model

Eringen theory has been shown to lose the non-locality effects in some situations, where the solutions assume the same values as the classical local case, e.g. an EB cantilever beam with a point load at the free end [76].

This paradox has been overcome by Challamel and Wang [234] using an hybrid gradient elastic model or integral non-local elastic models, which combines the local and non-local curvatures using two regularizing positive independent length-scale parameters $\{\ell_\sigma; \ell_\varepsilon\}$, leading to a distinct non-local elasticity effect in the static response. In the proposed model, the hybrid non-local beam equation is obtained starting from the hybrid constitutive model [19].
where both stress and strain gradient terms are introduced, along with the corresponding microstructural parameters $\ell_\sigma$ and $\ell_\varepsilon$. Following the same approach, as in the previous cases, leads to:

$$\Upsilon_{I0} [M(x) - \ell_\sigma^2 M(x)'''] = \chi(x) - \ell_\varepsilon^2 \chi''(x),$$

(4.49)

where $\Upsilon_{I0}$ is once again the bending flexibility of the undamaged beam. The so defined hybrid non-local model can be rewritten in an equivalent differential equations system:

$$\begin{cases}
\Upsilon_{I0} M(x) = \tilde{\chi}(x) - \ell_\varepsilon^2 \tilde{\chi}''(x); \\
\chi(x) = \tilde{\chi}(x) - \ell_\sigma^2 \tilde{\chi}''(x),
\end{cases}$$

(4.50a)

(4.50b)

where $\tilde{\chi}(x)$ is the effective non-local curvature, defined as the spatial weight average of the local curvature $\chi(x)$ [234, 250]. Eq. (4.50a) is formally equivalent to the constitutive law of a slender beam in bending with gradient strain, in which however the effective curvature $\tilde{\chi}(x)$ replaces $\chi(x)$ [9].

Challamel and Wang [234] investigated the influences of different microstructural parameters and boundary conditions using the hybrid non-local model; while Zhang et al. [250] studied the bending, buckling and vibration of EB nanobeams, providing the closed form solution for the hybrid Euler beam under uniform load and simply supported, clamped-clamped and cantilever beams. The solutions show that the length scales in the hybrid model affect the deflection in all the cases of study. As expected, in statically determined structures, like a cantilever beam or a simply supported beam, the bending moment and shear force are the same of the local solution. Buckling solutions of non-local beam are also investigated and the buckling loads for three different boundary conditions configuration (pinned-pinned, clamped-clamped and clamped-free) are proposed. Furthermore, the governing equation for free hybrid non-local beam vibrations is solved and the frequencies of the non-local beam are showed.

### 4.6.3.1 Internal energy for the hybrid non-local beam model

Challamel et al. [234] proposed an internal potential energy for the hybrid non-local model where an additional term $\ell_\varepsilon^2 \chi'(x) \tilde{\chi}'_H(x)$ is introduced into the non-local energy potential of Eq. (4.30) as follows:

$$\mathcal{U}_H = \frac{1}{2} \int_0^L EI \left[ \chi(x) \tilde{\chi}(x) + \ell_\varepsilon^2 \chi'(x) \tilde{\chi}'(x) \right] dx.$$  

(4.51)
After a double integration by parts, the equation for the variation of the potential energy assumes the following form:

$$\delta U_H = \int_0^L M(x) \delta \chi(x) \, dx + EI \frac{\ell}{\ell_s^2} \left[ \frac{\tilde{\chi}''(x)}{\ell_s} \delta \chi(x) \right]_0^L +$$

$$- \frac{EI}{2} \frac{\ell}{\ell_s^2} \left\{ \left[ \frac{\tilde{\chi}'(x)}{\ell_s} \delta \chi(x) \right]_0^L - \left[ \frac{\tilde{\chi}'(x)}{\ell_s} \delta \chi(x) \right]_0^L \right\} +$$

$$+ \frac{EI}{2} \frac{\ell}{\ell_s^2} \frac{\ell}{\ell_s^2} \left\{ \left[ \frac{\tilde{\chi}''(x)}{\ell_s} \delta \chi(x) \right]_0^L - \left[ \frac{\tilde{\chi}''(x)}{\ell_s} \delta \chi(x) \right]_0^L \right\},$$

with $M(x) = EI \left[ \tilde{\chi}(x) - \ell_s^2 \tilde{\chi}''(x) \right]$ and $\tilde{\chi}'(0) = \tilde{\chi}'(L) = 0$ are the BCs derived from the variational principles.

Eq. (4.52) can be rewritten as combination of local and non-local curvature as:

$$M(x) = EI \left[ \left( 1 - \frac{\ell_s^2}{\ell_s^2} \right) \tilde{\chi}_H(x) + \frac{\ell_s^2}{\ell_s^2} \chi(x) \right].$$

Further interpretation brings to the following equation:

$$M(x) - \ell_s^2 M''(x) = EI \left[ \chi(x) - \ell_s^2 \chi''(x) \right].$$

### 4.6.3.2 Analogy with others beam theories

Interestingly, the so defined hybrid non-local constitutive model Eq. (4.49) has the same form of the models used to study totally different problems [49, 63, 50] e.g. the beam deflection of composite beam with interlayer slip or the two parameters elastic foundation beam [115];

**Composite beam with interlayer slip.** Composite beam with interlayer slip represents a class of structures with applications in many engineering fields, as for example layered beams (i.e. with a mechanical joint), shear connections (lap joints mechanically connected or using adhesive) and sandwich constructions with a weak shear core [89, 49].

The differential equations system for the partially composite beam with interlayer slip (or sandwich beam), as represented in Fig. 4.7, subjected to a uniform bending moment assumes...
4. Phenomenological approaches

Figure 4.7: Geometric parameters of the composite beam

The following form [49]:

\[
\begin{cases}
EI u''''(x) - b h_0 \tau'(x) - q(x) = 0; \\
N_1'(x) + b \tau(x) = 0; \\
N_2'(x) - b \tau(x) = 0.
\end{cases}
\] (4.55)

while the corresponding BCs are:

\[
\begin{cases}
\left[(EI_0 u'''_b(x) + b h_0 \tau(x)) \delta u_b\right]_0^L = 0; \\
\left[(EI_0 u''_b(x) + M_0) \delta u_b\right]_0^L = 0; \\
[N_1(x) \delta u_b]_0^L = 0; \\
[N_2(x) \delta u_b]_0^L = 0.
\end{cases}
\] (4.56)

where the subscripts “1” and “2” identify the two sub-elements of different geometry and materials; \( b \) is the width of the beam; \( e \) is the core/interlayer thickness; \( G \) is the shear modulus of the interlayer/core; \( N_1 = E_1 A_1 u_{b1}(x), N_2 = E_2 A_2 u_{b2}(x) \) and \( \tau(x) = G \gamma(x) \) are the constitutive equations of each sub-element identity; while \( \gamma(x) = \frac{u_{b2}(x) - u_{b1}(x)}{e} + \frac{h_0}{e} u'_b(x) \). In the proposed formulation, the centre of gravity of the fully composite section \((c_{grc})\) identifies the centre of the coordinate system \((u_b, y)\). Eq. (4.55) has been derived using the variational method where the transversal partial interaction has been neglected. As consequence the flexible connection brings only shear interaction between the two sub-elements and then they can be assumed to have the same curvature [49].

For the case of clamped beam with uniform bending moment the BCs are: \( u_b(0) = u_b(L) = N(0) = N(0) = 0; \) and \( EI u''_b(x) = -M(x); \) the differential equations system Eq. (4.55),
written in terms of curvature \( \chi(x) = -u''_b(x) \), assumes the following form:

\[
\frac{\alpha_T^2}{EI_\infty} M(x) - \frac{1}{EI_0} M''(x) = \alpha_T^2 \chi(x) - \chi''(x),
\]

(4.57)

where \( M''(x) = -q(x) \) while \( \alpha_T = K_T \left( \frac{1}{E_1 A_1} + \frac{1}{E_2 A_2} + \frac{h_0^2}{EI_0} \right) \) relates the each sandwiches beam stiffness with the constant slip modulus \( K_T = \frac{b G}{e} \) of the partially composite beam.

Eq. (4.57) is analogue to the hybrid non-local theory of Eq. (4.49) where the beam model is a combination of non-local constitutive model with a local model:

\[
M(x) - \ell^2 \chi''(x) = EI_\infty \left[ \chi(x) - \ell^2 \chi''(x) \right],
\]

(4.58)

The equivalence among the parameters is as follow:

\[
\begin{cases}
\ell_\varepsilon = \frac{1}{\alpha_T}; \\
\ell_\sigma = \frac{1}{\alpha_T} \sqrt{\frac{EI_\infty}{EI_0}}.
\end{cases}
\]

(4.59)

Interestingly, in composite beams the stress distribution is statically indeterminate. As a consequence the partially composite beam can be solved if the global equation are known, while cannot be solved with only the local constitutive equations. Although the differential governing equation of the composite beam and gradient elasticity beam are analogue, the BCs for the two models are different. For the composite beam the BCs are those on Eqs. (4.56).

**Two parameters elastic foundation beam.** Further analogy with the hybrid non-local model can be found in models of elastic foundation as for example the Reissner foundation model [185] which differential equation assumes the following form:

\[
p(x) - \ell_f^2 p''(x) = k_0 \left[ w(x) - a_f^2 w''(x) \right],
\]

(4.60)

where \( p(x) \) is the foundation reaction and \( w(x) \) is the foundation deflection; while \( \ell_f \) and \( a_f \) are two constants related with the axial \( k_0 \) and shear stiffness \( k_{c0} \) of the spring layers. Indeed, the two parameter Pasternak foundation model [169, 214, 252] is a special case of the Reissner model obtained when \( \ell_f = 0 \) (Fig. 4.8).
4. Phenomenological approaches

4.6.4 Summary of the gradient beam theories

Table 4.2 summarises the main characteristics, internal potential energy and non-local curvature definitions, of the three non-local models described in the previous Subsections.

<table>
<thead>
<tr>
<th>Model</th>
<th>Internal potential energy $U$</th>
<th>Non-local curvature definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-local elastic beam (NL)</td>
<td>$\frac{1}{2} \int_0^L EI \chi \tilde{\chi}_{NL} , dx$</td>
<td>$\chi = \tilde{\chi}<em>{NL} - \ell_s^2 \tilde{\chi}''</em>{NL}$</td>
</tr>
<tr>
<td>Gradient elastic beam (GR)</td>
<td>$\frac{1}{2} \int_0^L EI \left( \chi^2 + \ell_s^2 \chi''^2 \right) , dx$</td>
<td>$\tilde{\chi}_{GR} = \chi - \ell_s^2 \chi''$</td>
</tr>
<tr>
<td>Hybrid non-local beam (H)</td>
<td>$\frac{1}{2} \int_0^L EI \left( \chi \tilde{\chi}_H + \ell_s^2 \chi' \tilde{\chi}_H' \right) , dx$</td>
<td>$\chi = \tilde{\chi}_H - \ell_s^2 \tilde{\chi}_H''$</td>
</tr>
</tbody>
</table>

Table 4.2: Internal potential energy and non-local curvature definition for the three investigated higher-order beam formulations

4.7 Reasons of the proposed model

In order to provide a model able to better represent the behaviour of the beam around the crack, and to take into account the distortion of the stress/strain field around the crack as well as to take into account the microstructure of the material without introducing any additional crack-related parameters a phenomenological approach based upon a convenient form of non-local elasticity theory has been proposed in this thesis.

Comparatively, only a few authors have attempted the analysis of cracked beams considering non-local elastic models and, to the best of the author’s knowledge, the problem of a gradient elastic beam with an arbitrary number of cracks has not been treated yet. Loya et al. [136] have studied the free flexural vibration of cracked EB nano-beams including Eringen’s non-local elasticity model to take into account the size-dependant effects. In their model, the beam has been splitted at the cracks sections, defining different undamaged segments, which
have been linked together at the cracked section using a simple rotational spring to simulate the presence of the crack. The extension of this formulation to the case of Timoshenko’s nano-beams has been proposed by Torabi and Dastgerdi [220] a few years later.

It should be noted here that the adopted Eringen’s model, although leading to less cumbersome problems (i.e. the order of governing equations does not increase), can overlook size-dependent effects in some situations (e.g. the “cantilever paradox” highlighted by Challamel and Wang [234]), and for this reason a more versatile gradient elasticity model is desirable. This is the case of the so-called “hybrid” formulation of gradient elasticity, which effectively combines and generalises the two theories of Eringen and Aifantis [19, 234, 250], using two length scale parameters for stress and strain, both related to the microstructure.

Motivated by the above considerations and aimed at addressing an existing gap in the dedicated literature, a new mathematical formulation is proposed in Chapter 5, which allows deriving exact closed-form solutions for the linear static analysis of multi-cracked gradient-elastic slender beams in flexure. The proposed approach builds upon the hybrid stress-strain gradient-elastic constitutive law (which allows recovering classical, Eringen’s and Aifantis’ theories as particular cases), and embeds the “flexibility crack” model [162, 68] (equivalent to the classical DS model) to represent the localised increase in the bending flexibility through a convenient Dirac’s delta function at the location of each crack. The solutions so obtained account for the microstructural effects, and show a smoother profile of rotations where the crack occurs. The extension of this model to dynamic problems is presented in Chapter 6.
The multi-cracked beam model presented in Section 3.1 focuses on the increased curvature at a single abscissa, meaning that a jump discontinuity appears in the rotations’ profile. In this Chapter, in order to overcome this effect, a simple yet effective model is proposed for the non-local curvature of multi-cracked beams using the hybrid non-local elasticity theory, and the resulting higher-order equations for the bending deflections are then solved in a general exact form via the Laplace’s transformation method. This study presents an improved representation of cracked slender beams, based on a general class of gradient elasticity with both stress and strain gradient, which allows to include both static and dynamic gradient models as well as smoothing the singularities in the flexibility crack model. Two numerical applications to statically determinate and indeterminate beams are presented and thoroughly discussed within Section 5.2, which allow highlighting the effects of the microstructural parameters. All the numerical results are also compared with those obtained by using the classical elasticity theory.
5. Non-local multi-damaged beam

Figure 5.1: EB multicracked beam with microstructure representing a carbon nanotube with localised defects

5.1 Proposed multi-cracked hybrid non-local EB beam

In the proposed model, in order to take into account the presence of cracks along the beam, the hybrid non-local beam model of Eq.(4.50a) is here represented as follows:

\[
\begin{align*}
\Upsilon_I^0 \left(1 + F_\beta(x)\right) M(x) &= \tilde{\chi}(x) - \ell_2 \tilde{\chi}''(x) ; \\
\chi(x) &= \tilde{\chi}(x) - \ell_2 \tilde{\chi}''(x),
\end{align*}
\]

(5.1a)

(5.1b)

where \(\Upsilon_I^0\) is the bending flexibility of the undamaged section; while the function \(F_\beta(x)\) is the same as defined in Eq. (3.9b) and describes mathematically the localised effect of the damages (see Fig. 5.1). Noteworthy, differently from Eq.(4.50a) the bending flexibility is now a function of the abscissa \(x\).

Similar to the case of the classical elasticity, the static response of a multi-cracked EB beam, equipped with the hybrid gradient elasticity and subjected to transverse loads, can be obtained by deriving the differential equation ruling the problem and applying the pertinent BCs. Combining equilibrium Eqs. (3.44a) and the constitutive Eq. (5.1a), yields:

\[
\begin{align*}
\Upsilon_I^0 [1 + F_\beta(x)] \left[q^{[2]}(x) + C_2 + C_3 x\right] &= \tilde{\chi}(x) - \ell_2^2 \tilde{\chi}''(x) ; \\
\chi(x) &= \tilde{\chi}(x) - \ell_2^2 \tilde{\chi}''(x),
\end{align*}
\]

(5.2a)

(5.2b)

where \(C_2\) and \(C_3\) are the two integration constants. Eq. (5.2a) can be solved for the effective non-local curvature \(\tilde{\chi}(x)\) by imposing the BCs \(\tilde{\chi}'(0) = 0\) and \(\tilde{\chi}'(L) = 0\), i.e. by assuming the stationarity of the non-local curvature at the beam’s ends, a condition which can be rigorously derived from variational principles [250]. Eq. (5.2b) allows then evaluating the local curvature.
\( \chi(x) \), whose integration delivers the rotation of the beam’s cross section:

\[
\varphi(x) = \chi^{[1]}(x) + C_1 = \int_0^x \chi(\xi) \, d\xi + C_1 ;
\] (5.3)

while in turn the transverse displacement is obtained by integrating the rotation:

\[
u_b(x) = -\varphi^{[1]}(x) + C_0 = -\int_0^x \varphi(\xi) \, d\xi + C_0 .
\] (5.4)

As in the classical elasticity theory, the two integration constants \( C_0 \) and \( C_1 \) are equivalent to \( u_b(0) \) and \( \varphi(0) \), respectively, and can be evaluated by imposing the BCs.

It should be noted, however, that differently from the classical elasticity theory, six BCs are required overall for the hybrid gradient-elastic model: four BCs are needed to evaluate the four integration constants \( (C_0, C_1, C_2 \text{and} C_3) \) associated with displacement, rotation, bending moment and shear force (see Eqs. (3.44), (5.3) and (5.4)); two additional conditions \( \tilde{\chi}'(0) = 0 \) and \( \tilde{\chi}'(L) = 0 \) are also required while integrating Eq. (5.2a).

### 5.1.1 Exact solutions

In order to derive the mathematical solution for the problem in hand, let us apply the Laplace’s transform operator \( \mathcal{L}\langle \cdot \rangle \) to both sides of Eq. (5.2a), which yields:

\[
\mathcal{L}\langle \tilde{\chi}(x) \rangle \left( 1 - s^2 \ell_\varepsilon^2 \right) + \ell_\varepsilon^2 \left( C_5 + C_4 s \right) = \mathcal{L}\langle q^{[2]}(x) \rangle + \left[ \sum_{j=1}^{n} \beta_j e^{-s\bar{x}_j} \left[ q^{[2]}(\bar{x}_j) + C_2 + C_3 \bar{x}_j \right] \right] .
\] (5.5)

where \( s \) is the Laplace’s variable associated with the abscissa \( x \); while \( C_4 \) and \( C_5 \) are the two additional integration constants, which do not appear with the classical elasticity theory. Taking now the inverse Laplace’s transform of Eq. (5.5), we obtain the closed-form expression for the effective non-local curvature:

\[
\tilde{\chi}(x) = \tilde{\chi}_0(x) + \sum_{j=1}^{n} \Delta \tilde{\chi}_i(x) ,
\] (5.6)
in which $\tilde{\chi}_0(x)$ is the contribution due to the undamaged response:

$$
\tilde{\chi}_0(x) = C_4 \cosh \left( \frac{x}{\ell_\varepsilon} \right) + C_5 \sinh \left( \frac{x}{\ell_\varepsilon} \right) \ell_\varepsilon +
+ Y_1 0 \left\{ \left[ 1 - \cosh \left( \frac{x}{\ell_\varepsilon} \right) \right] C_2 + \left[ x - \sinh \left( \frac{x}{\ell_\varepsilon} \right) \ell_\varepsilon \right] C_3 + Q(x) \right\}; \tag{5.7}
$$

while $\Delta \tilde{\chi}_i(x)$ is the individual term associated to the $i$th damage:

$$
\Delta \tilde{\chi}_i(x) = Y_1 0 \frac{\beta_j}{\ell_\varepsilon} \sinh \left( \frac{x - \bar{x}_j}{\ell_\varepsilon} \right) \left[ q^{[2]}(\bar{x}_j) + C_2 + C_3 \bar{x}_j \right] H(x - \bar{x}_j). \tag{5.8}
$$

The external load $q(x)$ appears indirectly in both contributions, namely through the second anti-derivative $q^{[2]}(x)$ evaluated at $x = \bar{x}_j$ in Eq. (5.8) and through the associated transformed function $Q(x)$ in Eq. (5.7), being:

$$
Q(x) = \mathcal{L}^{-1} \left\{ \mathcal{L} \left\langle q^{[2]}(x) \right\rangle \right\}. \tag{5.9}
$$

Combining Eqs. (5.6) to (5.9) reveals that the sought function $\tilde{\chi}(x)$ depends also on the flexural stiffness of the undamaged cross section $EI_0$, the strain length scale parameter $\ell_\varepsilon$, position $\bar{x}_j$ and severity parameter $\beta_j$ of the generic damage, along with the four integration constants $C_2, C_3, C_4$ and $C_5$. The stress length scale parameter $(\ell_\sigma)$ is introduced in the solution when the local curvature $\chi(x)$ is evaluated (see Eq. (5.2b)), while the other two integration constants $C_0$ and $C_1$ come out from the double integration of $\chi(x)$ to get the transverse displacement (see Eq. (5.3) and Eq. (5.4)).

Overall, only six integration constants are required, without any further BC needed where the crack is located. Importantly, for statically determinate beams, these six integration constants do not have to be evaluated altogether, that is:

- $C_2$ and $C_3$ only depend on the equilibrium conditions (see Eq. (3.44));
- once $C_2$ and $C_3$ are known, Eq. (5.6) allows evaluating $C_4$ and $C_5$, imposing that $\tilde{\chi}'(x) = 0$ at the two ends of the beam (the resulting expressions are offered within Section 5.1.2 as Eqs. (5.14) and (5.13), respectively);
- $C_0$ and $C_1$ then depend on compatibility considerations only (through Eq. (5.3) and Eq. (5.4)).
5.1 Proposed multi-cracked hybrid non-local EB beam

Table 5.1: Solution procedure

1. Collect all the data for the beam and load (namely, \(E, I_0, L, n, \bar{x}_j, \beta_j, q(x)\), classical BCs)

2. Evaluate:
   (a) \(q^2(x) = \int dx f^2 q(x) dx\)
   (b) \(Q(x)\) via Eq. (5.9)

3. Is the beam statically determinate?
   (a) Yes, then go to point 4
   (b) No, then go to point 8

4. Evaluate the integration constants \(C_2\) and \(C_3\) from equilibrium equations:
   (a) \(C_2 = M(0)\)
   (b) \(C_3 = V(0)\)

5. Evaluate the integration constants \(C_4\) and \(C_5\) by setting \(\tilde{\chi}'(0) = 0\) and \(\tilde{\chi}'(L) = 0\)

6. Evaluate \(\chi(x)\) via Eq. (5.2b), and then \(\varphi(x)\) and \(u(x)\) via Eq. (5.3) and (5.4), respectively

7. Evaluate the integration constants \(C_0\) and \(C_1\) based on the kinematic constraints imposed by the BCs (e.g. for a cantilever beam fixed at \(x = 0\), \(C_0 = C_1 = 0\)).

8. Evaluate the six integration constants by applying simultaneously the six BCs.

On the contrary, this sort of cascade approach for the BCs cannot be used for statically indeterminate beams and the six integration constants have to be evaluated altogether, i.e. by solving a set of six algebraic equations with six unknown variables.

It then follows that, for both statically determinate and indeterminate beams, well known mathematical tools such as the Laplace’s transform and the inverse Laplace’s transform suffice to evaluate the analytical solution for multi-cracked slender beams with hybrid gradient elasticity, provided that the function \(Q(x)\) of Eq. (5.9) can be computed in closed form. It may be worth stressing here that Eqs. (3.44), introduced for the classical elasticity theory, are valid also for the hybrid model of gradient elasticity; moreover Eqs. (3.27) and (3.40) can be also used, although the terms in the right-hand sides are different (See Section 5.1.3, for \(q(x) = q_0\)) and Section 5.1.4, for \(q(x) = q_0 \delta(x - x_k)\)). The complete procedure to obtain the static response of a slender multi-damaged EB beam is summarised in Table 5.1.

For illustration purposes, two numerical applications of the proposed approach are presented and discussed in the next section, while Section 5.1.3 provides the full mathematical solution in the case where \(q(x) = q_0\) (i.e. uniformly distributed transverse load) and Section 5.1.3 provides the full mathematical solution in case of point load where \(q(x) = q_0 \delta(x - x_k)\).
5. Non-local multi-damaged beam

It may be worth emphasising here that the proposed approach allows, for the first time in the technical literature, to solve the static problem for cracked gradient-elastic slender beams in bending, independently of number and location of cracks, and adopting two microstructural parameters of stress gradient and strain gradient.

5.1.2 Integration constants $C_4$ and $C_5$

This section offers the closed-form expressions for the additional integration constants $C_4$ and $C_5$, which appears in the solution of the multi-damaged EB beam when the hybrid gradient-elastic constitutive law is adopted.

To derive such expressions, the BCs $\tilde{\chi}'(0) = 0$ and $\tilde{\chi}'(L) = 0$ (i.e. stationary conditions at the beam’s ends) have to be imposed to Eq. (5.2a). Eqs. (5.6) to (5.8) provide the general solution for this equation and, taking the derivative with respect to the abscissa $x$, the rate of variation of the effective non-local curvature can be expressed as:

$$\tilde{\chi}'(x) = \tilde{\chi}'_0(x) + \sum_{j=1}^{n} \Delta \tilde{\chi}'_j(x), \quad (5.10)$$

where:

$$\tilde{\chi}'_0(x) = \frac{C_4}{\ell_e} \sinh \left( \frac{x}{\ell_e} \right) + C_5 \cosh \left( \frac{x}{\ell_e} \right)$$

$$+ \mathcal{T}_{10} \left\{ C_3 \left[ 1 - \cosh \left( \frac{x}{\ell_e} \right) \right] - \frac{C_2}{\ell_e} \sinh \left( \frac{x}{\ell_e} \right) + Q'(x) \right\}; \quad (5.11)$$

$$\Delta \tilde{\chi}'_j(x) = -\mathcal{T}_{10} \beta_j \left[ C_3 \bar{x}_j + C_2 + q^{[2]}(\bar{x}_j) \right]$$

$$\times \left[ \frac{1}{\ell_e^2} \cosh \left( \frac{x - \bar{x}_j}{\ell_e} \right) H(x - \bar{x}_j) + \frac{1}{\ell_e} \sinh \left( \frac{x - \bar{x}_j}{\ell_e} \right) \delta(x - \bar{x}_j) \right]. \quad (5.12)$$

Combining the expressions above and enforcing the BCs yields in closed form to the sought integration constants $C_4$ and $C_5$:

$$C_5 = -\mathcal{T}_{10} Q'(0); \quad (5.13)$$
5.1 Proposed multi-cracked hybrid non-local EB beam

\[ C_4 = \ell \left[ \frac{C_2}{\ell} + C_3 \tanh \left( \frac{L}{2\ell} \right) + Q'(0) \coth \left( \frac{L}{\ell} \right) \right. \]
\[
- Q'(L) \csch \left( \frac{L}{\ell} \right) + \csch \left( \frac{L}{\ell} \right) \sum_{j=1}^{n} \Delta \tilde{\chi}'_j(L) \Bigg] .
\]

(5.14)

It follows that the constant \( C_4 \) depends on the other two integration constants \( C_2 \) and \( C_3 \), which represent the bending moment \( M(0) \) and the shear force \( V(0) \) at the left end of the beam, respectively. As already pointed out in Subsection 5.1.1, these values can be then evaluated using equilibrium equations only for statically determinate beams, while they are appear to be fully coupled with all the other integration constants when the beam is statically indeterminate.

5.1.3 Solution for uniformly distributed load (UDL)

Aim of this section is to provide, as a way of example, the exact closed-form mathematical expressions of curvature \( \chi(x) \), rotation \( \varphi(x) \) and displacement \( u_b(x) \) functions for a multi-cracked EB beam under the UDL \( q_0 \).

To derive the sought solution, Eq. (5.7) and (5.8) can be particularised for the load case \( q(x) = q_0 \):

\[
\tilde{\chi}_0(x) = \tau_0 C_2 + \tau_0 C_3 \left[ \frac{x}{\ell} + \text{sech} \left( \frac{L}{2\ell} \right) \sinh \left( \frac{L - 2x}{2\ell} \right) \right]
\]
\[
+ \tau_0 \frac{q_0 \ell^2}{2} \left\{ \left( \frac{x}{\ell} \right)^2 + 2 \left[ 1 - \frac{L}{\ell} \csch \left( \frac{L}{\ell} \right) \cosh \left( \frac{x}{\ell} \right) \right] \right\} ;
\]

(5.15a)

\[
\Delta \tilde{\chi}_j(x) = \tau_0 \left[ \frac{\beta_j}{\ell} \left( C_2 + C_3 \bar{x}_j + \frac{q_0 \bar{x}_j^2}{2} \right) \right]
\]
\[
\times \left[ \csch \left( \frac{L}{\ell} \right) \cosh \left( \frac{x}{\ell} \right) \cosh \left( \frac{L - \bar{x}_j}{\ell} \right) \right]
\]
\[
- \sinh \left( \frac{x - \bar{x}_j}{\ell} \right) H(x - \bar{x}_j) \right] ,
\]

(5.15b)

in which it has been taken into account that for the UDL \( q_0 \): i) the second anti-derivative is \( q^{[2]}(x) = q_0 x^2 / 2 \); and ii) the function \( Q(x) \), defined by Eq. (5.9), takes the expression:

\[
Q(x) = \frac{q_0 \ell^2}{2} \left\{ \left( \frac{x}{\ell} \right)^2 + 2 \left[ 1 - \cosh \left( \frac{x}{\ell} \right) \right] \right\} .
\]

(5.16)
According to Eq. (5.6), the effective non-local curvature is given by the superposition of the \( n + 1 \) contributions of Eqs. (5.15), and the local curvature \( \chi(x) = -u''_\ell(x) \) can be successively obtained using Eq. (5.2b), which can be applied to the undamaged contribution \( \chi_0(x) \) and to each of the additional terms associated with the concentrated damages \( \Delta \chi_j(x) \). The local curvature can be then written in a form similar to Eq. (5.6):

\[
\chi(x) = \chi_0(x) + \sum_{j=1}^{n} \Delta \chi_j(x),
\]

(5.17)

where:

\[
\chi_0(x) = \bar{\chi}_0(x) - \ell_\sigma^2 \tilde{\chi}_0''(x) = \text{\( Y \)}_{10} \left( C_2 + C_3 x + \frac{q_0 \bar{x}_j^2}{2} \right) + \text{\( Y \)}_{10} (\ell_\varepsilon^2 - \ell_\sigma^2) \left\{ \frac{C_3}{\ell_\varepsilon} \text{sech} \left( \frac{L}{2 \ell_\varepsilon} \right) \sinh \left( \frac{L - 2x}{2 \ell_\varepsilon} \right) \right. \\
+ \left. q_0 \left[ 1 - \frac{L}{\ell_\varepsilon} \cosh \left( \frac{x}{\ell_\varepsilon} \right) \text{csch} \left( \frac{L}{\ell_\varepsilon} \right) \right] \right\};
\]

(5.18a)

\[
\Delta \chi_j(x) = \Delta \bar{\chi}_j(x) - \ell_\sigma^2 \Delta \tilde{\chi}_j''(x) = \text{\( Y \)}_{10} \beta_j \left( C_2 + C_3 \bar{x}_j + \frac{q_0 \bar{x}_j^2}{2} \right) \left\{ \frac{\ell_\varepsilon^2 - \ell_\sigma^2}{\ell_\varepsilon^3} \text{csch} \left( \frac{L}{\ell_\varepsilon} \right) \cosh \left( \frac{x}{\ell_\varepsilon} \right) \cosh \left( \frac{L - \bar{x}_j}{\ell_\varepsilon} \right) \\
- \sinh \left( \frac{x - \bar{x}_j}{\ell_\varepsilon} \right) H(x - \bar{x}_j) \right\} \\
+ \frac{\ell_\sigma^2}{\ell_\varepsilon} \delta(x - \bar{x}_j) + \frac{\ell_\sigma^2}{\ell_\varepsilon} \frac{d}{dx} \left[ \text{\( \sinh \)} \left( \frac{x - \bar{x}_j}{\ell_\varepsilon} \right) \delta(x - \bar{x}_j) \right].
\]

(5.18b)

The last two terms in Eq. (5.18b) deserve a special attention. Indeed, the first one is an impulsive term, centred at the position of the \( i \)th crack, \( x = \bar{x}_j \), whose intensity is proportional to \( \rho_\ell^2 = (\ell_\sigma / \ell_\varepsilon)^2 \), and is responsible for the finite jump in the rotations’ profile given by Eq. (5.27). The second term, on the contrary, can be neglected when rotations and displacements of the non-local beam are evaluated by means of successive integrations of Eq. (5.17). This can be rigorously demonstrated, and depends on the mathematical characteristics of the function \( \text{\( \sinh \)}(\cdot) \) that multiplies the Dirac’s delta within the square brackets to be differen-
tiated, namely \(\sinh(\cdot)\) has a zero value at the centre of the delta function \((x = \bar{x}_j)\) and is anti-symmetric with respect to this point.

Similarly to the case of classical elasticity (see Eq. (3.40)), the rotations’ profile along the beam’s axis can be posed as the superposition of the undamaged term \(\varphi_0(x)\) and \(n\) further contributions \(\Delta \varphi_j(x)\) due to the singularities in the beam’s bending stiffness, where each term is obtained by integrating the corresponding curvature. The two expressions are given by:

\[
\varphi_0(x) = C_1 + \gamma_{10} C_2 x + \gamma_{10} \frac{C_3 x^2}{2} + \gamma_{10} \frac{q_0 x^3}{6} + \gamma_{10} \left( \ell_e^2 - \ell_\sigma^2 \right) \left\{ - C_3 \cosh \left( \frac{L - 2x}{2\ell_e} \right) \text{sech} \left( \frac{L}{2\ell_e} \right) \right. \\
+ \left. q_0 \left[ x - L \text{csch} \left( \frac{L}{\ell_e} \right) \sinh \left( \frac{x}{\ell_e} \right) \right] \right\} ;
\]

\[
\Delta \varphi_j(x) = \gamma_{10} \beta_j \left( C_2 + C_3 \bar{x}_j + \frac{q_0 \bar{x}_j^2}{2} \right) \left\{ H(x - \bar{x}_j) \\
+ \frac{\ell_e^2 - \ell_\sigma^2}{\ell_e^2} \left[ \text{csch} \left( \frac{L}{\ell_e} \right) \sinh \left( \frac{x}{\ell_e} \right) \cosh \left( \frac{L - \bar{x}_j}{\ell_e} \right) \right] \\
- \cosh \left( \frac{x - \bar{x}_j}{\ell_e} \right) H(x - \bar{x}_j) \right\} ,
\]

and the additional integration constant is \(C_1 = \varphi(0)\).

Finally, integrating Eq. (5.19a) and Eq. (5.19b) provides the mathematical expressions for the functions \(u_{b0}(x)\) and \(\Delta u_{bj}(x)\) appearing in the right-hand side of Eq. (3.27), which then describes the deformed shape of the multi-damaged beam with hybrid gradient elasticity:

\[
u_{b0}(x) = C_0 - C_1 x - \gamma_{10} C_2 x^2 \frac{C_3 x^2}{2} \gamma_{10} \frac{C_3 x^3}{6} - \gamma_{10} \frac{q_0 x^4}{24} - \gamma_{10} \left( \ell_e^2 - \ell_\sigma^2 \right) \left\{ C_3 \ell_e \text{sech} \left( \frac{L}{2\ell_e} \right) \sinh \left( \frac{L - 2x}{2\ell_e} \right) \right. \\
+ \left. q_0 \left[ \frac{x^2}{2} - \ell_e \text{csch} \left( \frac{L}{\ell_e} \right) \cosh \left( \frac{x}{\ell_e} \right) \right] \right\} ;
\]
\[ \Delta u_{bj}(x) = -\mathcal{Y} I_0 \beta_j \left( C_2 + C_3 \bar{x}_j + \frac{q_0 \bar{x}_j^2}{2} \right) \left\{ (x - \bar{x}_j) H(x - \bar{x}_j) \right. \\
+ \frac{\ell_e^2 - \ell_{\sigma}^2}{\ell_e} \left[ \text{csch} \left( \frac{L}{\ell_e} \right) \cosh \left( \frac{x}{\ell_e} \right) \cosh \left( \frac{L - \bar{x}_j}{\ell_e} \right) \right] \left( \frac{x - \bar{x}_j}{\ell_e} \right) H(x - \bar{x}_j) \left\} \right. \\
+ \left. \sinh \left( \frac{x - \bar{x}_j}{\ell_e} \right) H(x - \bar{x}_j) \right\}, \] (5.20b)

and the additional integration constant is \( C_0 = u(0) \).

As expected, Eqs. (5.19a) and (5.20a) give rotations’ profile and deformed shape of the undamaged beam with classical elasticity when \( \ell_{\sigma} = \ell_e \).

### 5.1.3.1 Integration constants for cantilever beam

This section offers the closed-form expressions for the additional integration constants \( C_0, C_1, C_2 \) and \( C_3 \), which appears in the solution of the multi-damaged EB cantilever beam under UDL when the hybrid gradient-elastic constitutive law is adopted.

To derive such expressions, the BCs \( u(0) = 0 \), \( \varphi(0) = 0 \), \( V(L) = 0 \) and \( M(L) = 0 \) have be imposed, yielding:

\[ C_0^{UDL} = q_0 L \mathcal{Y}_0 \left( \ell_{\varepsilon}^2 - \ell_{\sigma}^2 \right) - \ell_e \coth \left( \frac{L}{\ell_e} \right) \]  
\[ + \sum_{j=1}^{n} \beta_j (L - \bar{x}_j)^2 \left[ \text{csch} \left( \frac{L}{\ell_e} \right) \cosh \left( \frac{L - \bar{x}_j}{\ell_e} \right) \right] \]  
\[ = q_0 L \mathcal{Y}_0 \left( \ell_{\varepsilon}^2 - \ell_{\sigma}^2 \right) - \ell_e \coth \left( \frac{L}{\ell_e} \right) \]  
\[ + \sum_{j=1}^{n} \beta_j (L - \bar{x}_j)^2 \text{csch} \left( \frac{L}{\ell_e} \right) \cosh \left( \frac{L - \bar{x}_j}{\ell_e} \right) \]  
\[ = \frac{1}{2} L^2 q_0 \]  
\[ = -L q_0. \] (5.21a) (5.21b) (5.21c) (5.21d)

### 5.1.4 Solution for point load (PL)

Aim of this section is to provide, as second example, the exact closed-form mathematical expressions of curvature \( \chi(x) \), rotation \( \varphi(x) \) and displacement \( u_b(x) \) functions for a multi-cracked EB beam under the PL \( P_0 \).
To derive the sought solution, Eq. (5.7) and Eq. (5.8) can be particularised for the load case $q(x) = q_0 \delta(x - x_k)$, yielding:

$$\tilde{\chi}_0(x) = \gamma_1 C_2 + \gamma_1 C_3 \left[ x + \ell_\varepsilon \sech \left( \frac{1}{2 \ell_\varepsilon} \right) \sinh \left( \frac{1 - 2 x}{2 \ell_\varepsilon} \right) \right] +$$

$$+ 2 \gamma_1 q_0 \ell_\varepsilon \csch \left( \frac{1}{\ell_\varepsilon} \right) \sinh^2 \left( \frac{x_k - 1}{2 \ell_\varepsilon} \right) \cosh \left( \frac{x}{\ell_\varepsilon} \right) +$$

$$+ \gamma_1 q_0 H(x - x_k) \left[ x - x_k - \ell_\varepsilon \sinh \left( \frac{x - x_k}{\ell_\varepsilon} \right) \right];$$

(5.22a)

$$\Delta \tilde{\chi}_j(x) = \gamma_1 \frac{\beta}{\ell_\varepsilon} \left[ C_2 + \bar{x}_j C_3 + q_0 (\bar{x}_j - x_k) H(x_j - x_k) \right]$$

$$\times \left[ \cosh \left( \frac{x}{\ell_\varepsilon} \right) \cosh \left( \frac{1 - \bar{x}_j}{\ell_\varepsilon} \right) \csch \left( \frac{1}{\ell_\varepsilon} \right) - H(x - \bar{x}_j) \sinh \left( \frac{x - \bar{x}_j}{\ell_\varepsilon} \right) \right],$$

(5.22b)

in which it has been taken into account that for the PL $q(x) = q_0 \delta(x - x_k)$: i) the second anti-derivative is $q^{(2)}(x) = \frac{1}{2} q_0 (x - x_k) H(x - x_k)$; and ii) the function $Q(x)$, defined by Eq. (5.9), takes the expression:

$$Q(x) = q_0 \ell_\varepsilon H(x - x_k) \left[ \frac{x - x_k}{\ell_\varepsilon} - \sinh \left( \frac{x - x_k}{\ell_\varepsilon} \right) \right].$$

(5.23)

According to Eq. (5.17), the curvature is given by the sum of the undamaged contribution $\chi_0(x)$ and each terms associated with the concentrated damages $\Delta \chi_j(x)$. The local curvature can be then written in a form similar to Eq. (5.6):

$$\chi_0(x) = \tilde{\chi}_0(x) - \ell_\sigma^2 \tilde{\chi}_0''(x) =$$

$$\gamma_1 C_2 + \gamma_1 C_3 x + \gamma_1 C_3 \ell_\sigma^2 \left( \frac{1}{2 \ell_\varepsilon} \right) \sinh \left( \frac{1 - 2 x}{2 \ell_\varepsilon} \right) +$$

$$2 \gamma_1 q_0 \ell_\varepsilon \left( \frac{x}{\ell_\varepsilon} \right) \csch \left( \frac{1}{\ell_\varepsilon} \right) \sinh^2 \left( \frac{x_k - 1}{2 \ell_\varepsilon} \right) +$$

$$\gamma_1 q_0 \left[ H(x - x_k) - \ell_\sigma^2 \delta'(x - x_k) \right] \left[ x - x_k - \ell_\varepsilon \sinh \left( \frac{x - x_k}{\ell_\varepsilon} \right) \right] +$$

$$- 2 \gamma_1 q_0 \ell_\sigma^2 \delta(x - x_k) \left[ 1 - \cosh \left( \frac{x - x_k}{\ell_\varepsilon} \right) \right] +$$

$$\gamma_1 q_0 \ell_\sigma^2 H(x - x_k) \sinh \left( \frac{x - x_k}{\ell_\varepsilon} \right);$$

(5.24a)
\[ \Delta \chi_j(x) = \Delta \tilde{\chi}_j - \ell_2^2 \Delta \tilde{\chi}_j'' = \mathcal{Y}_0 \frac{\beta_j}{\ell_e} \left[ C_2 + \bar{x}_j C_3 + q_0 (\bar{x}_j - x_k) H(\bar{x}_j - x_k) \right] \]

\[
\left\{ \frac{\ell_2^2 - \ell_2^2_\sigma}{\ell_2^2} \left[ \cosh \left( \frac{x}{\ell_e} \right) \cosh \left( \frac{1 - \bar{x}_j}{\ell_e} \right) \cosh \left( \frac{1}{\ell_e} \right) \right] - H(x - \bar{x}_j) \sinh \left( \frac{x - \bar{x}_j}{\ell_e} \right) \right\} + \ell_2^2 \frac{\beta_j}{\ell_e} \delta(x - \bar{x}_j) \sinh \left( \frac{x - \bar{x}_j}{\ell_e} \right) \sinh \left( \frac{\ell_2^2 - \ell_2^2_\sigma}{\ell_2^2} \right) \delta'[x - \bar{x}_j] \right\}. 
\]

(5.24b)

Similarly to the case of classical elasticity (see Eq. (3.40)), the rotations’ profile along the beam’s axis can be posed as the superposition of the undamaged term \( \varphi_0(x) \) and further contributions \( \Delta \varphi_j(x) \) due to the singularities in the beam’s bending stiffness, where each term is obtained by integrating the corresponding curvature. The two expressions are:

\[
\varphi_0(x) = C_1 + \mathcal{Y}_0 x C_2 + \mathcal{Y}_0 \frac{C_3 x^2}{2} - \mathcal{Y}_0 C_3 \left( \ell_2^2 - \ell_2^2_\sigma \right) \left( \frac{1 - 2x}{2\ell_e} \right) \text{sech} \left( \frac{1}{2\ell_e} \right) +
\]

\[
+ 2 \mathcal{Y}_0 q_0 \left( \frac{\ell_2^2 - \ell_2^2_\sigma}{\ell_2^2} \right) \left( \frac{1}{\ell_e} \right) \sinh \left( \frac{x}{\ell_e} \right) \sinh \left( \frac{x_k - 1}{2\ell_e} \right) \left( \frac{x_k}{\ell_e} \right) \ell_e^2 +
\]

\[
- \mathcal{Y}_0 q_0 \ell_2^2_\sigma \delta(x - x_k) \left[ x - x_k - \sinh \left( \frac{x - x_0}{\ell_e} \right) \ell_e \right] +
\]

\[
+ \mathcal{Y}_0 \frac{q_0}{2} H(x - x_k) \left\{ (x - x_k)^2 + 2 \left( \frac{\ell_2^2 - \ell_2^2_\sigma}{\ell_2^2} \right) \left[ 1 - \cosh \left( \frac{x - x_k}{\ell_e} \right) \right] \right\};
\]

(5.25a)

\[
\Delta \varphi_j(x) = \mathcal{Y}_0 \frac{\beta_j}{\ell_e^2} \left[ C_2 + \bar{x}_j C_3 + q_0 (\bar{x}_j - x_k) H(\bar{x}_j - x_k) \right] \]

\[
\left\{ \cosh \left( \frac{1 - \bar{x}_j}{\ell_e} \right) \text{sech} \left( \frac{1}{\ell_e} \right) \sinh \left( \frac{\ell_2^2 - \ell_2^2_\sigma}{\ell_2^2} \right) + \left[ \ell_2^2 - \cosh \left( \frac{x - \bar{x}_j}{\ell_e} \right) \left( \frac{\ell_2^2 - \ell_2^2_\sigma}{\ell_2^2} \right) \right] \right\} \left( \ell_2^2 - \ell_2^2_\sigma \right) \delta(x - \bar{x}_j) \sinh \left( \frac{x - \bar{x}_j}{\ell_e} \right) \left( \ell_2^2 - \ell_2^2_\sigma \right)
\]

(5.25b)

and the additional integration constant is \( C_1 = \varphi(0) \).
Finally, integrating Eq. (5.25a) and Eq. (5.25b) provides the mathematical expressions for the functions $u_{b0}(x)$ and $\Delta u_{bj}(x)$ appearing in the right-hand side of Eq. (3.27):

$$u_{b0}(x) = C_0 - x C_1 - \gamma_{10} x^2 C_2 - \gamma_{10} x^3 C_3 + \gamma_{10} C_3 \ell_e \left( \frac{L}{2\ell_e} \right) \sinh \left( \frac{L - 2x}{2\ell_e} \right) +$$

$$- 2 \gamma_{10} q_0 \ell_e \left( \ell_e^2 - \ell_\sigma^2 \right) \cosh \left( \frac{x}{\ell_e} \right) \sinh \left( \frac{L - 2x}{2\ell_e} \right) +$$

$$\gamma_{10} \frac{q_0}{2} H(x - x_k) \left\{ - \frac{(x - x_k)^3}{3} + 2 \left( \ell_e^2 - \ell_\sigma^2 \right) \left[ -x + x_k + \sinh \left( \frac{x - x_k}{\ell_e} \right) \ell_e \right] \right\};$$

(5.26a)

$$\Delta u_{bj}(x) = - \gamma_{10} \frac{\beta_j}{\ell_e} \left[ C_2 + \bar{x}_j C_3 + q_0 (\bar{x}_j - x_k) H(\bar{x}_j - x_k) \right]$$

$$\left\{ \cosh \left( \frac{1}{\ell_e} \right) \cosh \left( \frac{x - \bar{x}_j}{\ell_e} \right) \cosh \left( \frac{L - \bar{x}_j}{\ell_e} \right) \left( \ell_e^2 - \ell_\sigma^2 \right) + \right.$$

$$H(x - \bar{x}_j) \left[ - \left( \ell_e^2 - \ell_\sigma^2 \right) \sinh \left( \frac{x - \bar{x}_j}{\ell_e} \right) + \ell_e (x - \bar{x}_j) \right] \right\}. \quad (5.26b)$$

5.2 Numerical examples

In this section, two numerical applications are detailed to demonstrate the validity of the proposed strategy of modelling and solution, also highlighting and quantifying the effects of the non-local length scale parameters $\ell_e$ and $\ell_\sigma$ on the static response of the objective beams. Since the parameters are linked to the beam’s microstructure, their values cannot exceed the beam’s length, i.e. $0 \leq \ell_e < L$ and $0 \leq \ell_\sigma < L$.

In the first example, a cantilever beam with a single concentrated damage and subjected to a uniformly distributed load (UDL) is considered; while the second example investigates the statically indeterminate case of a clamped-clamped beam with three cracks under a concentrated force. The results obtained with the non-local formulation have been compared with those of the classical elasticity theory, which is equivalent to the adopted hybrid gradient-elastic theory when the two length scale parameters take the same value ($\ell_e = \ell_\sigma \geq 0$).

For each example, the results are displayed in terms of non-dimensional quantities, e.g. $EI_0 \chi(x)/(P L)$ (or $EI_0 \chi(x)/(q L^2)$) for the curvature and $EI_0 \varphi(x)/(P L^2)$ (or $EI_0 \varphi(x)/(q L^3)$) for the rotation, being $P$ a concentrated force and $q$ a UDL.
5. Non-local multi-damaged beam

5.2.1 Example one – Cantilever beam with uniformly distributed load $q_0$

In the first application, the beam is clamped at $x = 0$; the UDL $q_0$ is pointing downwards; a single crack is assumed at the position $\bar{x}_1 = 0.4L$, modelled with a rotational spring with dimensionless parameters $\beta_1 = 0.1$ (see Eq. (3.10b)).

Following the procedure presented in Section 5.1, the local curvature $\chi(x)$, the rotation $\varphi(x)$ and the displacement $u_b(x)$ are obtained for a generic set of microstructural parameters $\{\ell_\varepsilon, \ell_\sigma\}$. The constants $C_0$, $C_1$, $C_2$ and $C_3$ are evaluated by solving the four equations that result from applying the BCs for the internal forces at the free end ($V(L) = M(L) = 0$) and
the kinematic quantities at the fixed end \((u_b(0) = \varphi(0) = 0)\); the other constants \(C_4\) and \(C_5\) are evaluated by assuming the stationarity of the effective non-local curvature at both ends of the beam \(\tilde{X}'(0) = \tilde{X}'(L) = 0\).

Figure 5.2(a) shows the effects of the length scale parameters \(\{\ell_\varepsilon, \ell_\sigma\}\) on the beam’s deformed shape. For \(\ell_\varepsilon > \ell_\sigma > 0\) (dotted line and dashed line), the tip displacement reduces with respect to the classical continuum theory \((\ell_\varepsilon = \ell_\sigma > 0\) solid line); while if \(\ell_\varepsilon < \ell_\sigma\) (dot-dashed line), the beam experiences larger displacements. In the latter case, however, the behaviour of the non-local beam is physically inconsistent, as clearly evidenced by the counter-load deformations in Fig. 5.2(b), where the beam’s deformed shape is magnified in the neighbourhood of the crack. Indeed, for \(\ell_\varepsilon = L/50\) and \(\ell_\sigma = L/5\), the beam shows an upward cusp centred at the crack position, despite the fact that the load is downward.

To quantify the effects of length scale parameters on the apparent stiffness of the beam, Fig. 5.3 plots the displacement ratio \(u_b(L)/u_{b,\text{ref}}\) against the length scale ratio \(\rho_\ell = \ell_\sigma/\ell_\varepsilon\) for four different values of \(\ell_\sigma\), being \(u_{b,\text{ref}}\) the reference value of the tip (free-end) displacement obtained with the classical elasticity theory. For \(0 < \rho_\ell < 1\), the tip displacement is less than the reference value, independently of the micro-structural parameter \(\ell_\sigma\) (and, for a given ratio \(\rho_\ell\), the larger is the value of \(\ell_\sigma\), the greater is the discrepancy among local and non-local beam). It is possible to conclude that, for the physically consistent case where \(0 < \rho_\ell < 1\), the non-local beam always appears stiffer than the same beam with classical elasticity. Conversely, the non-local beam is more flexible for \(\rho_\ell > 1\).

Another way to show the inconsistency of the hybrid gradient model for \(\rho_\ell > 1\) is to evaluate the finite jump occurring in the profile of rotations at the position of the concentrated damage. According to Eq. (3.40), the rotation \(\varphi(x)\) is given in this case by the superposition of two terms, namely \(\varphi_0(x)\), which is continuous (see Eq. (3.41a)), and the function \(\Delta \varphi_1(x)\) due to the presence of the crack at \(x = \bar{x}_1\), which is discontinuous at that position (see Eq. (3.41b)). Combining now Eq. (3.44a) and Eqs. (5.19b), it can be shown that:

\[
|\varphi(\bar{x}_1^+) - \varphi(\bar{x}_1^-)| = Y_{10} \beta_1 |M(\bar{x}_1)| \rho_\ell^2, \tag{5.27}
\]

where the superscripted signs + and − in the argument of the rotation \(\varphi(x)\) stand for the limit from the right and from the left, respectively.

Equation (5.27) then reveals that the amplitude of the finite concentrated rotation predicted by the proposed model at the position of the crack increases with \(\rho_\ell^2\). For \(\rho_\ell = 0\) (i.e. \(\ell_\sigma = 0\) and \(\ell_\varepsilon > 0\)), the localised jump in the rotations’ profile vanishes completely, meaning that the beam’s curvature becomes smooth at the position of the crack. For \(\ell_\sigma > 0\) and \(\ell_\varepsilon \geq \ell_\sigma\) (i.e.
0 < \rho \leq 1), the beam’s deformed shape always shows a finite jump at the damage position, whose amplitude ranges from zero (the limiting case when \ell_\sigma \rightarrow 0) to the classical continuum solution (obtained for \ell_\sigma = \ell_\varepsilon > 0). On the contrary, the condition \ell_\sigma > \ell_\varepsilon > 0 (i.e. \rho > 1) aggravates the singularity, and therefore this combination of values for the microstructural parameters should be avoided.

Based on the above considerations, the stress length \ell_\sigma appears as a sensible lower limit for the strain length \ell_\varepsilon, and in the following the chain of inequalities 0 \leq \ell_\sigma \leq \ell_\varepsilon < L is always satisfied in order to avoid physically inconsistent results. To the best of author’s knowledge, no previous attempts have been successfully made to identify the physically acceptable range of variations of the two microstructural parameters \ell_\varepsilon and \ell_\sigma for cracked non-local beams.

Within the above conditions, further analyses have been carried out to investigate the effects of the two individual microstructural parameters \ell_\varepsilon and \ell_\sigma.

Figure 5.4(a) shows the rotations’ profiles for a fixed value of the strain length, \ell_\varepsilon = L/50, and four different values of the stress length, \ell_\sigma \leq \ell_\varepsilon. It can be seen that changing the parameter \ell_\sigma mainly affects the rotations in the neighbourhood of the cracks, as demonstrated by the magnified view of Fig. 5.4(b). This comparison confirms the prediction of Eq. (5.27), as the finite jump in the rotations reduces with \ell_\sigma.

Figures 5.5(a) and (b) show the continuous part of the curvature functions \chi(x) for a fixed stress length \ell_\sigma = L/50 and four different values of the strain length \ell_\varepsilon \geq \ell_\sigma (a Dirac’s delta of intensity \left| \varphi(\bar{x}_1^+) - \varphi(\bar{x}_1^-) \right| also appears at x = \bar{x}_1 in the mathematical expression of
5.2 Numerical examples

\[ E_0 L^2 \chi \]

\[ q L^2 \chi \]

\[ x/L \]

\[ (a) \]

\[ -0.5 \]

\[ -0.4 \]

\[ -0.3 \]

\[ -0.2 \]

\[ -0.1 \]

\[ 0 \]

\[ 0 \]

\[ 0.2 \]

\[ 0.4 \]

\[ 0.6 \]

\[ 0.8 \]

\[ 1 \]

\[ -0.35 \]

\[ -0.3 \]

\[ -0.25 \]

\[ -0.2 \]

\[ -0.15 \]

\[ -0.1 \]

\[ 0.3 \]

\[ 0.35 \]

\[ 0.4 \]

\[ 0.45 \]

\[ 0.5 \]

\[ -0.35 \]

\[ -0.3 \]

\[ -0.25 \]

\[ -0.2 \]

\[ -0.15 \]

\[ -0.1 \]

\[ 0.3 \]

\[ 0.35 \]

\[ 0.4 \]

\[ 0.45 \]

\[ 0.5 \]

Figure 5.5: Example one – Curvature function for a fixed value of stress length scale (\( \ell_\sigma = L/50 \)) and four different values of the stress length scale \( \ell_\varepsilon \) (a); magnification at the position of the crack (b)

\[ \chi(x), \] but has been hidden to make the graphical representation simpler. One can observe that, in comparison with the classical elasticity (solid line), the hybrid gradient model potentially provides a more accurate description of the discontinuity due to a concentrated damage. In fact, while the proposed non-local model for the cracked beam shows a cusp in the curvature function at the position of the damage (whose width and height depend on the microstructural parameters \( \ell_\sigma \) and \( \ell_\varepsilon \)), such cusp disappears in the solution obtained with classical elasticity theory.

To point out the effects of the non-local length scale parameters on the discontinuity due to the crack, the damage terms \( \Delta \varphi_1(x) \) of Eq. (5.25b) have been investigated. In particular Fig. 5.6 shows the ratio \( \Delta \varphi_1(x)/\left| \varphi(x^+_1,loc) - \varphi(x^-_1,loc) \right| \) between only the jump on the rotations of the non-local model \( \Delta \varphi_j(x) \) and the jump on the rotation for the classical continuum model \( \Delta \varphi_j,loc(x) \) along the beam. Three different values of the length scale parameters (\( \ell_\varepsilon = L/20, L/40, L/80 \)) and constant ratio \( \rho_\ell = 1/2 \) have been considered. The figure shows as expected that the discontinuity on the rotations at the crack position do not change (see Eq. 5.27), then when \( \ell_\sigma/\ell_\varepsilon \) is constant the solutions show the same finite jump in the rotations curve but the distance required to match the local solution becomes shorter.
5. Non-local multi-damaged beam

Figure 5.6: Example one – Damage effect in terms of rotations of the cantilever cracked beam for three different values of $\ell_\varepsilon$ and with the same ratio $\rho_\ell = 1/2$ compared with classical continuum solution $\ell_\varepsilon = \ell_\sigma$.

Figure 5.7: Example two – Deformed shape of the multi-cracked clamped-clamped beam for a fixed value of stress length scale ($\ell_\sigma = L/50$) and four different values of the strain length scale $\ell_\varepsilon$.

5.2.2 Example two – Clamped-clamped beam with a point load

A similar trend of results can be observed in the second example, in which a slender beam is clamped at both ends ($x = 0$ and $x = L$), and is loaded at $\bar{x}_P = 0.6 L$ with a concentrated force $P$, pointing downwards. The corresponding loading function can be then expressed as $q(x) = P \delta(x - \bar{x}_P)$. The beam has three cracks with the same depth, and hence the same damage factor $\beta_1 = \beta_2 = \beta_3 = 0.1$, located at $\bar{x}_1 = 0.1 L$, $\bar{x}_2 = 0.4 L$ and $\bar{x}_3 = 0.8 L$, respectively. This example thus demonstrates that the proposed approach enables the analytical
solution for statically indetermined non-local slender beams in bending with multiple concentrated damages (i.e. the most general case considered as part of this Chapter).

Figure 5.7 compares the deformed shape of the beam when the stress length scale takes a given value \( \ell_\sigma = L/50 \) and the strain length scale \( \ell_\varepsilon \) varies. As in the previous example, it appears that the overall stiffness of the beam increases with \( \ell_\varepsilon \), as the maximum deflection tends to reduce in comparison with the classical elasticity theory (solid line).

Fig. 5.8(a) shows the curvature \( \chi(x) \) for the same microstructural parameter as in Figure 5.7(a). Interestingly, it can be observed that at the position of the point load, \( \bar{x}_P = 0.6L \), due to the discontinuity in the shear force, the curvature of the classical continuum solution shows a sudden change in the slope, while using the non-local models the transition appears smoothed. Moreover, likewise the previous example, the curvature has a cusp at each crack position \( x = \bar{x}_j \), whose intensity depends on the length scale parameters (and one of such cusps is magnified within Fig. 5.8(b), to better appreciate the effect of the strain length scale \( \ell_\varepsilon \) in this circumstance).

The smoothing effect of the strain gradient parameter is further highlighted by Figs. 5.9(a) and (b), where the rotations’ profiles corresponding to three different values of \( \ell_\varepsilon \), with \( \ell_\sigma = 0 \), are plotted and compared with the classical continuum solution (solid line, obtained for \( \ell_\varepsilon = \ell_\sigma \)). It can be seen that, being \( \rho_\ell = \ell_\sigma / \ell_\varepsilon = 0 \) for the three non-local beams, the finite jumps typical of the local solution disappear (see Eq. 5.27); moreover, the length of the beam in which the effect of the crack is distributed increases with \( \ell_\varepsilon \).
5. Non-local multi-damaged beam

Figure 5.9: Example two – Profile of rotations for the multi-cracked clamped-clamped beam with fixed value of stress length scale ($\ell_\sigma = 0$) and four different values of the strain length scale $\ell_\varepsilon$ (a); magnification at the position of the second crack (b)

5.3 Concluding remarks

The Chapter has offered a new formulation for the linear-elastic analysis of non-local slender beams with multiple cracks under static transverse forces. The proposed approach is based on the flexibility crack model [162] (equivalent to the well-known discrete spring model), which has been extended to the hybrid constitutive law with both stress [76] and strain [9] gradient. Similar to the case of classical elasticity, each crack is conveniently represented by means of a Dirac’s delta function, whose intensity increases with the severity of the damage; in contrast, adopting the gradient elasticity allows spreading the effects of the cracks in the neighbourhood of the damaged abscissa, therefore providing a more realistic profile of rotations (as confirmed with the numerical examples).

Using the Laplace’s transformation method, the exact solution has been derived for the effective non-local curvature of the multi-cracked beam under a generic transverse load. This expression has been then used to evaluate in closed form the curvature of the beam and, through successive integrations, the profile of rotations and the deformed shape. It has been shown that, for a given load function $q(x)$, the solution of the static problem depends on six integration constants only, namely: i) the classical four constants $C_0$, $C_1$, $C_2$ and $C_3$, which are typically associated with the support conditions of the beam; ii) two additional constants $C_4$ and $C_5$, which appear in the general expression of the effective non-local curvature. Explicit closed-
form expressions have been provided in Section 5.1.2 for these two integration constants, assuming the stationarity of the non-local effective curvature at the beam’s ends.

Two different solution procedures have been delineated for statically determinate and indeterminate beams (see Table 5.1). As in the classical elasticity, the former case enjoys the favourable condition that equilibrium, constitutive and kinematic equations are decoupled, which then can be solved in cascade (with the gradient-elastic model, however, the integrations constants $C_4$ and $C_5$ have to be determined when solving the constitutive equations); while the governing equations become fully coupled for the latter case.

Importantly, since the adopted flexibility model treats the cracks as concentrated inhomogeneities in the bending flexibility of the beam, the size of the computational problem (i.e. the number of unknowns) is independent of the number $n$ of cracks, for both statically determinate and indeterminate beams.

The proposed formulation has been applied to analyse a cantilevered (statically determinate) beam, and a clamped-clamped (statically indeterminate) beam and the numerical results have allowed to quantify the effects of the two microstructural parameters, $\ell_\sigma$ and $\ell_\varepsilon$, associated with stress and strain gradient, also in comparison with the reference solution provided by the classical elasticity theory (recovered for $\ell_\sigma = \ell_\varepsilon$).

In particular, it has been theoretically found and numerically verified that the finite jump in the profile of rotations at the crack position is proportional to $\rho^2\ell_\varepsilon$, being $\rho \ell_\varepsilon = \ell_\sigma / \ell_\varepsilon$ a dimensionless length scale ratio (i.e. the less $\rho \ell_\varepsilon$, the smoother the profile of rotations, the more widespread the effects on the curvature of the beam); and it has been suggested that, for the purposes of analysing cracked beams in bending, physically consistent results are only obtained if $0 \leq \ell_\sigma \leq \ell_\varepsilon \ll L$, being $L$ the length of the beam.

It has been also shown that: i) differences among predictions of local and non-local elasticity theory increase with the dimensionless quantity $\ell_\varepsilon / L$; ii) the overall flexibility of the beam increases with the length scale ratio $\rho \ell_\varepsilon$ and, for a given value of $\rho \ell_\varepsilon$, with the microstructural length scales.

These new findings allow a better understanding of how gradient elasticity theories can be effectively adopted for the analysis of cracked slender beams. The extension of the closed-form solutions presented as part of this study to solve dynamic problems will be presented in the following chapter.

A journal paper summarising the key results of the work presented in this Chapter has been published for publication in International Journal of Solids & Structures [70].
This chapter deals with the free vibration analysis of multi-cracked beams modelled using hybrid non-local elasticity theory and Euler-Bernoulli (EB) beam theory. The microstructural effects are taken into consideration via the stress and strain gradients theory, while cracks are mathematically represented through Dirac’s delta functions centred at the damage position [236]. A novel computational method is presented, which builds upon the theory developed in the previous Chapter for static analysis of gradient-elastic slender beams in bending. Before proceeding with the derivation of the proposed approach, a short literature review is offered, focusing on the technical papers which have addressed the dynamic problem.

### 6.1 Position of the problem

Several papers have been published on the vibration of beams using non-local/gradient elasticity theory for both EB and Timoshenko beam. Solution of the dynamic problem can be found for example in Wang and Varadan [240], who have developed a model to study the vibration of both single walled nanotubes and double-walled nanotubes. While Wang et al. [236] have investigated the dynamics of non-local Timoshenko beams, using the Hamilton’s principle to
obtain a variational consistent solution and boundary conditions for the free vibration of beams using Eringen’s non-local elasticity theory ($\ell = 0$). Murmu and Adhikari have studied the transverse [153] and longitudinal [152] vibration of nanobeams, nanorods double-systems connected with distributed longitudinal/ transverse springs employing the Eringen stress gradient. Their works, which coupled together the fourth order one-dimensional equation of motion for the two beams to provide the analytical solution for the beam system, highlight the strong influence of the microstructural parameters on the vibration of micro-structures, as well as the influence of the effect of the spring stiffness on both out-of-phase and in-phase vibrations. The approach has been also extended to double-layered nanoplates [154]. Furthermore the non-local theory has been adopted to study the vibration of different structure configuration systems, like carbon nanotubes with attached buckyballs at tip [155, 156], where the additional mass appendix significantly influences both torsional and axial vibrations. In these works the results, which clearly show the importance of the non-local effects, have been validated with molecular dynamic simulations.

Looking at all the technical literature dedicated to non-local models, it can be noticed that the majority of them deal with strain or stress gradient non-local theory, while only a few papers consider the two-parameters hybrid non-local beam theory. Among them, Zhang et al. [251] have studied the free transverse vibration of the hybrid non-local EB beam by applying the classical separation of variables method, where the transverse displacement is assumed as:

$$u_b(x, t) = U_b(x) \exp(i \omega t), \quad (6.1)$$

in which $U_b(x)$ is the amplitude of the transverse displacement; $\omega$ is the natural frequency of vibrations; $i = \sqrt{-1}$ is the imaginary unit. The resulting equation of motion gives:

$$EI \left[U_b'''(x) - \ell^2 \sigma U_b''''(x)\right] = \omega^2 \rho A \left[U_b(x) - \ell^2 \sigma U_b''(x)\right], \quad (6.2)$$

where $EI$ is the flexural stiffness and $\rho A$ is the linear mass. The authors have only provided the analytical solution for the simply supported beam, as in this case the sixth-order differential equation can be converted into three second-order differential equations, while for other boundary conditions they suggested to resort numerical methods.

To the best of the author’s knowledge, the hybrid non-local formulation has not been used to study damage beam yet. Indeed, the problem of stress-gradient slender beams with multiple concentrated damages has been addressed in the technical literature by Loya et al. [136]. They used the governing equations of the dynamics of the non-local EB beams to describe the be-
haviour of the two undamaged segments obtained by splitting the beam at the crack position. Similarly to Zhang et al. [251], in order to solve the differential equation, they used the classical separation-of-variables method for each undamaged segment, assuming for the displacement the following exponential equation:

\[ u_b(x) = U_{b,j}(x) \exp(i \omega t), \] (6.3)

where \( U_{b,j}(x) \) is the amplitude of the transverse displacement for the \( j \)th segment. Using this expression for the displacement, the general solution of the differential equation can be written as:

\[ U_{b,j}(x) = A_{1,j} \sinh(\beta_r x) + A_{2,j} \cosh(\beta_r x) + A_{3,j} \sin(\beta_e x) + A_{4,j} \cos(\beta_e x). \] (6.4)

Each segment has the same parameters \( \beta_r \) and \( \beta_e \), which depend on the dimensions and material properties only, while the unknown constants \( A_{i,j} \) are different and can be determined by applying the four external BCs and the four compatibility conditions at the crack location:

\[ U_{b,j}(x_j) = U_{b,j+1}(x_j); \] (6.5)

\[ \Delta \varphi_j = U'_{b,j}(x_j) - U'_{b,j+1}(x_j) = k_j U''_{b,j}(x_j); \] (6.6)

\[ M_j(x_j) = M_{j+1}(x_j); \] (6.7)

\[ V_j(x_j) = V_{j+1}(x_j), \] (6.8)

where \( k_j \) is the flexibility constant for the spring representing the \( j \)th crack; the derivatives of the displacement amplitude \( U'_{b,j}(x) \) and \( U''_{b,j}(x) \) are the amplitude of the rotation and curvature respectively; finally, \( M_j(x) \) and \( V_j(x) \) are the bending moment and shear force of the \( j \)th segment. Eq. (6.4) can be rewritten in terms of the displacement \( U_0 = U(0) \), the rotation \( \varphi_0 = \varphi(0) \), the bending moment \( M_0 = M(0) \) and the shear force \( V_0 = V(0) \) at the position \( x = 0 \) of the beam, giving the following equations:

\[ U_{b,1}(x) = U_0 g_1(x) + \varphi_0 g_2(x) + M_0 g_3(x) + V_0 g_4(x); \quad 0 \leq x \leq x_j; \] (6.9)

\[ U_{b,j+1}(x) = U_{b,j}(x) + \Delta \varphi_j g_{j+1}(x - x_j); \quad x_j \leq x \leq x_{j+1}, \] (6.10)

where \( U_{b,j}(x) \) is the displacement function of the \( j \)th segment, while \( g_i(x), i = 1, 2, 3, 4 \) are the shape functions, whose detailed definition can be found in [136]. This formulation satisfies
the compatibility conditions at the cracked section. Two of the unknown constants can be obtained directly from the BCs at $x = 0$, while the remaining two constants are determined by solving a set of two simultaneous equations. Torabi and Dastgerdi [220] have used the same approach to evaluate the free vibrations of a Timoshenko beam with gradient elasticity.

Even though this approach appears to be effective, it can not be applied for the hybrid non-local theory. Indeed, in this case the governing equation is a sixth order differential equation (not a fourth order differential equation as in the case of stress gradient elasticity), therefore requires that extra continuity conditions have to be enforced at the interface between two adjacent undamaged segments of the beam. Apart from the increased size of the problem, the analytical expressions becomes more complicated, and make this approach unappealing. The same difficulty is encountered to develop a two-node FE beam for the hybrid non-local model where, differently from the FE beam with classical elasticity represented by two DoFs (displacement and rotation) at the end nodes, an additional DoF is required for each node, increasing the complexity of the system. The two-node beam element with hybrid non-local elasticity would then have six DoFs and, when connected with adjacent elements, the continuity of the first derivative of the non-local curvature $\tilde{\chi}'(x)$ should also be enforced along with displacements $u_T(x)$ and rotations $\varphi(x)$.

In order to avoid complexity such problems, a viable solution is to adopt a Galerkin-type approximation. In the following, the Galerkin method is initially reviewed, and then a convenient set of shape functions for the problem in hand are derived, allowing to compute the stiffness and mass matrices for multi-damaged hybrid gradient elastic EB beams.

6.2 Review of the Galerkin method

Differential equations which govern most of the practical problem are rarely solvable in a closed form. The alternative approach is to use approximate techniques, and among them the method of weighted residual (MWR) is one of the most widely implemented, as it is the basis for the FE method [104]. Boundary value problems can be solved using the MWR, which minimises in an average sense, i.e. in an integral form over the domain, the residual between the exact solution and an approximate solution.
For illustration purpose, let us consider the case of a EB cantilever beam, whose governing equation with homogeneous boundary conditions can be written as:

\[
\begin{cases}
[ EI(x) u''_b(x) ]'' - q(x) = 0; \\
u_b(0) = 0; u'_b(0) = 0; \\
u''_b(L) = 0; u'''_b(L) = 0,
\end{cases}
\]

where the imposition on the transverse displacement \( u_b(x) \) and its derivatives \( u'_b(x) \), \( u''_b(x) \) and \( u'''_b(x) \) are the BCs at the two ends 0 and \( L \) of the domain for the independent field variable \( x \).

The approximate solution \( u^*_b(x) \) in the MWR is assumed as:

\[
u^*_b(x) = \sum_{r=1}^{N_\phi} \theta^{(r)} \phi^{(r)}(x),
\]

which represents a linear combination of \( N_\phi \) shape (or trial) functions \( \phi^{(r)}(x) \), conveniently scaled by unknown constant parameters \( \theta^{(r)} \), independent of the field variable \( x \). Importantly, the trial functions are required to be continuous over the domain of interest and to satisfy the given boundary conditions. Eq. (6.12) can also be expressed in a matrix form as:

\[
u^*_b(x) = \phi^\top(x) \theta,
\]

where \( \phi = [\phi^{(1)}; \ldots \phi^{(N_\phi)}]^\top \) is the vector collecting the \( N_\phi \) shape functions, while \( \theta = [\theta^{(1)}; \ldots \theta^{(N_\phi)}]^\top \) collects the unknown constant parameters associated with the assumed shape functions.

The approximate solution so defined in Eq. (6.12) is in general different from the exact solution for the differential equation Eq. (6.11), and then a residual error \( R(x) \) exists. For the proposed example, the residual will be defined as:

\[
R(x) = [ EI(x) u^{''}_b(x) ]'' - q(x) \neq 0.
\]

The MWR requires the sum of the weighted residual to be null over the domain for \( N_\phi \) arbitrary weighting functions \( w^{(r)}(x) \). In a mathematical form, it results in a system of \( N_\phi \) algebraic equations, where each equation can be written as follow:

\[
\int_0^L w^{(r)}(x) R(x) \, dx = 0, \quad r = 1, \ldots, N_\phi.
\]
The \( N_\phi \) unknown constant parameters \( \theta^{(r)} \) can be evaluated as solution of the algebraic linear system. Due to the assumptions for the trial function, the solution will be exact at the boundaries, while in general, at any inner point there will be a non null residual.

The weighting function \( w^{(r)}(x) \) can assume different forms, defining several MWR techniques. The most common ones are: point collocation method; sub-domain method; least squares method; Galerkin’s method. Each of these are briefly explained in the next subsections.

### 6.2.1 Point collocation method

This method takes weighting functions on the form of Dirac’s delta functions, i.e. \( w^{(r)}(x) = \delta(x - x_r) \) which equals zeros everywhere \( \delta(x - x_r) = 0 \) for \( x \neq x_r \), except at zero \( x = x_r \) where it assumes infinite value, and it has the property that its integral over the domain equals one \( \int_0^L \delta(x - x_r) dx = 1 \) (see Fig. 6.1). Hence the integration of the weighted functions Eq. (6.15) is equivalent to force the residual to be null in the \( n \) specific points of the domain \( (R(x_r) = 0) \).

![Figure 6.1: Shape functions for Point Collocation](image)

### 6.2.2 Sub-domain collocation method

This method is a modification of the point collocation method, where the weighted residual is forced to zero not just at fixed points but on average over different subsections of the domain (see Fig. 6.2). The weighting function for the generic interval \([x_r, x_{r+1}]\) is a window function,

\[
 w^{(r)}(x) = H(x - x_r) - H(x - x_{r-1}),
\]

![Figure 6.2: Shape functions for Sub-domain collocation method](image)
leading to:

\[ \int_0^L w^{(r)}(x) R(x) \, dx = \int_{x_r}^{x_{r+1}} R(x) \, dx = 0 ; \quad r = 1, ..., N_\phi . \] (6.17)

### 6.2.3 Least Squares method

This method minimise the Euclidean norm of the residual \( R(x) \) with respect to all the unknown parameters \( \theta^{(r)} \), which is equivalent to impose the derivatives of the Euclidean norm with respect to the unknowns to be null. In a mathematical form this can be written in the following form:

\[ \frac{\partial}{\partial \theta^{(r)}} \int_0^L R^2(x) \, dx = 2 \int_0^L R(x) \frac{\partial R(x)}{\partial \theta^{(r)}} \, dx = 0 ; \quad r = 1, ..., N_\phi . \] (6.18)

The weight functions can be clearly identify by comparing of Eq. (6.18) with Eq. (6.15), yielding:

\[ w^{(r)}(x) = \frac{\partial R(x)}{\partial \theta^{(r)}} ; \quad r = 1, ..., N_\phi , \] (6.19)

where the coefficient 2 has been dropped since it cancels out in the equations.

### 6.2.4 Galerkin method

In this approach, the weighting functions are assumed to be equal to the shape functions \( \phi^{(r)}(x) \) introduced in the approximate solution of Eq. (6.12) (see Fig. 6.3):

\[ w^{(r)}(x) = \phi^{(r)}(x) ; \quad r = 1, ..., N_\phi . \] (6.20)

Similarly to the Least Square method it may be viewed as a minimisation of the product of the
residual and the approximating function with respect to the unknown parameters \( \theta^{(r)} \) [254]:

\[
\frac{\partial}{\partial \theta^{(r)}} \int_0^L \theta^{(r)} \phi^{(r)}(x) R(x) \, dx = \int_0^L R(x) \frac{\partial u^*_\theta(x)}{\partial \theta^{(r)}} \, dx = 0; \quad r = 1, \ldots, N_\phi.
\] (6.21)

In detail, using the Galerkin method, the \( N_\phi \) algebraic equations of Eq. (6.15) can be rewritten as:

\[
\int_0^L w^{(r)}(x) R(x) \, dx = \int_0^L \phi^{(r)}(x) R(x) \, dx = 0; \quad r = 1, \ldots, N_\phi.
\] (6.22)

6.2.4.1 Example: Euler-Bernoulli cantilever beam

For the aforementioned example of the cantilever beam Eq. (6.11) with classical elasticity, the residual as defined in Eq. (6.14) can be expressed as:

\[
R(x) = \sum_{s=1}^{N_\phi} \theta^{(s)} \left[ EI(x) \phi^{(r)}''(x) \right]'' - q(x).
\] (6.23)

The requirement of the weighted residual to be null over the domain for \( N_\phi \) arbitrary weighting functions \( w^{(r)}(x) = \phi^{(r)}(x) \) assumes the following form:

\[
\int_0^L \phi^{(r)}(x) \left\{ \sum_{s=1}^{N_\phi} \theta^{(s)} \left[ EI(x) \phi^{(s)}''(x) \right]'' - q(x) \right\} \, dx = 0; \quad r = 1, \ldots, N_\phi.
\] (6.24)

Moving out of the integrals the coefficients \( \theta^{(s)} \), which do not depend on the spatial variable \( x \), yields:

\[
\sum_{s=1}^{N_\phi} \theta^{(s)} \int_0^L \phi^{(r)}(x) \left[ EI(x) \phi^{(s)}''(x) \right]'' \, dx = \int_0^L \phi^{(r)}(x) q(x) \, dx; \quad r = 1, \ldots, N_\phi.
\] (6.25)

This system of \( N_\phi \) equations can be rewritten using a more compact form:

\[
\sum_{s=1}^{N_\phi} K_{rs} \theta^{(s)} = F_r; \quad r = 1, \ldots, N_\phi.
\] (6.26)

or using the matrix notation:

\[
K \theta = F,
\] (6.27)
6.2 Review of the Galerkin method

where

\[ K_{rs} = \int_0^L \phi^{(r)}(x) \left[ EI(x) \phi^{(s)''}(x) \right]'' \, dx \quad (6.28) \]

is the generic element of the stiffness matrix \( K \); while

\[ F_r = \int_0^L \phi^{(r)}(x) q(x) \, dx, \quad (6.29) \]

is the \( r \)th force vector coefficient. It is worth noting here that the generic stiffness matrix term of Eq. (6.28), after double integration by parts, can be rewritten in an equivalent form:

\[ K_{rs} = \int_0^L EI(x) \phi^{(r)''}(x) \phi^{(s)''}(x) \, dx, \quad (6.30) \]

where the order of differentiation has been reduced from four to two and the additional terms, resulting from the integration by parts, at the ends 0 and \( L \) of the integral are null accordingly with the BCs.

6.2.5 Energy approach

Structural problems can also be solved using an energy approach; e.g. the static equilibrium solution can be found by imposing the total potential energy \( \Pi(u_b) \), defined as the sum of the elastic internal strain energy \( \mathcal{U}(u_b) \) and the potential energy due to the external forces \( \mathcal{W}(u_b) \), to be stationary with respect to arbitrary changes \( \delta u_b \). Thus at the equilibrium the variation of the total potential energy \( \delta \Pi(u_b) \) is null for any \( \delta u_b \):

\[ \delta \Pi(u_b) = \left. \frac{\partial \Pi(u_b)}{\partial u_b} \right|_{u_b} = \left. \frac{\partial}{\partial u_b} \left[ \mathcal{U}(u_b) + \mathcal{W}(u_b) \right] \right|_{u_b} = 0, \quad (6.31) \]

where \( \mathcal{U}(u_b) \) and \( \mathcal{W}(u_b) \) are defined as follow:

\[ \mathcal{U}(u_b) = \frac{1}{2} \int_0^L M(x) \kappa(x) \, dx = \frac{1}{2} \int_0^L M(x) u_b''(x) \, dx; \quad (6.32a) \]

\[ \mathcal{W}(u_b) = -\frac{1}{2} \int_0^L q(x) u_b(x) \, dx. \quad (6.32b) \]
In case of the classical elasticity theory, the constitutive equation Eq. (3.5b) allows the strain energy to be rewritten in terms of transverse displacement $u_b(x)$ only:

$$U(u_b) = \frac{1}{2} \int_0^L EI(x) u_b''^2(x) \, dx. \quad (6.33a)$$

Assuming for the displacement functions $u_b(x)$ the trial function as defined in Eq. (6.12), the variation of the total potential energy of Eq. (6.31) can be rewritten in terms of the unknown parameters $\delta \Pi(u_b) = \delta \Pi(\theta^{(r)})$, yielding to the following equations system:

$$\frac{\partial \Pi(\theta^{(r)})}{\partial \theta^{(r)}} = \frac{\partial}{\partial \theta^{(r)}} \left[ V(\theta^{(r)}) + W(\theta^{(r)}) \right] = 0; \quad r = 1, \ldots, N_\phi, \quad (6.34)$$

The substitution of the energy potentials as defined in Eqs. (6.32) and (6.33) yields:

$$\sum_{s=1}^{N_\phi} \theta^{(s)} \int_0^L EI(x) \phi^{(r)''}(x) \phi^{(s)''}(x) \, dx - \int_0^L q(x) \phi^{(r)}(x) \, dx = 0; \quad r = 1, \ldots, N_\phi, \quad (6.35)$$

where can be easily recognised the stiffness matrix terms and the force component of Eqs. (6.29) and (6.30). We can then write the equations system using a matrix notation:

$$K \theta = F, \quad (6.36)$$

which is equivalent to Eq. (6.27) but it has been obtained using a completely different procedure.

### 6.3 Proposed method for the dynamic analysis of hybrid non-local beams

In this Section, a novel computational solution for the dynamic problem of the hybrid non-local beam model is proposed, based on a convenient set of shape functions, which allow achieving high accuracy, independently on the number and location of concentrated damage. The procedure requires three main steps:

1. Definition of the shape functions;
2. Definition of stiffness matrix and mass matrix;
3. Solution of the eigenvalue problem,
6.3 Proposed method for the dynamic analysis of hybrid non-local beams

which are then presented in the following subsections.

6.3.1 Definition of the shape functions

At the first instance, the domain is split in equal intervals $\Delta x$, defined by $N_\phi$ different positions (dummy load point positions) along the beam. Depending on the BCs, it would be:

- $\Delta x = x_{r+1} - x_r = \frac{L}{N_\phi + 1}$, if transverse displacements are prevented at both ends;
- $\Delta x = x_{r+1} - x_r = \frac{L}{N_\phi}$, for the cantilever beam.

Then the approximate solutions for bending moment $M^*(x, t)$, curvature $\chi^*(x, t)$ and displacement $u^*(x, t)$ are assumed to be a linear combination of $N_\phi$ different sets of shape functions:

\[
M^*(x, t) = \sum_{r=1}^{N_\phi} \theta^{(r)}(t) m^{(r)}(x) ;
\]
\[
\chi^*(x, t) = \sum_{r=1}^{N_\phi} \theta^{(r)}(t) c^{(r)}(x) ;
\]
\[
u^b_h(x, t) = \sum_{r=1}^{N_\phi} \theta^{(r)}(t) d^{(r)}(x) ,
\]

where $\theta^{(r)}(t)$ are unknown parameters depending on time only while $m^{(r)}(x)$, $c^{(r)}(x)$ and $d^{(r)}(x)$ are the assumed shape functions. In detail the $r$th transverse displacement shape function $d^{(r)}(x)$ is the closed-form solution of the beam under a unit point load $P^{(r)} = 1$ applied at the dummy load point position $x_r$ (see Fig. 6.4). For the EB beam with classical elasticity the differential equation will be:

\[
[EI(x) d^{(r)}''(x)]'' = \delta(x - x_r) ,
\]

where $x_r = r \Delta x$. Knowing the displacement $d^{(r)}(x)$, all the others static (shear force $v^{(r)}(x)$; bending moment $m^{(r)}(x)$) and kinematic (rotation $b^{(r)}(x)$; local curvature $c^{(r)}(x)$) variables can be computed as following relationships:

\[
b^{(r)}(x) = - \frac{d}{dx} d^{(r)}(x) ;
\]
\[
c^{(r)}(x) = \frac{d}{dx} b^{(r)}(x) ;
\]
6. Non-local multi-damaged beam: Dynamic

Figure 6.4: Displacement shape functions for the dynamic analysis

\[ m^{(r)}(x) = C_2^{(r)} + C_3^{(r)}(x - x_r)H(x - x_r); \]  
\[ v^{(r)}(x) = C_3^{(r)} + H(x - x_r). \]

It is worth noting that using shape functions generated by the application of point loads leads to a similar shear force and bending moment diagrams (piecewise constant and piecewise linear, respectively) as in the classical beam elements, where the cubic shape function for the displacement integrated two/three times leads to linear/constant value of the shear diagram along the beam segment. This is conceptually the same diagram obtained using the proposed approach where the combination of the \( N_\phi \) shear shape functions leads to constant shear diagram among the generic points \( x_r \) and \( x_{r+1} \).

Moreover, the proposed approach, applied to the hybrid gradient elastic beam, let us to study the dynamic problem without considering the continuity of the first derivative of the non-local curvature, which would represent an additional complication.

Furthermore a different load function, with two point loads with opposite sign and applied to two consecutive nodes, generating a localised bending moment has also been considered. The so obtained shape functions even though were more twisted and then able to easily capture the articulated mode shapes did not show to increase the accuracy of the results once compared with those obtained using shape functions from single point load which then has been assumed in the following examples due to their easier implementation.
6.3.2 Definition of stiffness matrix and mass matrix

In dynamics, for an ideal elastic system with no energy dissipation, the $N\phi$ Lagrange’s equation of motion with external forces can be written as:

$$\frac{d}{dt} \left[ \frac{\partial}{\partial \dot{\theta}^{(r)}(t)} \mathcal{L}(t) \right] - \frac{\partial}{\partial \theta^{(r)}(t)} \mathcal{L}(t) = \frac{\partial}{\partial \dot{\theta}^{(r)}} \mathcal{W}(t) \quad r = 1, \ldots, N\phi, \quad (6.40)$$

where $\theta^{(r)}(t)$, $\dot{\theta}^{(r)}(t)$ are the generalised displacements and generalised velocities, respectively; while $\mathcal{L}(t)$ is the so-called Lagrangian, defined as difference between the kinetic energy $\mathcal{T}(t)$ and the potential energy $\mathcal{U}(t)$:

$$\mathcal{L}(t) = \mathcal{T}(t) - \mathcal{U}(t), \quad (6.41)$$

where the internal strain energy $\mathcal{U}(t)$ and the potential energy due to the external forces $\mathcal{W}(t)$ are defined in Eqs. (6.32) while the kinetic energy $\mathcal{T}(t)$ assumes the following form:

$$\mathcal{T}(t) = \frac{1}{2} \int_0^L \rho A \dot{u}^2(x,t) \, dx. \quad (6.42)$$

By using the approximate expressions of the field variables Eqs. (6.37) the energy quantities can be rewritten in terms of the generalised quantities, as follow:

$$\mathcal{U}(t) = \frac{1}{2} \sum_{r=1}^{N\phi} \sum_{s=1}^{N\phi} \theta^{(r)}(t) \theta^{(s)}(t) \int_0^L m^{(r)}(x) e^{(s)}(x) \, dx; \quad (6.43a)$$

$$\mathcal{T}(t) = \frac{1}{2} \sum_{r=1}^{N\phi} \sum_{s=1}^{N\phi} \dot{\theta}^{(r)}(t) \dot{\theta}^{(s)}(t) \int_0^L \rho A d^{(r)}(x) d^{(s)}(x) \, dx; \quad (6.43b)$$

$$\mathcal{W}(t) = \frac{1}{2} \sum_{r=1}^{N\phi} \theta^{(r)}(t) \int_0^L q(x) d^{(r)}(x) \, dx. \quad (6.43c)$$

After some algebra, the application of Eqs. (6.43) to Eq. (6.40) yields to a system of ordinary differential equations, which can be posed in a compact matrix form as:

$$M \ddot{\theta}(t) + K \theta(t) = F(t), \quad (6.44)$$
where \( \theta(t) = [\theta^{(1)}(t); \cdots; \theta^{(N_\phi)}(t)]^\top \) is the vector of the Lagrange variables, \( F(t) = [F^{(1)}(t); \cdots; F^{(N_\phi)}(t)]^\top \) is the vector of the equivalent external load, where each term is expressed as:

\[
P_r(t) = \int_0^L q(x,t) d^{(r)}(x) \, dx ; \tag{6.45}
\]

while \( K \) and \( M \) are stiffness matrix, whose elements are so defined:

\[
K_{rs} = \int_0^L m^{(r)}(x) c^{(s)}(x) \, dx ; \tag{6.46}
\]
\[
M_{rs} = \int_0^L \rho A d^{(r)}(x) d^{(s)}(x) \, dx . \tag{6.47}
\]

### 6.3.3 Eigenvalue problem

In order to study the free vibration of the system, Eq. (6.44) is studied in absence of external forces \( F(t) = 0 \). The resulting system of homogeneous, linear differential equations becomes:

\[
M \ddot{\theta}(t) + K \theta(t) = 0 , \tag{6.48}
\]

whose \( i \)th harmonic solution \( \theta_i(t) \) can be expressed as function of the natural circular frequencies of vibration \( \omega_i \) and the associated vector \( a_i = [a_i^{(1)}; \cdots; a_i^{(N_\phi)}]^\top \) containing the generalised coordinates of the \( i \)th mode shape of the structural system:

\[
\theta_i(t) = a_i \sin(\omega_i t) ; \quad i = 1, \cdots, N_\phi . \tag{6.49}
\]

Substituting Eq. (6.49) into Eq. (6.48), leads to the generalised eigenproblem:

\[
K a_i = \omega_i^2 M a_i ; \quad i = 1, \cdots, N_\phi . \tag{6.50}
\]

Once the solution vector \( a_i \) is known the \( i \)th modal shape would be:

\[
\Phi_i(x) = a_i^\top d(x) , \tag{6.51}
\]

where \( d(x) = [d^{(1)}; \cdots; d^{(N_\phi)}]^\top \) is the vector of the shape functions introduced to approximate the solution.
For the classical elasticity theory, the constitutive law linearly relates bending moment $M(x)$ and curvature $\chi(x)$ through the flexibility function $\Upsilon_I(x)$ along the beam. The internal potential energy $\mathcal{U}(t)$ can then be defined as the integral over the domain of the product of bending moment and curvature:

$$\mathcal{U}(t) = \frac{1}{2} \int_0^L M(x, t) \chi(x, t) \, dx. \quad (6.53)$$

Analogously, for the proposed non-local hybrid elasticity model, where the constitutive law can be expressed as:

$$\Upsilon_I(x, t) M(x) = \bar{\chi}(x, t) - \ell_\varepsilon^2 \bar{\chi}''(x, t), \quad (6.54)$$

Once the $\omega_i$ are known the natural frequencies $f_i$ can be easily obtained.

$$f_i = \frac{\omega_i}{2\pi}. \quad (6.52)$$

The whole explained procedure is summarised in Table 6.1.

### 6.4 Considerations on the energy definitions in the proposed non-local cracked beam

The table below summarises the solution procedure for the dynamic problem:

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Collect all the data for the beam (namely, $E, I_0, L, n, \bar{x}<em>j, \beta_j, q(x)$, classical BCs) including the microstructural parameters ${\ell</em>\varepsilon, \ell_\sigma}$</td>
</tr>
<tr>
<td>2.</td>
<td>Choose the $N_\phi$ positions of the point loads ($N_\phi =$ number of shape functions)</td>
</tr>
<tr>
<td>3.</td>
<td>For each position $x_r$ of the point load evaluate:</td>
</tr>
<tr>
<td></td>
<td>(a) displacement functions $d^{(r)}(x)$</td>
</tr>
<tr>
<td></td>
<td>(b) curvature functions $c^{(r)}(x)$</td>
</tr>
<tr>
<td></td>
<td>(c) bending moment functions $m^{(r)}(x)$</td>
</tr>
<tr>
<td>4.</td>
<td>Evaluate:</td>
</tr>
<tr>
<td></td>
<td>(a) each term $K_{rs}$ of the stiffness matrix as integral all over the domain of the product of $c^{(r)}$ and $m^{(r)}$ (see Eq. (6.46))</td>
</tr>
<tr>
<td></td>
<td>(b) each term $M_{rs}$ of the mass matrix as integral all over the domain of the product of $d^{(r)}$ and $d^{(s)}$ (see Eq. (6.47))</td>
</tr>
<tr>
<td>5.</td>
<td>Evaluate the eigenvalue of the linear system $</td>
</tr>
<tr>
<td>6.</td>
<td>Evaluate the frequencies of the dynamic problem $f = \frac{\omega}{2\pi}$</td>
</tr>
</tbody>
</table>
the potential energy assumes the following form:

\[
U(t) = \frac{1}{2} \int_0^L M(x, t) \left[ \tilde{\chi}(x, t) - \ell_e^2 \tilde{\chi}''(x, t) \right] \, dx \\
= \frac{1}{2} \int_0^L M(x, t) \tilde{\chi}(x) \, dx ,
\]

(6.55)

where \(\tilde{\chi}(x) = \tilde{\chi}(x, t) - \ell_e^2 \tilde{\chi}''(x, t)\) will be here called “work-conjugate” curvature.

### 6.4.1 Energy in terms of local quantities

Interestingly, one can prove that the potential energy, expressed in Eq. (6.55) using local and non-local terms, can be rewritten in a different form using local quantities only. To show this, let us consider the following work-conjugate kinematics quantities, associated with the static quantities \(q(x)\), \(V(x)\) and \(M(x)\):

\[
\tilde{\chi}(x) = \tilde{\chi}(x) - \ell_e^2 \tilde{\chi}''(x) = \chi(x) + (\ell_e^2 - \ell_e^2) \tilde{\chi}''(x) = \tilde{\varphi}'(x) = -\tilde{\varphi}''(x); \tag{6.56a}
\]

\[
\tilde{\varphi}(x) = \tilde{\varphi}(x) - \ell_e^2 \tilde{\varphi}''(x) = \varphi(x) - (\ell_e^2 - \ell_e^2) \tilde{\chi}'(x) = -\tilde{u}'(x); \tag{6.56b}
\]

\[
\tilde{u}(x) = \tilde{u}(x) - \ell_e^2 \tilde{u}''(x) = u(x) - (\ell_e^2 - \ell_e^2) \tilde{u}''(x). \tag{6.56c}
\]

A double integration by parts of Eq. (6.55), expressed using the just defined work-conjugate quantities, gives:

\[
\int_0^L M(x) \tilde{\chi}(x) \, dx = [M(x) \tilde{\varphi}(x)]_0^L + [V(x) \tilde{u}(x)]_0^L + \int_0^L q(x) \tilde{u}(x) \, dx. \tag{6.57}
\]

It should be noted that the product \(M(x) \tilde{\varphi}(x)\) is always zero at the beam’s ends, as at such points the beam can either be free to rotate, and in this case \(M(0, L) = 0\), or fixed, and in this case \(\tilde{\varphi}(0, L) = 0\) (i.e. \(\varphi(0, L) = 0\), and also \(\tilde{\chi}'(0, L) = 0\), see Eq. (6.56b)). On the contrary the quantity \([V(x) \tilde{u}(x)]_0^L\) it is not always null. Substituting the second definition of
Eq. (6.56a) into the internal work (left-hand side of Eq. (6.57)) leads to:

\[
\int_0^L M(x) \tilde{\chi}(x) \, dx = \int_0^L M(x) \left[ \chi(x) + (\ell_\sigma^2 - \ell_\varepsilon^2) \tilde{\chi}''(x) \right] \, dx
\]

\[= \int_0^L M(x) \chi(x) \, dx + (\ell_\sigma^2 - \ell_\varepsilon^2) \int_0^L M(x) \tilde{\chi}''(x) \, dx. \tag{6.58}
\]

Substituting now Eq. (6.56b) and Eq. (6.56c) into the two terms in the right part of Eq. (6.57) gives:

\[
\left[ V(x) \tilde{u}(x) \right]_0^L = \left[ V(x) u(x) \right]_0^L - (\ell_\sigma^2 - \ell_\varepsilon^2) \left[ V(x) \tilde{\chi}(x) \right]_0^L; \tag{6.59a}
\]

\[
\int_0^L q(x) \tilde{u}(x) \, dx = \int_0^L q(x) u(x) \, dx - (\ell_\sigma^2 - \ell_\varepsilon^2) \int_0^L q(x) \tilde{\chi}(x) \, dx. \tag{6.59b}
\]

Recalling that \( q(x) = V'(x) \), after a double integration by parts of the last integral of Eq. (6.59b), one obtains:

\[
\int_0^L V'(x) \tilde{\chi}(x) \, dx = \left[ V(x) \tilde{\chi}(x) \right]_0^L - \left[ M(x) \tilde{\chi}'(x) \right]_0^L + \int_0^L M(x) \tilde{\chi}''(x) \, dx. \tag{6.60}
\]

Finally, combining Eqs. (6.58), (6.59a), (6.59b) and (6.60), after few simplifications, yields:

\[
\int_0^L M(x) \chi(x) \, dx = \int_0^L q(x) u(x) \, dx, \tag{6.61}
\]

which is the equivalence between the internal work and the external work expressed using only local quantities. The main differences with respect to the energy balance of Eq. (6.55) is that by using local quantities only the resulting stiffness matrix would be not symmetric.

### 6.4.2 Work conjugate

As part of this study, it has been verified the validity of the Maxwell-Betti reciprocal work theorem for the non-local beam model. It is well known that this theorem ensures that, for a linear elastic structure subjected to two sets of forces \( P_1 \) and \( P_2 \), the work done by the first load set \( P_1 \) for the displacement produced by the second load \( u_2(x_1) \) is equal to the work done by the second load set \( P_2 \) on the displacement due to \( u_1(x_2) \), i.e.:

\[ P_1 u_2(x_1) = P_2 u_1(x_2), \tag{6.62} \]
being \( x_1 \) and \( x_2 \) the abscissas where the loads \( P_1 \) and \( P_2 \) are applied. Eq. (6.62) is valid for a beam featuring a classical (local) elastic constitutive law, and (due to the equivalence of internal and external work) can also be expressed as:

\[
\int_0^L M_1(x) \chi_2(x) \, dx = \int_0^L M_2(x) \chi_1(x) \, dx ,
\]

(6.63)

where \( M_r(x) \) is the bending moment associated with the load \( P_r \) and \( \chi_r(x) = -u''_r(x) \), with \( r = 1, 2 \).

It is also well known, in the context of the classical (local) elasticity, that the external work of Eq. (6.62) and the internal work of Eq. (6.63) are equivalent, that is:

\[
\int_0^L M_1(x) \chi_2(x) \, dx = \int_0^L P_1(x) u_2(x) \, dx ;
\]

(6.64a)

\[
\int_0^L M_2(x) \chi_1(x) \, dx = \int_0^L P_2(x) u_1(x) \, dx ,
\]

(6.64b)

For the proposed EB hybrid non-local beam under two sets of transverse loads \( q_1(x) \) and \( q_2(x) \) the energy balance between internal and external forces of Eqs. (6.64) assumes the following expressions:

\[
\int_0^L M_1(x) \tilde{\chi}_2(x) \, dx = P_1 u_2(x_1) + \left[ V_1(x) \tilde{u}_2(x) \right]_0^L ;
\]

(6.65a)

\[
\int_0^L M_2(x) \tilde{\chi}_1(x) \, dx = P_2 u_1(x_2) + \left[ V_2(x) \tilde{u}_1(x) \right]_0^L ,
\]

(6.65b)

where the \( M_i(x) \), \( \tilde{\chi}_i(x) \), \( u_i(x) \) and \( \tilde{u}_i(x) \) are the \( i \)th \( (i = 1, 2) \) bending moment, work-conjugate curvature, local displacement and work-conjugate displacement due to the load \( P_i(x) \). After introducing \( \Upsilon_i M_1(x) = \tilde{\chi}_1(x) \) and \( \Upsilon_i M_2(x) = \tilde{\chi}_2(x) \), it can be easily seen that the two above equations are equivalent:

\[
\int_0^L \Upsilon_1(x) \tilde{\chi}_1(x) \tilde{\chi}_2(x) \, dx = \int_0^L \Upsilon_2(x) \tilde{\chi}_2(x) \tilde{\chi}_1(x) \, dx .
\]

(6.66)

One of the main consequences of the theorem is the symmetry of the stiffness matrices as each terms of \( K_{rs} \) defined as Eq. (6.46) is equal to the symmetric one \( K_{sr} \). Different considerations have to be done if the energy quantities are expressed in terms of local quantities Eqs. (6.61), in this case bending moment and local curvature are not linearly related as for
mending moment and work-conjugate curvature Eq. (6.54), then the stiffness matrix generated by using Eqs. (6.61) will not be symmetric.

6.5 Numerical examples

The dynamic analysis of three beams with different BCs has been carried out by implementing the proposed energy approach within an in house code built using the software Mathematica [244]. The exact dynamic response of the hybrid non-local multicrocked beam element is not yet available in the literature, and so to validate the proposed model the case of undamaged non-local beam has been taken into account, as well as the multi-damage local beam solution. The convergence for an increasing number of shape functions has also been investigated. The numerical examples have also been used to highlight and quantify the effects of the non-local length scale parameters $\ell_\varepsilon$ and $\ell_\sigma$ on the dynamic response of the damaged beams. Since the parameters are linked to the beam’s microstructure, their values cannot exceed the beam’s length, i.e. $0 \leq \ell_\varepsilon < L$ and $0 \leq \ell_\sigma < L$. The three examples studied are: a cantilever beam with a single concentrate damage; a simply supported beam with two cracks; a clamped-clamped beam with three damages.

For sake of generality the results are presented in terms of the dimensionless parameters, e.g. $\omega_i/\omega_{i,\text{loc}}$ is the ratio between the $i$th frequency of the non-local beam and the $i$th frequency of the beam using classical (local) elasticity.

6.5.1 Example one – Cantilever beam with a single damage

A cantilever beam of length $L$ clamped at $x = 0$ is studied in the first example (see Fig. 6.5). The beam has a single crack at the position $x_1 = L/3$, modelled with a rotational spring with dimensionless parameters $\beta_1 = 0.1$ (see Eq. (3.10b)).

![Figure 6.5: Example one – Cantilever beam with a single damage ($\beta_1 = 0.1$)](image)

Following the procedure presented in Table 6.1, a set of $N_{\phi}$ different concentrated loads pointing downwards is applied along the beam, each one giving a different shape function for the displacement $d^{(r)}(x)$, the local curvature $c^{(r)}(x)$ and the bending moment $m^{(r)}(x)$,
whose closed form solution can be found by using the procedure mentioned in Section 5.1. The integration constants $C_0^{(r)}$, $C_1^{(r)}$, $C_2^{(r)}$ and $C_3^{(r)}$ are evaluated by solving the four equations that result from applying the BCs for the internal forces at the free end ($v^{(r)}(L) = m^{(r)}(L) = 0$) and the kinematic quantities at the fixed end ($d^{(r)}(0) = b^{(r)}(0) = 0$); the other constants $C_4^{(r)}$ and $C_5^{(r)}$ are evaluated by assuming the stationarity of the effective non-local curvature at both ends of the beam ($\tilde{c}'^{(r)}(0) = \tilde{c}'^{(r)}(L) = 0$).

Following the procedure of Table 5.1, the BCs on the internal forces $m^{(r)}(L) = 0$ and $v^{(r)}(L) = 0$ due to the $r$th load position yield to:

\[ C_2^{(r)} = x_r; \quad C_3^{(r)} = -1. \quad (6.67a) \]

The other two constants $C_0^{(r)}$ and $C_1^{(r)}$ are defined by the BCs $d^{(r)}(0) = 0$ and $b^{(r)}(0) = 0$

\[ d^{(r)}(0) = d_0^{(r)}(0) + \Delta d_1^{(r)}(0) = 0, \quad (6.68) \]

where

\[
d_0^{(r)}(0) = C_0^{(r)} - \mathcal{Y}_I_0 C_3^{(r)} \ell_\varepsilon (\ell_2^2 - \ell_\sigma^2) \tanh \left( \frac{1}{2 \ell_\varepsilon} \right) + - 2 \mathcal{Y}_I_0 \ell_\varepsilon (\ell_2^2 - \ell_\sigma^2) \text{csch} \left( \frac{1}{\ell_\varepsilon} \right) \sinh \left( \frac{x_r - 1}{2 \ell_\varepsilon} \right), \quad (6.69)\]

and

\[ \Delta d_1^{(r)}(0) = - \mathcal{Y}_I_0 \beta_1 \ell_\varepsilon \left[ x_r - \bar{x}_1 + (\bar{x}_1 - x_r)H(\bar{x}_1 - x_r) \right] \left[ \text{csch} \left( \frac{1}{\ell_\varepsilon} \right) \cosh \left( \frac{1 - \bar{x}_1}{\ell_\varepsilon} \right) \left( \ell_2^2 - \ell_\sigma^2 \right) \right]. \quad (6.70) \]

While the rotation at the position $x = 0$ is given by:

\[ b^{(r)}(0) = b_0^{(r)}(0) + \Delta b_1^{(r)}(0) = 0, \quad (6.71) \]

where the undamaged contribution $b_0^{(r)}(0)$ is

\[ b_0^{(r)}(0) = C_1^{(r)} + \mathcal{Y}_I_0 \left( \ell_2^2 - \ell_\sigma^2 \right), \quad (6.72a) \]
and the additional contribution due to the damage $\Delta b_0^{(r)}(0)$ is:

$$\Delta \varphi_1^{(r)}(0) = 0.$$  \hspace{1cm} (6.72b)

The resulting integration constants are then:

$$C_1^{(r)} = -\mathcal{T}_I \ell \left( \ell^2 \varepsilon - \ell^2 \sigma \right);$$  \hspace{1cm} (6.73)

$$C_0^{(r)} = -\mathcal{T}_I \ell \left( \ell^2 \varepsilon - \ell^2 \sigma \right) \left\{ \tanh \left( \frac{1}{2\ell \varepsilon} \right) + 
\right.$$  

$$\left. + 2 \csch \left( \frac{1}{\ell \varepsilon} \right) \sinh^2 \left( \frac{x_r - 1}{2\ell \varepsilon} \right) \right\} - \Delta d^{(r)}(0).$$  \hspace{1cm} (6.74)

Furthermore, the additional integration constants $C_4^{(r)}$ and $C_5^{(r)}$ are those defined in Eqs. (5.14) and (5.13), obtained by imposing the BCs $\bar{c}'(0) = 0$ and $\bar{c}'(L) = 0$. For the $r$th loading position, the functions $Q(x)$ and $Q'(x)$ particularise as:

$$Q^{(r)}(x) = H(x - x_r) \left[ \frac{x - x_r}{\ell \varepsilon} - \sinh \left( \frac{x - x_r}{\ell \varepsilon} \right) \right],$$  \hspace{1cm} (6.75)

and

$$Q'(x) = \ell \varepsilon H(x - x_r) \left[ 1 - \cosh \left( \frac{x - x_r}{\ell \varepsilon} \right) \right],$$  \hspace{1cm} (6.76)

leading to:

$$C_5^{(r)} = 0;$$  \hspace{1cm} (6.77a)

$$C_4^{(r)} = \mathcal{T}_I \ell \left\{ \frac{x_r}{\ell \varepsilon} - \tanh \left( \frac{L}{2\ell \varepsilon} \right) - \ell \varepsilon \left[ 1 - \cosh \left( \frac{L - x_r}{\ell \varepsilon} \right) \right] \csch \left( \frac{L}{\ell \varepsilon} \right) \right\}. $$  \hspace{1cm} (6.77b)

### 6.5.1.1 Shape functions for the cantilever beam

Once the values of the integration constants are known, it is possible to evaluate the shape functions for each position of the PL. Figure 6.6 shows the positions of the PL forces for the case $N_\phi = 8$ and the correspondent eight shape functions for transverse displacement, curvature and bending moment for $\{\ell \varepsilon = L/20; \ell \sigma = 0\}$.

As expected, the eight displacement shape functions $d^{(r)}(x)$ of Fig. 6.6(a) show a higher deflection when the load is applied farther from the fixed end. Since the crack is located within the interval $[x_2, x_3]$, its presence does not affect the first two shape functions (as for $x_r \leq \bar{x}_1$ the cracked section is unloaded). The presence of the crack is hardly noticeable
from the deformed shapes $d^{(r)}(x)$ of Fig. 6.6(a), while it becomes evident in the associated curvature functions $c^{(r)}(x)$ of Fig. 6.6(b), in which a cusp appears for $r \geq 3$; the larger $x_r$, the higher the bending moment $m^{(r)}(x)$, the bigger the cusp. Since the cantilever beam is statically determinate, the bending moment functions $m^{(r)}(x)$ of Fig. 6.6(c) do not depend on constitutive law of the material and are insensitive to the presence of any crack. Indeed, they are represented by inclined straight lines having their maximum value at $x = 0$ and linearly reducing up to zero at the position where the force $P_r$ is applied, and $m^{(r)}(x) = 0$ for $x \geq x_r$. 
6.5 Numerical examples

<table>
<thead>
<tr>
<th>$\omega_{1,\text{loc}}$</th>
<th>$\omega_{2,\text{loc}}$</th>
<th>$\omega_{3,\text{loc}}$</th>
<th>$\omega_{4,\text{loc}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.72</td>
<td>107.47</td>
<td>290.98</td>
<td>607.63</td>
</tr>
</tbody>
</table>

Table 6.2: Example one – First four natural frequencies for the local case solution $\omega_{i,\text{loc}}$ for the cantilever beam

Figure 6.7: Example one – Natural frequency $\omega_i$ ($i = 1, 2, 3, 4$) of the cracked cantilever beam for $\ell_\sigma = 0$ against the strain length scale parameter $\ell_\varepsilon$ (the horizontal line represents the local case solutions $\ell_\sigma = \ell_\varepsilon$) ($N_\phi = 16$)

6.5.1.2 Effects of the length scale parameters

The results in terms of natural frequencies and modal shapes obtained with the non-local formulation for different combinations of the length scale parameters $\ell_\varepsilon$ and $\ell_\sigma$ have been compared with those of the classical elasticity theory. As pointed out in the previous Chapter, the latter are equivalent to those offered by the hybrid gradient-elastic theory when the two length scale parameters take the same value ($\ell_\varepsilon = \ell_\sigma \geq 0$).

Figure 6.7 shows with a log-log plot the variation of the first four natural frequencies ($\omega_i$, $i = 1, 2, 3, 4$) without stress gradient effects ($\ell_\sigma = 0$) against the dimensionless strain length scale parameter $\ell_\varepsilon/L$ (between $1/100$ and $1/5$), while the horizontal lines represent the limiting values of the classical elasticity, also listed in Table 6.2. For the fourth frequency, the increment is larger than 100\% of the local elasticity solution when $\ell_\varepsilon/L = 1/5$, as it can be clearly seen in Fig. 6.8(a), where the ratio between the non-local and local solutions $\omega_i/\omega_{i,\text{loc}}$ for the first four frequencies is plotted against the strain length scale parameter. It is possible to notice how the relative effect is higher for higher frequencies, independently of the strain length scale parameter.
6. Non-local multi-damaged beam: Dynamic

\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\{i = 4; \ell_\sigma = 0\}
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\{i = 3; \ell_\sigma = 0\}
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\{i = 2; \ell_\sigma = 0\}
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\{i = 1, \ell_\sigma = 0\}
\]
\[
\ell_\sigma = \ell_\varepsilon
\]

\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\{i = 4; \ell_\sigma = L/20\}
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\{i = 3; \ell_\sigma = L/20\}
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\{i = 2; \ell_\sigma = L/20\}
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\{i = 1; \ell_\sigma = L/20\}
\]
\[
\ell_\sigma = \ell_\varepsilon
\]

Figure 6.8: Example one – Normalised natural frequency \(\frac{\omega_i}{\omega_{i,loc}}(i = 1, 2, 3, 4)\) of the cracked cantilever beam for \(\ell_\sigma = 0\) (a) and \(\ell_\sigma = L/20\) (b) (the solid horizontal line represents the local case solutions \(\ell_\sigma = \ell_\varepsilon\)).

Similarly to Fig. 6.8(a), Fig. 6.8(b) shows the trend against \(\ell_\varepsilon/L\) of the ratio \(\frac{\omega_i}{\omega_{i,loc}}\) for the first four natural frequencies when the stress length scale parameter is taken as \(\ell_\sigma = L/20\). All the curves intersect at \(\ell_\varepsilon/L = 1/20\), where the two length scale parameters are equal, and at this point the solution coincide with the classical continuum solution, i.e. \(\frac{\omega_i}{\omega_{i,loc}} = 1\). In this circumstance maximum ratio for the fourth frequency is obtained again at \(\ell_\varepsilon/L = 1/5\), where \(\omega_4/\omega_{4,loc}\) is slightly less than 2, i.e. less than the maximum value for \(\ell_\sigma = 0\). This phenomenon can be better understood looking at Figs. 6.9(a) and (b), where the trend for the

\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\ell_\sigma = 0
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\ell_\sigma = L/50
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\ell_\sigma = L/20
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\ell_\sigma = L/10
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(a)\quad\ell_\sigma = \ell_\varepsilon
\]

\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\ell_\sigma = 0
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\ell_\sigma = L/50
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\ell_\sigma = L/20
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\ell_\sigma = L/10
\]
\[
\frac{\omega_i}{\omega_{i,loc}}(b)\quad\ell_\sigma = \ell_\varepsilon
\]

Figure 6.9: Example one – Normalised natural frequency \(\frac{\omega_i}{\omega_{i,loc}}\) for the first \((i = 1)\) (a) and fourth \((i = 4)\) (b) natural frequencies of the cracked cantilever beam for four different values of the stress length scale parameter \(\ell_\sigma\) \((N_\phi = 16)\).
first and fourth frequencies are plotted for different values of \( \ell_\sigma = \{0; \ L/50; \ L/20; \ L/10; \ \ell_\epsilon\} \). All the different curves intersect the local case solution when \( \ell_\epsilon = \ell_\sigma \) and increasing the stress length scale parameter \( \ell_\sigma \) reduces the frequencies. Fig. 6.9(a) allows quantifying the effects of the length scale parameters for the first and the fourth frequency. In both cases we observe that for \( \ell_\epsilon < \ell_\sigma \) the modal frequency reduces, i.e. \( \omega_i < \omega_{i,\text{loc}} \), meaning that the gradient-elastic beam appears less stiff than the local counterpart. The opposite happens for \( \ell_\epsilon > \ell_\sigma \), as in this case the modal frequencies increase, i.e. \( \omega_i > \omega_{i,\text{loc}} \). For a given value of the stress-gradient length scale parameter \( \ell_\sigma \), the larger \( |\ell_\epsilon - \ell_\sigma| \), the more significant is the variation observed in the modal frequencies.

Moreover, for a given set of length scale parameters \( \{\ell_\epsilon; \ \ell_\sigma\} \), the higher the mode of vibration, the larger tends to be the variation of the modal frequency. For the objective beam under consideration and the range of selected parameters there is a maximum reduction of first modal frequency \( \omega_1 \) of about 6\% for \( \ell_\epsilon = L/100 \) and \( \ell_\sigma = L/10 \), while the maximum increase of about 5\% is seen for \( \ell_\epsilon = L/5 \) and \( \ell_\sigma = 0 \) (see Fig. 6.9(a)). Larger variation are observed for the fourth modal frequency (see Fig. 6.9(b)), in which for the same values of length scale parameters \( \omega_4 \) reduces up to about 30\% less than \( \omega_{4,\text{loc}} \) (for \( \ell_\epsilon = L/100 \) and \( \ell_\sigma = L/10 \)) and increases up to about 120\% more than \( \omega_{4,\text{loc}} \) (for \( \ell_\epsilon = L/5 \) and \( \ell_\sigma = 0 \)).

Figure 6.10 shows the trend for the first four natural frequencies for a constant value of the strain length scale parameter, \( \ell_\epsilon = L/20 \), against the stress length scale \( \ell_\sigma \). As usual, the intersection point represents the solution for \( \ell_\epsilon = \ell_\sigma = L/20 \) and is equivalent to the classical continuum solution. For both left and right side of the intersection point \( (\ell_\sigma = L/20) \), correspond-
Figure 6.11: Example one – Normalised natural frequency $\omega_i/\omega_{i,\text{loc}}$ ($i = 1, 2, 3, 4$) of the cracked cantilever beam for several combinations of the ratio $\rho_\ell = \{0; 1/4; 1/2; 3/4; 1; 3/2; 2\}$ (the solid horizontal line represents the local case solution $\rho_\ell = \ell_\sigma/\ell_\varepsilon = 1$) ($N_\phi = 16$)

ent to length scale ratio $\rho_\ell < 1$ and $\rho_\ell > 1$, respectively, the higher the frequency, the larger are the non-local effects. In particular, for the fourth frequency and $\{\ell_\varepsilon = L/20; \ell_\sigma = L/100\}$ the frequency ratio $\omega_4/\omega_{4,\text{loc}}$ is about 1.15, meaning that the non-local frequency is 15% higher than the corresponding local one; while for $\{\ell_\varepsilon = L/20; \ell_\sigma = L/5\}$ the non-local frequency is about 50% lower than the corresponding local one.

Figures 6.11 show the trend for the first four modal frequencies for different values of the length scale ratio $\rho_\ell = \{0; 1/4; 1/2; 3/4; 1; 3/2; 2\}$. The curves show that the non-local effects vanish for $\rho_\ell = 1$, while the more the length scale ratio $\rho_\ell \geq 0$ differs from 1, the larger the variation in terms of modal frequencies, and such variation is comparatively more significant for higher modes of vibration. The effects of the damage parameter $\beta_1$ on the first four natural frequencies are illustrated with Figs. 6.12 assuming $\ell_\varepsilon = L/20$ and five different values of the strain length parameter $\ell_\varepsilon = \{L/40; L/30; L/20; L/10; L/5\}$. As expected the numerical results, when compared with the local beam counterpart ($\ell_\sigma = \ell_\varepsilon = L/20$), show
6.5 Numerical examples

\[ \ell_\varepsilon = \frac{L}{40}; \cdots; \ell_\varepsilon = \frac{L}{30}; \ell_\sigma = \ell_\varepsilon; \cdots; \ell_\varepsilon = \frac{L}{10}; \cdots; \ell_\varepsilon = \frac{L}{5} \]

\[ \ell_\varepsilon = \frac{L}{5} \]

\[ \ell_\varepsilon = \frac{L}{40}; \cdots; \ell_\varepsilon = \frac{L}{30}; \ell_\sigma = \ell_\varepsilon; \cdots; \ell_\varepsilon = \frac{L}{10}; \cdots; \ell_\varepsilon = \frac{L}{5} \]

\[ \ell_\varepsilon = \frac{L}{5} \]

Figure 6.12: Example one – Diagram of the first four frequencies \( \omega_i \) \( (i = 1, 2, 3, 4) \) against the damage parameter \( \beta_1 \) for the cantilever beam for \( \ell_\sigma = L/20 \) and four different values of the strain length scale parameter \( \ell_\varepsilon = \{ L/40; L/30; L/20; L/10; L/5 \} \) \( (N_\phi = 16) \)

how the material appears to soften for \( \ell_\varepsilon < \ell_\sigma \), leading to lower frequencies; on the contrary, the material becomes stiffer for \( \ell_\varepsilon > \ell_\sigma \). Focusing now on the effects of the damage parameter on the modal frequencies, the higher the damage, the more flexible is the structure, the lower tend to be the frequencies. This is particularly evident with the first three modal frequencies in which, independently of the microstructural parameters \( \ell_\varepsilon \) and \( \ell_\sigma \), a significant reduction of the frequencies \( \omega_1, \omega_2 \) and \( \omega_3 \) is observed when \( \beta_1 \) increases. A much less pronounced reduction is seen in the fourth modal frequency \( \omega_4 \) (Fig. 6.12(d)), as the associated modal shape is such that the bending moment is very small at the cracked cross section \( x = L/3 \), and for this reason the concentrated damage has very little effect on the fourth mode.

6.5.1.3 Modal shapes

The above analysis of the modal frequencies proceeds with studying the effects of the microstructural parameters \( \ell_\varepsilon \) and \( \ell_\sigma \) on the modal shapes of the cracked cantilever beam.
6. Non-local multi-damaged beam: Dynamic

\[ \ell_\varepsilon = \frac{L}{40} \; ; \; \ell_\varepsilon = \frac{L}{20} \; ; \; \ell_\varepsilon = \frac{L}{10} \; ; \; \ell_\varepsilon = \frac{L}{5} \]

\[ \ell_\sigma = \frac{L}{20} \]

Figure 6.13: Example one – First four mode shapes for the cantilever beam for \( \ell_\sigma = L/20 \) and four different values of the strain length scale parameter \( \ell_\varepsilon = \{L/40; L/20; L/10; L/5\} \) \((N\phi = 8)\)

Figs. 6.13 then show the first four modal shapes for four different combinations of the length scale parameters \( \ell_\sigma = L/20 \) and \( \ell_\varepsilon = \{L/40; L/20; L/10; L/5\} \). As usual, the solid line represents the local case, retrieved for \( \ell_\varepsilon = \ell_\sigma = L/20 \). The first observation is that the first mode shape \( \Phi_1(x) \) is scarcely affected by the microstructural parameters, while their effects become increasingly more significant in the higher modes of vibration (see Fig. 6.13(a)). Interestingly, Fig. 6.13(b), (c) and (d) reveal that the \( i \)th modal shape (with \( i \geq 2 \)) has \( i - 1 \) nodes, i.e. points (other than the fixed end at \( x = 0 \)) where the family of curves \( \Phi_i(x) \) tends to pass, independently of the strain gradient parameter \( \ell_\varepsilon \).

### 6.5.1.4 Convergence

In order to check the efficiency of the proposed method, the convergence of the frequency error against the number of shape functions has been investigated. Since the exact dynamic solution of the cracked hybrid non-local beam model is not available, the frequency obtained using a higher number of shape functions \((N\phi = 64)\) has been assumed as a reference value for this
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Figure 6.14: Example one – Convergence diagram of the normalised frequency error $\epsilon_i$ ($i = 1, 2, 3, 4$) for the first four frequencies for the cantilever beam for $\ell_\sigma = 0$ and four different values of the strain length scale parameter $\ell_\varepsilon = \{0; L/50; L/20; L/10\}$ study. The dimensionless measure of frequency error for the $i$th mode has been defined as:

$$
\epsilon_i(N_\phi) = \frac{|\omega_i(N_\phi) - \omega_{i,\text{ref}}|}{\omega_{i,\text{ref}}},
$$

where $\omega_{i,\text{ref}} = \omega_i(64\text{el})$, and the trend of $\epsilon_i(N_\phi)$ is shown for the first four modal frequency as part of Fig. 6.14. First of all, one can observe that the relative error assumes values lower than 2% for all the cases investigated where the number of elements range from 6 to 64 elements, it can then be considered highly acceptable in most of the engineering simulations. All the graphs show the convergence to the reference solution for an increasing number of shape functions. Interestingly, the relative error of the first three modal frequencies is already lower than 1% when six shape functions are adopted, while for the fourth frequency it gradually reduces from 2% to 1% with $N_\phi = 16$. Moreover, the error does not decreases monotonically for third and fourth modal frequency, due to the fact that different positions of the dummy point loads lead to different shape functions and, unless the interval $\Delta x$ is split into an integer number of smaller intervals (e.g. $\Delta x/2, \Delta x/3$, etcetera), then the discretisation with more shapes function can be less accurate. However, for this example it is clear that using 16 points (i.e. $\Delta x = L/16$) is enough to ensure that the results are enough accurate to study the first four natural frequencies.
6.5.2 Example two – Simply supported beam with two cracks

For the second numerical example, a simply supported beam of length $L$ pinned at $x = 0$ and $x = L$ has been considered (see Fig. 6.15). The beam has two cracks at $x_1 = L \times 3/10$ and $x_2 = L \times 4/7$, modelled with two rotational springs with the same dimensionless damage parameter $\beta_1 = \beta_2 = 0.1$ (see Eq. (3.10b)).

![Figure 6.15: Example two – Simply supported beam with two damages ($\beta_1 = \beta_2 = 0.1$)](image)

6.5.2.1 Shape functions

The number of shape functions has been selected as $N_\phi = 7$, corresponding to a discretisation interval $\Delta x = L/8$. The $r$th shape function $d^{(r)}(x)$ is the deformed shape caused by a unit point load applied at $x_r = r \Delta x$, as illustrated within Fig. 6.16(a), while the corresponding curvature $c^{(r)}(x)$ and bending moment $m^{(r)}(x)$ are shown with Figs. 6.16(b) and (c), respectively. As expected, the seven displacement shape functions $d^{(r)}(x)$ show a higher deflection when the load is applied close to the mid-span of the beam ($x = L/2$). As in the first example, the presence of the damage is clearly evidenced with the curvature functions, which show a cusp at each crack position. The increase in the curvature is higher for higher values of the bending moment around the crack position. Interestingly, although the beam is simply supported, the curvature does not go to zero because of the gradient-elastic constitutive law.

6.5.2.2 Effects of the length scale parameters

The results in terms of natural frequencies and modal shapes obtained with the adopted non-local formulation for different combinations of the length scale parameters $\ell_\varepsilon$ and $\ell_\sigma$ have been compared with those of the classical elasticity theory. For comparison purpose Tab. 6.3 gives the first four frequencies for the classical elasticity solution $\omega_{i,\text{loc}}$ ($i = 1, 2, 3, 4$).

The log-log plot of Fig. 6.17 shows for the Aifantis’ model ($\ell_\sigma = 0$), how the first four natural frequencies increase with the strain length scale parameter $\ell_\varepsilon$. 

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Figure 6.16: Example two – Transversal displacement (a), curvature (b) and bending moment (c) shape functions for the multi cracked simply supported beam for \( \ell = L/20; \ell_\sigma = 0 \) \( (N_\phi = 7) \)

A comparison of the rate of increase of the natural frequencies for the Aifantis’ model is offered with Fig. 6.18(a), revealing that the higher the mode, the larger the effects. For instance, the first natural frequency rises of about 5% when the ratio \( \ell_c/L \) increases from 1/100 up to 1/5, while the fourth natural frequency rises of more than 200% for the same variation of \( \ell_c/L \). This indicates that the length scale parameter \( \ell_c \) exerts a more significant effect at the higher frequency. Fig. 6.18(b) displays the results of the same analysis done with the hybrid gradient-elastic model, with stress length scale parameters \( \ell_\sigma = L/20 \). As in Fig. 6.18(a), higher frequencies are more affected by the strain length scale; furthermore for \( \ell_c = \ell_\sigma = L/20 \), the
Table 6.3: Example two – First four natural frequencies for the local case solution $\omega_{i,\text{loc}}$ for the simply supported beam

<table>
<thead>
<tr>
<th>$\omega_{1,\text{loc}}$</th>
<th>$\omega_{2,\text{loc}}$</th>
<th>$\omega_{3,\text{loc}}$</th>
<th>$\omega_{4,\text{loc}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>43.21</td>
<td>179.79</td>
<td>421.45</td>
<td>741.61</td>
</tr>
</tbody>
</table>

Figure 6.17: Example two – First four natural frequencies of the cracked simply supported beam for $\ell_\sigma = 0$ and several combinations of strain length scale parameter $\ell_\varepsilon$ (the horizontal lines represent the local case solutions $\ell_\sigma = \ell_\varepsilon$) ($N_\varphi = 15$)

frequencies equal their local counterpart; for $\ell_\varepsilon < \ell_\sigma$ the normalised frequency is lower than the local one; whereas the opposite happens for $\ell_\varepsilon > \ell_\sigma$.

In general, the higher the microstructural parameters $\{\ell_\varepsilon/L; \ell_\sigma/L\}$, the larger their effects, the more significant will be the non-local theory. In this context, Fig. 6.19 shows that in the first mode the effects of the microstructure start to be relevant (higher than 1% if compared with the classical elasticity theory) only for the strain length scale $\ell_\varepsilon/L \geq 1/20$; in the second mode instead a smaller microstructure $\ell_\varepsilon/L \simeq 1/45$ is enough to show microstructural effects of about 1%; for the third and fourth frequencies the effects can be seen even for a smaller microstructure, $\ell_\varepsilon/L \simeq 1/65$ for the third mode and $\ell_\varepsilon/L \simeq 1/85$ for the fourth mode. Tab. 6.4 shows the ratio between the average modal wave lengths $\lambda_i = L/i$ and the strain length scale parameter $\ell_\varepsilon(1\%)$ associated with an effect of about 1% if compared with the classical solution, for the first four natural frequencies ($i = 1; 2; 3; 4$) of the simply supported beam $\ell_\sigma = 0$. Interestingly the table highlights how, independently of the frequency considered, the microstructural parameters starts to be noticeable (effects higher than 1%) when the ratio $\lambda_i/\ell_\varepsilon$ is less than about 20. For this reason higher frequencies are more affected by the microstructural
6.5 Numerical examples

6.5.2.3 Modal shapes

The dynamic analysis of the doubly-cracked simply-supported beam of Fig. 6.15 proceeds with studying the first four modal shapes, looking at the effects of different microstructural parameters. Fig. 6.22 shows the modal shapes for four different combinations of the length scale parameters $\ell_\sigma = L/20$ and $\ell_\epsilon = \{L/40; L/20; L/10; L/5\}$. The solid line represents the local case solution as $\ell_\epsilon = \ell_\sigma = L/20$. It can be observed that, as in the previous example, the trends in the results are consistent with the previous examples, leading to similar considerations on the effects of varying $\ell_\epsilon$ and $\ell_\sigma$. 

Parameters, as they correspond to wave lengths comparable with the length scales. Figures 6.20 and 6.21 show similar trends of results as those presented in Figs. 6.20 and 6.11 for the cantilever beam analysis in the previous example, leading to similar consideration on the effects of varying $\ell_\epsilon$ and $\ell_\sigma$. 

Table 6.4: Example two – Relation between wave lengths $\lambda_i$ and the microstructural parameters for the first four natural frequencies of the simply supported beam for the case $\ell_\sigma = 0$.
Figure 6.19: Example two – Normalised natural frequency $\omega_i/\omega_{i,loc}$ ($i = 1, 2, 3, 4$) on the range $\{1 - 1.02\}$ of the cracked simply supported beam for $\ell_\sigma = 0$ (the solid horizontal line represents the local case solutions $\ell_\sigma = \ell_\varepsilon$) ($N_\phi = 16$)

Figure 6.20: Example two – Normalised natural frequency $\omega_i/\omega_{i,loc}$ for the first ($i = 1$) (a) and fourth ($i = 4$) (b) natural frequencies of the cracked simply supported beam for several combinations of the stress length scale parameter $\ell_\sigma$ (the solid horizontal line represents the local case solution $\ell_\sigma = \ell_\varepsilon$) ($N_\phi = 15$)

(see Fig. 6.13), independently of the strain gradient parameter $\ell_\varepsilon$, the modal shape $\Phi_i(x)$ shows $(i - 1)$ modes, i.e. internal points where all the different modal shapes tend to pass. Figs. 6.23, 6.24, 6.25 and 6.26 display the first four modal shapes not only in terms of transverse displacement (a), but also rotation $\phi_i$ (b), bending moment $M_i$ (c) and shear force $V_i$ (d). All the results are obtained using $N_\phi = 7$ dummy load points. In particular, the rotations profiles for the different microstructures highlight the smoothing effect due to the inclusion of the non-local theory when $0 < \rho_\ell < 1$ (i.e. when $\ell_\varepsilon > \ell_\sigma$) while when $\rho_\ell > 1$ (i.e. when $\ell_\varepsilon < \ell_\sigma$) the jump on the rotations is even amplified if compared with the classical solution, leading to an angular point at the deformation profiles.
As a direct consequence of the proposed computational approach, the shear force diagrams of the modal shapes are piecewise constant functions, while the bending moment are piecewise linear functions. This is consistent with the application of dummy point loads at \( x_r = r \Delta x \) \((r = 1, 2, \ldots, N_\phi)\), which induces finite jumps in the shear force diagram and sudden changes in the slope of the bending moment diagram. It is worth stressing here that, the so obtained diagrams are similar to those that would be obtained if a classical FE beam method was adopted where displacement cubic functions leads to quadratic rotations, linear curvature and bending moment and constant shear force within each beam element. The proposed method has then the advantage to ensure the same type of approximation grade as the classical FE method.
The values of the generalised coordinates $a_i^{(r)}$ as introduced in Eq. (6.49) for the first four modes are listed in Tab. 6.5 for the case $\ell_\sigma = \ell_\varepsilon = L/20$. Focusing on the first mode the table clearly shows higher values of the coordinates $a_i^{(r)}$ at the midspan of the beam, where the corresponding shape function has the maximum displacement; analogously, for the higher modes the coefficients $a_i^{(r)}$ show their larger values (either positive or negative) for positions $r$ close to the valleys and peaks of each mode.

### 6.5.2.4 Convergence

Figure 6.27 shows a convergence study for the second example with results similar to those presented with Fig. 6.14 for the first example.
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\[ \ell_{\varepsilon} = \frac{L}{40}; \quad \ell_{\varepsilon} = \frac{L}{20}; \quad \ell_{\varepsilon} = \frac{L}{10}; \quad \ell_{\varepsilon} = \frac{L}{5} \]

\[ \phi_1, \gamma_1, M_1, V_1 \]

Figure 6.23: Example two – Displacement, rotations, shear force and bending moment first mode shapes for the simply supported beam for \( \ell_{\sigma} = \frac{L}{20} \) and four different values of the strain length scale parameter \( \ell_{\varepsilon} = \{0; \frac{L}{20}; \frac{L}{10}; \frac{L}{5}\} (N_\phi = 7) \)

<table>
<thead>
<tr>
<th>Mode</th>
<th>( a_i^{(1)} )</th>
<th>( a_i^{(2)} )</th>
<th>( a_i^{(3)} )</th>
<th>( a_i^{(4)} )</th>
<th>( a_i^{(5)} )</th>
<th>( a_i^{(6)} )</th>
<th>( a_i^{(7)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st mode</td>
<td>0.358</td>
<td>0.751</td>
<td>0.918</td>
<td>1</td>
<td>0.957</td>
<td>0.654</td>
<td>0.371</td>
</tr>
<tr>
<td>2nd mode</td>
<td>0.610</td>
<td>1</td>
<td>0.674</td>
<td>-0.109</td>
<td>-0.720</td>
<td>-0.861</td>
<td>-0.595</td>
</tr>
<tr>
<td>3rd mode</td>
<td>0.728</td>
<td>0.634</td>
<td>-0.264</td>
<td>-1</td>
<td>-0.439</td>
<td>0.827</td>
<td>0.974</td>
</tr>
<tr>
<td>4th mode</td>
<td>1</td>
<td>0.013</td>
<td>-0.873</td>
<td>0.0247</td>
<td>0.736</td>
<td>-0.0764</td>
<td>-0.636</td>
</tr>
</tbody>
</table>

Table 6.5: Example two – Mode shape dummy load points coefficients \( a_i^{(r)} \) \( r = 1 \ldots 7 \) of the first four natural frequencies \( i = 1, 2, 3, 4 \) of the simply supported beam for the case \( \ell_{\sigma} = \ell_{\varepsilon} = \frac{L}{20} \)
ℓ_ε = L/40; ℓ_ε = L/20; ℓ_ε = L/10; ℓ_ε = L/5

Figure 6.24: Example two – Displacement, rotations, shear force and bending moment second mode shapes for the simply supported beam for ℓ_σ = L/20 and four different values of the strain length scale parameter ℓ_ε = {0; L/20; L/10; L/5} (N_φ = 7)

Figure 6.25: Example two – Displacement, rotations, shear force and bending moment third mode shapes for the simply supported beam for ℓ_σ = L/20 and four different values of the strain length scale parameter ℓ_ε = {0; L/20; L/10; L/5} (N_φ = 7)
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Figure 6.26: Example two – Displacement, rotations, shear force and bending moment fourth mode shapes for the simply supported beam for $\ell_{\sigma} = L/20$ and four different values of the strain length scale parameter $\ell_{\varepsilon} = \{0; \ L/20; \ L/10; \ L/5\}$

Figure 6.27: Example two – Convergence diagram of the normalised frequency error $\epsilon_i \ (i = 1, 2, 3, 4)$ for the simply supported beam for $\ell_{\sigma} = 0$ and four different values of the strain length scale parameter $\ell_{\varepsilon} = \{0; \ L/50; \ L/20; \ L/10\}$
6.5.3 Example three – Clamped-clamped beam with three cracks

A beam of length \( L \) clamped at the position \( x = 0 \) and \( x = L \) is studied in the third and final example (Fig. 6.28). The beam has three cracks at \( x_1 = L/3 \), \( x_2 = 3L/7 \) and \( x_3 = 5L/7 \), modelled with three rotational springs with the same dimensionless parameter \( \beta_1 = \beta_2 = \beta_3 = 0.1 \) (see Eq. (3.10b)). This example demonstrates that the proposed approach enables the dynamic solution for statically undetermined non-local slender beams in bending with multiple concentrated damages.

6.5.3.1 Shape functions

As in the previous example, \( N_\phi = 7 \) shape functions have been used for the dynamic analysis, corresponding to the seven dummy point loads \( P_r \) shown as part of Fig. 6.29, with interval \( \Delta x = L/(N_\phi + 1) = L/8 \).

Displacements \( d^{(r)}(x) \), curvatures \( c^{(r)}(x) \) and bending moments \( m^{(r)}(x) \) for each of the seven loads \( P_r \) are also shown within Fig. 6.29 and, similarly to the previous examples, the effects of the cracks is clearly visible in the local curvatures, with cusps appearing at \( x_1 \), \( x_2 \) and \( x_3 \).

6.5.3.2 Effects of the length scale parameters

The same type of investigation as for the previous two examples have been carried out on the clamped-clamped beam of Fig. 6.28, showing similar trends of results, as summarised with Figs. 6.30 to 6.33, while Table 6.6 lists the reference values of the first four modal frequencies, obtained for the local elasticity theory.

Finally, the first four modal shapes and the convergence study for relevant values of the gradient elastic parameters and are presented within Figs. 6.34, and 6.35, which lend them-
Figure 6.29: Example three – Transversal displacement (a), curvature (b) and bending moment (c) shape functions for the multi-cracked clamped-clamped beam for \( \{ \ell_x = L/20; \ell_\sigma = 0 \} \) \((N_\phi = 7)\)

<table>
<thead>
<tr>
<th>(\omega_{1,loc})</th>
<th>(\omega_{2,loc})</th>
<th>(\omega_{3,loc})</th>
<th>(\omega_{4,loc})</th>
</tr>
</thead>
<tbody>
<tr>
<td>104.52</td>
<td>266.92</td>
<td>558.42</td>
<td>939.99</td>
</tr>
</tbody>
</table>

Table 6.6: Example three – First four natural frequencies for the local case \(\omega_{i,loc}\) solution for the clamped-clamped beam
Figure 6.30: Example three – First four natural frequencies of the clamped-clamped beam for \( \ell_{\sigma} = 0 \) against the strain length scale parameter \( \ell_{\varepsilon} \) (the horizontal lines represent the local case solutions \( \ell_{\sigma} = \ell_{\varepsilon} \)(\( N_\phi = 15 \))

Figure 6.31: Example three – Normalised natural frequency \( \omega_i/\omega_{i,\text{loc}} \) \( (i = 1, 2, 3, 4) \) of the cracked clamped-clamped beam for (a) \( \ell_{\sigma} = 0 \) and (b) \( \ell_{\sigma} = L/20 \) (the horizontal lines represent the local case solutions \( \ell_{\sigma} = \ell_{\varepsilon} \)(\( N_\phi = 15 \))

selves to similar considerations as for the statically determined beams considered in the previous examples.
6.5 Numerical examples

\begin{align*}
\rho_ℓ &= 0; \quad \rho_ℓ = 1/4; \quad \rho_ℓ = 1/2; \quad \rho_ℓ = 3/4; \quad \rho_ℓ = 1; \quad \rho_ℓ = 3/2; \\
\ell_ε/L &= \{0; 1/4; 1/2; 3/4; 1; 3/2; 2\} \text{ (the solid horizontal line represents the local case solution } \rho_ℓ = \epsilon_σ/\ell_ε = 1)(N_φ = 15)\
\end{align*}

Figure 6.32: Example three – Normalised natural frequency \(\omega_i/\omega_{i,loc}\) \((i = 1, 2, 3, 4)\) of the cracked clamped-clamped beam for several combinations of the ratio \(\rho_ℓ = \{0; 1/4; 1/2; 3/4; 1; 3/2; 2\}\) (the solid horizontal line represents the local case solution \(\rho_ℓ = \epsilon_σ/\ell_ε = 1)(N_φ = 15)\)

\begin{align*}
\omega_1/\omega_{1,loc} &\quad \omega_2/\omega_{2,loc} \\
\ell_ε/L &\quad \ell_ε/L \\
(a) &\quad (b)
\end{align*}

Figure 6.33: Example three – Normalised natural frequency \(\omega_i/\omega_{i,loc}\) for the first \((a)\) and fourth \((b)\) natural frequencies of the clamped-clamped beam for several combinations of the stress length scale \(\epsilon_σ\) (the horizontal solid line represents the local case solution \(\ell_ε = \epsilon_σ)(N_φ = 15)\)

\begin{align*}
\ell_ε/\Omega &\quad \ell_ε/\Omega \\
(ε_σ = 0; \quad \ell_ε = L/10; \quad \ell_ε = L/20; \quad \ell_ε = L/50) &\quad (ε_σ = 0; \quad \ell_ε = L/10; \quad \ell_ε = L/20; \quad \ell_ε = L/50)
\end{align*}
Figure 6.34: Example three – First four mode shapes for the clamped-clamped beam for \( \ell_\sigma = L/20 \) and four combination of the length scale parameters \( \ell_\varepsilon = \{ L/40; L/20; L/10; L/5 \} \) \((N_\phi = 7)\)

Figure 6.35: Example three – Convergence diagram of the normalised frequency error \( \epsilon_i \) \((i = 1, 2, 3, 4)\) for the clamped-clamped beam for \( \ell_\sigma = 0 \) and four different values of the strain length scale parameter \( \ell_\varepsilon = \{ 0; L/50; L/20; L/10 \} \)
6.6 Concluding remarks

In this Chapter, a new computational meshless method has been proposed for the dynamic analysis of multi-cracked beams with hybrid gradient elastic constitutive law. In details, a Galerkin approximation has been used, in which the shape functions are conveniently chosen as the closed-form solutions of the beam under unit point loads applied at positions spaced along the beam. Differently than the standard FE beam model, no continuity conditions on the field variables are required between the internal points, and this is particularly important as otherwise hybrid non-local elasticity would require additional continuity conditions at each node. Furthermore, the proposed method ensures the same grade of approximation as the traditional beam element for the classical (i.e. local) elasticity. The method has been used to derive natural frequencies and modal shapes, by solving the eigenvalue problem of the discretised system.

As for the static analysis presented in the previous Chapter, the size of the computational problem is independent of the number \( n \) of cracks on the beam due to the adopted flexibility model which treats the cracks as concentrated inhomogeneities. The adopted hybrid non-local beam model includes two length scale parameters, which enables one to describe a variety of microstructural materials. Three numerical examples have been deeply analysed: i) a cantilever beam with one crack; ii) a simply supported beam with two cracks; and iii) a clamped-clamped beam three cracks. Several combinations of the two microstructural parameters \( \{ \ell_c, \ell_\sigma \} \) have been studied and compared with the classical elasticity theory solution (recovered for \( \ell_c = \ell_\sigma \geq 0 \)). The results highlight how the microstructure effects are more relevant at higher frequencies and how these effects are related to the wave length over length scale ratio, e.g. for the simply supported beam they start to be significant from an engineering point of view (solution differs by more than 1% if compared with the classical solution) when the wave lengths are less than 20 times the microstructural parameters.
Cracks and other forms of damage can have a significant impact on the performance of structures under static and dynamic loading. Physically, a crack represents a local reduction of the section, which corresponds to an increase of the local flexibility or, analogously, a reduction of the local rigidity. One of the most used methods to represent this discontinuity is through the introduction of a local element, a rotational spring, at the damage position. As a consequence, the beam/column results to be split in two parts, linked by the spring element, which is able to transfer from one side to the other all the internal forces. In this method, called discrete spring (DS) method, the spring has stiffness related to the damage intensity. This idealisation ensures a model for the cracked structure able to accurately take into account the location of the crack and therefore can be successfully implemented in iterative processes.

The main shortcoming with the DS model is that, if a conventional beam element is used, two additional Finite Element (FE) nodes must be placed at the location of each concentrated damage, i.e. one node on each side. This could be particularly cumbersome if the DS model is used for the purposes of damage identification, as it would also require re-meshing the beam during the identification process. The need then arises for an accurate and computationally efficient beam element able to account for the presence of multiple cracks, without increasing the size of the FE assembly for the structural frame under investigation.
Furthermore, by using the DS model an abrupt finite jump appears between the rotations of
the sections just before and immediately after the crack. This phenomenon does not respect
the real behaviour of the material, which instead shows a smooth transition of the rotations along
the beam.

To overcome these inefficiencies, the aim of the proposed work was to “develop an efficient
computational method for the static and dynamic analysis of beams with local and non-local
elasticity in presence of concentrated damages”. Various goals were achieved during this re-
search, so to fulfil the aim of this project, and they are summarised in the next section.

7.1 Achievements of this Research Project

The results of this work are summarised according to the objectives defined in Section 1.1:

1. To review the existing models for the analysis of damaged structures.

   The literature review carried out allowed the identification of two main classes of damage
   model: the 2D/3D Finite Element model and the one dimensional/beam damage model. Moreover, among the one dimensional/beam damage model, three categories have been
   recognised: local stiffness reduction; continuous models; and DS models.

   The “local stiffness reduction” consists of reducing the stiffness of a whole element to sim-
   ulate the presence of a crack in the finite element model of the damaged beam. It requires a
   fine mesh, and also the derivation of the effects of a crack on the stiffness of the whole ele-
   ment (i.e. an equivalent smeared damage must be defined). The “continuous cracked model”
   requires the assumption of the stress distribution around the crack through the definition of
   an opportune decay function related with crack location and depth. In the “DS model”, in-
   stead, the damage is concentrated at a single position, as the beam-type structure is divided
   at the damage position where the new elements are pinned together and the residual stiff-
   ness is simulated by the addition of a rotational spring, which can be linear or bi-linear for
   “always open” or “breathing” cracks, respectively.

   It clearly emerged that, if the goal of the structural analysis is the damage detection, the
   “DS models” provide the best trade-off between accuracy and computational effort. The
   major difficulties with this approach are that a finite element node must be placed at the
   crack location, requiring re-meshing during the identification process. Many formulations
   for these models have been developed, including the exact closed-form solutions for the
   “rigidity modelling” [33, 34], where the singularities in the flexural stiffness, correspond-
7.1 Achievements of this Research Project

ing to the damage, are represented by Dirac’s delta functions, which appears in the model as negative impulses in the flexural stiffness of the beam. Although expedient, this is not consistent with the definite-positive nature of the beam’s stiffness. Aimed at overcoming this intrinsic theoretical flaw, Palmeri and Cicirello [162] have presented a “flexibility modelling” of crack using positive Dirac’s delta functions in the beam’s bending flexibility, i.e. the inverse of the flexural stiffness.

Furthermore, since the DS representation concentrates the increased curvature in the neighbourhood of the damage at a single abscissa, a jump discontinuity appears in the field of rotations experienced by the beam.

The absence of a computationally efficient beam element for studying damaged frame structures and the inaccuracies of the DS model, revealed from the literature review, justify the attempt of this Thesis to improve the available beam models both in terms of including shear effects and rotatory inertia as well as in terms of a more accurate representation of the beam behaviour around the crack.

2. **To develop a two-node multi-cracked beam element with classical elasticity.**

In order to achieve this objective, a consistent two-node finite element for multi-damaged slender/short beams (MDBs) has been developed. In the first stage, the exact closed-form solutions for MDBs, including shear effects, subjected to both axial and transverse loads have been derived. Differently from the flexibility modelling proposed by Palmeri and Cicirello [162], this formulations includes axial and shear deformations concentrated at the position of the damage. Subsequently, the components of the \(6 \times 6\) stiffness matrix for a planar multi-cracked beam has been obtained using the unit displacement method. The consistent mass matrix has been also computed, by adopting the same shape functions to represent the inertial forces on the MDB element, including the rotatory inertia. The proposed method has been implemented in an house-made code in Matlab and it has been validated by comparing the results of several different models realised with the commercial FE software SAP2000, where the damage has been represented by splitting the beam and introducing an axial-rotational-translational spring element to simulate the stiffness reduction due to the damage. The outcomes of this objective have been presented at an international conference, and a paper has been accepted for publication in Computers & Structures.

3. **To review the higher-order gradient theories in particular when applied to the beam model.**
Several references have been reviewed in order to identify the main higher-order enrichment approaches available in the literature. Two main classes have been identified, a phenomenological and a microstructural approach, which are both able to overcome inefficiencies of the classical continuum theory. Indeed, the enrichment with Laplacians (i.e. gradients) has several advantages, including to capture size effects and to study problems with singularities in the stress and strain fields.

Using the literature review, it was possible to identify the hybrid non-local beam model as the most promising one to overcome the DS crack inadequacy on the rotations’s profile at the damage position. Interestingly, this model always accounts for non-local effects, differently than other non-local theories (as the Aifantis’ or Eringen’s models), which lose the higher gradient effects in the static response of some structural configurations. Furthermore, it has been shown that the hybrid model is analogous to others beam theories, as the composite beam with interlayer slip and the two parameters elastic foundation (Reissner foundation model), extending the applicability of the proposed formulation. Moreover, the review revealed that only few authors have attempted the analysis of cracked beams using higher order theories, even though their potential importance, especially in problems where microstructural effects cannot be neglected.

4. **To develop a theoretical model for the beam with multiple concentrated damage using higher order gradients.**

Aimed at overcoming physical inconsistencies on the rotations’s profile and to develop an improved representation of cracked slender beams, a new mathematical formulation has been proposed, which allows deriving the closed-form solutions, in terms of kinematic variables and internal forces, for the linear static analysis of multi-cracked hybrid non-local slender beams in bending. The proposed approach builds upon the hybrid stress-strain gradient-elastic constitutive law (which allows recovering classical, Eringen’s and Aifantis’ theories as particular cases) and embeds the “flexibility crack” model (equivalent to the classical DS model) to represent the localised increase in the bending flexibility through a convenient Dirac’s delta function at the location of each crack.

The sixth order differential equation for the multi-damaged beam under a generic transverse load has been solved using the Laplace’s transformation method, providing the solution for the non-local curvature. Local curvature, rotation, and transverse displacement have been obtained following their definitions. Differently from the classical theory, where the fourth order differential equation requires only four boundary conditions, the general solution of
the sixth order differential equations for the static response of the hybrid non-local beam model depends on six integration constants, requiring two additional boundary conditions (which can be usually derived from variational principles).

The proposed model shows the ability to spread the effects of the concentrated crack along the beam, therefore reducing or even totally removing the finite jump in the rotations, as well as accounting for the microstructural heterogeneity of the material. The results of this objective have been presented at an international conference, and have been included as part of a paper accepted for publication in the International Journal of Solids and Structures.

5. To develop an efficient method to solve the dynamic problem of the multi-cracked beam enriched with higher order gradients.

The numerical solution in terms of natural frequencies and modal shapes for the free vibrations of hybrid non-local Euler-Bernoulli beams, including the DS representation of damage, has been derived using a new Galerkin-type method, where the closed form-solutions of the beam (in terms of displacement, bending moment and curvature) under unit point load have been chosen as shape functions. The proposed method, following an energy approach, allows the computation of the stiffness matrix and mass matrix. Hence, the dynamic properties of the beam are obtained as a solution of the resulting eigenvalue problem. A third journal paper based on the results of this work is planned, with the target journal agreed with the supervisors being Computer Methods in Applied Mechanics and Engineering.

6. To study the effects of the non-local parameters on the static and dynamic structural response.

The comparisons between the solutions of the hybrid non-local beam model and the beam with classical elasticity allowed to quantify the effects of the two microstructural parameters $\ell_\varepsilon$ and $\ell_\sigma$ in terms of static and dynamic behaviour. The comparisons highlighted that the non-local effects increases with increasing value of the dimensionless microstructural parameters. An important role is played by the ratio $\rho_\ell$ between the length scale parameters ($\rho_\ell = \ell_\sigma / \ell_\varepsilon$); if $\rho_\ell$ belongs to the range $0 < \rho_\ell < 1$, the overall flexibility of the beam is higher than its local counterpart ($\rho_\ell = 1$); the opposite if $\rho_\ell > 1$. Furthermore, the finite jump in the rotations’ profile at the crack position resulted to be proportional to the square of the ratio $\rho_\ell^2$. In particular, the less $\rho_\ell$ the smoother the rotations’ profile, the more widespread the effects on the curvature of the beam. These new findings allow a better understanding of how gradient elasticity theories can be effectively adopted for the analysis of cracked slender beam.
7.2 Novelties and Key Findings

7.2.1 Novelties

The main novelties of this Thesis have been summarise below:

- closed-form analytical solutions for multi-damaged beams under static loads, accounting for the shear deformations, rotatory inertia and an arbitrary number $n$ of axial, rotational and shear springs placed anywhere along the beam;

- stiffness matrix, consistent mass matrix and vector of equivalent nodal forces for a two-node MDB element with concentrated damage represented using a fully articulated set of axial, rotational and shear springs at the position of each discontinuity;

- the stiffness matrix only depends on a few damage parameters $(a_0, b_0, b_1, b_2, c_0)$ which summarise the properties of the $n$ damages;

- inclusion of the rotatory inertia in the consistent mass matrix and the shear effects leading to an effective model for the dynamic analysis of short beams, particularly for the higher modes;

- a MDB element which embeds the effects of the concentrated damages without enlarging the size of the FE model and is able to deliver the same static solution, as the conventional DS model, with just a single FE for each beam and column in the frame structures;

- a multi-cracked Euler-Bernoulli beam model with hybrid non-local elasticity, where both stress gradient and strain gradient are considered along with two independent length scales. Also noteworthy is the adopted flexibility model, which treats the cracks as concentrated inhomogeneities, allows to study the multi-cracked beam independently of the number $n$ of cracks;

- two length scale parameters are included in the adopted hybrid non-local beam model, enabling to study a variety of microstructural materials;

- closed-form analytical solutions for the multi-cracked Euler-Bernoulli beams with hybrid non-local elasticity under static loads $q(x)$, including the formulations for the two additional integration constants $C_4$ and $C_5$ appearing in the six-order differential, as well as the two different solutions procedure for statically determinate and indeterminate beams;
• the adopted hybrid gradient elasticity allows spreading the effects of the cracks in the neighbour- 
bhood of the damaged abscissa, therefore providing a more realistic rotations’ profile;

• a meshless computational approach for static and dynamic solution of the hybrid non-local 
multicracked beam systems, without requiring any continuity conditions on the field vari- 
ables between the internal points, and ensuring the same grade of approximation as the 
traditional beam element with classical (i.e. local) elasticity.

• free vibration analysis of multi-cracked Euler-Bernoulli beams modelled using hybrid non-
local elasticity theory.

7.2.2 Findings

The main findings of this Thesis have been summarised below, divided between the two main 
outcomes: the MDB element and the hybrid non-local multicracked beam model. Regarding 
the MDB element the main findings are:

• the results obtained with the proposed MDB method have been compared with an equivalent 
DS SAP2000 model, showing how the two models give exactly the same static results;

• from a dynamic point of view, the proposed MDB element guarantees an improved per- 
formance in comparison with other approximate approaches, as the LSR one, placing it 
as extremely suitable for the dynamic analysis of planar frame structures with concentrated 
damages. The comparison between the MDB model and the equivalent DS SAP2000 method 
gives the same eigenproperties if a lumped mass is adopted. Meanwhile, using the proposed 
consistent mass matrix, the approximate eigenproperties converge more rapidly to the exact 
solution. Further static and dynamic comparison with the LSR model shows how the DS 
model is more accurate and able to provide a better estimation of the dynamic properties 
with few elements;

• a parametric study has quantified the relative impact of shear effects and rotatory inertia 
when included in the computational dynamic analysis of MDBs, highlighting their import- 
ance.

Regarding the hybrid non-local multicracked beam model the main findings are:

• the finite jump in the rotations’ profile at the crack position resulted to be proportional to the 
square of the ratio $\frac{\rho}{\ell}$. In particular, the less $\rho/\ell$, the smoother the rotations’ profile, the more 
widespread the effects on the curvature of the beam;
• for $\rho_\ell > 1$ the structure shows a physically inconsistent behaviour at the crack’s position, where the displacements assumes opposite direction with respect to the applied load;

• higher frequencies are more affected by the microstructural parameters;

• the effects of the two microstructural parameters on the natural frequencies have been analysed against the local case solutions and the non-local case solution of the undamaged counterpart. If $\rho_\ell < 1$, higher frequencies are obtained and the opposite happens if $\rho_\ell > 1$.

• effects of the higher order terms are strictly related with the ratio between wavelength and length scale parameters. This is, the higher the ratio, the bigger the influence, e.g. for wavelengths lower than 20 times the microstructural parameters, the effects start to be noticeable (solution differs by more than $1\%$ if compared with the classical solution).

### 7.2.3 Recommendations for Future Work

Further extension of the proposed formulation can be:

• to extend the proposed MDB model to the three dimensional case.

  The formulation for the MDB valid for $y$-plane can be extended to include the effects on the $z$-plane by adding the corresponding kinematic and static variables. In particular, displacements and rotations in the $y$-plane (see subsection 3.1.3), could be written as $u_{T,z}$ and $\varphi_y$, where an additional letter has been introduced in the subscript identifying the direction of the corresponding vector. Displacement in the $z$-plane and rotation around the $y$-axis, appearing when the third dimension is considered, could be written as $u_{T,y}$ and $\varphi_z$ and could be obtained using the same procedure as proposed in Chapter 3. The corresponding internal forces would be identified as $V_z$, $M_y$, $V_y$, $M_z$. Furthermore two additional damage parameters, $\beta_y$ and $\gamma_y$, for the rotational and transverse springs in the $y$ direction have to be defined.

• to include crack initiation and propagation phenomena into the MDB element.

  Non-linear behaviours of the crack can be included in the formulation, e.g. non-linear opening and closing effect could be introduced by relating the damaged parameter to the curvature of the beam at the crack section. Furthermore, phenomena as the crack propagation or crack initiation can be studied using a multi-scale approach where the whole frame is first investigated and the resulting internal forces are introduced in another model, accurately representing the beam around the crack only, using 2D/3D finite elements, .
• to experimentally validate the hybrid non-local multicracked beam model under both static and dynamic loads.

Before validating the proposed hybrid non-local multi-cracked beam model, the microstructural parameters have to be identified as solution of an inverse problem where the analytical solutions are compared with the experimental results of several cracked beam samples with relevant microstructure (i.e. aluminium honeycomb beam or concrete beam with different size of the aggregates). The comparisons could focus on the displacement, rotations and curvature as well as frequencies and modal shapes. The effects of static or dynamic excitations could be measured using lasers optical sensors or accelerometers. Once the non-local parameters have been obtained, the hybrid multi-cracked model can be validated by comparing the behaviour of damaged samples with the analytical model, especially by looking at the profile of rotations in the neighbourhood of the crack.

• to extend the hybrid multicracked Euler-Bernoulli beam model to composite beam with interlayer slip problems or two-parameter foundation models by considering different boundary conditions.

Composite beam with interlayer slip or the two parameters foundation models are analogues to the hybrid non-local theory for particular boundary conditions (see subsection 4.6.3.2). Therefore, the corresponding sixth order differential equation system could be solved using the same approach as section 5.1 where different boundary conditions related to the new physical problem have to be considered.

• to extend the hybrid multicracked Euler-Bernoulli beam to the Timoshenko beam.

The extension to Timoshenko beam can be provided by considering the additional contribution of the shear effect \( u_s \) on the displacement \( u_T = u_b + u_s \) and providing the corresponding solution in a closed form. The non-local effect on the shear strain could be neglected as suggested in the work of Wang et al. [235]; or it could be included into the non-local effects on the shear strain as defined in Eq.(4.43) (see the work of Ke et al. [114]).

• to develop a FE non-local hybrid beam element including singularities.

In order to provide the FE beam element, the kinematic variable and corresponding internal forces at the two beam ends need to be identified; in particular, the non-local curvature \( \tilde{\chi} \) and its static counterpart which can be obtained by solving the variational principle. It will follow the definition of the shape functions and the calculation of each term for stiffness
matrix, mass matrix and equivalent load vectors for different load distributions. Key point will be the continuity conditions among the non-local curvature at the nodes of the FE beam.

### 7.2.4 Closing Remarks

The damage models available in the literature have been reported to be in many cases not efficient enough to reach an adequate grade of accuracy especially at higher frequencies. This thesis addresses some of the related issues and proposes an efficient new formulation for the analysis of damaged beams using classical elasticity theory, as well as the enriched hybrid non-local theory, that might help future researches to develop more reliable technologies for monitoring the structural integrity or study the behaviour of nano-structures (for which size effects are crucial).
Exact closed-form solutions for the static analysis of multi-cracked gradient-elastic beams in bending. Marco Donà, Alessandro Palmeri, Mariateresa Lombardo

Journal Articles


Conference Proceedings


- Donà, M., Palmeri, A., Lombardo, M. and Cicirello, A., A Two-Node Multi-Cracked Beam Element for Static and Dynamic Analysis of Planar Frames, in B.H.V. Topping, (Editor),

Future publications


[103] T. C. Huang, W. A. Nash, University of Florida. Engineering, and Industrial Experiment Station. *Effect of Rotatory Inertia and Shear on the Vibration of Beams Treated by the Approximate Methods of Ritz and Galerkin*. Engineering progress at the University of Florida. Florida Engineering and Industrial Experiment Station, College of Engineering, University of Florida, 1959.


