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EULER CHARACTERS AND SUPER JACOBI POLYNOMIALS

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Abstract. We prove that Euler supercharacters for orthosymplectic Lie superalgebras can be obtained as a certain specialization of super Jacobi polynomials. A new version of Weyl type formula for super Schur functions and specialized super Jacobi polynomials play a key role in the proof.

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1. Introduction

The main purpose of this paper is to develop further the link between the theory of the deformed Calogero-Moser systems and representation theory of Lie superalgebras [1, 2]. Recall that in the \( BC(m,n) \) case the deformed Calogero-Moser systems depend on 3 parameters \( k, p, q \). For generic values of these parameters they have polynomial eigenfunctions \( S\lambda(u, v; k, p, q) \) called super Jacobi polynomials [3]. The special case \( k = p = -1, q = 0 \) corresponds to the orthosymplectic Lie superalgebra \( \mathfrak{osp}(2m + 1, 2n) \).

It turns out that this case is singular in the sense that the corresponding limit does not always exist. However if we consider first the limit when \( k \to -1 \) with generic \( p, q \) and then let \( (p, q) \to (-1,0) \) then this limit does always exist and gives what we call \textit{specialized super Jacobi polynomials} \( S\lambda(u, v; -1, -1, 0) \). A natural question is what do they correspond to in the representation theory of orthosymplectic Lie superalgebras. We show
that the answer is given by the so-called Euler characters studied by Penkov and Serganova [4, 5].

There is a classical construction due to Borel, Weil and Bott of the irreducible representations of the complex semisimple Lie groups $G$ in terms of the cohomology of the holomorphic line bundles over the corresponding flag varieties $G/B$ (see e.g. [6], section 23.3). Such line bundles $L_\lambda$ are determined by the weight $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra of Lie algebra of $G$ and $B$ is Borel subgroup of $G$. The cohomology groups $H^i(G/B, L_\lambda)$ are finite-dimensional and have natural actions of $G$ on them. By Kodaira vanishing theorem all of them are zero except one depending on the Weyl chamber the weight $\lambda$ belongs to (see details in [6, 7]). In particular, for a dominant weight $\lambda$ the space of sections $H^0(G/B, L_\lambda)$ gives the irreducible representation with highest weight $\lambda$.

In the Lie supergroup case in general there is no vanishing property, so this construction does not work [4]. The idea is to consider the virtual representation given by the Euler characteristic

$$E_\lambda = \sum_i (-1)^i H^i(G/B, \mathcal{O}_\lambda)$$

for certain sheaf cohomology groups (see [5]). For the generic (typical) highest weights $\lambda$ this leads to the Kac character formula [8].

One can generalize this construction for any parabolic subgroup $P$ of $G$ in a natural way. In the orthosymplectic case $G = OSP(2m + 1, 2n)$ there is a natural choice of $P$ with the reductive part $GL(m,n)$. The corresponding supercharacter $E_\lambda$ (called Euler supercharacter) can be given by the general explicit formula due to Serganova [5]. Our main result is that $E_\lambda$ up to a constant factor coincides with the specialized super Jacobi polynomials $S_{J\lambda}(u,v;-1,-1,0)$. A similar result holds for the Lie superalgebra $osp(2m,2n)$ and super Jacobi polynomials $S_{J\lambda}(u,v;-1,0,0)$.

The proof is based on a new formula for super Schur polynomials and the super version [3] of Okounkov’s formula for Jacobi polynomials [9]. We prove also the Pieri and Jacobi–Trudy formulas for the corresponding specialized super Jacobi polynomials and Euler supercharacters.

It turns out that we can simplify the relations with super Jacobi polynomials if we choose a different Borel subalgebra and the corresponding parabolic subalgebras following recent work by Gruson and Serganova [10]. In that case the Euler supercharacters coincide with specialized super Jacobi polynomials without non-trivial factor (see the last section for the details).

This shows that the super Jacobi polynomials can be considered as a natural deformation of the Euler supercharacters and gives one more evidence of a close relationship between quantum integrable systems and representation theory.
2. WEYL TYPE FORMULAS FOR SUPER SCHUR POLYNOMIALS

We start with the new formula for super Schur polynomials, which will play an important role in this work.

Let \( H(m,n) \) be the set of partitions with \( \lambda_{m+1} \leq n \), which means that the corresponding Young diagram belongs to the \( \text{fat} (m,n) \)-hook. Let \( \lambda \) be such a partition and let \( d = m - n \) be the superdimension. Introduce the following quantities

\[
i(\lambda) = \max\{i \mid \lambda_i + d - i \geq 0, \ 1 \leq i \leq m\}, \quad (1)
\]

\[
j(\lambda) = \max\{j \mid \lambda'_j - d - j \geq 0, \ 1 \leq j \leq n\}. \quad (2)
\]

If all \( \lambda_i + d - i < 0 \) then by definition \( i(\lambda) = 0 \) (and similarly for \( j(\lambda) \)). It is easy to verify that in all cases \( m - i(\lambda) = n - j(\lambda) \), so \( i(\lambda) - j(\lambda) = d \).

We should mention that Moens and van der Jeugt introduced a similar quantity \( k(\lambda) \), which they called \( (m,n) \)-index of \( \lambda \) (see Definition 2.2 in [19]). They were motivated by Kac-Wakimoto formula [11]. It is related to our \( i(\lambda) \) by \( i(\lambda) = k(\lambda) - 1 \). Moens and van der Jeugt used this quantity to write down a new determinantal formula for super Schur polynomials different from Sergeev-Pragacz formula [12].

Our formula (7) below is another new formula of Weyl type, which generalizes Sergeev-Pragacz formula. Let us denote by \( \pi_\lambda \) the set of pairs \( (i,j) \) such that \( i \leq i(\lambda) \) or \( j \leq j(\lambda) \) and fix a partition \( \nu \) such that \( \lambda \cap \Pi_{m,n} \subseteq \nu \subseteq \pi_\lambda \), where \( \Pi_{m,n} \) is the rectangle of the size \( m \times n \). When \( \nu = \pi_\lambda \) the formula (7) coincides with Serganova’s formula (11) for a special choice of parabolic subalgebra depending on the weight (although that was not the way we came to this).

Introduce the following quantities by

\[
l_i = \lambda_i + m - \nu_i - i, \ 1 \leq i \leq i(\lambda), \quad l_i = m - i, \ i(\lambda) < i \leq m, \quad (3)
\]

\[
k_j = \lambda'_j + n - \nu'_j - j, \ 1 \leq j \leq j(\lambda), \quad k_j = n - j, \ j(\lambda) < j \leq n. \quad (4)
\]

Now we can formulate the main result of this section. Recall that super Schur polynomial is the supercharacter of the polynomial representation \( M = V^\lambda \) of \( \mathfrak{gl}(m,n) \) determined by a Young diagram \( \lambda \) from the fat hook \( H(m,n) \). It can be given by the following Jacobi–Trudy formula (see [13]):

\[
SP_\lambda(x,y) = \begin{vmatrix}
    h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+l-1} \\
    h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+l-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{\lambda_l-t+1} & h_{\lambda_l-t+2} & \cdots & h_{\lambda_l}
\end{vmatrix}, \quad (5)
\]

where \( l = l(\lambda) \) is the number of non-zero parts in the partition \( \lambda \), \( h_k = h_k(x,y) \) are determined by

\[
\prod_{j=1}^{m}(1 - ty_j) \prod_{i=1}^{n}(1 - tx_i) = \sum_{a=0}^{\infty} h_a(x,y)t^a \quad (6)
\]
and \( h_a = 0 \) if \( a < 0 \). One can check that the highest coefficient of \( \text{SP}_\lambda(x, y) \) is equal to \((-1)^b \), where \( b = \sum_{j>m} \lambda_j \). This explains the appearance of this sign below (see also [14]).

**Theorem 2.1.** The super Schur polynomial \( \text{SP}_\lambda(x_1, \ldots, x_m, y_1, \ldots, y_n) \) can be expressed by the following Weyl type formula for any choice of partition \( \nu \) such that \( \lambda \cap \Pi_{m,n} \subseteq \nu \subseteq \pi_\lambda \):

\[
\text{SP}_\lambda(x_1, \ldots, x_m, y_1, \ldots, y_n) = \frac{(-1)^b \sum_{w \in S_m \times S_n} w \left[ \prod_{(i,j) \in \nu} (x_i - y_j) \prod_{i<j} x_i^{k_i} y_j^{k_j} \Delta(x) \Delta(y) \right]}{\Delta(x) \Delta(y)},
\]

where \( \Delta(x) = \prod_{i<j} (x_i - x_j), \Delta(y) = \prod_{i<j} (y_i - y_j), b = \sum_{j>m} \lambda_j \).

**Proof.** For any function \( f(x, y) \) define the following alternation operations

\[
\{f(x, y)\} = \sum_{w \in S_m \times S_n} \varepsilon(w)w(f(x, y))
\]

and

\[
\{f(x, y)\}_x = \sum_{w \in S_m} \varepsilon(w)w(f(x, y)),
\]

where \( S_m \) and \( S_n \) permute \( x_i \) and \( y_j \) respectively. Introduce also the notations

\[
x^{\rho_m} = x_1^{m-1}x_2^{m-2} \ldots x_m^0, \quad y^{\rho_n} = y_1^{n-1}y_2^{n-2} \ldots y_n^0.
\]

First let us prove the following equality

\[
\{h_a(x, y)x^{\rho_m}y^{\rho_n}\} = \left\{ \prod_{j=1}^{n} (x_1 - y_j)x_1^{a+d-1}x_2^{m-2} \ldots x_m^0y_1^{n-1} \ldots y_n^0 \right\}, \quad (8)
\]

where \( a \) is an integer such that \( a + d - 1 \geq 0 \). Indeed, we have from the usual Weyl formula for \( a \geq 0 \)

\[
\{h_a(x)x_1^{m-1}x_2^{m-2} \ldots x_m^0 \}_x = \{x_1^{a+m-1}x_2^{m-2} \ldots x_m^0 \}_x
\]

This is true also for all \( a \geq 1 - m \) because the left-hand side for negative \( a \) is zero by definition. From (6) we have

\[
h_k(x, y) = \sum_{j=0}^{n} (-1)^j h_{k-j}(x)e_j(y),
\]

where \( h_k \) and \( e_j \) are complete symmetric and elementary symmetric polynomials respectively. Note now that if \( a + d - 1 \geq 0 \) and \( 0 \leq j \leq n \) we have \( a - j \geq 1 - m \). Therefore in that case

\[
\{h_a(x, y)x_1^{m-1} \ldots x_m^0 \}_x = \sum_{j=0}^{n} \{h_{a-j}(x)x_1^{m-1} \ldots x_m^0 \}_x (-1)^j e_j(y) =
\]
Lemma 2.2. If \( \lambda_1 + d - 1 \geq 0 \) we have the following equality

\[
\{SP_\lambda(x, y)x^{\rho_m}y^{\rho_n}\} = \left\{ \prod_{j=1}^{n} (x_1 - y_j)x_1^{\lambda_1 + d - 1}SP_\lambda(x, y)x_2^{m-2} \ldots x_n^{0}y^{\rho_n} \right\}
\]

(9)

This implies the formula (8).

We prove now the theorem by induction. For this we need the following

We prove now the theorem by induction. For this we need the following

**Proof.** To prove it we use the Jacobi-Trudy formula (5). Let \( \lambda_1 + d - 1 \geq 0 \), then we have

\[
\{SP_\lambda(x_1, \ldots, x_n, y_1, \ldots, y_n)x_1^{m-1} \ldots x_n^{0}y_1^{n-1} \ldots y_n^{0}\} =
\]

\[
\left[ \begin{array}{c}
\left\{x_1^{\lambda_1} \ldots y_n^{0}\right\} & \left\{x_1^{\lambda_1} \ldots y_n^{0}\right\} & \left\{x_1^{\lambda_1} \ldots y_n^{0}\right\} \\
\left\{h_{\lambda_1} \ldots h_{\lambda_1} + 1 \ldots h_{\lambda_1 + 1} \right\} & \left\{h_{\lambda_1} \ldots h_{\lambda_1} + 1 \right\} & \left\{h_{\lambda_1} \ldots h_{\lambda_1} + 1 \right\} \\
\left\{h_{\lambda_1} \ldots h_{\lambda_1} + 1 \right\} & \left\{h_{\lambda_1} \ldots h_{\lambda_1} + 1 \right\} & \left\{h_{\lambda_1} \ldots h_{\lambda_1} + 1 \right\} \\
\left\{h_{\lambda_1} \ldots h_{\lambda_1} \right\} & \left\{h_{\lambda_1} \ldots h_{\lambda_1} \right\} & \left\{h_{\lambda_1} \ldots h_{\lambda_1} \right\}
\end{array} \right]
\]

(9)

Now multiplying every column except the last one by \( x_1 \) and subtracting it from the next column taking into account the equality

\[h_{a-1}(x, y) = \frac{1}{x_1}h_a(x, y)\]

we get

\[
\left\{ \prod_{j=1}^{n} (x_1 - y_j)x_1^{\lambda_1 + d - 1} \right\} = \left\{ \prod_{j=1}^{n} (x_1 - y_j)x_1^{\lambda_1 + d - 1} \right\}
\]

where \( \hat{h}_k = h_k(\hat{x}, y) \) and \( \hat{x} = (x_2, \ldots, x_m) \) as before. This implies the formula (9).
Lemma 2.3. Let $\lambda_1 < n$, then we have the following equality

$$\{SP\lambda(x, y_1, \ldots, y_n)x^{\rho_m}y^{\rho_n}\} = \{SP\lambda(x, y_1, \ldots, y_{n-1})x^{\rho_m}y^{\rho_n}\}.$$  

Proof. We have (see formulas (5.9) in [18] and (25) in [14])

$$SP\lambda(x, y) = \sum_{\mu \subseteq \lambda} (-1)^{|\mu|}S_{\lambda/\mu}(x)S_{\mu'}(y)$$

So, if $\lambda_1 < n$ then $\mu'_n = 0$ and lemma follows.

Now we finish the proof by induction in $m + n$. Note first that the case $n = 0$ follows from the classical Weyl formula, while in the case $m = 0$ we have also the sign $(-1)^{\lambda_{m+1} + \lambda_{m+2} + \ldots}$ determined as in the formula (25) in [14]. Let us assume now that $m > 0$. Rewrite formula (7) in the form

$$\{SP\lambda(x, y)x^{\rho_m}y^{\rho_n}\} = (-1)^b \left\{ \prod_{(i, j) \in \nu} (x_i - y_j)x_1^{l_1} \ldots x_m^{l_m} y_1^{k_1} \ldots y_n^{k_n} \right\}. \quad (10)$$

If $\nu_1 < n$ (and hence $\lambda_1 < n$) by induction assumption we have

$$\{SP\lambda(x, y_1, \ldots, y_{n-1})x^{\rho_m}y^{\rho_n-1}\} = (-1)^b \left\{ \prod_{(i, j) \in \nu} (x_i - y_j)x_1^{l_1} \ldots x_m^{l_m} y_1^{k_1} \ldots y_{n-1}^{k_{n-1}} \right\}.$$  

Multiplying both sides by the product $y_1 \ldots y_n$ and using Lemma 2.3, we prove the theorem in this case.

If $\nu_1 = n$ then the box $(1, n)$ belongs to $\pi_{\lambda}$ and therefore by definition either $i(\lambda) > 0$ or $j(\lambda) = n$. If $i(\lambda) = 0$ then $j(\lambda) = n - m < n$ since $m > 0$. Thus, $i(\lambda) \geq 1$, which implies that $\lambda_1 + d - 1 \geq 0$. This means that we can apply Lemma 2.2. By induction we have

$$\{SP\lambda(x_2, \ldots, x_m)x_2^{m-2} \ldots x_m^0 y^{\rho_n}\} = (-1)^b \left\{ \prod_{(i, j) \in \hat{\nu}} (x_i - y_j)x_2^{l_2} \ldots x_m^{l_m} y_1^{k_1} \ldots y_n^{k_n} \right\},$$

where $\hat{\nu}$ is the partition $\nu$ without the first row. Now the theorem follows from Lemma 2.2. \qed

3. Euler supercharacters for Lie superalgebra $osp(2m + 1, 2n)$

In this section we consider the case of orthosymplectic Lie superalgebra $osp(2m + 1, 2n)$, the case of $osp(2m, 2n)$ is considered in Section 7.

Recall the description of the root system of Lie superalgebra $g = osp(2m + 1, 2n)$. We have $g = g_0 \oplus g_1$, where $g_0 = so(2m + 1) \oplus sp(2n)$ and $g_1 = V_1 \oplus V_2$ where $V_1$ and $V_2$ are the identical representations of $so(2m + 1)$ and $sp(2n)$.
respectively. Let $\pm \varepsilon_1, \ldots, \pm \varepsilon_m, \pm \delta_1, \ldots, \pm \delta_n$ be the non-zero weights of the standard representation of $\mathfrak{g}$. The root system of $\mathfrak{osp}(2m+1, 2n)$ consists of

$$R_0 = \{\pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_i, \pm \delta_p \pm \delta_q, \pm 2\delta_p, i \neq j, 1 \leq i, j \leq m, p \neq q, 1 \leq p, q \leq n\},$$

$$R_1 = \{\pm \varepsilon_i \pm \delta_p, \pm \delta_p\}, \quad R_{iso} = \{\pm \varepsilon_i \pm \delta_p\},$$

where $R_0, R_1$ and $R_{iso}$ are even, odd and isotropic parts respectively. The invariant bilinear form is given by

$$(\varepsilon_i, \varepsilon_i) = 1, (\varepsilon_i, \varepsilon_j) = 0, i \neq j, (\delta_p, \delta_p) = -1, (\delta_p, \delta_q) = 0, p \neq q, (\varepsilon_i, \delta_p) = 0.$$ 

The Weyl group $W_0 = (S_m \times \mathbb{Z}_2^m) \times (S_n \times \mathbb{Z}_2^n)$ acts on the weights by separately permuting $\varepsilon_i, i = 1, \ldots, m$ and $\delta_p, p = 1, \ldots, n$ and changing their signs. A distinguished system of simple roots can be chosen as

$$B = \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}.$$ 

Introduce the variables $x_i = e^{\varepsilon_i}, x_i^{1/2} = e^{\varepsilon_i/2}, u_i = x_i + x_i^{-1}, i = 1, \ldots, m$ and $y_p = e^{\delta_p}, v_p = y_p + y_p^{-1}, p = 1, \ldots, n$.

For any parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and any finite-dimensional representation $M$ of $\mathfrak{p}$ by a super version of Borel-Weil-Bott construction one can define the corresponding Euler supercharacter $E^\rho(M)$. According to the general formula due to Serganova [5]

$$E^\rho(M) = \sum_{w \in W_0} w \left( \frac{D e^{\rho \operatorname{sch} M}}{\prod_{\alpha \in \mathfrak{p} \cap \mathfrak{r}_1^+} (1 - e^{-\alpha})} \right)$$(11)

with

$$D = \frac{\prod_{\alpha \in \mathfrak{r}_1^+} (e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha \in \mathfrak{r}_1^+} (e^{\alpha/2} - e^{-\alpha/2})}.$$ 

Here $\rho$ is the half-sum of the even positive roots minus the half-sum of odd positive roots, $R_\rho$ is the set of roots $\alpha$ such that $\mathfrak{g}_{\pm \alpha} \subset \mathfrak{p}$ (see formula (3.1) in [5]). Note that we use here the supercharacter rather than the character used by Serganova. This leads simply to the change of signs in some places.

Consider now the parabolic subalgebra $\mathfrak{p}$ with

$$R_\mathfrak{p} = \{\varepsilon_i - \varepsilon_j, \delta_p - \delta_q, \pm (\varepsilon_i - \delta_p)\},$$

where $1 \leq i, j \leq m, i \neq j$ and $1 \leq p, q \leq n, p \neq q$. The algebra $\mathfrak{p}$ is isomorphic to the sum of the Lie superalgebra $\mathfrak{gl}(m, n)$ and some nilpotent Lie superalgebra. In that case Serganova’s formula (11) has the form

$$E(M) = \sum_{w \in W_0} w \left( \frac{\prod_{(i,j) \in \Pi_{m,n}} (1 - x_i^{-1} y_j^{-1}) x_1^{m-1} \cdots x_i^{1} y_1^{n-1} \cdots y_n^{1} \operatorname{sch} M}{\Delta(u) \Delta(v) \prod_{i=1}^m (x_i^{1} - x_i^{-1}) \prod_{j=1}^n (y_j^{1} + y_j^{-1})} \right),$$

where $\Pi_{m,n}$ is the rectangle of the size $m \times n$, $\Delta(u) = \prod_{i<j}^m (u_i - u_j), \Delta(v) = \prod_{i<j}^n (v_i - v_j)$ and $u_i = x_i + x_i^{-1}, v_j = y_j + y_j^{-1}$ as before.
The following proposition explains the appearance of the powers of 2 in the later considerations. As far as we know the appearance of a power of 2 coefficient was first noticed by Cheng and Wang [17]. It should have a geometrical explanation but we give here a direct algebraic proof.

**Proposition 3.1.** For the trivial even representation \( M \) we have

\[
E(M) = 2^{\min(m,n)}.
\]

**Proof.** It is well known (see e.g. [18], Chapter 1, formula (4.3')) that

\[
\prod_{i,j} (1 - x_i^{-1} y_j^{-1}) = \sum (-1)^{\lambda} S_\lambda(x_1^{-1}, \ldots, x_m^{-1}) S_{\lambda'}(y_1^{-1}, \ldots, y_n^{-1}),
\]

where \( S_\lambda \) is the Schur polynomial and the sum is over all the Young diagrams \( \lambda \), which are contained in the \((m \times n)\) rectangle, \( \lambda' \) is the diagram transposed to \( \lambda \). Using the Weyl formula for Schur polynomials we can replace in the formula (12) the product \( \prod_{i,j} (1 - x_i^{-1} y_j^{-1}) \) by the sum

\[
\sum (-1)^{\lambda} x_1^{-\lambda_m + m - 1/2} \ldots x_m^{-\lambda_1 + 1/2} y_1^{-\lambda'_n + n - 1/2} \ldots y_n^{-\lambda'_1 + 1/2}.
\]

One can check that the only non-vanishing terms correspond to the symmetric Young diagrams \( \lambda = \lambda' \). Now the proposition follows from the fact that the number of the symmetric diagrams contained in the \((m \times n)\) rectangle is equal to \( 2^{\min(m,n)} \), which can be easily proved by induction. \( \square \)

Consider now the polynomial representations \( M = V^\lambda \) of \( \mathfrak{gl}(m,n) \) determined by a Young diagram \( \lambda \in H(m,n) \) with the supercharacter given by the super Schur polynomial:

\[
sch V^\lambda = SP_\lambda(x_1, \ldots, x_m, y_1, \ldots, y_n).
\]

The next step is to rewrite the formula for the Euler supercharacter

\[
E_\lambda = E(V^\lambda),
\]

using the formula (7) for the super Schur polynomials in the case when \( \nu = \pi_\lambda \). In this case \( k_i, l_j \) are defined by

\[
l_i = \lambda_i + d - i, \quad 1 \leq i \leq \ell(\lambda), \quad l_i = m - i, \quad i(\lambda) < i \leq m,
\]

\[
k_j = \lambda'_j - d - j, \quad 1 \leq j \leq \ell(\lambda), \quad k_j = n - j, \quad j(\lambda) < j \leq n,
\]

where \( d = m - n \).

Introduce the following polynomials (which are particular cases of classical Jacobi polynomials, see the next section)

\[
\varphi_a(z) = \frac{w^{a+1/2} - w^{-a-1/2}}{w^{1/2} - w^{-1/2}}, \quad \psi_a(z) = \frac{w^{a+1/2} + w^{-a-1/2}}{w^{1/2} + w^{-1/2}},
\]

where \( z = w + w^{-1} \). Define also \( \Pi_\lambda(u, v) \) as

\[
\Pi_\lambda(u, v) = \prod_{(i,j) \in \pi_\lambda} (u_i - v_j).
\]
Theorem 3.2. The Euler supercharacters can be given by the following formula

\[ E_\lambda(u, v) = C(\lambda) \sum_{w \in S_m \times S_n} w \left[ \Pi_\lambda(u, v) \frac{\varphi_{l_1}(u_1) \cdots \varphi_{l_m}(u_m) \psi_{k_1}(v_1) \cdots \psi_{k_n}(v_n)}{\Delta(u) \Delta(v)} \right], \tag{18} \]

where \( C(\lambda) = (-1)^b 2^{n-i(\lambda)} (-1)^b 2^n-j(\lambda) \), \( b = \sum_{j>m} \lambda_j \) and \( i(\lambda), j(\lambda), l_i, k_j \) are defined by (1), (2), (15), (16).

Proof. According to Theorem 2.1 for \( M = V^\lambda \) the corresponding supercharacter \( \text{sch} M = SP_\lambda \) can be given by (7), or, equivalently, by (10). Substituting this into Serganova’s formula (12) we get

\[ E(M) = (-1)^b \sum_{w \in W_0} w \left( \prod_{(i,j) \in \Pi_{m,n}} \frac{(1 - x_i^{-1} y_j^{-1}) (x_i - y_j) x_i^{l_i} y_j^{l_j} y_i^{k_i} y_j^{k_j}}{\Delta(u) \Delta(v) \prod_{i=1}^m (x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}}) \prod_{j=1}^n (y_j^{\frac{1}{2}} + y_j^{-\frac{1}{2}})} \right), \]

where we use the notations

\[ x_i^{l_i} = x_i^{l_1} \cdots x_i^{l_m} \quad y_j^{k_j} = y_j^{k_1} \cdots y_j^{k_n}. \]

Since

\[ \prod_{(i,j) \in \pi_{\lambda}} \frac{(1 - x_i^{-1} y_j^{-1}) (x_i - y_j) x_i^{l_i} y_j^{l_j} y_i^{k_i} y_j^{k_j}}{\Delta(u) \Delta(v) \prod_{i=1}^m (x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}}) \prod_{j=1}^n (y_j^{\frac{1}{2}} + y_j^{-\frac{1}{2}})} = \prod_{(i,j) \in \pi_{\lambda}} (u_i - v_j) = \Pi_\lambda(u, v), \]

we have

\[ E(M) = (-1)^b \sum_{w \in W_0} w \left( \prod_{(i,j) \in \Pi_{m,n} \setminus \pi_{\lambda}} \frac{(1 - x_i^{-1} y_j^{-1}) x_i^{l_i} y_j^{l_j} y_i^{k_i} y_j^{k_j}}{\Delta(u) \Delta(v) \prod_{i=1}^m (x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}}) \prod_{j=1}^n (y_j^{\frac{1}{2}} + y_j^{-\frac{1}{2}})} \right). \]

Now Proposition 3.1 and Weyl’s formula for the \( BC_n \) root system (which is the root system of the Lie superalgebra \( \mathfrak{osp}(1, 2n) \)) allow us to replace here \( \prod_{(i,j) \in \Pi_{m,n} \setminus \pi_{\lambda}} (1 - x_i^{-1} y_j^{-1}) \) by \( 2^{m-i(\lambda)} = 2^n-j(\lambda) \) to come to

\[ E(M) = C(\lambda) \sum_{w \in W_0} w \left( \frac{\Pi_\lambda(u, v) x_i^{l_i} y_j^{l_j} y_i^{k_i} y_j^{k_j}}{\Delta(u) \Delta(v) \prod_{i=1}^m (x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}}) \prod_{j=1}^n (y_j^{\frac{1}{2}} + y_j^{-\frac{1}{2}})} \right). \tag{19} \]

Summing now over the subgroup \( \mathbb{Z}_2^m \times \mathbb{Z}_2^n \subset W_0 \) and using (17) we have the claim. \( \square \)

We will use this formula now to show the relation with super Jacobi polynomials.
4. SUPER JACOBI POLYNOMIALS FOR $k = -1$

The main result of this section is the following Weyl type formula for the super Jacobi polynomials [3] with $k = -1$. Let us introduce the following polynomials $f_l(z, p, q)$, which are certain normalized versions of the classical Jacobi polynomials $P^{\alpha, \beta}_l(z)$ with $\alpha = -p - q - \frac{1}{2}$, $\beta = q - \frac{1}{2}$:

$$f_l(z, p, q) = \sum_{i=0}^{l} C_{l,i}(z - 2)^i,$$

where $C_{l,l} = 1$ and

$$C_{l,i} = \frac{4^{l-i}(i+1)\ldots(l-1)!(i+1-p-q-1/2)\ldots(l-p-q-1/2)}{(l-i)!(l+i-p-2q)\ldots(2l-1-p-2q)}$$

for $i < l$. Introduce also

$$g_k(w, p, q) = f_k(w, -p, -1-q).$$

Note that the polynomials $\varphi(z), \psi(z)$ from the previous section are the particular cases:

$$\varphi_l(z) = f_l(z, -1, 0), \quad \psi_k(z) = g_k(z, -1, 0) = f_k(z, 1, -1).$$

Later we drop the parameters for brevity, writing simply $f_k(z), g_k(z)$.

Let $\lambda \in H(m, n)$ be a partition from the fat hook and $\pi_{\lambda}, l_i, k_j$ be the same as in the previous section (see formulas (15),(16) above).

**Theorem 4.1.** The super Jacobi polynomials for special value of parameter $k = -1$ can be given by the following formula

$$SJ_{\lambda}(u, v, -1, p, q) = (-1)^b \sum_{w \in S_m \times S_n} w \left[ \frac{\Pi_\lambda(u, v)}{\Delta(u)\Delta(v)} f_{l_1}(u_1)\ldots f_{l_m}(u_m)g_{k_1}(v_1)\ldots g_{k_n}(v_n) \right], \quad (20)$$

where as before $b = \sum_{j>m} \lambda_j$ and $\Pi_\lambda(u, v) = \prod_{(i,j) \in \pi_\lambda} (u_i - v_j)$.

We prove this first in the particular case when $\lambda$ contains the $m \times n$ rectangle, i.e.

$$\lambda_m \geq n.$$

The corresponding formula for super Jacobi polynomials can be considered as a natural analogue of Berele-Regev factorization formula for super Schur polynomials [15]. For such a diagram $\lambda$ one can consider its sub-diagram $\mu$, which is the diagram $\lambda$ without first $n$ columns. Define

$$w_i(\lambda) = \mu_i, \quad i = 1, \ldots, m, \quad z_j(\lambda) = \lambda'_j, \quad j = 1, \ldots, n. \quad (21)$$

In other words, $w_i$ is the length of $i$-th row of $\mu$ and $z_j$ is the length of $j$-th column of $\lambda \in H_{m,n}$. 

10
The super Jacobi polynomials [3] in the special case $k = -1$ can be defined for generic $p, q$ in terms of super Schur polynomials $SP_{\lambda}$ by Okounkov's formula

$$SJ_{\lambda}(u, v, -1, p, q) = \sum_{\tilde{\lambda} \subseteq \lambda} K_{\lambda, \tilde{\lambda}} SP_{\tilde{\lambda}}(\tilde{u}, \tilde{v})$$

(22)

where $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_n)$, $\tilde{u} = (u_1 - 2, \ldots, u_m - 2)$, $\tilde{v} = (v_1 - 2, \ldots, v_n - 2)$ and

$$K_{\lambda, \tilde{\lambda}} = 4^{|\lambda| - |\tilde{\lambda}|} \frac{C_{\lambda}^0(d) C_{\lambda}^0(d - p - q - \frac{1}{2}) I_{\lambda}(w(\lambda), z(\lambda), -1, h)}{C_{\lambda}^0(d) C_{\lambda}^0(d - p - q - \frac{1}{2}) C_{\lambda}^0(2h - 1)}.$$  

(23)

Here $d = m - n$ is the superdimension, $h = d - \frac{1}{2}p - q$,

$$C_{\lambda}^+(x) = \prod_{(ij) \in \lambda} (\lambda_i + j - (\lambda'_j + i) + x),$$

(24)

$$C_{\lambda}^-(x) = \prod_{(ij) \in \lambda} (\lambda_i - j + (\lambda'_j - i) + x),$$

(25)

$$C_{\lambda}^0(x) = \prod_{(ij) \in \lambda} (j - 1 - (i - 1) + x),$$

(26)

$I_{\lambda}(w, z, -1, h)$ is the specialization of the deformed interpolation $BC$ polynomial $I_{\lambda}(w, z, k, h)$ (see Proposition 6.3 in [3]), $w(\lambda)$ and $z(\lambda)$ are defined by (21) above. We should note that we are using here more convenient variables

$$u_i = x_i + x_i^{-1}, v_j = y_j + y_j^{-1},$$

rather than $u_i = \frac{1}{2}(x_i + x_i^{-1} - 2), v_j = \frac{1}{2}(y_j + y_j^{-1} - 2)$ used in [3].

**Theorem 4.2.** Let $\lambda \in H(m, n)$ contain the $m \times n$ rectangle. Then the super Jacobi polynomials $SJ_{\lambda}(u, v, -1, p, q)$ can be expressed in terms of the usual Jacobi polynomials as

$$SJ_{\lambda}(u, v, -1, p, q) = (-1)^{|\nu|} \prod_{i=1}^{m} \prod_{j=1}^{n} (u_i - v_j) J_{\mu}(u, -1, p, q) J_{\nu}(v, -1, -p, -1 - q),$$

(27)

where

$$\mu_i = \lambda_i - n, \ i = 1, \ldots, m, \ \nu_j = \lambda'_j - m, \ j = 1, \ldots, n.$$  

(28)

**Proof.** To prove this we need the following factorization formula for the deformed interpolation $BC$ polynomials. We should mention that in the special case $k = -1$ these polynomials are the particular case of the factorial super Schur functions considered by Molev in [16], but in our particular case one can give a simple direct proof.
Lemma 4.3. If $\lambda \in H(m, n)$ contains the $m \times n$ rectangle, then we have the following formula for deformed interpolation BC polynomials

$$I_\lambda(w, z, -1, h) = (-1)^{|\nu|} \prod_{i=1}^m \prod_{j=1}^n \left[(w_i + n + h - i)^2 - (z_j - h - j + 1)^2\right] \times$$

$$I_\mu(w_1, \ldots, w_m, -1, h + n)I_\nu(z_1 - m, \ldots, z_n - m, -1, 1 - h + m),$$

where $I_\mu(w, k, h)$ is the usual interpolation BC polynomial and $\mu, \nu$ are the same as in the theorem.

Proof. Let us denote by $\hat{I}_\lambda$ the right-hand side of the previous equality. According to Proposition 6.3 from [3] it is enough to prove that $\hat{I}_\lambda$ satisfies the following properties:

1) $\hat{I}_\lambda$ is a polynomial in variables $(w_i + h + n - i)^2$, $i = 1, \ldots, m$ and $(z_j - h + 1 - j)^2$, $j = 1, \ldots, n$;
2) the degree of this polynomial is $2|\lambda|$;
3) $\hat{I}_\lambda(w(\lambda), z(\lambda)) = 0$ if $\lambda \in H(m, n)$ and $\lambda \not\subseteq \tilde{\lambda}$;
4) $\hat{I}_\lambda(w(\lambda), z(\lambda)) = \prod_{(i,j) \in \lambda} (1 + \lambda_i - j + \lambda_j' - i)(2h - 1 + \lambda_i + j - \lambda_j' - i)$.

First two statements are obvious from the explicit form of $\hat{I}_\lambda$. The fourth statement can be checked directly. Let us prove the third property.

Let us suppose that $\lambda \not\subseteq \tilde{\lambda}$. Consider two possible cases depending on whether $\tilde{\lambda}$ contains the $m \times n$ rectangle or not. In the first case we have $\mu, \nu \not\subseteq \tilde{\mu}$, $\nu \not\subseteq \tilde{\nu}$. Therefore by definition of the interpolation BC polynomials [9]

$$I_\mu(\mu, -1, h + n)I_\nu(\nu, -1, 1 - h + m) = 0.$$ 

In the second case consider the box $(i, n)$, $1 \leq i \leq m$ such that $(i, n) \not\subseteq \tilde{\lambda}$, but $(i - 1, n) \in \tilde{\lambda}$ (if $i = 1$, we require only first condition). Note that from our assumptions on $\lambda$ it follows that such a box does exist. Then we have $(w_i + n + h - i) + (z_n - h - n + 1) = \mu_i + n - i + \nu_n - n + 1 = 0 + n - i + (i - 1) - n + 1 = 0$, which means that $\hat{I}_\lambda(w(\lambda), z(\lambda)) = 0$. The lemma is proved. \qed

Now we can prove Theorem 4.2. First we note that in Okounkov’s formula we can always assume that the diagram $\tilde{\lambda}$ contains the $m \times n$ rectangle. Indeed, otherwise $K_{\lambda, \tilde{\lambda}} = 0$ since in that case one can easily see that

$$\frac{C_{\lambda}^0(d)}{C_{\lambda}^d(d)} = 0.$$ 

Therefore by Lemma 4.3

$$I_\lambda(w(\lambda), z(\lambda), -1, h) = (-1)^{|\nu|} \prod_{i=1}^m \prod_{j=1}^n (\lambda_i + \lambda_j' - i - j + 1)(\lambda_i - \lambda_j' - j - i + 2h - 1) \times$$

$$I_\mu(\mu, -1, h + n)I_\nu(\nu, -1, 1 - h + m),$$

\[12\]
where \( \tilde{\mu}, \tilde{\nu} \) are defined by \( \tilde{\lambda} \) as in (28). We rewrite now the coefficient \( K_{\lambda, \tilde{\lambda}} \) in terms of the diagrams \( \mu \) and \( \nu \). We have

\[
C_{\lambda}^{-}(1) = \prod_{i=1}^{m} \prod_{j=1}^{n} (\lambda_i + \lambda'_j - i - j + 1)C_{\mu}^{-}(1)C_{\nu'}^{-}(1),
\]

\[
C_{\lambda}^{+}(2h-1) = \prod_{i=1}^{m} \prod_{j=1}^{n} (\lambda_i - \lambda'_j + i + 2h-1)C_{\mu}^{+}(2(h+n)-1)C_{\nu'}^{+}(2(h-m)-1),
\]

\[
\frac{C_{\lambda}^{0}(m - n)}{C_{\lambda}^{0}(m - n)} = \frac{C_{\mu}^{0}(m)}{C_{\mu}^{0}(m)} \frac{C_{\nu'}^{0}(-n)}{C_{\nu'}^{0}(-n)},
\]

\[
\frac{C_{\lambda}^{0}(m - n - p - q - \frac{1}{2})}{C_{\lambda}^{0}(m - n - p - q - \frac{1}{2})} = \frac{C_{\mu}^{0}(m - p - q - \frac{1}{2})}{C_{\mu}^{0}(m - p - q - \frac{1}{2})} \frac{C_{\nu'}^{0}(-n - p - q - \frac{1}{2})}{C_{\nu'}^{0}(-n - p - q - \frac{1}{2})}.
\]

Now we use the Berele-Regev factorization formula [15] for super Schur polynomials

\[
SP_{\lambda}(u, v, -1) = (-1)^{|\nu|} \prod_{i=1}^{m} \prod_{j=1}^{n} (u_i - v_j)P_{\mu}(u)P_{\nu}(v),
\]

where \( P_{\lambda}(u) \) are usual Schur polynomials. Comparing (22) and (27) and using Okounkov’s formula for usual Jacobi polynomials [9] we see that in order to prove the theorem we need to show that

\[
J_{\nu}(v_1, \ldots, v_n, -1, -p, -1 - q) = (-1)^{|\nu|} \sum_{\tilde{\nu} \subseteq \nu} D_{\nu, \tilde{\nu}} P_{\tilde{\nu}}(\tilde{v}),
\]

where

\[
D_{\nu, \tilde{\nu}} = 4^{|\nu| - |\tilde{\nu}|} \frac{C_{\nu}^{0}(-n)}{C_{\nu}^{0}(-n)} \frac{C_{\nu'}^{0}(-n - p - q - \frac{1}{2})}{C_{\nu'}^{0}(-n - p - q - \frac{1}{2})} \frac{I_{\tilde{\nu}}(\nu, -1, 1 - h + m)}{I_{\tilde{\nu}}(\nu, -1, 1 - h + m)} \frac{C_{\lambda}^{+}(1)}{C_{\lambda}^{+}(1)} \frac{C_{\lambda}^{+}(2(h-m)-1)}{C_{\lambda}^{+}(1)},
\]

Since

\[
C_{\nu}^{0}(x) = (-1)^{|\nu|} C_{\nu'}^{0}(-x), \quad C_{\nu}^{-}(x) = C_{\nu'}^{-}(-x), \quad C_{\nu}^{+}(x) = (-1)^{|\lambda|} C_{\lambda}^{+}(-x),
\]

we have

\[
(-1)^{|\nu|} D_{\nu, \tilde{\nu}} = 4^{|\nu| - |\tilde{\nu}|} \frac{C_{\nu}^{0}(n)}{C_{\nu}^{0}(n)} \frac{C_{\nu'}^{0}(n + p + q + \frac{1}{2})}{C_{\nu'}^{0}(n + p + q + \frac{1}{2})} \frac{I_{\tilde{\nu}}(\nu, -1, 1 - h + m)}{I_{\tilde{\nu}}(\nu, -1, 1 - h + m)} \frac{C_{\lambda}^{+}(1)}{C_{\lambda}^{+}(1)} \frac{C_{\lambda}^{+}(1 - 2(h-m))}{C_{\lambda}^{+}(1 - 2(h-m))}
\]

and the proof now follows from Okounkov’s formula. \(\square\)

Let’s come to the proof of the main theorem 4.1. We need the following result. Denote the right-hand side of the formula (20) as \(RHS\).

**Proposition 4.4.** The right-hand side of the formula (20) can be written in terms of super Schur functions as

\[
RHS = \sum_{\mu \subseteq \lambda} C_{\mu}(d, p, q)SP_{\mu}(u_1 - 2, \ldots, u_m - 2, v_1 - 2, \ldots, v_n - 2),
\]

where the coefficients \( C_{\mu}(d, p, q) \) are some rational functions of \( \mu, d, p, q \).
To prove this let us denote \( r = i(\lambda) \), \( s = j(\lambda) \) and expand every polynomial \( f_i \), \( 1 \leq i \leq r \) in terms of the powers of \( u_i - 2 \) and every polynomial \( g_k \), \( 1 \leq j \leq s \) in terms of the powers of \( v_j - 2 \). Then RHS is the sum of terms

\[
\sum_{w \in S_m \times S_n} \frac{w \left[ \Pi(\lambda, u) (u_1 - 2) \cdots (u_r - 2) \delta v (u_1 - 2) \cdots (v_r - 2) \delta F \cdot G \right]}{\Delta(u) \Delta(v)}
\]

with some constant factors depending on \( d, p, q \), where

\[
F_{\lambda} = (u_{r+1} - 2)^{m-r} \cdots (u_m - 2) G_{\lambda} = (v_{s+1} - 2)^{m-s} \cdots (v_n - 2)
\]

and \( T = (\tilde{t}_1, \ldots, \tilde{t}_r, \tilde{k}_1, \ldots, \tilde{k}_s) \) satisfy the conditions \( 0 \leq \tilde{t}_i \leq \tilde{k}_i \), \( 1 \leq i \leq i(\lambda) \) and \( 0 \leq \tilde{k}_j \leq k_j \), \( 1 \leq j \leq j(\lambda) \). Since \( \pi(\lambda) \) is symmetric with respect to \( u_1, \ldots, u_r \) and \( v_1, \ldots, v_s \) we may assume that both \( \tilde{t}_i \) and \( \tilde{k}_j \) are pairwise different. Now the proposition follows from theorem 2.1 because of the following

**Lemma 4.5.** Let \( a_1 > \cdots > a_n \) be a sequence of non-negative integers and \( b_1, \ldots, b_n \) sequence of pairwise different non-negative integers such that \( a_i \geq b_i \), \( i = 1, \ldots, n \). Let us reorder sequence \( \{b_i\} \) in decreasing order \( b'_1 > b'_2 > \cdots > b'_n \). Then for any \( 1 \leq i \leq n \) we have \( b'_i \leq a_i \).

**Proof.** We will prove the lemma by induction on \( n \). The case \( n = 1 \) is obvious. Assume that the lemma is true for some \( n \) and consider the sequences \( a_1 > \cdots > a_n > a_{n+1}, b_1, \ldots, b_{n+1} \), which satisfy the conditions of the lemma and the corresponding sequence \( b'_1 > b'_2 > \cdots > b'_{n} > b'_{n+1} \). Let \( c = \min\{b_1, \ldots, b_{n+1}\} \). If \( c = b_{n+1} \) we can apply inductive assumption to \( a_1, \ldots, a_n, b'_1, \ldots, b'_n \). If \( c = b_i \neq b_{n+1} \) then we can apply inductive assumption to \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_i-1, b_{i+1}, b_{i+2}, \ldots, b_n \). The lemma is proved. \( \square \)

Let us finish the proof of Theorem 4.1. The proof is by induction on \( m+n \). First note that if \( \lambda \) contains the \( m \times n \) rectangle then the theorem follows from Theorem 4.2 and Proposition 7.1 from Okounkov and Olshanski [9]. In particular, this is always true when either \( m \) or \( n \) is zero.

Assume now that both \( m \) and \( n \) are positive and \( \lambda \) does not contain the \( m \times n \) rectangle. Make the substitution \( u_m = v_n = t \) in both sides of the formula (20). The result is independent on \( t \) and reduces to the case with smaller number of variables \( u_1, \ldots, u_{m-1} \) and \( v_1, \ldots, v_{n-1} \); for the left-hand side this follows from Okounkov’s formula, for the right-hand side from Proposition 4.4. Thus by induction we have that the difference between the right-hand side and left-hand side of the formula (20) is divisible by the product \( \prod_{i=1}^{m} \prod_{j=1}^{n} (u_i - v_j) \). Note that both sides are linear combinations of the super Schur polynomials \( SP_\lambda \) with \( \tilde{\lambda} \subseteq \lambda \), which follows from Okounkov’s formula and Proposition 4.4. However the ideal generated by \( \prod_{i=1}^{m} \prod_{j=1}^{n} (u_i - v_j) \) is known to be linearly spanned by the super Schur polynomials \( SP_\mu \) with \( \mu \) containing the \( m \times n \) rectangle. This means that
the difference is actually zero since by assumption $\lambda$ does not contain it. This completes the proof of Theorem 4.1.

As a corollary we have one of our main results.

**Theorem 4.6.** The limit of the super Jacobi polynomials $SJ_\lambda(u, v; -1, p, q)$ as $(p, q) \to (-1, 0)$ is well defined and coincides up to a constant factor with the Euler supercharacter for the Lie superalgebra $osp(2m + 1, 2n)$:

$$SJ_\lambda(u, v; -1, -1, 0) = 2^{d(\lambda) - m} E_\lambda(u, v).$$  

(29)

The proof follows from comparison of formulas (18) and (20).

5. **Pieri formula**

One of the problems with the special value $k = -1$ is that the spectrum of the corresponding ring of quantum integrals is not simple. This means that we need more information to characterize the specialized super Jacobi polynomials in this case. In this section we derive the Pieri formula for super Jacobi polynomials in the case when $k = -1$ and show that it allows to characterize them uniquely.

This formula can be deduced from the Pieri formula [3] for general $k$ by taking the limit $k \to -1$. However the corresponding calculations are quite long, so we will do this in a different way.

Let us introduce some notations. Let $P_{m,n}$ be the set of all partitions $\lambda$ from the fat $(m,n)$ hook $H(m,n)$. Let us call a box $\square = (i, j)$ of Young diagram $\lambda$ special if

$$i - j = d,$$

where $d = m - n$ as before is superdimension. We will write $\mu \sim \lambda$ if the Young diagram $\mu$ can be obtained from $\lambda$ by removing or adding one box.

Introduce the following functions:

$$a_d(\mu, \lambda) = \begin{cases} 0, & \mu = \lambda \setminus \square \text{ for special } \square \\ 1, & \text{otherwise}, \end{cases}$$  

(30)

$$b_d(\lambda) = \begin{cases} -1, & \text{if } \lambda \text{ has a removable special box} \\ 1, & \text{if there is a special box which can be added to } \lambda \\ 0, & \text{otherwise}. \end{cases}$$  

(31)

**Theorem 5.1.** Let $SJ_\lambda(u, v; -1, -1, 0) = SJ_\lambda(u, v)$ be specialized super Jacobi polynomials, then the following Pieri formula holds

$$\left( \sum_{i=1}^{m} u_i - \sum_{j=1}^{n} v_j + 1 \right) SJ_\lambda(u, v) =$$

$$\sum_{\mu \sim \lambda, \mu \in P_{m,n}} a_d(\mu, \lambda) SJ_\mu(u, v) + b_d(\lambda) SJ_\lambda(u, v)$$  

(32)
Proof. We need the following Pieri formula for the Jacobi symmetric functions $J_\lambda(u, k, p, q, h) \in \Lambda$, where $\Lambda$ is the algebra of symmetric functions [18], $h$ is an additional parameter (see [3]). First note that the limit of $J_\lambda(u, k, p, q, h)$ when $k \to -1$ for generic $p, q$ is well defined as it follows from Okounkov’s formula. We denote this limit by $J_\lambda$. By $\lambda \pm \varepsilon_i$ we denote the sets $(\lambda_1, \ldots, \lambda_i-1, \lambda_i \pm 1, \lambda_{i+1}, \ldots)$ respectively.

Lemma 5.2. For $k = -1$ and generic $p, q$ the Jacobi symmetric functions satisfy the following Pieri formula

$$p_1 J_\lambda = \sum_{i: \lambda + \varepsilon_i \in P_{m,n}} J_{\lambda + \varepsilon_i} + \left( \sum_{i=1}^{\ell(\lambda)} a(\lambda_i + d - i) \right) J_\lambda$$

$$+ \left( p + \frac{p(p + 2q + 1)}{2h - 2h(\lambda) - 1} \right) J_\lambda + \sum_{i: \lambda - \varepsilon_i \in P_{m,n}} b(\lambda_i + d - i) J_{\lambda - \varepsilon_i}$$

where $p_1 = u_1 + u_2 + \ldots$, $d = h + \frac{1}{2}p + q$ and

$$a(l) = -\frac{2p(p + 2q + 1)}{(2l - p - 2q - 1)(2l - p - 2q + 1)},$$

$$b(l) = \frac{2l(2l - 2q - 1)(2l - 2p - 2q - 1)(2l - 2p - 4q - 2)}{(2l - p - 2q - 1)^2(2l - p - 2q)}.$$

To prove the lemma we use the following formula for the Jacobi polynomials with $k = -1$ in $m$ variables (see Proposition 7.1 in Okounkov and Olshanski [9]):

$$J_\lambda(u_{i-1}, p, q) = \frac{1}{\Delta(u)} \sum_{w \in S_m} \varepsilon(w)w [f_{\lambda_1 + m-1}(u_1)f_{\lambda_2 + m-2}(u_2) \ldots f_{\lambda_m}(u_m)],$$

where as before $f_i(z) = f_i(z, p, q)$ are the classical normalized Jacobi polynomials in one variable with parameters $p, q$. They satisfy the following three-term recurrence relation (see e.g. [21]):

$$zf_i(z) = f_{i+1}(z) + a(l)f_i(z) + b(l)f_{i-1}(z)$$

with $a(l), b(l)$ given by (34),(35). Therefore we have

$$\left( \sum_{i=1}^{m} u_i \right) J_\lambda(u) = \frac{1}{\Delta(u)} \sum_{i=1}^{m} \sum_{w \in S_m} \varepsilon(w)w [u_i f_{\lambda_1 + m-1}(u_1) \ldots f_{\lambda_m}(u_m)]$$

$$= \sum_{i: \lambda + \varepsilon_i \in P_{m,n}} J_{\lambda + \varepsilon_i}(u) + \left( \sum_{i=1}^{\ell(\lambda)} a(\lambda_i + m - i) + \sum_{i=\ell(\lambda)+1}^{m} a(m - i) \right) J_\lambda(u)$$

$$+ \sum_{i: \lambda - \varepsilon_i \in P_{m,n}} b(\lambda_i + m - i) J_{\lambda - \varepsilon_i}(u),$$
where $J_{\lambda}(u) = J_{\lambda}(u, -1, p, q)$ and $l(\lambda)$ is the number of non-zero parts in partition $\lambda$. Since

$$a(x) = \frac{p(p + 2q + 1)}{2x - p - 2q + 1} - \frac{p(p + 2q + 1)}{2x - p - 2q - 1}$$

we have

$$\sum_{i = l(\lambda) + 1}^{m} a(m - i) = p + \frac{p(p + 2q + 1)}{2m - 2l(\lambda) - p - 2q - 1}.$$ 

Comparing this with (33) we see that this formula is true after the natural homomorphism $\Lambda \to \Lambda_m$, $\Lambda_m$ is the algebra of symmetric polynomials of $m$ variables, if we specialize $d$ to $m$. Since this is valid for all $m$, the lemma follows.

The super Jacobi polynomials $SJ_{\lambda}(u, v, -1, p, q)$ are defined as the image of Jacobi symmetric functions $J_{\lambda}$ under the homomorphism

$$\varphi : \Lambda \to \Lambda_{m,n}, \quad \varphi(p_l) = \sum_{i = 1}^{m} u_i - \sum_{j = 1}^{n} v_j,$$

where $d$ is specialized to $m - n$ and $\Lambda_{m,n}$ is the algebra of supersymmetric polynomials (see e.g. [18]). Computing the limits when $(p, q) \to (-1, 0)$

$$\lim_{(p, q) \to (-1, 0)} a(l) = \delta(l + 1) - \delta(l), \quad \lim_{(p, q) \to (-1, 0)} b(l) = 1 - \delta(l)$$

$$\lim_{(p, q) \to (-1, 0)} \left( p + \frac{p(p + 2q + 1)}{2d - 2l(\lambda) - p - 2q - 1} \right) = -1 + \delta(d - l(\lambda)),$$

we have the following formula

$$\left( \sum_{i = 1}^{m} u_i - \sum_{j = 1}^{n} v_j \right) SJ_{\lambda}(u, v) =$$

$$\sum_{i: \lambda + \varepsilon_i \in \mathcal{P}_{m,n}} SJ_{\lambda + \varepsilon_i}(u, v) + \sum_{i: \lambda - \varepsilon_i \in \mathcal{P}_{m,n}} [1 - \delta(\lambda_i - i + d)] SJ_{\lambda - \varepsilon_i}(u, v)$$

$$+ \sum_{i = 1}^{l(\lambda)} [\delta(\lambda_i - i + d + 1) - \delta(\lambda_i - i + d)] SJ_{\lambda}(u, v) + [\delta(d - l(\lambda)) - 1] SJ_{\lambda}(u, v),$$

where $d = m - n$ and

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}$$

One can check that it is equivalent to the Pieri formula (32). □

**Remark 5.3.** The form of the factor on the left-hand side of Pieri formula (32), which was chosen by convenience, has a clear representation-theoretic meaning: $\sum_{i = 1}^{m} u_i - \sum_{j = 1}^{n} v_j + 1$ is the supercharacter of the standard representation of $osp(2m + 1, 2n)$. 17
Remark 5.4. One can possibly use this for the alternative proof of the main theorem. For this one should only prove that the Euler supercharacters satisfy the corresponding version of the Pieri formula. This is related to the translation functors in representation theory (see e.g. [20]).

6. Jacobi–Trudy formula for Euler supercharacters

In this section we give one more formula for specialized super Jacobi polynomials (and hence for the Euler supercharacters) of Jacobi–Trudy type.

We start with the Jacobi–Trudy formula for Jacobi symmetric functions following [22]. Let \( J_\lambda \in \Lambda \) be the Jacobi symmetric functions with \( k = -1 \) and generic \( p \) and \( q \) (see [3]). They depend also on the additional parameter \( d \) replacing the dimension of the space. Let \( h_i = J_\lambda \) for the partition \( \lambda = (i) \) consisting of one part for positive \( i \) and \( h_i \equiv 0 \) if \( i < 0 \). Define recursively the sequence \( h_i^{(r)} \in \Lambda, i \in \mathbb{Z} \) by the relation

\[
h_i^{(r+1)} = h_i^{(r)} + a(i + d - 1)h_i^{(r)} + b(i + d - 1)h_{i-1}^{(r)}
\]

for \( r = 0, 1, \ldots \) with initial data \( h_i^{(0)} = h_i \) and

\[
a(x) = -\frac{2p(p + 2q + 1)}{(2x - p - 2q - 1)(2x - p - 2q + 1)}, \\
b(x) = \frac{2x(2x - 2q - 1)(2x - 2p - 2q - 1)(2x - 2p - 4q - 2)}{(2x - p - 2q)(2x - p - 2q - 1)^2(2x - p - 2q - 2)}.
\]

Theorem 6.1. [22] The Jacobi symmetric functions with \( k = -1 \) have the following Jacobi-Trudy representation:

\[
J_\lambda = \begin{vmatrix}
    h_{\lambda_1} & h_{\lambda_1}^{(1)} & \cdots & h_{\lambda_1}^{(l-1)} \\
    h_{\lambda_2-1} & h_{\lambda_2-1}^{(1)} & \cdots & h_{\lambda_2-1}^{(l-1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{\lambda_l-t+1} & h_{\lambda_l-t+1}^{(1)} & \cdots & h_{\lambda_l-t+1}^{(l-1)}
\end{vmatrix},
\]

where \( l = l(\lambda) \).

Taking the limit \( p \to -1, q \to 0 \) and the homomorphism \( \phi_{m,n} : \Lambda \to \Lambda_{m,n} \) we have the following

Corollary 6.2. The specialized super Jacobi polynomials \( SJ_\lambda \) satisfy the following Jacobi–Trudy formula

\[
SJ_\lambda(u,v;-1,-1,0) = \begin{vmatrix}
    h_{\lambda_1} & h_{\lambda_1}^{(1)} & \cdots & h_{\lambda_1}^{(l-1)} \\
    h_{\lambda_2-1} & h_{\lambda_2-1}^{(1)} & \cdots & h_{\lambda_2-1}^{(l-1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{\lambda_l-t+1} & h_{\lambda_l-t+1}^{(1)} & \cdots & h_{\lambda_l-t+1}^{(l-1)}
\end{vmatrix},
\]

where \( \lambda \in H_{m,n} \), \( d = m - n \) and \( h_i^{(r)} \) are defined recursively by (37) with

\[
a(x) = \delta(x+1) - \delta(x), \quad b(x) = 1 - \delta(x)
\]
and $h_i^{(0)} = h_i = S J_{\lambda}(u, v; -1, -1, 0)$ for $\lambda = (i)$ for positive $i$ and $h_i^{(0)} = h_i \equiv 0$ if $i < 0$.

7. **Euler supercharacters for different choice of Borel subalgebra**

It turns out that the relation with super Jacobi polynomials can be made more direct if we choose, following to Gruson and Serganova [10], a different Borel subalgebra and suitable parabolic subalgebras.

We consider again the case $\mathfrak{g} = \mathfrak{osp}(2m + 1, 2n)$, assuming for convenience at the beginning that $m \geq n$. Choose the following set of simple roots

$$B = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-n+1} - \delta_1, \delta_1 - \varepsilon_{m-n+2}, \varepsilon_{m-n+2} - \delta_2, \ldots, \varepsilon_m - \delta_n, \delta_n\}$$

This choice is special since this set contains the maximal possible number of isotropic roots. The corresponding set of even positive roots is

$$R_0^+ = \{\varepsilon_i \pm \varepsilon_j, \varepsilon_k, \delta_p \pm \delta_q, 2\delta_r\}$$

with $1 \leq i < j \leq m, 1 \leq k \leq m, 1 \leq p < q \leq n, 1 \leq r \leq n$ and the half-sum

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in R_0^+} \alpha = (m - \frac{1}{2})\varepsilon_1 + (m - \frac{3}{2})\varepsilon_2 + \cdots + \frac{1}{2}\varepsilon_m + n\delta_1 + (n-1)\delta_2 + \cdots + \delta_n.$$ 

The set of positive odd roots is

$$R_1^+ = \{\varepsilon_i \pm \delta_j, i - j \leq m - n, \delta_j \pm \varepsilon_i, i - j > m - n, \delta_j, j = 1, \ldots, n\}$$

with the half-sum

$$\rho_1 = n(\varepsilon_1 + \cdots + \varepsilon_{m-n+1}) + (n-1)\varepsilon_{m-n+2} + \cdots + \varepsilon_n + (n - \frac{1}{2})\delta_1 + \cdots + \frac{1}{2}\delta_n,$$

so

$$\rho = \rho_0 - \rho_1 = \sum_{i=1}^{m-n} (m - n - i + \frac{1}{2})\varepsilon_i - \sum_{i=m-n+1}^{m} \varepsilon_i + \frac{n}{2} \sum_{j=1}^{n} \delta_j.$$ 

The following lemma gives a description of the highest weights with respect to our choice of simple roots $B$ in terms of partitions. We need this to establish the relation with super Jacobi polynomials.

**Lemma 7.1.** The weight

$$\chi = a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m + b_1\delta_1 + b_2\delta_2 + \cdots + b_n\delta_n$$

is a highest weight of an irreducible finite dimensional $\mathfrak{osp}(2m + 1, 2n)$-module if and only if there exists a partition $\lambda \in H(m, n)$ from the fat hook such that

$$\chi + \rho = \sum_{i=1}^{i(\lambda)} (\lambda_i + d - i + \frac{1}{2})\varepsilon_i - \frac{1}{2} \sum_{i > i(\lambda)} \varepsilon_i + \sum_{j=1}^{j(\lambda)} (\lambda'_j - d - j + \frac{1}{2})\delta_j + \frac{1}{2} \sum_{j > j(\lambda)} \delta_j. \quad (40)$$
The proof is a geometric reformulation of the conditions on the highest weights given in [10], Corollary 3.

Let $\lambda$ be a partition from $H(m,n)$ and $\chi$ be the corresponding highest weight. Consider the parabolic subalgebra $p = p(\lambda)$ with

$$R_p = \{ \pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_k, \pm \delta_p \pm \delta_q, \pm \delta_r, \pm 2\delta_r, \pm 2\varepsilon_k \pm \delta_r \}$$

where $i(\lambda) < i < j \leq m$, $i(\lambda) < k \leq m$, $j(\lambda) < p < q \leq n$, $j(\lambda) < r \leq n$ and $i(\lambda), j(\lambda)$ are defined by (1), (2) respectively.

Note that $R_p$ is the root system of the subalgebra $osp(2l + 1, 2l) \subset osp(2m+1, 2n)$ with $l = m-i(\lambda) = n-j(\lambda)$. One can check that $\lambda'_{j(\lambda)} \geq i(\lambda)$.

In the case when $\lambda'_{j(\lambda)} > i(\lambda)$ the subalgebra $p$ is the maximal parabolic one such that $\chi$ can be extended to $p$ as a one-dimensional representation.

Let $E^p(\chi)$ be the corresponding Euler supercharacter given by (11). Let $\pi_\lambda$ be the same as in section 2 and define $t(\lambda)$ as the number of pairs $(i, j) \in \pi(\lambda)$ with $i - j > d = m - n$. Similarly let $s(\lambda)$ be the number of pairs $(i, j) \in \lambda$ with $i - j > d$.

**Theorem 7.2.** Let $p$ and $\chi$ be the parabolic subalgebra and its one-dimensional representation determined by a partition $\lambda \in H(m,n)$, then the specialized Jacobi polynomial $SP_\lambda(u,v,-1,-1,0)$ coincides up to a sign with the corresponding Euler supercharacter:

$$SP_\lambda(u,v,-1,-1,0) = (-1)^s(\lambda)E^p(\chi).$$  (41)

**Proof.** According to general formula (11)

$$E^p(\chi) = \sum_{w \in W_0} w \left( \frac{\prod_{\alpha \in R_1^+} (e^{\alpha/2} - e^{-\alpha/2}) e^{\chi+\rho}}{\prod_{\alpha \in R_0^+} (e^{\alpha/2} - e^{-\alpha/2}) \prod_{\alpha \in R_p \cap R_1^+} (1 - e^{-\alpha})} \right) =$$

$$\sum_{w \in W_0} w \left( \frac{\prod_{\alpha \not\in R_p \cap R_1^+} (e^{\alpha/2} - e^{-\alpha/2}) e^{\chi+\rho+\tau}}{\prod_{\alpha \in R_0^+} (e^{\alpha/2} - e^{-\alpha/2})} \right),$$

where

$$\tau = \frac{1}{2} \sum_{\alpha \in R_p \cap R_1^+} \alpha.$$

A simple calculation shows that

$$\prod_{\alpha \not\in R_p \cap R_1^+} (e^{\alpha/2} - e^{-\alpha/2}) = (-1)^{t(\lambda)} \prod_{(i,j) \in \pi(\lambda)} (u_i - v_j) \prod_{j < j(\lambda)} (y_j^{\frac{1}{2}} - y_j^{-\frac{1}{2}}),$$

where $u_i = x_i + x_i^{-1}$, $x_i = e^{\varepsilon_i}$, $i = 1, \ldots, m$, $v_j = y_j + y_j^{-1}$, $y_j = e^{\delta_j}$, $j = 1, \ldots, n$. One can check also that

$$\chi + \rho + \tau = \sum_{i=1}^{m} \left( l_i + \frac{1}{2} \right) \varepsilon_i + \sum_{j=1}^{j(\lambda)} \left( k_j + \frac{1}{2} \right) \delta_j + \sum_{j > j(\lambda)}^{n} (n - j + 1) \delta_j.$$
where \( l_i, k_j \) are defined by (15), (16). Therefore we have

\[
E^p(\chi) = (-1)^{t(\lambda)} \sum_{w \in W_0} \left( \prod_{(i,j) \in \pi_\lambda} (u_i - v_j) \prod_{j \leq j(\lambda)} \left( y_j^{\frac{1}{2}} - y_j^{-\frac{1}{2}} \right) e^{\chi + \rho + \tau} \right) = \\
(-1)^{t(\lambda)} \sum_{w \in W_0} \left( \prod_{(i,j) \in \pi_\lambda} (u_i - v_j) e^{\chi + \rho + \tau} \right)
\]

Now we use the identities

\[
\sum_{w \in W_i} w \left( \frac{y_1^{l_1} y_2^{l_2} \cdots y_l^{l_l}}{\Delta(v) \prod_{j=1}^m (y_j - y_j^{-1})} \right) = 1 = \sum_{w \in W_2} w \left( \frac{y_1^{l_1} y_2^{l_2} \cdots y_l^{l_l}}{\Delta(v) \prod_{j=1}^m (y_j^{\frac{1}{2}} + y_j^{-\frac{1}{2}})} \right),
\]

where \( W_i \) is the Weyl group of type \( C_l \approx BC_l \), which is a semi-direct product of permutation group \( S_l \) and \( \mathbb{Z}_2^l \). These identities follow from the Weyl (super)character formula for trivial representations of \( \mathfrak{sp}(2l) \) and \( \mathfrak{osp}(1, 2l) \). This leads to

\[
E^p(\chi) = (-1)^{t(\lambda)} \sum_{w \in W_0} \left( \prod_{(i,j) \in \pi_\lambda} (u_i - v_j) x_1^{l_1 + \frac{1}{2}} \cdots x_m^{l_m + \frac{1}{2}} y_1^{k_1 + \frac{1}{2}} \cdots y_n^{k_n + \frac{1}{2}} \right) \Delta(u) \Delta(v) \prod_{i=1}^m (x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}}) \prod_{j=1}^n (y_j^{\frac{1}{2}} + y_j^{-\frac{1}{2}})
\]

Comparing this with Theorem 4.1 and using the obvious relation

\[
sl(\lambda) = tl(\lambda) + bl(\lambda), \quad bl(\lambda) = \sum_{i > m} \lambda_i
\]

we have the claim. \( \square \)

8. The case of \( \mathfrak{osp}(2m, 2n) \)

In this section we present the results in the even orthosymplectic case \( \mathfrak{g} = \mathfrak{osp}(2m, 2n) \).

It would be instructive to start with the special case \( n = 0 \), i.e. with the usual orthogonal Lie algebra \( \mathfrak{o}(2m) \). It has the root system of type \( D_m \) with simple roots

\[
\{ \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m \}.
\]

We have a symmetry \( \varepsilon_m \rightarrow -\varepsilon_m \), corresponding to the (outer) automorphism \( \theta \) of this Lie algebra. This automorphism appears in the description of the representations of the corresponding Lie group \( O(2m) \), which consists of two connected components.

Recall (see [6]) that the highest weights \( \mu = \mu_1 \varepsilon_1 + \cdots + \mu_m \varepsilon_m \) of irreducible finite-dimensional representations of Lie algebra \( \mathfrak{o}(2m) \) have the
following form: all \( \mu_i \) are either integer or half-integer and satisfy the inequalities:

\[
\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m-1} \geq |\mu_m|.
\]

The half-integer \( \mu \) correspond to the spinor representations and can not be extended to the representations of the orthogonal group \( SO(2m) \), so we restrict ourselves by integer \( \mu \). Corresponding representation \( V^\mu \) of \( o(2m) \) can be extended to the full orthogonal group \( O(2m) \) if and only if it is invariant under the automorphism \( \theta \), which is equivalent to \( \mu_m = 0 \). If \( \mu_m \neq 0 \) then one should consider the direct sum

\[
W^\mu = V^\mu \oplus V^{\theta(\mu)},
\]

which gives an irreducible representation of \( O(2m) \).

It is interesting that the sum of the corresponding Euler supercharacters appears also in the general orthosymplectic case \( g = \mathfrak{osp}(2m, 2n) \) as the limit of super Jacobi polynomials (see below), so these limits are natural to link with supergroup \( OSP(2m, 2n) \) rather than Lie superalgebra \( \mathfrak{osp}(2m, 2n) \).

Now let us give precise formulation of the results. We have \( g = g_0 \oplus g_1 \), where \( g_0 = \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n) \) and \( g_1 = V_1 \otimes V_2 \) where \( V_1 \) and \( V_2 \) are the standard representations of \( \mathfrak{so}(2m) \) and \( \mathfrak{sp}(2n) \) respectively. Let \( \pm \varepsilon_1, \ldots, \pm \varepsilon_m, \pm \delta_1, \ldots, \pm \delta_n \) be the non-zero weights of the standard representation of \( g \). The root system of \( \mathfrak{osp}(2m, 2n) \) consists of

\[
R_0 = \{ \pm \varepsilon_i \pm \varepsilon_j, \pm \delta_p \pm \delta_q, \pm 2\delta_r, 1 \leq i < j \leq m, 1 \leq p < q \leq n, 1 \leq r \leq n \},
\]

\[
R_1 = R_{iso} = \{ \pm \varepsilon_i \pm \delta_p, 1 \leq i \leq m, 1 \leq p \leq n \},
\]

where \( R_0, R_1 \) and \( R_{iso} \) are even, odd and isotropic parts respectively. The Weyl group \( W_0 = \left( S_m \ltimes \mathbb{Z}_2^{(m-1)} \right) \times \left( S_n \ltimes \mathbb{Z}_2^{n} \right) \) acts on the weights by separately permuting \( \varepsilon_i, j = 1, \ldots, m \) and \( \delta_p, p = 1, \ldots, n \) and changing their signs such that the total number of signs of \( \varepsilon_i \) is even. A distinguished system of simple roots can be chosen as

\[
B = \{ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \varepsilon_1 - \varepsilon_2, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m \}.
\]

We have again a symmetry \( \varepsilon_m \rightarrow -\varepsilon_m \), corresponding to the automorphism of Lie superalgebra \( \mathfrak{osp}(2m, 2n) \), which is also denoted by \( \theta \). It acts also in a natural way on the Grothendieck ring and supercharacters.

Consider the parabolic subalgebra \( \mathfrak{p} \) with

\[
R_p = \{ \varepsilon_i - \varepsilon_j, \delta_p - \delta_q, \pm(\varepsilon_i - \delta_p), 1 \leq i, j \leq m, i \neq j, 1 \leq p, q \leq n, p \neq q \},
\]

which is isomorphic to the sum of the Lie superalgebra \( \mathfrak{gl}(m, n) \) and some nilpotent Lie superalgebra. For any finite-dimensional representation \( M \) of \( \mathfrak{p} \) the corresponding Euler supercharacter \( E^p(M) \) is given by (11), which in
this case has the form

\[ E^p(M) = \sum_{w \in W_0} w \left( \prod_{(i,j) \in \Pi_{m,n}} \frac{(1-x_i^{-1}y_j^{-1})x_1^{m-1} \ldots x_m^{i}y_1^{n} \ldots y_n^{i} \text{schM}}{\Delta(u)\Delta(v)\prod_{j=1}^{n}(y_j - y_j^{-1})} \right). \]

Here \( \Pi_{m,n} \) is the rectangle of the size \( m \times n \), \( \Delta(u) = \prod_{i<j}^{m}(u_i - u_j) \), \( \Delta(v) = \prod_{i<j}^{n}(v_i - v_j) \) and \( u_i = x_i + x_i^{-1}, v_j = y_j + y_j^{-1}, y_j = e^y \) as before.

**Proposition 8.1.** For the trivial even representation \( M \) we have

\[ E^p(M) = 2^{\min(m-1, n)}. \]  

**Proof.** The proof is similar to Proposition 3.1 and is based on the formula

\[ \prod_{i,j}(1 - x_i^{-1}y_j^{-1}) = \sum (-1)^{|\lambda|} S_\lambda(x_1^{-1}, \ldots, x_m^{-1})S_\lambda(y_1^{-1}, \ldots, y_n^{-1}), \]

where \( S_\lambda \) is the Schur polynomial and the sum is over all the Young diagrams \( \lambda \), which are contained in the \((m \times n)\) rectangle. Replacing in the formula (42) as before the product \( \prod_{i,j}(1 - x_i^{-1}y_j^{-1}) \) by the sum

\[ \sum_{\lambda \in \Pi_{m,n}} (-1)^{|\lambda|} x_1^{-\lambda_1-m-1} \ldots x_m^{-\lambda_m} y_1^{-\lambda_1+n} \ldots y_n^{-\lambda_n+1}, \]

one can check that

\[ \sum_{w \in W_0} w \left( \frac{x_1^{-\lambda_1-m-1} \ldots x_m^{-\lambda_m} y_1^{-\lambda_1+n} \ldots y_n^{-\lambda_n+1}}{\Delta(u)\Delta(v)\prod_{j=1}^{n}(y_j - y_j^{-1})} \right) = 0 \]

unless \( \lambda \) is either empty or partition of the form

\( \lambda = (a_1, \ldots, a_r | a_1 + 1, \ldots, a_r + 1) \)

in Frobenius notations (see e.g. [18]), in which case it is equal to \((-1)^{|\lambda|}\).

Now the proposition follows from the fact that the number of such diagrams contained in the \((m \times n)\) rectangle is equal to \( 2^{\min(m-1, n)} \), which can be proved by induction or reduced to the odd case.

Let \( M = V^\lambda \) be the polynomial representation of \( \mathfrak{gl}(m,n) \) determined by a Young diagram \( \lambda \in H(m,n) \) and define the Euler supercharacters as \( E_\lambda = E(V^\lambda) \). Introduce the following polynomials

\[ \varphi_a(z) = w^a + w^{-a}, \quad a > 0, \quad \varphi_0(z) = 1, \quad \psi_a(z) = \frac{w^{a+1} - w^{-a-1}}{w - w^{-1}}, \quad (44) \]

where \( z = w + w^{-1} \). They are particular case of the Jacobi polynomials known as Chebyshev polynomials of the first and second kind respectively.

Let \( i(\lambda), j(\lambda), k_i, l_j \) be defined as before by (1),(2),(15),(16). Define the following modification of \( i(\lambda) \) by

\[ i^*(\lambda) = \max \{ i \mid \lambda_i + d - i > 0, \quad 1 \leq i \leq m \}. \]
It is clear that $i^*(\lambda)$ is equal to $i(\lambda) - 1$ or to $i(\lambda)$ depending whether $l_i(\lambda) = 0$ or not.

Similarly to Theorem 3.2 one can prove that

**Theorem 8.2.** If $l_m = 0$ then the Euler supercharacters for $\text{osp}(2m, 2n)$ can be given by the following formula

$$E_\lambda = C(\lambda) \sum_{w \in S_m \times S_n} w \left[ \frac{\Pi(u, v) \varphi_{l_1}(u_1) \cdots \varphi_{l_m}(u_m) \psi_{k_1}(v_1) \cdots \psi_{k_n}(v_n)}{\Delta(u) \Delta(v)} \right],$$

where $C(\lambda) = (-1)^b 2^{m-i^*(\lambda)-1}$, $b = \sum_{j>m} \lambda_j$.

If $l_m > 0$ then we have a similar formula for the sum of Euler supercharacters

$$E_\lambda + \theta(E_\lambda) = (-1)^b \sum_{w \in S_m \times S_n} w \left[ \frac{\Pi(u, v) \varphi_{l_1}(u_1) \cdots \varphi_{l_m}(u_m) \psi_{k_1}(v_1) \cdots \psi_{k_n}(v_n)}{\Delta(u) \Delta(v)} \right].$$

**Theorem 8.3.** The limit of the super Jacobi polynomials $SJ_\lambda(u, v; -1, p, q)$ as $(p, q) \to (0, 0)$ is well defined and coincides up to a constant factor with the Euler supercharacter or sum of two Euler supercharacters:

$$SJ_\lambda(u, v; -1, 0, 0) = 2^{i^*(\lambda) - m + 1} E_\lambda(u, v)$$

if $l_m = 0$, and

$$SJ_\lambda(u, v; -1, 0, 0) = E_\lambda + \theta(E_\lambda)$$

if $l_m > 0$.

The proof follows from comparison of formulas (45),(46) with (20).

The Pieri formula for the Euler supercharacters of $\text{osp}(2m, 2n)$ follows from Pieri formula for super Jacobi polynomials with $p = q = 0$. Introduce the following functions:

$$a_d(\mu, \lambda) = \begin{cases} 0, & \mu = \lambda \setminus \square \text{ and } j - i = -d \\ 2, & \mu = \lambda \setminus \square \text{ and } j - i = 1 - d \\ 1, & \text{otherwise,} \end{cases}$$

As before $\mu \sim \lambda$ means that the Young diagram $\mu$ can be obtained from $\lambda$ by removing or adding one box.

**Theorem 8.4.** Let $SJ_\lambda(u, v; -1, 0, 0) = SJ_\lambda(u, v)$ be the specialized super Jacobi polynomials, then the following Pieri formula holds

$$\left( \sum_{i=1}^{m} u_i - \sum_{j=1}^{n} v_j \right) SJ_\lambda(u, v) = \sum_{\mu \sim \lambda, \mu \in P_{m,n}} a_d(\mu, \lambda) SJ_\mu(u, v)$$

The Jacobi-Trudy formula in this case follows directly from (38).
Proposition 8.5. The specialized super Jacobi polynomials satisfy the following Jacobi–Trudy formula

\[ \mathcal{S}J_\lambda(u, v; -1, 0, 0) = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1}^{(1)} & \ldots & h_{\lambda_1}^{(l-1)} \\ h_{\lambda_2-1} & h_{\lambda_2-1}^{(1)} & \ldots & h_{\lambda_2-1}^{(l-1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-i+1} & h_{\lambda_l-i+1}^{(1)} & \ldots & h_{\lambda_l-i+1}^{(l-1)} \end{vmatrix}, \tag{51} \]

where \( \lambda \in H_{m,n} \), \( d = m - n \) and \( h_i^{(r)} \) are defined recursively by (37) with

\[ a(x) = 0, \quad b(x) = 1 + \delta(x - 1) - \delta(x) \]

and \( h_i^{(0)} = h_i = \mathcal{S}J_\lambda(u, v; -1, 0, 0) \) for \( \lambda = (i) \) for positive \( i \) and \( h_i^{(0)} = h_i \equiv 0 \) if \( i < 0 \).

Consider now a different choice of Borel subalgebra and suitable parabolic subalgebras, following Gruson and Serganova [10].

We assume for convenience that \( m > n \). Choose the set of simple roots with maximal number of isotropic roots:

\[ B = \{ \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-n} - \delta_1, \delta_1 - \varepsilon_{m-n+1}, \varepsilon_{m-n+1} - \delta_2, \ldots, \delta_n - \varepsilon_m, \delta_n + \varepsilon_m \}. \]

The corresponding set of even positive roots is

\[ R_0^+ = \{ \varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq m, \ \delta_p \pm \delta_q, 1 \leq p < q \leq n, 2\delta_r, 1 \leq r \leq n \} \]

with the half-sum

\[ \rho_0 = \frac{1}{2} \sum_{\alpha \in R_0^+} \alpha = (m-1)\varepsilon_1 + (m-2)\varepsilon_2 + \cdots + \varepsilon_{m-1} + n\delta_1 + (n-1)\delta_2 + \cdots + \delta_n. \]

The set of positive odd roots is

\[ R_1^+ = \{ \varepsilon_i \pm \delta_p, i-p < m-n, \ \delta_p \pm \varepsilon_i, i-p \geq m-n, 1 \leq i \leq m, 1 \leq p \leq n \} \]

with the half-sum

\[ \rho_1 = n(\varepsilon_1 + \cdots + \varepsilon_{m-n}) + (n-1)\varepsilon_{m-n+1} + \cdots + \varepsilon_{m-1} + n\delta_1 + \cdots + \delta_n, \]

so

\[ \rho = \rho_0 - \rho_1 = \sum_{i=1}^{m-n} (m-n-i)\varepsilon_i. \]

Lemma 8.6. The weight

\[ \chi = a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m + b_1\delta_1 + b_2\delta_2 + \cdots + b_n\delta_n \]

with \( a_m \geq 0 \) is a highest weight of an irreducible finite-dimensional \( \mathfrak{osp}(2m, 2n) \)-module if and only if there exists a partition \( \lambda \in H(m, n) \) from the fat hook such that

\[ \chi + \rho = \sum_{i=1}^{i(\lambda)} (\lambda_i + d - i)\varepsilon_i + \sum_{j=1}^{j(\lambda)} (\lambda'_j - d - j + 1)\delta_j \tag{52} \]
The proof again follows from comparison with Corollary 3 from [10].

Let $\lambda$ be a partition from $H(m, n)$ and $\chi$ be the corresponding highest weight. Consider the parabolic subalgebra $p = p(\lambda)$ with

$$R_p = \{ \pm \varepsilon_i \pm \varepsilon_j, \pm \delta_p \pm \delta_q, \pm 2\delta_r, \pm \varepsilon_k \pm \delta_r \}$$

where $i(\lambda) < i < j \leq m$, $j(\lambda) < p < q \leq n$, $i(\lambda) < k \leq m$, $j(\lambda) < r \leq n$ and $i(\lambda)$, $j(\lambda)$ as before are defined by (1), (2). $R_p$ is the root system of the subalgebra $osp(2l, 2l) \subset osp(2m, 2n)$ with $l = m - i(\lambda) = n - j(\lambda)$.

Let $E^p(\chi)$ be the corresponding Euler supercharacter given by (11). Let $\pi_\lambda$ be the same as in section 2 and define $t(\lambda)$ as the number of pairs $(i, j) \in \pi(\lambda)$ with $i - j \geq d = m - n$. Similarly let $s(\lambda)$ be the number of pairs $(i, j) \in \lambda$ with $i - j \geq d$.

**Theorem 8.7.** Let $p$ and $\chi$ be the parabolic subalgebra and its one-dimensional representation determined by a partition $\lambda \in H(m, n)$, then

$$S J_\lambda(u, v; -1, 0, 0) = (-1)^{s(\lambda)} 2^{t(\lambda) - i(\lambda)} E^p(\chi)$$

(53)

if $\lambda_m \leq n$, and

$$S J_\lambda(u, v; -1, 0, 0) = (-1)^{s(\lambda)} (E^p(\chi) + \theta(E^p(\chi)))$$

(54)

if $\lambda_m > n$.

**Proof.** According to general formula (11)

$$E^p(\chi) = \sum_{w \in W_0} w \left( \prod_{\alpha \notin R_p \cup R_1^+} (e^{\alpha/2} - e^{-\alpha/2}) e^{\chi + \rho + \tau} \right)$$

where

$$\tau = \frac{1}{2} \sum_{\alpha \in R_p \cap R_1^+} \alpha = \sum_{i > i(\lambda)} (m - i) \varepsilon_i + \sum_{j > j(\lambda)} (n - j + 1) \delta_j.$$

One can check that

$$\prod_{\alpha \notin R_p \cap R_1^+} (e^{\alpha/2} - e^{-\alpha/2}) = (-1)^{t(\lambda)} \prod_{(i, j) \in \pi_\lambda} (u_i - v_j)$$

and that

$$\chi + \rho + \tau = \sum_{i=1}^{m} l_i \varepsilon_i + \sum_{j=1}^{n} (k_j + 1) \delta_j,$$

where $l_i$, $k_j$ are defined by (15), (16). Therefore we have

$$E^p(\chi) = (-1)^{t(\lambda)} \sum_{w \in W_0} w \left( \frac{\Pi_\lambda(u, v)e^{\chi + \rho + \tau}}{\Delta(u)\Delta(v) \prod_{j=1}^{n} (y_j - y_j^{-1})} \right) =$$

$$(-1)^{t(\lambda)} \sum_{w \in W_0} w \left( \frac{\Pi_\lambda(u, v) x_1^{l_1} \ldots x_m^{l_m} y_1^{k_1+1} \ldots y_n^{k_n+1}}{\Delta(u)\Delta(v)} \right).$$
If $l_m = 0$ then it is easy to see that the average over $\mathbb{Z}_{2}^{m-1} \times \mathbb{Z}_{2}^{n}$ gives
\[
E^{p}(\chi) = D(\lambda) \sum_{w \in S_m \times S_n} w \left( \frac{\Pi_{\lambda}(u, v) \varphi_{l_1}(u_1) \ldots \varphi_{l_m}(u_m) \psi_{k_1+1}(v_1) \ldots \psi_{k_n+1}(v_n)}{\Delta(u)\Delta(v)} \right)
\]
with $D(\lambda) = (-1)^{t(\lambda)} 2^{i(\lambda)-i^*(\lambda)}$. If $l_m > 0$ then we should consider the sum
\[
E^{p}(\chi) + \theta(E^{p}(\chi)) = (-1)^{t(\lambda)} \sum_{w \in S_m \times S_n} w \left( \frac{\Pi_{\lambda}(u, v) \varphi_{l_1}(u_1) \ldots \varphi_{l_m}(u_m) \psi_{k_1+1}(v_1) \ldots \psi_{k_n+1}(v_n)}{\Delta(u)\Delta(v)} \right).
\]
Now the claim follows from Theorem 4.1 and the relation
\[
t(\lambda) = s(\lambda) + b(\lambda).
\]

\[\square\]

**Remark 8.8.** The coefficient $2^{i(\lambda)-i^*(\lambda)}$ (which can be either 1 or 2) can be eliminated by a different choice of parabolic subalgebra. Indeed, in the case when $i^*(\lambda) = i(\lambda) - 1$ our choice of parabolic subalgebra $p$ is not the maximal one, which in that case corresponds to $\mathfrak{osp}(2l+2, 2l)$.

9. **Concluding remarks**

We have shown that the Euler supercharacters for orthosymplectic Lie superalgebras coincide with the specialized super Jacobi polynomials. This fact seems to be very important and should have more conceptual proof. One possibility is to show that the Euler supercharacters can be uniquely characterized by the Pieri formula and the fact that they are the common eigenfunctions of the corresponding algebra of quantum integrals of deformed Calogero–Moser problem. From representation theory point of view this gives the action of the translation functors on Euler supercharacters.

Another interesting question is about geometry of the genuine parameter space of the (super) Jacobi polynomials. We have seen that the specialization process requires some blowing-up procedure. The calculations in the special case of $\mathfrak{osp}(3, 2)$ indicate that this may lead to a description of the characters of important classes of finite-dimensional representations.

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