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Metadata Record: https://dspace.lboro.ac.uk/2134/15517

Version: Accepted for publication

Publisher: © Elsevier

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Random Periodic Solutions of SPDEs via Integral Equations and Wiener-Sobolev Compact Embedding

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Summary. In this paper, we study the existence of random periodic solutions for semilinear stochastic partial differential equations on a bounded domain \( D \subset R^d \) with a smooth boundary in the space \( L^2(D) \). We identify them as the solutions of coupled forward-backward infinite horizon stochastic integral equations on \( L^2(D) \) in general cases. For this we use Mercer’s Theorem and eigenvalues and eigenfunctions of the second order differential operators in the infinite horizon integral equations. We then use the argument of the relative compactness of Wiener-Sobolev spaces in \( C_0([0,T], L^2(\Omega \times D)) \) and generalized Schauder’s fixed point theorem to prove the existence of a solution of the coupled stochastic forward-backward infinite horizon integral equations. The results are also valid for stationary solutions as a special case when the period \( \tau \) can take an arbitrary number. This is the first paper in literature to study random periodic solutions of stochastic partial differential equations. Our result is also new in finding semi-stable stationary solution for non-dissipative stochastic partial differential equations, while in literature the classical method is to use the pull-back technique so researchers were only able to find stable stationary solutions for dissipative systems.

Keywords: random periodic solution, semilinear stochastic partial differential equation, Wiener-Sobolev compactness, Malliavin derivative, coupled forward-backward infinite horizon stochastic integral equations.

1 Introduction

Dynamics of nonlinear differential equations, both deterministic and stochastic, are complex. It is of great importance to understand these complexities. Mathematicians have made enormous progress in understanding these complexities for deterministic systems, both of finite dimensional and infinite dimensional. Understanding the complexities of stochastic systems are far from clear even for stationary solutions. The concept of stationary solutions is the stochastic counterpart of fixed points to deterministic dynamical systems. A fixed point is the simplest equilibrium and large time limiting set of a deterministic dynamical system. A periodic solution is a more complicated limiting set. The theory of periodic solutions has played a central role in the study of the complex behaviour of a dynamical system. They are relatively simple trajectories themselves. However, their existence and construction is a challenging problem in the study of dynamical systems. The study has occupied a central role in the theory of dynamical system since the seminal work Henri Poincaré [25]. Periodic solutions of partial differential equations of parabolic type has been studied by a number of authors, Vejvoda [31], Fife [13], Hess [15], Lieberman [17], [18], to name but a few. From periodic solutions, more complicated solutions can be built in. Since the theory of the existence of the solution of the stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs) become better understood (Da Prato and Zabczyk [8], Prévôt and Röckner [27]) we need to study more detailed question about the behaviour of solutions of SDEs and SPDEs. Mathematicians have been very much interested in the study of the existence of solutions of SDEs and SPDEs, and invariant manifolds near stationary solutions. For results about SPDEs, see Sinai [28], [29], Mattingly [21], E, Khanin, Mazel and Sinai [11], Caraballo, Kloeden and Schmalfuss [3], Liu and Zhao [20], Zhang and Zhao [32], [33], Duan,
Lu and Schmalfuss [9], [10], Mohammed, Zhang and Zhao [22], Lian and Lu [19], though there are still many problems that need to be understood. In literature, there were only few works on periodicity of stochastic systems. For linear stochastic differential equations with periodic coefficients in the sense of distribution, see Chojnowska-Michalik [5], [6], and for one-dimensional random mappings, see Klünger [16]. We began to address the problem of pathwise random periodic solutions to SDEs in Zhao and Zheng [34], Feng, Zhao and Zhou [12]. In this context, first we would like to motivate the reader with the following question. Consider a deterministic evolution equation on a Hilbert space $H$

\[ \frac{du}{dt} = Au + f(u). \]  

(1.1)

Assume it has a periodic solution of periodic $\tau$, $Z : (-\infty, \infty) \to H$ such that $Z(t + \tau) = Z(t)$, for any $t \in (-\infty, \infty)$. Now we consider the following stochastic differential equation, which can be regarded formally as the random perturbation of (1.1) with a white noise perturbation:

\[ du = (Au + f(u))dt + g(u)dW(t). \]  

(1.2)

Here $W$ is a two-sided Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ valued in a Hilbert space $K$ and $g : H \to L_2(K, H)$ taking values in the space of Hilbert-Schmidt operators. Assume the solution of such an equation with a given initial condition exists and is unique. Such an equation has been considered in literature for many SDEs and SPDEs. The question to ask is: does equation (1.2) still possess a periodic solution? Of course the answer is definitely no in general if we think periodic solution a close trajectory as in the deterministic sense. But a close trajectory is not the right notion of random periodic solution to stochastic systems, just like the deterministic fixed point is not a right notion for stochastic systems. One can not expect that, in general, equation (1.2) has a solution such that $u(t + \tau) = u(t)$ unless in a very special situation. There is an interaction between the periodic solution and the noise. Intuitively, the periodic solution has tendency to make trajectories of the random dynamical system following a periodic circle, at least in the dissipative case. The noise tends to make trajectories spreading out. Understanding of this kind of phenomenon was attempted by considering first linear approximation in physics literature, assuming the deterministic macroscopic equation has a periodic solution (see e.g. [30]). Note the following observation: let

\[ u(t) = Z(t) + v(t). \]

Then $v$ satisfies

\[ dv(t) = (Av(t) + b(t, v(t)))dt + \sigma(t, v(t))dW(t), \]  

(1.3)

where

\[ b(t, v) = f(Z(t) + v) - f(Z(t)), \]

\[ \sigma(t, v) = g(Z(t) + v). \]

Note $b, \sigma$ are periodic function in $t$, i.e. $b(t + \tau, v) = b(t, v)$ and $\sigma(t + \tau, v) = \sigma(t, v)$ for any $t \in R$ and $v \in H$. Now the question is reduced to the study of the random periodic solution of equation (1.3) with periodic coefficients. In fact, this kind of stochastic differential equations with periodic coefficients arises in modelling many physical problems. For example, it was considered in climate dynamics literature that mid-latitude oceans can be modelled by time periodic wind forcing when one takes into account
the seasonal cycles in winds. But a more realistic model should include a stochastic effects ([4]). The periodic solution is naturally extended to the notion of the random periodic solution to equation such as equation (1.3) with periodic coefficients by [12]. If the periodic solution $Z$ of Equation (1.1) is exponentially stable and the noise is reasonably small in Equation (1.3) ($g(u)$ is Lipschitz in $u$ and the Lipschitz constant is reasonably small), we can construct a stable random periodic solution to equation (1.3) therefore obtain a random periodic solution of equation (1.2). But in the non-dissipative case that equation (1.1) has a periodic solution $Z$ of period $\tau$, not stable but semi-stable, the situation is more complicated. Pull-back and Poincaré mapping approaches do not seem working easily in this situation.

In [12], we proved in the case that $H = R^d$ and $A$ is hyperbolic the existence of random periodic solution of Equation (1.3) is equivalent to the existence of a solution of an infinite horizon $(-\infty, \infty)$ integral equation. In fact, the result holds in both finite and infinite dimensional spaces, though we only gave the proof in the $R^d$ case. Furthermore, we extended the Schauder fixed point theorem to the case when the subspace of the Banach space is not closed and the Wiener-Sobolev compactness theorem to the relative compactness on the space $C([0, T], L^2(dP))$. Then we proved the existence of a solution of the infinite horizon integral equation.

In this paper, we continue to push this new idea to the following stochastic partial differential equation of parabolic type on a bounded domain $D \subset R^d$ with a smooth boundary:

$$
\begin{align*}
du(t, x) &= Lu(t, x) dt + F(t, u(t, x)) dt + \sum_{k=1}^{\infty} \sigma_k(t) \phi_k(x) dW^k(t), \quad t \geq s, \\
& \quad u(s) = \psi \in L^2(D), \\
& \quad u(t)|_{\partial D} = 0.
\end{align*}
$$

(1.4)

Here $L$ is the second order differential operator with Dirichlet boundary condition on $D$,

$$
Lu = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + c(x)u.
$$

(1.5)

Assume

**Condition (L):** the coefficients $a_{ij}, c$ are smooth functions on $\bar{D}$, $a_{ij} = a_{ji}$, and there exists $\gamma > 0$ such that $\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq \gamma |\xi|^2$ for any $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in R^d$.

Under the above conditions, $L$ is a self-adjoint uniformly elliptic operator and has discrete real-valued eigenvalues $\mu_1 \geq \mu_2 \geq \cdots$ such that $\mu_k \to -\infty$ when $k \to \infty$. Denote by $\{\phi_k \in L^2(D), \ k \geq 1\}$ a complete orthonormal system of eigenfunctions of $L$ with corresponding eigenvalues $\mu_k$, $k \geq 1$. Here the space $L^2(D)$ is a standard square integrable measurable function space vanishing on the boundary with norm $|| \cdot ||_{L^2(D)}$. A standard notation $H^1_0(D)$ denotes a standard Sobolev space of the square integrable measurable functions having the first order weak derivative in $L^2(D)$ and vanishing at the boundary $\partial D$. This is a Hilbert space with inner product $(u, v) = \int_D u(x)v(x)dx + \int_{\partial D}(Du(x), Dv(x))dx$, for any $u, v \in H^1_0(D)$. From the uniformly elliptic condition, it’s not difficult to know that $\phi_k \in H^1_0(D)$ and there exists a constant $C$ such that

$$
||\nabla \phi_k||_{L^2(D)} \leq C\sqrt{|\mu_k|}.
$$

(1.6)

We will use it in the proof of our main theorem.
We assume the driving noise $W^k$ are mutually independent one-dimensional two-sided standard Brownian motions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $\sum_{k=1}^{\infty} \sigma_k^2(t) < \infty$. Denote $\Delta := \{(t, s) \in R^2, s \leq t\}$. Equation (1.4) generates a semi-flow $u : \Delta \times H \times \Omega \to \Omega$ when the solution exists uniquely in the space $H = L^2(D)$. Define $\theta : (-\infty, \infty) \times \Omega \to \Omega$ by $\theta_t \omega^k(s) = W^k(t + s) - W^k(t)$. Therefore $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in R})$ is a metric dynamical system. Function $F : R \times R \to R$ is a continuous function. Without causing confusion of notation, we define Nemytskii operator $F : R \times L^2(D) \to L^2(D)$ with the same notation

$$F(t, u(t))(x) = F(t, u(t,x)), \quad F^i(t, u(t))(x) = \int_D F(t, u(t))(y)\phi_i(y)dy \phi_i(x), \quad x \in D, \; u \in L^2(D).$$

Assume $F$ and $\sigma_k$ satisfy:

**Condition (P)** There exists a constant $\tau > 0$ such that for any $t \in R$, $u \in L^2(D)$

$$F(t, u) = F(t + \tau, u), \; \sigma_k(t) = \sigma_k(t + \tau).$$

First, we give the definition of the random periodic solution

**Definition 1.1** A random periodic solution of period $\tau$ of a semi-flow $u : \Delta \times L^2(D) \times \Omega \to L^2(D)$ is an $\mathcal{F}$-measurable map $\varphi : (-\infty, \infty) \times \Omega \to L^2(D)$ such that

$$u(t + \tau, \varphi(t, \omega), \omega) = \varphi(t + \tau, \omega) = \varphi(t, \theta_{\tau} \omega), \quad (1.7)$$

for any $t \in R$ and $\omega \in \Omega$.

Instead of following the traditional geometric method of establishing the Poincaré mapping and finding its fixed point, in this paper, we will push the new analysis method of coupled infinite horizon forward-backward integral equations to the stochastic partial differential equations. This is the first paper dealing with the important question of periodic solution to stochastic partial differential equations. We apply our result to the perturbation problem (1.1) and (1.2) we posed in the case when $H = R^d$, and the case when $H = L^2(D)$, $A = \mathcal{L}$ a second order differential operator (1.5) on a smooth bounded domain $D$. Assume the deterministic system has a periodic solution $Z$ which is hyperbolic. Denote by $G$ the graph of the periodic solution in $H$. Let $N$ be large enough such that the open ball with center 0 and radius $N$ covers $G$. One can then define a differentiable function (assuming $f$ is differentiable) such that

$$f_N(u) = \chi(\frac{||u||^2}{N^2})f(u).$$

Here $\chi : R^1 \to R^1$ is a smooth function such that

$$\chi(z) = \begin{cases} 1, \text{ when } |z| \leq 1, \\ 0, \text{ when } |z| \geq 4. \end{cases}$$

It is easy to see that the truncated system

$$\frac{du}{dt} = Au + f_N(u) \quad (1.8)$$

has the same periodic solution $Z$ as Equation (1.1). Our results imply that the perturbed system to Equation (1.8) by an additive noise considered in [12] and in this paper respectively has a random periodic solution.
2 Forward-backward infinite horizon stochastic integral equations

We consider the semilinear stochastic partial differential equation (1.4). Denote the solution by
\[ u(t, s, \omega, x). \]
Throughout this paper, we suppose that \( \mathcal{L} \) is hyperbolic, i.e. none of the eigenvalues of \( \mathcal{L} \) is zero, and \( T_t = e^{\mathcal{L}t} \) is a hyperbolic linear flow induced by \( \mathcal{L} \). So \( L^2(D) \) has a direct sum decomposition:
\[ L^2(D) = E^s \oplus E^u, \]
where
\[ E^s = \text{span}\{v : v \text{ is a generalized eigenvector for an eigenvalue } \mu \text{ with } \mu < 0\}, \]
\[ E^u = \text{span}\{v : v \text{ is a generalized eigenvector for an eigenvalue } \mu \text{ with } \mu > 0\}. \]

Denote \( \mu_m \) is the smallest positive eigenvalue of \( \mathcal{L} \), and \( \mu_{m+1} \) is the largest negative one. We also define the projections onto each subspace by
\[ P^+ : L^2(D) \to E^u, \quad P^- : L^2(D) \to E^s. \]

Define \( \mathcal{F}_t := \sigma(W_u - W_v, s \leq v \leq u \leq t) \) and \( \mathcal{F}^t := \vee_{s \leq t} \mathcal{F}_s \). The solution of the initial value problem (1.4) is given by the following variation of constant formula:
\[
\begin{align*}
\left( T_t \phi \right)(x) &= \int_D K(t, x, y) \phi(y) dy \\
&= \int_D K(t-s, x,y) \psi(y) dy + \int_s^t \int_D K(t-r, x,y) F(s, u(r, s, \psi, \omega))(y) dy dr \\
&\quad + \sum_{k=1}^{\infty} \int_s^t \int_D K(t-r, x,y) \sigma_k(r) \phi_k(y) dy dW^k(r),
\end{align*}
\]
where \( K(t, x, y) \) is the heat kernel of the second order differential operator \( \mathcal{L} \),
\[ \left( T_t \phi \right)(x) = \int_D K(t, x, y) \phi(y) dy, \]
defines a linear operator \( T_t : L^2(D) \to L^2(D) \) and \( \int_s^t \sigma_k(r) (T_{t-r} \phi_k)(\cdot) dW^k(r) \) is an \( L^2(D) \)-valued stochastic integral. Because \( \mathcal{L} \) is a compact self-adjoint operator under the condition of this paper, so by Mercer’s theorem (Chapter 3, Theorem 17, [14]), we have
\[ K(t, x, y) = \sum_{i=1}^{\infty} e^{\mu_i t} \phi_i(x) \phi_i(y). \]

We consider a solution of the following coupled forward-backward infinite horizon stochastic integral equation, which is a \( B(R) \otimes B(D) \otimes \mathcal{F} \)-measurable map \( Y : (-\infty, \infty) \times \Omega \to L^2(D) \) satisfying
\[
\begin{align*}
Y(t, \omega) &= \int_{-\infty}^t T_{t-s} P^- F(s, Y(s, \omega)) ds - \int_t^\infty T_{t-s} P^+ F(s, Y(s, \omega)) ds \\
&\quad + (\omega) \sum_{k=1}^{\infty} \int_{-\infty}^t \sigma_k(s) T_{t-s} P^- \phi_k dW^k(s) - (\omega) \sum_{k=1}^{\infty} \int_t^\infty \sigma_k(s) T_{t-s} P^+ \phi_k dW^k(s) \quad (2.2)
\end{align*}
\]
for all \( \omega \in \Omega, \ t \in (-\infty, \infty). \) The value of \( Y(t, \omega) \in L^2(D) \) at \( x \) is \( Y(t, \omega)(x). \) Sometimes we write as \( Y(t, \omega, x) \) when there is no confusing. We will give the following general theorem which identifies the solution of the equation (2.2) and a random periodic solution of stochastic differential equation (1.4). First, we recall the definition of a tempered random variable (Definition 4.1.1 in [1]):
**Definition 2.1** A random variable \( X : \Omega \rightarrow L^2(D) \) is called tempered with respect to the dynamical system \( \theta \) if

\[
\lim_{r \to \pm \infty} \frac{1}{|r|} \log \|X(\theta_r, \omega)\|_{L^2(D)} = 0.
\]

The random variable is called tempered from above (below) if in the above limit, the function \( \log^+ \) (\( \log^- \)) is replaced by \( \log^+ \) (\( \log^- \)) part of the function \( \log \).

**Theorem 2.1** Assume Condition (P). If Cauchy problem (1.4) has a unique solution \( u(t, s, \omega, x) \) and the coupled forward-backward infinite horizon stochastic integral equation (2.2) has one solution \( Y : (-\infty, +\infty) \times \Omega \rightarrow L^2(D) \) such that \( Y(t + \tau, \omega) = Y(t, \theta_r \omega) \) for any \( t \in R \text{ a.s.} \), then \( Y \) is a random periodic solution of equation (1.4) i.e.

\[
u(t + \tau, t, Y(t, \omega), \omega) = Y(t + \tau, \omega) = Y(t, \theta_r \omega) \quad \text{for any} \quad t \in R \quad \text{a.s.}
\]

(2.3)

Conversely, if equation (1.4) has a random periodic solution \( Y : (-\infty, +\infty) \times \Omega \rightarrow L^2(D) \) of period \( \tau \) which is tempered from above for each \( t \), then \( Y \) is a solution of the coupled forward-backward infinite horizon stochastic integral equation (2.2).

**Proof:** Similar to the proof of Theorem 2.1 in [12].

We will need the following generalized Schauder’s fixed point theorem to prove our theorem. The proof was refined from the proof of Schauder’s fixed point theorem and was given in [12].

**Theorem 2.2** (Generalized Schauder’s fixed point theorem) Let \( H \) be a Banach space, \( S \) be a convex subset of \( H \). Assume a map \( T : H \rightarrow H \) is continuous and \( T(S) \subset S \) is relatively compact in \( H \). Then \( T \) has a fixed point in \( H \).

The generalized Schauder’s fixed point theorem requires us to check the relative compactness. Since the equation can be transformed to an \( \omega \)-wise equation, one could be tempted to treat \( \omega \) as a parameter and to try to define \( \omega \)-parameterised Banach space and subspace, and then to use Rellich-Kondrachov compactness embedding theorem to check the relative compactness. The problem with this approach is that, we get one solution with a parameter \( \omega_1 \) and one solution with a parameter \( \omega_2 \), but no priori relation between these solutions may be known. They may indeed belong to two different families of random periodic solutions due to the non-uniqueness of the solutions of the infinite horizon integral equation. Assume \( \omega_2 = \theta_r \omega_1 \). It is desirable to have \( Y(t + \tau, \omega_1) = Y(t, \omega_2) \) for all \( t \geq 0 \). But this is beyond what the analytic method can offer to us immediately. To overcome this difficulty, we use Malliavin calculus, Wiener-Sobolev compact embedding theorem to get the relatively compactness of a sequence in \( C^0([0, T], L^2(\Omega \times D)) \) with Sobolev norm being bounded in \( L^2(\Omega) \) and Malliavin derivative being bounded and equicontinuous in \( L^2(\Omega \times D) \) uniformly in time.

We denote by \( C_p^\infty(R^n) \) the set of infinitely differentiable functions \( f : R^n \rightarrow R \) such that \( f \) and all its partial derivatives have polynomial growth. Let \( S \) be the class of smooth random variables \( F \) such that \( F = f(W(h_1), \ldots, W(h_n)) \) with \( n \in N, \ h_1, \ldots, h_n \in L^2([0, T]) \) and \( f \in C_p^\infty(R^n) \), \( W(h_i) = \int_0^T h_i(s)dW(s) \). The derivative operator of a smooth random variable \( F \) is the stochastic process \( \{D_tF, \ t \in [0, T]\} \) defined by (c.f. [23])

\[
D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i(t).
\]
We will denote $D^{1,2}$ the domain of $D$ in $L^2(\Omega)$, i.e. $D^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm
$$
\|F\|_{1,2}^2 = E|F|^2 + E\|D_tF\|_{L^2([0,T])}^2.
$$
Denote $C^0([0,T], L^2(\Omega \times D))$ the set of continuous functions $f(\cdot, \cdot, \omega)$ with the norm
$$
\|f\|^2 = \sup_{t \in [0,T]} \int_D E|f(t,x)|^2 dx < \infty.
$$
It’s easy to check the following refined version of relative compactness of Wiener-Sobolev space in Bally-Saussereau [2] also holds. This kind of compactness as a purely random variable without including time and space variables was investigated by Da Prato, Malliavin and Nualart [7] and Bally-Saussereau [2] also holds. This kind of compactness as a purely random variable version with- out including time and space variables was investigated by Da Prato, Malliavin and Nualart [7] and Pesat [24] first. Bally-Saussereau considered the convergence in $L^2([0,T] \times \Omega \times D)$.

**Theorem 2.3** Let $D$ be a bounded domain in $\mathbb{R}^d$. Consider a sequence $(v_n)_{n \in N}$ of $C^0([0,T], L^2(\Omega \times D))$. Suppose that:

1. $\sup_{n \in N} \sup_{t \in [0,T]} E\|v_n(t, \cdot)\|_{H^1(D)}^2 < \infty$.
2. $\sup_{n \in N} \sup_{t \in [0,T]} \int_D E\|v_n(t,x, \cdot)\|_{L^2(D)}^2 dx < \infty$.
3. There exists a constant $C > 0$ such that for any $t_1, t_2 \in [0,T]$
$$
\sup_n \int_D E|v_n(t_1,x) - v_n(t_2,x)|^2 dx < C|t_1 - t_2|.
$$
4. (4i) There exists a constant $C$ such that for any $0 < \alpha < \beta < T$, and $h \in R$ with $|h| < \min(\alpha, T-\beta)$, and any $t_1, t_2 \in [0,T]$,
$$
\sup_n \int_D \int_0^h E|D_{\theta + h} v_n(t_1,x) - D_{\theta} v_n(t_2,x)|^2 d\theta dx < C(|h| + |t_1 - t_2|).\nonumber
$$
(4ii) For any $\epsilon > 0$, there exist $0 < \alpha < \beta < T$ such that
$$
\sup_n \sup_{t \in [0,T]} \int_0^{|\alpha, \beta|} \int_D E|D_{\theta} v_n(t,x)|^2 d\theta dx < \epsilon.
$$
Then $(v_n, n \in N)$ is relatively compact in $C^0([0,T], L^2(\Omega \times D))$.

**Proof:** Recall the Wiener chaos expansion
$$
v_n(t, \omega, x) = \sum_{m=0}^\infty I_m(f_n^m(\cdot, t, x))(\omega),
$$
where $f_n^m(\cdot, t, x)$ are symmetric elements of $L^2([0,T]^{m} \times D)$ for each $m \geq 0$. When $m = 0$, $f_n^0(t,x) = E v_n(t,x)$, and
$$
\sup_n \|f_n^0(t, \cdot)\|_{H^1(D)}^2 \leq \sup_n E\|v_n(t, \cdot)\|_{H^1(D)}^2 < \infty.
$$
So $f_n^0(t,x)$ is relatively compact in $L^2(D)$ for fixed $t \in [0,T]$ by Rellich-Kondrachov compact embedding theorem. But for any $t_1, t_2 \in [0,T]$,
$$
\sup_n \sup_{t \in [0,T]} \|f_n^0(t, \cdot)\|_{L^2(D)}^2 \leq \sup_n E\|v_n(t, \cdot)\|_{L^2(D)}^2 < \infty,
$$
$$
\sup_n \|f_n^0(t_1, \cdot) - f_n^0(t_2, \cdot)\|_{L^2(D)}^2 \leq \sup_n E\|v_n(t_1) - v_n(t_2)\|_{L^2(D)}^2 \leq C|t_1 - t_2|.
$$
So by Arzela-Ascoli lemma, \( \{f_n^m\}_{n=1}^\infty \) is relatively compact in \( C^0([0, T], L^2(D)) \). For each \( m \geq 1 \), using the same argument as in Bally-Saussereau [2], we conclude for each fixed \( t \), \( \{f_n^m(\cdot, t, x)\}_{n \in \mathbb{N}} \) is relatively compact in \( L^2([0, T]^m \times D) \). Moreover, for each \( t_1, t_2 \in [0, T] \), consider

\[
\sup_n \int_D \|f_n^m(\cdot, t_1, x) - f_n^m(\cdot, t_2, x)\|^2_{L^2([0, T]^m)} dx \\
\leq \sup_n \int_D \int_0^T E|D_\theta v_n(t_1, x) - D_\theta v_n(t_2, x)|^2 d\theta dx \\
\leq C|t_1 - t_2|,
\]

and

\[
\sup_n \sup_{t \in [0, T]} \int_D \|f_n^m(\cdot, t, x)\|^2_{L^2([0, T]^m)} dx \leq \sup_n \sup_{t \in [0, T]} \int_D \int_0^T E|D_\theta v_n(t, x)|^2 d\theta dx < \infty.
\]

Then by Arzela-Ascoli lemma, we know that \( \{f_n^m\}_{n=1}^\infty \) is relatively compact in \( C^0([0, T], L^2([0, T]^m \times D)) \). Thus we can conclude \( \{v_n\}_{n=1}^\infty \) is relatively compact in \( C^0([0, T], L^2(\Omega \times D)) \) using the same argument as in [2].

Now we are going to prove that equation (2.2) has a solution under some conditions. So according to Theorem 2.1, this gives the existence of the random periodic solution for the stochastic evolution equation (1.4).

**Theorem 2.4** Assume the coefficients of the second order differential operator \( \mathcal{L} \) satisfy condition (L) and the operator \( \mathcal{L} \) is hyperbolic. Let \( F : (-\infty, \infty) \times R \to R \) be a continuous map, globally bounded and \( \nabla F(t, \cdot) \) being globally bounded, and \( F \) and \( \sigma_k \) also satisfy Condition (P) and \( \sum_{k=1}^\infty |\sigma_k(t)|^2 < \infty \), and there exists a constant \( L_1 > 0 \) such that \( \sum_{k=1}^\infty |\sigma_k(s_1) - \sigma_k(s_2)|^2 \leq L_1 |s_1 - s_2| \). Then there exists at least one \( \mathcal{B}(R) \otimes \mathcal{F} \)-measurable map \( Y : (-\infty, +\infty) \times \Omega \to L^2(D) \) satisfying equation (2.2) and \( Y(t + \tau, \omega) = Y(t, \theta, \omega) \) for any \( t \in R, \omega \in \Omega \).

The proof of the theorem is very complex and is based on the following observation and a series of lemmas. Define the \( \mathcal{B}(R) \otimes \mathcal{F} \)-measurable map \( Y_1 : (-\infty, +\infty) \times \Omega \to L^2(D) \) by

\[
Y_1(t, \omega) = (\omega) \sum_{k=1}^\infty \int_{-\infty}^t \sigma_k(s) T_{t-s} P^- \phi_k dW^k(s) - (\omega) \sum_{k=1}^\infty \int_{-\infty}^\infty \sigma_k(s) T_{t-s} P^- \phi_k dW^k(s). \tag{2.4}
\]

Then by changing of variable and periodicity of \( \sigma_k \), we have

\[
Y_1(t, \theta, \omega) \\
= (\theta, \omega) \sum_{k=1}^\infty \int_{-\infty}^t \sigma_k(s) T_{t-s} P^- \phi_k dW^k(s) - (\theta, \omega) \sum_{k=1}^\infty \int_{-\infty}^\infty \sigma_k(s) T_{t-s} P^- \phi_k dW^k(s) \\
= (\omega) \sum_{k=1}^\infty \int_{-\infty}^{t+\tau} \sigma_k(s) T_{t+\tau-s} P^- \phi_k dW^k(s) - (\omega) \sum_{k=1}^\infty \int_{t+\tau}^{\infty} \sigma_k(s) T_{t+\tau-s} P^- \phi_k dW^k(s) \\
= Y_1(t + \tau, \omega). \tag{2.5}
\]

On the other hand,
\[ Y_1(t, \omega, x) = \sum_{k=1}^{\infty} \sum_{i=m+1}^{\infty} \int_{-\infty}^{t} e^{\mu_i(t-s)} \sigma_k(s) \int_{D} \phi_i(y) \phi_k(y) \, dy \phi_i(x) \, dW^k(s) \]

\[ - \sum_{k=1}^{m} \sum_{i=1}^{m} \int_{t}^{\infty} e^{\mu_i(t-s)} \sigma_k(s) \int_{D} \phi_i(y) \phi_k(y) \, dy \phi_i(x) \, dW^k(s) \]

\[ = \sum_{i=m+1}^{\infty} \int_{t}^{\infty} e^{\mu_i(t-s)} \sigma_i(s) \, dW^i(s) \phi_i(x) - \sum_{i=1}^{m} \int_{t}^{\infty} e^{\mu_i(t-s)} \sigma_i(s) \, dW^i(s) \phi_i(x), \]

as \( \{ \phi_i \} \) is the basis of \( L^2(D) \), so \( \int_{D} \phi_i(y) \phi_j(y) \, dy = 0 \), when \( i \neq j \) and \( \int_{D} \phi_i^2(y) \, dy = 1 \). Moreover, we can calculate

\[ ||Y_1||^2 = \sup_{t} E \int_{D} |Y_1(t, y)|^2 \, dy \]

\[ \leq 2 \sup_{t} E \int_{D} \left| \sum_{i=m+1}^{\infty} \int_{-\infty}^{t} e^{\mu_i(t-s)} \sigma_i(s) \, dW^i(s) \phi_i(y) \right|^2 \, dy \]

\[ + 2 \sup_{t} E \int_{D} \left| \sum_{i=1}^{m} \int_{t}^{\infty} e^{\mu_i(t-s)} \sigma_i(s) \, dW^i(s) \phi_i(y) \right|^2 \, dy \]

\[ = 2 \sup_{t} E \int_{D} \sum_{i=m+1}^{\infty} \int_{-\infty}^{t} e^{2\mu_i(t-s)} |\sigma_i(s)|^2 \, ds |\phi_i(y)|^2 \, dy \]

\[ + 2 \sup_{t} E \int_{D} \sum_{i=1}^{m} \int_{t}^{\infty} e^{2\mu_i(t-s)} |\sigma_i(s)|^2 \, ds |\phi_i(y)|^2 \, dy \]

\[ \leq 2 \sup_{t} E \sum_{i=m+1}^{\infty} \int_{-\infty}^{t} e^{2\mu_{m+1}(t-s)} |\sigma_i(s)|^2 \, ds \]

\[ + 2 \sup_{t} E \sum_{i=1}^{m} \int_{t}^{\infty} e^{2\mu_{m+1}(t-s)} |\sigma_i(s)|^2 \, ds \]

\[ \leq \left( \frac{1}{\mu_{m+1}} + \frac{1}{\mu_{m}} \right) \sup_{s \in (-\infty, \infty)} \sum_{i=1}^{\infty} |\sigma_i(s)|^2 < \infty. \]

Secondly, we need to solve the equation

\[ Z(t, \omega) = \int_{-\infty}^{t} T_{t-s} P^- F(s, Z(s, \omega) + Y_1(s, \omega)) \, ds \]

\[ - \int_{t}^{\infty} T_{t-s} P^+ F(s, Z(s, \omega) + Y_1(s, \omega)) \, ds. \]

For this we define

\[ C^0_{\tau}((-\infty, +\infty), L^2(\Omega \times D)) \]

\[ := \{ f \in C^0((-\infty, +\infty), L^2(\Omega \times D)) : \text{for any} \ t \in (-\infty, \infty), \ f(t + t, \omega, x) = f(t, \theta_\tau \omega, x) \}, \]

and for any \( z \in C^0_{\tau}((-\infty, +\infty), L^2(\Omega \times D)) \), define

\[ M(z)(t, \omega, x) \]

\[ := \int_{-\infty}^{t} T_{t-s} P^- F(s, z(s, \omega) + Y_1(s, \omega))(x) \, ds - \int_{t}^{\infty} T_{t-s} P^+ F(s, z(s, \omega) + Y_1(s, \omega))(x) \, ds. \]
The idea is to find a fixed point to $\mathcal{M}$ in $C^0_{\nu}((\infty, +\infty), L^2(\Omega \times D))$ using the generalized Schauder’s fixed point Theorem 2.2.

**Lemma 2.1** Under the conditions of Theorem 2.4, the map

$$\mathcal{M} : C^0_{\nu}((\infty, +\infty), L^2(\Omega \times D)) \rightarrow C^0_{\nu}((\infty, +\infty), L^2(\Omega \times D))$$

is a continuous map. Moreover $\mathcal{M}$ maps $C^0_{\nu}((\infty, +\infty), L^2(\Omega \times D))$ into $C^0_{\nu}((\infty, +\infty), L^2(\Omega \times D)) \cap L^\infty((\infty, +\infty), L^2(\Omega, H^1_0(D)))$.

**Proof:** Firstly, for any $z \in C^0_{\nu}((\infty, +\infty), L^2(\Omega \times D))$, from $\{\phi_i\}$ is the basis of $L^2(D)$, Cauchy-Schwarz inequality and the linear growth of $F$ with respect to the second variable, we have

$$E \int_D |\mathcal{M}(z)(t,x)|^2 dx$$

$$\leq 2 \int_D E \left[ \int_{-\infty}^t \int_D \sum_{i=m+1}^{\infty} e^{\mu_i(t-s)} \phi_i(x) \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right]^2 dx$$

$$+ 2 \int_D E \left[ \int_{-\infty}^t \int_D \sum_{i=1}^{m} e^{\mu_i(t-s)} \phi_i(x) \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right]^2 dx$$

$$= 2E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^t \int_D e^{\mu_i(t-s)} \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right]^2$$

$$+ 2E \sum_{i=1}^{m} \left[ \int_{-\infty}^t \int_D e^{\mu_i(t-s)} \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right]^2$$

$$\leq 2E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^t \int_D e^{\mu_i(t-s)} |\phi_i(y)|^2 dy ds \cdot \int_{-\infty}^t \int_D e^{\mu_i(t-s)} |F^i(s, z(s) + Y_1(s))(y)|^2 dy ds \right]$$

$$+ 2E \sum_{i=1}^{m} \left[ \int_{-\infty}^t \int_D e^{\mu_i(t-s)} |\phi_i(y)|^2 dy ds \cdot \int_{t}^{\infty} \int_D e^{\mu_i(t-s)} |F^i(s, z(s) + Y_1(s))(y)|^2 dy ds \right]$$

$$\leq \left( \frac{2}{\mu_{m+1}} \right) \sum_{i=m+1}^{\infty} E \int_{-\infty}^t \int_D e^{\mu_{m+1}(t-s)} |F^i(s, z(s) + Y_1(s))(y)|^2 dy ds$$

$$+ \frac{2}{\mu_{m}} \sum_{i=1}^{m} \int_{-\infty}^t \int_D e^{\mu_m(t-s)} |F^i(s, z(s) + Y_1(s))(y)|^2 dy ds$$

$$\leq 2||F||^2_{\infty} \left( \frac{1}{\mu_{m+1}} + \frac{1}{\mu_{m}} \right) vol(D)$$

$$< \infty.$$
For the first term, considering \( \{ \phi_i \} \) is the basis of \( L^2(D) \), and noting the following simple computation, for \( \ell \geq m+1 \),
\[
\int_{-\infty}^{t_1} |e^{\ell(t_1-s)} - e^{\ell(t_2-s)}| ds = -\int_{-\infty}^{t_1} e^{\ell(t_1-s)}|1 - e^{\ell(t_2-t)}| ds \leq (t_2 - t_1) \int_{-\infty}^{t_1} e^{\ell(t_1-s)}|\mu_1| ds = t_2 - t_1,
\]
we have the following estimate,
\[
\begin{align*}
&\int_D E\left| \int_{-\infty}^{t_1} T_{t_1-s} P^- F(s, z(s) + Y_1(s))(x) ds - \int_{-\infty}^{t_2} T_{t_2-s} P^- F(s, z(s) + Y_1(s))(x) ds \right|^2 dx \\
&\leq 2 \int_D E\left| \int_{-\infty}^{t_1} \sum_{i=m+1}^{\infty} (e^{\ell(t_1-s)} - e^{\ell(t_2-s)}) \phi_i(x) \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right|^2 dx \\
&+2 \int_D E\left| \int_{t_1}^{t_2} \sum_{i=m+1}^{\infty} e^{\ell(t_2-s)} \phi_i(x) \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right|^2 dx \\
&\leq 2E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t_1} \left| \int_D (e^{\ell(t_1-s)} - e^{\ell(t_2-s)}) \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy \right|^2 ds \\
&\quad \cdot \int_{-\infty}^{t_1} \left| \int_D (e^{\ell(t_1-s)} - e^{\ell(t_2-s)}) (F^i(s, z(s) + Y_1(s))(y) \right|^2 dy ds \\
&+2E \sum_{i=m+1}^{\infty} \int_{t_1}^{t_2} \int_D |\phi_i(y)|^2 dy ds \cdot \int_{t_1}^{t_2} \int_D |F^i(s, z(s) + Y_1(s))(y)|^2 dy ds \\
&\leq 2E \sum_{i=m+1}^{\infty} (t_2 - t_1) \int_{-\infty}^{t_1} \int_D e^{\ell(t_1-s)}|F^i(s, z(s) + Y_1(s))(y)|^2 dy ds \\
&\quad +2(t_2 - t_1)^2 \| F \|^2_{L^2(\Omega \times D)} vol(D) \\
&\leq \left( -\left( \frac{2}{\mu_{m+1}} \right) t_2 - t_1 \right) \| F \|^2_{L^2(\Omega \times D)} + 2(t_2 - t_1)^2 \| F \|^2_{L^2(\Omega \times D)} \\
&\leq C|t_2 - t_1|.
\end{align*}
\]
And by a similar argument to the second part, we have
\[
E\left| \int_{t_1}^{t_2} T_{t_1-s} P^+ F(s, z(s) + Y_1(s)) ds - \int_{t_2}^{t_1} T_{t_2-s} P^+ F(s, z(s) + Y_1(s)) ds \right|^2 \leq C|t_2 - t_1|.
\]
Therefore, by combining two parts, we have
\[
E \int_D |\mathcal{M}(z)(t_2, x) - \mathcal{M}(z)(t_1, x)|^2 dx \leq C|t_2 - t_1|.
\]
Therefore we have \( \mathcal{M} \) also maps \( C^0_\tau((-\infty, +\infty), L^2(\Omega \times D)) \) into itself. To see the continuity, for any \( z_1, z_2 \in C^0_\tau((-\infty, +\infty), L^2(\Omega \times D)) \),
\[
\int_D E|\mathcal{M}(z_1)(t, x) - \mathcal{M}(z_2)(t, x)|^2 dx
\]
\[
\leq 2E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t} \int_{D} e^{\mu_i(t-s)} \phi_i(y) (F^i(s, z_1(s) + Y_1(s))(y) - F^i(s, z_2(s) + Y_1(s))(y)) dy ds \right]^2
\]

\[
+ 2E \sum_{i=1}^{m} \left[ \int_{-\infty}^{t} \int_{D} e^{\mu_i(t-s)} \phi_i(y) (F^i(s, z_1(s) + Y_1(s))(y) - F^i(s, z_2(s) + Y_1(s))(y)) dy ds \right]^2
\]

\[
\leq 2E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t} \int_{D} e^{\mu_i(t-s)} |\phi_i(y)|^2 dy ds \right.
\]

\[
\cdot \int_{-\infty}^{t} \int_{D} e^{\mu_i(t-s)} |F^i(s, z_1(s) + Y_1(s))(y) - F^i(s, z_2(s) + Y_1(s))(y)|^2 dy ds \left. \right]
\]

\[
+ 2E \sum_{i=1}^{m} \left[ \int_{-\infty}^{t} \int_{D} e^{\mu_i(t-s)} |\phi_i(y)|^2 dy ds \cdot \int_{-\infty}^{t} \int_{D} e^{\mu_i(t-s)} |F^i(s, z_1(s) + Y_1(s))(y) - F^i(s, z_2(s) + Y_1(s))(y)|^2 dy ds \right]
\]

\[
\leq 2 \left( \frac{1}{\mu_{m+1}} \right) \int_{-\infty}^{t} \int_{D} e^{\mu_{m+1}(t-s)} |F(s, z_1(s) + Y_1(s))(y) - F(s, z_2(s) + Y_1(s))(y)|^2 dy ds
\]

\[
+ 2 \left( \frac{1}{\mu_{m}} \right) \int_{t}^{\infty} \int_{D} e^{\mu_{m}(t-s)} |F(s, z_1(s) + Y_1(s))(y) - F(s, z_2(s) + Y_1(s))(y)|^2 dy ds
\]

\[
\leq 2 \| \nabla F \|^2 \left( \frac{1}{\mu_{m+1}} + \frac{1}{\mu_{m}} \right) \sup_{t \in (-\infty, +\infty)} \int_{D} E |z_1(t, x) - z_2(t, x)|^2 dx,
\]

where

\[
\| \nabla F \|^2 := \sup_{t \in (-\infty, +\infty), u \in R} |\nabla F(t, u)|^2 = \sup_{t \in (-\infty, +\infty), u \in R} \sum_{i=1}^{\infty} |\nabla F^i(t, u)|^2.
\]

That is to say that \( M : C^0_\tau(((-\infty, +\infty), L^2(\Omega \times D)) \to C^0_\tau(((-\infty, +\infty), L^2(\Omega \times D)) \) is a continuous map.

Secondly, we need to prove \( M(z) \in L^\infty(((-\infty, \infty), L^2(\Omega, H^1(D))) \) for \( z \in C^0_\tau(((-\infty, +\infty), L^2(\Omega \times D)). \)

Note

\[
E \int_{D} \| \nabla_x M(z)(t, x) \|^2 dx
\]

\[
\leq 2E \int_{D} \left[ \int_{-\infty}^{t} \int_{D} \sum_{i=m+1}^{\infty} e^{\mu_i(t-s)} \nabla_x \phi_i(x) \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right]^2 dx
\]

\[
+ 2E \int_{D} \left[ \int_{-\infty}^{t} \int_{D} \sum_{i=1}^{m} e^{\mu_i(t-s)} \nabla_x \phi_i(x) \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right]^2 dx
\]

\[
:= A_1 + A_2.
\]

For \( A_1 \), by Cauchy-Schwarz inequality and (1.6), we have

\[
A_1 = 2E \int_{D} \left[ \sum_{i,j=m+1}^{\infty} \int_{-\infty}^{t} \int_{D} e^{\mu_i(t-s)} \nabla_x \phi_i(x) \phi_i(y) F^i(s, z(s) + Y_1(s))(y) dy ds \right.
\]

\[
\cdot \int_{-\infty}^{t} \int_{D} e^{\mu_j(t-s)} \nabla_x \phi_j(x) \phi_j(y) F^j(s, z(s) + Y_1(s))(y) dy ds \right] dx
\]
Similarly, 

\[
\frac{\alpha}{\alpha M C_\alpha M} = \frac{\alpha}{\alpha M C_\alpha M} 
\]

Therefore, we can see that:

\[
\int_t^\infty \int_{-\infty}^\infty e^{\mu_i(t-s)}|\phi_i(y)||F^i(s, z(s) + Y_1(s))(y)| dy ds \\
\int_t^\infty \int_{-\infty}^\infty e^{\mu_j(t-s)}|\phi_j(y)||F^j(s, z(s) + Y_1(s))(y)| dy ds \\
\leq 2CE \left[ \sum_{i=m+1}^{\infty} \left( \int_t^\infty \int_{-\infty}^\infty e^{\mu_i(t-s)}|\mu_i|^2|\phi_i(y)||F^i(s, z(s) + Y_1(s))(y)| dy ds \right)^2 \right] \frac{1}{\mu_{m+1}} \\
\leq 2C \left[ \int_t^\infty \int_{-\infty}^\infty e^{\mu_{m+1}(t-s)}|F^i(s, z(s) + Y_1(s))(y)|^2 dy ds \right] \frac{1}{\mu_{m+1}} \\
\leq 2C ||F||^2_2 \left( \frac{1}{\mu_{m+1}} \right) \text{vol}(D) < \infty.
\]

Similarly,

\[
A_2 \leq 2C ||F||^2_2 \left( \frac{1}{\mu_{m+1}} \right) \text{vol}(D) < \infty.
\]

Therefore, we can see \( \mathcal{M} \) maps \( C_\alpha^0((-\infty, +\infty), L^2(\Omega \times D)) \) into \( L^\infty((-\infty, +\infty), L^2(\Omega, H_0^1(D))) \).

Now let us define a subset of \( C^0_\alpha((-\infty, +\infty), L^2(\Omega \times D)) \) as follows:

\[
C_\alpha^0((-\infty, +\infty), L^2(D, D_1^1, D_1^1)) := \{ f \in C_\alpha^0((-\infty, +\infty), L^2(\Omega \times D)) : f_{[0, \tau]} \in C_\alpha^0([0, \tau), L^2(D, D_1^1, D_1^1)) \},
\]

i.e. \( ||f||^2 = \sup_{t \in [0, \tau]} \int_{D} ||f(t, x)||^2_{L^2(D)} < \infty \), and for any \( t, r \in [0, \tau], \ i \in \{0, \pm 1, \pm 2, \ldots \} \)

\[
\int_{D} E|D_r f(t, \theta_{1r}, x)|^2 dx \leq \alpha_r(t) \sup_{s, r_1, r_2 \in [0, \tau]} \int_{D} E|D_{r_1} f(s, \theta_{1r}, x) - D_{r_2} f(s, \theta_{1r}, x)|^2 dx \left| r_1 - r_2 \right| < \infty.
\]

Here \( \alpha_r(t) \) is the solution of integral equation (see page 324 in [26])

\[
\alpha_r(t) = A \int_{r-2^r}^{r+2^r} e^{-\beta(t-s)} \alpha_r(s) ds + B,
\]

(2.8)
where
\[
A = C \| \nabla F \|_\infty^2 \left( - \frac{1}{\mu_{m+1}} \sum_{i=0}^{\infty} e^{\mu_{m+1} i \tau} + \frac{1}{\mu_m} \sum_{i=0}^{\infty} e^{-\mu_m i \tau} \right),
\]
\[
B = C \| \nabla F \|_\infty^2 \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} \sigma_j^2(s) \left( \frac{1}{\mu_{m+1}} + \frac{1}{\mu_m} \right), \quad \beta = \min \{-\mu_{m+1}, \mu_m\}.
\]

This is a convex set.

**Lemma 2.2** Under the conditions of Theorem 2.4, \( \mathcal{M} \) maps \( C^0_{\tau, \alpha}((-\infty, +\infty), L^2(D, \mathcal{D}^{1,2})) \) into itself.

**Proof:** The Malliavin derivatives of \( Y_1(t, \omega, x) \) and \( \mathcal{M}(z)(t, \omega, x) \) can be calculated as:

\[
D_r Y_1(t, \omega, x) = \begin{cases} 
\sum_{i=m+1}^{\infty} e^{\mu_i(t-r)} \phi_i(x) \sigma_i(r), & \text{if } r \leq t, \\
- \sum_{i=1}^{m} e^{\mu_i(t-r)} \phi_i(x) \sigma_i(r), & \text{if } r > t.
\end{cases} \tag{2.9}
\]

When \( r \leq t \), it is easy to see that

\[
D_r \mathcal{M}(z)(t, \omega, x)
\]

\[
= \sum_{i=m+1}^{\infty} \left( \int_{-\infty}^{t} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) D_r z(s, \omega, y) dy ds \right) \phi_i(x)
\]

\[
- \sum_{i=1}^{m} \left( \int_{t}^{\infty} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) D_r z(s, \omega, y) dy ds \right) \phi_i(x)
\]

\[
+ \sum_{i=m+1}^{\infty} \left( \int_{-\infty}^{t} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) \sum_{j=1}^{m} \left( - e^{\mu_j(s-r)} \phi_j(y) \sigma_j(r) \right) dy ds \right) \phi_i(x)
\]

\[
+ \sum_{i=m+1}^{\infty} \left( \int_{t}^{\infty} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) \sum_{j=m+1}^{\infty} \left( e^{\mu_j(s-r)} \phi_j(y) \sigma_j(r) \right) dy ds \right) \phi_i(x)
\]

\[
- \sum_{i=1}^{m} \left( \int_{t}^{\infty} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) \sum_{j=m+1}^{\infty} \left( e^{\mu_j(s-r)} \phi_j(y) \sigma_j(r) \right) dy ds \right) \phi_i(x).
\]

Similarly, when \( r > t \), we have

\[
D_r \mathcal{M}(z)(t, \omega)
\]

\[
= \sum_{i=m+1}^{\infty} \left( \int_{-\infty}^{t} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) D_r z^j(s, \omega, y) dy ds \right) \phi_i(x)
\]

\[
- \sum_{i=1}^{m} \left( \int_{t}^{\infty} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) D_r z^j(s, \omega, y) dy ds \right) \phi_i(x)
\]

\[
+ \sum_{i=m+1}^{\infty} \left( \int_{-\infty}^{t} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) \sum_{j=1}^{m} \left( - e^{\mu_j(s-r)} \phi_j(y) \sigma_j(r) \right) dy ds \right) \phi_i(x)
\]

\[
- \sum_{i=1}^{m} \left( \int_{t}^{\infty} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \omega) + Y_1(s, \omega))(y) \sum_{j=1}^{m} \left( - e^{\mu_j(s-r)} \phi_j(y) \sigma_j(r) \right) dy ds \right) \phi_i(x)
\]
So using Cauchy-Schwarz inequality, we have for any $k = 0, \pm 1, \pm 2, \cdots$, $z \in C^0_{r,\alpha}((\infty, +\infty), L^2(D, D^{1,2}))$, when $0 \leq r \leq t < \tau,$

$$E \int_D \left| D_r M(z)(t, \theta_{k\tau}, x) \right|^2 \, dx \leq C E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \theta_{k\tau}) + Y_1(s, \theta_{k\tau}))(y) D_r z(s, \theta_{k\tau}, y) \, dy \, ds \right]^2$$

$$+ C E \sum_{i=1}^{m} \left[ \int_{-\infty}^{t} \int_D e^{\mu_i(t-s)} \phi_i(y) \nabla F^i(s, z(s, \theta_{k\tau}) + Y_1(s, \theta_{k\tau}))(y) D_r z(s, \theta_{k\tau}, y) \, dy \, ds \right]^2$$

$$+ C E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t} \int_D e^{\mu_1(t-s)} \phi_1(y) \nabla F^1(s, z(s, \theta_{k\tau}) + Y_1(s, \theta_{k\tau}))(y) D_r Y_1(s, \theta_{k\tau}, y) \, dy \, ds \right]^2$$

$$\leq C E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t} \int_D e^{\mu_i(t-s)} |\phi_i(y)|^2 \, dy \, ds \right]^2$$

$$+ C E \sum_{i=1}^{m} \left[ \int_{-\infty}^{t} \int_D e^{2\mu_i(t-s)} |\phi_i(y)|^2 \, dy \, ds \right]^2$$

$$+ C E \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t} \int_D e^{2\mu_1(t-s)} |\phi_1(y)|^2 \, dy \, ds \right]^2$$

$$\leq C \left( \frac{1}{\mu_{m+1}} \right)^2 \frac{|| \nabla F ||_\infty^2}{t} \int_{-\infty}^{t} \int_D e^{\mu_{m+1}(t-s)} E|D_r z(s, \theta_{k\tau}, y)|^2 \, dy \, ds$$

$$+ C \frac{1}{\mu_m} || \nabla F ||_\infty^2 \int_{-\infty}^{t} \int_D e^{\mu_m(t-s)} E|D_r z(s, \theta_{k\tau}, y)|^2 \, dy \, ds$$

$$+ C \left( \frac{2}{\mu_{m+1}} \right)^2 || \nabla F ||_\infty^2 \int_{-\infty}^{t+k\tau} \int_D E|D_r Y_1(s, \cdot, y)|^2 \, dy \, ds$$

$$+ C \frac{2}{\mu_m} || \nabla F ||_\infty^2 \int_{t+k\tau}^{\infty} \int_D E|D_r Y_1(s, \cdot, y)|^2 \, dy \, ds.$$

Let us first deal with the third and the fourth terms. When $k = 0, 1, 2, \cdots$, we have $t + k\tau \geq r$ and
Therefore, we have

$$
\int_{-\infty}^{t+k\tau} \int_D E|D_r Y_1(s, \cdot, y)|^2 \, dy \, ds
\leq \int_{-\infty}^{t+k\tau} \int_D E|D_r Y_1(s, \cdot, y)|^2 \, dy \, ds + \int_{-\infty}^{t+k\tau} \int_D E|D_r Y_1(s, \cdot, y)|^2 \, dy \, ds
\leq \int_{-\infty}^{t+k\tau} \int_D \left| \sum_{j=1}^{m} e^{i\mu_j (s-r)} \phi_j(y) \sigma_j(r) \right|^2 \, dy \, ds + \int_{-\infty}^{t+k\tau} \int_D \left| \sum_{j=m+1}^{\infty} e^{i\mu_j (s-r)} \phi_j(y) \sigma_j(r) \right|^2 \, dy \, ds
\leq \left( \frac{1}{2\mu_m} - \frac{1}{2\mu_{m+1}} \right) s \sum_{j=1}^{\infty} \sigma_j^2(s).
$$

When $k = -1, -2, \cdots$, we have $t + k\tau < r$ and

$$
\int_{-\infty}^{t+k\tau} \int_D E|D_r Y_1(s, \cdot, y)|^2 \, dy \, ds = \int_{-\infty}^{t+k\tau} \int_D \left| \sum_{j=1}^{m} e^{i\mu_j (s-r)} \phi_j(y) \sigma_j(r) \right|^2 \, dy \, ds
= \int_{-\infty}^{t+k\tau} \int_D \left| \sum_{j=1}^{m} e^{i\mu_j (s-r)} \phi_j(y) \sigma_j(r) \right|^2 \, dy \, ds
\leq \left( \frac{1}{2\mu_m} - \frac{1}{2\mu_{m+1}} \right) s \sum_{j=1}^{\infty} \sigma_j^2(s).
$$

So,

$$
\int_{-\infty}^{t+k\tau} \int_D E|D_r Y_1(s, \cdot, y)|^2 \, dy \, ds \leq \left( \frac{1}{\mu_m} - \frac{1}{\mu_{m+1}} \right) s \sum_{j=1}^{\infty} \sigma_j^2(s).
$$

Similarly,

$$
\int_{t+k\tau}^{\infty} \int_D E|D_r Y_1(s, \cdot, y)|^2 \, dy \, ds \leq \left( \frac{1}{2\mu_m} - \frac{1}{2\mu_{m+1}} \right) s \sum_{j=1}^{\infty} \sigma_j^2(s).
$$

Therefore, we have

$$
E \int_D |D_r M(z(t, \theta_{k\tau}, t, x))|^2 \, dx
\leq C(-\frac{1}{\mu_{m+1}}) ||\nabla F||^2 \int_{t-\tau}^{t} \int_D \left| \sum_{i=0}^{\infty} e^{i\mu_{m+1} (t-s) + i\tau} E|D_r z(s-i\tau, \theta_{k\tau}, y)|^2 \, dy \, ds + C(-\frac{1}{\mu_{m+1}}) ||\nabla F||^2 \int_{t-\tau}^{t} \int_D \left| \sum_{i=0}^{\infty} e^{i\mu_{m+1} (t-s) + i\tau} E|D_r z(s-i\tau, \theta_{k\tau}, y)|^2 \, dy \, ds
+ C \frac{1}{\mu_m} ||\nabla F||^2 \int_{t-\tau}^{t+2\tau} \int_D e^{i\mu_m (t-s)} E|D_r z(s, \theta_{k\tau}, y)|^2 \, dy \, ds
+ C \frac{1}{\mu_m} ||\nabla F||^2 \int_{t-\tau}^{t+2\tau} \int_D e^{i\mu_m (t-s)} E|D_r z(s, \theta_{k\tau}, y)|^2 \, dy \, ds
+ C \frac{1}{\mu_m} ||\nabla F||^2 \int_{t-\tau}^{t+2\tau} \int_D \left| \sum_{i=0}^{\infty} e^{i\mu_{m+1} (t-s-i\tau)} E|D_r z(s+i\tau, \theta_{k\tau}, y)|^2 \, dy \, ds
\leq \left( \frac{1}{\mu_m} - \frac{1}{\mu_{m+1}} \right) s \sum_{j=1}^{\infty} \sigma_j^2(s) \left( \frac{1}{\mu_m^2} + \frac{1}{\mu_{m+1}^2} \right).
$$
\[
\leq C(-\frac{1}{\mu_{m+1}})\|\nabla F\|_\infty^2 \sum_{i=0}^{\infty} e^{\mu_{m+1}\tau} \int_{t+r-2\tau}^{t+r+2\tau} e^{-\beta|t-s|} \int_D E|\mathcal{D}_r z(s, \theta_{-ir+k\tau}, y)|^2 dy ds \\
+ C(-\frac{1}{\mu_{m+1}} + \frac{1}{\mu_m})\|\nabla F\|_\infty^2 \int_{t-r-2\tau}^{t-r+2\tau} e^{-\beta|t-s|} E|\mathcal{D}_r z(s, \theta_{k\tau}, y)|^2 dy ds \\
+ C\frac{1}{\mu_m}\|\nabla F\|_\infty^2 \sum_{i=0}^{\infty} e^{-\mu_{m+1}i\tau} \int_{t-r-2\tau}^{t+r+2\tau} e^{-\beta|t-s|} \int_D E|\mathcal{D}_r z(s, \theta_{i\tau+k\tau}, y)|^2 dy ds \\
+ C\|\nabla F\|_\infty^2 \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} \sigma_j^2(s)(\frac{1}{\mu_{m+1}^2} + \frac{1}{\mu_m^2}) \\
\leq A \int_{t-r-2\tau}^{t+r+2\tau} e^{-\beta|t-s|}\alpha_r(s) ds + B \\
= \alpha_r(t).
\]

Similarly, when \(0 \leq t < r < \tau\),
\[
E \int_D |\mathcal{D}_r \mathcal{M}(z)(t, \theta_{k\tau}, x)|^2 dx \\
\leq -C\frac{1}{\mu_{m+1}}\|\nabla F\|_\infty^2 \int_{t-r-2\tau}^{t-2\tau} \sum_{i=0}^{\infty} e^{\mu_{m+1}(t-s+i\tau)} \int_D E|\mathcal{D}_r z(s, s - i\tau, \theta_{k\tau}, y)|^2 dy ds \\
- C\frac{1}{\mu_m}\|\nabla F\|_\infty^2 \int_{t-2\tau}^{t-\tau} e^{\mu_{m+1}(t-s)} \int_D E|\mathcal{D}_r z(s, \theta_{k\tau}, y)|^2 dy ds \\
+ C\frac{1}{\mu_m}\|\nabla F\|_\infty^2 \int_{t-\tau}^{t} \sum_{i=0}^{\infty} e^{\mu_{m}(t-s-i\tau)} \int_D E|\mathcal{D}_r z(s, s + i\tau, \theta_{k\tau}, y)|^2 dy ds \\
+ C\|\nabla F\|_\infty^2 \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} \sigma_j^2(s)(\frac{1}{\mu_{m+1}^2} + \frac{1}{\mu_m^2}) \\
\leq A \int_{t-r-2\tau}^{t+r+2\tau} e^{-\beta|t-s|}\alpha_r(s) ds + B \\
= \alpha_r(t).
\]

Therefore, for any \(k = 0, \pm 1, \pm 2, \cdots\), we have
\[
E \int_D |\mathcal{D}_r \mathcal{M}(z)(t, \theta_{k\tau}, x)|^2 dx \leq \alpha_r(t).
\]

Moreover, the solution \(\alpha_r(t)\) of equation (2.25) is continuous in \(t\), so for \(z \in C^0_{r, \alpha}((-\infty, +\infty), L^2(D, \mathcal{D}^{1,2}))\), there exists a constant \(\alpha_1\) such that for any \(t, r \in [0, \tau), k = 0, \pm 1, \pm 2, \cdots\),
\[
E \int_D |\mathcal{D}_r z(t, \theta_{k\tau}, x)|^2 dx \leq \alpha_1, \text{ and } E \int_D |\mathcal{D}_r \mathcal{M}(z)(t, \theta_{k\tau}, x)|^2 dx \leq \alpha_1,
\]

Now suppose there exists \(L_2 \geq 0\) such that for any \(r_1, r_2, s \in [0, \tau), k = 0, \pm 1, \pm 2, \cdots\),
\[
\frac{1}{|r_1 - r_2|} \int_D E|\mathcal{D}_{r_1} z(s, \theta_{k\tau}, x) - \mathcal{D}_{r_2} z(s, \theta_{k\tau}, x)|^2 dx \leq L_2.
\]

Then we have when \(0 \leq r_1 < r_2 \leq t < \tau, k = 0, \pm 1, \pm 2, \cdots\)
Similarly,

\[
\frac{1}{|r_1 - r_2|} \int_D E|D_{r_1} M(z)(t, \theta_{k r^*}, x) - D_{r_2} M(z)(t, \theta_{k r^*}, x)|^2 \, dx
\]

\[
\leq \frac{C}{|r_1 - r_2|} \int_D \left\{ E \right\} \int_{-\infty}^{t} \int_D \sum_{i=m+1}^{\infty} e^{\mu_i(t-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s, \theta_{k r^*}) + Y_1(s, \theta_{k r^*}))(y) \cdot (D_{r_1} z(s, \theta_{k r^*}, y) - D_{r_2} z(s, \theta_{k r^*}, y)) \, dyds \right. \\
\left. + E \int_{t}^{\infty} \int_D \sum_{i=1}^{m} e^{\mu_i(t-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s, \theta_{k r^*}) + Y_1(s, \theta_{k r^*}))(y) \cdot (D_{r_1} z(s, \theta_{k r^*}, y) - D_{r_2} z(s, \theta_{k r^*}, y)) \, dyds \right|^2 \\
\leq C \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t} \int_D e^{\mu_i(t-s)}|\phi_i(y)|^2 \, dyds \right] \\
\cdot \left[ \int_{-\infty}^{t} \int_D e^{\mu_{m+1}(t-s)} |\nabla F^i(s, z(s, \theta_{k r^*}) + Y_1(s, \theta_{k r^*}))(y)|^2 |D_{r_1} z(s, \theta_{k r^*}, y) - D_{r_2} z(s, \theta_{k r^*}, y)|^2 \, dyds \right] \\
\leq C \sum_{i=m+1}^{\infty} \left[ \int_{-\infty}^{t} \int_D e^{\mu_{m+1}(t-s)} E|D_{r_1} z(s, \theta_{k r^*}, y) - D_{r_2} z(s, \theta_{k r^*}, y)|^2 \, dyds \right] \\
\leq \frac{C}{|r_1 - r_2|} \int_D \sum_{i=m+1}^{\infty} \left[ \int_{0}^{t} \int_D e^{\mu_{m+1}(t-s)} E|D_{r_1} z(s, \theta_{k r^*}, y) - D_{r_2} z(s, \theta_{k r^*}, y)|^2 \, dyds \right] \\
\leq \frac{C}{\mu_{m+1}} \|\nabla F\|_{L_2}^2 \left[ 1 + \sum_{i=0}^{\infty} e^{\mu_{m+i+1}\tau} \right].
\]

We will estimate them in the following. We first have that

\[
A_1 \leq \frac{C}{|r_1 - r_2|} \int_D E \left\{ \int_{-\infty}^{t} \int_D \sum_{i=m+1}^{\infty} e^{\mu_i(t-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s, \theta_{k r^*}) + Y_1(s, \theta_{k r^*}))(y) \cdot (D_{r_1} z(s, \theta_{k r^*}, y) - D_{r_2} z(s, \theta_{k r^*}, y)) \, dyds \right. \\
\left. + E \int_{t}^{\infty} \int_D \sum_{i=1}^{m} e^{\mu_i(t-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s, \theta_{k r^*}) + Y_1(s, \theta_{k r^*}))(y) \cdot (D_{r_1} z(s, \theta_{k r^*}, y) - D_{r_2} z(s, \theta_{k r^*}, y)) \, dyds \right|^2 \\
:= A_1 + A_2 + A_3 + A_4 + A_5.
\]
\[
A_2 \leq \frac{C}{\mu_m^2} ||\nabla F||_\infty^2 L_2[1 + \sum_{i=0}^{\infty} e^{-\mu_m \cdot r}].
\]

For \( A_3 \), using Cauchy-Schwarz inequality again, we have

\[
A_3 = \frac{C}{|r_1 - r_2|} \int_D E \left| \int_r^{r_1} \int_{D_i} E_{i=m+1} \sum_{i=m+1}^{\infty} e^{|\mu_i(t-s)|} \phi_i(x) \phi_i(y) \right| \nabla F^i(s, z(s, \theta_{k\tau}) + Y_1(s, \theta_{k\tau}))(y)D_{r_1}Y_1(s, \theta_{k\tau}, y)dyds \right|^2 dx
\]

\[
+ \frac{C}{|r_1 - r_2|} \int_D E \left| \int_r^{t} \int_{D_i} E_{i=m+1} \sum_{i=m+1}^{\infty} e^{|\mu_i(t-s)|} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \right| D_{r_1}Y_1(s, \theta_{k\tau}, y)dyds \right|^2 dx
\]

\[
\leq \frac{C}{|r_1 - r_2|} \int_D E \left| \int_r^{t} \int_{D_i} E_{i=m+1} \sum_{i=m+1}^{\infty} e^{|\mu_i(t-s)|} \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \right| D_{r_1}Y_1(s, \theta_{k\tau}, y)dyds \right|^2 dx
\]

Note that

\[
\int_r^{r_1} \int_{D_i} E \left| D_{r_1}Y_1(s, \theta_{k\tau}, y) \right|^2 dyds = \int_{r_1}^{r_1+k\tau} \int_{D_i} E \left| D_{r_1}Y_1(s, \theta_{k\tau}, y) \right|^2 dyds,
\]

so when \( k = 0, 1, 2, \ldots \), we have

\[
\int_{r_1+k\tau}^{r_1+2k\tau} \int_{D_i} E \left| D_{r_1}Y_1(s, \theta_{k\tau}, y) \right|^2 dyds = \int_{r_1+k\tau}^{r_1+2k\tau} \int_{D_i} E \left| \sum_{j=m+1}^{\infty} e^{j(s-r_1)} \phi_j(y) \sigma_j(r_1) \right|^2 dyds
\]

\[
\leq |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]

When \( k = -1, -2, \ldots \), we have \( r_2 + k\tau < r_1 \) and

\[
\int_{r_1+k\tau}^{r_2+k\tau} \int_{D_i} E \left| D_{r_1}Y_1(s, \theta_{k\tau}, y) \right|^2 dyds = \int_{r_1+k\tau}^{r_2+k\tau} \int_{D_i} E \left| \sum_{j=1}^{m} e^{j(s-r_1)} |\phi_j(y)| |\sigma_j(r_1)|^2 dyds
\]

\[
\leq |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]
Therefore

\[
\int_{r_1}^{r_2} \int_{D} E|D_{r_1} Y_1(s, \theta_{\kappa r}, y)|^2 dyds \leq |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]

Similarly,

\[
\int_{r_2}^{t} \int_{D} E|D_{r_2} Y_1(s, \theta_{\kappa r}, y) - D_{r_2} Y_1(s, \theta_{\kappa r}, y)|^2 dyds
= \int_{r_2+k\tau}^{t+k\tau} \int_{D} E|D_{r_2} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dyds.
\]

When \( k = 0, 1, 2, \ldots \), we have

\[
\int_{r_2+k\tau}^{t+k\tau} \int_{D} E|D_{r_2} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dyds
= \int_{r_2+k\tau}^{t+k\tau} \int_{D} \left| \sum_{j=m+1}^{\infty} (\epsilon^{\mu_j(s-r_1)} \sigma_j(r_1) - \epsilon^{\mu_j(s-r_1)} \sigma_j(r_2) + \epsilon^{\mu_j(s-r_1)} \sigma_j(r_2) - \epsilon^{\mu_j(s-r_2)} \sigma_j(r_2)) \phi_j(y) \right|^2 dyds
\]

\[
\leq 2 \int_{r_2+k\tau}^{t+k\tau} \int_{D} \sum_{j=m+1}^{\infty} |\sigma_j(r_1) - \sigma_j(r_2)|^2 |\phi_j(y)|^2 dyds
+ 2 \int_{r_2+k\tau}^{t+k\tau} \int_{D} \sum_{j=m+1}^{\infty} |\epsilon^{\mu_j(s-r_1)} - \epsilon^{\mu_j(s-r_2)}|^2 |\phi_j(y)|^2 |\sigma_j(r_2)|^2 dyds
\]

\[
\leq 2L_1 |r_2 - r_1|(t - r_2) + |r_2 - r_1|(t - r_2) \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]

When \( k = -1, -2, \ldots \), we have \( r_2 + k\tau < t + k\tau < r_1 < r_2 \leq t < \tau \) and

\[
\int_{r_2+k\tau}^{t+k\tau} \int_{D} E|D_{r_2} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dyds
= \int_{r_2+k\tau}^{t+k\tau} \int_{D} \left| \sum_{j=1}^{m} (\epsilon^{\mu_j(s-r_1)} \sigma_j(r_1) - \epsilon^{\mu_j(s-r_1)} \sigma_j(r_2) + \epsilon^{\mu_j(s-r_1)} \sigma_j(r_2) - \epsilon^{\mu_j(s-r_2)} \sigma_j(r_2)) \phi_j(y) \right|^2 dyds
\]

\[
\leq 2 \int_{r_2+k\tau}^{t+k\tau} \int_{D} \sum_{j=1}^{m} |\sigma_j(r_1) - \sigma_j(r_2)|^2 |\phi_j(y)|^2 dyds
+ 2 \int_{r_2+k\tau}^{t+k\tau} \int_{D} \sum_{j=1}^{m} |\epsilon^{\mu_j(s-r_1)} - \epsilon^{\mu_j(s-r_2)}|^2 |\phi_j(y)|^2 |\sigma_j(r_2)|^2 dyds
\]

\[
\leq 2L_1 |r_2 - r_1|(t - r_2) + |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]

Therefore,
\[
\int_{r_2}^{t} \int_{D} E|D_{r_2} Y_{1}(s, \theta_{k \tau}, y) - D_{r_2} Y_{1}(s, \theta_{k \tau}, y)|^2 dy ds \\
\leq 2L_1 |r_2 - r_1| (t - r_2) + |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2
\] (2.13)

With the estimate (2.12) and (2.13), we have

\[
A_3 \leq \frac{C}{|r_1 - r_2|} ||\nabla F||^2_{\infty} (r_2 - r_1)^2 \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 \\
+ \frac{C}{|r_1 - r_2|} \left[ ||\nabla F||^2_{\infty} (t - r_2)^2 2L_1 |r_2 - r_1| + ||\nabla F||^2_{\infty} |t - r_2||r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} \sigma_j^2(s) \right] \\
\leq C ||\nabla F||^2_{\infty} \tau \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 + C ||\nabla F||^2_{\infty} (2L_1 \tau^2 + \tau) \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 \\
< \infty.
\]

About \( A_4 \),

\[
A_4 = \frac{C}{|r_1 - r_2|} \int_{D} \left[ \int_{-\infty}^{r_1} \int_{D} e^{\mu_i(t-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \\
\cdot (D_{r_1} Y_1(s, \theta_{k \tau}, y) - D_{r_2} Y_1(s, \theta_{k \tau}, y)) dy ds \right]^2 dx \\
+ \frac{C}{|r_1 - r_2|} \int_{D} \left[ \int_{r_1}^{r_2} \sum_{i=m+1}^{\infty} e^{\mu_i(t-s)} \phi_i(x) \phi_i(y) \\
\cdot \nabla F^i(s, z(s) + Y_1(s))(y) D_{r_2} Y_1(s, \theta_{k \tau}, y) dy ds \right]^2 dx
\]

\[
\leq \frac{C}{|r_1 - r_2|} \sum_{i=m+1}^{\infty} \int_{D} \int_{-\infty}^{r_1} \int_{D} e^{2\mu_i(t-s)} |\phi_i(y)|^2 dy ds \\
\cdot \int_{-\infty}^{r_1} \int_{D} |\nabla F^i(s, z(s) + Y_1(s))(y)|^2 |D_{r_1} Y_1(s, \theta_{k \tau}, y) - D_{r_2} Y_1(s, \theta_{k \tau}, y)|^2 dy ds \\
+ \frac{C}{|r_1 - r_2|} \sum_{i=m+1}^{\infty} \int_{r_1}^{r_2} \int_{D} e^{2\mu_i(t-s)} |\phi_i(y)|^2 dy ds \\
\cdot \int_{r_1}^{r_2} \int_{D} |\nabla F^i(s, z(s) + Y_1(s))(y)|^2 |D_{r_2} Y_1(s, \theta_{k \tau}, y)|^2 dy ds \\
\leq \frac{C}{|r_1 - r_2|} (-\frac{1}{2\mu_{m+1}})||\nabla F||^2_{\infty} \int_{-\infty}^{r_1} \int_{D} E|D_{r_1} Y_1(s, \theta_{k \tau}, y) - D_{r_2} Y_1(s, \theta_{k \tau}, y)|^2 dy ds \\
+ \frac{C}{|r_1 - r_2|} (r_2 - r_1)||\nabla F||^2_{\infty} \int_{r_1}^{r_2} \int_{D} E|D_{r_2} Y_1(s, \theta_{k \tau}, y)|^2 dy ds.
\]

Similar to (2.12),

\[
\int_{r_1}^{r_2} \int_{D} E|D_{r_2} Y_1(s, \theta_{k \tau}, y)|^2 dy ds \leq |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\] (2.14)

Secondly,
\[
\int_{-\infty}^{r_1} \int_D E|D_{r_1} Y_1(s, \theta_{k\tau}, y) - D_{r_2} Y_1(s, \theta_{k\tau}, y)|^2 dy ds
= \int_{-\infty}^{r_1+k\tau} \int_D E|D_{r_1} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dy ds.
\]

When \( k = 0, -1, -2, \ldots \), we have
\[
\int_{-\infty}^{r_1+k\tau} \int_D E|D_{r_1} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dy ds
= \int_{-\infty}^{r_1+k\tau} \int_D \left| \sum_{j=1}^{m} (e^{\mu_j(s-r_1)} \sigma_j(r_1) - e^{\mu_j(s-r_1)} \sigma_j(r_2) + e^{\mu_j(s-r_1)} \sigma_j(r_2) - e^{\mu_j(s-r_2)} \sigma_j(r_2) ) \phi_j(y) \right|^2 dy ds
= \int_{-\infty}^{r_1+k\tau} \int_D \left| \sum_{j=1}^{m} (e^{\mu_j(s-r_1)} (\sigma_j(r_1) - \sigma_j(r_2)) + (e^{\mu_j(s-r_1)} - e^{\mu_j(s-r_2)}) \sigma_j(r_2) ) \phi_j(y) \right|^2 dy ds
\leq 2 \int_{-\infty}^{r_1+k\tau} \int_D \sum_{j=1}^{m} e^{\mu_j(s-r_1)} |\sigma_j(r_1) - \sigma_j(r_2)|^2 |\phi_j(y)|^2 dy ds
\]
\[
+ 2 \int_{-\infty}^{r_1+k\tau} \int_D \sum_{j=1}^{m} |e^{\mu_j(s-r_1)} - e^{\mu_j(s-r_2)}|^2 |\phi_j(y)|^2 |\sigma_j(r_2)|^2 dy ds
\leq 2 L_1 \frac{1}{\mu_m} |r_2 - r_1| + |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]

When \( k = 1, 2, \ldots \), we have \( r_1 + k\tau > r_2 \) and
\[
\int_{-\infty}^{r_1+k\tau} \int_D E|D_{r_1} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dy ds
= \int_{-\infty}^{r_1} \int_D E|D_{r_1} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dy ds + \int_{r_1}^{r_2} \int_D E|D_{r_1} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dy ds + \int_{r_2}^{r_1+k\tau} \int_D E|D_{r_1} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dy ds
\]
Let us estimate them separately. About the first term,
\[
\int_{-\infty}^{r_1} \int_D E|D_{r_1} Y_1(s, \cdot, y) - D_{r_2} Y_1(s, \cdot, y)|^2 dy ds
= \int_{-\infty}^{r_1} \int_D \left| \sum_{j=1}^{m} (e^{\mu_j(s-r_1)} \sigma_j(r_1) - e^{\mu_j(s-r_1)} \sigma_j(r_2) + e^{\mu_j(s-r_1)} \sigma_j(r_2) - e^{\mu_j(s-r_2)} \sigma_j(r_2) ) \phi_j(y) \right|^2 dy ds
= \int_{-\infty}^{r_1} \int_D \left| \sum_{j=1}^{m} (e^{\mu_j(s-r_1)} (\sigma_j(r_1) - \sigma_j(r_2)) + (e^{\mu_j(s-r_1)} - e^{\mu_j(s-r_2)}) \sigma_j(r_2) ) \phi_j(y) \right|^2 dy ds
\leq 2 \int_{-\infty}^{r_1} \int_D \sum_{j=1}^{m} e^{\mu_j(s-r_1)} |\sigma_j(r_1) - \sigma_j(r_2)|^2 |\phi_j(y)|^2 dy ds
\]
\[
+ 2 \int_{-\infty}^{r_1} \int_D \sum_{j=1}^{m} |e^{\mu_j(s-r_1)} - e^{\mu_j(s-r_2)}|^2 |\phi_j(y)|^2 |\sigma_j(r_2)|^2 dy ds
\leq 2 L_1 \frac{1}{\mu_m} |r_2 - r_1| + |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]
About the second term,
\[
\int_{r_1}^{r_2} \int_D |E[D_{r_1}Y_1(s, r_1, y) - D_{r_2}Y_1(s, r_2, y)]|^2 dy ds \\
\leq 2 \int_{r_1}^{r_2} \int_D |E[D_{r_1}Y_1(s, r_1, y)]|^2 dy ds + 2 \int_{r_1}^{r_2} \int_D |E[D_{r_2}Y_1(s, r_2, y)]|^2 dy ds \\
\leq 2|r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2
\]

About the third term,
\[
\int_{r_2}^{r_1 + k\tau} \int_D |E[D_{r_1}Y_1(s, r_1, y) - D_{r_2}Y_1(s, r_2, y)]|^2 dy ds \\
= \int_{-\infty}^{r_1 + k\tau} \int_D \left| \sum_{j=m+1}^{\infty} (e^{\mu_j(s-r_1)} \sigma_j(r_1) - e^{\mu_j(s-r_1)} \sigma_j(r_2) + e^{\mu_j(s-r_1)} \sigma_j(r_2) - e^{\mu_j(s-r_2)} \sigma_j(r_2)) \phi_j(y) \right|^2 dy ds \\
= \int_{-\infty}^{r_1 + k\tau} \int_D \sum_{j=m+1}^{\infty} |e^{\mu_j(s-r_1)} \sigma_j(r_1) - \sigma_j(r_2)|^2 |\phi_j(y)|^2 dy ds \\
+ 2 \int_{-\infty}^{r_1 + k\tau} \int_D \sum_{j=m+1}^{\infty} |e^{\mu_j(s-r_1)} - e^{\mu_j(s-r_2)}|^2 |\phi_j(y)|^2 |\sigma_j(r_2)|^2 dy ds \\
\leq 2L_1(\frac{1}{\mu_{m+1}})|r_2 - r_1| + |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]

Therefore,
\[
\int_{-\infty}^{r_1} \int_D |E[D_{r_1}Y_1(s, \theta_{k\tau}, y) - D_{r_2}Y_1(s, \theta_{k\tau}, y)]|^2 dy ds \\
\leq 2L_1(\frac{1}{\mu_m} - \frac{1}{\mu_{m+1}})|r_2 - r_1| + |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2. \tag{2.15}
\]

With (2.14) and (2.15), we have
\[
A_4 \leq \frac{C}{|r_1 - r_2|} \|\nabla F\|_{\infty}^2 \left[ (-\frac{1}{2\mu_{m+1}}) (2L_1(\frac{1}{\mu_m} - \frac{1}{\mu_{m+1}}) |r_2 - r_1| + |r_2 - r_1| \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2) \\
+ |r_2 - r_1|^2 \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 \right] \\
\leq C \|\nabla F\|_{\infty}^2 \left( \frac{1}{\mu_{m+1}^2} - \frac{1}{\mu_{m+1} \mu_m} \right) L_1 + (-\frac{1}{2\mu_{m+1}}) \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 + \tau \cdot \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 \\
< \infty.
\]

As for $A_5$, similarly to $A_4$, we have
\[
A_5 \leq C \|\nabla F\|_{\infty}^2 \left( \frac{1}{\mu_m^2} - \frac{1}{\mu_{m+1} \mu_m} \right) L_1 + \frac{1}{2\mu_m} \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 < \infty.
\]
So, when $0 \leq r_1 < r_2 \leq t < \tau$,

$$\frac{1}{|r_1 - r_2|} \int_D E|\mathcal{D}_r \mathcal{M}(z)(t, \theta_{k\tau}, x) - \mathcal{D}_{2r} \mathcal{M}(z)(t, \theta_{k\tau}, x)|^2 dx \leq \hat{C}.$$ 

When $0 \leq r_1 < t < r_2 < \tau$,

$$\frac{1}{|r_1 - r_2|} \int_D E|\mathcal{D}_{2r} \mathcal{M}(z)(t, \theta_{k\tau}, x) - \mathcal{D}_{r_1} \mathcal{M}(z)(t, \theta_{k\tau}, x)|^2 dx$$

\[\leq \frac{C}{|r_1 - r_2|} \int_D E \left| \int_{-\infty}^t \sum_{i=m+1}^{\infty} e^{\mu_i (t-s)} \phi_i(x) \phi_i(y) \cdot \nabla F^i(s, z(s, \theta_{k\tau}) + Y_1(s, \theta_{k\tau}))(y) \right|^2 dy ds

+ E \left| \int_{-\infty}^t \sum_{i=m+1}^{\infty} e^{\mu_i (t-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s, \theta_{k\tau}) + Y_1(s, \theta_{k\tau}))(y) \right|^2 dy ds

+ E \left| \int_{-\infty}^t \sum_{i=m+1}^{\infty} e^{\mu_i (t-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s, \theta_{k\tau}) + Y_1(s, \theta_{k\tau}))(y) \right|^2 dy ds

+ E \left| \int_{-\infty}^t \sum_{i=m+1}^{\infty} e^{\mu_i (t-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s, \theta_{k\tau}) + Y_1(s, \theta_{k\tau}))(y) \right|^2 dy ds

Thus using a similar method as before, we can see that

$$\frac{1}{|r_1 - r_2|} \int_D E|\mathcal{D}_{2r} \mathcal{M}(z)(t, \theta_{k\tau}, x) - \mathcal{D}_{r_1} \mathcal{M}(z)(t, \theta_{k\tau}, x)|^2 dx$$

\[\leq C||\nabla F||_{\infty}^2 \left\{ \left( \frac{1}{\mu_{m+1}} + \frac{1}{\mu_m} \right) L_2 (1 + \sum_{i=0}^{\infty} e^{\mu_{m+1} i \tau} + \sum_{i=0}^{\infty} e^{-\mu_{m} i \tau} + \frac{1}{\mu_{m+1}} + \frac{1}{\mu_m} - \frac{1}{\mu_m \mu_{m+1}} + 2\tau^2) L_1 \right. 

\left. + \left( \frac{1}{2\mu_{m+1}} + \frac{1}{2\mu_m} + 4\tau \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 \right) \right\}

:= \tilde{C}.\]
When $0 \leq t \leq t_1 < t_2 < \tau$, similar to the case when $0 \leq r_1 < r_2 \leq t < \tau$. Therefore, $\mathcal{M}$ maps $C^0_{r,\alpha}((-\infty, +\infty), L^2(D, \mathcal{D}^{1,2}))$ to itself.

Define the set

$$S := C^0((0, \infty), L^2(\Omega \times D)) \cap L^\infty((0, \infty), L^2(\Omega, H^1_0(D))) \cap C^0_{r,\alpha}((0, \infty), L^2(D, \mathcal{D}^{1,2})).$$

Define

$$\mathcal{M}(S)\|_{[0,\tau]} := \{f\|_{[0,\tau]} : f \in \mathcal{M}(S)\}.$$

**Lemma 2.3** The set $\mathcal{M}(S)\|_{[0,\tau]}$ is relatively compact in $C^0([0, \tau), L^2(\Omega \times D))$.

**Proof:** With what we have proved in Lemma 2.2, we also need to prove that $\mathcal{D}_r \mathcal{M}(z)(t)$ is equicontinuous in $t$ in the space $L^2(D, \mathcal{D}^{1,2})$. We will consider several cases. When $0 \leq r \leq t_1 < t_2 < \tau$, for $z \in S$,

$$\int_D E|\mathcal{D}_r \mathcal{M}(z)(t_2, x) - \mathcal{D}_r \mathcal{M}(z)(t_1, x)|^2 dx \leq C \int_D \left\{ E \left| \int_{-\infty}^{t_1} \sum_{i=m+1}^{\infty} (e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)}) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \mathcal{D}_r z(s, y) dy ds \right|^2 
+ E \left| \int_{t_1}^{t_2} \sum_{i=m+1}^{\infty} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \mathcal{D}_r z(s, y) dy ds \right|^2 
+ E \left| \int_{t_1}^{\infty} \sum_{i=1}^{m} (e^{\mu_i(t_1-s)} - e^{\mu_i(t_2-s)}) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \mathcal{D}_r Y_1(s, y) dy ds \right|^2 
+ E \left| \int_{t_1}^{t_2} \sum_{i=m+1}^{\infty} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \mathcal{D}_r Y_1(s, y) dy ds \right|^2 
+ E \left| \int_{-\infty}^{t_1} \sum_{i=m+1}^{\infty} (e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)}) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \mathcal{D}_r Y_1(s, y) dy ds \right|^2 
+ E \left| \int_{t_1}^{\infty} \sum_{i=1}^{m} (e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)}) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \mathcal{D}_r Y_1(s, y) dy ds \right|^2 
+ E \left| \int_{t_1}^{t_2} \sum_{i=1}^{m} (e^{\mu_i(t_1-s)} - e^{\mu_i(t_2-s)}) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \mathcal{D}_r Y_1(s, y) dy ds \right|^2 \right\} dx 
: = B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8 + B_9.$$

We will estimate them in the following steps. First, we have

$$B_1 \leq C \int_D E \left| \int_{-\infty}^{t_1} \sum_{i=m+1}^{\infty} (e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)}) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \mathcal{D}_r z(s, y) dy ds \right|^2 dx.$$
\[ \begin{align*}
&\leq C \sum_{i=m+1}^{\infty} E \int_{-\infty}^{t_1} \int_D |e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)}| \cdot |\phi_i(y)|^2 dy ds \\
&\cdot \int_{-\infty}^{t_1} \int_D |e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)}| \| \nabla F^i(s, z(s) + Y_1(s))(y) \|^2 |D_r z(s, y)|^2 dy ds \\
&\leq C|t_2 - t_1| \cdot \| \nabla F \|^2_\infty \int_{-\infty}^{t_1} \int_D e^{\mu_{m+1}(t_1-s)} E|D_r z(s, y)|^2 dy ds \\
&\leq C|t_2 - t_1| \cdot \| \nabla F \|^2_\infty \left[ \int_0^T \sum_{i=0}^{\infty} e^{\mu_{m+1}(t_1-s+\tau+i\tau)} E|D_r z(s, \theta_{-(i+1)\tau}, y)|^2 dy ds \\
&\quad + \int_{t_1}^{t_2} \int_D e^{\mu_{m+1}(t_1-s)} E|D_r z(s, y)|^2 dy ds \right] \\
&\leq - \frac{C}{\mu_{m+1}} |t_2 - t_1| \cdot \| \nabla F \|^2_\infty \alpha_1 \left( \sum_{i=0}^{\infty} e^{-\mu_{m+i}\tau} + 1 \right).
\end{align*} \]

About \( B_2 \), we have
\[ B_2 \leq C \sum_{i=m+1}^{\infty} E \int_{t_1}^{t_2} \int_D \left( e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)} \right) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) |D_r z(s, y)|^2 dy ds dx \]

\[ \leq C \sum_{i=m+1}^{\infty} E \int_{t_1}^{t_2} \int_D |\phi_i(y)|^2 dy ds \cdot \int_{t_1}^{t_2} \int_D \| \nabla F^i(s, z(s) + Y_1(s))(y) \|^2 |D_r z(s, y)|^2 dy ds \]

\[ \leq C|t_2 - t_1| \cdot \| \nabla F \|^2_\infty \tau \alpha_1. \]

Similar to \( B_1 \), we have
\[ B_3 \leq \frac{C}{\mu_{m}} |t_2 - t_1| \cdot \| \nabla F \|^2_\infty \alpha_1 \left( \sum_{i=0}^{\infty} e^{-\mu_{m+i}\tau} + 1 \right). \]

Similar to \( B_2 \), we have
\[ B_4 \leq C|t_2 - t_1| \cdot \| \nabla F \|^2_\infty \tau \alpha_1. \]

About \( B_5 \),
\[ B_5 \leq C \int_D E \left| \int_{t_1}^{t_1} \int_D \sum_{i=m+1}^{\infty} \left( e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)} \right) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \right|^2 dx \]

\[ \cdot \sum_{j=m+1}^{\infty} |\phi_j(y)\sigma_j(r)|^2 dy ds \]

\[ \leq C \sum_{i=m+1}^{\infty} E \int_{t_1}^{t_2} \int_D |e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)}| \cdot |\phi_i(y)|^2 dy ds \\
\cdot \int_{t_1}^{t_2} \int_D |e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)}| \| \nabla F^i(s, z(s) + Y_1(s))(y) \|^2 \sum_{j=m+1}^{\infty} |\phi_j(y)\sigma_j(r)|^2 dy ds \]

\[ \leq C|t_2 - t_1| \cdot \| \nabla F \|^2_\infty \int_{t_1}^{t_2} \int_D \sum_{j=m+1}^{\infty} |\phi_j(y)|^2 |\sigma_j(r)|^2 dy ds \]

\[ \leq C|t_2 - t_1| \cdot \| \nabla F \|^2_\infty \tau \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} \sigma_j^2(s). \]
About $B_6$,
\[
B_6 \leq C \int_D E \left| \int_{t_1}^{t_2} \int_D \sum_{i=m+1}^{\infty} \left( e^{\mu_i(t_2-s)} - e^{\mu_i(t_1-s)} \right) \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) \right| \phi_j(y) \sigma_j(r) dy ds \right|^2 dx
\]
\[
\leq C \int_{t_1}^{t_2} \int_D |\phi_i(y)|^2 dy ds \cdot \int_{t_1}^{t_2} \int_D \left| \nabla F^i(s, z(s) + Y_1(s))(y) \right| \sum_{j=m+1}^{\infty} \phi_j(y) \sigma_j(r) dy ds \right|^2 dy ds
\]
\[
\leq C|t_2 - t_1| \cdot \|\nabla F\|_\infty^2 \int_{t_1}^{t_2} \int_D \sum_{j=m+1}^{\infty} |\phi_j(y)|^2 |\sigma_j(r)|^2 dy ds
\]
\[
\leq C|t_2 - t_1| \cdot \|\nabla F\|_\infty^2 \tau \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]
Similarly, we have
\[
B_7 \leq C|t_2 - t_1| \cdot \|\nabla F\|_\infty^2 \left( -\frac{1}{\mu_m+1} \right) \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} \sigma_j^2(s),
\]
\[
B_8 \leq C|t_2 - t_1| \cdot \|\nabla F\|_\infty^2 \frac{1}{\mu_m} \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} \sigma_j^2(s),
\]
\[
B_9 \leq C|t_2 - t_1| \cdot \|\nabla F\|_\infty^2 \tau \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2.
\]
Therefore, for any $z \in S$ and $0 \leq r \leq t_1 < t_2 < \tau$, we have
\[
\int_D E[|D_r \mathcal{M}(z)(t_2, x) - D_r \mathcal{M}(z)(t_1, x)|^2 dx \leq \tilde{C}|t_2 - t_1|.
\]
When $0 \leq t_1 < r < t_2 < \tau$, $z \in S$, similar as before, we can compute that
\[
\int_D E[|D_r \mathcal{M}(z)(t_2, x) - D_r \mathcal{M}(z)(t_1, x)|^2 dx
\]
\[
\leq C \int_D \left\{ E \left| \int_{t_1}^{t_2} \int_D \sum_{i=m+1}^{\infty} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) D_r z(s, y) dy ds \right|^2 
\right.
\]
\[
+ E \left| \int_{t_1}^{t_2} \int_D \sum_{i=1}^{m} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) D_r z(s, y) dy ds \right|^2 
\]
\[
+ E \left| \int_{t_1}^{t_2} \int_D \sum_{i=1}^{m} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) D_r Y_1(s, y) dy ds \right|^2 
\]
\[
+ E \left| \int_{t_1}^{t_2} \int_D \sum_{i=m+1}^{\infty} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) D_r Y_1(s, y) dy ds \right|^2
\]
+E\| - \int_{-\infty}^{t_1} \int_{D}^{\infty} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) D_r Y_1(s, y) dy ds \\
+ \int_{-\infty}^{t_1} \int_{D}^{\infty} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) D_r Y_1(s, y) dy ds^2 \\
+E\| - \int_{t_2}^{\infty} \int_{D}^{\infty} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) D_r Y_1(s, y) dy ds \\
+ \int_{t_2}^{\infty} \int_{D}^{\infty} e^{\mu_i(t_2-s)} \phi_i(x) \phi_i(y) \nabla F^i(s, z(s) + Y_1(s))(y) D_r Y_1(s, y) dy ds^2 \} dx \\
\leq C\left\{ \left( \frac{1}{\mu_{m+1}} \right) \left( 1 + \int_{-\infty}^{\infty} e^{\mu_{m+1} t_1^2} \| \nabla F \|_2^2 \omega_1(t_2 - t_1) \right) + \frac{1}{\mu_m} \left( 1 + \int_{-\infty}^{\infty} e^{-\mu_{m-1} t_1^2} \| \nabla F \|_2^2 \omega_1(t_2 - t_1) \right) \\
+ 2|t_2 - t_1| \| \nabla F \|_2^2 \tau \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 \\
+ \left( \frac{1}{\mu_{m-1}} + \tau \right) \| \nabla F \|_2^2 \sum_{j=1}^{\infty} \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 (t_2 - t_1) \\
+ \left( \frac{1}{\mu_m} + \tau \right) \| \nabla F \|_2^2 \sum_{j=1}^{\infty} \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} |\sigma_j(s)|^2 (t_2 - t_1) \right\} \\
\leq \hat{C}|t_2 - t_1|.

The case when 0 \leq t_1 < t_2 < r < \tau is similar to the case when 0 \leq r \leq t_1 < t_2 < \tau. Thus, from the above arguments, by Theorem 2.3, M(S)|_{[0, \tau)} is relatively compact in C^0([0, \tau), L^2(\Omega \times D)).

From the periodicity of M(z)(t), we can prove

**Lemma 2.4** The set M(S) is relatively compact in C^0_{\infty}((\infty, \infty), L^2(\Omega \times D)).

**Proof:** From Lemma 2.3, we know for any sequence M(z_n) \in S, there exists a subsequence, still denoted by M(z_n) and Z^* \in C^0_{\infty}([0, \tau), L^2(\Omega \times D)) such that

\[
\sup_{t \in [0, \tau)} \int_D |E[M(z_n)(t, \cdot, x) - Z^*(t, \cdot, x)|^2 dx \to 0
\]

as n \to \infty. Set for \tau \leq t < 2\tau,

\[Z^*(t, \omega, x) = Z^*(t - \tau, \theta_\tau, \omega, x).\]

Noting

\[M(z_n)(t, \theta_\tau, x) = M(z_n)(t + \tau, \omega, x),\]

from (2.16), and the probability preserving property of \theta, we have

\[
\sup_{t \in [\tau, 2\tau)} \int_D |E[M(z_n)(t, \cdot, x) - Z^*(t, \cdot, x)|^2 dx = \sup_{t \in [0, \tau)} \int_D |E[M(z_n)(t + \tau, \cdot, x) - Z^*(t + \tau, \cdot, x)|^2 dx \\
= \sup_{t \in [0, \tau)} \int_D |E[M(z_n)(t, \theta_\tau, \cdot, x) - Z^*(t, \theta_\tau, \cdot, x)|^2 dx \\
= \sup_{t \in [0, \tau)} \int_D |E[M(z_n)(t, \cdot, x) - Z^*(t, \cdot, x)|^2 dx \\
\to 0.
\]
Similarly one can prove that
\[
\sup_{t \in [0, \tau]} \int_D E[|\mathcal{M}(z_n)(t + m\tau, \cdot, x) - Z^*(t + m\tau, \cdot, x)|^2] \, dx \quad (2.17)
\]
for any \( m \in \{0, \pm 1, \pm 2, \cdots\} \). Therefore
\[
\sup_{t \in [-\infty, +\infty]} \int_D E[|\mathcal{M}(z_n)(t, \cdot, x) - Z^*(t, \cdot, x)|^2] \, dx \to 0,
\]
as \( n \to \infty \). Therefore \( \mathcal{M}(S) \) is relatively compact in \( C^0_0((-\infty, +\infty), L^2(\Omega \times D)) \).

\textbf{Proof of Theorem 2.4:} From the above four lemmas, according to the generalized Schauder’s fixed point theorem, \( \mathcal{M} \) has a fixed point in \( C^0_0((-\infty, +\infty), L^2(\Omega \times D)) \). That is to say there exists a solution \( Z \in C^0_0((-\infty, +\infty), L^2(\Omega \times D)) \) of equation (2.6) such that for any \( t \in (-\infty, +\infty) \), \( Z(t + \tau, \omega, x) = Z(t, \theta \omega, x) \). Then \( Y = Z + Y_1 \) is the desired solution of (2.2). Moreover, \( Y(t + \tau, \omega, x) = Y(t, \theta \omega, x) \).

Now we consider the semilinear stochastic differential equations with the additive noise of the form
\[
du(t, x) = [\mathcal{L}u(t, x) + F(u(t, x))] \, dt + \sum_{k=1}^{\infty} \sigma_k \phi_k(x) W^k(t), \quad (2.19)
\]
\[
u(0) = \psi \in L^2(D),
\]
\[
u(t)|_{\partial D} = 0,
\]
for \( t \geq 0 \). Here \( F \) and \( \sigma_k \) do not depend on time \( t \), that is to say, \( \tau \) in Condition (P) can be chosen as an arbitrary real number. We have a similar variation of constant representation to (2.2). The difference is that for this equation, we have a cocycle. Similar to Theorem 2.1, we can prove the following theorem. But we do not give the proof here.

\textbf{Theorem 2.5} Assume Cauchy problem (2.19) has a unique solution \( u(t, \omega, x) \) and the coupled forward-backward infinite horizon stochastic integral equation
\[
Y(\omega) = \int_{-\infty}^0 T_{-s} P^- F(Y(\theta_s \omega)) \, ds - \int_0^\infty T_{-s} P^+ F(Y(\theta_s \omega)) \, ds + (\omega) \sum_{k=1}^{\infty} \int_{-\infty}^0 \sigma_k T_{-s} P^- \phi_k W^k(s) - (\omega) \sum_{k=1}^{\infty} \int_0^\infty \sigma_k T_{-s} P^+ \phi_k W^k(s) \quad (2.20)
\]
has one solution \( Y : \Omega \to L^2(D) \), then \( Y \) is a stationary solution of equation (2.19) i.e.
\[
u(t, Y(\omega), \omega) = Y(\theta_t \omega) \quad \text{for any} \ t \geq 0 \quad \text{a.s.} \quad (2.21)
\]
Conversely, if equation (2.19) has a stationary solution \( Y : \Omega \to L^2(D) \) which is tempered from above, then \( Y \) is a solution of the coupled forward-backward infinite horizon stochastic integral equation (2.20).

\textbf{Theorem 2.6} Assume the same conditions on \( \mathcal{L} \) as in Theorem 2.4 and \( \sum_{k=1}^{\infty} \sigma_k^2 < \infty \). Let \( F : R \to R \) be a continuous map, globally bounded and \( \nabla F \) being globally bounded. Then there exists at least one \( \mathcal{F} \)-measurable map \( Y : \Omega \to L^2(D) \) satisfying (2.20).
Proof: Set the $\mathcal{F}$-measurable map $Y_1 : \Omega \to L^2(D)$

$$Y_1(\omega) = (\omega) \sum_{k=1}^{\infty} \int_{-\infty}^{0} \sigma_k T^{-s} P^- \phi_k W^k(s) - (\omega) \sum_{k=1}^{\infty} \int_{0}^{\infty} \sigma_k T^{-s} P^+ \phi_k W^k(s).$$

Then we have

$$Y_1(\theta_t \omega) = (\theta_t \omega) \sum_{k=1}^{\infty} \int_{-\infty}^{0} \sigma_k T^{-s} P^- \phi_k W^k(s) - (\theta_t \omega) \sum_{k=1}^{\infty} \int_{0}^{\infty} \sigma_k T^{-s} P^+ \phi_k W^k(s)$$

$$= (\omega) \sum_{k=1}^{\infty} \int_{-\infty}^{t} \sigma_k T^{-s} P^- \phi_k W^k(s) - (\omega) \sum_{k=1}^{\infty} \int_{0}^{\infty} \sigma_k T^{-s} P^+ \phi_k W^k(s).$$

We need to solve the equation

$$Z(t, \omega)$$

$$= \int_{-\infty}^{t} T_{t-s} P^- F(Z(s, \omega)) + Y_1(\theta_s \omega))ds - \int_{t}^{\infty} T_{t-s} P^+ F(Z(s, \omega) + Y_1(\theta_s \omega))ds. \quad (2.23)$$

For this, define

$$C^0_s((\infty, +\infty), L^2(\Omega \times D))$$

$$:= \{ f \in C^0((\infty, +\infty), L^2(\Omega \times D)) : \text{ for any } t \in (-\infty, \infty), f(t, \omega, x) = f(0, \theta_t \omega, x) \},$$

We now define for any $z \in C^0_s((\infty, +\infty), L^2(\Omega \times D))$,

$$\mathcal{M}(z)(t, \omega) = \int_{-\infty}^{t} T_{t-s} P^- F(z(s, \omega) + Y_1(\theta_s \omega))ds$$

$$- \int_{t}^{\infty} T_{t-s} P^+ F(z(s, \omega) + Y_1(\theta_s \omega))ds. \quad (2.24)$$

It’s easy to see that

$$\mathcal{M}(z)(0, \theta_t \omega)$$

$$= \int_{-\infty}^{0} T_{s+} P^- F(z(s, \theta_t \omega) + Y_1(\theta_{s+t} \omega))ds - \int_{0}^{\infty} T_{s+} P^+ F(z(s, \theta_t \omega) + Y_1(\theta_{s+t} \omega))ds$$

$$= \int_{-\infty}^{0} T_{s+} P^- F(z(s + t, \omega) + Y_1(\theta_{s+t} \omega))ds - \int_{0}^{\infty} T_{s+} P^+ F(z(s + t, \omega) + Y_1(\theta_{s+t} \omega, x))ds$$

$$= \int_{-\infty}^{t} T_{t-s} P^- F(z(s, \omega) + Y_1(\theta_s \omega))ds - \int_{t}^{\infty} T_{t-s} P^+ F(z(s, \omega) + Y_1(\theta_s \omega))ds$$

$$= \mathcal{M}(z)(t, \omega).$$

By the similar method in the proof of Lemma 2.1, we can see that the $\mathcal{M}$ defined in (2.24) maps

$$C^0_s((\infty, +\infty), L^2(\Omega \times D) \to C^0_s((\infty, +\infty), L^2(\Omega \times D))$$

is a continuous map. Moreover $\mathcal{M}$ maps

$$C^0_s((\infty, +\infty), L^2(\Omega \times D))$$

into $C^0_s((\infty, +\infty), L^2(\Omega \times D)) \cap L^\infty((-\infty, +\infty), L^2(\Omega, H^1_0(D)))$. For a fixed $T > 0$, define

$$C^{0}_{T, \alpha}((\infty, +\infty), L^2(D, D^{1,2}))$$

$$:= \{ f \in C^0((\infty, +\infty), L^2(\Omega \times D)) : f|_{[0,T]} \in C^0([0,T), L^2(D, D^{1,2})) \},$$
And similar to Lemma 2.2 we can get
\[ M \]
\[ \text{set} \]
\[ \text{Here} \]
\[ \text{Similar to Lemma 2.3 we can prove the set} \]
\[ \text{where} \]
\[ \text{need to prove that} \]
\[ \int \]
\[ \text{such that} \]
\[ \theta \]
\[ \text{and by the probability preserving property of} \]
\[ \text{and} \]
\[ \text{Z} \]
\[ \text{that there exists} \]
\[ \int \]
\[ \theta \]
\[ \text{is relatively compact in} \]
\[ \text{is the solution of integral equation (see page 324 in [26])} \]
\[ \alpha_r(t) = A \int_{r-2T}^{t+2T} e^{-\beta |t-s|} \alpha_r(s) \, ds + B, \quad (2.25) \]
where
\[ A = C \| \nabla F \|_2^2 \left( -\frac{1}{\mu_{m+1}} \sum_{i=0}^{\infty} e^{\mu_{m+1}iT} + \frac{1}{\mu_{m}} \sum_{i=0}^{\infty} e^{-\mu_{m}iT} \right), \]
\[ B = C \| \nabla F \|_2^2 \sup_{s \in (-\infty, \infty)} \sum_{j=1}^{\infty} \sigma_j^2(s) \left( \frac{1}{\mu_{m+1}} + \frac{1}{\mu_{m}} \right), \]
\[ \beta = \min\{\mu_{m+1}, \mu_{m}\}. \]

And similar to Lemma 2.2 we can get \( \mathcal{M} \) maps \( C^0_{T,\alpha}((-\infty, \infty), L^2(D, D^{1/2})) \) into itself. Define the set
\[ S := C^0_T((-\infty, \infty), L^2(\Omega \times D)) \cap L^\infty((-\infty, \infty), L^2(\Omega, H^1_0(D))) \cap C^0_{T,\alpha}((-\infty, \infty), L^2(D, D^{1/2})). \]

Similar to Lemma 2.3 we can prove the set \( \mathcal{M}(S)|_{0, T} \) is relatively compact in \( C^0([0, T], L^2(\Omega \times D)) \). We need to prove that \( \mathcal{M}(S) \) is relatively compact in \( C^0_s((-\infty, +\infty), L^2(\Omega \times D)) \). Note also for any sequence \( \mathcal{M}(z_n) \in \mathcal{M}(S) \), there exists a subsequence, still denoted by \( \mathcal{M}(z_n) \) and \( Z^* \in C^0([0, T], L^2(\Omega \times D)) \) such that
\[ \int_D E|\mathcal{M}(z_n)(0, \cdot, x) - Z^*(\cdot, x)|^2 \, dx \to 0, \quad \text{as} \quad n \to \infty. \]

Define
\[ Z^*(t, \omega, x) = Z^*(0, 0, \omega, x). \]

Noting
\[ \mathcal{M}(z_n)(0, \theta \omega, x) = \mathcal{M}(z_n)(t, \omega, x), \]
and by the probability preserving property of \( \theta \), we have
\[ \sup_{t \in (-\infty, \infty)} \int_D E|\mathcal{M}(z_n)(t, \cdot, x) - Z^*(\theta t \cdot, x)|^2 \, dx = \sup_{t \in (-\infty, \infty)} \int_D E|\mathcal{M}(z_n)(0, \theta t \cdot) - Z^*(\theta t \cdot)|^2 \, dx \]
\[ = \int_D E|\mathcal{M}(z_n)(0, \cdot, x) - Z^*(\cdot, x)|^2 \, dx \]
\[ \to 0, \quad \text{as} \quad n \to \infty. \]

So \( \mathcal{M}(S) \) is relatively compact in \( C^0_s((-\infty, +\infty), L^2(\Omega \times D)) \). Therefore, according to generalized Schauder’s fixed point theorem, \( \mathcal{M} \) has a fixed point in \( C^0_s((-\infty, +\infty), L^2(\Omega \times D)) \). That is to say that there exists \( Z \in C^0_s((-\infty, +\infty), L^2(\Omega \times D)) \) such that for any \( t \in (-\infty, +\infty), Z(t, \omega) = Z(0, \theta t \omega) \)
and
\[ Z(0, \theta t \omega) = \int_{-\infty}^t T_{t-s} P^- F(Z(0, \theta s \omega) + Y_1(\theta s \omega)) \, ds - \int_t^{+\infty} T_{t-s} P^+ F(Z(0, \theta s \omega) + Y_1(\theta s \omega)) \, ds. \]
Finally, we add $Y_1$ defined by the integral equation (2.22) to the above equation and also assume

$$Y(\omega) := Z(0, \omega) + Y_1(\omega).$$

It’s easy to see that $Y(\omega, x)$ satisfies (2.20).

Acknowledgements. We would like to thank the referee for very useful comments and pointing out to us the references [5, 6], and [16].

References

20. Y. Liu, H. Z. Zhao, Representation of pathwise stationary solutions of stochastic Burgers equations, Stochastics and Dynamics, Vol. 9 (2009), No. 4, 613-634