Shape is as important as size: roundness and the science of measurement

This item was submitted to Loughborough University’s Institutional Repository by the/author.

Citation: SANGWIN, C.J., 2014. Shape is as important as size: roundness and the science of measurement. School Science Review, 353, pp. 115-122.

Additional Information:

- This article was published in the journal, School Science Review [ASE © the author]. The website is at: http://www.ase.org.uk/journals/school-science-review/

Metadata Record: https://dspace.lboro.ac.uk/2134/15882

Version: Published

Publisher: ASE © the author

Rights: This work is made available according to the conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) licence. Full details of this licence are available at: https://creativecommons.org/licenses/by-nc-nd/4.0/

Please cite the published version.
Shape is as important as size: roundness and the science of measurement

Christopher J. Sangwin

ABSTRACT Science relies on standards for measurement. This article describes assessment of geometric shape, rather than size. We consider the problem of establishing departure from roundness. In doing this we examine some curious geometric shapes, including non-circular shapes of constant width and shapes that rotate smoothly inside triangles. This has led to the recent reassessment of the standard testing procedures using V-blocks.

Background

Instruments are the arbiters of science. No scientific experiment can be conducted without reliable and accurate instrumentation. In order for experiments to be repeatable, both over time and by a wide variety of experimenters, standards that specify basic scientific units are needed. Time, mass and length are among the first measurable quantities we appreciate. Not only are these standards essential for experimentation in science but they are also essential in technology; for example, quality control also relies on instrumentation. Many people appreciate the need for these standards. What is, perhaps, less obvious is a similar need for standards for geometric shape or form, not just for size.

How would you judge a freehand circle-drawing competition? That is, given two simple closed convex two-dimensional curves, which is the best approximation to a circle?

A circle is defined as the set of points equidistant from a given centre. Knowing that a shape is a circle enables us to use the well-known properties from geometry. In particular, we can bisect a chord with a perpendicular line and be confident that the line passes through the centre. If, on the other hand, we have a shape that we need to test, which point should we choose as the ‘centre’? If the shape is not a perfect circle then the point of intersection of perpendicular bisectors of two chords will move around as the chords move. For some ellipses, these lines might intersect outside the ellipse! This is illustrated in Figure 1. Before we address this question, we shall see why the answer is so important.

On 28 January 1986, the NASA space shuttle Challenger broke apart 73 seconds into its flight, leading to the deaths of its seven crew members. An O-ring seal in its right solid rocket booster failed, causing a flare to ignite the external fuel tank. The Presidential Commission on the Space Shuttle Challenger Accident, also known as the Rogers Commission (after its chairman), was formed to investigate the disaster. Their findings included the following.

5. Launch site records show that the right Solid Rocket Motor segments were assembled using approved procedures. However, significant out-of-round conditions existed between the two segments joined at the right Solid Rocket Motor aft field joint (the joint that failed). (Rogers, 1986)
The Nobel Prize-winning theoretical physicist Richard Feynman was a member of the Rogers Commission and later explained the testing procedure as follows:

NASA gave me all the numbers on how far out of round the sections can get, so I tried to figure out how much the resulting squeeze was and where it was located—maybe the minimum squeeze was where the leak occurred. The numbers were measurements taken along diameters, every 60 degrees. But three matching diameters won’t guarantee that things will fit; six diameters, or any other number of diameters, won’t do either. [...] So the numbers NASA gave me were useless. (Feynman, 1989)

Feynman interviewed technicians who assembled the rocket motors:

‘I have a question: when you measure the three diameters and all the diameters match, do the sections really fit together? It seems to me that you could have some bumps on one side and some flat areas directly across, so the three diameters would match, but the sections wouldn’t fit.’

‘Yes, yes!’ they say. ‘We get bumps like that. We call them nipples.’ [...] ‘We get nipples all the time,’ they continued. ‘We’ve been tryin’ to tell the supervisor about it, but we never get anywhere!’ (Feynman, 1989)

The geometry of roundness and testing for departures from roundness thus played a significant contributory role in this disaster. But what did Feynman mean when he said ‘three matching diameters won’t guarantee that things will fit; six diameters, or any other number of diameters, won’t do either’?

Before reading on, you might like to perform the following experiment. Take two UK 50p pieces, and two straight edges. Place the coins on a flat surface and trap them between two parallel straight edges, as shown in Figure 2. For different orientations of the 50p coins, what is the distance between the rulers?

**Figure 1** Perpendicular bisectors of chords meet in the centre of a circle; for an ellipse, they do not

![Figure 1](image)

**Figure 2** An experiment with 50p coins

**Diameter and width**

Imagine we have a pair of calipers, designed to measure the width of a shape. The width of a unit square varies between 1 and $\sqrt{2}$, depending on the orientation. If we have a circle then the width is constant. The previous sentence has the form ‘if $A$ then $B$’. Using our calipers we measure the width of a given shape being tested in every direction and find it does not vary; that is, the width is constant. If the width is constant, then do we have a circle? This statement has the form ‘if $B$ then $A$’. Notice the relationship between these two statements, the hypothesis and conclusion are reversed. Mathematicians describe these statements as the *converse* of each other. They are easily confused and, most importantly, they are independent of each other. We know the first is true. All circles have constant width – it is simply the diameter. What can we conclude about the second?
Take, for example, a 50p piece. Measure the diameter of this in every direction, and you may be surprised to find the diameter does not, within the accuracy you have available on bench calipers, appear to vary. Trapping two or more coins between parallel 30 cm rulers is a very effective qualitative experiment. The 50p coin is manifestly not circular, however! Actually, the geometry underlying the design of this coin guarantees the width is constant. How can this happen?

The simplest non-circular shape of constant width is an equilateral triangle with circular arcs centred at each vertex and passing through the other two. It is known as Reuleaux’s rotor, after Franz Reuleaux (1829–1905), and is shown in Figure 3. It is easy to see that Reuleaux’s rotor also has constant width. A tangent to one of the circular arcs will be of constant distance from a parallel line through the centre of this arc.

This shape is by no means unique; indeed, there are very many shapes of ‘constant width’. In this article we define a shape to be round if it has a circular cross section. Using the word round enables us to talk about both two-dimensional and three-dimensional shapes. Again, Reuleaux’s rotor is simple to make, and two rotors can be trapped between parallel straight edges or made into a roller by creating an axle. However, note that the axle will necessarily move up and down. If the axle also stays at a constant height then we must have a circular wheel. Non-circular shapes of constant width can be used as rollers but not as wheels. The difference between the two is a key idea. In measuring the width we did not choose a centre. Instead, we relied on a property of the circle, not on its defining characteristic. Shapes of constant width appear round if we confound constant width with roundness. The procedure described by Feynman is exactly this – measuring the diameter in a number of orientations. For many other engineering applications it is essential to establish and quantify departure from roundness since many devices depend on rotation, or a correct fit.

There are a whole variety of families of two-dimensional shapes of constant width. In each there is a continuous spectrum with Reuleaux’s rotor at one extreme and the circle at the other. Either the number of sides in a polygon increases, or there is continuous change of an arc’s radius or construction angle to create this spectrum. These are shown in Figure 4, and all these shapes can be drawn simply with a ruler and compass. Shapes of constant width can be created in a variety of other ways, and circular arcs are not necessary. Some of these are described in detail in Bryant and Sangwin (2008).

We can also create solid objects with constant width. The simplest of these is to rotate the Reuleaux rotor about an axis of symmetry. Actually, any of the shapes of constant width in Figure 4 can be rotated about their axis of symmetry. A selection of such solids made by John Bryant are shown in Figure 5. The simple
geometry of these shapes makes them ideal 3D printing projects. Solids of constant width do not need to have an axis of rotational symmetry. The best known is based on the regular tetrahedron and is sometimes called Meissner’s tetrahedron (Chakerian and Groemer, 1983).

Applications

These surprising shapes have many useful applications beyond decorative coins such as the 50p. The essential characteristic for applications is that we have one shape that rotates smoothly inside another. Two examples of such simple pairs of shapes are two circles of identical size that can rotate together and form a mechanical bearing, and a circle that rotates smoothly inside a square.

One of the most remarkable applications is drilling a square hole. This can be done in a number of ways, one of which is to use Reuleaux’s rotor and remove parts to create cutting faces that allow the escape of the cut material. A wonderful animation of this is online at www.etudes.ru/ru/mov/mov017/index.php.

However, this does not get right into the corners of the square, only removing about 98% of the material. A drill that can cut a perfect square is based on a shape of constant width derived from a right-angled isosceles triangle. This is shown on the right of Figure 4 where we use the case with $t=45^\circ$. If a cam of this shape is housed in a square hole of side length $w$ then part of the quadrant from $D$ to $E$ must always touch one of the sides. Hence, the locus of the point $C$ must follow a square path as the shape is rotated inside a square. This path is shown in Figure 6.

Another common application of these shapes occurs as part of a cam. Because the shapes rotate in a square, they keep contact with both sides of the cam-follower without the need for a spring. An example cam, from Reuleaux’s famous book (1876), is shown in Figure 7. As the cam $a$ rotates about the point in the middle of the dashed circle, the rectangular follower moves vertically. For part of the time the rectangular box is stationary, and this part of the motion is often called a ‘dwell’. This is particularly useful in mechanisms such as printing presses where part of the machine should

![Figure 5](image5.png) Three-dimensional shapes of constant width

![Figure 6](image6.png) This rotor forms the basis of a drill that cuts a square hole

![Figure 7](image7.png) A cam incorporating Reuleaux’s mechanism
be still, while another mechanism is brought into play.

Reuleaux’s rotor is also used in the design of the rotary engine car. While a traditional engine uses a piston that reciprocates, a rotary engine generates rotation directly. In this design, the Reuleaux rotor rolls around a circular inner gear and the corners trace out the shape we see in Figure 8. There are many other pairs of shapes that can be used in this way. Pumps and engines are, in some sense, the inverse of each other. It is therefore not surprising that such geometry finds applications here also. Two examples are shown in Figure 9. On the left is a Roots blower. This is commonly used to move large volumes of air at low pressures. On the right is a pump by Evard. It is only a small step from shapes such as these to gears, which also rotate in pairs touching each other and transmitting force. Reuleaux’s book (1876) examined and catalogued a huge variety of pumps and engines and all kinds of pairs of shapes that move together. Rotors for other regular polygons (Goldberg, 1960) also have applications.

Standards for assessment of roundness

We have seen examples that illustrate why having constant width is not sufficient to guarantee that the shape is a circle. This still leaves the practical question of what we can measure to decide whether a shape is round, and to quantify any departure from roundness. Measuring width relies on two points of contact. However, a circle is uniquely defined by three points of contact. That is, through any three points there is a unique circle. We take a slight liberty here, by defining a straight line as a circle of infinite radius. Can we create an effective test based on three points of contact instead of two?

Imagine a circle resting in the solid V-shaped block shown in Figure 10. The internal angle is $a$ and, by symmetry, the top of the circle will be found on the angle bisector of the two faces of the V-block. Imagine a measuring device used along this line. If a circle is rotated in the block then the position of the measuring device stays constant. Can we conclude the converse? That is to say, if a shape rotates smoothly inside a V-block and the
position of the measurer remains constant, is the shape round?

This is exactly what is proposed in part 3 of the British Standard for Assessment of Departures from Roundness, BS 3730 (British Standards Institution, 1987), which defines ‘Methods for determining departures from roundness using two- and three-point measurement’. Now, testing width is a thoroughly sensible thing to do. If the width varies, then the shape is not round, and width is cheap and very accurate to measure. However, on its own it is simply not sufficient, as we have seen.

The three-point measurement certainly seems like an improvement but are we about to make the same logical mistake? A circle rotates smoothly inside the V-block but if a shape rotates smoothly is it a circle? The reliability of the test depends on the answer to this question. If we can construct non-circular shapes via pure geometry that are indistinguishable from a perfect circle under test conditions using a V-block then we will have deceived the test.

Imagine we can construct geometric shapes that, when rotated inside a triangle of fixed size, remain in contact with all three sides. Such a shape is called a rotor of the triangle. It is important to acknowledge a subtle difference between the test proposed in Figure 10 and a shape that rotates in a triangle. A rotor of a triangle measures the relative distances between three tangent lines. Figure 10 shows two fixed tangent lines and we measure by finding the points on the curve that lie on the angle bisector of these tangent lines. We know nothing particular about the direction of the tangent line at this point. The test shown in Figure 11 uses a small circle as the contact point, which is different again. Note that the point of contact in Figure 11 is not generally the point of contact in Figure 10. These are very subtle differences, but ones that are not acknowledged in applications such as that proposed by Goho, Kimiyuki and Hayashi (1999).

Despite these subtleties, if we can find a shape that rotates in a triangle then it casts serious doubt on the efficacy of the standard tests for assessment of roundness using V-blocks.

Rotors of a triangle

Every triangle has a circle that could rotate inside it. This is called the incircle and we can find this by pure geometry. The centre of the incircle must lie on the angle bisector of a pair of sides. In fact, the angle bisectors all meet at a point (called the incentre), which is shown in Figure 12. The basic question that concerns us is for which triangles can we find non-circular shapes that rotate inside the triangle? Can we characterise all such shapes, and what are their properties?

To do this, we take a family of lines. The region enclosed by this family is known as the envelope. This is a natural way of defining our

**Figure 10** A circle in a V-block

**Figure 11** The symmetrical British Standard summit method

**Figure 12** The incircle of a triangle
shapes because it creates the shape through the tangent lines, and it is precisely the properties of the tangents we need: rotating inside a specified triangle is all about tangent lines lying in a particular configuration. An example is shown in Figure 13, where Reuleaux’s rotor is defined by an envelope of lines. The precise details of the lines needed are given in Kearsley (1952).

An envelope of lines can be transformed into a parametric equation for the curve using calculus. The parametric curve

\[x = -\sin(t) + \frac{1}{2}\sin(4t) + \frac{1}{12}\sin(6t)\]

\[y = \cos(t) + \frac{1}{2}\cos(4t) - \frac{1}{12}\cos(6t)\]

is plotted in Figure 14. This shape has both constant width, and so is a rotor of a square, and rotates inside an equilateral triangle. It is, perhaps, not entirely surprising that a regular triangle has a non-circular shape that rotates inside it.

The three angles inside a triangle uniquely define the shape, but not the size. To give a clean answer to our question, we have to think about how to measure angles. There are 360 degrees in one rotation (although in mathematics the alternative unit of radians is commonly used: \(1 \text{ radian} = 180/\pi\)). A somewhat neglected unit of angle is the fraction of a rotation. That is, a whole rotation is one unit and thus 60° would be 1/6 of a rotation.

The second issue we need to raise is that of rational numbers. A real number \( x \) is rational if there exist integers \( p \) and \( q \) such that \( x = p/q \). In mathematics the distinction between rational and irrational numbers is important; for example, ‘a half’ is clearly rational as it can be written as 1/2 or 2/4. It is harder to prove that a number is not rational. Famously, \( \sqrt{2} \) is irrational. It is easier to see that \( \log_2 3 \log_3 2 \) is irrational. If not, then \( 2^{\log_2 3} = 3 \), or raising both sides to the power \( q \) then \( 2^q = 3^q \). Whatever integer values \( p \) and \( q \) take, the left-hand side is even and the right-hand side odd. Thus \( \log_2 3 \neq p/q \) so it must be irrational. \( \pi \) and \( e \) are also irrational numbers. These rather abstract distinctions appear to be only interesting to pure mathematics. However, in this situation nature cares very much about irrational numbers.

**Theorem 5.1** A triangle has a non-circular rotor if and only if all angles are rational.

This is demonstrated in Bryant and Sangwin (2008), with the underlying research and full details in Sangwin (2009). The greatest possible departure from roundness of a rotor can be quantified by taking the lowest common multiple \( n \) of the denominators of all three angles (where the angle unit is fraction of a rotation). The maximum departure from roundness is approximately \( 1/n^2 \), so the larger the denominator of the fraction (in lowest terms), the smaller the departure from roundness. Having only angles of 60° (i.e. 1/6 in units of fraction of a rotation) yields the smallest number possible in the denominator, i.e. 6, and is therefore the largest possible departure from roundness. The testing regime proposed by Goho et al. (1999) used angles including 113/360. Having the large denominator 360 is a very sensible practical choice.

Given this analysis, we can reconsider the standard testing procedure BS 3730 (British Standards Institution, 1987), which only uses particularly convenient angles within the triangles that can result in shapes with potentially larger departure from roundness. So, the results of this research show that, while V-blocks...
are not hopeless, there are shapes that will rotate in triangles with rational angles. Using multiple V-blocks with angles having co-prime denominators radically reduces the extent to which they can be fooled. Indeed, the theoretical maximum error is rapidly reduced below a practical limit, resurrecting the usefulness of the test. That said, it is not entirely clear that the authors of the standard appreciated this mathematics. Had they done so, they are very likely to have made quite different choices for the angles specified to be used in standard V-blocks!

**Conclusion**

Standards for measurement are essential for science and engineering. Unfortunately, the geometry described in this article shows why assessing roundness is not a simple matter of establishing constant width. Although much of this geometry has been known for hundreds of years, a failure to appreciate its implications can have disastrous, even fatal, consequences. Intriguingly, the standard tests, or common implemented variations, can still be ‘deceived’ by shapes that rotate within a triangle. Yet, by appreciating rational and irrational numbers, we can improve this simple testing procedure dramatically.

**Acknowledgement**

Figures 4, 5, 8, 10 and 13 are reproduced from *How Round is Your Circle?* (Bryant and Sangwin, 2008) with permission from Princeton University Press.

**References**


Where can I get some of these shapes?

The classic book on hands-on mathematical models by Cundy and Rollet (1961) opens by saying that ‘the main use of a model is the pleasure derived from making it’. Shapes of constant width can be easily made from cardboard, plywood and other basic materials, and two can easily be fastened together to make rollers. Solid shapes make excellent 3D printing projects.

Plastic solids and two-dimensional shapes of constant width can be purchased from Maths Gear: mathsgear.co.uk.

Metal solids of constant width can be purchased from Grand Illusions: www.grand-illusions.com. Both sites also have a range of other very interesting toys and mathematical curiosities. Please note the companies listed here are independent of both the author and publisher.

---

Chris Sangwin is a Senior Lecturer in Mathematics Education in the Mathematics Education Centre at Loughborough University. From 2000 to 2011 he was seconded half time to the UK Higher Education Academy ‘Maths Stats and OR Network’ to promote learning and teaching of university mathematics. In 2006, he was awarded a National Teaching Fellowship. His learning and teaching interests include (i) automatic assessment of mathematics using computer algebra, in particular the development of the STACK system, and (ii) problem solving using the Moore method and similar student-centred approaches. He is the author of a number of books, including *How Round is Your Circle*, which illustrates and investigates many links between mathematics and engineering using physical models.

Email: C.J.Sangwin@lboro.ac.uk