Towards the answer to the second part of Hilbert’s 16th problem

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**Citation:** WANG, S., 2004. Towards the answer to the second part of Hilbert’s 16th problem. World Journal of Engineering, 2, pp.32-40.

**Metadata Record:** [https://dspace.lboro.ac.uk/2134/16230](https://dspace.lboro.ac.uk/2134/16230)

**Version:** Published

**Publisher:** © Multi-Science Publishing

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Towards the answer to the second part of Hilbert’s 16th problem

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(Received 26 September 2004; accepted 2 November 2004)

Abstract

A method is given for the estimation of limit cycles in general planar polynomial vector fields. The estimated Hilbert number is \( H(n) = N_v(n^2 - 1) \) where \( n \) is the order of the polynomial vector field and \( N_v \) is the number of critical points. A variety of examples including the well known Shi’s second order system (Shi, 1979) are presented to verify the method by comparing with existing results and Poincaré phase portraits from computer simulations. This development is of considerable significance towards the answer to the second part of Hilbert’s 16th problem.

Key words: Keywords: Planar polynomial vector field, Limit cycles, Phase portraits, Hilbert

1. Introduction

The second part of Hilbert’s 16th problem (Hilbert, 2000) concerns with the number and locations of limit cycles of a planar polynomial vector field. Since it’s birth in 1900 numerous attempts have been made to solve it. Recently, Ilyashenko published an excellent paper (Ilyashenko, 2002) about the history of the problem in which many useful references can be found. He concluded in the work that after more than one hundred years our knowledge on the problem was almost the same as at the time when it was stated. It appears to be the most persistent in the famous Hilbert’s list, second only to the Riemann ξ-function conjecture. Many other research work on the problem can also be found in the books by Gaiko (2003), Chow et al. (1994), and Ye et al. (1986). Lloyd has published a series papers on related problems (Lloyd, 1988; Lloyd, 1996). A very latest study is reported by Gaiko and Horssen (Wang, 2004a) on a generalized Lienard system. The present paper reports a method for the estimation of limit cycles in general planar polynomial vector fields, which has been presented in the 4th world congress of nonlinear analysis as an invited lecture (Gaiko and Horssen, 2004). The theoretical development is presented in Section 2. A variety of examples are presented in Section 3 to verify the method. Some predictions are compared with computed Poincaré phase trajectories from the excellent work by Nayfeh and Balachandran (1995). Conclusions are made in Section 4.

2. Theory

A general planar vector field (or autonomous
system) can be expressed as

\begin{align}
\dot{x}_1 &= -x_2 + \varepsilon f_1(x_1, x_2) \\
\dot{x}_2 &= \omega^2 x_1 - \varepsilon f_2(x_1, x_2) \\
\end{align}

(1a, b)

where \(\omega\) and \(\varepsilon\) are non-zero numerical parameters. The second part of Hilbert’s 16th problem concerns with the limit cycles of the field in the case of that \(f_1(x_1, x_2)\) and \(f_2(x_1, x_2)\) are polynomials. This paper studies the limit cycles of the field nested around critical points. Without losing any generality the origin of the filed is assumed to be a critical point since other locations can be made to be new origins by transformations. Therefore, the solutions of the field can then be expressed in the following general form

\begin{align}
x_1 &= a \cos \omega t + \varepsilon \int_0^t \sin \omega (t - \tau) f(x_1, x_2, x_1, x_2) \, d\tau \\
x_2 &= a \omega \sin \omega t - \varepsilon \int_0^t \cos \omega (t - \tau) f(x_1, x_2, x_1, x_2) \, d\tau + \varepsilon f_1(x_1, x_2) \\
\end{align}

(2)

where

\begin{align}
f(x_1, x_2, x_1, x_2) &= f_2(x_1, x_2) + \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2 \\
\end{align}

(3)

It is noted that \(x_1 = a\) and \(x_2 = 0\) at \(t = 0\) are assumed as initial conditions when deriving expressions (2) and (3). For a closed orbit with period \(T = T(a, \varepsilon)\), we have

\begin{align}
x_1(T) - x_1(0) = 0 \\
x_2(T) - x_2(0) = 0 \\
\end{align}

(5)

(6)

Substituting expressions (2) and (3) into equations (5) and (6), respectively yields

\begin{align}
\varepsilon F_1(a, \varepsilon) &= \varepsilon \left( a \left( \cos \omega T - 1 \right) \right) / \varepsilon + \int_0^T \sin \omega (T - \tau) f(x_1, x_2, x_1, x_2) \, d\tau = 0 \\
\varepsilon F_2(a, \varepsilon) &= \varepsilon \left( a \omega \sin \omega T \right) / \varepsilon - \int_0^T \cos \omega (T - \tau) f(x_1, x_2, x_1, x_2) \, d\tau + f_1(x_1(T), x_2(T)) - f_1(a, x_2(0)) = 0 \\
\end{align}

(7)

(8)

It is noted that the third and fourth terms in the bracket in equation (8) disappear if \(f_1(x_1, x_2)\) is not present. Since \(\varepsilon \neq 0\), equations (5) and (6) are equivalent to

\begin{align}
F_1(a, \varepsilon) &= 0 \\
F_2(a, \varepsilon) &= 0 \\
\end{align}

(9)

(10)

Since \(\varepsilon\) can have any non-zero values we consider the situation when it is small. Equations (9) and (10) become

\begin{align}
F_1(a, \varepsilon) &= -a \omega \sin \omega T \frac{\partial T}{\partial \varepsilon} + \int_0^T \sin \omega (T - \tau) f(x_1, x_2, x_1, x_2) \, d\tau = 0 \\
F_2(a, \varepsilon) &= a \omega^2 \cos \omega T \frac{\partial T}{\partial \varepsilon} - \int_0^T \cos \omega (T - \tau) f(x_1, x_2, x_1, x_2) \, d\tau + f_1(x_1(T), x_2(T)) - f_1(a, x_2(0)) = 0 \\
\end{align}

(11)

(12)

In these two equations

\begin{align}
x_1 &= x_{10} + \varepsilon \int_0^T \sin \omega (T - \tau) f(x_{10}, x_{20}, x_{10}, x_{20}) \, d\tau \\
x_2 &= x_{20} - \varepsilon \int_0^T \cos \omega (T - \tau) f(x_{10}, x_{20}, x_{10}, x_{20}) \, d\tau + \varepsilon f_1(x_{10}, x_{20}) \\
\end{align}

(13)

(14)

\begin{align}
x_1 &= x_{10} + \varepsilon \int_0^T \cos \omega (T - \tau) f(x_{10}, x_{20}, x_{10}, x_{20}) \, d\tau \\
x_2 &= x_{20} + \varepsilon \int_0^T \sin \omega (T - \tau) f(x_{10}, x_{20}, x_{10}, x_{20}) \, d\tau - \varepsilon f_1(x_{10}, x_{20}) \\
\end{align}

(15)

(16)

In equations (13)—(16)

\begin{align}
x_{10} &= a \cos \omega t \\
\dot{x}_{10} &= -a \omega \sin \omega t \\
x_{20} &= a \omega \sin \omega t \\
\dot{x}_{20} &= a \omega^2 \cos \omega t \\
\end{align}

(17)

(18)

(19)

(20)

Since for small \(\varepsilon\), \(T\) is close to \(2\pi / \omega\). Therefore, equations (11) and (12) become
Therefore, the values of $\alpha$ can be determined from equation (21), which govern the nature of the solutions (2) and (3). Three situations can arise, i.e. (a) $\alpha$ equals to zero or complex value, which suggests no closed orbit; (b) $\alpha$ equals to continuous values, which suggests continuous closed orbits; (c) $\alpha$ equals to isolated non-zero real values, which suggests limit cycles. Substituting these $\alpha$ values into equation (22) gives the value of which shows the trend of $\partial T / \partial \varepsilon$ frequency variation of limit cycles with respect to the parameter $\varepsilon$. Now, we can conclude this finding in the following theorem.

**Limit Cycle Theorem (LCT):** The planar polynomial vector field (1) has limit cycles nested around the critical point $(0,0)$ when equation (21) has isolated non-zero real $\alpha$ values. The number of limit cycles equals to the number of non-zero $\alpha$ values.

Now, the details of equation (21) are considered. Let

\[
f_1(x_1, x_2) = \sum_{i=1}^{n} q_i(x_1, x_2) = \sum_{i=1}^{n} \sum_{j=1}^{i} \beta_i x_1^{i-j+1} x_2^{j-1} \tag{23}
\]

\[
f_2(x_1, x_2) = \sum_{i=1}^{n} p_i(x_1, x_2) = \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_i x_1^{i-j+1} x_2^{j-1} \tag{24}
\]

Thus, we have

\[
f(x_1, x_2, \dot{x}_1, \dot{x}_2) = \sum_{i=1}^{n} \left[ p_i(x_1, x_2) + x_1 \frac{\partial q_i(x_1, x_2)}{\partial x_1} + x_2 \frac{\partial q_i(x_1, x_2)}{\partial x_2} \right] \tag{25}
\]

where $x_1, x_2, \dot{x}_1$ and $\dot{x}_2$ are given in equations (13)—(16), respectively with

\[
f_1(x_{10}, x_{20}) = \sum_{i=1}^{n} q_i(x_{10}, x_{20}) = \sum_{i=1}^{n} \sum_{j=1}^{i} \beta_i x_{10}^{i-j+1} x_{20}^{j-1} \tag{26}
\]

\[
f_2(x_{10}, x_{20}) = \sum_{i=1}^{n} p_i(x_{10}, x_{20}) = \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_i x_{10}^{i-j+1} x_{20}^{j-1} \tag{27}
\]

and

\[
f(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = \sum_{i=1}^{n} \left[ p_i(x_{10}, x_{20}) + x_{10} \frac{\partial q_i(x_{10}, x_{20})}{\partial x_{10}} + x_{20} \frac{\partial q_i(x_{10}, x_{20})}{\partial x_{20}} \right] \tag{28}
\]

Therefore, substituting these equations into equation (21) leads to

\[
\sum_{i=1}^{n} \sum_{m=i}^{n} \varepsilon^m c_{im} a^{m-1} = 0 \tag{29}
\]

where $c_{im}$ are constant coefficients and $c_0 = 0$. By neglecting the higher order $\varepsilon$ terms for same $a^{m-1}$ terms, equation (29) becomes

\[
\sum_{i=0}^{n} \sum_{m=i}^{(n+1)\varepsilon^{m-1}} \varepsilon^m c_{im} a^{m-1} = 0 \tag{30}
\]

It can be seen that equation (30) is a polynomial equation in terms of $\alpha$. The highest order is $n^2 - 1$ where $n$ is the order of the planar polynomial vector field (1). Therefore, the maximum number of limit cycles of the field (1) is $N_\alpha (n^2 - 1)$ where $N_\alpha$ is the number of the critical points. It needs to be noted that although the equation (21) is derived for small $\varepsilon$, it is still a good estimate condition for use to determine the existence of limit cycles for large $\varepsilon$ case. This will be demonstrated by examples in Section 3. Poincare phase trajectories are given by the expressions (13) and (15). The integral parts in (13) and (15) represent the contributions from higher order frequencies and can be approximated by their mean values as

\[
a_0 = \frac{\omega \varepsilon}{2 \pi} \int_0^{2\pi/\omega} \sin \omega(t - \tau) \int_0^{2\pi/\omega} \sin \omega(t - \tau) \]

\[ f(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) \, dx \, dt \]  

(31)

\[ b_0 = \frac{\omega \varepsilon}{2\pi} \int_{0}^{\frac{2\pi}{\omega}} \int_{0}^{t} \cos(\omega(t - \tau)) \, f(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) \, dx \, dt \]  

(32)

These expressions are expected to give good predictions for small \( \varepsilon \) values. Experimental examples of next section show that they can even give good predictions for lower order limit cycles for large \( \varepsilon \) values.

3. Experiment

3.1. Liénard equation

The first experiment concerns with the study of the limit cycles of the Liénard equation, i.e.

\[ \dot{x}_1 = -x_2 \]  

(33)

\[ \dot{x}_2 = \omega^2 x_1 - f_2(x_2) \]  

(34)

where

\[ f_2(x_2) = \sum_{n}^{\infty} m \mu_n \]  

(35)

and \( m \) is an integer and \( \mu_n \)'s are numerical parameters. It has only one critical point, i.e. the origin \((0,0)\). All possible limit cycles nest around this point. For this problem, equation (21) becomes

\[ \frac{\partial T}{\partial \varepsilon} = \frac{1}{2\omega} \int_{0}^{2\pi/\omega} \cos(\omega t) f_2(x_2) \, dx \, dt = 0 \]  

(36)

where \( f_2(x_2) \) is the odd terms of \( f_2(x_2) \), that is

\[ \int_{0}^{2\pi/\omega} \sin(\omega \varepsilon) \sum_{n}^{\infty} \mu_{2i-1} (\omega \varepsilon \sin(\omega \varepsilon))^{2i-1} \, dx = \]  

\[ = \int_{0}^{2\pi/\omega} \sum_{n}^{\infty} \mu_{2i-1} (\omega \varepsilon)^{2i-1} (\sin(\omega \varepsilon))^{2i} \, dx = 0 \]  

(37)

where \( m = 2n - 1 \) or \( 2n \). It is seen form equation (37) that the maximum number of isolated non-zero real \( (\omega \varepsilon)^2 \), i.e. the maximum number of limit cycles is \( n \) when \( m = 2n - 1 \) and \( n - 1 \) when \( m = 2n \). It is seen that the existence of limit cycles depends on the odd terms only. From equation (22) it is found that

\[ \frac{\partial T}{\partial \varepsilon} = \frac{1}{2\omega} \int_{0}^{2\pi/\omega} \cos(\omega \varepsilon) f_2(x_2) \, dx \, dt = 0 \]  

(38)

It suggests that there is no variation of frequencies when \( \varepsilon \) is very small. To know the variation trend of \( T \) for not very small, an improved formula needs to be developed in this particular case. From equation (12) we have

\[ a_0^2 \cos(\omega T) \frac{\partial T}{\partial \varepsilon} - \int_{0}^{\frac{r}{\omega}} \cos(\omega T) f_2(x_2) \, dx \, dt = 0 \]  

(39)

where we take

\[ T = 2\pi/\omega_a \]  

(40)

and \( \omega_a \) is close to but not equal to \( \omega \). Therefore,

\[ \frac{\partial T}{\partial \varepsilon} = \frac{a_0}{\omega^3 \cos(\omega T)} \sum_{n}^{\infty} \frac{1}{j+1} \mu_j (a_0 \omega)^{j-1} \sin(j+1) \omega T \]  

(41)

Since the factor \( a_0/(\omega^3 \cos(\omega T)) > 0 \), the sign of \( \partial T/\partial \varepsilon \) depends on the sign of the summation term only. It can be found that the expressions for \( a_0 \) and \( b_0 \), i.e. (31) and (32) become

\[ a_0 = \frac{\varepsilon \omega}{2\pi} \int_{0}^{2\pi/\omega} \sin(\omega \varepsilon) f_2(x_2) \, dx \, dt \]  

(42)

\[ b_0 = 0 \]  

(43)

In equation (42) \( f_2(x_2) \) is the even terms of \( f_2(x_2) \). It is seen that depends on the even terms only. Three specific forms of \( f_2(x_2) \) are considered here.

(a) \( f_2(x_2) = \mu_3 x_2^3 \) and \( \omega = 1 \)

From equation (37), we have

\[ (a_0 \omega_a)^2 = -\frac{4\mu_1}{3\mu_3} \]  

(44)

It is seen that if \( \mu_1 \mu_3 < 0 \), \( (a_0)^2 \) has one isolated non-zero value, and hence a limit cycle exists. For \( \mu_3 = -\mu_1 = 1 \), \( (a_0)^2 = 4/3 \). Hence, the predicted amplitude \( a = 1.155 \) by taking From equations (42) and (43) it is found that \( a_0 = b_0 = 0 \). The results from computer simulation are \( a = 1.258 \), \( \omega_a = 0.9425 \), \( a_0 = b_0 = 0 \), which
are read from graphs in E. Oxenhölm (On the second part of Hilbert's 16th problem). The amplitude of the velocity, i.e. \((\omega_0)\) equals to 1.155. The computer simulation result is 1.164 (measured from graph in Oxenhölm, 2004). It can be seen that the present predictions are in a good agreement with the simulations although \(\varepsilon\), in fact \(\mu_j\)'s, is not a small value. It should be noted that from the current theory, i.e. equation (44), the existence of limit cycle does not depend on the \(\varepsilon\) value. The value of \((\omega_0)^2\) which approaches to \((\omega_0)^2\) for small \(\varepsilon\) solely depends on the relative values of \(\mu_1\) and \(\mu_3\). However, the amplitude \(\alpha\) and frequency \(\omega_0\) depend on the value of \(\varepsilon\). The trend of variation of \(\omega_0\) can be predicted by using equation (41). Substituting equation (44) into (41) yields

\[
\frac{\partial T}{\partial \varepsilon} \approx \frac{\omega_0}{\omega^3 \cos \omega T} \mu_1 \left( \frac{1}{2} - \frac{1}{3} \sin^2 \omega T \right) \sin^2 \omega T
\]

\[(45)\]

It can be seen that the sign of \(\partial T/\partial \varepsilon\) only depends on the sign of \(\mu_1\), i.e. the sign of \(\varepsilon\). In this example, \(\mu_1 < 0\), and hence, the period of the limit cycle, \(T\) decreases, that is, the frequency \(\omega_0\) increases for non-small \(\varepsilon\). This has been seen in computer simulations. It is noted that \(\partial T/\partial \varepsilon\) approaches to zero for small \(\varepsilon\).

(b) \(f_2(x_2) = \mu_3 x_2^3 + \mu_2 x_2^3 + \mu_1 x_2\) with \(\mu_3 = -\mu_1 = 1, \mu_2 = 0.5\) and \(\omega = 1\)

The predicted amplitude \(a = 1.155\) and frequency \(\omega_0 = \omega = 1\) for the limit cycle with \(a_0 = 0.6667, b_0 = 0\). The results from computer simulation are \(a = 1.300, \omega_0 = 0.9146, a_0 = 0.6667\) and \(b_0 = 0\) (measured from graph in Oxenhölm, 2004). The amplitude of the velocity, i.e. \((\omega_0)\) equals to 1.155. The computer simulation result is 1.182 (measured from graph in Oxenhölm, 2004). Again, the predictions from the present theory are in a good agreement with computer simulations.

(c) \(f_2(x_2) = \mu_3 x_2^5 + \mu_3 x_2^3 + \mu_1 x_2\) with \(\mu_3 = \mu_1 = 0.25, \mu_3 = -2.0\) and \(\omega = 1\)

From equation (37) we have

\[
5 \mu_3 (\omega_0)^4 + \frac{3}{4} \mu_3 (\omega_0)^2 + \mu_1 = 0 \quad (46)
\]

Two roots of \((\omega_0)^2\) are obtained from this equation. They are \((\omega_0)^2 = 0.1697, 9.4303\). Therefore, there are two limit cycles. For the first one, \(\alpha = 0.4119, \omega_0 = \omega = 1\). The computer simulation gives \(\alpha = 0.4500, \omega_0 = 0.9896\) (measured from the graphs in Oxenhölm, 2004). The magnitude of velocity, i.e. \((\omega_0) = 0.4453\) (measured from the graphs in Oxenhölm, 2004). The agreement is good. For the second one, \(\alpha = 3.071, \omega_0 = \omega = 1\). The computer simulation gives \(\alpha = 9.412\) and \(\omega_0 = 0.4328\) (measured from the graphs in Oxenhölm, 2004) which are quite different from the predictions. The measured magnitude of velocity, i.e. \((\omega_0) = 3.000\) (Oxenhölm, 2004). It is interesting to notice that for both limit cycles the magnitudes of velocity, i.e. are in a good agreement between the predictions and simulations. This fact suggests that the condition (21) for the existence of limit cycles is valid for any values of \(\varepsilon\). Now, the trend of variation of \(\omega_0\) with respect to \(\varepsilon\) is studied. From equation (41), it is obtained that

\[
\frac{\partial T}{\partial \varepsilon} \approx \frac{\omega_0}{\omega^3 \cos \omega T} \left( \frac{1}{6} \mu_5 (\omega_0)^4 \sin^4 \omega T + \frac{1}{4} \mu_3 (\omega_0)^2 \sin^2 \omega T + \frac{1}{2} \mu_1 \sin^2 \omega T \right) \quad (47)
\]

It is seen that the sign of only depends on the bracket part in the right hand side. It can be seen from equation (47) that for both limit cycles the bracket part is positive. Therefore, the period will increase with respect to \(\varepsilon\). This explains the small value of \(\omega_0\) 0.4328 for the second limit cycles.

3.2. Extended Lienard equation

In this sub-section the Lienard equation (33) and (34) is extended to include non-linear stiffness part. That is

\[
x_1 = -x_2 \quad (48)
\]

\[
x_2 = \omega^2 x_1 - f_2(x_1, x_2) \quad (49)
\]

where

\[
f_2(x_1, x_2) = \sum_{i} a_i x_i^4 + \sum_{i} \mu_i x_i^2 \]

\[
= f_{2a}(x_1) + f_{2b}(x_2) \quad (50)
\]
and L is an integer and's are numerical parameters. It has at least one critical point, i.e. the origin (0, 0). Without losing any generality we consider limit cycles nesting around this point. For this problem, equation (21) becomes the same as that of equation (37). That is, the non-linear stiffness effect does not determine the limit cycles around a critical point. But, it should be noted that it can increase the number of critical points and consequently the total number of limit cycles. This point will be illustrated later in an example. The equation (22) in this case takes the form

\[ a_0 \omega^2 \frac{\partial T}{\partial x} - \int_0^{2\pi/\omega} \cos \omega t f_{2\omega}(x_1) \, dt = 0 \] (51)

where \( f_{2\omega}(x_1) \) represents the odd terms of \( f_\omega(x_1) \). \( a_0 \) and \( b_0 \) are given by

\[ a_0 = \frac{\omega c}{2\pi} \int_0^{2\pi/\omega} \int_0^t \sin \omega (t - \tau) (f_{2\omega}(x_1) + f_{3\omega e}(x_2)) \, d\tau dt \] (52)

\[ b_0 = \frac{\omega c}{2\pi} \int_0^{2\pi/\omega} \int_0^t \cos \omega (t - \tau) (f_{2\omega}(x_1) \, d\tau dt \] (53)

Now, we consider the following specific example (Nayfeh and Balachandran, 1995)

\[ x_1 = -x_2 \] (54)

\[ x_2 = \omega^2 x_1 - \omega^2 (0.271 x_1^5 - 1.402 x_1^3) - (\mu_3 x_3^3 + \mu_1 x_2) \] (55)

This vector field has five critical points (Nayfeh and Balachandran, 1995). They are at C (0, 0), A and E (± 2.0782, 0) and (± 0.9243, 0). The latter two are saddle points around which no limit cycles nest. Hence, we only consider the possibility of existence of limit cycles around the first three critical points. Three situations are studied.

(a) \( \omega = 5.278, \mu_3 = \mu_1 = 0 \)

In this case, equation (37) gives continuous values of a for all the three critical points. Therefore, continuous closed orbits occur around the points.

(b) \( \omega = 5.278, \mu_3 = 0.108, \mu_1 = 0.172 \)

In this case, equation (37) gives zero value of a for all the three critical points. Therefore, no closed orbits occur around the points. The three critical points become stable foci.

(c) \( \omega = 5.278, \mu_3 = 0.108, \mu_1 = -0.172 \)

In this case, equation (37) gives (aωc)C = (aωc)A = (aωc)E = 1.457. There exists one limit cycle around each point. Since \( \omega_{ac} = 5.278 \) and \( \omega_{ac} = 15.032 \), hence \( a_0 = 0.276 \) and \( a_A = a_E = 0.0969 \). Computer simulation gives \( a_0 = 0.29 \) and \( a_A = a_E = 0.102 \); and \( \omega_{ac} = 5.207 \) and \( \omega_{ac} = 14.804 \) (measured from graphs in Nayfeh and Balachandran, 1995).

It is seen that the predictions from the current theory are in a good agreement with simulations.

3.3. General planar polynomial vector field

First example (Nayfeh and Balachandran, 1995) considered in this sub-section is

\[ x_1 = -x_2 + \mu x_1 + (\alpha x_1 - \beta x_2)(x_1^2 + x_2^2) \] (56)

\[ x_2 = x_1 + \mu x_2 + (\beta x_1 + \alpha x_2)(x_1^2 + x_2^2) \] (57)

By using the following transformation

\[ x_1 = r \cos \theta \] (58)

\[ x_2 = r \sin \theta \] (59)

Equations (58) and (59) become

\[ r = r(\mu + ar^2) \] (60)

\[ \dot{\theta} = 1 + \beta r^2 \] (61)

From equation (60), it seems that when \( \mu r < 0 \), there is a limit circle nested around the critical point \( r = 0 \) with the amplitude and frequency as

\[ r = (-\mu/a)^{1/2} \] (62)

\[ \theta = 1 - \beta \mu / a \] (63)

Now, the current theory is applied to the
problem (56) and (57). The field has a critical point at its origin \((0, 0)\). This point is a neutral one at which Poincare-Andronov-Hopf bifurcation occurs known as flutter in aeroelasticity (Wang, 1999; Wang, 2004b). The current prediction gives

\[ a^2 = -\frac{2\mu}{3\alpha} \]  

(64)

Therefore, we have

\[ a = 0.8165\left(-\frac{\mu}{\alpha}\right)^{1/2} \]  

(65)

which is close to the exact solution (62). The predicted frequency is \(\omega_c = 1\) which is close to the exact solution (63) when is small.

The second example (Nayfeh and Balachandran, 1995) considered here is given as

\[ x_1 = -x_2 + x_1^2 - x_1x_2 \]  

(66)

\[ x_2 = x_1 + x_1x_2 \]  

(67)

The origin \((0, 0)\) is a critical point. Bendixon divergence theorem suggests that a closed orbit may exist around this point. Therefore, a limit cycle may be possible. However, the current theory gives \(a = 0\). Therefore, there is no closed orbit, and consequently no limit cycles.

The next example considered is given as (Shi, 1979)

\[ x_1 = -x_2 + \epsilon(\alpha x_1x_2 - \gamma x_2^2) \]  

(68)

\[ x_2 = x_1 - \epsilon(\lambda x_2 + \rho x_2^3 + \beta x_1x_2 - \xi x_2^2) \]  

(69)

where \(\lambda\) is a very small parameter. There are two critical points, i.e. \((0, 0)\) and \((1/\epsilon\rho, 0)\). Shi (Shi, 1979) shows that there are three and one limit cycles nested around the two critical points, respectively. Shi’s work (Shi, 1979) once caused a shocking wave in the field of limit cycle world. His four limit cycles in quadratic systems still hold the world record. Now, the current theory is put on a test by studying this problem. The limit cycles nested around the critical point \((0, 0)\) is studied first. By applying equation (21) it is obtained that

\[ c_0 + \epsilon c_1 a + \epsilon^2 c_2 a^2 + \epsilon^2 c_3 a^3 = 0 \]  

(70)

where

\[ c_0 = \lambda - \epsilon\pi\lambda^2/2 \]  

(71)

\[ c_1 = -\lambda(17\alpha + 10\xi + 7\rho)/3 \]  

(72)

\[ c_2 = (\alpha\beta + \alpha\gamma + \beta\rho - \beta\xi + 2\xi\gamma)/4 \]  

(73)

\[ c_3 = \{4pq(-a - 8\rho + 7\xi) - 2\beta(q^2 + p - 2s)(5p - 7s)/4q(-3\alpha + 4p - 2s) + 4aq(-3\alpha + 2p + 4p - 2s) - 18\alpha\gamma(p - 2s) + 4q(-3\alpha + 2p + 28s + 6\rho - 18\xi) + 6a\beta(p - 2s) - 2\gamma[-2q^2 - 6\betaq + 18\gammaq + (p - 2s)(-6a - 10p + 14s + 6\rho - 18\xi)]/72 \]  

(74)

where

\[ p = a + \rho, q = 2\gamma - \beta, s = a + \xi \]  

(75)

Now, we let

\[ c_2 = -1900\lambda/\epsilon \]  

(76)

\[ c_3 = 2004\lambda/\epsilon^2 \]  

(77)

Substituting equations (76) and (77) into equation (70) gives

\[ c_0 + \epsilon c_1 a - 1900\lambda a^2 + 2004\lambda a^3 = 0 \]  

(78)

It can be seen that the term \(\epsilon c_1 a\) in equation (78) and the term \(\epsilon^2 a^2/2\) in \(c_0\) can be neglected. Thus equation (78) becomes

\[ 1 - 1900a^2 + 2004a^3 = 0 \]  

(79)

Three real roots of equation (78) are found to be

\[ a_1 = 0.9476 \]  

(80)

\[ a_2 = 0.02323 \]  

(81)

\[ a_1 = -0.02267 \]  

(82)

Therefore, we can conclude that there are
three limit cycles nested around the critical point \((0, 0)\). The phase portrait is given by equations (13) and (15). Now, we determine the specific forms of the parameters \(\alpha, \beta, \gamma, \rho, \zeta\) and \(\lambda\). The parameter \(\lambda\) must be very small in order to eliminate the resonance effect at the critical point \((0, 0)\). To eliminate the resonance effect at the critical point \((1/\varepsilon \rho, 0)\) the following two conditions should be satisfied, i.e.

\[
\frac{2}{\varepsilon} + \alpha/\varepsilon \rho = \delta_1 \tag{83}
\]

\[
\lambda + \beta/\varepsilon \rho = \delta_2 \tag{84}
\]

where \(\delta_1\) and \(\delta_2\) are very small parameters, and here are taken to be zero. Thus, we have

\[
\alpha = -2\rho \tag{85}
\]

\[
\beta = -\varepsilon \lambda \rho \tag{86}
\]

and equations (76) and (77) become

\[
2(\zeta - \rho) \gamma - \beta \gamma = -4 \ast 1900\lambda / \varepsilon + \beta \rho \tag{87}
\]

\[
2(12\rho \zeta - 8\zeta^2 + 21\rho^2) \gamma - 56\gamma^3 + \\
\beta(18\gamma^2 + 27\rho \zeta - 10\zeta^2 - 38\rho^2 - \beta^2) \\
= 2004 \ast 36\lambda / \varepsilon^2 \tag{88}
\]

Neglecting the \(\beta\) terms in the above two equations and the \(56\gamma^3\) term in the second assuming \(|\gamma| << 1\) gives

\[
(\zeta - \rho) \gamma = -2 \ast 1900\lambda / \varepsilon \tag{89}
\]

\[
(12\rho \zeta - 8\zeta^2 + 21\rho^2) \gamma = 2004 \ast 18\lambda / \varepsilon^2 \tag{90}
\]

For small \(\varepsilon\) the above equations give

\[
\zeta^2 - b\zeta + b\rho = 0 \tag{91}
\]

where \(b = 4509/3800\varepsilon\). Therefore, we have

\[
\zeta_{1,2} = b[1 \pm (1 - 4\rho / b)^{0.5}] / 2 \\
= b[1 \pm (1 - 2\rho / b - 2\rho^2 / b^2 + o(1\rho, b\rho)^3)] / 2 \tag{92}
\]

The \(\zeta_2\) is neglected and \(\zeta_1\) is given as

\[
\zeta_1 = \rho + \rho^2 / b + o(1\rho^3 / b^2) \tag{93}
\]

Substituting \(\zeta_1\) into equation (89) gives

\[
\gamma = -4509\lambda / (\varepsilon \rho)^2 \tag{94}
\]

In summary, the specific forms of the parameters are that \(\lambda = \lambda, \rho = \rho, \alpha = -2\rho, \beta = -\varepsilon \lambda \rho, \zeta = \rho + \rho^2 / b\) and \(\gamma = -4509\lambda / (\varepsilon \rho)^2\).

Next, we consider the limit cycles nested around the critical point \((1/\varepsilon \rho, 0)\). Again, by applying equation (21) it is obtained that

\[
(\lambda - 1900\lambda \alpha^2 + 2004\lambda \alpha^3) + [-2 + 4(\rho - 1503\lambda / (\varepsilon \rho)^2)\alpha - 49599\lambda / (\varepsilon \rho) a^2] = 0 
\]

Neglecting the first two terms in the first bracket leads to

\[
2 - 4(\rho - 1503\lambda / (\varepsilon \rho)^2)\alpha + \\
49599\lambda / (\varepsilon \rho) a^2 - 2004\lambda \alpha^3 = 0 \tag{96}
\]

Let

\[
\alpha = u / (\varepsilon \rho) \tag{97}
\]

\[
k = (\varepsilon \rho)^2 / \lambda \tag{98}
\]

Thus, equation (96) becomes

\[
u^2 - 24.75u^2 - (3 - k / (501\varepsilon))u - k / 1002 = 0 \tag{99}
\]

It can be shown that

\[
k / \varepsilon < 744.57 \tag{100}
\]

is the sufficient condition for equation (99) to have three real roots. Now, taking \(k / \varepsilon = 501\) into equation (99) leads to

\[
u^2 - 24.75u^2 - 2u + \varepsilon / 2 = 0 \tag{101}
\]

Three roots are found to be

\[
u_1 = \varepsilon / 4, \nu_2 = -0.0805, \nu_3 = 24.83 \tag{102}
\]
Therefore, we have
\[ a_1 = \frac{1}{(4\rho)}, \quad a_3 = -0.0805/(\varepsilon\rho), \]
\[ a_3 = 24.83/(\varepsilon\rho) \]  \hspace{1cm} (103)

The represents one limit cycle of finite amplitude and represent limit cycles with infinite amplitude, which are usually not considered to be limit cycles. Thus, we conclude there is one limit cycle nested around the critical point \((1/(\varepsilon\rho), 0)\).

In total, there are four limit cycles in the system (68) and (69). This result agrees with the Shi (1979).

4. Conclusion

The developed method works well for the estimation of limit cycles in all the planar polynomial vector fields considered in the paper including the well known shi’s second order system (Shi, 1979). The method is very general and can be comfortably extended to study high order and high dimensional systems. It is of considerable significance in both science and engineering.

References


Warf, S., 2004b. Towards the answer to the second part of Hilbert’s 16th problem, An invited lecture in the Fourth World Congress of Nonlinear Analysis, June 30 - July 7, Orlando, Florida, USA.