Active control of transient vibration of thin-walled composite beams

This item was submitted to Loughborough University’s Institutional Repository by the/an author.


Metadata Record: https://dspace.lboro.ac.uk/2134/16234

Version: Published

Publisher: © Multi-Science Publishing

Rights: This work is made available according to the conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) licence. Full details of this licence are available at: https://creativecommons.org/licenses/by-nc-nd/4.0/

Please cite the published version.
Active control of transient vibration of thin-walled composite beams

Simon S Wang, Xiaohu Wang and Joseph Loughlan

Department of Aeronautical and Automotive Engineering,
Loughborough University, LE11 3TU, England, UK.
E-mail: s.wang@lboro.ac.uk

(Received 26 February 2004; accepted 25 April 2004)

Abstract

This paper reports a further study on active control of transient vibration of thin-walled composite beams by using piezoelectric materials. A theorem of sensibility and three corollaries are proved to establish the absolute sensibility of modal vibration by distributed sensors. In the case of non-exact sensing, a sensing residual is defined to quantify the error in motion sensing. Distributed actuators are designed based on the principle of moment excitation. Actuating spill over is also defined to quantify the error in motion actuation. A theorem of sensing residual is discovered, which reveals the physical nature and the relationship between sensing residual and actuating spill over. Extensive applications are conducted to validate the developed theory.

Key words: Actuator, Composite control, Piezoelectric, Sensor, Vibration

1. Introduction

Owing to dynamic operation or external disturbance deployable space frame structures may undergo transient vibration, which adversely affects their normal functions. An active control strategy can be more attractive than a passive damping design when considering suppression of the vibration. This is particularly true when the frame structures are made of fibre reinforced composite materials which have relatively lower internal damping capability than metal materials. Extensive research work has been conducted in this area and comprehensive literature reviews can be found in references (Rao and Sunar, 1994; Crawley, 1994; Loewy, 1997; Chee et al., 1998). Following the earlier works (Bailey and Hubbard, 1985; Hubbard, 1986) in which polyvinylidene fluoride (PVDF) films are used as actuators to control beam vibrations, numerous studies on analytical (Crawley and de Luis, 1987; Baz and Poh, 1988; Tzou and Gadre, 1989; Crawley and Anderson, 1990; Lee, 1990; Lee and Moon, 1990; Lee et al., 1991; Wang and Rogers, 1991a, 1991b; Tzou and Tseng, 1991; Tzou and Fu, 1994) and numerical (Chandrashekhar, and Agarwal, 1993; Hwang and Park, 1993; Samanta, 1996) modelling have been reported of active vibration control of intelligent beams and plates with piezoelectric sensors and actuators. Among many others, several studies are briefly reviewed here. A selective modal transducer theory is developed for piezoelectric laminated composite
plates that are capable of sensing and exciting any specified set of vibration modes according to a specified set of modal participation factors (Miller et al., 1996a, 1996b). A three-dimensional theory of elasticity is adopted to study the vibration of rectangular laminated plates with embedded PZT layers (Batra and Liang, 1997). Solutions of close form are obtained. Recently, Sun et al. (1997) have reported a series of study on vibration control of smart beams and plates (Sun et al., 1999; 2001; 2002) using distributed piezoelectric elements. Sun et al. (1997) gave a brief study of vibration control of smart beams using a modal approach. In a latest study (Sun and Tong, 2002), Dongchang Sun and Liyong Tong have investigated the control stability of smart beams with debonded piezoelectric actuators. Also recently, Kim et al. (2001) report a study on the design of modal transducers by optimising spatial distribution of discrete gain weights. Saravanos and Christoforou (2002) gave a latest study on active control of dynamic response of adaptive piezoelectric laminated plates. An exact Ritz method is developed based on a mixed-field piezoelectric laminate theory that encompasses both displacement and electric displacement fields through the laminate.

This paper reports a further study on active control of transient vibration of thin-walled composite beams by using bonded piezoelectric sensor and actuator layers. Since motion sensing and actuating are two crucial operations in an active control process a theorem of sensibility and three corollaries are proved to establish the absolute determinacy of transient modal vibration using distributed piezoelectric sensors. It should be noted that such a theorem of sensibility has not been reported in previous work although distributed piezoelectric sensors were used to sense a motion. In the case of non-exact sensing a sensing residual is defined to measure the sensing error. Distributed actuators are designed based on the principle of moment excitation. Actuating spill over is also defined to measure the actuating error. A theorem of sensing residual is discovered, which reveals the physical nature of the sensing residual and the quantitative relationship between sensing residual and actuating spill over. The arrangement of the paper is as follows: Dynamic equations are presented in Section 2 of a thin-walled laminated composite beam bonded with piezoelectric layers. In Sections 3 and 4, theories on sensing, actuating and velocity feedback control are presented. A theorem of sensing residual is given in Section 5. Extensive applications are presented in Section 6. Conclusions are drawn in Section 7.

2. Dynamic equations

Thin-walled fibre reinforced composite beams are considered of uniform cross section. A Cartesian coordinate system is used in the analysis with \( \alpha \) being the longitudinal elastic centroidal axis and \( \alpha y \) and \( \alpha z \) the two principal axes. It is assumed that the transient vibration of the beam is in the plane \( \alpha x \alpha \), i.e. axis \( \alpha y \) is the neutral axis of the bending motion. To maximise their functions two piezoelectric layers are bonded onto the beam at the furthest distances from the neutral axis, one of which is the actuator layer and the other the sensor layer. It should also be noted that off angle fibre lay-ups are not present in the beam to avoid coupling between torsion and bending motions. Under applied voltage \( V(x,t) \) on the actuator the governing equation for flexural vibration can be written as

\[
\rho A \ddot{w}(x,t) + E I \dot{w}^{(4)}(x,t) = -e_{31}^a \overline{A}_a \dot{F}(x,t)
\]

(1)

Where \( e_{31}^a \) is the piezoelectric stress coefficient of the actuator, \( t_a \) the thickness, \( \overline{A}_a \) the first moment of area about the neutral axis \( \alpha y \), \( \rho A \) and \( E I \) have their usual meanings. It is noted that the internal material damping is not considered in eq.1. In modal expression the flexural motion \( w(x,t) \) takes the form

\[
w(x,t) = \sum_{j=1}^{n} W_j(t) \Psi_j(x)
\]

(2)

Where \( W_j(t) \) and \( \Psi_j(x) \) are the respective time domain modal coordinate and space domain modal shape function of the \( j \)th mode. In eq.2 it is assumed that the first \( n_a \) modes are present. By using eq.2 the modal form of eq.1 reads as

\[
\ddot{w}_j(t) + \omega_j^2 W_j(t) = -e_{31}^a \overline{A}_a / \]

\[( t_\alpha \rho A \int_0^l \Psi_j^2(x) \, dx \big) \int_0^1 \Psi_j^{(2)}(x,t) \, \Psi_j(x) \, dx \]  
\[\text{(3)}\]

Where
\[\omega_j = \left( \frac{EI}{\rho A} \right)^{(1/2)} k_j^2\]  
\[\text{(4)}\]
is the \(j\)th angular natural frequency, \(k_j\) the wave number and \(l\) the length of the beam.

3. Sensing of motion

A sensing theory is presented in this Section, which mainly concerns with the sensibility of the vibration represented by eq. 3. To realise sensing the sensor layer is divided into a number of electronically independent patches, each of which acts as a sensor. Due to the converse piezoelectric effect the charge output of each sensor is

\[q_i(t) = e_{31}^i \int_\varepsilon \varepsilon_n^i(x,t) \, dA \quad (i = 1, 2, \ldots, n_s)\]  
\[\text{(5)}\]

Where \(e_{31}^i\) is the piezoelectric stress coefficient of the sensor layer, \(A_i\) the mid-surface area of the sensor and \(\varepsilon_n^i(x,t)\) the normal strain in the \(x\)-direction within the \(i\)th sensor. It is also assumed that there are \(n_s\) sensors. By neglecting the axial extension part, \(\varepsilon_n^i(x,t)\) contains bending strain only, i.e.

\[\varepsilon_n^i(x,t) = -x_3 \omega^{(2)}(x,t)\]  
\[\text{(6)}\]

Where \(x_3\) is the \(z\) coordinate of the mid-surface of the sensor layer. Thus, by using eqs. 2 and 6, eq. 5 is rewritten in the following matrix form

\[\{ q(t) \}_{n_{s \times 1}} = [S]_{n_{s \times n_m}} \{ W(t) \}_{n_m \times 1}\]  
\[\text{(7)}\]

Each entry of the sensing matrix \([S]\) is defined as

\[S_{ij} = e_{31}^i \int_\varepsilon (-x_3) \Psi_j^{(2)}(x) \, dA\]  
\[\text{(8)}\]
representing the charge output of the \(i\)th sensor caused by the \(j\)th mode. The following theorem establishes the absolute determinacy of the modal vibration by using eq. 7.

**Theorem of sensibility**: The modal vibration of eq. 3 can always be exactly sensed by using uniformly distributed piezoelectric sensors which cover the whole length of the beam.

Proof: By pre-multiplying \([S]^T\) at both sides of eq. 7, it is obtained that

\[\{ q(t) \}_{n_{s \times 1}} = [S]_{n_{s \times n_m}} \{ W(t) \}_{n_m \times 1} = [S]_{n_{s \times n_m}} \{ q(t) \}_{n_{s \times 1}}\]  
\[\text{(9)}\]

By using eq. 8 the entries \((j, k) (j, k = 1, 2, \ldots, n_m)\) in the square matrix \([S]^T[S]\) can be expressed as

\[S_j^T S_k = (e_{31}^i \int_\varepsilon \varepsilon_n^i(x,t) \Psi_j^{(2)}(x) \, dx \int_\varepsilon \Psi_j^{(2)}(x) \, dx)\]  
\[\text{(10)}\]

Where \(S_j\) is the \(j\)th column of the sensing matrix and \(b\) the arc-width of the \(i\)th sensor. For any given small positive value of \(\varepsilon\), there always exists a number \(n_0 > n_m\), when \(n_s > n_0\), eq. 10 can be written as

\[e S_j^T S_k = (e_{31}^i \int_\varepsilon \varepsilon_n^i(x,t) \Psi_j^{(2)}(x) \, dx \int_\varepsilon \Psi_j^{(2)}(x) \, dx)\]

\[L \Psi_j^{(2)}(x_j) L_{\Psi} + \text{error}\]  
\[= L \Psi_j^{(2)}(x_j) + \text{error}\]  
\[= L \Psi_j^{(2)}(x_j) + \text{error}\]  
\[+ \text{error}\]  
\[\text{(11)}\]

Where \(L\) is the length of the beam, \(L_{\Psi}\) the length of the \(i\)th sensor and \(|\text{error}| < \varepsilon\). Equation 11 shows that the square matrix \([S]^T[S]\) approximates a diagonal matrix and hence its inverse exists. Partic Therefore, the modal coordinates can be found from eq. 9, i.e.

\[\{ W(t) \}_{n_m \times 1} = ([S]^T[S]_{n_{s \times n_m}}^{-1}) \{ q(t) \}_{n_{s \times 1}}\]  
\[\text{(12)}\]

The theorem is proved, from which following three corollaries are developed.

**Corollary 1**: The modal vibration of eq. 3 can always be exactly sensed by using \(n_m\) distributed
piezoelectric sensors.

Prove: since \([ S ]^T [ S ]\) in eq. 12 is not singular the rank of the sensing matrix \([ S ]\) must be \(n_m\). Therefore, there must exist at least one group of \(n_m\) rows in \([ S ]\), which forms a non-singular matrix \([ S_1 ]\). Thus, from a simpler sensing relationship

\[
\{ q_1(t) \}_{n \times 1} = [ S_1 ]_{n \times n} \{ W(t) \}_{n \times 1} \tag{13}
\]

the modal \(\{ w(t) \}\) coordinates is obtained as

\[
\{ W(t) \}_{n \times 1} = [ S_1 ]_{n \times n}^{-1} \{ q_1(t) \}_{n \times 1} \tag{14}
\]

The corollary 1 is proved.

**Corollary 2:** The modal coordinates given by eq. 14 are identical to that given by eq. 12.

Prove: By rearranging the order of the \(n_s\) sensors in eq. 7 the non-singular sensing matrix \([ S_1 ]\) can be the first \(n_m\) rows of \([ S ]\). Thus, eq. 12 becomes

\[
\{ W(t) \}_{n \times 1} = \begin{bmatrix} S_1^T S_1 + S_2^T S_2 \end{bmatrix}_{n \times n}^{-1} \begin{bmatrix} q_1(t) \\ S_2^T S_2 \end{bmatrix}_{n \times 1} \tag{15}
\]

By substituting \([ S_2 ] = [ S_1 ]^T [ T ]\) and \([ q_2 ] = [ S_2 ] [ W(t) ]\) into eq. 15, eq. 14 will be obtained where \([ T ]\) is the linear coefficient matrix of \([ S_2 ]\) when expressed in terms of \([ S_1 ]\). Therefore, the corollary 2 is proved.

Due to practical restrictions the design of sensing system with many small sensors implied by eq. 7 may not be an optimum choice in reality. Sensors of large size should be considered. Particularly we consider the design of \(n_m\) large sensors covering the whole length of the beam. In fact these large sensors can be regarded as the combination of the small ones used in eq. 7 and the sensing relationship changes to be

\[
\{ Q(t) \}_{n \times 1} = [ S ]_{n \times n} \{ W(t) \}_{n \times 1} \tag{16}
\]

When \([ S ]\) in eq. 16 is non-singular the modal coordinates can be sensed, i.e.

\[
\{ W(t) \}_{n \times 1} = [ S ]^{-1}_{n \times n} \{ Q(t) \}_{n \times 1} \tag{17}
\]

**Corollary 3:** The modal coordinates given by eq. 17 are identical to that given by eq. 12.

Prove: Substituting eq. 17 into eq. 7 gives

\[
\{ q(t) \}_{n \times 1} = [ S ]_{n \times n} \{ S \}_{n \times n}^{-1} \{ Q(t) \}_{n \times 1} \tag{18}
\]

and the use of eqs. 18 and 12 gives eq. 17. This proves corollary 3.

It should be noted that in practice the number of sensors can be less than the number of vibration modes, i.e. \(n_s < n_m\). In this case modal coordinates can only be sensed in an approximate manner. In this study, it is assumed that it is the first \(n_s\) modes that are sensed. Therefore, from eq. 17

\[
\{ W^*(t) \}_{n \times 1} = [ S_3 ]_{n \times n}^{-1} \{ Q(t) \}_{n \times 1} \tag{19}
\]

Where \( W^*(t)\) represents the sensed first \(n_s\) modal coordinates and \([ S_3 ]\) the first \(n_s\) column of the sensing matrix \([ S ]\). Substituting eq. 7 into eq. 19 gives

\[
\{ W^*(t) \}_{n \times 1} = \{ W(t) \}_{n \times 1} - [ R ]_{n \times n} \{ W(t) \}_{n \times 1} \tag{20}
\]

Where \(n_r = n_m - n_s\) is the number of residual modes and

\[
[R]_{n \times n} = [ S_3 ]_{n \times n}^{-1} [ S_4 ]_{n \times n} \tag{21}
\]

representing the residual sensing matrix. It is clear that the second term on the right hand side of eq. 20 represents the sensing error termed as sensing residual here.

4. Actuating of motion and control equation

Under applied voltage \(V(x, t)\) the moment curvature relationship of the vibrating beam is
\[
\bar{E}Iw^{(2)}(x, t) = -M(x, t) - e_{31}^a \bar{A}t_aV(x, t)
\]

(22)

Where \( M(x, t) \) is the bending moment which varies linearly along the beam and vanishes in the case of statically determinate beams, such as cantilevers, simply supported beams. It is clear from eq. 22 that the applied voltage \( V(x, t) \) acts like a distributed bending moment. Therefore, it is designed to take the form (Sun et al., 1997)

\[
V(x, t) = - \sum_{k=1}^{n_c} \sum_{k=1}^{n_c} p_k(t) \bar{E} \bar{A} \Psi_k^{(2)}(x)
= \sum_{k=1}^{n_c} \sum_{k=1}^{n_c} p_k(t) M_k(x)
\]

(23)

Where \( n_c \) is the number of modes to be controlled, \( M_k(x) \) the \( k \)th modal control moment and \( p_k(t) \) termed the \( k \)th ideal modal control force. Due to difficulties involved in the application of continuous voltage the actuator layer is divided into a number of electronically independent patches, each of which acts as an actuator. The voltage on each actuator is designed to be the average of that given in eq. 23, that is

\[
V_i(t) = L_{\alpha_i}^{-1} \int_{\alpha_i} V(x, t) \, dx \quad (i = 1, 2, \ldots, n_a)
\]

(24)

Where \( L_{\alpha_i} \) is the length of the \( i \)th actuator and \( n_a \) the number of actuators. Substituting eq. 23 into eq. 24 gives

\[
\left| V(t) \right|_{n_a \times 1} = \left[ L \right]_{n_a \times n_a} \left[ M \right]_{n_a \times 1} \left| p(t) \right|_{n_a \times 1}
\]

(25)

Where \( \left[ L \right] \) is a diagonal matrix of actuator length and each entry of the actuating matrix \( \left[ M \right] \) is defined as

\[
M_k = \int_{\alpha_i} M_k(x) \, dx
\]

(26)

representing the actuating moment of the \( i \)th actuator designed to control the \( k \)th mode. When this stepwise voltage \( \tilde{V}(x, t) \) is applied on the actuators other vibration modes than the \( n_c \) designed controlling modes can be excited. This actuating spill over needs to be considered. To this aim the modal series expansion of \( \tilde{V}(x, t) \) is developed as

\[
\tilde{V}(x, t) = \sum_{j \geq 1} P_j(t) M_j(x)
\]

(27)

It should be noted that \( P_j(t) \) here is the \( j \)th real modal control force when \( \tilde{V}(x, t) \) is applied. Also, it is assumed that the number of spill over modes is \( n_\psi = n_m - n_c \). Using eqs. 25 and 27 and orthogonality of modal functions, the relationship between \( P_j(t) \) and \( p_j(t) \) is found to be

\[
\left[ P(t) \right]_{n_\psi \times 1} = \left[ \alpha \right]_{n_m \times n_\psi} \left[ M \right]_{n_m \times n_m} \left[ L \right]_{n_m \times n_m} \left[ M \right]_{n_m \times n_m} \left[ p(t) \right]_{n_m \times 1}
\]

(28)

Where \( \left[ \alpha \right] \) is a diagonal matrix with

\[
a_j = \int_{\alpha} M_j^2(x) \, dx
\]

(29)

Assuming that a negative velocity feedback control loop is designed, the ideal modal control force is then

\[
\left| p(t) \right|_{n_\psi \times 1} = -\left[ G \right]_{n_m \times n_\psi} \left| \tilde{W}^* (t) \right|_{n_\psi \times 1}
\]

(30)

Where \( \left[ G \right] \) is the gain matrix. Therefore, using eqs. 20 and 30 the real modal control force in eq. 28 becomes

\[
\left| P(t) \right|_{n_\psi \times 1} = -\left[ \alpha \right]_{n_m \times n_\psi} \left[ M \right]_{n_m \times n_m} \left[ L \right]_{n_m \times n_m} \left[ M \right]_{n_m \times n_m} \left[ G \right]_{n_m \times n_m} \left[ I, R \right]_{n_m \times n_m} \left| \tilde{W} (t) \right|_{n_\psi \times 1}
\]

(31)

Finally, the close loop modal control equation is obtained by substituting eqs. 27 and 31 into eq. 3. It is

\[
\left\{ \tilde{W}^{\delta} (t) \right\} + [C] \left[ \tilde{W} (t) \right] + [\Omega] \left[ W (t) \right] = [0]
\]

(32)

Where \( \left[ \Omega \right] \) is the diagonal matrix of the square of angular natural frequencies and the active damping matrix \( [C] \) is found to be
\[
[C]_{n \times n} = e^{\eta_1 \mathbf{A}^{(1)}_{a \alpha}} \left[ \Omega \right]_{n \times n} \left[ \alpha \right]_{n \times n}^{(-1)}
\]
\[
\left[ M \right]_{n \times n} \left[ L \right]_{n \times n}^{(-1)} \left[ M \right]_{n \times n}
\]
\[
\left[ G \right]_{n \times n} \left[ I, R \right]_{n \times n}
\]
(33)

5. Sensing residual and actuating spill over

In the case of exact sensing or complete filtering the sensing residual matrix \([ R ]\) degenerates to a null matrix and the real modal control force in eq. 31 becomes

\[
\left[ P(t) \right]_{n \times 1} = -\left[ \alpha \right]^{(-1)}_{n \times n} \left[ M \right]_{n \times n} \left[ L \right]^{(-1)}_{n \times n} \left[ M \right]_{n \times n}
\]
\[
\left[ G \right]_{n \times n} \left[ \dot{W}(t) \right]_{n \times 1}
\]
(34)

It is seen from eq. 34 that there is no coupling between the spill over and controlling modes. Moreover, as long as a positive definite gain matrix \([ G ]\) is selected the real modal control forces do not provide any negative damping to the \(n_c\) controlling modes. Of course, \(n_{sp}\) spill over modes will be excited. However, in the case of non-exact sensing or non-complete filtering it is seen from eq. 31 that the sensing residual couples the spill over and controlling modes together. To reveal the nature of the sensing residual a special situation is considered in which \(n_a = n_s = n_c\), \(L_1 = L_2\), and \([ G ] = g[I]\). The real modal control force in eq. 31 becomes

\[
\left[ P(t) \right]_{n \times 1} = -g[\alpha]^{(-1)}_{n \times n} \left[ M \right]_{n \times n} \left[ L \right]^{(-1)}_{n \times n}
\]
\[
\left[ M \right]_{n \times n} \left[ S \right]^{(-1)}_{n \times n} \left[ S \right]_{n \times n}
\]
\[
\left\{ \dot{W}(t) \right\}_{n \times 1}
\]
\[
= -g[\alpha]^{(-1)}_{n \times n} \left[ M \right]_{n \times n} \left[ L \right]^{(-1)}_{n \times n}
\]

Now, it becomes evident that the real modal control forces in eq. 35 are identical to that in eq. 34 when \(n_c = n_m\) and \([ G ] = g[I]\) in the exact sensing case. Therefore, the conclusion can be drawn as follows.

**Theorem of sensing residual:** The nature of sensing residual does not destabilise the controlling process by providing negative damping to the controlling modes. Instead, its effect is equivalent to increase the number of controlling modes to \(n_m\) as if in an exact sensing case and consequently results in high actuating voltage requirement.

In the next Section extensive applications are conducted to validate the theory described above.

6. Applications

The first application example concerns with the control of the transient vibration of a cantilever steel beam with two bonded PVDF layers as respective sensor and actuator as shown in Figure 1. The physical properties of the materials and dimensions of the beam are recorded in Table 1. It is assumed that the vibration of the beam is excited by an impulse \(I_0 = 5 \times 10^{-4} Ns\) at the midpoint of \(x = 200 mm\) and at time \(t = 0\). The first ten modes are considered to be significant and their harmonic modal time histories within the initial 0. 3s are shown in Figure 2. The corresponding tip displacement is shown in Figure 3. Active control starts at time \(t = 0.3 s\). Three control strategies are employed to test the theory presented in previous Sections, in which sensors and actuators of uniform size are used.

![Figure 1. A steel beam with bonded PVDF layers.](image-url)
Fig. 2. Initial first ten transient modal histories of the steel beam.

Fig. 3. Initial transient tip response of the steel beam.

Table 1. Material properties and dimensions of the steel beam.

<table>
<thead>
<tr>
<th>Materials</th>
<th>( l ) mm</th>
<th>( h ) mm</th>
<th>( E ) GPa</th>
<th>( \rho ) kg/m³</th>
<th>( \varepsilon_{31} ) N/Vm</th>
</tr>
</thead>
<tbody>
<tr>
<td>PVDF</td>
<td>400</td>
<td>0.3</td>
<td>2</td>
<td>1780</td>
<td>0.06</td>
</tr>
<tr>
<td>Steel</td>
<td>400</td>
<td>1</td>
<td>210</td>
<td>8000</td>
<td>-</td>
</tr>
</tbody>
</table>
6.1. Exact sensing and selected control

It is noted that this control strategy is equivalent to the strategy of non-exact sensing with filtering. The purpose of this exercise is to demonstrate the controllability of selected modes and spill over effect as suggested by eq. 34. Since the first ten modes are considered to be significant, i.e. \( n_m = 10 \), for exact sensing the number of sensors needs to be no less than ten, i.e. \( n_s \geq 10 \) as suggested by the theorem of sensibility and the corollary 1. Here, ten sensors are used, i.e. \( n_s = 10 \). In the first case designated as a1 ore actuator is used, i.e. \( n_a = 1 \) and the first mode is selected to be controlled, i.e. \( n_c = 1 \). The feedback gain is taken to be 1200, i.e. \( g_{11} = 1200 \). Figure 4 shows the ten modal time histories. It is observed that the selected first mode is actively damped away and the rest are not controlled. Also, the spill over effect is obvious when comparing Figure 4 with Figure 2 which shows the harmonic motion. Figure 5 shows the applied voltage on the actuator, which resembles the variation of the first mode as expected. To strengthen the validation a second case designated as a2 is similarly studied in which seven actuators are used and the first seven modes are selected to be controlled, i.e. \( n_a = n_c = 7 \). Feedback gains are taken to be \( g_{11} = 1200 \), \( g_{22} = 100 \), \( g_{33} = 70 \), \( g_{44} = g_{55} = g_{66} = g_{77} = 5 \) and \( g_{ij} = 0 \, (i \neq j) \). Figure 6 shows the ten modal time histories. Again, it is seen that the selected first seven modes are actively suppressed and the rest still persists. The spill over effect is evident. Figure 7 shows the applied voltage on the seven actuators. As anticipated, it contains a major contribution from the most pronounced first mode whilst retaining the signatures of the rest.

6.2. Exact sensing and complete control

This exercise is the direct extension of exercise (a) to demonstrate the controllability of all modes of the cantilever beam using different number of actuators. Two cases considered here are designated as b1 and b2 respectively and are the same as the respective a1 and a2 cases except that all the ten modes are selected to be controlled here, i.e. \( n_c = 10 \). In the b1 case, the feedback gains are taken to be \( g_{11} = 1800 \), \( g_{22} = 300 \), \( g_{33} = 100 \), \( g_{44} = 50 \), \( g_{55} = 20 \), \( g_{66} = 20 \), \( g_{77} = g_{88} = g_{99} = g_{10} \), and \( g_{ij} = 0 \, (i \neq j) \). Figure 8 shows the gradually contained ten vibration modes. The applied voltage on the actuator is shown in Figure 9. It is observed that higher voltage is required here to suppress all the modes compared with that in Fi-

![Fig. 4. Real time modal histories in the a1 case (n_s = n_a = 1, n_c = n_m = 10)](image-url)
Fig. 5. Real time applied voltage in the a1 case ($n_a = n_c = 1, n_e = n_m = 10$).

Fig. 6. Real time modal histories in the a2 case ($n_a = n_c = 7, n_e = n_m = 10$).
Fig. 7. Real time applied voltages in the a2 case \((n_x = n_c = 7, n_s = n_m = 10)\).

Fig. 8. Real time modal histories in the b1 case \((n_x = 1, n_c = n_s = n_m = 10)\).
Figure 5 where only one mode is under control. Moreover, the high frequency fluctuation of the voltage shows the participation of all modes. In the b2 case, the feedback gains are taken to be \( g_{11} = 1200, g_{22} = 100, g_{33} = 70, g_{44} = g_{55} = g_{66} = g_{77} = g_{88} = g_{99} = g_{10} = 5 \) and \( g_{ij} = 0(i \neq j) \). Figure 10 shows that all the ten modes are nicely controlled. The applied voltages on the seven actuators are shown in Figure 11. By comparing Figure 10 and Figure 8 it can be observed that it is quicker to suppress vibration by using more actuators.

![Graph](image1)

**Fig. 9.** Real time applied voltage in the b1 case \((n_a = 1, n_c = n_s = n_m = 10)\).

![Graph](image2)

**Fig. 10.** Real time modal histories in the b2 case \((n_a = 7, n_c = n_s = n_m = 10)\).
6.3. Non-exact sensing with no filtering

The purpose of this exercise is to demonstrate the effect of sensing residual as discussed in Section 5. Two cases considered here are designated as c1 and c2 respectively and again the same as the respective a1 and a2 cases except that the number of sensors used here are one and seven, respectively. Therefore, sensing residuals exist. Feedback gain \( g_{11} \) is taken to be 1200 in c1 case, which is the same as that in a1 case. Figure 12 shows the ten modal time histories and Figure 13 shows the voltage applied on the actuator. It can be seen that the ten modes are more quickly controlled compared with the b1 case. This is expected since the c1 case is identical to the b1 case when uniform gains are used, i.e. \( g_{ii} = 1200, g_{ij} = 0(i \neq j), i, j = 1, 2, \ldots, 10 \), as stated by the theorem of sensing residual in Section 5. However, it should be noted from Figure 13 that extremely high voltage is required. In the c2 case the feedback gains are taken to be the same as that in a2 case, i.e. \( g_{11} = 1200, g_{22} = 100, g_{33} = 70, g_{44} = g_{55} = g_{66} = g_{77} = 5 \) and \( g_{ij} = 0(i \neq j) \). Figures 14 and 15 show the modal time histories and the applied voltages, respectively. It is seen from Figures 14 and 10 that the modal time histories in c2 and b2 cases are quite similar. However, the sensing residual in c2 case results in high voltage requirement as shown in Figure 15 as again suggested by the theorem of sensing residual.

The second application example concerns with a cantilever composite beam as shown in Figure 16. Two PVDF layers are bonded at the top and bottom surfaces as actuator and sensor layers. The physical properties and dimensions of the composite beam are summarised in Table 2. Again, it is assumed that the beam is excited by an impulse \( I_0 = 5 \times 10^{-4} \text{ Ns} \) at the midpoint of \( x = 200 \text{mm} \). The first ten modes are considered to be significant, i.e. \( n_s = 10 \) and exact sensing is achieved by using ten sensors, i.e. \( n_s = 10 \). Two actuators are utilised to control the ten modes, i.e. \( n_a = 2 \) and \( n_e = 10 \). The feedback gains are taken to be \( g_{ii} = 600, g_{ij} = 0(i \neq j) \) and \( i, j = 1, 2, \ldots, 10 \). Control starts at \( t = 0.3 \) s after the impulse and the time histories of the first ten modal time histories are shown in Figure 17. It can be seen that the transient vibration is nicely damped away. The applied voltages on the two actuators
are shown in Figure 18. The voltage on the second actuator is high due to the fact that only two actu-
ators are utilised to control all the ten modes.

Fig. 12. Real time modal histories in the c1 case \((n_k = 1, n_e = n_s = 1, n_m = 10)\).

Fig. 13. Real time applied voltages in the c1 case \((n_k = 1, n_e = n_s = 1, n_m = 10)\).
Fig. 14. Real time modal histories in the c2 case \((n_a = 7, n_c = n_b = 7, n_m = 10)\).

Fig. 15. Real time applied voltages in the c2 case \((n_a = 7, n_c = n_b = 7, n_m = 10)\).
Fig. 16. A composite beam with bonded PVDF layers.

Fig. 17. Real time modal histories in the composite beam case \( (n_2 = 2, n_3 = n_4 = n_5 = 10) \).

Table 2
Material properties and dimensions of the composite beam.

<table>
<thead>
<tr>
<th>Materials</th>
<th>( l ) mm</th>
<th>( h ) mm</th>
<th>( E ) GPa</th>
<th>( \rho ) kg/m³</th>
<th>( e_{51} ) N/mm²</th>
</tr>
</thead>
<tbody>
<tr>
<td>PVDF</td>
<td>400</td>
<td>0.3</td>
<td>2</td>
<td>1780</td>
<td>0.06</td>
</tr>
<tr>
<td>layer 3</td>
<td>400</td>
<td>0.3</td>
<td>2.5</td>
<td>1200</td>
<td>-</td>
</tr>
<tr>
<td>layer 2</td>
<td>400</td>
<td>0.4</td>
<td>3.0</td>
<td>1600</td>
<td>-</td>
</tr>
<tr>
<td>layer 1</td>
<td>400</td>
<td>0.3</td>
<td>2.5</td>
<td>1200</td>
<td>-</td>
</tr>
</tbody>
</table>

7. Conclusions

A theorem of sensibility is rigorously proved to show that the transient vibration of thin-walled laminated composite beams can always be exactly sensed by using distributed piezoelectric sensors which may either cover the whole length of the beam or properly locate at discrete positions. A theorem of sensing residual is discovered. The sensing residual in non-exact sensing couples controlling and spill over modes together. It does not destabilise the designed controlling modes by providing negative damping and its effect is equivalent to control all the modes as if in the case of
exact sensing and consequently results in high voltage requirement. A series of application examples are presented to validate the theory. Although only single beams are considered in this paper the presented theory has a wide general application.

References


