Higher-order airy functions of the first kind and spectral properties of the massless relativistic quartic anharmonic oscillator

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HIGHER-ORDER AIRY FUNCTIONS OF THE FIRST KIND AND SPECTRAL PROPERTIES OF THE MASSLESS RELATIVISTIC QUARTIC ANHARMONIC OSCILLATOR

Samuel Ogechukwu DURUGO

A Doctoral Thesis

submitted in partial fulfilment of the requirements for the award of the degree of
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of

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This thesis consists of two parts. In the first part, we study a class of special functions $A_{ik}(y)$, $k = 2, 4, 6, \cdots$ generalising the classical Airy function $Ai(y)$ to higher orders and in the second part, we apply expressions and properties of $Ai_4(y)$ to spectral problem of a specific operator. The first part is however motivated by latter part.

We establish regularity properties of $A_{ik}(y)$ and particularly show that $A_{ik}(y)$ is smooth, bounded, and extends to the complex plane as an entire function, and obtain pointwise bounds on $A_{ik}(y)$ for all $k$. Some analytic properties of $A_{ik}(y)$ are also derived allowing one to express $A_{ik}(y)$ as a finite sum of certain generalised hypergeometric functions. We further obtain full asymptotic expansions of $A_{ik}(y)$ and their first derivative $A_{ik}'(y)$ both for $y > 0$ and for $y < 0$. Using these expansions, we derive expressions for the negative real zeroes of $A_{ik}(y)$ and $A_{ik}'(y)$.

Using expressions and properties of $Ai_4(y)$, we extensively study spectral properties of a non-local operator $H$ whose physical interpretation is the massless relativistic quartic anharmonic oscillator in one dimension. Various spectral results for $H$ are derived including estimates of eigenvalues, spectral gaps and trace formula, and a Weyl-type asymptotic relation. We study asymptotic behaviour, analyticity, and uniform boundedness properties of the eigenfunctions $\psi_n(x)$ of $H$. The Fourier transforms of these eigenfunctions are expressed in two terms, one involving $Ai_4(y)$ and another term derived from $Ai_4(y)$ denoted by $\widetilde{Ai}_4(y)$. By investigating the small effect generated by $\widetilde{Ai}_4(y)$ this work shows that eigenvalues $\lambda_n$ of $H$ are exponentially close, with increasing $n \in \mathbb{N}$, to the negative real zeroes of $Ai_4(y)$ and those of its first derivative $Ai'_4(y)$ arranged in alternating and increasing order of magnitude. The eigenfunctions $\psi_n(x)$ are also shown to be exponentially well-approximated by the inverse Fourier transform of $Ai_4(|y| - \lambda_n)$ in its normalised form.
To Benedicta Akuezi Durugo, my beloved mother, for her endless love, support and encouragement
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Therefore, the flight shall perish from the swift, and the strong shall not strengthen his force, neither shall the mighty deliver himself: neither shall he stand that handles the bow; and he that is swift of foot shall not deliver himself: neither shall he that rides the horse deliver himself.

— Amos 2:14-15 (KJV).

I returned, and saw under the sun, that the race is not to the swift, nor battle to the strong, neither yet bread to the wise, nor riches to men of understanding, nor yet favour to men of skill; but time and chance happen to them all.

— Ecclesiastes 9:11 (KJV).

Special thanks to almighty God – the one and only true God and most merciful father; with whom there is no variableness. His banner over me is love! He preserves my soul from they that are too strong for me, and though I walk through the most gruesome thorns of life, He lay me down in green pastures.

In the course of studying for my PhD, I have enjoyed a tuition-only research studentship from Loughborough University without which this work would not have been completed. Many have trod this academic journey of life for ages yet it was very rewarding studying under Dr. József Lörinczi as every step was a painstaking, rigorous, and daunting task. I say a BIG thank you to every member of staff of the School of Mathematical Sciences, Loughborough University for their help, no matter how little, while this piece of work was being developed.

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My family stood by me all the way encouraging and providing me with pieces of advice from time to time. I have also benefitted from their continuous prayers and love.

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October, 2014                        S. O. D.
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CHAPTER 1

Introduction

1.1. Background review and motivation

This work is concerned with special functions related to a family of non-local (pseudo-differential) operators in one dimension of the form

\[ H_k := \sqrt{-\frac{d^2}{dx^2}} + x^k \quad (k = 2, 4, 6, \cdots) \]  

acting on \( L^2(\mathbb{R}) \). These operators are specific cases of the one-dimensional massless relativistic Schrödinger operators with potentials \( V(x) = x^k \). They have the physical interpretation of a massless relativistic quantum harmonic oscillator when exponent \( k = 2 \) and massless relativistic quantum anharmonic oscillators when \( k \geq 4 \). In probability, \( H_k \) is the generator of a Cauchy process under the influence of the potential \( V(x) = x^k \).

Under Fourier transform, (1.1) translates into a \( k \)th-order differential operator \( \tilde{H}_k \) given by

\[ \tilde{H}_k = \gamma_k \frac{d^k}{dy^k} + |y| \quad (\gamma_k := (-1)^{k/2}). \]  

For \( k = 2 \), (1.2) is associated with the Schrödinger’s equation for a particle confined within a triangular potential well in one dimension whose solution is expressed in terms of the Airy function (historically denoted as \( \text{Ai}(y) \)):

\[ \text{Ai}(y) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{t^3}{3} + yt \right) \, dt \quad (y \in \mathbb{R}). \]  

(1.3)

But for \( k \geq 4 \), (1.2) is now associated with certain \( k \)th-order differential equations whose solutions we found in terms of \( \text{Ai}_k(y) \) formally defined as

\[ \text{Ai}_k(y) = \frac{1}{2\pi i} \int_{\mathcal{G}} e^{-\gamma_k s - \frac{k+1}{s+1}} \, ds \]  

where \( \mathcal{G} \) is any contour extending from \( \infty e^{-\frac{k\pi i}{2k+1}} \) to \( \infty e^{\frac{k\pi i}{2k+1}} \) in the complex \( s \)-plane cutting through the negative real semi-axis. The integral is convergent due to the decay behaviour of \( e^{-\frac{k+1}{s+1}} \) at the extreme points \( \infty e^{-\frac{k\pi i}{2k+1}} \) and \( \infty e^{\frac{k\pi i}{2k+1}} \) of the contour \( \mathcal{G} \). When \( k = 2 \), (1.4) reduces to (1.3). Hence, we name the function \( \text{Ai}_k(y) \) (for each \( k = 2, 4, 6, \cdots \)) \( k \)th-order Airy function of the first kind. Integrals of the form (1.4) are little known in the literature. Therefore, our first goal in this work is to contribute to filling
1.1. BACKGROUND REVIEW AND MOTIVATION

in this gap. Amongst other things, we study regularity and specific analytic properties of the class of functions $A_{ik}(y)$ and those of their first derivatives $A_{ik}'(y)$.

The Airy function of the first kind $Ai(y)$ is a special function that has been widely utilised and well documented [1, 28, 38, 59, 61, 81]. The function is named after the British astronomer and mathematician, Sir George Biddell Airy (1801 - 1892), who discovered this function in a bid to study light intensity in the neighbourhood of a caustic [3]. The function $Ai(y)$ together with a related function $Bi(y)$ called the Airy function of the second kind (or Bairy function as it is informally called) and formally defined by

$$Bi(y) = \frac{1}{\pi} \int_{0}^{\infty} \left[ e^{yt - \frac{t^3}{3}} + \sin \left( \frac{t^3}{3} + yt \right) \right] dt \quad (y \in \mathbb{R}),$$

forms a fundamental set of solutions to the differential equation

$$\frac{d^2}{dy^2} \chi(y) - y \chi(y) = 0 \quad (1.5)$$

known as the Airy equation (or the Stokes equation). It is the simplest second-order linear ordinary differential equation (ODE) that admits solutions that change their behaviour from oscillatory to exponential — a behavioural change often referred to as the ‘Stokes phenomenon’, and used in the mathematics literature to describe a differential equation with a turning point [82] Chap. VIII. The integral (1.3) defining $Ai(y)$ is conditionally convergent but absolutely convergent if one takes analytic continuation into the upper half region of the complex plane. Along $y > 0$, $Ai(y)$ is positive, convex, and has an exponential decay but exhibits a damp oscillation (i.e., ever-increasing frequency and ever-decreasing amplitude) along $y < 0$. The function $Ai(y)$ has the following series expression:

$$Ai(y) = \frac{1}{3^{2/3} \Gamma \left( \frac{2}{3} \right)} \sum_{l=0}^{\infty} \frac{1}{l! \left( \frac{4}{3} \right)_l} \left( \frac{y^3}{9} \right)^l - \frac{y}{3^{1/3} \Gamma \left( \frac{1}{3} \right)} \sum_{l=0}^{\infty} \frac{1}{l! \left( \frac{4}{3} \right)_l} \left( \frac{y^3}{9} \right)^l \quad (1.6)$$

with the notation $(\beta)_l = \Gamma(\beta + l)/\Gamma(\beta)$ for $\beta \in \mathbb{C}$ and $l \in \mathbb{N}$, which is valid for $-\infty < y < \infty$ indicating that $Ai(y)$ has a radius of convergence of $\infty$, and extends to an entire function on $\mathbb{C}$. Many more properties of $Ai(y)$ as well as those of $Bi(y)$ including their asymptotic expansions, relationships with other (special) functions, and details about the distribution and computations of their zeroes are extensively covered in [1, 61, 81].

The Airy function $Ai(y)$ is also identified as the solution to Schrödinger’s equation for a particle moving in a one-dimensional homogeneous gravitational (or electrical) force field [30] Vol. I, pp. 101 - 107]. This has also further motivated its use in providing uniform semiclassical approximations near a turning point in the WKB method of approximations [30] Vol. I, Chap. II.E]. Recently, $Ai(y)$ has found applications in studies of spectral properties of the massless relativistic harmonic oscillator.
1.1. BACKGROUND REVIEW AND MOTIVATION

Also, Ai(y) has recently been extensively applied in many studies on determinantal point processes (DPPs) arising mainly in random matrix theory (especially in the GUE and/or GOE-random matrix models) \[12, 79\]. Also, Ai(y) has recently been extensively applied in many studies on determinantal point processes (DPPs) arising mainly in random matrix theory (especially in the GUE and/or GOE-random matrix models) \[12, 79\], quantum mechanics of interacting particles \[2, 42\], and machine learning \[52\]. The Airy function Ai(y) has also appeared in a number of studies on polynuclear growth (PNG) models \[43, 64, 65\]. Another area that has employed the Airy function is the totally asymmetric simple exclusion processes (TASEPs) \[13, 29\].

Following the study by G.B. Airy in \[3\], there have been many studies focusing especially on the general behaviour of solutions to the ODE (1.5) with various extensions to higher order cases; see for instance, \[58\] for a historical survey of these studies. In \[37, 48, 63, 80\], results concerning the asymptotic behaviour and Stokes phenomenon of solutions to the ODE in higher orders were obtained. While these studies have presented remarkable results about the nature and behaviour of the solutions considered for the ODE in higher orders, no specific attempts were made to obtain expressions in a form similar to Ai_k(y) or study integrals of the form given specifically by (1.3). Furthermore, by exploring Ai_k(y) more extensively, applications in recent areas such as the determinantal point processes extended to higher-order cases may be of interest. To the best of our knowledge, higher-order Airy functions Ai_k(y) have never been used in the context of studies of spectral problems of non-local operators as Ai(y) has been used.

Recently, there is a growing interest in the spectral theory of pseudo-differential operators (with the non-local operators class (1.1) as a specific case) and in related non-local boundary-value problems. Due to the multiple roles of operators such as (1.1) and many others of interest, there are strong connections and a rich interplay between their properties studied in potential theory \[10, 17, 33, 44\], stochastic processes \[4, 11, 70\], pseudo-differential operators \[39, 44, 72, 77\] and mathematical physics \[15, 40\]. Although there is an increasing literature in this regard, there is a shortage of detailed understanding of examples that can stand as benchmark cases to the general theory. Therefore, our second goal in this work is to contribute to filling in this gap by providing a concrete example and studying this in great detail.

The potential \(V(x) = x^k\) (k even and positive) is non-negative and divergent at \(\pm \infty\). For this class of potentials, operators \(T_{k,t} := e^{-tH_k}\) for all \(t > 0\) are compact so that the essential spectrum \(\sigma_{ess}(H_k)\) of \(H_k\) is empty. Hence, \(H_k\) possesses purely discrete spectrum and there exists an infinite sequence of functions \(\{\psi_n : n \in \mathbb{N}\} \subseteq L^2(\mathbb{R})\) as eigenfunctions of \(H_k\), which are continuous and bounded such that \(T_{k,t}\psi_n = e^{-t\lambda_n}\psi_n\), where \(0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty\). We wish to derive various spectral and
analytic results for the perturbed operators (1.1). These include, but are not limited to, estimates of eigenvalues, spectral gap, and expressions and properties of eigenfunctions.

Studying $H_k$ in complete generality (i.e. any $k$) poses much computational difficulty. In this thesis, we limit ourselves to $k = 4$ from which the difficulties of extending even to $k = 6$ will be seen. However, while the finite $k > 6$ cases appear to be difficult to handle, the $k \to \infty$ case is more manageable — this problem is related to the killed Cauchy process, which has been studied by other authors [51] using other methods and obtaining various approximations of eigenvalues and eigenfunctions — our approach based on the $k \to \infty$ limit provides a new way, which is outside the scope of this thesis and will be further explored elsewhere.

The quadratic case (when $k = 2$) has been investigated in [57]. There, the authors analysed spectral properties of the operator and established a number of analytic properties. In fact, spectral properties of the non-local operator of the quadratic harmonic oscillator were studied from the solutions of the eigenproblem of the local Schrödinger operator $H := -\frac{d^2}{dx^2} + |y|$ obtained by using Fourier transform to turn the eigenproblem of the non-local operator into an ODE problem, which will be the main step in the second part of this thesis too. A particularity of $k = 2$ is that the so obtained ODE is a Schrödinger equation related to the Airy function $\text{Ai}(y)$, and this observation allowed a closed-form solution of the eigenproblem of a non-local Schrödinger operator and a detailed investigation of the spectrum and properties of the eigenfunctions.

In particular, in [57], full asymptotic expansions for all the eigenfunctions $\psi_n(x)$ for large argument $x$ were obtained showing leading terms in the order of $x^{-4}$ for $n = 2j - 1$, and an improvement to $x^{-5}$ for $n = 2j$, for each $j \in \mathbb{N}$. Specifically, the asymptotics of the ground state was obtained as follows:

$$\psi_1(x) = \sqrt{\frac{2}{-a_1'}} \left( \frac{p_3(a_1')}{x^4} - \frac{p_5(a_1')}{x^6} + \cdots + (-1)^N \frac{p_{2N-1}(a_1')}{x^{2N}} \right) + O\left( \frac{1}{x^{2(N+1)}} \right),$$

where $a_1' \approx -1.01879$ denotes the first negative real zero of the derivative of the Airy function $\text{Ai}(y)$, and $p_l$ denotes an $l$-th order polynomial defined recursively with another $l$-th order polynomial $q_l$ as follows: $p_{l+1}(x) = p_l'(x) + xq_l(x)$ and $q_{l+1}(x) = p_l(x) + q_l'(x)$ with $p_0(x) \equiv 1$ and $q_0(x) \equiv 0$. They also derived Maclaurin series expansions for each of the eigenfunctions, proved uniform boundedness and analytic properties of the eigenfunctions, obtained information about the shape of the ground state $\psi_1(x)$, and derived estimates for the integral kernel $u(t, x, y)$:

$$\frac{1}{C(1 + x^4)(1 + y^4)} \leq u(t, x, y) \leq \frac{Ce^{\alpha_1't}}{(1 + x^4)(1 + y^4)},$$

for sufficiently large $t > 1$ and all $x, y \in \mathbb{R}$, and constant $C > 0$. 
Furthermore, eigenvalues $\lambda_n$ ($n \in \mathbb{N}$) were expressed in terms of the negative real zeroes of the Airy function $\text{Ai}(y)$ and those of the first derivative $\text{Ai}'(y)$ arranged in alternating and increasing order of magnitude; in particular, they obtained $\lambda_n = [3(2n-1)\pi/8]^{2/3}(1 + o(1))$ as $n \to \infty$. Lower bound estimate of the spectral gap was given by $\lambda_2 - \lambda_1 \geq (3^{2/3} - 1)(3\pi/8)^{2/3}$. Trace formula asymptotics were also derived: $\sum_{n \geq 1} e^{-t\lambda_n} = O(t^{-3/2})$ as $t \to 0^+$. 

For the quartic case (when $k = 4$), we consider similar eigenproblem involving now a local differential operator of the form $\tilde{H} := \frac{d^4}{dy^4} + |y|$ obtained by Fourier transform. The quadratic and quartic cases differ on several counts, and in the quartic case a fascinating phenomenon can be observed, which does not occur in the quadratic case. This difference is due to the fact that while in the quadratic case, the Fourier transform of the eigenfunctions can be expressed in a single term (dependent on $\text{Ai}(y)$), in the quartic case, it is expressed in two terms involving the fourth-order Airy function of the first kind $\text{Ai}_4(y)$ and another term derived from $\text{Ai}_4(y)$, which we denote as $\tilde{\text{Ai}}_4(y)$ and has an integral representation of the form

$$\tilde{\text{Ai}}_4(y) = \frac{1}{\pi} \int_0^{\infty} \left[ e^{-yt} - \frac{y^2}{2} - \sin \left( \frac{t^5}{5} + yt \right) \right] dt \quad (y \in \mathbb{R}).$$

The function $\tilde{\text{Ai}}_4(y)$ is comparable to $y^{-3/8}e^{-\frac{2\sqrt{2}}{3}\frac{y^{5/4}}{4}}$ for $y > 0$ and comparable to $|y|^{-3/8}e^{\frac{4}{5} \frac{y^{5/4}}{4}}$ for $y < 0$. 

Unlike the quadratic case, the eigenvalues are no longer the negative real zeroes of $\text{Ai}_4(y)$ and $\text{Ai}_4'(y)$ but the negative real zeroes of certain higher transcendental functions $\Phi_1(y)$ and $\Phi_2(y)$ expressed in terms of the generalised Fresnel’s sine and cosine integral functions (i.e., $\text{si}(a,z)$ and $\text{ci}(a,z)$, respectively [61] (8.21.4) – (8.21.5)). The linear combination of $\text{Ai}_4(y)$ and $\tilde{\text{Ai}}_4(y)$ provides an expression mostly dominated by $\text{Ai}_4(y)$, which then reduces the effect of $\tilde{\text{Ai}}_4(y)$ significantly. This sub-dominant role played by $\tilde{\text{Ai}}_4(y)$ results in a small effect, which is reflected both in the expressions and properties of eigenvalues and eigenfunctions, and will be studied in detail below.

1.2. Synopsis

The remainder of this work is structured into two main chapters with appendices at the end of the work.

In Chapter 2, the object of interest is the higher-order Airy functions of the first kind $\text{Ai}_k(y)$. This class of special functions, which we define in terms of certain integral representations of Laplace-type is rigorously studied in this chapter. In Theorems 2.1 and 2.2, we establish some regularity results for $\text{Ai}_k(y)$ and subsequently show that $\text{Ai}_k(y)$ satisfies a $k$th-order Airy-type differential equation.
then study some analytic properties of $\text{Ai}_k(y)$ and $\text{Ai}'_k(y)$. In fact, some of these results are non-trivial generalisations of those known for the classical Airy function of the first kind $\text{Ai}(y)$ and its first derivative $\text{Ai}'(y)$. In particular, we derive an analytic series expansion for $\text{Ai}_k(y)$ in Theorem 2.3 and hence, in Corollary 2.1 we express $\text{Ai}_k(y)$ as a finite sum of the generalised hypergeometric function of the form $\text{F}_{k-1}(a; b; z)$. Full asymptotic expansions for $\text{Ai}_k(y)$ and $\text{Ai}'_k(y)$ are also derived in Theorems 2.4 and 2.5 with explicit expressions of the coefficients of these expansions in terms of Bell polynomials in Theorem 2.8 and Corollary 2.2. In Theorem 2.10 we compute asymptotic expansions of the negative real zeroes of $\text{Ai}_k(y)$ and $\text{Ai}'_k(y)$. Finally, $\text{Ai}_k(y)$ is examined non-rigorously for its large $k$ limit. It turns out that this limit is expressed as the well-known $\text{sinc}(y)$ function.

In Chapter 3, we study spectral properties of the massless relativistic operator $H$ of the quartic anharmonic oscillator in one dimension. Under Fourier transform, the eigenproblem associated with $H$ is transformed into another eigenproblem associated with a fourth-order differential operator $\tilde{H}$. We construct solutions to this fourth-order ODE in terms of the fourth-order Airy function of the first kind $\text{Ai}_4(y)$. In particular, four solutions are obtained, two of which are ruled out by the $L^1$ condition of the domain of $\tilde{H}$ leaving us with $\text{Ai}_4(y)$ and $\tilde{\text{Ai}}_4(y)$ (expressed in terms of $\text{Ai}_4(y)$ by some rotations of its argument). The application of the boundary conditions derived for the ODE to a linear combination of $\text{Ai}_4(y)$ and $\tilde{\text{Ai}}_4(y)$ allows us to identify the spectrum of $H$ as the zeroes of higher transcendental functions $\Phi_1(y)$ and $\Phi_2(y)$ expressed in terms of the generalised Fresnel's sine and cosine integral functions (refer to Theorem 3.1 and Corollary 3.1). In Corollary 3.2 we also derive asymptotic relations for $\Phi_1(y)$ and $\Phi_2(y)$ for $y < 0$ using asymptotic expansions obtained for $\text{Ai}_4(y)$ and $\tilde{\text{Ai}}_4(y)$. In Corollary 3.3, approximation formulae for $\lambda_n$ are obtained showing, more explicitly, the dependence of $\lambda_n$ on $n \in \mathbb{N}$. In Theorem 3.2, upper and lower bounds on the full sequence of the spectral gaps $\lambda_{n+1} - \lambda_n$ are derived.

We also derive in Theorem 3.3 an expression describing the behaviour of the heat trace around $t = 0$ and a Weyl-type asymptotic relation in Theorem 3.4. We obtain expressions for the Fourier transform of the $L^2$-normalised eigenfunctions in Theorem 3.5, full asymptotic expansions of all the eigenfunctions $\psi_n(x)$ for large argument $x$ in Theorem 3.6, analytic series expansions and a uniform boundedness property of all the eigenfunctions $\psi_n(x)$ in Theorems 3.7 and 3.8 respectively. Furthermore, we investigate the small effect generated by $\tilde{\text{Ai}}_4(y)$ and in Theorem 3.9 establish exponential bounds on the differences between the eigenvalues $\lambda_n$ and the zeroes $\mu_n$ of $\text{Ai}_4(y)$ and $\text{Ai}'_4(y)$. This investigation is also carried out at the level of eigenfunctions, and we show that the eigenfunctions $\psi_n(x)$
are exponentially well-approximated by the inverse Fourier transform of $\text{Ai}_4(|y| - \lambda_n)$ in its normalised form both in $L^2$ sense and in $L^\infty$ sense (see Theorems 3.10 and 3.11 respectively).
CHAPTER 2

Higher-Order Airy Functions

In this chapter, we introduce the higher-order Airy functions of the first kind and present the general theory of this class of special functions. We also derive some analytic properties of these functions including analytic and asymptotic series expansions. Furthermore, we obtain qualitative information about their negative real zeroes.

2.1. Definition, integral representations, and regularity properties

Here we start by formally defining the higher-order Airy functions of the first kind in the form of integral representations of Laplace-type. Next, we establish regularity properties of $\text{Ai}_k(y)$ and particularly show that these functions are smooth, bounded, and extend to entire functions on the complex plane $\mathbb{C}$. We also derive pointwise bounds on $\text{Ai}_k(y)$ for each $k$. Then, we conclude the section by showing that this class of functions indeed satisfies a certain higher order differential equation.

For $y \in \mathbb{R}$, we consider the following contour integral:

$$
\frac{1}{2\pi i} \int_{\gamma} e^{-\gamma_k y s - \frac{s^{k+1}}{k+1}} \, ds \quad \text{with} \quad \gamma_k := (-1)^{k/2} \text{ for each } k = 2, 4, 6, \cdots
$$

where $\gamma$ is any infinite contour in the complex $s$-plane that starts at infinity in the sector $-\frac{k\pi}{k+1} - \delta < \arg(s) < \frac{k\pi}{k+1}$ and ends at infinity in the sector $\frac{k\pi}{k+1} - \delta < \arg(s) < \frac{k\pi}{k+1} + \delta$ for $0 \leq \delta < \frac{(k-1)\pi}{2(k+1)}$, cutting through the negative real semi-axis. The integral is convergent since $\Re e(s^{k+1}) \to +\infty$ as $|s| \to \infty$ within $|\arg(s) \pm \frac{k\pi}{k+1}| < \delta$. But it is also possible to continuously deform the contour in (2.1) to align with the imaginary axis without altering the value of the integral. We now rigorously give the definition of the higher-order Airy functions of the first kind.

**Definition 2.1** (Higher-order Airy functions). Let $k = 2, 4, 6, \cdots$ and write $\gamma_k := (-1)^{k/2}$. For every $y \in \mathbb{R}$, we call the function

$$
\text{Ai}_k(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-\gamma_k y s - \frac{s^{k+1}}{k+1}} \, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\left(\frac{k+1}{k+1} + ty\right) dt} = \frac{1}{\pi} \int_{0}^{\infty} \cos \left(\frac{t^{k+1}}{k+1} + ty\right) \, dt
$$

$k$th-order Airy function of the first kind.
The above integrals are conditionally convergent. By integration-by-parts and \( k < \infty \), we have
\[
\int_{0}^{\infty} \cos \left( \frac{t^{k+1}}{k+1} + ty \right) dt = \int_{0}^{\infty} \frac{1}{t^{k}} \frac{d}{dt} \left( \sin \left( \frac{t^{k+1}}{k+1} + ty \right) \right) dt
\]
\[
\leq k \int_{0}^{\infty} \frac{t^{k-1}}{(t^{k} + y)^2} dt < \infty \quad \text{for every } y \neq 0.
\]

But one can show that these integrals are absolutely and uniformly convergent when analytically continued to the upper half region of the complex plane. To see this, we consider
\[
\text{Ai}_k(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{i(p_k(t)+iz)} dt, \quad p_k(t) := \frac{t^{k+1}}{k+1}, \quad C \ni z := y + i\sigma. \tag{2.3}
\]

Let \( \tau > 0 \), and consider
\[
\text{Ai}_k(\tau, z) = \frac{1}{2\pi} \int_{-\infty+\pi i}^{\infty+\pi i} e^{i(p_k(\zeta)+z\zeta)} d\zeta \quad (\Im(\zeta) = \eta > 0, \ C \ni z := y + i\sigma). \tag{2.4}
\]

It is easy to show that the value of this integral remains invariant for any choice of \( \tau > 0 \) (see the proof of Theorem \ref{thm1} below). Furthermore, we observe that
\[
\Re e \left[ i(p_k(t + i\tau) + (t + i\tau)z) \right] = -\tau^k - (\tau y + i\sigma) + O(\tau^3k^{-2}) \quad (k = 2, 4, 6, \cdots).
\]

Hence, given that \( \tau^k \) grows faster than any linear function of \( t \) as \( |t| \to \infty \), the integral in \ref{2.4} converges absolutely for all \( |z| < \infty \). Moreover, we have that \( e^{i(p_k(t+i\tau))} \to e^{i(p_k(t))} \) as \( \tau \to 0^+ \), and hence, \( \text{Ai}_k(\tau, z) \) tends to \( \text{Ai}_k(z) \) as \( \tau \to 0^+ \). This convergence is also true if \( z \in \mathbb{C} \) is replaced with \( y \in \mathbb{R} \). Moreover, one can show that \( \text{Ai}_k(z) \) is an entire function of \( z \in \mathbb{C} \). We now provide a formal proof of the above claims.

**Theorem 2.1.** Let \( k = 2, 4, 6, \cdots \), we have that each \( \text{Ai}_k(y) \) is smooth, bounded, and extends to the complex plane \( \mathbb{C} \) as an entire function.

**Proof.** From \ref{2.4}, we have
\[
\text{Ai}_k(\tau, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t + i\tau, z) dt \tag{2.5}
\]
where \( G(t + i\tau, z) := e^{i(p_k(t+i\tau) + (t+i\tau)z)} \) with \( C \ni z := y + i\sigma \), and we have
\[
|G(t + i\tau, z)| = e^{\Re e[i(p_k(t+i\tau) + (t+i\tau)\zeta))] = e^{-\Im p_k(t+i\tau) - (\tau y + i\sigma)} = e^{-\tau^k + O(\tau^3k^{-2}) - (\tau y + i\sigma)}
\]
for all \( |z| < \infty \) and \( t \in \mathbb{R} \). Consequently, the integral \ref{2.5} defining \( \text{Ai}_k(\tau, z) \) is absolutely and uniformly convergent for all \( |z| < \infty \). Furthermore, there exists a constant \( C_1 > 0 \) such that for any \( N \in \mathbb{N} \), we have
\[
|\partial_{\tau}^N G(t + i\tau, z)| \leq C_1(1 + |t|)^N e^{-\tau^k + O(\tau^3k^{-2}) - (\tau y + i\sigma)} \quad \text{for all } |z| < \infty \text{ and } t \in \mathbb{R}.
\]
Hence, it is possible to differentiate under the integral in (2.5) with respect to \( z \); moreover, it is easy to show that \( \text{Ai}_k(\tau, z) \) does not depend on \( \tau \). Thus, given that
\[
G(t + i\tau, z) = e^{-\frac{t}{\pi} - \frac{\pi}{2} i \tau^2} (t + i\tau)^{-\frac{1}{2}} e^{\tau^2 t - \frac{1}{2} i \tau^2} - \frac{1}{\pi} \int_{\tau}^{\infty} e^{\tau^2 t - \frac{1}{2} i \tau^2} (t + i\tau)^{-\frac{1}{2}} e^{\tau^2 t - \frac{1}{2} i \tau^2} - d\tau 
\]
as \( t \to \pm \infty \) for every \( |z| < \infty \) and \( \tau > 0 \), it follows from the Cauchy-Riemann equations that
\[
\frac{\partial}{\partial \tau} \text{Ai}_k(\tau, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial \tau} G(t + i\tau, z) \, dt = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial \tau} G(t + i\tau, z) \, dt = 0.
\]
Consequently, the integral (2.5) (or (2.4)) is invariant under the translation \( t \mapsto t + i\tau, \tau > 0 \). In essence, we can write
\[
\text{Ai}_k(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t + i\tau, z) \, dt, \tag{2.6}
\]
and for any \( \tau > 0 \), \( \text{Ai}_k(z) \) given by (2.6) (or (2.3)) defines an entire function of \( z \in \mathbb{C} \).

Finally, we show that \( \text{Ai}_k(y) \) is smooth for \( y \in \mathbb{R} \), and the derivatives are, at most, of polynomial growth. Thus, we replace \( z \) with \( y \) in (2.6) so that for any \( N \in \mathbb{N} \) we have that
\[
\left| \frac{d^N}{dy^N} \text{Ai}_k(y) \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |(t + i\tau)^N G(t + i\tau, y)| \, dt.
\]
Now, for \( |y| \geq 1 \), we set \( \tau = |y|^{-1} \). Then, there exists a constant \( C_2 > 0 \) such that
\[
|(t + i\tau)^N G(t + i\tau, y)| = |t + i|^{-\bar{N}} e^{-\tau|y| + O(\tau|y|^{-1} - \bar{y})} 
\]
\[
\leq C_2(1 + |t|)^N e^{-\tau|y|} = C_2(1 + |t|)^N e^{-|\bar{y}|^\frac{1}{N}}.
\]
Hence, using the change of variable \( u = |y|^{-1/2}t \) in a subsequent step below, we have that
\[
\int_{-\infty}^{\infty} |(t + i\tau)^N G(t + i\tau, y)| \, dt \leq C_2 \int_{-\infty}^{\infty} (1 + |t|)^N e^{-|\bar{y}|^\frac{1}{N}} \, dt 
\]
\[
\leq C_2 |y|^\frac{1}{N} \int_{-\infty}^{\infty} (1 + |t|)^N e^{-u^\frac{1}{N}} \, du 
\]
\[
\leq C_3 |y|^\frac{1}{N}
\]
for \( |y| \geq 1 \). Thus, we conclude that \( \text{Ai}_k(y) \) is smooth and bounded. \( \square \)

In the next statement we derive pointwise bounds for the higher-order Airy functions \( \text{Ai}_k(y) \).

**Theorem 2.2.** Let \( k = 2, 4, 6, \ldots \), then there exist constants \( C_1, C_2 > 0 \) such that
\[
|\text{Ai}_k(y)| \leq \min \left\{ C_1 e^{-2|y|}, C_2 |y|^{-\frac{k+1}{2}} \right\} \quad y \in \mathbb{R}. \tag{2.7}
\]
Moreover, we have that \( \text{Ai}_k \in L^1(0, \infty) \cap L^2(0, \infty) \).
2.1. Definition, Integral Representations, and Regularity Properties

Proof. We only need to show the proof for (2.7); that \( A_{i,k} \in L^1(0,\infty) \cap L^2(0,\infty) \) follows from (2.7).

First, we recall that the integral in (2.3) is invariant under the translation \( t \mapsto t + 2i \). Therefore, for \( y \gg -1 \), we consider

\[
\left| \int_{-\infty}^{\infty} e^{i(p(t)+ty)} \, dt \right| = \left| \int_{-\infty}^{\infty} e^{i(p(t+2i)+(t+2i)y)} \, dt \right|
\leq \int_{-\infty}^{\infty} e^{i(p(t+2i)+(t+2i)y)} \, dt
= \int_{-\infty}^{\infty} e^{\eta i((t+2i)+(t+2i)y)} \, dt = Ce^{-2y},
\]

where \( C := \int_{-\infty}^{\infty} e^{-2t} + O(8^{d^2}) \, dt < \infty \) (since \( k = 2, 4, 6, \cdots \)). Therefore, we have that

\[
|A_{i,k}(y)| \leq \frac{C}{2\pi} e^{-2y}, \quad y \gg -1. \tag{2.8}
\]

Now, for \( y \ll 1 \), we use the following map: \( y \mapsto -y \), then it follows from (2.3) that

\[
A_{i,k}(-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(p(t)-ty)} \, dt, \quad w > 0.
\]

Set \( t = \frac{1}{2} u \) and \( \eta = \frac{y+1}{2} \), subsequently. Then it follows that we can consider

\[
I(\eta) = \frac{\eta^{1/2}}{2\pi} \int_{-\infty}^{\infty} e^{i\phi(u)} \, du, \quad \eta > 0,
\]

where \( h(u) = \frac{u^{k+1}}{k+1} - u \), which is smooth and has local absolute minimum at \( u = \pm 1 \) (i.e. \( h'(\pm 1) = 0 \) and \( |h''(\pm 1)| \gg k \neq 0 \)); these are identified as interior critical points of the integrand of \( I(\eta) \), and lie on the interval of integration \((-\infty, \infty)\). Hence, we decompose the integral \( I(\eta) \) as follows:

\[
\int_{-\infty}^{\infty} e^{i\phi(u)} \, du = \left\{ \int_{-\infty}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty} \right\} e^{i\eta h(u)} \, du. \tag{2.9}
\]

Our aim is to show that the integral in (2.9) decays as \( O(\eta^{-1/2}) \) for \( \eta \gg -1 \). Clearly, each of the integrals on the right-hand side of (2.9) takes the following general form:

\[
\int_{a}^{b} e^{i\eta h(u)} \, du, \quad \text{or} \quad \int_{a}^{b} e^{-i\eta h(u)} \, du,
\]

where \( a = 1 \), and \( b = 0 \) or \( \infty \); both integrals will yield the same result.

In the remainder of the proof, we employ the idea of sub-level set estimation procedure by Van der Corput [32 Section 2.6.2]. We proceed as follows: we assume that \( \tau \in C^2(\mathbb{R}) \), and in particular that \( h \in C^2(\mathbb{R}) \). Then, we define \( \tau(u) \equiv h(u) - h(1) \) (where the dot equality “\( \equiv \)” implies that \( h(1) \) is defined by its Taylor’s series expansion about \( u = 1 \) but truncated after the \( \frac{h''(1)}{2}(u-1)^2 \) term), and write

\[
\int_{a}^{b} e^{i\eta h(u)} \, du = e^{i\eta h(1)} \int_{1}^{b} e^{i\eta \tau(u)} \, du.
\]
Then, it follows that \(|\tau''(u)| = |h''(1)| > 1\) for all \(u \in [0, \infty)\). Next, we define the following two sub-(and super-)level sets: \(E_1 := \{u \in (1, b) : |\tau''(u)| \leq \beta\}\) and \(E_2 := \{u \in (1, b) : |\tau''(u)| > \beta\}\), respectively, and write

\[
\left| \int_1^b e^{i\eta\tau(u)} \, du \right| \leq \left| \int_{E_1} e^{i\eta\tau(u)} \, du \right| + \left| \int_{E_2} e^{i\eta\tau(u)} \, du \right|.
\]

Furthermore, since \(\tau''(u) > 1\), we let \(\nu = \tau'\) so that \(|\nu'(u)| > 1\); and hence, it follows from Proposition 2.6.7(a) that

\[
\left| \int_{E_1} e^{i\eta\nu(u)} \, du \right| \leq \left| \int_{E_1} |e^{i\eta\nu(u)}| \, du \right| = |E_1| \leq 4\epsilon\beta \leq 12\beta. \tag{2.10}
\]

For the corresponding estimate over \(E_2\), we note that since \(|\tau''(u)| > 1\), then the set \(E_2\) is the union of at most four intervals on each of which \(\tau'(u)\) is monotone. Suppose \((c_1, c_2)\) is one of such intervals on which \(\tau'(u)\) also has a fixed sign. Then, on \((c_1, c_2)\), we have that

\[
\left| \int_{c_1}^{c_2} e^{i\eta\nu(u)} \, du \right| = \left| \int_{c_1}^{c_2} \frac{d}{du} \left( e^{i\eta\nu(u)} \right) \frac{1}{i\eta\tau'(u)} \, du \right|
\]

\[
\leq \frac{1}{\eta} \left| \frac{e^{i\eta\nu(c_2)}}{\tau'(c_2)} \right| - \frac{e^{i\eta\nu(c_1)}}{\tau'(c_1)} \left| + \frac{1}{\eta} \left| \int_{c_1}^{c_2} \frac{d}{du} \left( \frac{1}{\tau'(u)} \right) \, du \right| \right|
\]

\[
\leq \frac{2}{\eta} \max \left\{ \frac{1}{\tau'(c_2)}, \frac{1}{\tau'(c_1)} \right\} + \frac{1}{\eta} \left| \int_{c_1}^{c_2} \frac{d}{du} \left( \frac{1}{\tau'(u)} \right) \, du \right| + \frac{1}{\eta} \left| \int_{c_1}^{c_2} \frac{d}{du} \left( \frac{1}{\tau'(u)} \right) \, du \right| \leq \frac{3}{\beta\eta},
\]

where we have used the monotonicity of \(\frac{1}{\tau'(u)}\) to move the absolute value from inside the integral to outside. Therefore, we have that

\[
\left| \int_{E_2} e^{i\eta\nu(u)} \, du \right| \leq \frac{12}{\beta\eta}. \tag{2.11}
\]

By adding (2.10) and (2.11) and choosing \(\beta = \eta^{1/2}\) to optimize the estimate, we deduce that

\[
\left| \int_1^b e^{i\eta\nu(u)} \, du \right| \leq 12\eta^{-1/2}.
\]

Hence, using \(\eta = y^{-\frac{1}{2}}\), it follows that

\[
|\text{Ai}_k(y)| \leq \frac{6}{\pi}|y|^{-\frac{1}{2k}} \epsilon_{\frac{y}{2}}, \quad y \ll 1, \tag{2.12}
\]

which completes the required proof.

To conclude this section, we now show that \(\text{Ai}_k(y)\) indeed satisfies the following \(k\)th-order differential equation:

\[
\frac{d^k}{dy^k} \chi(y) + \gamma_k \chi'(y) = 0. \tag{2.13}
\]

Let us consider

\[
\text{Ai}_k(y) = \frac{1}{2\pi i} \int_{\gamma} e^{-i\frac{y^2}{\pi \tau}} \, ds. \tag{2.14}
\]
2.2. ANALYTIC SERIES EXPANSIONS AND CONNECTION WITH GENERALISED HYPERGEOMETRIC FUNCTIONS

where \( \mathcal{G} \) is the infinite contour described above. The integral (2.14) is absolutely and uniformly convergent, and from the result of Theorem 2.1 above, \( \text{Ai}_k(y) \) is analytic, which implies that its derivatives of all orders can be obtained by differentiating under the integral sign. Therefore, we find that

\[
\frac{d^k}{dy^k} \text{Ai}_k(y) + \gamma_k y \text{Ai}_k(y) = \frac{1}{2\pi i} \int_{\mathcal{G}} (s^k + \gamma_k y) e^{-\frac{s y \gamma_k}{\pi}} \, ds
\]

But the term

\[
e^{-\gamma k y \frac{s}{\pi}} \]

is decaying as \( |s| \to \infty \) at \( \arg(s) = 2\pi r/(k + 1) \) for any integer \( r \in \mathbb{Z} \), and hence, vanishes at the endpoints of \( \mathcal{G} \). Therefore, the \( k \)th-order differential equation (2.13) is satisfied by \( \chi(y) = \text{Ai}_k(y) \).

2.2. Analytic series expansions and connection with generalised hypergeometric functions

In this section, we obtain analytic series expansions of the higher-order Airy functions \( \text{Ai}_k(y) \) valid for every \( -\infty < y < \infty \). With these expansions, \( \text{Ai}_k(y) \) (for each \( k = 2, 4, 6, \cdots \)) is expressed in terms of certain generalised hypergeometric functions.

The equation (2.13) admits analytic series solutions. Thus, consider the power series

\[
y^p \sum_{l=0}^{\infty} a_l y^{(k+1)l} \quad \text{with} \quad a_0 = 1.
\]

By direct substitution into (2.13), we obtain the following equation:

\[
a_0 \frac{p!}{(p-k)!} y^{p-k} + \sum_{l=0}^{\infty} a_{l+1} \frac{(k+1)(l+1) + p)!}{((k+1)l + p)!} + \gamma_k a_l \right] y^{(k+1)l+p+1} = 0
\]

satisfied for every \( -\infty < y < \infty \). Thus, we require

\[
p(p-1)(p-2) \cdots (p-k+1) = 0
\]

(2.16)

\[
a_{l+1} = -\gamma_k a_l \frac{((k+1)l + p + 1)!}{((k+1)(l+1) + p)!}
\]

(2.17)

and for fixed \( k \) and \( p \), and any value of \( l \geq 0 \), \( |a_{l+1}|/|a_l| \) expressed in terms of the right-hand side of (2.17) is finite. Therefore, the solution of (2.17) is given by

\[
a_l = \frac{(-\gamma_k)^l}{(k+1)!} \frac{(p + 2)!((k + 1)l + p + 1)! \cdots ((k+1)l + p - k)!}{((k+1)l + p + 1)!((k+1)l + p)! \cdots ((k+1)l + p)!}.
\]

(2.18)
Furthermore, we observe that
\[
\frac{(p + 1)!}{(k + p + 1)!} = \frac{1}{(k + 1)^k \left(1 + \frac{p}{k+1}\right) \left(1 + \frac{p-1}{k+1}\right) \cdots \left(1 + \frac{p-k+1}{k+1}\right)}
\]
\[
\frac{(k + p + 2)!}{(2k + 1)!} = \frac{1}{(k + 1)^k \left(2 + \frac{p}{k+1}\right) \left(2 + \frac{p-1}{k+1}\right) \cdots \left(2 + \frac{p-k+1}{k+1}\right)}
\]
\[
\frac{((k + 1)l + p - k)!}{((k + 1)l + p)!} = \frac{1}{(k + 1)^k \left(l + \frac{p}{k+1}\right) \left(l + \frac{p-1}{k+1}\right) \cdots \left(l + \frac{p-k+1}{k+1}\right)}.
\]

Thus, using the identity
\[
\frac{1}{(\beta + 1)(\beta + 2) \cdots (\beta + l)} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + l + 1)},
\]

it follows from (2.18) that
\[
a_l = (-\gamma_k)^l \prod_{j=0}^{k-1} \frac{\Gamma \left(\frac{p-j}{k+1} + 1\right)}{\Gamma \left(\frac{p-j}{k+1} + l + 1\right)} \frac{1}{(k + 1)^l},
\] (2.19)

and from (2.16), we have that \(p = 0, 1, 2, \cdots, k - 1\). Moreover, we have that
\[
\lim_{l \to \infty} \frac{a_l}{a_{l+1}} = \lim_{l \to \infty} \left[ (k + 1)^l \prod_{j=0}^{k-1} \frac{\Gamma \left(\frac{p-j}{k+1} + 1\right)}{\Gamma \left(\frac{p-j}{k+1} + l + 1\right)} \right] = \infty
\]
implying that the power series (2.15) with \(a_l\) given by (2.19) is convergent for every \(-\infty < y < \infty\).

Therefore, the analytic series solutions of (2.13) denoted by \(u_p(y)\) are given by
\[
u_p(y) = y^p \sum_{l=0}^{\infty} \frac{1}{l!} \prod_{j=0}^{k-1} \frac{\Gamma \left(\frac{p-j}{k+1} + 1\right)}{\Gamma \left(\frac{p-j}{k+1} + l + 1\right)} \left[ -\gamma_k y^{k+1} \right]^l
\]
for each \(p = 0, 1, 2, \cdots, k - 1\). Using these series solutions, we obtain the analytic series expansions for the higher-order Airy functions \(\text{Ai}_k(y)\) valid for every \(-\infty < y < \infty\).

**Theorem 2.3.** For each \(k = 2, 4, 6, \cdots\), we have
\[
\text{Ai}_k(y) = \sum_{p=0}^{k-1} \frac{\text{Ai}_k^{(p)}(0)}{p!} y^p \sum_{l=0}^{\infty} \frac{1}{l!} \prod_{j=0}^{k-1} \frac{\Gamma \left(\frac{p-j}{k+1} + 1\right)}{\Gamma \left(\frac{p-j}{k+1} + l + 1\right)} \left[ -\gamma_k y^{k+1} \right]^l
\]
\[\ast\] (2.20)

with \(\gamma_k := (-1)^{k/2}\); where
\[
\text{Ai}_k^{(p)}(0) = \frac{\cos \left(\frac{\pi}{2} \frac{p+1}{k+1} + \frac{p\pi}{2}\right)}{(k + 1)^{l_p} \Gamma \left(\frac{k-p}{k+1}\right) \sin \left(\frac{p+1}{k+1} \pi\right)}
\]
\[\ast\] (2.21)

**Proof.** Let
\[
\text{Ai}_k(y) = \sum_{p=0}^{k-1} \beta_p u_p(y)
\]
(2.22)

for some constants \(\beta_p\) to be determined. Then we differentiate both sides of (2.22) \(p\) times and set \(y = 0\) to obtain \(\text{Ai}_k^{(p)}(0) = p! \beta_p\) for each \(p = 0, 1, \cdots, k - 1\). Hence, (2.20) follows.
Next, we derive the values of \( \text{Ai}_k^{(p)}(0) \) expressed in (2.21). Hence, we recall the integral representation

\[
\text{Ai}_k(y) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{\nu^{k+1}}{k+1} + yt \right) \, dt.
\]  
(2.23)

By differentiating (2.23) \( p \) times and setting \( y = 0 \), we obtain

\[
\text{Ai}_k^{(p)}(0) = \frac{1}{\pi} \int_0^\infty t^p \cos \left( \frac{\nu^{k+1}}{k+1} + \frac{p\pi}{2} \right) \, dt.
\]  
(2.24)

Next, we make the following change of variable: \( v = \frac{\nu^{k+1}}{k+1} \), which reduces (2.24) to

\[
\text{Ai}_k^{(p)}(0) = \frac{1}{\pi(k+1)^{\frac{k+1}{p+1}}} \int_0^\infty v^{\frac{k+1}{p}} \cos \left( v + \frac{p\pi}{2} \right) \, dv.
\]  
(2.25)

Using the Mellin transforms of \( \cos(v) \) and \( \sin(v) \) in (2.25), we obtain

\[
\text{Ai}_k^{(p)}(0) = \frac{1}{\sqrt{\pi(k+1)^{\frac{k+1}{p+1}}}} \left[ \cos \left( \frac{p\pi}{2} \right) \Gamma \left( Z + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} - Z \right) - \sin \left( \frac{p\pi}{2} \right) \Gamma \left( Z + \frac{1}{2} \right) \Gamma \left( 1 - Z \right) \right],
\]  
(2.26)

where \( Z = \frac{1}{2} + \frac{1}{2} \frac{p+1}{k+1} \). Observe that the first and second terms in (2.26) vanish for odd and even \( p \)'s, respectively. Now, for the first term we apply the duplication formula:

\[
\Gamma \left( Z + \frac{1}{2} \right) \Gamma \left( Z \right) = 2^{1-2Z} \sqrt{\pi} \Gamma (2Z),
\]

and obtain

\[
\frac{\Gamma \left( Z \right)}{2^{1-2Z} \sqrt{\pi} \Gamma \left( \frac{1}{2} - Z \right)} = \frac{\Gamma (2Z)}{\Gamma \left( \frac{1}{2} - Z \right) \Gamma \left( Z + \frac{1}{2} \right)};
\]  
(2.27)

subsequent application of the following identity:

\[
\Gamma(c) \Gamma(1-c) = \pi / \sin(c\pi) \quad \text{for } 0 < c < 1,
\]  
(2.28)

on (2.27) with \( c = Z + \frac{1}{2} \) now yields

\[
\frac{\Gamma (2Z)}{\Gamma \left( \frac{1}{2} - Z \right) \Gamma \left( Z + \frac{1}{2} \right)} = \frac{\Gamma (2Z) \sin \left( Z\pi + \frac{\pi}{2} \right)}{\pi} = \frac{\Gamma \left( \frac{p+1}{k+1} \right) \sin \left( \frac{\pi}{2} \frac{p+1}{k+1} + \frac{\pi}{2} \right)}{\pi} \\
= \frac{\Gamma \left( \frac{p+1}{k+1} \right) \cos \left( \pi \frac{p+1}{k+1} \right)}{\pi}.
\]  
(2.29)

For the second term, we apply the identity (2.28) directly with \( c = Z \), and the duplication formula, to obtain

\[
\frac{\Gamma \left( Z + \frac{1}{2} \right)}{\sqrt{2^{1-2Z} \Gamma (1 - Z)}} = \frac{\Gamma \left( Z + \frac{1}{2} \right) \Gamma (Z) \sin(Z\pi)}{\pi^{1/2} 2^{1-2Z}} = \frac{\Gamma (2Z) \sin(Z\pi)}{\pi} \\
= \frac{\Gamma \left( \frac{p+1}{k+1} \right) \sin \left( \frac{\pi}{2} \frac{p+1}{k+1} \right)}{\pi}.
\]  
(2.30)
2.2. ANALYTIC SERIES EXPANSIONS AND CONNECTION WITH GENERALISED HYPERGEOMETRIC FUNCTIONS

We now insert (2.29) and (2.30) into (2.26), and simplify further using angle addition formulas to obtain

\[ \text{Ai}_{k}(^{(p)})(0) = \frac{\Gamma\left(\frac{p+1}{k+1}\right) \cos \left(\frac{\pi \cdot p+1}{2(k+1)} + \frac{\pi \cdot p}{2}\right)}{\pi(k+1)^{\frac{k-p}{2}}} \].

Subsequent application of the identity (2.28) with \( c = \frac{p+1}{k+1} \) then yields the required result:

\[ \text{Ai}_{k}(^{(p)})(0) = \frac{\cos \left(\frac{\pi \cdot p+1}{2(k+1)} + \frac{\pi \cdot p}{2}\right)}{(k+1)^{\frac{k-p}{2}} \Gamma \left(\frac{k-p}{k+1}\right) \sin \left(\frac{p+1}{k+1}\pi\right)} \text{ for each } 0 \leq p \leq k-1. \] (2.31)

This completes the proof. \( \square \)

**Remark 2.1.** For \( p = 0, 1, \) and \( k = 2, \) we have from (2.21) the following usual values for the standard Airy function of first kind: \( \text{Ai}(0) = \frac{1}{3^{\frac{3}{2}} \Gamma(\frac{1}{2})}; \) \( \text{Ai}'(0) = -\frac{1}{3^{\frac{1}{2}} \Gamma(\frac{1}{2})}, \) and from (2.20) we have the expression in (1.6).

Using the following definition of the generalised hypergeometric functions (cf. [61] (16.2.1)):

\[ _pF_q \left( a_1, \ldots, a_p; b_1, \ldots, b_q; z \right) = \sum_{l=0}^{\infty} \frac{\langle a_1 \rangle \ldots \langle a_p \rangle \langle b_1 \rangle \ldots \langle b_q \rangle}{l!} \frac{z^l}{\langle l \rangle}, \]

together with the identity \( \frac{1}{\langle a \rangle} = \frac{\Gamma(a)}{\Gamma(a+1)} \) (where \( \langle a \rangle := a(a+1)(a+2) \cdots (a+l-1) \) and \( \langle a \rangle_0 = 1 \) denote the Pochhammer’s symbol of the rising factorial), we can express \( \text{Ai}_k(y) \) defined in (2.20) as a finite sum of the generalised hypergeometric functions. Observe that

\[ \frac{\Gamma\left(\frac{p-j}{k+1} + 1\right)}{\Gamma\left(\frac{p-j}{k+1} + l + 1\right)} = \frac{1}{\langle \frac{p-j}{k+1} + 1 \rangle_l}. \]

Therefore, we have that

\[ \sum_{j=0}^{\infty} \frac{1}{l!} \prod_{j=0}^{k-1} \frac{\Gamma\left(\frac{p-j}{k+1} + 1\right)}{\Gamma\left(\frac{p-j}{k+1} + l + 1\right)} \left[-\gamma_k y^{k+1}\right]^l \left(\frac{\gamma_k y^{k+1}}{(k+1)^{k}}\right)^l = _0F_{k-1} \left( -, \frac{p+2}{k+1}, \ldots, \frac{p+k+1}{k+1}; -\gamma_k y^{k+1} \right) \right], \]

where * denotes that values of the entries of \( _0F_{k-1} \) for which \( j = p \) are omitted; for instance, if \( p = 0 \) the last term \( \frac{p+k+1}{k+1} \) is omitted, and if \( p = k-1 \) the first term \( \frac{p+2}{k+1} \) is omitted. Furthermore, we set

\[ U_{k,p}(y) := _0F_{k-1} \left( -, \frac{p+2}{k+1}, \ldots, \frac{p+k+1}{k+1}; -\gamma_k y^{k+1} \right) \right), \]

and state as follows:

**Corollary 2.1.** For each \( k = 2, 4, 6, \ldots \), we have that

\[ \text{Ai}_k(y) = \sum_{p=0}^{k-1} \frac{\text{Ai}_{k}(^{(p)})(0)}{p!} U_{k,p}(y). \] (2.32)
2.2. ANALYTIC SERIES EXPANSIONS AND CONNECTION WITH GENERALISED HYPERGEOMETRIC FUNCTIONS

By the uniqueness theorem of solutions of linear differential equations, (2.20) defines \( A_i_k(y) \) as a (unique) linear combination of the series solutions \( u_p(y) \), and hence, a particular solution of (2.13). Note that the equation (2.13) is invariant under the rotation \( y \mapsto ye^{\frac{n\pi i}{2}} \) for any \( n \in \mathbb{Z} \). Therefore, by phase shift of the argument, other solutions of equation (2.13) are obtained accordingly. Thus, for any \( r \in \mathbb{Z} \), we have

\[
A_i_k(ye^{\frac{n\pi i}{2}}) = \sum_{p=0}^{k-1} \frac{A_i_k(p)}{p!} e^{\frac{2npi}{k+1}} y^p \sum_{l=0}^{\infty} \frac{1}{l!} \prod_{j=0}^{k-1} \Gamma \left( \frac{p-j}{k+1} + 1 \right) \left[ -y_k^k \right]^l.
\]

We now show a linear dependence relation. We seek a relation of the form

\[
\sum_{r=0}^{k} e^{\frac{n\pi i}{2}} A_i_k(\chi e^{\frac{n\pi i}{2}}) \equiv 0
\]

valid for every \(-\infty < y < \infty\), where \( X \) is to be determined. By substituting (2.20) into this relation, we obtain

\[
\sum_{r=0}^{k} e^{\frac{n\pi i}{2}} \sum_{q=1}^{k} \frac{A_i_k(q-1)(0)}{(q-1)!} e^{\frac{2npi(q-1)}{k+1}} u_{q-1}(y) \equiv 0.
\]

Since all the \( k \) power series \( u_{q-1}(y) \) \( (q = 1, 2, \cdots, k) \) are independent, the multiplication of these \( k \) series by the coefficients in relation (2.33) must vanish. Thus, we have

\[
\sum_{r=0}^{k} e^{\frac{n\pi i}{2}} \left( \frac{\chi}{\pi} + \frac{2q-1}{k+1} \right)^r = 0
\]

for each \( q = 1, 2, \cdots, k \), which is just a geometric progression with common ratio \( e^{\frac{n\pi i}{2}} \left( \frac{\chi}{\pi} + \frac{2q-1}{k+1} \right) \). Hence, the sum to the \((k+1)\)-st term involves the numerator

\[
e^{\frac{n\pi i}{2}} \left( \frac{\chi}{\pi} + \frac{2q-1}{k+1} \right)^l = e^{\frac{(k+1)\pi i}{2}} - 1.
\]

If this must vanish as expected, we must have that \((k+1)X = 2lk \) for any \( l \in \mathbb{Z} \). Thus, we have that \( X = 2lk/(k+1) \). We fix \( l = 1 \), and have that \( X = 2k/(k+1) \). Hence, we obtain

\[
\sum_{r=0}^{k} e^{\frac{n\pi i}{2}} A_i_k(\chi e^{\frac{n\pi i}{2}}) = 0
\]

from which we now have that

\[
A_i_k(y) = -\sum_{r=1}^{k} e^{\frac{n\pi i}{2}} A_i_k(\chi e^{\frac{n\pi i}{2}}).
\]

Let the integers \( r_1, r_2, \cdots, r_k \) be mutually distinct mod \((k+1)\) and \( \rho^{k+1} = 1 \). We can compute the Wronskian \( W[A_i_k(\rho^{r_1}y), A_i_k(\rho^{r_2}y), \cdots, A_i_k(\rho^{r_k}y); y] \). Note that

\[
W[A_i_k(\rho^{r_1}y), A_i_k(\rho^{r_2}y), \cdots, A_i_k(\rho^{r_k}y); y] \neq 0 \quad \text{for any } y \in X
\]
2.3. ASYMPTOTIC SERIES EXPANSIONS

If \( W[A_i_k(\rho^1 y), A_i_k(\rho^2 y), \ldots, A_i_k(\rho^k y); y_0] \neq 0 \) for some \( y_0 \in X \) (where \( X \) is any open sub-interval of \( \mathbb{R} \)), which follows from the well-known Abel identity given by

\[
W[A_i_k(\rho^1 y), A_i_k(\rho^2 y), \ldots, A_i_k(\rho^k y); y] = W[A_i_k(\rho^1 y), A_i_k(\rho^2 y), \ldots, A_i_k(\rho^k y); y_0]
\]

for the particular case (2.13) in which the coefficient of the \((k - 1)\)th derivative \( \frac{d^{k-1}}{dy^{k-1}} \) is zero (refer to [19 p. 83]). Hence, it is convenient to compute \( W \equiv W[A_i_k(\rho^1 y), A_i_k(\rho^2 y), \ldots, A_i_k(\rho^k y); y_0] \), the Wronskian evaluated at any given point \( y_0 \in X \subseteq \mathbb{R} \), for which \( A_i_k^{(p)}(y_0) \neq 0 \) for all \( p = 0, 1, 2, \ldots, k - 1 \). Thus, we have

\[
W = \begin{vmatrix}
A_i_k(y_0) & A_i_k'(y_0) & \cdots & A_i_k^{(k-1)}(y_0) \\
\rho^1 A_i_k'(y_0) & \rho^2 A_i_k''(y_0) & \cdots & \rho^k A_i_k^{(k-2)}(y_0) \\
\vdots & \vdots & \ddots & \vdots \\
\rho^{(k-1)r_1} A_i_k^{(k-1)}(y_0) & \rho^{(k-1)r_2} A_i_k^{(k-1)}(y_0) & \cdots & \rho^{(k-1)r_k} A_i_k^{(k-1)}(y_0)
\end{vmatrix}
\]

\[
= \prod_{p=0}^{k-1} A_i_k^{(p)}(y_0) \prod_{1 \leq i < j \leq k} (\rho^i - \rho^j)
\]

\[
= \prod_{p=0}^{k-1} A_i_k^{(p)}(y_0) \prod_{1 \leq i < j \leq k} e^{\left\langle \frac{\pi r_j}{k+1} \right\rangle} \left[ e^{\left\langle \frac{\pi r_i}{k+1} \right\rangle} - e^{\left\langle \frac{\pi r_i}{k+1} \right\rangle} \right]
\]

\[
= \prod_{p=0}^{k-1} A_i_k^{(p)}(y_0) \prod_{1 \leq i < j \leq k} e^{\left\langle \frac{\pi r_j}{k+1} + \frac{1}{2} \right\rangle} \sin \left( \frac{r_i - r_j}{k+1} \pi \right) \neq 0.
\] (2.35)

In particular, we have that \( A_i_k^{(p)}(0) \neq 0 \) for all \( p = 0, 1, 2, \ldots, k - 1 \) (refer to (2.21)). Therefore, we have that the set \( \{A_i_k(\rho^j y) : j = 1, 2, \ldots, k\} \) forms a fundamental set of solutions for the \( k \)th-order differential equation (2.13).

2.3. Asymptotic series expansions

In this section, we study asymptotic behaviour of \( A_i_k(y) \) and that of its first derivative \( A_i_k'(y) \) for large values of their argument \( y \). Full asymptotic expansions are obtained for \( y \to \pm \infty \). With the knowledge of these expansions, we observe that \( A_i_k(y) \) and \( A_i_k'(y) \) both oscillate on the positive as well as the negative real semi-axis for \( k \geq 4 \). Moreover, we observe that expansions for \( y > 0 \) soon become asymptotically equal to expansions for \( y < 0 \) as \( k \to \infty \). Thus, this fascinating property of these expansions supports our claim that as \( k \) gets larger, the positive and negative real zeroes close up more in absolute value up to negligibly small error. More so, these expansions for \( A_i_k(y) \) become
asymptotically equal to \( \sin(y)/\pi y \) as \( k \to \infty \). In the next section, we study the negative real zeroes of \( \text{Ai}_k(y) \) and \( \text{Ai}'_k(y) \). The negative zeroes are of particular importance in this work: the results we obtain for these zeroes (especially for the case \( k = 4 \)) are used extensively in a later part of this work.

2.3. ASYMPTOTIC SERIES EXPANSIONS

For the large argument behaviour of \( \text{Ai}_k(y) \) and \( \text{Ai}'_k(y) \) on the positive real semi-axis, we have the following statement:

**Theorem 2.4.** For each \( k = 2, 4, 6, \ldots \), we have

\[
\text{Ai}_k(y) = \frac{y^{3k/4}}{\pi} e^{-\sin(\xi)\xi} \left[ \cos \left( \cos \left( \frac{\pi}{k} \right) \xi - \eta_k \right) \varphi_1(\xi) + \sin \left( \cos \left( \frac{\pi}{k} \right) \xi - \eta_k \right) \varphi_2(\xi) \right] \quad (2.36)
\]

and

\[
\text{Ai}'_k(y) = \frac{y^{3k/4}}{\pi} e^{-\sin(\xi)\xi} \left[ \sin \left( \cos \left( \frac{\pi}{k} \right) \xi - \eta_k \right) \varphi'_1(\xi) - \cos \left( \cos \left( \frac{\pi}{k} \right) \xi - \eta_k \right) \varphi'_2(\xi) \right] \quad (2.37)
\]

as \( y \to +\infty \); where

\[
\varphi_1(\xi) \sim \sum_{r=0}^{\infty} (-1)^r \frac{U_{2r}(k)}{\xi^{2r}}, \quad \varphi_2(\xi) \sim \sum_{r=0}^{\infty} (-1)^r \frac{U_{2r+1}(k)}{\xi^{2r+1}}
\]

and

\[
\varphi'_1(\xi) \sim \sum_{r=0}^{\infty} (-1)^r \frac{V_{2r}(k)}{\xi^{2r}}, \quad \varphi'_2(\xi) \sim \sum_{r=0}^{\infty} (-1)^r \frac{V_{2r+1}(k)}{\xi^{2r+1}}
\]

with \( \xi = \left( \frac{k+1}{k} \right)^{1/4} y \) and \( \eta_k := \left( \frac{k-2\pi}{4k} \right) \); and each \( U_r(k) \) and \( V_r(k) \) are real constants depending only on \( k \), and expressed in terms of Bell polynomials.

**Proof.** We start by deriving the expression (2.36). We consider the integral representation

\[
\text{Ai}_k(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \left( \frac{k+1}{k} \pi t + \xi \right) \xi} \, dt, \quad (2.38)
\]

By setting \( t = y^{1/4} v \) in (2.38), we obtain a Fourier-type integral with phase function \( h_k(v) = \frac{k+1}{k} \pi + v \):

\[
\text{Ai}_k(y) = \frac{w^{1/4}}{2\pi} \int_{-\infty}^{\infty} e^{iwh_k(v)} \, dv
\]

where \( w = y^{k+1} \) is real and positive. To apply the stationary phase technique on this Fourier-type integral, we are required to obtain those values of \( v \) for which the phase function is stationary (i.e. where \( h'_k(v) \) vanishes), which by assertion, give the major asymptotic contributions to the Fourier-type integral. Hence, by differentiating \( h_k \) once and setting it to zero, we obtain the stationary points \( v_j = e^{(2j+1)i\pi} / y, \; 0 \leq j \leq k-1 \) of \( h_k(v) \). But we observe that none of these stationary points lies on the interval of integration \((-\infty, \infty)\). Furthermore, since the integral (2.38) is invariant under the translation
2.3. ASYMPTOTIC SERIES EXPANSIONS

$t \mapsto t + it$ for $\tau > 0$, we therefore fix $v_0 = e^{\pi i/k}$ (corresponding to $j = 0$) and take $\tau = \Im(v_0)$. Thus, by analytic continuation, we have

$$
\text{Ai}_k(y) = \frac{1}{2\pi} \int_{-\infty+it}^{\infty+it} e^{i\left(\frac{k+1}{\tau + it} + \zeta\right)} d\zeta = \frac{1}{\pi} \Re e\left(\int_{it}^{\infty+it} e^{i\left(\frac{k+1}{\tau + it} + \zeta\right)} d\zeta\right). \quad (2.39)
$$

By setting $\zeta = y^2 u$ in (2.39), we then consider the integral

$$
J(w) = w^{\frac{1}{k+1}} \int_{it}^{\infty+it} e^{iwh_k(u)} du \quad (2.40)
$$

where also $w = y^{k+1}$ is real and positive and $h_k(u) = \frac{u^{k+1}}{k+1} + u$. We split the integral (2.40) at $u_0 = e^{\pi i/k}$ and obtain

$$
\int_{it}^{\infty+it} e^{iwh_k(u)} du = \int_{it}^{u_0} e^{iwh_k(u)} du + \int_{u_0}^{\infty+it} e^{iwh_k(u)} du =: J_1(w) + J_2(w). \quad (2.41)
$$

Let $u_\delta = e^{i(\delta + \pi/k)}$ for $0 < \delta < \pi/4k$. Set $T_\delta = h_k(u_\delta) - h_k(u_0)$ and introduce the new variable

$$
s = h_k(u) - h_k(u_0) = \left(\frac{u^{k+1}}{k+1} + u - \frac{k}{k+1} u_0\right) = -\frac{k}{2u_0}(u - u_0)^2 G_k(u). \quad (2.42)
$$

To obtain the last expression in (2.42), we have expanded $\frac{u^{k+1}}{k+1} + u - \frac{k}{k+1} u_0$ near the point $u = u_0$ as a convergent power series with zero coefficients for higher terms above the $(k + 1)$st term, and we have

$$
G_k(u) = 1 + \sum_{p=1}^{k-1} g_{k,p}(u - u_0)^p
$$

with

$$
g_{k,p} := \frac{1}{(p + 2)!} \left. \frac{d^{p+2} h_k(u)}{du^{p+2}} \right|_{u = u_0} = \frac{2(k - 1)^p}{(p + 2)!} u_0^p \quad \text{for each } p = 1, 2, \cdots, k - 1, \quad (2.43)
$$

and $(k - 1)_p := (k - 1)(k - 2) \cdots (k - p)$ with $(k - 1)_0 = 1$ (since $g_{k,p} = 0$ for $p = k, k + 1, \cdots$). We observe that $\Re e(h'_k(u))$ is positive and continuous in $(u_0, u_0)$ and since $\Re e(h_k(u))$ is monotone in $(u_0, u_0)$, a one-to-one relationship exists between $u - u_0$ and $s$, and from (2.42) we get

$$
\pm s^{1/2} = i \left(\frac{k}{2u_0}\right)^{1/2} (u - u_0)^1 (G_k(u))^{1/2}. \quad (2.44)
$$

Hence, on reversion, we obtain

$$
u^\pm = u_0 + \sum_{r=1}^{\infty} b_{k,r}(\mp is^{1/2})^r,
$$

where $b_{k,r}$ is obtained using Lagrange reversion formula (see (21) p. 125):

$$
b_{k,r} := \left. \frac{1}{r!} \frac{d^{r-1}}{du^{r-1}} \left[ \frac{(u - u_0)^2}{h_k(u) - h_k(u_0)} \right] \right|_{u = u_0}
= \left(\frac{2u_0}{k}\right)^{r/2} \left. \frac{1}{r!} \frac{d^{r-1}}{du^{r-1}} [G_k(u)]^{-r/2} \right|_{u = u_0}. \quad (2.45)
$$
2.3. ASYMPTOTIC SERIES EXPANSIONS

We take $u^-$ for the first integral $J_1(w)$ and take $u^+$ for the second integral $J_2(w)$ in (2.41), respectively. By a straightforward transformation using the change of variable (2.42), we obtain

$$J_1(w) := - \int_{u_0}^{it} e^{iwh u} du \sim - \int_{u_0}^{it} e^{iwh u} du$$

$$= e^{iwh u_0} \int_0^{T_{\delta_1}} e^{iws f_1(s)} ds = e^{iwh u_0} \left( \int_0^{\infty} - \int_{T_{\delta_1}}^{\infty} \right) e^{iws f_1(s)} ds,$$  \hspace{1cm} (2.46)

in which for each $N \geq 1$, we have

$$f_1(s) = - \frac{du^-}{ds} = \sum_{r=1}^{N-1} i q_{1,k,r} s^{\frac{r}{2}-1} + F_{1,N}(s) \quad \text{for } 0 < s < T_{\delta_1},$$

and set $f_1(s) = 0$ for $s \geq T_{\delta_1}$; where $q_{1,k,r} := rb_{k,r}/2$, and $F_{1,N}(s) = O(s^{\frac{N}{2}-1})$ as $s \to 0^+$. Similarly, we have

$$J_2(w) := \int_{u_0}^{iu_2} e^{iwh u} du \sim \int_{u_0}^{iu_2} e^{iwh u} du$$

$$= e^{iwh u_0} \int_0^{T_{\delta_2}} e^{iws f_2(s)} ds = e^{iwh u_0} \left( \int_0^{\infty} - \int_{T_{\delta_2}}^{\infty} \right) e^{iws f_2(s)} ds,$$  \hspace{1cm} (2.47)

in which for each $N \geq 1$, we have

$$f_2(s) = \frac{du^+}{ds} = \sum_{r=1}^{N-1} (-i)^r q_{2,k,r} s^{\frac{r}{2}-1} + F_{2,N}(s) \quad \text{for } 0 < s < T_{\delta_2},$$

and set $f_2(s) = 0$ for $s \geq T_{\delta_2}$; where $q_{2,k,r} := rb_{k,r}/2$, and $F_{2,N}(s) = O(s^{\frac{N}{2}-1})$ as $s \to 0^+$. Combining (2.46) and (2.47), we then have

$$J(w) \sim w^{\frac{1}{2+it}} e^{iwh_0} \left( \int_0^{\infty} - \int_{T_{\delta}}^{\infty} \right) e^{iws f(s)} ds,$$  \hspace{1cm} (2.48)

in which for each $N \geq 1$, we have

$$f(s) = \frac{du^+}{ds} - \frac{du^-}{ds} = -i \sum_{r=0}^{N-1} (-1)^r q_{k,2r} s^{\frac{2r+1}{2}} + F_{N}(s) \quad \text{for } 0 < s < T_{\delta},$$

and set $f(s) = 0$ for $s \geq T_{\delta}$ (with $T_{\delta} = T_{\delta_1} \equiv T_{\delta_2}$); where $q_{k,l} := (l + 1)b_{k,l+1}$, and $F_{N}(s) = O(s^{\frac{2N-1}{2}})$ as $s \to 0^+$.

By applying the extended version of Watson’s lemma [60, Theorem 2], we have

$$\int_0^{\infty} e^{iws f(s)} ds = -i \sum_{r=0}^{N-1} (-1)^r q_{k,2r} \lim_{\eta \to 0^+} \int_0^{\infty} e^{-(\eta - iw)s} s^{\frac{2r+1}{2}} ds + E_N(w)$$

$$= -i \sum_{r=0}^{N-1} (-1)^r q_{k,2r} e^{-\frac{r+1}{2}} \Gamma \left( \frac{2r+1}{2} \right) + E_N(w),$$  \hspace{1cm} (2.49)
where
\[ E_N(w) = \lim_{\eta \to 0^+} \int_0^\infty e^{-(\eta - iw)x} F_N(x) \, dx \]
\[ = O \left( \lim_{\eta \to 0^+} \int_0^\infty e^{-(\eta - iw)x} s^{2k-1} \, ds \right) = O(w^{-2k+1}). \]

Therefore, by inserting (2.49) into (2.48), equation (2.40) now becomes
\[ J(w) \sim -i\omega^{k+1} e^{\frac{j\omega}{\kappa+\frac{k}{k+1}}} \sum_{r=0}^\infty (-1)^r q_{k,2r} e^{\frac{(2r+1)\pi}{4}} \Gamma \left( \frac{2r+1}{2} \right) w^{-2k+1} \]
\[ = w^{k+1} e^{-w^{k+1}/\kappa} \frac{\xi}{\kappa} \left[ \cos \left( \frac{k \cos \left( \frac{\pi}{4} \right) y^{k+1}}{k+1} \right) \right. \]
\[ + i \sin \left( \frac{k \cos \left( \frac{\pi}{4} \right) y^{k+1}}{k+1} \right) \left. \right] \sum_{r=0}^\infty (-1)^r q_{k,2r} e^{\frac{2r\pi}{4}} \Gamma \left( \frac{2r+1}{2} \right) \]
\[ = w^{k+1} e^{-w^{k+1}/\kappa} \frac{\xi}{\kappa} \sum_{r=0}^\infty (-1)^r q_{k,2r} e^{\frac{2r\pi}{4}} \Gamma \left( \frac{2r+1}{2} \right) \]
\[ \text{as } w \to \infty; \text{ where} \]
\[ q_{k,2r} := \left( \frac{2}{k} \right)^{r+1} \frac{u_0}{(2r)!} \frac{d^{2r}}{du^{2r}} \left[ G_k(u) \right]^{\frac{2r+1}{2}} \bigg|_{u=\eta}. \]

The result follows by combining (2.50) and (2.39) obtaining \( U_i(k) := \left( \frac{k}{k+1} \right)^r q_{k,2r} \Gamma \left( \frac{2r+1}{2} \right) \) after further very trivial simplifications.

The same procedure yields (2.37) by first differentiating (2.38), and follow the steps above with very slight changes in the expression of the coefficients \( q_{k,2r}' \). In this case, we obtain
\[ q_{k,2r}' := \left( \frac{2}{k} \right)^{r+1} \frac{u_0'}{(2r)!} \frac{d^{2r}}{du^{2r}} \left( \frac{\chi(u)}{|G_k(u)|^{\frac{2r+1}{2}}} \right) \bigg|_{u=\eta}. \]
\[ \text{where } \chi(u) = u \text{ with } V_i(k) := \left( \frac{k}{k+1} \right)^r q_{k,2r}' \Gamma \left( \frac{2r+1}{2} \right). \]

**2.3.2. Expansions for large negative argument**

For the behaviour of \( Ai_k(y) \) and \( Ai_k'(y) \) on the negative real semi-axis, we also have the following results:

**Theorem 2.5.** For each \( k = 2, 4, 6, \ldots \), we have
\[ Ai_k(-y) = \frac{y^{\frac{1+k}{2}}}{\pi} \left\{ \cos \left( \xi - \frac{\pi}{4} \right) P(\xi) + \sin \left( \xi - \frac{\pi}{4} \right) Q(\xi) \right\} \quad \text{as } y \to +\infty, \]
\[ \text{and} \]
\[ Ai'_k(-y) = \frac{y^{\frac{1+k}{2}}}{\pi} \left\{ \sin \left( \xi - \frac{\pi}{4} \right) R(\xi) - \cos \left( \xi - \frac{\pi}{4} \right) S(\xi) \right\} \quad \text{as } y \to +\infty; \]
\[ \text{where} \]
\[ P(\xi) \sim \sum_{r=0}^\infty (-1)^r \frac{j^{2r}}{\xi^{2r}} \quad \text{and} \quad Q(\xi) \sim \sum_{r=0}^\infty (-1)^r \frac{j^{2r+1}}{\xi^{2r+1}}, \]
By setting $t$ obtain stationary point lying on the range of integration ($-u$ has required results. Thus, by differentiating (2.53) we again recall the integral representation

$$R(\xi) = \sum_{r=0}^{\infty} (-1)^{r} \frac{\bar{V}_{2r}(k)}{\xi^{2r}}$$

and

$$S(\xi) = \sum_{r=0}^{\infty} (-1)^{r} \frac{\bar{V}_{2r+1}(k)}{\xi^{2r+1}}$$

with $\xi = \frac{k}{k+1} \frac{\bar{V}_{2r}}{\xi^{2r}}$; while $\bar{U}_{2r}(k), \bar{U}_{2r+1}(k), \bar{V}_{2r}(k), \bar{V}_{2r+1}(k)$, are real constants depending only on $k$, and expressed in terms of Bell polynomials.

**Proof.** To prove (2.53) we again recall the integral representation

$$A_{i} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \left( \frac{k+1}{k+1} \right) y} \, dt = \frac{1}{\pi} \Re \left( \int_{0}^{\infty} e^{i \left( \frac{k+1}{k+1} \right) y} \, dt \right).$$

(2.55)

For $y < 0$, we make the following transformation: $y \leftrightarrow -y$, so that

$$A_{i}(-y) = \frac{1}{\pi} \Re \left( \int_{0}^{\infty} e^{i \left( \frac{k+1}{k+1} \right) y} \, dt \right) \quad (y > 0).$$

(2.56)

By setting $t = y^{\frac{1}{k+1}}(u+1)$ in (2.56), we consider the integral

$$J(w) = w^{\frac{1}{k+1}} \int_{-1}^{\infty} e^{iwh(u)} \, du,$$

(2.57)

where $w = y^{\frac{1}{k+1}}$ and $h_{k}(u) = \frac{1}{k+1}(u+1)^{k+1} - u - 1$. The integrand in (2.57) is highly oscillatory and has $h_{k}(u)$ as its phase function. Hence, we also employ the stationary phase method to derive the required results. Thus, by differentiating $h_{k}$ once and setting it to zero, we obtain the stationary points $u_{j} = e^{\frac{2\pi}{k+1}j} - 1$, $0 \leq j \leq k - 1$ of $h_{k}$. Next, we choose $u_{0} = 0$ (corresponding to $j = 0$), which is the only stationary point lying on the range of integration $(-1, \infty)$, and split the integral (2.57) at this point to obtain

$$\int_{-1}^{\infty} e^{iwh(u)} \, du = \int_{0}^{\infty} e^{iwh_{k}(u)} \, du - \int_{0}^{-1} e^{iwh_{k}(u)} \, du =: J_{1}(w) + J_{2}(w).$$

(2.58)

We can choose $b > 0$ close to $u = 0$, and observe that $h_{k}(u)$ is infinitely continuously differentiable in $(0, b]$, and the inequality $h_{k}'(u) > 0$ holds in this interval. Let

$$\tilde{g}_{k,p} := \frac{(k-1)p}{(p+2)!} \quad \text{for each } p = 1, 2, \ldots, k - 1,$$

(2.59)

and $(k-1)p := (k-1)(k-2)\cdots(k-p)$ with $(k-1)_{0} = 1$. By binomial expansion, we have

$$h_{k}(u) = h_{k}(0) + \frac{k}{2} u^{2} \tilde{G}_{k}(u),$$

(2.60)

where $G_{k}(u) = 1 + \sum_{p=1}^{k-1} \tilde{g}_{k,p} u^{p}$.

For the first integral $J_{1}(w)$ in (2.58), we set $T_{1} := h_{k}(b) - h_{k}(0)$ and make the change of variables

$$s = h_{k}(u) - h_{k}(0),$$

(2.61)
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and since $h_k(u)$ is continuous and monotone increasing on $(0, b)$, a one-to-one relationship exists between $u$ and $s$. Hence, by reversing (2.60) using (2.61), we have

$$u = \sum_{r=1}^{\infty} c_k r s^{r/2} \quad \text{(for } s \text{ near } 0),$$

where $c_k$ is obtained using Lagrange reversion formula (see [21] p. 125):

$$c_k := \frac{1}{r!} \frac{d^{r-1}}{du^{r-1}} \left[ \frac{u^2}{h_k(u) - h_k(0)} \right]^{r/2} \bigg|_{u=0}$$

By employing the change of variables (2.61), we obtain

$$\int_0^b e^{iu h_k(u)} du = e^{iu h_k(0)} \int_0^{T_1} e^{iws} f_1(s) ds = e^{-iws} \left( \int_0^\infty - \int_{T_1}^{\infty} \right) e^{iws} f_1(s) ds,$$  \hspace{1cm} (2.63)

in which for each $N \geq 1$, we have

$$f_1(s) = \frac{du}{ds} = \sum_{r=0}^{N-1} \bar{q}_{k,r} s^{r-1} + F_{1,N}(s) \quad \text{for } 0 < s < T_1,$$

and set $f_1(s) = 0$ for $s \geq T_1$; where $\bar{q}_{k,r} := \frac{r+1}{r} c_{k,r+1}$, and $F_{1,N}(s) = O(s^{\frac{N-1}{2}})$ as $s \to 0^+$. Now, for $\eta > 0$, we have

$$\int_0^\infty e^{iws} f_1(s) ds = \sum_{r=0}^{N-1} \bar{q}_{k,r} \lim_{\eta \to 0^+} \int_0^\infty e^{-\eta s^{r+1}} s^{\frac{r-1}{2}} ds + E_{1,N}(w)$$

$$= \sum_{r=0}^{N-1} \frac{e^{\left(\frac{r+1}{4}\right) \eta r} \bar{q}_{k,r} \Gamma \left( \frac{r+1}{2} \right)}{\eta^{\frac{r+1}{2}}} + E_{1,N}(w),$$  \hspace{1cm} (2.64)

where

$$E_{1,N}(w) = \lim_{\eta \to 0^+} \int_0^\infty e^{-\eta s^{r+1}} F_{1,N}(s) ds$$

$$= O \left( \lim_{\eta \to 0^+} \int_0^\infty e^{-\eta s^{r+1}} s^{\frac{N-1}{2}} ds \right) = O(w^{-\frac{N-1}{2}}).$$

Therefore, by inserting (2.64) into (2.63), we have

$$J_{1}(w) \sim e^{-i \left( \frac{\eta}{r+1} \frac{s^{r+1}}{2} \right)} \sum_{r=0}^{\infty} \frac{e^{i ws} \bar{q}_{k,r} \Gamma \left( \frac{r+1}{2} \right)}{\eta^{\frac{r+1}{2}}} \quad \text{as } w \to \infty,$$  \hspace{1cm} (2.65)

where $\bar{q}_{k,r} := \left( \frac{2}{\pi} \right)^{\frac{r+1}{2}} \frac{1}{2} \frac{w^{r+1}}{d u} \left[ G_k(u) \right]^{-\frac{1}{2}} \bigg|_{u=0}^{-\frac{1}{2}}$.

Similarly, for the integral $J_2(w)$ in (2.58), we set $T_2 := h_k(-1) - h_k(0)$ and make the change of variables

$$s = h_k(0) - h_k(u),$$  \hspace{1cm} (2.66)
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and since \( h_k(u) \) is continuous on \((-1, 0)\) and from \( u = 0 \) to \( u = -1 \), \( h_k(u) \) is monotone increasing, a one-to-one relationship also exists between \( u \) and \( s \). Hence, by reversing (2.60) using (2.66), we have

\[
u = \sum_{r=1}^{\infty} (-1)^r c_k r^{r/2}, \quad \text{(for } s \text{ near } 0),
\]

where \( c_k \) is the same as the last expression in (2.62) obtained using the Lagrange reversion formula. By employing the change of variables (2.66), we then obtain

\[
\int_{-1}^{1} e^{ivh_k(u)} \, du = e^{ivh_k(0)} \int_{0}^{-T_2} e^{-iws} f_2(s) \, ds = -e^{-\frac{i\pi}{4}} \left( \int_{0}^{\infty} - \int_{T_2}^{\infty} \right) e^{iws} f_2(s) \, ds, \tag{2.67}
\]

in which for each \( N \geq 1 \), we have

\[
f_2(s) = \frac{du}{ds} = \sum_{r=0}^{N-1} (-1)^r \bar{q}_{k,r} s^{r+\frac{1}{2} - 1} + F_{2,N}(s) \quad \text{for } 0 < s < T_2,
\]

and set \( f_2(s) = 0 \) for \( s \geq T_2 \); and \( F_{2,N}(s) = O(s^{\frac{N+1}{2}}) \) as \( s \to 0^+ \). Now, for \( \eta > 0 \), we have

\[
\int_{0}^{\infty} e^{iws} f_2(s) \, ds = \sum_{r=0}^{N-1} (-1)^r \bar{q}_{k,r} \lim_{\eta \to 0^+} \int_{0}^{\infty} e^{-\eta q + iws} s^{r+\frac{1}{2} - 1} \, ds + E_{2,N}(w)
\]

\[
= \sum_{r=0}^{N-1} (-1)^r e^{\frac{i\pi}{4+r}} \bar{q}_{k,r} \Gamma \left( \frac{r+1}{2} \right) + E_{2,N}(w), \tag{2.68}
\]

where

\[
E_{2,N}(w) = \lim_{\eta \to 0^+} \int_{0}^{\infty} e^{-\eta q + iws} F_{2,N}(s) \, ds
\]

\[
= O \left( \lim_{\eta \to 0^+} \int_{0}^{\infty} e^{-\eta q + iws} s^{\frac{N-1}{2}} \, ds \right) = O(w^{-\frac{N+1}{4}}).
\]

Therefore, by inserting (2.68) into (2.67), we have

\[
J_2(w) = -\int_{-1}^{1} e^{ivh_k(u)} \, du \sim e^{-i(\frac{\pi}{4} - \frac{1}{2})} \sum_{r=0}^{\infty} (-1)^r e^{\frac{i\pi}{4+r}} \bar{q}_{k,r} \Gamma \left( \frac{r+1}{2} \right) \quad \text{as } w \to \infty, \tag{2.69}
\]

where \( \bar{q}_{k,r} := \left( \frac{2}{\pi} \right)^{\frac{r+1}{2}} \frac{1}{2^r} \left. \frac{d^r}{du^r} (\bar{G}_k(u)) \right|_{u=0}^{-\frac{r+1}{2}} \right|_{u=0}^{-\frac{r+1}{2}} \). Therefore, by combining (2.65) and (2.69) into (2.58), we obtain

\[
\int_{-1}^{1} e^{ivh_k(u)} \, du = e^{-i(\frac{\pi}{4} - \frac{1}{2})} \sum_{r=0}^{\infty} e^{\frac{i\pi}{4+r}} \bar{q}_{k,2r} \Gamma \left( r + \frac{1}{2} \right) \quad \text{as } w \to \infty, \tag{2.70}
\]

where

\[
\bar{q}_{k,2r} := \left( \frac{2}{\pi k} \right)^{r+\frac{1}{2}} \frac{1}{(2r)!} \left. \frac{d^{2r}}{du^{2r}} (\bar{G}_k(u)) \right|_{u=0}^{-\frac{r+1}{2}} \right|_{u=0}^{-\frac{r+1}{2}}. \tag{2.71}
\]

Further expansion of (2.70) and equating real parts using (2.27) together with \( \bar{U}_{2r}(k) \), we obtain (2.53) with \( \bar{U}_{2r}(k) := (\frac{k}{k+1})^r \bar{q}_{k,2r} \Gamma \left( 2r + \frac{1}{2} \right) \) and \( \bar{U}_{2r+1}(k) := (\frac{k}{k+1})^r \bar{q}_{k,2r+1} \Gamma \left( 2r + \frac{3}{2} \right) \).
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A very slight modification of the above procedure yields (2.54). Hence, we differentiate in (2.55) and make the following transformation: \( y \mapsto -y \), to obtain

\[
Ai'_k(-y) = \frac{1}{\pi} \Re \left( i \int_0^\infty t e^{i \frac{y}{k+1} t} \, dt \right), \quad (y > 0).
\]  

(2.72)

By the same change of variable \( t = y^{\frac{1}{k+1}} (u + 1) \), we may consider the integral

\[
J'(y) = iy^{\frac{1}{k+1}} \int_{-1}^\infty e^{i w h_k(u)} \chi(u) \, du,
\]

where \( w = y^{\frac{1}{k+1}} \), \( h_k(u) = \frac{(u+1)^{k+1}}{k+1} - u - 1 \), and \( \chi(u) = u + 1 \). By similar argument leading to (2.70), we also obtain

\[
J'(y) = iy^{\frac{1}{k+1}} \int_{-1}^\infty e^{i w h_k(u)} \, du \sim iy^{\frac{1}{k+1}} e^{-i \left( \frac{k+1}{k+1} \chi(u) \right)} \sum_{r=0}^\infty \frac{e^{i w} \tilde{q}_{k,2r} \Gamma \left( r + \frac{1}{2} \right)}{w^{r+\frac{1}{2}}} \text{ as } w \to \infty,
\]  

(2.73)

where

\[
\tilde{q}_{k,2r} := \left( \frac{2}{k} \right)^{r+\frac{1}{2}} \frac{1}{(2r)!} \frac{d^{2r}}{d u^{2r}} \left( \frac{\chi(u)}{(G_k(u))^{\frac{1}{2}}} \right) \bigg|_{u=0}.
\]  

(2.74)

Hence, by further expansion of (2.73) and equating real parts using (2.72), we obtain (2.54) with \( \tilde{V}_{2r}(k) := \left( \frac{k}{k+1} \right)^r \tilde{q}_{k,2r} \Gamma \left( 2r + \frac{3}{2} \right) \) and \( \tilde{V}_{2r+1}(k) := \left( \frac{k}{k+1} \right)^r \tilde{q}_{k,2r+2} \Gamma \left( 2r + \frac{5}{2} \right) \), which completes the proof.

\[\square\]

2.3.3. Coefficients of the asymptotic expansions

In this subsection, we derive explicit expressions for the coefficients of the asymptotic expansions obtained for \( Ai_k(y) \) and \( Ai'_k(y) \) in Theorems 2.4 and 2.5, respectively. These explicit expressions are given in terms of Bell polynomials (refer to the results of Theorem 2.8 and Corollary 2.2 below) obtained by applying Faà di Bruno’s formula (for computing the repeated derivative of a composite function) on the Lagrange formula expressions given by (2.51), (2.52), (2.71), and (2.74). But before showing these results, we put together various tools used in the derivation, which are garnered from combinatorics.

Adopting notations of [20], for any \( \sigma \in \mathbb{C} \), we use \( C_{\sigma'} [G(u)]^{\sigma'} \) to denote the coefficient of \( u^{\sigma'} \) in the formal expansion of \([G(u)]^{\sigma'} \), where \( G(u) \) is a formal unitary power series about the origin; that is, \( G(0) = 1 \). For each \( r, l \geq 0 \), \( \mathcal{B}_{r,l}(g_1, \ldots) \) denotes the ordinary incomplete/partial Bell polynomial defined as \( \mathcal{B}_{r,l} := C_{\sigma'} [g(u) - 1]^{\sigma'} \) (a brief review of Bell polynomials is presented in the appendix; see [20], pp. 133-137 for more details). We define as follows:
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**Definition 2.2** (Ordinary Potential Polynomials). Let \( G \) be a formal unitary power series \( G(u) = 1 + g_1 u + g_2 u^2 + \cdots \), for any \( \sigma \in \mathbb{C} \) we define the ordinary potential polynomial \( \mathcal{P}_r^{(\sigma)} \) by

\[
[G(u)]^\sigma = \sum_{r \geq 0} \mathcal{P}_r^{(\sigma)} u^r \quad \text{with} \quad \mathcal{P}_0^{(\sigma)} = 1.
\]

(2.75)

In other words, the ordinary potential polynomial \( \mathcal{P}_r^{(\sigma)} \) is a combinatorial function acting on the finite set \( \{g_1, \cdots, g_r\} \) given by

\[
\mathcal{P}_r^{(\sigma)} \equiv \mathcal{P}_r^{(\sigma)}(g_1, \cdots, g_r) := C_{\sigma r}[G(u)]^\sigma.
\]

(2.76)

Furthermore, let \( G \) be some differentiable function of class \( C^\infty \), we consider the following expansion:

\[
[G(u)]^\sigma = \left( \sum_{p \geq 0} g_p u^p \right)^\sigma = \left( 1 + \sum_{p \geq 1} g_p u^p \right)^\sigma = 1 + \sum_{l \geq 1} \binom{\sigma}{l} \left( \sum_{p \geq 1} g_p u^p \right)^l
\]

\[
= 1 + \sum_{l \geq 1} \binom{\sigma}{l} \sum_{r \geq l} B_r,l(g_1, g_2, \cdots, g_{r-l+1}) u^r
\]

\[
= 1 + \sum_{r \geq l} \binom{\sigma}{l} B_r,l(g_1, g_2, \cdots, g_{r-l+1}) u^r \quad \text{with} \quad g_0 = 1.
\]

(2.77)

Hence, using the following relation, the ordinary potential polynomial \( \mathcal{P}_r^{(\sigma)} \) can be computed as the \( r \)-th repeated derivative of the composite function \( [G(u)]^\sigma \) evaluated at the point \( u = 0 \):

**Theorem 2.6.** Let \( r = 1, 2, \cdots \), and \( \sigma \in \mathbb{C} \), we have

\[
\mathcal{P}_r^{(\sigma)} \equiv \mathcal{P}_r^{(\sigma)}(g_1, g_2, \cdots, g_r) = \sum_{l=1}^r \binom{\sigma}{l} B_r,l(g_1, g_2, \cdots, g_{r-l+1}),
\]

(2.78)

where \( g_p := \frac{d^p}{du^p}G(u)|_{u=0} \) for each \( p = 0, 1, 2, \cdots \) with \( \mathcal{P}_0^{(\sigma)} = 1 \).

**Remark 2.2.** Observe that if we replace \( g_p \) with \( g_p/p! \) for each \( p \geq 0 \) in (2.77), then using (B-4), we obtain

\[
\left( \sum_{p \geq 0} \frac{g_p}{p!} \right)^\sigma = 1 + \sum_{r \geq 1} \mathcal{P}_r^{(\sigma)}(g_1, \cdots, g_r) \frac{u^r}{r!} = 1 + \sum_{r \geq 1} \left( \sum_{l=1}^r \binom{\sigma}{l} B_r,l(g_1, g_2, \cdots, g_{r-l+1}) \right) \frac{u^r}{r!},
\]

(2.79)

where \( B_r,l \) is now the exponential partial Bell polynomial and \( \mathcal{P}_r^{(\sigma)} \) is its associated potential polynomial; one can compare (2.79) with [20] Theorem B, p.141.

The best way to compute the repeated derivative of a composite function is by the use of Faà di Bruno’s formula. This is stated as follows:
Therefore, by substituting (2.85) and (2.86) into (2.80), we obtain
\[ \frac{d^j}{du^j} H(g(u)) = \sum \frac{j!}{\epsilon_1! \epsilon_2! \cdots \epsilon_j!} H^{(\epsilon_1+\epsilon_2+\cdots+\epsilon_j)}(g(u)) \prod_{p=1}^{j} \left( \frac{g^{(p)}(u)}{p!} \right)^{\epsilon_p}, \tag{2.80} \]
where the sum is taking over all sets of non-negative integer solutions \( \epsilon_1, \ldots, \epsilon_j \) of \( \epsilon_1 + 2\epsilon_2 + \cdots + j\epsilon_j = j \) subject to \( \epsilon_1 + \epsilon_2 + \cdots + \epsilon_j = l \) for each non-negative integer \( l \geq 0 \).

**Proof.** See [20, Section 3.4]. \( \square \)

Now, for some \( a \in \mathbb{R} \), we consider the following formal unitary power series:
\[ G(u) = 1 + g_1(u - a) + g_2(u - a)^2 + \cdots \quad \text{with} \quad G(a) = 1, \tag{2.81} \]
where \( g_p := G^{(p)}(a) \) for each \( p = 0, 1, 2, \cdots \) with \( g_0 = 1 \), and let
\[ g(u) := G(u) - 1 = g_1(u - a) + g_2(u - a)^2 + \cdots \quad \text{with} \quad g(a) = 0. \tag{2.82} \]
Furthermore, let \( H(w) = w^l \), and define the composite function of \( g \) by \( H \) by the following formal power series expansion:
\[ H(g(u)) := H \circ g = [g(u)]^l = b_1, l(u - a) + b_2, l(u - a)^2 + b_3, l(u - a)^3 + \cdots. \tag{2.83} \]
Our aim is to determine the coefficients \( b_{j,l} \). This can be obtained by repeated differentiation of (2.83) using the Faà di Bruno’s formula (Theorem 2.7); and we have
\[ b_{j,l} = \frac{1}{j!} \left. \frac{d^j}{du^j} [g(u)]^l \right|_{u=a}. \tag{2.84} \]
But since \( g(a) = 0 \) (refer to (2.82)), we have that
\[ \frac{d^j}{dg^j} [g(u)]^l \bigg|_{u=a} = \begin{cases} l! & j = l \\ 0 & \text{otherwise}. \end{cases} \tag{2.85} \]
It also follows from (2.82) that
\[ g^{(p)}(a) = p! g_p. \tag{2.86} \]
Therefore, by substituting (2.85) and (2.86) into (2.80), we obtain
\[ \left. \frac{1}{j!} \frac{d^j}{du^j} [g(u)]^l \right|_{u=a} = \sum \frac{l!}{\epsilon_1! \epsilon_2! \cdots \epsilon_j!} \prod_{p=1}^{j} g_p^{\epsilon_p}, \]
which, by (2.84), gives the expression for the coefficients \( b_{j,l} \) for each \( j, l \geq 0 \). By definition, we may now set
\[ \mathcal{B}_{j,l}(g_1, g_2, \ldots, g_{j-l+1}) := b_{j,l}. \tag{2.87} \]
Let \( \chi(u) \) be some function with sufficient number of derivatives, and consider
\[ q_{\sigma} := \frac{1}{r!} \frac{d^r}{du^r} \left\{ [G(u)]^{-\sigma} \chi(u) \right\} \bigg|_{u=a} \quad \text{for} \quad \sigma \in \mathbb{C}. \tag{2.88} \]
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By putting together the results of Definition 2.2, (2.84), and (2.87), we prove the next important result, which will be applied directly to compute the constant coefficients explicitly, as required.

**Theorem 2.8.** Let \( \sigma \in \mathbb{C} \) and \( r \in \mathbb{N}_0 \), we have

\[
q_r = \sum_{j=0}^{r} a_{r-j} \sum_{l=0}^{j} \left( -\frac{\sigma}{l} \right) \mathcal{R}_{j,l}(g_1, g_2, \cdots, g_{j-l+1}),
\]

where

\[
a_{r-j} := \frac{1}{(r-j)!} \left. \frac{d^{r-j}}{du^{r-j}} \chi(u) \right|_{u=a}.
\]

**Proof.** By Leibniz formula, we have

\[
\frac{1}{r!} \left. \frac{d^r}{du^r} \left\{ [G(u)]^{-\sigma} \chi(u) \right\} \right|_{u=a} = \sum_{j=0}^{r} \frac{1}{r!} \left( \begin{array}{c} r \\ j \end{array} \right) \left. \left[ \frac{d^j}{du^j} [G(u)]^{-\sigma} \right] \right|_{u=a} \left[ \left. \frac{d^{r-j}}{du^{r-j}} \chi(u) \right|_{u=a} \right]
\]

\[
= \sum_{j=0}^{r} a_{r-j} \frac{1}{j!} \left. \frac{d^j}{du^j} [G(u)]^{-\sigma} \right|_{u=a},
\]

where \( a_{r-j} \) is given by (2.90). From (2.75), it follows that

\[
[G(u)]^{-\sigma} = \sum_{j=0}^{r} C_{wj} [G(u)]^{-\sigma} u^j = 1 + \sum_{j=1}^{\infty} \mathcal{R}_{j}^{(\sigma)}(g_1, \cdots, g_r) u^j \quad \text{with} \quad \mathcal{R}_0^{(\sigma)} = 1
\]

where \( C_{wj} [G(u)]^{-\sigma} \) is given by (2.90). From (2.75), it follows that

\[
[G(u)]^{-\sigma} = [1 + (G(u) - 1)]^{-\sigma} = \sum_{j=0}^{r} \left( -\frac{\sigma}{l} \right) [G(u) - 1]^l = 1 + \sum_{j=1}^{\infty} \left( -\frac{\sigma}{l} \right) \sum_{j=1}^{\infty} C_{wj} [G(u) - 1]^l u^j
\]

\[
= 1 + \sum_{j=1}^{\infty} \left( \sum_{l=1}^{j} \left( -\frac{\sigma}{l} \right) C_{wj} [G(u) - 1]^l \right) u^j,
\]

where

\[
C_{wj} [G(u) - 1]^l := \left. \frac{1}{j!} \frac{d^j}{du^j} [G(u) - 1]^l \right|_{u=a}.
\]

By comparing (2.92) and (2.93), we also have that

\[
\mathcal{R}_{j}^{(\sigma)}(g_1, \cdots, g_r) = \sum_{l=1}^{j} \left( -\frac{\sigma}{l} \right) C_{wj} [G(u) - 1]^l.
\]

But from (2.82), we have that \( g(u) = G(u) - 1 \); therefore, using (2.84) together with (2.87), it follows that

\[
C_{wj} [G(u) - 1]^l := C_{wj} [g(u)]^l = \left. \frac{1}{j!} \frac{d^j}{du^j} [g(u)]^l \right|_{u=a} = \mathcal{R}_{j,l}(g_1, g_2, \cdots, g_{j-l+1}),
\]

which when combined with (2.94) together with (2.92), yields

\[
\frac{1}{j!} \left. \frac{d^j}{du^j} [G(u)]^{-\sigma} \right|_{u=a} = \sum_{l=1}^{j} \left( -\frac{\sigma}{l} \right) \mathcal{R}_{j,l}(g_1, g_2, \cdots, g_{j-l+1}).
\]
2.4. Negative Real Zeros of $A_{ik}(y)$ and $A_{ik}'(y)$

The result (2.89) then follows by inserting (2.95) into (2.91), which completes the required proof. □

Using the result of Theorem 2.8, we are now ready to derive explicit forms of the constant coefficients as required. Recall from Theorems 2.4 and 2.5 that

$$U_r := \left( \frac{k}{k + 1} \right)^r q_{k,2r}; \quad V_r := \left( \frac{k}{k + 1} \right)^r q_{k,2r}', \quad \text{for each } r = 0, 1, 2, \ldots \quad (2.96)$$

$$\bar{U}_r := \left( \frac{k}{k + 1} \right)^r \bar{q}_{k,2r}; \quad \bar{V}_r := \left( \frac{k}{k + 1} \right)^r \bar{q}_{k,2r}', \quad \text{for each } r = 0, 1, 2, \ldots, \quad (2.97)$$

where $q_{k,t}$, $q_{k,l}$, $\bar{q}_{k,t}$, and $\bar{q}_{k,l}$ are given by (2.51), (2.52), (2.71), and (2.74), respectively. Hence, we deduce the following results by direct application of Theorem 2.8 given (2.96):

**Corollary 2.2.** We have that $U_0 = \sqrt{\frac{2\pi}{k}}$ and $V_0 = \sqrt{\frac{2\pi}{k}}$, and for each $r = 1, 2, \ldots$, we have

$$U_r = \left( \frac{k}{k + 1} \right)^r \Gamma \left( r + \frac{1}{2} \right) \sum_{j=1}^{2r} \left( \frac{-r+1}{r} \right)^j \bar{B}_{2j}(g_{1,k}, \ldots, g_{2r-j+1,k}) \quad (2.98)$$

and

$$V_r = \left( \frac{k}{k + 1} \right)^r \Gamma \left( r + \frac{1}{2} \right) \sum_{j=0}^{2r} \sum_{l=0}^{j} \left( \frac{-r+1}{l} \right) \bar{B}_{2j}(g_{1,k}, \ldots, g_{2r-j+1,k}) \quad (2.99)$$

where $a_{2r-1} = a_{2r} = 1$, and $a_{2r-j} = 0$ for each $j = 1, 2, \ldots, 2r - 2$, and

$$g_{p,k} := \frac{2(k-1)!}{(p+2)!} \quad \text{for each } p = 1, 2, \ldots, k - 1, \text{ and } g_{p,k} = 0 \text{ for } p \geq k.$$

**Remark 2.3.** Similar expressions also hold for $\bar{U}_r$ and $\bar{V}_r$ by directly applying Theorem 2.8 on (2.97) with the only difference being that the entries $g_{p,k}$ for $\bar{B}_{2j}$ and $\bar{B}_{2r,l}$ are now replaced by $\bar{g}_{p,k} := \frac{2(k-1)!}{(p+2)!}$ for each $p = 1, 2, \ldots, k - 1$, and $\bar{g}_{p,k} = 0$ for $p \geq k$.

2.4. Negative real zeros of $A_{ik}(y)$ and $A_{ik}'(y)$

In this section, we present asymptotic expansions of the negative real zeros of the higher-order Airy functions $A_{ik}(y)$ and those of their first derivatives $A_{ik}'(y)$. We employ the idea of reversion of asymptotic expansions (cf. [28]). First, we recall some background study on the reversion of asymptotic expansions (or power series expansion) taken from [28].

Let us consider the following asymptotic expansion relation in a certain sector of the complex $z$-plane:

$$f(z) \sim z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \quad \text{as } z \to \infty \quad (2.100)$$

Specifically, we consider a closed annular sector denoted by $S$ whose vertex subtends an angle less than $2\pi$ at the origin and contains two closed annular sectors $S_1$ and $S_2$ with vertices at the origin;
where \( S_1 \) is a proper interior to \( S \) and \( S_2 \) is a proper interior to \( S_1 \). We will assume that \( f(z) \) is analytic in a region containing the annular sector \( S \). The following important statement is taken from [28 pp. 764-765]:

**Theorem 2.9** (Reversion of asymptotic expansions). Let \( f(z) \) be analytic in a region containing \( S \), and \( f(z) \) be given by \((2.100)\) as \( z \to \infty \) in \( S \) uniformly with respect to \( \arg z \). If the boundary arcs of \( S_1 \) and \( S_2 \) subtend sufficiently large radii. Then:

(1) \( \zeta = f(z) \) has exactly one root \( z = F(\zeta) \) in \( S_1 \) for each \( \zeta \in S_2 \);

(2) \( F(\zeta) \) is analytic in \( S_2 \); and

(3) as \( \zeta \to \infty \) in \( S_2 \)

\[ z \sim \zeta - b_0 - b_1 - b_2 \frac{1}{\zeta} - \cdots . \quad (2.101) \]

Moreover, by Lagrange’s theorem, we have for each \( n = 1, 2, \cdots \) that \( nb_n \) is the coefficient of \( z^{-1} \) in the asymptotic expansion of \( (f(z))^n \) in descending powers of \( z \). The first four coefficients may be verified to be (cf. [59 p. 22])

\[ b_0 = a_0, \quad b_1 = a_1, \quad b_2 = a_0a_1 + a_2, \quad b_3 = a_0^2a_1 + a_1^2 + 2a_0a_2 + a_3. \]

We will now apply the idea enunciated above leading to Theorem 2.9 to obtain the asymptotic expansions of the negative real zeroes of the higher-order Airy functions \( \text{Ai}_k(y) \) and the first derivatives \( \text{Ai}_k'(y) \). We take \( \alpha_{k,j} \) and \( \alpha'_{k,j} \) (for \( j \in \mathbb{N} \)) to denote the negative real zeroes of \( \text{Ai}_k \) and \( \text{Ai}_k' \), respectively. Furthermore, we set the following expressions:

\[ A_1 := \sigma_0; \quad A_2 := \left[ \frac{\sigma_2^2}{2} (1 - \rho(k)) \right]; \quad A_3 := \frac{\sigma_2^3}{3} - \sigma_0 \sigma_2 \rho(k) - \frac{\sigma_0^3}{6} \rho(k)(3 - \rho(k)), \]

and

\[ \tilde{A}_1 := \tilde{\sigma}_0; \quad \tilde{A}_2 := \left[ \frac{\tilde{\sigma}_2^2}{2} (1 - \rho(k)) \right]; \quad \tilde{A}_3 := \frac{\tilde{\sigma}_2^3}{3} - \tilde{\sigma}_0 \tilde{\sigma}_2 \rho(k) - \frac{\tilde{\sigma}_0^3}{6} \rho(k)(3 - \rho(k)); \]

where

\[ \sigma_2 := \sigma_0^2 + \frac{1}{3} (3 \sigma_1 + \sigma_0^3); \quad \sigma_3 := 2\sigma_0^3 + \frac{4}{3} \sigma_0 (3 \sigma_1 + \sigma_0^3) + \frac{1}{5} (5 \sigma_2 + 5 \sigma_0^2 \sigma_1 + \sigma_0^5), \]

\[ \tilde{\sigma}_2 := \tilde{\sigma}_0^2 + \frac{1}{3} (3 \tilde{\sigma}_1 + \tilde{\sigma}_0^3); \quad \tilde{\sigma}_3 := 2\tilde{\sigma}_0^3 + \frac{4}{3} \tilde{\sigma}_0 (3 \tilde{\sigma}_1 + \tilde{\sigma}_0^3) + \frac{1}{5} (5 \tilde{\sigma}_2 + 5 \tilde{\sigma}_0^2 \tilde{\sigma}_1 + \tilde{\sigma}_0^5), \]

with

\[ \sigma_r := \sum_{l=0}^{r} (-1)^l \frac{U_{2l+1}(k)}{U_0(k)} \left( \frac{1}{U_0(k)} \right)^{(r-1)} \quad \text{for each } r = 1, 2, \cdots \quad \text{and } \sigma_0 := \frac{U_1(k)}{U_0(k)} \text{ and } \left( \frac{1}{U_0(k)} \right)^{(r-1)} = 1 \]

\[ \tilde{\sigma}_r := \sum_{l=0}^{r} (-1)^l \frac{\tilde{V}_{2l+1}(k)}{\tilde{V}_0(k)} \left( \frac{1}{\tilde{V}_0(k)} \right)^{(r-1)} \quad \text{for each } r = 1, 2, \cdots \quad \text{and } \tilde{\sigma}_0 := \frac{\tilde{V}_1(k)}{\tilde{V}_0(k)} \text{ and } \left( \frac{1}{\tilde{V}_0(k)} \right)^{(r-1)} = 1, \]
and $\mathcal{P}_{m}^{(-1)}$ and $\mathcal{P}_{m}^{(-1)}$ are the potential polynomials obtained by expanding $[P(\xi)]^{-1}$ and $[R(\xi)]^{-1}$, respectively, expressed as

$$\mathcal{P}_{m}^{(-1)} := \sum_{j=1}^{m} \left( -1 \right)^{j} \mathcal{B}_{m,j} \left( -\frac{\bar{U}_{1}(k)}{U_{0}(k)} \ldots, \frac{(-1)^{m-j+1} \bar{U}_{2(m-j+1)}(k)}{U_{0}(k)} \right),$$

$$\mathcal{P}_{m}^{(-1)} := \sum_{j=1}^{m} \left( -1 \right)^{j} \mathcal{B}_{m,j} \left( -\frac{\bar{V}_{1}(k)}{V_{0}(k)} \ldots, \frac{(-1)^{m-j+1} \bar{V}_{2(m-j+1)}(k)}{V_{0}(k)} \right).$$

All of $\bar{U}_{r}(k), \bar{V}_{r}(k)$ for each $r = 0, 1, 2, \cdots, P(\xi)$, and $R(\xi)$ appear in the expansions of $\text{Ai}_{k}$ and $\text{Ai}_{k}'$ in Theorem 2.5 above, and $\mathcal{B}_{m,j}(a_{1}, a_{2}, \cdots)$ denotes ordinary incomplete Bell polynomials. Therefore, we have as follows:

**Theorem 2.10.** For each $k = 2, 4, 6, \cdots$, we have

$$\alpha_{k,j} = -f \left( \frac{4j - 1}{4\varrho(k)} \pi \right); \quad \alpha'_{k,j} = -g \left( \frac{4j - 3}{4\varrho(k)} \pi \right), \quad (2.102)$$

where

$$f(\tau) \sim \tau^{\varrho(k)} \left[ 1 + \frac{A_{1}}{\varrho(k)} \frac{1}{\tau^{1}} - \frac{A_{2}}{[\varrho(k)]^{2}} \frac{1}{\tau^{2}} + \frac{A_{3}}{[\varrho(k)]^{3}} \frac{1}{\tau^{3}} - \cdots \right], \quad (2.103)$$

$$g(\tau) \sim \tau^{\varrho(k)} \left[ 1 + \frac{\bar{A}_{1}}{\varrho(k)} \frac{1}{\tau^{1}} - \frac{\bar{A}_{2}}{[\varrho(k)]^{2}} \frac{1}{\tau^{2}} + \frac{\bar{A}_{3}}{[\varrho(k)]^{3}} \frac{1}{\tau^{3}} - \cdots \right], \quad (2.104)$$

as $\tau \to \infty$; with $\varrho(k) := \frac{k}{k+1}$.

**Proof.** We only show the steps leading to (2.103) as similar steps yield (2.104). For $y > 0$, it follows from (2.53) that

$$\pi y^{\frac{1}{k+1}} \text{Ai}_{k}(-y) = \cos \left( \xi - \frac{\pi}{4} \right) P(\xi) + \sin \left( \xi - \frac{\pi}{4} \right) Q(\xi), \quad \xi = \frac{k}{k+1} y^{\frac{1}{k+1}}, \quad (2.105)$$

with

$$P(\xi) \sim U_{0}(k) \left[ 1 + \sum_{r=1}^{\infty} (-1)^{r} \frac{\bar{U}_{2r}(k)}{U_{0}(k)} \frac{1}{\xi^{2r}} \right], \quad Q(\xi) \sim \frac{U_{1}(k)}{\xi} \left[ 1 + \sum_{r=1}^{\infty} (-1)^{r} \frac{\bar{U}_{2r+1}(k)}{U_{1}(k)} \frac{1}{\xi^{2r}} \right].$$

For large $y$, the dominant term on the right-hand side of (2.105) is $\cos \left( \xi - \frac{\pi}{4} \right) P(\xi)$, and we can set

$$P(\xi) = G(\xi) \cos(\theta) \quad \text{and} \quad Q(\xi) = G(\xi) \sin(\theta),$$

with

$$G(\xi) = \sqrt{[P(\xi)]^{2} + [Q(\xi)]^{2}} \quad \text{and} \quad \theta = \arctan \left( \frac{Q(\xi)}{P(\xi)} \right).$$

Then, we have $\pi y^{\frac{1}{k+1}} \text{Ai}_{k}(-y) = G(\xi) \cos \left( \xi - \frac{\pi}{4} - \theta \right)$ equivalent to (2.105), and by setting $\text{Ai}_{k}(-y) = 0$, we have

$$\xi = \beta_{j} + \theta \quad \text{with} \quad \beta_{j} = \left( j - \frac{1}{4} \right) \pi \text{ for each } j \in \mathbb{N}. \quad (2.106)$$
Next, we obtain the series of \( \tan(\theta) \) in the powers of \( \xi^{-1} \). Thus, we have
\[
\tan(\theta) = \frac{Q(\xi)}{P(\xi)} \sim \frac{\dot{U}_1(k)}{U_0(k)} \xi \left( 1 + \sum_{r \geq 1} (-1)^r \frac{\dot{U}_{2r+1}(k)}{U_1(k)} \xi^{2r} \right) \left( 1 + \sum_{r \geq 1} (-1)^r \frac{\dot{U}_{2r}(k)}{U_0(k)} \xi^{2r} \right)^{-1}
\]
\[
= \frac{\dot{U}_1(k)}{U_0(k)} \xi \left( 1 + \sum_{r \geq 1} (-1)^r c_r \xi^{2r} \right),
\]
where \( c_r := \sum_{m=0}^{r} (-1)^{m-r} \frac{\dot{U}_{2m+1}(k)}{U_1(k)} \mathcal{P}_{r-m}^{(-1)} \) and \( \mathcal{P}_{l}^{(-1)} \) is the potential polynomial obtained by expanding \( \left( 1 + \sum_{r \geq 1} (-1)^r \frac{\dot{U}_{2r}(k)}{U_0(k)} \xi^{2r} \right)^{-1} \).

We now proceed to obtain the series of \( \theta \) in powers of \( \xi^{-1} \). Recall that
\[
\theta = \sum_{l \geq 1} (-1)^{l+1} \frac{[\tan(\theta)]^{2l-1}}{2l-1},
\]
and from (2.107), we set \( \sigma_0 = \dot{U}_1(k)/U_0(k) \) and \( z = \xi^{-2} \), then from (2.108) we obtain
\[
\theta \sim \sum_{l \geq 1} \frac{(-1)^{l+1} \sigma_0^{2l-1}}{2l-1} \left( 1 + \sum_{r \geq 1} (-1)^r c_r z^r \right)^{2l-1}
\]
\[
= \sum_{l \geq 1} \frac{(-1)^{l+1} \sigma_0^{2l-1}}{2l-1} \xi^{2l-1} \left[ 1 + \sum_{r \geq 1} \left( \sum_{m=1}^{2l-1} \frac{2l-1}{m} \mathcal{P}_{p,m}(-c_1, \ldots, (-1)^{p-m+1}c_{p-m+1}) \right) z^m \right]
\]
\[
= \frac{\sigma_0}{\xi} - \frac{(3\sigma_0 c_1 + \sigma_0^3)}{3} \frac{1}{\xi^3} + \left( \frac{5\sigma_0 c_2 + 5\sigma_0^3 c_1 + \sigma_0^5}{5} \right) \frac{1}{\xi^5} - \cdots.
\]
We take \( \sigma_l = \sigma_0 c_l \) in (2.109) for each \( l = 1, 2, \ldots \), and using (2.106), we obtain
\[
\xi \sim \beta_j + \frac{\sigma_0}{\xi} - \frac{(3\sigma_1 + \sigma_0^3)}{3} \frac{1}{\xi^3} + \left( \frac{5\sigma_2 + 5\sigma_0^3 \sigma_1 + \sigma_0^5}{5} \right) \frac{1}{\xi^5} - \cdots.
\]
Now, we obtain series for \( \xi \) in powers of \( \beta_j^{-1} \). Thus, consider an expansion of the form
\[
\xi \sim \beta_j + \frac{a_1}{\xi} + \frac{a_3}{\xi^3} + \frac{a_5}{\xi^5} + \cdots,
\]
it follows from (2.101) that
\[
\xi \sim \beta_j - \frac{a_1}{\beta_j} - \frac{(a_1^2 + a_3)}{\beta_j^3} - \frac{2a_1^3 + 4a_1 a_3 + a_5}{\beta_j^5} + \cdots.
\]
Therefore, by equating coefficients, we obtain \( a_1 = \sigma_0; a_3 = -\frac{1}{4}(3\sigma_1 + \sigma_0^3); a_5 = \frac{1}{4}(5\sigma_2 + 5\sigma_0^3 \sigma_1 + \sigma_0^5) \), and since \( y = (\tilde{\varphi}(k))^{-1} \varphi^{(k)} \) (where \( \varphi^{(k)} = \frac{k}{\varphi} \)), we conclude that for large \( j \) the equation \( \text{Ai}_k(-y) = 0 \) is satisfied by \( y = f(\tau) \) with \( \tau = [\varphi^{(k)}]^{-1} \beta_j \), and by binomial expansion, \( f(\tau) \) is given by (2.103). We then conclude by an application of the principle of argument [28, Theorem 2.5] that \( a_{k,j} = -y \). \( \Box \)

The leading term of the above expansions are specifically good approximations of large negative real zeroes of \( \text{Ai}_k(y) \) and \( \text{Ai}_k'(y) \). Hence, we state the result more succinctly.
Hence, it follows that the function $\cos \left( \frac{4j - 3\pi}{4g(k)} \right)$ as $j \to \infty$. Furthermore, we observe that the right-hand side integral in (2.114) is absolutely and uniformly convergent outside $[-1, 1]$ under the integral.

Corollary 2.4. For each $n \in \mathbb{N}$, we have

$$\alpha_{4,j} = -f \left( \frac{5(4j - 1)\pi}{16} \right); \quad \alpha'_{4,j} = -g \left( \frac{5(4j - 3)\pi}{16} \right),$$

(2.111)

where

$$f(\tau) \sim \tau^{4/5} \left( 1 + \frac{0.1586783204}{\tau^2} - \frac{0.03595263992}{\tau^4} + \frac{0.01511323043}{\tau^6} - \cdots \right), \quad (2.112)$$

and

$$g(\tau) \sim \tau^{4/5} \left( 1 - \frac{0.05289277344}{\tau^2} - \frac{0.00379437462}{\tau^4} - \frac{0.0005042991615}{\tau^6} - \cdots \right), \quad (2.113)$$

as $\tau \to \infty$.

2.5. On the large $k$ limit of $A_{i_k}(y)$

In this section, we comment and heuristically show with supported graphical evidence that the large $k$ limit of $A_{i_k}(y)$ is exactly $\text{sinc}(y)/\pi$. One would observe that for all $k$ and every $y \in \mathbb{R}$, the function $\cos \left( \frac{k + 1}{k + 1} + yt \right)$ behaves like $\cos(yt)$ inside the interval $[-1, 1]$ but for sufficiently large $k$, the function $\cos \left( \frac{k + 1}{k + 1} + yt \right)$ is highly oscillatory outside this interval for every $y \in \mathbb{R}$. These high oscillations result to fast cancellations outside $[-1, 1]$ under the integral $\int_{-\infty}^{\infty} e^{i(p_k(t)+yt)} dt$ with $p_k(t) := t^{k+1}/(k+1)$. Hence, for sufficiently large $k$, one has that

$$A_{i_k}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(p_k(t)+yt)} dt \approx \frac{1}{2\pi} \int_{-1}^{1} e^{i(p_k(t)+yt)} dt. \quad (2.114)$$

Furthermore, we observe that the right-hand side integral in (2.114) is absolutely and uniformly convergent, and hence, we can take limit under the integral. Moreover, we have that

$$\lim_{k \to \infty} p_k(t) = \lim_{k \to \infty} \frac{t^{k+1}}{k+1} = \lim_{k \to \infty} \frac{e^{(k+1)\ln(t)}}{k+1} = \begin{cases} 0 & t \in (0, 1] \\ \infty & t \not\in (0, 1]. \end{cases}$$

Hence, it follows that

$$\lim_{k \to \infty} \frac{1}{2\pi} \int_{-1}^{1} e^{i(p_k(t)+yt)} dt = \frac{1}{2\pi} \int_{-1}^{1} e^{iyt} dt = \frac{1}{\pi} \sin(y) y = \frac{1}{\pi} \text{sinc}(y).$$

We can rigorously show that $A_{i_k}(y)$ indeed tends to $\text{sinc}(y)/\pi$. One way of doing this is to reformulate $A_{i_k}(y)$ in tempered distribution setting, and hence, show the convergence of $A_{i_k}(y)$ as a sequence of functions to $\text{sinc}(y)/\pi$ in the weak sense. We do not explore the details in this work as it is outside
2.5. ON THE LARGE $k$ LIMIT OF $Ai_k(y)$

the scope of this study. However, we make very brief comments here and show the details elsewhere. Thus, let $g \in C^\infty(\mathbb{R})$ with compact support in $[-1, 1]$ and define $G(y) = (\mathcal{F}g)(y)$, hence, $G \in S(\mathbb{R})$. Then, it easily follows from Fubini’s theorem that

$$\int_{-\infty}^{\infty} Ai_k(y)G(y) \, dy = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{itk}g(t) \, dt. \quad (2.115)$$

Similarly, we have that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \text{sinc}(y)G(y) \, dy = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} g(t) \, dt. \quad (2.116)$$

Therefore, one can rigorously show using the right-hand expressions in (2.115) and (2.116) that given an arbitrary small $\epsilon > 0$, we have that

$$\left| \int_{-\infty}^{\infty} \left( Ai_k(y) - \frac{1}{\pi} \text{sinc}(y) \right) G(y) \, dy \right| < \epsilon$$

for sufficiently large $k$ to conclude that the $\text{sinc}(y)/\pi$ is the weak limit of $Ai_k(y)$.

![Figure 2.1](image-url)

**Figure 2.1.** Plots of $Ai_k(y)$ (blue dotted line), $Ai_{12}(y)$ (in gray), $Ai_{22}(y)$ (in purple), $Ai_{50}(y)$ (in green), $Ai_{150}(y)$ (in blue), and $Ai_\infty(y)$ (in brick red)

Plots in Figure 2.1 above show the convergence of $Ai_k(y)$ to $\text{sinc}(y)/\pi$ graphically. We observe that as $k$ increases, the amplitudes of oscillations of each higher order $Ai_k(y)$ reduce to those of $Ai_\infty(y)$ := $\text{sinc}(y)/\pi$ on the left axis while on the right axis, the amplitudes are seen to increase to quickly align with the amplitudes of $Ai_\infty(y)$. Also, as $k$ increases the negative and positive real zeroes of $Ai_k(y)$ become more numerically equal in magnitude up to negligibly small errors.
CHAPTER 3

Spectral Analysis of the Relativistic Quartic Quantum Oscillator

In this chapter, we study in great detail spectral properties of the massless relativistic operator of the quartic anharmonic oscillator. In particular, we obtain details about the spectrum of this operator and derive various expressions and properties for the eigenfunctions of the operator.

3.1. Statement of the problem and regularity properties

We consider the non-local operator of the quartic anharmonic oscillator

\[ H := H_0 + x^4 \]  \hspace{1cm} (3.1)

acting on \( L^2(\mathbb{R}) \); where \( H_0 := \sqrt{-\frac{d^2}{dx^2}} \). The operator \( H_0 \) is positive, non-local, and self-adjoint. For \( \psi \in C_0^\infty(\mathbb{R}) \), we have

\[ H_0\psi(x) = \frac{1}{\pi} \text{p.v.} \int_\mathbb{R} \frac{\psi(x) - \psi(y)}{(x-y)^2} \, dy = \frac{1}{2\pi} \int_0^\infty \frac{2\psi(x) - \psi(x+z) - \psi(x-z)}{z^2} \, dz, \quad x \in \mathbb{R}, \quad (3.2) \]

where \text{p.v.} denotes the Cauchy principal value. It is well-known that the definition of \( H_0 \) can be reformulated via Fourier transform derived directly from (3.2) (see for instance, \cite{24}, Proposition 3.3 and references therein). Thus, by applying Fourier inversion and Plancherel’s theorem on (3.2), we have

\[ H_0\psi(x) = (\mathcal{F}^{-1}[y|\mathcal{F}\psi](x) \quad \text{for all } \psi \in \mathcal{D}(H_0). \]

Wherefore, the domain \( \mathcal{D}(H_0) \) of \( H_0 \) coincides with \( \mathcal{H}^1(\mathbb{R}) \) (the \( L^2 \)-Sobolev space on \( \mathbb{R} \) of order one), and hence, \( \mathcal{D}(H_0) = \mathcal{H}^1(\mathbb{R}) \). In addition, \( H_0 \) is essentially self-adjoint on \( C_0^\infty(\mathbb{R}) \) and its spectrum \( \sigma(H_0) \) is given as \( \sigma(H_0) \subset [0, \infty) \).

Consider the multiplication operator \( V \), whose action on the elements of its domain is defined as follows:

\[
\begin{cases}
(V\psi)(x) = x^4\psi(x) & \text{for every } \psi \in \mathcal{D}(V) \\
\mathcal{D}(V) := \{ \psi \in L^2(\mathbb{R}) : x^4\psi \in L^2(\mathbb{R}) \}.
\end{cases}
\]

The domain \( \mathcal{D}(V) \) of the operator \( V \) is dense in \( L^2(\mathbb{R}) \) since it contains the subspace of all functions of \( L^2(\mathbb{R}) \) each of which vanishes outside a bounded open interval in \( \mathbb{R} \), which is also known to be dense.
in $L^2(\mathbb{R})$. We also note that $\mathcal{S}(\mathbb{R}) \subset H^1(\mathbb{R}) \cap \Sigma(V)$ (where $\mathcal{S}(\mathbb{R})$ is the usual Schwartz space of rapidly decreasing functions in $C^\infty(\mathbb{R})$). The operator $V$ with domain $\Sigma(V)$ is evidently an unbounded self-adjoint operator from the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ to itself. Moreover, $V$ is essentially self-adjoint on $C^\infty_0(\mathbb{R})$, and its spectrum $\sigma(V)$ is given as $\sigma(V) \subset [0, \infty)$. Since the function $V(x) = x^4$ is locally bounded and Borel measurable, the operator $H := H_0 + x^4$ defined as the sum of two self-adjoint operators is self-adjoint on $\Sigma(H) = \Sigma(H_0) \cap \Sigma(V) = H^1(\mathbb{R})$ (cf. [57]), and hence, the spectrum $\sigma(H)$ is given as $\sigma(H) \subset [0, \infty)$.

By [57] Theorem 2.2 there exists a self-adjoint operator $K$ bounded from below and identified as the self-adjoint operator $H_0 + x^4$ such that $e^{-tK} = T_t$ where $\{T_t : t \geq 0\}$ is a strongly continuous symmetric semigroup defined by an integral kernel $u(t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ for every $t > 0$ such that
\[
(T_t \psi)(x) = \int_{-\infty}^{\infty} u(t, x, y) \psi(y) \, dy \quad \psi \in L^2(\mathbb{R}),
\]
and for every $t > 0$, $u(t, x, y)$ is strictly positive, bounded, and jointly continuous in $x, y \in \mathbb{R}$ [46] Lemma 3]. In addition, one can show (see [46] Lemma 1]) that for all $t > 0$, the operator $T_t$ is compact. Therefore, by the general theory of semigroups (see [23]), there exists an orthonormal basis of eigenfunctions $\{\psi_n(x) : n \in \mathbb{N}\} \subset L^2(\mathbb{R})$ and corresponding eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ satisfying $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$ each having finite multiplicity such that $T_t \psi_n(x) = e^{-t \lambda_n} \psi_n(x)$ for all $n \in \mathbb{N}$. By the joint continuity and boundedness of the kernel $u(t, x, y)$, the eigenfunctions are bounded and continuous. In addition, the first eigenfunction $\psi_1(x)$ is strictly positive [46]. Furthermore, for every $\psi_n \in \Sigma(H)$, we have that
\[
-H \psi_n(x) = \lim_{t \to 0^+} \frac{T_t \psi_n(x) - \psi_n(x)}{t} = \lim_{t \to 0^+} \frac{e^{-t \lambda_n} - 1}{t} \psi_n(x) = -\lambda_n \psi_n(x)
\]
holds for every $x \in \mathbb{R}$, and hence, we define the eigenproblem
\[
\begin{cases}
H \psi_n(x) = \lambda_n \psi_n(x) \\
\psi_n \in \Sigma(H) = H^1(\mathbb{R}).
\end{cases}
\]
(3.4)
In this chapter, we study properties and derive expressions of the eigenpairs $(\lambda_n, \psi_n)$.

Although $\psi_n \in L^2(\mathbb{R})$, is bounded and continuous for each $n \in \mathbb{N}$, we establish further regularity properties of $\psi_n(x)$, which will be required in the next section. In particular, we show the following integrability properties.

**Lemma 3.1.** For each $n \in \mathbb{N}$, we have that $\psi_n \in L^1(\mathbb{R})$ and $x^4 \psi_n \in L^1(\mathbb{R})$.

**Proof.** First, we show the stronger integrability condition $x^4 \psi_n \in L^1(\mathbb{R})$. Then, using this result, we can easily show that $\psi_n \in L^1(\mathbb{R})$. Thus, we recall the domination property [47] Corollary 2.1] that
there exists a constant $C > 0$ such that $|\psi_n(x)| \leq C \|\psi_n\|_\infty \psi_1(x)$ for each $n \in \mathbb{N}$. But for all $x \in \mathbb{R}$, the result of [47] Example 4.8 yields $\psi_1(x) \leq \frac{C_V}{(1+x^2)(1+y^2)} \leq \frac{C_V}{1+y^2}$, $C_V > 0$. Therefore, it readily follows that

$$\int_{-\infty}^{\infty} x^4 |\psi_n(x)| \, dx \leq C_{V,n} \int_{-\infty}^{\infty} \frac{x^4}{1 + x^6} \, dx < \infty \quad (C_{V,n} := C_V C_n).$$

Next, we have that

$$\int_{-\infty}^{\infty} |\psi_n(x)| \, dx = \int_{|x| < 1} |\psi_n(x)| \, dx + \int_{|x| > 1} |\psi_n(x)| \, dx \leq \int_{|x| < 1} |\psi_n(x)| \, dx + \int_{|x| > 1} x^4 |\psi_n(x)| \, dx \leq \int_{|x| < 1} |\psi_n(x)| \, dx + \int_{-\infty}^{\infty} x^4 |\psi_n(x)| \, dx \leq \int_{|x| < 1} |\psi_n(x)| \, dx + \|\cdot\|_1^4 \psi_n||_1.$$

To complete the proof, we recall that $\psi_n(x)$ is continuous and bounded so that

$$\int_{|x| < 1} |\psi_n(x)| \, dx \leq \sup_{x \in \mathbb{R}} |\psi_n(x)| \int_{|x| < 1} \, dx = 2 \sup_{x \in \mathbb{R}} |\psi_n(x)| < \infty.$$

Equivalently, by Cauchy-Schwarz inequality, we have that

$$\int_{|x| < 1} |\psi_n(x)| \, dx \leq \sqrt{2} \left( \int_{-\infty}^{\infty} |\psi_n(x)| \, dx \right)^{1/2} = \sqrt{2} |\psi_n|_2 < \infty.$$

By either way, the proof is complete.

\[\square\]

**Remark 3.1.** From the results of Lemma [3.1] above, we conclude that $\psi_n \in \mathcal{H}^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Since the Schwartz space $S(\mathbb{R})$ is dense in both $\mathcal{H}^1(\mathbb{R})$ and $L^1(\mathbb{R})$, we also conclude that the space $\mathcal{H}^1(\mathbb{R}) \cap L^1(\mathbb{R})$ is not empty. Furthermore, from the inclusion $\mathcal{H}^1(\mathbb{R}) \cap L^1(\mathbb{R}) \subseteq L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, we have that $\mathcal{H}^1(\mathbb{R}) \cap L^1(\mathbb{R})$ is also a dense subspace of $L^2(\mathbb{R})$.

### 3.2. Transformation of the eigenproblem

Let $\phi_n$ denote the Fourier transform of the eigenfunctions $\psi_n$ of $H$:

$$\phi_n(y) := (\mathcal{F} \psi_n)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i xy} \psi_n(x) \, dx. \quad (3.5)$$

The integral (3.5) is well-defined since $\psi_n \in \mathcal{H}^1(\mathbb{R}) \cap L^1(\mathbb{R})$ (by Lemma [3.1] together with Remark [3.1]). By Cauchy-Schwarz inequality and the fact that $\psi_n \in \mathcal{H}^1(\mathbb{R})$, we have that

$$\int_{-\infty}^{\infty} |\phi_n(y)| \, dy \leq \left( \int_{-\infty}^{\infty} \frac{dy}{1 + y^2} \right)^{1/2} \left( \int_{-\infty}^{\infty} (1 + y^2) |\phi_n(y)|^2 \, dy \right)^{1/2} = \sqrt{\pi} \left( \int_{-\infty}^{\infty} (1 + y^2) |\mathcal{F} \psi_n(y)|^2 \, dy \right)^{1/2} = \sqrt{\pi} |\psi_n|_{\mathcal{H}^1(\mathbb{R})} < \infty. \quad (3.6)$$
Since $\phi_n \in L^1(\mathbb{R})$ (by (3.6)), the Fourier inversion theorem on $L^1(\mathbb{R})$ [69 Theorem 7.7(c)] allows us to recover $\psi_n$ from $\phi_n$. Thus, for $\phi_n \in L^1(\mathbb{R})$, we can define

$$(\mathcal{F}^{-1}\phi_n)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \phi_n(y) \, dy \quad (x \in \mathbb{R}),$$

and given that $\psi_n \in L^1(\mathbb{R})$, then by the Fourier inversion theorem on $L^1(\mathbb{R})$, we have

$$\psi_n(x) = (\mathcal{F}^{-1}\phi_n)(x) \quad \text{for almost every } x \in \mathbb{R}. \quad (3.7)$$

In addition, we already know that $\psi_n \in \mathcal{H}^1(\mathbb{R}) \cap L^1(\mathbb{R}) \subseteq L^2(\mathbb{R}) \cap L^1(\mathbb{R})$; hence, by Plancherel’s theorem $\phi_n \in L^2(\mathbb{R})$, and the identity (3.7) now holds for all $x \in \mathbb{R}$.

Furthermore, using the fact that $x^4\psi_n \in L^1(\mathbb{R})$ (from the result of Lemma 3.1) and noting the uniform boundedness of the integral in (3.3), for all $\psi_n \in \mathcal{H}^1(\mathbb{R}) \cap L^1(\mathbb{R})$, we have that

$$\left| \frac{d^4}{dy^4}\phi_n(y) \right| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} x^4 e^{-ixy} \psi_n(x) \, dx \right| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 |\psi_n(x)| \, dx < \infty \quad \text{for every } y \in \mathbb{R}. \quad (3.8)$$

Therefore, by combining (3.6) and (3.8), we conclude that $\phi_n \in C^4(\mathbb{R}) \cap L^1(\mathbb{R})$.

Furthermore, we observe that under Fourier transform, one obtains

$$(\mathcal{F} x^4 \psi_n)(y) = \frac{d^4}{dy^4} \phi_n(y) \quad \text{and} \quad \left(\mathcal{F} \sqrt{-\frac{d^2}{dx^2}} \psi_n\right)(y) = |y|\phi_n(y).$$

Therefore, by applying Fourier transform on (3.1), we obtain a local unbounded positive operator $\hat{H}$ in the Hilbert space $L^2(\mathbb{R})$ generated by the following fourth order differential expression $L$:

$$L\phi_n := \frac{d^4}{dy^4} \phi_n(y) + |y|\phi_n(y). \quad (3.9)$$

By virtue of $|y|$ appearing in (3.9), we observe that $L$ can be realised as the sum of two formal differential expressions denoted by $L^+$ and $L^-$ defined on the half-lines $y > 0$ and $y < 0$, respectively:

$$L\phi_n = L^+\phi_{n,1} + L^-\phi_{n,2}$$

such that

$$L^+\phi_{n,1}(y) := \frac{d^4}{dy^4} \phi_{n,1}(y) + y\phi_{n,1}(y) \quad \text{for } y > 0 \quad (3.10)$$

$$L^-\phi_{n,2}(y) := \frac{d^4}{dy^4} \phi_{n,2}(y) - y\phi_{n,2}(y) \quad \text{for } y < 0, \quad (3.11)$$

and

$$\phi_{n,1}(y) := \begin{cases} \phi_n(y) & y > 0 \\ 0 & y \leq 0 \end{cases} \quad \text{and} \quad \phi_{n,2}(y) := \begin{cases} 0 & y > 0 \\ \phi_n(y) & y < 0. \end{cases} \quad (3.12)$$

Observe that in (3.11), we can consider the map $y \mapsto -y$ and set $\phi_{n,1}(y) = \phi_{n,2}(-y)$, to obtain

$$L^-\phi_{n,1}(y) = \frac{d^4}{dy^4} \phi_{n,1}(y) + y\phi_{n,1}(y) =: L^+\phi_{n,1}(y) \quad \text{for } y > 0.$$
3.2. TRANSFORMATION OF THE EIGENPROBLEM

By similar transformation for (3.10), and setting \( \phi_{n,2}(y) = \phi_{n,1}(-y) \), we also obtain
\[
\mathcal{L}^+ \phi_{n,2}(y) = \frac{d^4}{dy^4} \phi_{n,2}(y) - y \phi_{n,2}(y) =: \mathcal{L}^- \phi_{n,2}(y) \quad \text{for} \ y < 0.
\]
Hence, we have that
\[
\mathcal{L} \phi_n(y) = \mathcal{L}^+ \phi_n(y) = \mathcal{L}^- \phi_n(-y) \quad \text{for} \ y > 0. \tag{3.13}
\]
The identity (3.13) also holds for \( y < 0 \) by replacing \( y \) with \( -y \) in the expression. Therefore, it is sufficient to study \( \mathcal{L} \) for \( y > 0 \). Then, by odd and even extensions, the analysis can be extended to the entire real line \( \mathbb{R} \).

Furthermore, by (3.8), the (global) decay property of \( \psi_n \) (i.e., the behaviour of \( \psi_n(x) \) as \( |x| \to \infty \)) is translated into the local smoothness property of \( \phi_n \) (i.e., the behaviour of \( \phi_n(x) \) near zero). In particular, we have that
\[
\frac{d^l}{dy^l} \phi_n(0) = \frac{(-i)^l}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^l \psi_n(x) \, dx, \quad \text{for} \ l = 0, 1, 2, 3, \tag{3.14}
\]
and by the orthonormality of \( \{\psi_n(x) : n \in \mathbb{N}\} \) and the strict positivity of \( \psi_1(x) \), we deduce that \( \psi_n(x) \) is symmetric (or even) for \( n = 1, 3, 5, \cdots \) and antisymmetric (or odd) for \( n = 2, 4, 6, \cdots \), it then necessarily follows from (3.14) that
\[
\phi_n(0^+) = \phi_n''(0^+) = 0 \quad \text{for each} \ n = 2, 4, 6, \cdots, \tag{3.15}
\]
(for odd extension), and
\[
\phi_n'(0^+) = \phi_n'''(0^+) = 0 \quad \text{for each} \ n = 1, 3, 5, \cdots \tag{3.16}
\]
(for even extension). Therefore, the eigenproblem obtained by performing Fourier transform on the eigenproblem of (3.4) is given by
\[
\tilde{H} \phi_n(y) := \frac{d^4}{dy^4} \phi_n(y) + |y| \phi_n(y) = \lambda_n \phi_n(y), \tag{3.17}
\]
for \( \phi_n \in C^4(\mathbb{R}) \cap L^1(\mathbb{R}) \) satisfying the boundary conditions (3.15) and (3.16) for odd and even solutions, respectively.

We note that the eigenvalues \( \lambda_n \), which are scalar multiplications of the eigenfunctions, are the same for both eigenproblems (3.4) and (3.17); this is because Fourier transform is unitary, and with unitary transforms (that is, transforms that preserve the inner product), the spectrum is invariant under scalar multiplication. Our aim in the remaining sections of this chapter, is to study the spectrum and obtain expressions for the eigenfunctions of \( H \) through the \( C^4(\mathbb{R}) \cap L^1(\mathbb{R}) \) solutions of (3.17) satisfying (3.15) and (3.16).
3.3. Construction of solutions to the ODE

We recall from (3.13) that if $\phi_n(y)$ is a solution to (3.17), then $\phi_n(-y)$ is also a solution to (3.17). Hence, we consider (3.17) only for $y > 0$, and extend the result to the whole of $\mathbb{R}$ by odd and even extensions. Thus, for $y > 0$, the eigenproblem of (3.17) can be written as

$$
\frac{d^4}{dy^4} \phi_n(y) + (y - \lambda_n)\phi_n(y) = 0. \quad (3.18)
$$

We set $\phi_n(y) \equiv \chi(y - \lambda_n)$ and make the linear transformation $y - \lambda_n \mapsto y$ to obtain

$$
\frac{d^4}{dy^4} \chi(y) + y\chi(y) = 0. \quad (3.19)
$$

A particular solution of (3.19) is the fourth-order Airy function $\text{Ai}_4(y)$:

$$
\text{Ai}_4(y) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{t^5}{5} + yt \right) \, dt. \quad (3.20)
$$

Note that equation (3.19) is invariant under the rotations $y \mapsto ye^{2\pi i/5}, \ l \in \mathbb{Z}$, thus we have four other solutions by rotation of the argument of $\text{Ai}_4(y)$. That is,

$$
\text{Ai}_4(ye^{-2\pi i/5}), \ \text{Ai}_4(ye^{-4\pi i/5}), \ \text{Ai}_4(ye^{4\pi i/5}), \ \text{Ai}_4(ye^{2\pi i/5}). \quad (3.21)
$$

Any four of the above five solutions can be shown to have a non-vanishing Wronskian (refer to (2.35)), and hence, independent. Thus, a particular solution to (3.19) can be constructed by taking a linear combination of any four of these solutions. To simplify notations we let

$$
A_1(y) := e^{-2\pi i/5} \text{Ai}_4(ye^{-2\pi i/5}), \ A_2(y) := e^{-4\pi i/5} \text{Ai}_4(ye^{-4\pi i/5});
$$

$$
A_3(y) := e^{4\pi i/5} \text{Ai}_4(ye^{4\pi i/5}); \ A_4(y) := e^{2\pi i/5} \text{Ai}_4(ye^{2\pi i/5}),
$$

then by Cauchy’s theorem on contour integration around a closed curve (or directly from (2.34)), we have that

$$
\text{Ai}_4(y) = -\sum_{r=1}^4 A_r(y). \quad (3.22)
$$

Three other solutions independent of $\text{Ai}_4(y)$ can also be constructed by taking other suitable linear combinations of $A_r(y)$ for $r = 1, 2, 3, 4$, which together with (3.22) form a complete set of independent solutions to the differential equation (3.19). They are given as follows:

$$
\overline{\text{Ai}}_4(y) = i \sum_{r=1}^4 (-1)^{r+1} A_r(y), \quad (3.23)
$$

which has an integral representation of the form

$$
\text{Ai}_4(y) = \frac{1}{\pi} \int_0^\infty \left[ e^{-yt} - \frac{t^5}{5} - \sin \left( \frac{t^5}{5} + yt \right) \right] \, dt, \quad (3.24)
$$

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and two other solutions with the expressions

\[ G_3(y) = A_4(y) + A_1(y) - [A_3(y) + A_2(y)] ; \quad G_4(y) = i [A_4(y) - A_1(y)] + i [A_3(y) - A_2(y)]. \]

Details about these expressions are presented in Appendices A.1 - A.3. Therefore, we have

\[ \chi(y) = c_1 Ai_4(y) + c_2 \tilde{Ai}_4(y) + c_3 G_3(y) + c_4 G_4(y). \] (3.25)

Since \( G_3(y) \) and \( G_4(y) \) diverge as \( y \to \infty \), the fact that \( \chi \in L^1(0, \infty) \cap L^2(0, \infty) \) implies that \( c_3 = c_4 = 0 \). Furthermore, we recall that \( \phi_n(y) = \chi(y - \lambda_n) \), and by extension to \( \mathbb{R} \), we obtain the even and odd solutions of (3.17) satisfying the given boundary conditions (3.15) and (3.16) as

\[ \phi_{2j-1}(y) = c_{1,2j-1} Ai_4(|y| - \lambda_{2j-1}) + c_{2,2j-1} \tilde{Ai}_4(|y| - \lambda_{2j-1}), \quad y \in \mathbb{R}, \ j \in \mathbb{N} \] (3.26)

\[ \phi_{2j}(y) = c_{1,2j} \text{sgn}(y) Ai_4(|y| - \lambda_{2j}) + c_{2,2j} \text{sgn}(y) \tilde{Ai}_4(|y| - \lambda_{2j}), \quad y \in \mathbb{R}, \ j \in \mathbb{N}, \] (3.27)

respectively.

### 3.4. Identification and approximation of the spectrum

The boundary conditions (3.15) and (3.16) applied to (3.26) and (3.27) yield

\[ \det \begin{pmatrix} Ai_4'(-\lambda_n) & \tilde{Ai}_4'(-\lambda_n) \\ Ai_4''(-\lambda_n) & \tilde{Ai}_4''(-\lambda_n) \end{pmatrix} = 0 \quad \text{and} \quad \det \begin{pmatrix} Ai_4''(-\lambda_n) & \tilde{Ai}_4''(-\lambda_n) \\ Ai_4''(-\lambda_n) & \tilde{Ai}_4''(-\lambda_n) \end{pmatrix} = 0 \] (3.28)

for \( n = 2, 4, \ldots \) and \( n = 1, 3, \ldots \), respectively. We define

\[ \Phi_1(y) := Ai_4(y)\tilde{Ai}_4''(y) - Ai_4'(y)\tilde{Ai}_4(y); \quad \Phi_2(y) := Ai_4''(y)\tilde{Ai}_4''(y) - Ai_4''(y)\tilde{Ai}_4'(y). \] (3.29)

and state as follows:

**Theorem 3.1.** The eigenvalues of \( H \) are given by

\[ \lambda_{2j-1} = -a_{2,j}, \quad \lambda_{2j} = -a_{1,j} \]

for each \( j \in \mathbb{N} \), where \( a_{1,j} \) and \( a_{2,j} \) are the negative real zeroes of the determinant functions \( \Phi_1(y) \) and \( \Phi_2(y) \), respectively, arranged in alternating and increasing order of magnitude.

It turns out that \( \Phi_1(y) \) and \( \Phi_2(y) \) are higher transcendental functions defined only by their integral representations. These integral representations, which are expressed in terms of generalised Fresnel sine and cosine integral functions, are derived in the statement that immediately follows.

**Corollary 3.1.** We have

\[ \Phi_1(y) = \frac{1}{2\pi^2} \int_{0}^{\infty} \left[ h_2(y) \sin \left( \frac{v^5}{20} - y v \right) - h_1(y) \cos \left( \frac{v^5}{20} - y v \right) \right] dv \] (3.30)
Therefore, from (3.35), we have

\[ \Phi_2(y) = \frac{1}{2\pi^2} \int_0^\infty \left[ h_4(v) \sin \left( \frac{v^5}{20} - yv \right) - h_3(v) \cos \left( \frac{v^5}{20} - yv \right) \right] dv \tag{3.31} \]

where

\[ h_1(v) := v^{1/2} \text{si} \left( \frac{1}{2}, \frac{v^5}{16} \right), \quad h_2(v) := v^{1/2} \text{ci} \left( \frac{1}{2}, \frac{v^5}{16} \right); \]

\[ h_3(v) := 2 \sin \left( \frac{v^5}{16} \right) - v^{5/2} \text{ci} \left( \frac{1}{2}, \frac{v^5}{16} \right), \quad h_4(v) := 2 \cos \left( \frac{v^5}{16} \right) + v^{5/2} \text{si} \left( \frac{1}{2}, \frac{v^5}{16} \right), \]

and \( \text{si}(a, z) \) and \( \text{ci}(a, z) \) are the usual generalised Fresnel sine and cosine integral functions, respectively (refer to (3.46) below).

**Proof.** We will only show the proof for (3.30); by similar procedure, (3.31) is established. Now, we consider

\[ \text{Ai}_a(y) \tilde{\text{Ai}}_a''(y) = I(y) + J(y), \tag{3.32} \]

where

\[ I(y) := \frac{1}{2\pi^2} \left( \int_{-\infty}^{\infty} e^{-\frac{y^5}{16}} \, dt \right) \left( \int_{-\infty}^{\infty} u^2 e^{-yu - \frac{y^5}{16}} \, du \right) \tag{3.33} \]

and

\[ J(y) := \frac{1}{\pi^2} \left( \int_{0}^{\infty} \cos \left( \frac{5}{5} + yt \right) \, dt \right) \left( \int_{0}^{\infty} u^2 \sin \left( \frac{u^5}{5} + yu \right) \, du \right). \tag{3.34} \]

From (3.33), we have

\[ I(y) := \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} u^2 e^{-y(t+u)-\frac{y^5}{16}(t^2+u^2)} \, du \, dt. \tag{3.35} \]

We make the following linear transformation: \( X = t + u \) and \( Y = t - u \) so that

\[ t = \frac{1}{2}(X + Y), \quad u = \frac{1}{2}(X - Y) \tag{3.36} \]

and

\[ \frac{1}{5}(t^5 + u^5) = -\frac{X^5}{20} + \frac{X}{16}(Y^2 + X^2)^2, \quad \frac{1}{5}(t^5 - u^5) = -\frac{Y^5}{20} + \frac{Y}{16}(X^2 + Y^2)^2, \tag{3.37} \]

while the Jacobian of the transform is given by \( du \, dt = \frac{1}{2} \, dY \, dX \). From (3.36), we have as follows:

1. when \( u = 0, Y = X \); and hence, \( X = t \) for \( -\infty < t < \infty \);
2. when \( t = 0, X = -Y \); and hence, \( Y = -u \) for \( 0 < u < \infty \).

Therefore, from (3.35), we have

\[ I(y) = \frac{1}{16\pi^2} \int_{-\infty}^{\infty} e^{-y \frac{X+Y}{20}} \left( \int_{0}^{\infty} (X-Y)^2 e^{-\frac{X}{16}(Y^2 + X^2)^2} \, dY \right) \, dX \]

\[ = -\frac{1}{16\pi^2} \int_{-\infty}^{\infty} e^{-y \frac{X+Y}{20}} \left( \int_{0}^{\infty} (X+Y)^2 e^{-\frac{Y}{16}(Y^2 + X^2)^2} \, dY \right) \, dX. \tag{3.38} \]
Next, from (3.34), we have that
\[ J(y) = \frac{1}{4\pi ^2i} \left( \int_0^\infty \left[ e^{\frac{y^2 + t^2}{2}} + e^{-i\left(\frac{y^2 + t^2}{2}\right)} \right] dt \right) \left( \int_0^\infty u^2 \left[ e^{i\left(\frac{y^2 + tu}{2}\right)} - e^{-i\left(\frac{y^2 + tu}{2}\right)} \right] du \right) \]
\[ = \frac{1}{4\pi ^2i} [J_1(y) - J_2(y) + J_3(y) - J_4(y)], \] (3.39)

where
\[ J_1(y) := \int_0^\infty \int_0^\infty u^2 e^{i\left(\frac{y^2 + tu}{2}\right)} dt \left( \int_0^{\infty} (X - Y)^2 e^{-\frac{X^2}{2} + X^2} dY \right) dX, \]
\[ J_2(y) := \int_0^\infty \int_0^\infty u^2 e^{-i\left(\frac{y^2 + tu}{2}\right)} dt \left( \int_0^{\infty} (X - Y)^2 e^{-\frac{X^2}{2} + X^2} dY \right) dX, \]
\[ J_3(y) := \int_0^\infty \int_0^\infty u^2 e^{-i\left(\frac{y^2 + tu}{2}\right)} dt \left( \int_0^{\infty} (Y - X)^2 e^{-\frac{X^2}{2} + X^2} dX \right) dY, \]
\[ J_4(y) := \int_0^\infty \int_0^\infty u^2 e^{i\left(\frac{y^2 + tu}{2}\right)} dt \left( \int_0^{\infty} (Y - X)^2 e^{-\frac{X^2}{2} + X^2} dX \right) dY. \]

But we observe that \( J_1(y) = J_4(y) \) and \( J_2(y) = J_3(y) \); hence, we immediately have that
\[ J_1(y) - J_2(y) + J_3(y) - J_4(y) = 0 \]
so that by combining (3.38) and (3.39), it follows from (3.32) that
\[ Ai_4(y) \tilde{Ai}_4(y) = -\frac{1}{16\pi ^2} \int_{-\infty}^{\infty} e^{-yX + \frac{X^5}{5}} \left( \int_0^{\infty} (X + Y)^2 e^{-\frac{X^2}{2} + X^2} dY \right) dX, \] (3.40)

Also, we consider
\[ Ai_4''(y) \tilde{Ai}_4(y) = \tilde{I}(y) + \tilde{J}(y) \] (3.41)

where
\[ \tilde{I}(y) := \frac{1}{2\pi ^2i} \left( \int_{-\infty}^{\infty} t^2 e^{-yt - \frac{t^2}{2}} dt \right) \left( \int_0^{\infty} e^{-yu - \frac{u^2}{2}} du \right) \]
and
\[ \tilde{J}(y) := \frac{1}{\pi ^2} \left( \int_0^{\infty} t^2 \cos \left( \frac{t^5}{5} + yt \right) dt \right) \left( \int_0^{\infty} \sin \left( \frac{u^5}{5} + yu \right) du \right) = 0. \]

We proceed with \( \tilde{I}(y) \) in similar manner as with \( I(y) \) and obtain
\[ \tilde{I}(y) = -\frac{1}{16\pi ^2} \int_{-\infty}^{\infty} e^{-yX + \frac{X^5}{5}} \left( \int_0^{\infty} (X - Y)^2 e^{-\frac{X^2}{2} + X^2} dY \right) dX; \]
again, we have that \( \tilde{J}(y) = 0 \). Hence, from (3.41), we have that
\[ Ai_4''(y) \tilde{Ai}_4(y) = -\frac{1}{16\pi ^2} \int_{-\infty}^{\infty} e^{-yX + \frac{X^5}{5}} \left( \int_0^{\infty} (X - Y)^2 e^{-\frac{X^2}{2} + X^2} dY \right) dX. \] (3.42)

But \((X + Y)^2 - (X - Y)^2 = 4XY\) so that by combining (3.40) and (3.42), we have
\[ \Phi_1(y) = -\frac{1}{4\pi ^2i} \int_{-\infty}^{\infty} Xb(X)e^{-yX + \frac{X^5}{5}} dX \]
with
\[ b(X) := \int_0^\infty Y e^{-\frac{X}{16}(Y^2 + X^2)^2} \, dY. \]

By further simplification we obtain
\[
\int_0^\infty Y e^{-\frac{X}{16}(Y^2 + X^2)^2} \, dY = \frac{1}{2} \int_0^\infty e^{-\frac{X}{16}(Y^2 + X^2)^2} \, dY
= \frac{2}{\sqrt{X}} \int_0^\infty e^{-Y^2} \, dY
= \sqrt{\frac{\pi}{X}} \text{erfc}\left(\frac{X^{5/2}}{4}\right)
= X^{-1/2} \Gamma\left(1, \frac{X^5}{16}\right).
\]

Therefore, we have that
\[
\Phi_1(y) = -\frac{1}{4\pi^2 i} \int_{-\infty}^{\infty} X^{1/2} \Gamma\left(1, \frac{X^5}{16}\right) e^{-\frac{Y}{16}X^5} e^{-YX + \frac{Y^3}{2}} \, dX,
\]
where \(X^{1/2}\) is understood as taking its principal part:
\[ X^{1/2} = \begin{cases} e^{\frac{i}{2} \ln|X| + \text{arg}X}, & \text{for } X \neq 0 \\ 0, & \text{for } X = 0. \end{cases} \]

Hence, we have that
\[
\Phi_1(y) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} |X|^{1/2} \Gamma\left(1, \frac{iX^5}{16}\right) e^{i\left(\frac{X^5}{16}X + \frac{Y^3}{2} \text{sgn}(X)\right)} \, dX
= -\frac{1}{2\pi^2} \Re e \int_{0}^{\infty} X^{1/2} \Gamma\left(1, \frac{iX^5}{16}\right) e^{i\left(\frac{X^5}{16}X + \frac{Y^3}{2}\right)} \, dX. \tag{3.44}
\]

Lastly, we employ the following well-known identity to complete the derivation for \(\Phi_1(y)\) as presented in (3.30):
\[
\Gamma\left(1, \frac{iX^5}{16}\right) = e^{\frac{\pi}{4}} \left[ \text{ci} \left(1, \frac{X^5}{16}\right) - \text{si} \left(1, \frac{X^5}{16}\right) \right], \tag{3.45}
\]
where \(\text{si}(a,z)\) and \(\text{ci}(a,z)\) are the generalised Fresnel sine and cosine integral functions, respectively, given by
\[
\text{si}(a,z) = \int_z^\infty t^{a-1} \sin(t) \, dt; \quad \text{ci}(a,z) = \int_z^\infty t^{a-1} \cos(t) \, dt \tag{3.46}
\]
valid for \(\Re e(a) < 1\) and \(\Re e(z) > 0\).

Unfortunately, the integral representations of the higher transcendental functions \(\Phi_1(y)\) and \(\Phi_2(y)\) given in Corollary 3.1 above cannot be further simplified making it much difficult to use these expressions for other applications in this work. However, we are able to obtain some asymptotic formulæ expressing \(\Phi_1(y)\) and \(\Phi_2(y)\) in terms of simple functions of sine and cosine for \(y < 0\). These expressions are obtained using the asymptotic formulæ derived for \(\text{Ai}_4(y)\) and \(\tilde{\text{Ai}}_4(y)\) for \(y < 0\) (refer to Appendix A.3). Thus, we state as follows:
3.4. IDENTIFICATION AND APPROXIMATION OF THE SPECTRUM

Corollary 3.2. Let \( y = -\lambda \) for \( \lambda > 0 \). There exits a real number \( 0 < \lambda_0 \leq 1/2 \) such that for every \( \lambda > \lambda_0 \), we have

\[
\Phi_1(-\lambda) = \frac{1}{\pi} \lambda^{-1/4} e^{\frac{4}{5} \lambda^{5/4}} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O \left( \frac{e^{\frac{4}{5} \lambda^{5/4}}}{\lambda^{7/8}} \right) \tag{3.47}
\]

and

\[
\Phi_2(-\lambda) = \frac{1}{\pi} \lambda^{1/4} e^{\frac{4}{5} \lambda^{5/4}} \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O \left( \frac{e^{\frac{4}{5} \lambda^{5/4}}}{\lambda^{3/8}} \right). \tag{3.48}
\]

Proof. In Appendix A.3 we show that

\[
\text{Ai}_4(-\lambda) = \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) \left( 1 + O(\lambda^{-5/8}) \right) \tag{3.49}
\]

and

\[
\tilde{\text{Ai}}_4(-\lambda) = \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} e^{\frac{4}{5} \lambda^{5/4}} \left( 1 + O(\lambda^{-5/8}) \right) \tag{3.50}
\]

for every \( \lambda > \lambda_0 \) for some \( 0 < \lambda_0 \leq 1/2 \). Since \( \text{Ai}_4(y) \) and \( \tilde{\text{Ai}}_4(y) \) are analytic functions, both expansions (3.49) and (3.50) can be differentiated. Thus, we have

\[
\text{Ai}_4(-\lambda)\tilde{\text{Ai}}_4'(-\lambda) = \left( \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O(\lambda^{-1}) \right) \times
\]

\[
\times \left( \frac{1}{\sqrt{2\pi}} \lambda^{1/8} e^{\frac{4}{5} \lambda^{5/4}} + O \left( \lambda^{-1/2} e^{\frac{4}{5} \lambda^{5/4}} \right) \right)
\]

\[
= \frac{1}{2\pi} \lambda^{-1/4} e^{\frac{4}{5} \lambda^{5/4}} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O \left( \frac{e^{\frac{4}{5} \lambda^{5/4}}}{\lambda^{7/8}} \right) \quad \text{for } \lambda > \lambda_0,
\]

and

\[
\text{Ai}_4''(-\lambda)\tilde{\text{Ai}}_4(-\lambda) = - \left( \frac{1}{\sqrt{2\pi}} \lambda^{1/8} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O(\lambda^{-1/2}) \right) \times
\]

\[
\times \left( \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} e^{\frac{4}{5} \lambda^{5/4}} + O \left( \lambda^{-1} e^{\frac{4}{5} \lambda^{5/4}} \right) \right)
\]

\[
= - \left[ \frac{1}{2\pi} \lambda^{-1/4} e^{\frac{4}{5} \lambda^{5/4}} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O \left( \frac{e^{\frac{4}{5} \lambda^{5/4}}}{\lambda^{7/8}} \right) \right] \quad \text{for } \lambda > \lambda_0.
\]

Then, by combining (3.51) and (3.52), we simply have that

\[
\Phi_1(-\lambda) := \text{Ai}_4(-\lambda)\tilde{\text{Ai}}_4''(-\lambda) - \text{Ai}_4''(-\lambda)\tilde{\text{Ai}}_4(-\lambda)
\]

\[
= \frac{1}{\pi} \lambda^{-1/4} e^{\frac{4}{5} \lambda^{5/4}} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O \left( \frac{e^{\frac{4}{5} \lambda^{5/4}}}{\lambda^{7/8}} \right) \quad \text{for } \lambda > \lambda_0.
\]
3.4. IDENTIFICATION AND APPROXIMATION OF THE SPECTRUM

Similarly, we have

\[ Ai'_4(-\lambda) \tilde{A}''_4(-\lambda) = \left( \frac{1}{\sqrt{2\pi}} \lambda^{-1/8} \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O(\lambda^{-3/4}) \right) \times \]

\[ \times \left( \frac{1}{\sqrt{2\pi}} \lambda^{3/8} e^{\frac{2}{5} \lambda^{5/4}} + O\left( \lambda^{-1/4} e^{\frac{2}{5} \lambda^{5/4}} \right) \right) \]

\[ = \frac{1}{2\pi} \lambda^{1/4} e^{\frac{2}{5} \lambda^{5/4}} \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O \left( \frac{e^{\frac{2}{5} \lambda^{5/4}}}{\lambda^{3/8}} \right) \]  

for \( \lambda > \lambda_0 \),

and

\[ Ai''_4(-\lambda) \tilde{A}'_4(-\lambda) = -\left( \frac{1}{\sqrt{2\pi}} \lambda^{3/8} \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O(\lambda^{-1/4}) \right) \times \]

\[ \times \left( \frac{1}{\sqrt{2\pi}} \lambda^{-1/8} e^{\frac{2}{5} \lambda^{5/4}} + O\left( \lambda^{-3/4} e^{\frac{2}{5} \lambda^{5/4}} \right) \right) \]

\[ = -\left[ \frac{1}{2\pi} \lambda^{1/4} e^{\frac{2}{5} \lambda^{5/4}} \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O \left( \frac{e^{\frac{2}{5} \lambda^{5/4}}}{\lambda^{3/8}} \right) \right] \]  

for \( \lambda > \lambda_0 \).

Therefore, by combining (3.53) and (3.54), we simply have that

\[ \Phi_2(-\lambda) := Ai'_4(-\lambda) \tilde{A}''_4(-\lambda) - Ai''_4(-\lambda) \tilde{A}'_4(-\lambda) \]

\[ = \frac{1}{\pi} \lambda^{1/4} e^{\frac{2}{5} \lambda^{5/4}} \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + O \left( \frac{e^{\frac{2}{5} \lambda^{5/4}}}{\lambda^{3/8}} \right) \]  

for \( \lambda > \lambda_0 \).

Hence, the proof is complete. 

\[ \square \]

Figure 3.1. Plots of \( \Phi_1(y) \) (in red) and (3.47) (in blue dash dot)

Figure 3.2. Plots of \( \Phi_2(y) \) (in red) and (3.48) (in blue dash dot)
3.4. IDENTIFICATION AND APPROXIMATION OF THE SPECTRUM

Note that $\Phi_1(y)$ and $\Phi_2(y)$ are positive, convex, and decay exponentially on $y > 0$ as shown in Figures [3.1 - 3.2] (red lines) above. However, on $y < 0$, these functions are oscillatory and have ever-increasing amplitudes. We also plot $\frac{1}{\pi}y^{-1/4}e^{\frac{5y}{4}} \sin \left( \frac{4}{5y} \frac{5y}{4} + \frac{3}{4} \right)$ and $\frac{1}{\pi}y^{1/4}e^{\frac{5y}{4}} \cos \left( \frac{4}{5y} \frac{5y}{4} + \frac{3}{4} \right)$ (i.e., the asymptotic relations (3.47) and (3.48), respectively) along with $\Phi_1(y)$ and $\Phi_2(y)$, respectively, for $y < 0$ indicated by the blue dash dotted lines in Figures [3.1] and [3.2] and showing very nice and accurate fits for $\Phi_1(y)$ and $\Phi_2(y)$, respectively, when $y < 0$.

Using the asymptotic relations of Corollary 3.2 we solve the equation $\Phi_1(-\lambda_2) = 0$ for $\lambda_2$ and the equation $\Phi_2(-\lambda_{2j-1}) = 0$ for $\lambda_{2j-1}$ for each $j \in \mathbb{N}$ to obtain expressions for the eigenvalues $\lambda_n$ showing explicit dependence on $n$.

**Corollary 3.3.** For each $n \in \mathbb{N}$, we have

$$
\lambda_n = \left( \frac{5(2n - 1)\pi}{16} \right)^{4/5} \left[ 1 + O \left( \frac{1}{n^{3/2}} \right) \right].
$$

Numerical computations of the eigenvalues $\lambda_n$ are presented in Table C.1 (see Appendix C). The last statement we prove in this section gives information about the bounds on the spectral gaps $\lambda_{n+1} - \lambda_n$ for each $n \in \mathbb{N}$; we see that the spectral gaps $\lambda_{n+1} - \lambda_n$ behave like $O \left( n^{-1/5} \right)$. In particular, the bounds we obtain here are two-sided (i.e., lower and upper bounds) with best-known constants.

**Theorem 3.2.** For each $n \in \mathbb{N}$, we have

$$
\frac{\pi}{2} \left( \frac{8}{15\pi} \right)^{1/5} \left( n - \frac{1}{2} \right)^{-1/5} \leq \lambda_{n+1} - \lambda_n \leq \frac{\pi}{2} \left( \frac{8}{5\pi} \right)^{1/5} \left( n - \frac{1}{2} \right)^{-1/5}.
$$

**Proof.** To derive the upper bound, we recall from (3.55) that

$$
\lambda_{n+1} - \lambda_n \approx \left( \frac{5(2n + 1)\pi}{16} \right)^{4/5} - \left( \frac{5(2n - 1)\pi}{16} \right)^{4/5}.
$$

We then employ the following inequality (cf: [35 (2.15.2)]):

$$
\sigma(\xi - \eta)\xi^{\sigma - 1} \leq \xi^{\sigma} - \eta^{\sigma} \leq \sigma(\xi - \eta)\eta^{\sigma - 1} \quad \text{for} \quad 0 < \sigma \leq 1, \text{ and } \xi \geq \eta \geq 0
$$

on (3.57) to obtain

$$
\left( \frac{5(2n + 1)\pi}{16} \right)^{4/5} - \left( \frac{5(2n - 1)\pi}{16} \right)^{4/5} \leq \frac{\pi}{2} \left( \frac{8}{5\pi} \right)^{1/5} \left( n - \frac{1}{2} \right)^{-1/5}.
$$

For the lower bound, we deduce from (2.111) - (2.113) that

$$
\mu_{2j-1} = -\alpha_{4, j} \leq \left( \frac{5(4j - 3)\pi}{16} \right)^{4/5} \quad \text{for each } j \in \mathbb{N}
$$

and

$$
\mu_{2j} = -\alpha_{4, j} \geq \left( \frac{5(4j - 1)\pi}{16} \right)^{4/5} \quad \text{for each } j \in \mathbb{N}.
$$
Therefore, by employing the inequality (3.58), we obtain
\[
\mu_{2j} - \mu_{2j-1} \geq \left( \frac{5(4j-1)\pi}{16} \right)^{4/5} - \left( \frac{5(4j-3)\pi}{16} \right)^{4/5} \geq \frac{\pi}{2} \left( \frac{4}{5\pi} \right)^{1/5} \left( 4j - \frac{1}{4} \right)^{-1/5}
\] (3.59)

We now set \( n = 2j - 1, \ j \in \mathbb{N} \) in (3.59) to obtain
\[
\mu_{n+1} - \mu_n \geq \frac{\pi}{2} \left( \frac{8}{5\pi} \right)^{1/5} \left( n + \frac{1}{2} \right)^{-1/5},
\]
and similarly, we set \( n = 2j, \ j \in \mathbb{N} \) to obtain
\[
\mu_n - \mu_{n-1} \geq \frac{\pi}{2} \left( \frac{8}{5\pi} \right)^{1/5} \left( n - \frac{1}{2} \right)^{-1/5} \geq \frac{\pi}{2} \left( \frac{8}{5\pi} \right)^{1/5} \left( n + \frac{1}{2} \right)^{-1/5}.
\]

We therefore conclude that
\[
\mu_{n+1} - \mu_n \geq \frac{\pi}{2} \left( \frac{8}{5\pi} \right)^{1/5} \left( n + \frac{1}{2} \right)^{-1/5} \quad \text{for all} \ n \in \mathbb{N}.
\] (3.60)

By assertion, we have that \( \lambda_n \approx \mu_n \) so that from (3.60) we find the optimum value of \( C > 0 \) for which
\[
\frac{\pi}{2} \left( \frac{8}{5\pi} \right)^{1/5} \left( n + \frac{1}{2} \right)^{-1/5} \geq C \left( n - \frac{1}{2} \right)^{-1/5}.
\]

Since \( 1 \leq \frac{n+1/2}{n-1/2} \leq 3 \) for all \( n \in \mathbb{N} \), we obtain
\[
C^{-5} = \frac{1}{\left( \frac{\pi}{2} \right)^3 \left( \frac{8}{5\pi} \right)} \sup_{n \in \mathbb{N}} \left( \frac{n+1/2}{n-1/2} \right) = \frac{3}{\left( \frac{\pi}{2} \right)^3 \left( \frac{8}{5\pi} \right)},
\]
which gives \( C = \frac{\pi}{2} \left( \frac{8}{15\pi} \right)^{1/5} \) as required. Hence, the proof is complete. \( \square \)

Plots of the spectral gaps together with the lower and upper bounds on these gaps are displayed in Figure 3.3 below.

Figure 3.3. Actual spectral gaps estimates (in red dots) with lower (in blue dots) and upper bounds (in green dots) of (3.56).
3.5. Heat trace estimates and Weyl-type theorem

Using the asymptotic approximation of Corollary 3.3, we are able to derive an expression for the behaviour of the trace of the semigroup operator $e^{-tH}$ for $t > 0$ in a neighbourhood of $t = 0$. Let the trace be defined by $Z(t) := \sum_{n=1}^{\infty} e^{-t\lambda_n}$, we have the following result:

**Theorem 3.3.** We have the following heat trace estimate:

$$\lim_{t \to 0} t^{5/4} Z(t) = \frac{2\Gamma \left( \frac{5}{4} \right)}{\pi}. \quad (3.61)$$

**Proof.** Let us consider

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} = F(t) + G(t),$$

where

$$F(t) := \sum_{j=1}^{\infty} e^{-\lambda_{2j-1} t}, \quad G(t) := \sum_{j=1}^{\infty} e^{-\lambda_{2j} t}.$$

Then, for every $0 \leq t \leq 1$, it follows from (3.55) that we can find a constant $C > 0$ such that

$$e^{-Ct} \sum_{j=1}^{\infty} e^{-\left( \frac{5(4j-3)\pi}{16} \right) t} \leq F(t) \leq e^{Ct} \sum_{j=1}^{\infty} e^{-\left( \frac{5(4j-3)\pi}{16} \right) t}.$$

Passing from summation to integration, we observe that $x \mapsto e^{-\left( \frac{5(4x-3)\pi}{16} \right) t}$ is non-negative and non-increasing on the interval $[1, \infty)$. Thus, we have that

$$e^{-Ct} \int_1^{\infty} e^{-\left( \frac{5(4x-3)\pi}{16} \right) t} dx \leq F(t) \leq e^{Ct} \int_1^{\infty} e^{-\left( \frac{5(4x-3)\pi}{16} \right) t} dx.$$

Next, by change of variables we set $y = \left( \frac{5(4x-3)\pi}{16} \right)^{4/5} t$ and have that

$$\int_1^{\infty} e^{-\left( \frac{5(4x-3)\pi}{16} \right) t} dx = \frac{1}{\pi\left( \frac{5}{4} \right)^{4/5}} \int_{\left( \frac{5}{16} \right)^{4/5} t}^{\infty} y^{1/2} e^{-y} dy$$

which implies that $\lim_{t \to 0} t^{5/4} F(t) = \frac{\Gamma \left( \frac{3}{4} \right)}{\pi}$; similarly, we have $\lim_{t \to 0} t^{5/4} G(t) = \frac{\Gamma \left( \frac{3}{4} \right)}{\pi}$. By combining these results, we obtain

$$\lim_{t \to 0} t^{5/4} \sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{2\Gamma \left( \frac{5}{4} \right)}{\pi}.$$

The required proof is complete. $\square$

Let $N(\lambda)$ denote the spectral counting function defining the number of eigenvalues $\lambda_n$ (for each $n \in \mathbb{N}$) of $H$ that are less than or equal to $\lambda > 0$. We prove the following Weyl-type result:

**Theorem 3.4.** We have

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{5/4}} = \frac{5}{2\pi}. \quad (3.62)$$
3.5. HEAT TRACE ESTIMATES AND WEYL-TYPE THEOREM

Proof. By definition, the spectral counting function $N(\lambda)$ is given by $N(\lambda) := | \{ n \in \mathbb{N} : \lambda_n \leq \lambda \} |$. Let $a_l$ count the multiplicity of $\lambda_l$ for each $l \in \mathbb{N}$. With $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \leq R$, for every $R > 0$ and finite, we have

$$N(\lambda) = \begin{cases} a_1 + a_2 + \cdots + a_l & \lambda_l \leq \lambda < \lambda_{l+1}; \ l = 1, 2, \ldots, n \\ 0 & 0 < \lambda < \lambda_1. \end{cases}$$

But $a_l = 1$ for all $l \in \mathbb{N}$, and hence, we have

$$N(\lambda) = \sum_{l=1}^{n} 1_{[\lambda_l, \lambda_{l+1})}(\lambda) \quad \text{for} \ 0 < \lambda \leq R,$$

with $N(\lambda) = 0$ for $\lambda \leq 0$. We observe that $N(\lambda)$ is a step-function with point of discontinuity at $\lambda = \lambda_n$ for each $n \in \mathbb{N}$ and right-continuous at this point with jump

$$d_n := N(\lambda_n^+) - N(\lambda_n^-) = 1 \quad \text{for each} \ n \in \mathbb{N}.$$

In addition, $N(\lambda)$ is of bounded variation in $(0, R)$, and the integral

$$\int_0^R e^{-\lambda t} \, dN(\lambda), \quad t > 0$$
exists. Furthermore, taking limit as $R \to \infty$, we have that

$$\int_0^\infty e^{-\lambda t} \, dN(\lambda) = \lim_{R \to \infty} \int_0^R e^{-\lambda t} \, dN(\lambda). \quad (3.63)$$

This limit exists for every $0 < t < \infty$ since by integration-by-parts, one gets

$$\lim_{R \to \infty} \int_0^R e^{-\lambda t} \, dN(\lambda) = t \int_0^\infty e^{-\lambda t} N(\lambda) \, d\lambda < \infty \quad \text{for every} \ 0 < t < \infty,$$

and hence, the integral on the left-hand side of $(3.63)$ defining the Laplace-Stieltjes transform of $N(\lambda)$ is convergent. Furthermore, we have that

$$t \int_0^\infty e^{-\lambda t} N(\lambda) \, d\lambda = t \left( \int_0^{\lambda_1} + \sum_{n \geq 1} \int_{\lambda_n}^{\lambda_{n+1}} \right) e^{-\lambda t} N(\lambda) \, d\lambda$$

$$= t \sum_{n \geq 1} \int_{\lambda_n}^{\lambda_{n+1}} e^{-\lambda t} \, d\lambda$$

$$= \sum_{n \geq 1} n \left( e^{-\lambda_n t} - e^{-\lambda_{n+1} t} \right).$$

By telescopic summation, it follows that

$$\sum_{n \geq 1} n \left( e^{-\lambda_n t} - e^{-\lambda_{n+1} t} \right) = \sum_{n \geq 1} e^{-\lambda_n t}.$$

Hence, we obtain

$$\int_0^\infty e^{-\lambda t} \, dN(\lambda) = \sum_{n \geq 1} e^{-\lambda_n t}. \quad (3.64)$$

By definition, we have $Z(t) := \sum_{n=1}^{\infty} e^{-t \lambda_n}$, and from Theorem 3.3 we have that

$$\int_0^\infty e^{-\lambda t} \, dN(\lambda) = Z(t) \sim \frac{2\Gamma\left(\frac{5}{4}\right)}{\pi} t^{-5/4} \quad \text{as} \ t \to 0^+.$$

(3.65)
We then apply a Karamata-Tauberian theorem [49, Theorem 15.3] on (3.65) to obtain the desired result (3.62), which completes the proof.

3.6. Expressions and properties of eigenfunctions

In this section, we start by deriving expressions for the Fourier transforms of the $L^2$-normalised eigenfunctions $\psi_n(x)$ of the operator $H$. Thus, let

$$
\Lambda_1(\xi) := \tilde{A}_4(\xi) A_4'(\xi) - A_4(\xi) \tilde{A}_4'(\xi)
$$

(3.66)

$$
\Lambda_2(\xi) := \tilde{A}_4'(\xi) A_4''(\xi) - A_4''(\xi) \tilde{A}_4(\xi)
$$

(3.67)

$$
\Lambda_3(\xi) := \tilde{A}_4''(\xi) A_4''(\xi) - A_4''(\xi) \tilde{A}_4(\xi).
$$

(3.68)

We state as follows:

**Theorem 3.5.** The Fourier transforms of the $L^2$-normalised eigenfunctions of $H$ is given by

$$(\mathcal{F}\psi_n)(y) =
\begin{cases}
  c_{1,n} A_4(|y| - \lambda_n) + c_{2,n} \tilde{A}_4(|y| - \lambda_n) & n = 2j - 1 \\
  c_{1,n} \text{sgn}(y) A_4(|y| - \lambda_n) + c_{2,n} \text{sgn}(y) \tilde{A}_4(|y| - \lambda_n) & n = 2j
\end{cases}
$$

(3.69)

for each $j = 1, 2, 3, \ldots$; where

$$
c_{1,n} := \begin{cases}
  -\frac{1}{\sqrt{2}} \frac{\tilde{A}_4(-\lambda_n)}{\sqrt{A_4'(-\lambda_n) + A_4''(-\lambda_n)}} & n = 1, 3, 5, \ldots \\
  \frac{1}{\sqrt{2}} \frac{\tilde{A}_4(-\lambda_n)}{A_4'(-\lambda_n) A_4''(-\lambda_n)} & n = 2, 4, 6, \ldots
\end{cases}
$$

(3.70)

and

$$
c_{2,n} := \begin{cases}
  \frac{1}{\sqrt{2}} \frac{A_4'(-\lambda_n)}{\sqrt{A_4'(-\lambda_n) + A_4''(-\lambda_n)}} & n = 1, 3, 5, \ldots \\
  -\frac{1}{\sqrt{2}} \frac{A_4''(-\lambda_n)}{A_4'(-\lambda_n) A_4''(-\lambda_n)} & n = 2, 4, 6, \ldots
\end{cases}
$$

(3.71)

**Proof.** We are only required to derive expressions for $c_{1,n}$ and $c_{2,n}$. Take $\phi_n(y) = \chi(y - \lambda_n)$ for $y > 0$ and recall from (3.19) that

$$
\frac{d^4}{dz^4} \chi(z) = -\zeta \chi(z).
$$

(3.72)

Hence, we consider

$$
\int_{0}^{\infty} \phi_n^2(y) \, dy = 2 \int_{0}^{\infty} \chi^2(y - \lambda_n) \, dy = 2 \int_{-\infty}^{\infty} \chi^2(z) \, dz.
$$

Using (3.72), we have the identity

$$
\frac{d}{dz} \left( \chi^2(z) + 2 \chi'(z) \chi'''(z) - [\chi''(z)]^2 \right) = \chi^2(z).
$$

Thus, we obtain

$$
\int_{-\infty}^{\infty} \phi_n^2(y) \, dy = 2 \left( \lambda_n \chi^2(-\lambda_n) + [\chi''(-\lambda_n)]^2 - 2 \chi'(-\lambda_n) \chi'''(-\lambda_n) \right).
$$

(3.73)
From the boundary conditions (3.15), we have that
\[ \chi'(-\lambda_n) = \chi'''(-\lambda_n) = 0 \quad \text{for each } n = 1, 3, 5, \ldots \] (3.74)

implying that
\[ c_2 = -c_1 \frac{\text{Ai}_4'(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} = -c_1 \frac{\text{Ai}_4'''(-\lambda_n)}{\text{Ai}_4''(-\lambda_n)} \quad \text{for each } n = 1, 3, 5, \ldots \] (3.75)

or
\[ c_1 = -c_2 \frac{\text{Ai}_4(-\lambda_n)}{\text{Ai}_4''(-\lambda_n)} = -c_2 \frac{\text{Ai}_4''(-\lambda_n)}{\text{Ai}_4'''(-\lambda_n)} \quad \text{for each } n = 1, 3, 5, \ldots . \] (3.76)

Plancherel’s theorem and combination of (3.73) and (3.74) yield
\[ 1 = \int_{-\infty}^{\infty} \phi_n^2(x) \, dx = \int_{-\infty}^{\infty} \phi_n^2(y) \, dy = 2 \left( \lambda_n \chi^2(-\lambda_n) + [\chi''(-\lambda_n)]^2 \right) \quad n = 1, 3, 5, \ldots . \] (3.77)

With (3.75), we have that
\[ \chi(-\lambda_n) = c_1 \text{Ai}_4(-\lambda_n) + c_2 \tilde{\text{Ai}}_4(-\lambda_n) = -\frac{c_1}{\text{Ai}_4(-\lambda_n)} \Lambda_1(-\lambda_n) \]
\[ \chi''(-\lambda_n) = c_1 \text{Ai}_4''(-\lambda_n) + c_2 \tilde{\text{Ai}}_4''(-\lambda_n) = -\frac{c_1}{\text{Ai}_4''(-\lambda_n)} \Lambda_2(-\lambda_n) . \]

Hence, from (3.77), we have
\[ c_1 = \pm \frac{1}{\sqrt{2}} \frac{\tilde{\text{Ai}}_4(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} \quad n = 1, 3, 5, \ldots . \] (3.78)

Similarly, with (3.76), we have that
\[ \chi(-\lambda_n) = c_1 \text{Ai}_4(-\lambda_n) + c_2 \tilde{\text{Ai}}_4(-\lambda_n) = \frac{c_2}{\text{Ai}_4'(-\lambda_n)} \Lambda_1(-\lambda_n) \]
\[ \chi''(-\lambda_n) = c_1 \text{Ai}_4''(-\lambda_n) + c_2 \tilde{\text{Ai}}_4''(-\lambda_n) = \frac{c_2}{\text{Ai}_4''(-\lambda_n)} \Lambda_2(-\lambda_n) . \]

Then, from (3.77), we have
\[ c_2 = \pm \frac{1}{\sqrt{2}} \frac{\text{Ai}_4'(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} \quad n = 1, 3, 5, \ldots , \] (3.79)

where \( \Lambda_1(\xi) \) and \( \Lambda_2(\xi) \) are given by (3.66) and (3.67), respectively. To satisfy the normalisation condition \( \| \phi_n \|_2 = 1 \), we observe that \( c_1 \) and \( c_2 \) must have opposite signs in their expressions. Furthermore, \( \tilde{\text{Ai}}_4(\xi) \) is strictly positive and monotone increasing on \( \xi < 0 \), and moreover, we have that \( (-1)^r \tilde{\text{Ai}}_4^{(r)}(\xi) > 0 \) on \( \xi < 0 \) for each \( r \in \mathbb{N} \), but \( \text{Ai}_4(\xi) \) and its higher derivatives are oscillatory. Therefore, to keep \( c_{1,n} \) positive for each \( n = 1, 3, 5, \ldots , \) which corresponds to the dominating term in the expression for \( \phi_n(y) \), then from (3.78) and (3.79), we have that
\[ c_{1,n} \equiv c_1 = -\frac{1}{\sqrt{2}} \frac{\tilde{\text{Ai}}_4(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} \]
\[ c_{2,n} \equiv c_2 = \frac{1}{\sqrt{2}} \frac{\text{Ai}_4(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} \]
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for each \( n = 1, 3, 5, \ldots \).

To obtain the results for \( n = 2, 4, 6, \ldots \), we employ the boundary conditions \((3.16)\) to obtain

\[
\chi(-\lambda_n) = \chi''(-\lambda_n) = 0 \quad \text{for each } n = 2, 4, 6, \cdots, \tag{3.80}
\]

which implies that

\[
c_2 = -c_1 \frac{\text{Ai}_4(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} \equiv -c_1 \frac{\text{Ai}_4''(-\lambda_n)}{\text{Ai}_4''(-\lambda_n)} \quad \text{for each } n = 2, 4, 6, \cdots \tag{3.81}
\]

or

\[
c_1 = -c_2 \frac{\tilde{\text{Ai}}_4(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} \equiv -c_2 \frac{\tilde{\text{Ai}}_4''(-\lambda_n)}{\text{Ai}_4''(-\lambda_n)} \quad \text{for each } n = 2, 4, 6, \cdots \tag{3.82}
\]

Plancherel’s theorem and combination of \((3.73)\) and \((3.80)\) yield

\[
1 = \int_{-\infty}^{\infty} \psi_n^2(x) \, dx = \int_{-\infty}^{\infty} \tilde{\phi}_n^2(y) \, dy = -4\chi'(-\lambda_n)\chi'''(-\lambda_n) \quad n = 2, 4, 6, \cdots. \tag{3.83}
\]

With \((3.81)\), we have that

\[
\chi'(-\lambda_n) = c_1 \frac{\text{Ai}_4'(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} + c_2 \frac{\tilde{\text{Ai}}_4'(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} = -\frac{c_1}{\text{Ai}_4(-\lambda_n)} \Lambda_2(-\lambda_n)
\]

\[
\chi'''(-\lambda_n) = c_1 \frac{\text{Ai}_4'''(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} + c_2 \frac{\tilde{\text{Ai}}_4'''(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} = \frac{c_1}{\text{Ai}_4''(-\lambda_n)} \Lambda_3(-\lambda_n).
\]

Thus, from \((3.83)\) we have

\[
c_1 = \frac{1}{2} \frac{\tilde{\text{Ai}}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} \quad n = 2, 4, 6, \cdots \tag{3.84}
\]

Similarly, with \((3.82)\), we have that

\[
\chi'(-\lambda_n) = c_1 \frac{\text{Ai}_4'(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} + c_2 \frac{\tilde{\text{Ai}}_4'(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} = \frac{c_1}{\text{Ai}_4''(-\lambda_n)} \Lambda_2(-\lambda_n)
\]

\[
\chi'''(-\lambda_n) = c_1 \frac{\text{Ai}_4'''(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} + c_2 \frac{\tilde{\text{Ai}}_4'''(-\lambda_n)}{\text{Ai}_4(-\lambda_n)} = -\frac{c_1}{\text{Ai}_4''(-\lambda_n)} \Lambda_3(-\lambda_n).
\]

It then follows from \((3.83)\) that

\[
c_2 = \frac{1}{2} \frac{\text{Ai}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} \quad n = 2, 4, 6, \cdots \tag{3.85}
\]

where \(\Lambda_3(\xi)\) is given by \((3.68)\). As earlier noted, \(c_1\) and \(c_2\) must have opposite signs in their expressions in order to satisfy the normalisation condition \(||\psi_n||_2 = 1\). Moreover, \(\tilde{\text{Ai}}_4''(\xi)\) is strictly positive and monotone increasing on \(\xi < 0\). Therefore, by keeping \(c_{1,n}\) positive for each \(n = 2, 4, 6, \cdots\), then from \((3.84)\) and \((3.85)\), we have that

\[
c_{1,n} \equiv c_1 = \frac{1}{2} \frac{\tilde{\text{Ai}}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}}
\]

\[
c_{2,n} \equiv c_2 = -\frac{1}{2} \frac{\text{Ai}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}}
\]

for each \(n = 2, 4, 6, \cdots\). This completes the required proof. \(\square\)
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Plots of the first four eigenfunctions $\psi_n(x)$ (for $n = 1, 2, 3, 4$) displayed in Figure 3.4 are obtained numerically by computing the corresponding inverse sine and cosine Fourier transforms of $\phi_n(y)$ expressed in (3.69) and displayed in Figure 3.5. As shown in Figure 3.4 below, the first eigenfunction (also known as the ground state) $\psi_1(x)$ of $H$ (red line), is strictly positive; this is expected in theory. The plots of the next three eigenfunctions (i.e., $\psi_n(x)$ for $n = 2, 3, 4$) show increasing number of nodes. Each successive eigenfunction $\psi_n(x)$ has a total number of nodes equal to the index value $n \in \mathbb{N}$. These results together with those obtained in [57] allow us to postulate that Courant’s famous nodal domain theorem for elliptic operators (see [22, Vol. 1, Secs. V.5, VI.6]) also holds for non-local operators of the form (1.1).

![Figure 3.4](image1.png)  
**Figure 3.4.** First four eigenfunctions $\psi_n(x)$ of the operator $H$ with first eigenfunction (or ground state) in red, and the next three eigenfunctions in purple, blue, and green, respectively.

![Figure 3.5](image2.png)  
**Figure 3.5.** Fourier transforms $\phi_n(y)$ of the eigenfunctions with the Fourier transform of first eigenfunction in red, and the next three in purple, blue, and green, respectively.

We now proceed with other properties of the eigenfunctions $\psi_n(x)$. First, we obtain full details about the asymptotic behaviour of these eigenfunctions. This is stated as follows:

**Theorem 3.6.** For every $j \in \mathbb{N}$ and $N = 2, 3, \ldots$ we have that

\[
\psi_{2j-1}(x) = \sum_{l=1}^{N-1} (-1)^{l+1} \frac{P(\lambda_{2j-1})}{x^{4l+2l}} + O\left(\frac{1}{x^{2N+4}}\right) 
\]

(3.86)

\[
\psi_{2j}(x) = \sum_{l=1}^{N-1} (-1)^{l+1} \frac{Q(\lambda_{2j})}{x^{5l+2l}} + O\left(\frac{1}{x^{2N+5}}\right) 
\]

(3.87)
as \( x \to \infty \); where

\[
\mathcal{P}(-\lambda_1) = \sqrt{\frac{2}{\pi}} [c_{1,1} Ai_4(-\lambda_1) + c_{2,1} \tilde{Ai}_4(-\lambda_1)], \quad \mathcal{Q}(-\lambda_2) = 2 \sqrt{\frac{2}{\pi}} [c_{1,2} Ai'_4(-\lambda_2) + c_{2,2} \tilde{Ai}'_4(-\lambda_2)];
\]

\[
\mathcal{P}(\lambda_{2j-1}) = \sqrt{\frac{2}{\pi}} [c_{1,2j-1} Ai_4^{(3+2j)}(-\lambda_{2j-1}) + c_{2,2j-1} \tilde{Ai}_4^{(3+2j)}(-\lambda_{2j-1})] \quad \text{for each } j = 2, 3, \cdots;
\]

\[
\mathcal{Q}(\lambda_{2j}) = \sqrt{\frac{2}{\pi}} [c_{1,2j-1} Ai_4^{(4+2j)}(-\lambda_{2j}) + c_{2,2j-1} \tilde{Ai}_4^{(4+2j)}(-\lambda_{2j})] \quad \text{for each } j = 2, 3, \cdots.
\]

**Proof.** We recall (3.69) for each \( n = 1, 3, 5, \cdots \), and by taking inverse cosine transform, we have that

\[
\psi_{2j-1}(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xy)\phi_{2j-1}(y) \, dy \quad \text{for each } j \in \mathbb{N}; \tag{3.88}
\]

where \( \phi_{2j-1}(y) = c_{1,2j-1} Ai_4(y - \lambda_{2j-1}) + c_{2,2j-1} \tilde{Ai}_4(y - \lambda_{2j-1}) \) for \( y > 0 \). Then, we apply integration-by-parts \( (2N + 4) \)-times in (3.88) together with the fact that \( \lim_{y \to \infty} \sin \left(\frac{\pi y}{2}\right) \phi^{(l)}_{2j-1}(y) = 0 \) for all \( r, l \in \mathbb{N}_0 \) to obtain

\[
\int_0^\infty \cos(xy)\phi_{2j-1}(y) \, dy = \sum_{r=0}^N (-1)^{r+1} \frac{\cos(xy)\phi^{(2r+1)}_{2j-1}(y)}{x^{2r+2}} \bigg|_{y=0} + E_{1,N}(x)
\]

\[
= \sum_{r=0}^N (-1)^{r+1} \frac{\phi^{(2r+1)}_{2j-1}(0)}{x^{2r+2}} + E_{1,N}(x); \tag{3.89}
\]

where

\[
E_{1,N}(x) := \frac{(-1)^{N+1}}{x^{2N+2}} \int_0^\infty \cos(xy)\phi^{(2N+2)}_{2j-1}(y) \, dy
\]

\[
= \frac{(-1)^N}{x^{2N+4}} \left[ \cos(xy)\phi^{(2N+3)}_{2j-1}(y) \bigg|_{y=0} + \int_0^\infty \cos(xy)\phi^{(2N+4)}_{2j-1}(y) \, dy \right]
\]

\[
= \frac{(-1)^N}{x^{2N+4}} \left( \phi^{(2N+3)}_{2j-1}(0) + \int_0^\infty \cos(xy)\phi^{(2N+4)}_{2j-1}(y) \, dy \right)
\]

for each \( N = 2, 3, 4, \cdots \). But for sufficiently large \( M_1, M_2 > 0 \) dependent only on \( N \), we have that

\[
|\phi^{(2N+3)}_{2j-1}(0)| \leq c_{1,2j-1} |Ai^{(2N+3)}_4(-\lambda_{2j-1})| + c_{2,2j-1} |\tilde{Ai}^{(2N+3)}_4(-\lambda_{2j-1})| \leq M_1
\]

and

\[
\int_0^\infty |\phi^{(2N+4)}_{2j-1}(y)| \, dy \leq c_{1,2j-1} \int_0^\infty |Ai^{(2N+4)}_4(y - \lambda_{2j-1})| \, dy + c_{2,2j-1} \int_0^\infty |\tilde{Ai}^{(2N+4)}_4(y - \lambda_{2j-1})| \, dy \leq M_2.
\]

Hence, there exists \( C_{1,j,N} > 0 \) such that \( |E_{1,N}(x)| \leq \frac{C_{1,j,N}}{|x|^{2N+4}} \). Now, we recall the initial conditions

\[
\phi'_{2j-1}(0) = \phi''_{2j-1}(0) = 0, \quad j = 1, 2, 3, \cdots.
\]

Moreover, differentiating (3.18) once, we have that \( \phi^{(5)}_{2j-1}(y) = -\phi_{2j-1}(y) - (y - \lambda_{2j-1})\phi'_{2j-1}(y) \), which reduces to \( \phi^{(5)}_{2j-1}(0) = -\phi_{2j-1}(0) \) with \( \phi_{2j-1}(0) > 0 \) for all \( j \in \mathbb{N} \). Therefore, using these pieces of information, we obtain (3.86) from (3.89) combined with (3.88).
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We follow similar line of argument to establish (3.87). Again, we recall (3.69) for each \( n = 2, 4, 6, \ldots \) so that by inverse sine transform, we have that
\[
\psi_2(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(xy) \phi_2(y) \, dy \quad \text{for each } j \in \mathbb{N};
\] (3.90)
where \( \phi_2(y) = c_{1,2j} \text{Ai}(y - \lambda_{2j}) + c_{2,2j} \text{Ai}(y - \lambda_{2j}) \) for \( y > 0 \). Then, we apply integration-by-parts \((2N+5)\)-times in (3.90) together with the fact that \( \lim_{y \to \infty} \sin \left( xy + \frac{\pi}{2} \right) \phi_2^{(j)}(y) = 0 \) for all \( r, l \in \mathbb{N}_0 \) to obtain
\[
\int_0^\infty \sin(xy) \phi_2(y) \, dy = \sum_{r=0}^{N+1} (-1)^r \frac{\cos(xy)\phi_2^{(2r)}(y)}{x^{2r+1}} \bigg|_{y=0} + E_{2,N}(x) = \sum_{r=0}^{N+1} (-1)^r \frac{\phi_2^{(2r)}(0)}{x^{2r+1}} + E_{2,N}(x); \] (3.91)
where
\[
E_{2,N}(x) := \frac{(-1)^{N+1}}{x^{2N+3}} \int_0^\infty \cos(xy)\phi_2^{(2N+3)}(y) \, dy = \frac{(-1)^N}{x^{2N+5}} \left[ \cos(xy)\phi_2^{(2N+4)}(y) \bigg|_{y=0} + \int_0^\infty \cos(xy)\phi_2^{(2N+5)}(y) \, dy \right] = \frac{(-1)^N}{x^{2N+5}} \left( \phi_2^{(2N+4)}(0) + \int_0^\infty \cos(xy)\phi_2^{(2N+5)}(y) \, dy \right),
\]
for each \( N = 2, 3, 4, \ldots \). Similarly, there exists \( C_{2,j,N} > 0 \) such that \( |E_{2,N}(x)| \leq \frac{C_{2,j,N}}{|x|^{2N+5}} \). Furthermore, we recall the initial conditions \( \phi_j(0) = \phi_j'(0) = 0 \), \( j = 1, 2, 3, \ldots \). Moreover, we have that \( \phi_2^{(4)}(0) = -\lambda_{2j} \phi_2(0) = 0 \), and by differentiating (3.18) twice, we have that \( \phi_2^{(6)}(y) = -2\phi_2'(y) - (y - \lambda_{2j})\phi_2''(y) \), which reduces to \( \phi_2^{(6)}(0) = -2\phi_2'(0) \) with \( \phi_2'(0) > 0 \) for all \( j \in \mathbb{N} \). Therefore, putting these pieces together, we obtain (3.87) from (3.91) combined with (3.90). \( \square \)

From the smoothing property of the evolution semigroup \( e^{-itH} \), the eigenfunctions \( \psi_n(x) \) are known to be bounded and continuous functions. Furthermore, the next two results show even stronger properties of these eigenfunctions.

**Theorem 3.7.** Every \( \psi_n(x) \) is analytic on \( \mathbb{R} \), and for each \( n \in \mathbb{N} \), \( \psi_n(x) \) has the following analytic expansions:

\[
\psi_n(x) = \sum_{r=0}^{\infty} (-1)^r a_{2r}(\lambda_n)x^{2r} \quad n = 2j - 1, \] (3.92)
\[
\psi_n(x) = \sum_{r=0}^{\infty} (-1)^r a_{2r+1}(\lambda_n)x^{2r+1} \quad n = 2j, \] (3.93)
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for each \( j = 1, 2, 3, \ldots \), and for all \( x \in \mathbb{R} \); where

\[
a_p(u) := \frac{1}{p!} \sqrt{\frac{2}{\pi}} \int_0^\infty y^p \left[ c_{1,n} \text{Ai}_4(y-u) + c_{2,n} \tilde{\text{Ai}}_4(y-u) \right] \, dy.
\]

**Proof.** We only consider the proof for the \( n = 2j - 1 \) case as the result is easily extended to the \( n = 2j \) case. Now, for each \( n = 2j - 1 \), \( j = 1, 2, 3, \ldots \), we have that

\[
\psi_n(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xy) \left[ c_{1,n} \text{Ai}_4(y-\lambda_n) + c_{2,n} \tilde{\text{Ai}}_4(y-\lambda_n) \right] \, dy
\]

\[
= \sqrt{\frac{2}{\pi}} \int_0^\infty \sum_{j=0}^\infty (-1)^j \frac{(xy)^{2j}}{(2j)!} \left[ c_{1,n} \text{Ai}_4(y-\lambda_n) + c_{2,n} \tilde{\text{Ai}}_4(y-\lambda_n) \right] \, dy. \quad (3.94)
\]

Let us define

\[
a_p(\lambda_n) := \frac{1}{p!} \sqrt{\frac{2}{\pi}} \int_0^\infty y^p \left[ c_{1,n} \text{Ai}_4(y-\lambda_n) + c_{2,n} \tilde{\text{Ai}}_4(y-\lambda_n) \right] \, dy, \quad \text{for each } p = 2r, \ r = 1, 2, 3, \ldots.
\]

We need to show that for each \( n = 2j - 1 \), \( j = 1, 2, 3, \ldots \),

\[
\sum_{r=0}^\infty a_{2r}(\lambda_n) |x|^{2r} < \infty \quad (3.95)
\]

for almost all \( x \in \mathbb{R} \), then we can conclude using the monotone and dominated convergence theorems that \((3.92)\) follows from \((3.94)\) for each \( n = 2j - 1 \), \( j \in \mathbb{N} \) by interchanging summation with integration.

It suffices to only show that the radius of convergence \( R \) of the power series in \((3.95)\) lies in the interval \((0, \infty)\). We recall that \(|\tilde{\text{Ai}}_4(y)| \leq |\text{Ai}_4(y)|\) for \( y > 0 \); moreover, we have \(\text{Ai}_4(y) \leq \frac{1}{2\sqrt{2\pi y}} y^{-\frac{3}{4}} e^{-\frac{4}{3}y^\frac{3}{2}}\) for \( y > 0 \), which implies that there exists a constant \(C_n > 0\) such that \(|c_{1,n} \text{Ai}_4(y-\lambda_n) + c_{2,n} \tilde{\text{Ai}}_4(y-\lambda_n)| < C_n e^{-\frac{4}{3}y^\frac{3}{2}}\) for \( y > 0 \). Therefore, we obtain

\[
a_{2r}(\lambda_n) \leq \frac{C_n}{(2r)!} \sqrt{\frac{2}{\pi}} \int_0^\infty y^{2r} e^{-\frac{4}{3}y^\frac{3}{2}} \, dy.
\]

Using the change of variable \( z = \frac{4}{3}y^\frac{3}{2} \), we can write

\[
\tilde{a}_{2r} := \frac{1}{(2r)!} \sqrt{\frac{2}{\pi}} \left( \frac{4}{3} \right)^{\frac{2r+1}{2}} \int_0^\infty z^{-\frac{2r+1}{2}} e^{-z} \, dz = \frac{1}{(2r)!} \sqrt{\frac{2}{\pi}} \left( \frac{4}{3} \right)^{\frac{2r+1}{2}} \Gamma \left( \frac{4(2r+1)}{3} \right),
\]

then using Stirling’s formula for large \( \alpha \) behaviour of \( \Gamma(\alpha) \), we obtain

\[
\frac{\tilde{a}_{2r}}{\tilde{a}_{2r+2}} = (2r+2)(2r+1) \left( \frac{4}{5} \right)^{\frac{2r+1}{2}} \frac{\Gamma \left( \frac{4(2r+1)}{3} \right)}{\Gamma \left( \frac{4(2r+3)}{3} \right)}
\]

\[
\approx (2r+2)(2r+1) \left( \frac{4}{5} \right)^{\frac{2r+1}{2}} e^{\frac{2r+1}{2r+3} \frac{4(2r+1)}{3} - \frac{5}{4(2r+3)}} \left( \frac{5}{2r+3} \right)^{\frac{3}{2}},
\]

which implies that \(\frac{\tilde{a}_{2r}}{\tilde{a}_{2r+2}} \to \infty\) as \( r \to \infty \).

\[\square\]

**Theorem 3.8.** For each \( n \in \mathbb{N} \), the functions \( \psi_n(x) \) are uniformly bounded. Moreover, each \( \psi_n(x) \) has a finite number of zeroes.
3.6. EXPRESSIONS AND PROPERTIES OF EIGENFUNCTIONS

**Proof.** Recall the domination property [47, Corollary 2.1] that there exists a constant $C > 0$ such that $|\psi_n(x)| \leq C||\psi_n||_{\infty} \psi_1(x)$ for each $n \in \mathbb{N}$. Hence, it suffices to show that the first eigenfunction $\psi_1(x)$ is uniformly bounded. Furthermore, there exists a constant $C' > 0$ such that

$$c_{1,1} \text{Ai}_1(y) + c_{2,1} \text{Ai}_i(y) \leq C'|\text{Ai}_i(y)| \quad \text{for} \quad y > -\lambda_i.$$

Hence, for all $x \in \mathbb{R}$, we consider

$$\psi_1(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xy) [c_{1,1} \text{Ai}_1(y - \lambda_i) + c_{2,1} \text{Ai}_i(y - \lambda_i)] \, dy$$

$$= \sqrt{\frac{2}{\pi}} \int_{-\lambda_i}^\infty \cos(x(u + \lambda_i)) [c_{1,1} \text{Ai}_i(u) + c_{2,1} \text{Ai}_i(u)] \, du$$

$$\leq C' \sqrt{\frac{2}{\pi}} \int_{-\lambda_i}^\infty \cos(x(u + \lambda_i)) |\text{Ai}_i(u)| \, du$$

$$= C' \sqrt{\frac{2}{\pi}} \left( \int_{-\lambda_i}^0 + \int_{-\lambda_i}^{5/8} + \int_{5/8}^\infty \right) \cos(x(u + \lambda_i)) |\text{Ai}_i(u)| \, du. \quad (3.96)$$

Now, we employ the result $|\text{Ai}_i(y)| \leq \min\{C_1 e^{-2u}, C_2 |u|^{-5/8}\}$ of Theorem 2.2 in the first integral for the interval $(-\lambda_i, 0)$ to obtain

$$\int_{-\lambda_i}^0 \cos(x(u + \lambda_i)) |\text{Ai}_i(u)| \, du \leq \int_{-\lambda_i}^0 \min\{C_1 e^{-2u}, C_2 |u|^{-5/8}\} \, du$$

$$= \frac{C_1}{2} (e^{-2u} - 1) + \frac{8C_2}{13} \left( \lambda_i^{13/8} - (-a)^{13/8} \right) < \infty \quad (3.97)$$

for some constants $C_1, C_2 > 0$ and $a \in (-\lambda_i, 0)$. Moreover, we know that $\text{Ai}_i(u)$ is strictly positive in the interval $(-\lambda_i, 5/8)$, and monotonically decreasing in the interval $(0, 5/8)$. Therefore, we simply have that

$$\int_{0}^{5/8} \cos(x(u + \lambda_i)) |\text{Ai}_i(u)| \, du \leq \text{Ai}_i(0) \int_{0}^{5/8} |\cos(x(u + \lambda_i))| \, du$$

$$= \frac{5}{8} \frac{\cos(\pi/10)}{5^2 \Gamma\left(\frac{4}{5}\right) \sin(\pi/5)} < \infty. \quad (3.98)$$

Lastly, using the fact that $\text{Ai}_i(u) \leq e^{-\frac{3}{4}u^\frac{3}{2}}$ for $u > 5/8$, we have that

$$\int_{5/8}^\infty \cos(x(u + \lambda_i)) |\text{Ai}_i(u)| \, du \leq \int_{5/8}^\infty |\text{Ai}_i(u)| \, du$$

$$\leq \int_{5/8}^\infty e^{-\frac{3}{4}u^\frac{3}{2}} \, du < \infty. \quad (3.99)$$

Therefore, putting (3.97), (3.98), and (3.99) together in (3.96), we have shown that for some positive constant $C$,

$$\psi_1(x) \leq C, \quad \text{for all} \quad x \in \mathbb{R},$$
and hence, $\psi_1$ is uniformly bounded. Therefore, using the domination property, we completely prove uniform boundedness for $\psi_n(x)$ for all $n \in \mathbb{N}$. This completes the proof. □

3.7. Further results: the small effect

In this section, we investigate the small effect arising from $\tilde{A}_i(y)$ in (3.26) - (3.27). This investigation will be carried out at two levels: at the level of the eigenvalues in Subsection 3.7.1 and at the level of the eigenfunctions in Subsection 3.7.2.

3.7.1. Small effect on the level of the spectrum

The result of this subsection tells us how well do the negative real zeroes of $A_4(y)$ and $A_4'(y)$, arranged in alternating and increasing order of magnitude, approximate the true eigenvalues $\lambda_n$ of $H$.

To proceed with the investigation, we define as follows:

$$\mu_n := \begin{cases} -\alpha_{4,j} & n = 2j - 1 \\ -\alpha_{4,j} & n = 2j \end{cases}$$

for each $j = 1, 2, 3, \cdots$; where $\alpha_{4,j}$ and $\alpha_{4,j}'$ are the negative real zeroes of $A_4(y)$ and $A_4'(y)$, respectively; refer to (2.111) - (2.113) for expressions of these zeroes. To prove the main result of this subsection, we will need a number of preliminary results. Thus, given that

$$\mathcal{L}\phi_n(y) := \frac{d^4}{dy^4}\phi_n(y) + |y|\phi_n(y), \quad y \in \mathbb{R}$$

so that $(\mathcal{L} - \lambda_n)\phi_n(y) = 0$, we prove the following result:

**Lemma 3.2.** For each $n \in \mathbb{N}$, we have

$$\| (\mathcal{L} - \mu_n)A_4(|y| - \lambda_n) \|_2^2 \leq \frac{4 \sqrt{2} \Gamma\left(\frac{4}{3}\right) |c_{2,n}|}{\pi^{2/3}} \frac{|c_{2,n}|}{c_{1,n}}.$$  

**Proof.** Given that $\mu_n \geq \lambda_n$ for some $n \in \mathbb{N}$, we consider

$$|(\mathcal{L} - \mu_n)A_4(|y| - \lambda_n)| = \frac{1}{c_{1,n}} \left|(\mathcal{L} - \mu_n)\phi_n(y) - c_{2,n}(\mathcal{L} - \mu_n)\tilde{A}_4(|y| - \lambda_n)\right|$$

$$\leq \frac{1}{c_{1,n}} \left|(\mathcal{L} - \lambda_n)\phi_n(y) - c_{2,n}(\mathcal{L} - \mu_n)\tilde{A}_4(|y| - \lambda_n)\right|$$

$$= \frac{|c_{2,n}|}{c_{1,n}} \left|(\mathcal{L} - \mu_n)\tilde{A}_4(|y| - \lambda_n)\right|$$

$$\leq \frac{|c_{2,n}|}{c_{1,n}} \left|(\mathcal{L} - \lambda_n)\tilde{A}_4(|y| - \lambda_n)\right|. \quad (3.103)$$
Hence, we have that
\[
\int_{-\infty}^{\infty} \left| (\mathcal{L} - \mu_n) \text{Ai}_4(y) - \lambda_n \right|^2 dy \leq \frac{|c_{2,n}|}{c_{1,n}} \int_{-\infty}^{\infty} \left| (\mathcal{L} - \lambda_n) \bar{\text{Ai}}_4(y) - \lambda_n \right|^2 dy \\
= \frac{2|c_{2,n}|}{c_{1,n}} \int_{0}^{\infty} \left| \left( \frac{d^4}{dy^4} + y - \lambda_n \right) \bar{\text{Ai}}_4(y - \lambda_n) \right|^2 dy. \tag{3.104}
\]
But
\[
\bar{\text{Ai}}_4(z) \approx \frac{1}{\pi} \int_{0}^{\infty} e^{-\gamma - \frac{\pi}{4}} dt \quad \text{for every } z \in \mathbb{R}.
\]
Therefore, it follows from (3.104) that
\[
\| (\mathcal{L} - \mu_n) \text{Ai}_4(| \cdot | - \lambda_n) \|_2^2 \leq \frac{2|c_{2,n}|}{\pi c_{1,n}} \int_{0}^{\infty} \left( \left| \int_{0}^{\infty} \left( t^4 + y - \lambda_n \right) e^{-\gamma - \frac{\pi}{4}} dt \right|^2 + \int_{0}^{\infty} \left( y - \lambda_n \right) e^{-\gamma - \frac{\pi}{4}} dt \right) \| \text{Ai}_4 \|_2 \| e^{-(y - \lambda_n)^2/\pi} \|_2^2 dy. \tag{3.105}
\]
We now want to estimate the inner integrals in (3.105). We proceed as follows: let \( t = (\lambda_n - y)^{1/4} u \), then we have that
\[
\int_{0}^{\infty} t^4 e^{-\gamma - \frac{\pi}{4}} dt = (\lambda_n - y)^{5/4} \int_{0}^{\infty} u^4 e^{-(\lambda_n - y)^{5/4} h(u)} du, \tag{3.106}
\]
where \( h(u) = \frac{u^5}{5} - u \). But we observe that \( h(u) \) has vanishing first derivative \( h'(u) \) at \( u = 1 \), which lies in the range of integration. Hence, the integral is maximum around a small neighbourhood of \( u = 1 \). Therefore, we restrict the integral to a small region around \( u = 1 \), expand \( h(u) \) around \( u = 1 \):
\[
h(u) = h(1) + \frac{1}{2} h''(1)(u - 1)^2 = -\frac{4}{5} + 2(u - 1)^2, \]
and write
\[
\int_{0}^{\infty} u^4 e^{-(\lambda_n - y)^{5/4} h(u)} du = e^{\frac{1}{5}(\lambda_n - y)^{5/4}} \int_{1-\epsilon}^{1+\epsilon} e^{-(\lambda_n - y)^{5/4} (u - 1)^2} du
\]
\[
= \frac{1}{\sqrt{2(\lambda_n - y)^{5/4}}} e^{\frac{1}{5}(\lambda_n - y)^{5/4}} \int_{\sqrt{2}(\lambda_n - y)^{5/8} \epsilon}^{\sqrt{2}(\lambda_n - y)^{5/8} \epsilon} e^{-z^2} dz
\]
\[
\leq \sqrt{\frac{\pi}{2(\lambda_n - y)^{5/4}}} e^{\frac{1}{5}(\lambda_n - y)^{5/4}}.
\]
Hence, from (3.106), we have that
\[
\left| \int_{0}^{\infty} t^4 e^{-(y - \lambda_n)^{5/4} \frac{\pi}{4}} dt \right| \leq \sqrt{\frac{\pi}{2}} |y - \lambda_n|^{5/8} e^{-\frac{2\pi^2}{5} |y - \lambda_n|^{5/4}}. \tag{3.107}
\]
Similarly, we found that
\[
\left| \int_{0}^{\infty} (y - \lambda_n) e^{-(y - \lambda_n)^{5/4} \frac{\pi}{4}} dt \right| \leq \sqrt{\frac{\pi}{2}} |y - \lambda_n|^{5/8} e^{-\frac{2\pi^2}{5} |y - \lambda_n|^{5/4}}. \tag{3.108}
\]
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Therefore, putting (3.107) and (3.108) together, it follows from (3.105) that
\[
\|(L - \mu_n) A_4(\cdot \mid \lambda_n)\|_2^2 \leq \frac{4 |c_{2,n}|}{\pi c_{1,n}} \int_0^{\infty} [y - \lambda_n]^{5/4} e^{-\frac{4\sqrt{5}}{5} (y - \lambda_n)^{5/4}} dy
\]
\[
= \frac{4 |c_{2,n}|}{\pi c_{1,n}} \left[ I_1(\lambda_n) + I_2(\lambda_n) \right], \tag{3.109}
\]
where
\[
I_1(\lambda_n) := \int_0^{\lambda_n} (\lambda_n - y)^{5/4} e^{-\frac{4\sqrt{5}}{5} (\lambda_n - y)^{5/4}} dy
\]
\[
= \frac{1}{\sqrt{2}} \left[ -\lambda_n e^{-\frac{4\sqrt{5}}{5} \lambda_n^{5/4}} + \int_0^{\lambda_n} e^{-\frac{4\sqrt{5}}{5} (y - \lambda_n)^{5/4}} dy \right]
\]
\[
\leq \frac{1}{\sqrt{2}} \left[ -\lambda_n e^{-\frac{4\sqrt{5}}{5} \lambda_n^{5/4}} + \int_0^{\infty} e^{-\frac{4\sqrt{5}}{5} z^{5/4}} dz \right]
\]
\[
= \frac{1}{\sqrt{2}} \left[ -\lambda_n e^{-\frac{4\sqrt{5}}{5} \lambda_n^{5/4}} + \frac{1}{51/5} \Gamma \left( \frac{4}{5} \right) \right],
\]
and
\[
I_2(\lambda_n) := \int_0^{\infty} (y - \lambda_n)^{5/4} e^{-\frac{4\sqrt{5}}{5} (y - \lambda_n)^{5/4}} dy
\]
\[
= \frac{1}{\sqrt{2}} \int_0^{\infty} e^{-\frac{4\sqrt{5}}{5} (y - \lambda_n)^{5/4}} dy
\]
\[
= \frac{1}{\sqrt{2}} \int_0^{\infty} e^{-\frac{4\sqrt{5}}{5} z^{5/4}} dz = \frac{1}{\sqrt{2}} \frac{1}{51/5} \Gamma \left( \frac{4}{5} \right).
\]
Therefore, from (3.109), we have that
\[
\|(L - \mu_n) A_4(\cdot \mid \lambda_n)\|_2^2 \leq \frac{4 |c_{2,n}|}{\pi \sqrt{2} c_{1,n}} \left[ \frac{2}{51/5} \Gamma \left( \frac{4}{5} \right) - \lambda_n e^{-\frac{4\sqrt{5}}{5} \lambda_n^{5/4}} \right].
\]
But \(\lambda_n e^{-\frac{4\sqrt{5}}{5} \lambda_n^{5/4}} > 0\) for all \(\lambda_n > \lambda_1 > \lambda_1^* \approx 0.3881\) (where \(\lambda_1^*\) is a critical point of \(\lambda_n e^{-\frac{4\sqrt{5}}{5} \lambda_n^{5/4}}\)); hence, further simplification yields
\[
\|(L - \mu_n) A_4(\cdot \mid \lambda_n)\|_2^2 \leq \frac{4 \sqrt{2}}{\pi 5^{1/5}} \Gamma \left( \frac{4}{5} \right) \frac{|c_{2,n}|}{c_{1,n}}.
\]
The result is also true for all \(n \in \mathbb{N}\) for which \(\mu_n < \lambda_n\). \(\square\)

Next, we obtain formulae expressing the normalisation constants \(c_{1,n}\) and \(c_{2,n}\) explicitly in terms of \(n\).

**Lemma 3.3.** For all \(n \in \mathbb{N}\), we have that
\[
c_{1,n} \approx \sqrt{\frac{\pi}{2}} n^{-1/8} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} \quad \text{and} \quad |c_{2,n}| \approx \sqrt{\frac{\pi}{2}} n^{-1/8} e^{-\frac{4\sqrt{5}}{5} \lambda_n^{5/4}} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} e^{-\frac{4\sqrt{5}}{5} \lambda_n^{5/4}}. \tag{3.110}
\]
3.7. FURTHER RESULTS: THE SMALL EFFECT

Proof. Recall from (3.70) and (3.71) that
\[
c_{1,n} := \begin{cases} 
-\frac{1}{\sqrt{2}} \frac{\tilde{\lambda}_n(\lambda_n)}{\Lambda_n(\lambda_n)} & n = 1, 3, 5, \ldots \\
\frac{1}{\sqrt{2}} \frac{\Lambda_n(\lambda_n)}{\alpha \beta_n(\lambda_n)} & n = 2, 4, 6, \ldots 
\end{cases} 
\]  
(3.111)

and
\[
c_{2,n} := \begin{cases} 
\frac{1}{\sqrt{2}} \frac{\Lambda_n(\lambda_n)}{\alpha \beta_n(\lambda_n)} & n = 1, 3, 5, \ldots \\
-\frac{1}{\sqrt{2}} \frac{\tilde{\lambda}_n(\lambda_n)}{\Lambda_n(\lambda_n)} & n = 2, 4, 6, \ldots 
\end{cases} 
\]  
(3.112)

where
\[
\Lambda_1(-\lambda) := \tilde{\Lambda}_4(-\lambda) \Lambda_4'(-\lambda) - \Lambda_4(-\lambda) \tilde{\Lambda}_4'(\lambda) \\
\Lambda_2(-\lambda) := \tilde{\Lambda}_4'-(-\lambda) \Lambda_4''(-\lambda) - \Lambda_4''(-\lambda) \tilde{\Lambda}_4''(-\lambda) \\
\Lambda_3(-\lambda) := \tilde{\Lambda}_4''(-\lambda) \Lambda_4''(-\lambda) - \Lambda_4''(-\lambda) \tilde{\Lambda}_4''(-\lambda)
\]

for \( \lambda > 0 \). But from (3.49) and (3.50), we have that
\[
\Lambda_4(-\lambda) = \frac{1}{\sqrt{2\pi}} e^{-3/8} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) \left( 1 + O(\lambda^{-5/8}) \right); \\
\tilde{\Lambda}_4(-\lambda) = \frac{1}{\sqrt{2\pi}} e^{-3/8} e^{\frac{4}{5}\lambda^{5/4}} \left( 1 + O(\lambda^{-5/8}) \right)
\]
for every \( \lambda > \lambda_0 \) for some \( 0 < \lambda_0 \leq 1/2 \); and given that \( \Lambda_4(y) \) and \( \tilde{\Lambda}_4(y) \) are analytic functions, both asymptotic relations can be differentiated. Thus, we obtain
\[
\Lambda_4'(-\lambda) \approx -\frac{1}{\sqrt{2\pi}} \lambda^{-1/8} \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right); \\
\tilde{\Lambda}_4'(-\lambda) \approx -\frac{1}{\sqrt{2\pi}} \lambda^{-1/8} e^{\frac{4}{5}\lambda^{5/4}}; 
\]  
(3.113)

\[
\Lambda_4''(-\lambda) \approx -\frac{1}{\sqrt{2\pi}} \lambda^{1/8} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right); \\
\tilde{\Lambda}_4''(-\lambda) \approx \frac{1}{\sqrt{2\pi}} \lambda^{1/8} e^{\frac{4}{5}\lambda^{5/4}}; 
\]  
(3.114)

\[
\Lambda_4'''(-\lambda) \approx \frac{1}{\sqrt{2\pi}} \lambda^{3/8} \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right); \\
\tilde{\Lambda}_4'''(-\lambda) \approx -\frac{1}{\sqrt{2\pi}} \lambda^{3/8} e^{\frac{4}{5}\lambda^{5/4}}. 
\]  
(3.115)

Hence, we have that
\[
\Lambda_1(-\lambda) \approx \frac{1}{2\pi} \left[ \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) - \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) \right] \lambda^{-1/2} e^{\frac{4}{5}\lambda^{5/4}} \\
\Lambda_2(-\lambda) \approx \frac{1}{2\pi} \left[ \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) + \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) \right] e^{\frac{4}{5}\lambda^{5/4}} \\
\Lambda_3(-\lambda) \approx \frac{1}{2\pi} \left[ \cos \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) - \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) \right] \lambda^{1/2} e^{\frac{4}{5}\lambda^{5/4}}. 
\]

Using these relations, we have that \( \lambda \Lambda_1''(-\lambda) + \Lambda_2''(-\lambda) \approx \frac{1}{2\pi} e^{\frac{4}{5}\lambda^{5/4}} \). Thus, in particular, we set \( \lambda = \lambda_n \) and obtain
\[
-\frac{1}{\sqrt{2}} \frac{\Lambda_4(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} \approx \sqrt{\frac{\pi}{2}} \lambda^{-1/8} \text{ for each } n = 1, 3, 5, \ldots.
\]

Similarly, we have
\[
\frac{1}{\sqrt{2}} \frac{\Lambda_4'(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} \approx -\sqrt{\frac{\pi}{2}} \lambda^{-1/8} \cos \left( \frac{4}{5} \lambda_n^{5/4} + \frac{\pi}{4} \right) e^{-\frac{4}{5}\lambda_n^{5/4}} \text{ for each } n = 1, 3, 5, \ldots.
\]
3.7. FURTHER RESULTS: THE SMALL EFFECT

We also have that

\[ \Lambda_2(-\lambda)\Lambda_3(-\lambda) \approx \frac{1}{4\pi^2} \left[ \sin^2 \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) - \cos^2 \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) \right] \lambda^{1/2} e^{\frac{1}{5} \lambda^{5/4}} \]

\[ = \frac{1}{4\pi^2} \sin \left( \frac{8}{5} \lambda^{5/4} \right) \lambda^{1/2} e^{\frac{1}{5} \lambda^{5/4}}. \]

In particular, we set \( \lambda = \lambda_n \) with \( \lambda_n = \left[ \frac{5}{16} (2n - 1) \pi \right]^{4/5} \) and observe that \( \sin \left( \frac{8}{5} \lambda_n^{5/4} \right) \equiv \cos(n\pi) = 1 \) for each \( n = 2, 4, 6, \cdots \). Hence, we obtain

\[ \Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n) \approx \frac{1}{4\pi^2} \lambda_n^{1/2} e^{\frac{1}{5} \lambda_n^{5/4}} \quad \text{for each } n = 2, 4, 6, \cdots. \]

Therefore, we have that

\[ \frac{1}{2} \frac{\tilde{\Lambda}_n''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} \approx \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} \quad \text{for each } n = 2, 4, 6, \cdots, \]

and

\[ \frac{1}{2} \frac{\tilde{\Lambda}_n''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n)\Lambda_3(-\lambda_n)}} \approx \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} \sin \left( \frac{4}{5} \lambda_n^{5/4} + \frac{\pi}{4} \right) e^{-\frac{1}{5} \lambda_n^{5/4}} \quad \text{for each } n = 2, 4, 6, \cdots. \]

The fact that \( \lambda_n \sim n^{4/5} \) establishes the results. \( \square \)

Let

\[ C_n := 1/\| \Lambda_4(| \cdot | - \lambda_n) \|_2 \quad (3.116) \]

and with the identity \( \left( z \Lambda_3'(z) + 2 \Lambda_4'(z) \Lambda_4''(z) - [\Lambda_4''(z)]^2 \right)' = \Lambda_3'(z) \), we have that

\[ \| \Lambda_4(| \cdot | - \lambda_n) \|_2 = \left( \int_{-\infty}^{\infty} \Lambda_4(|z| - \lambda_n) \, dz \right)^{1/2} = \left( \int_{-\infty}^{\infty} \Lambda_4(z) \, dz \right)^{1/2} \]

\[ = \sqrt{2} \lambda_n \Lambda_3'(-\lambda_n) + [\Lambda_4''(-\lambda_n)]^2 - 2 \Lambda_4'(-\lambda_n) \Lambda_4''(-\lambda_n) \right)^{1/2}. \quad (3.117) \]

Furthermore, we recall that \( \mu_n = -\alpha'_n \) for each \( n = 1, 3, 5, \cdots \) and \( \mu_n = -\alpha_n \) for each \( n = 2, 4, 6, \cdots \), where \( \alpha_n \) and \( \alpha'_n \) are the negative real zeroes of \( \Lambda_4 \) and \( \Lambda_4' \), respectively, and by assertion, \( \lambda_n \approx \mu_n \) so that from (3.116) together with (3.117), we have that

\[ C_n \approx \frac{1}{\sqrt{2}} \times \begin{cases} \lambda_n \frac{\Lambda_3'(-\lambda_n) + [\Lambda_4''(-\lambda_n)]^2}{}
\quad n = 2j - 1
\end{cases} \]

(3.118)

for each \( j \in \mathbb{N} \). We also recall from (3.49) that

\[ \Lambda_4(-\lambda) = \frac{1}{\sqrt{2\pi}} \lambda^{-3/8} \sin \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) \left[ 1 + O \left( \lambda^{-5/8} \right) \right] \quad (3.119) \]

for every \( \lambda > \lambda_0 \) given that \( 0 < \lambda_0 \leq 1/2 \). Since \( \Lambda_4 \) is analytic we can differentiate (3.119) to obtain expressions for \( \Lambda_4', \Lambda_4'', \) and \( \Lambda_4''' \) (refer to (3.113) - (3.115)). Thus, we have that

\[ \lambda \Lambda_4(-\lambda) + [\Lambda_4''(-\lambda)]^2 \approx \frac{1}{\pi} \lambda^{1/4} \sin^2 \left( \frac{4}{5} \lambda^{5/4} + \frac{\pi}{4} \right) \].
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We now set \( \lambda = \lambda_n \), and with \( \lambda_n \equiv [5(2n - 1)\pi/16]^{4/5} \), we found that

\[
\sin^2 \left( \frac{4}{5} \lambda_n^{5/4} + \frac{\pi}{4} \right) = \frac{1}{2} \left[ 1 + \sin \left( \frac{8}{5} \lambda_n^{5/4} \right) \right] \equiv 1 \quad \text{for each } n = 1, 3, 5, \ldots.
\]

Also, we have that

\[
[Ai_4''(-\lambda)]^2 - 2 Ai_4'(-\lambda) \ Ai_4'''(-\lambda) \approx \frac{1}{2\pi} \lambda^{1/4} \left[ 1 + \cos^2 \left( \frac{4}{5} \lambda_n^{5/4} + \frac{\pi}{4} \right) \right].
\]

We set \( \lambda = \lambda_n \) so that with \( \lambda_n \equiv [5(2n - 1)\pi/16]^{4/5} \), we also have that

\[
\cos^2 \left( \frac{4}{5} \lambda_n^{5/4} + \frac{\pi}{4} \right) = \frac{1}{2} \left[ 1 - \sin \left( \frac{8}{5} \lambda_n^{5/4} \right) \right] \equiv 1 \quad \text{for each } n = 2, 4, 6, \ldots.
\]

Hence, from (3.118) we immediately have that

\[
C_n \equiv \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} \quad \text{for each } n \in \mathbb{N}.
\]  

(3.120)

**Remark 3.2.** By comparing (3.110) with (3.120), we conclude that \( c_{1,n} \equiv C_n := 1/|| Ai_4(\cdot | - \lambda_n) ||_2 \).

We now prove the main result of this subsection. In particular, we show that, for each \( n \in \mathbb{N} \), the eigenvalues \( \lambda_n \) are exponentially close to the negative real zeroes of \( Ai_4(y) \) and \( Ai_4'(y) \) arranged in alternating and increasing order of magnitude given by \( \mu_n \) as defined in (3.100).

**Theorem 3.9.** For each \( n \geq 1 \), we have that

\[
|\lambda_n - \mu_n| \leq \frac{2^{5/4}}{5^{1/10}} \sqrt{\frac{\Gamma(4/5)}{\pi}} \gamma_n
\]

(3.121)

where \( \gamma_n := \sqrt{c_{1,n} c_{2,n}} \). Moreover, we have \( \gamma_n \approx \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} e^{-\frac{\pi}{4} \lambda_n^{3/4}} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} e^{-\frac{\pi}{4} n} \).

**Proof.** Our proof is inspired by the proof of Theorem 1 [53 pp. 2390 - 2393]. Since \( (\phi_l)_{l \in \mathbb{N}} \) forms a complete orthonormal set in \( L^2(\mathbb{R}) \), and \( Ai_4(|\cdot| - \lambda_n) \in L^2(\mathbb{R}) \), by the general theory on eigenfunctions expansion, we can write \( Ai_4(|\cdot| - \lambda_n) = \sum_{l \geq 1} b_l \phi_l(y) \) for each \( n \in \mathbb{N} \), where \( b_l := \langle Ai_4(|\cdot| - \lambda_n), \phi_l \rangle \), and the Parseval identity

\[
\sum_{l \geq 1} b_l^2 = \| Ai_4(|\cdot| - \lambda_n) \|_2^2
\]

holds. We then have that

\[
\| (\mathcal{L} - \mu_n) Ai_4(|\cdot| - \lambda_n) \|_2^2 = \sum_{l \geq 1} b_l^2 (\lambda_l - \mu_n)^2.
\]

(3.123)

Let \( \lambda_{q(n)} \) denote the eigenvalue closest to \( \mu_n \). It follows from (3.123) and application of (3.122) that

\[
\sum_{l \geq 1} b_l^2 (\lambda_l - \mu_n)^2 \geq (\lambda_{q(n)} - \mu_n)^2 \sum_{l \geq 1} b_l^2 = (\lambda_{q(n)} - \mu_n)^2 \| Ai_4(|\cdot| - \lambda_n) \|_2^2.
\]

(3.124)
Using (3.124) in (3.123), and together with (3.102), we obtain
\[ |\lambda_{q(n)} - \mu_n| \leq 2 \sqrt{\frac{\sqrt{2} \Gamma\left(\frac{3}{4}\right)}{\pi^{5/4} \sqrt{\|A_{2,n}\|}} \sqrt{\|\lambda_n\|_2}}. \]

But \( \|A_{14}(\cdot - \lambda_n)\|_2 \equiv \sqrt{2} \left( \sum_{r=0}^{2} (-1)^r \right) A_{14}^{(r)}(-\lambda_n) A_{14}^{(4-r)}(-\lambda_n) \) is \( \frac{1}{c_{1,n}} \) (by Remark 3.2). Therefore, we have
\[ |\lambda_{q(n)} - \mu_n| \leq \frac{2^{5/4}}{5^{1/4}} \frac{\Gamma(4/5)}{\pi} \gamma_n \tag{3.125} \]

with \( \gamma_n := \sqrt{c_{1,n} \|A_{2,n}\|} \), and using the results of Lemma 3.3, we have that \( \gamma_n \approx \sqrt{\pi} \lambda_n^{-1/8} e^{-3/4} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} e^{-\frac{3}{2} \pi}. \)

Take \( \delta = \frac{5\pi}{4} \left( \frac{\pi}{17} \right)^{5/4} \approx 0.475813 \) and let
\[ n_0 = \lceil 2 \ln (C/\delta) \rceil + 1 \tag{3.126} \]

where \( C = \frac{2^{5/4}}{5^{1/4}} \frac{\Gamma(4/5)}{\pi} \sup_{n \geq 1} \gamma_n n^{1/10} e^{2n/5} = 0.667134. \) We claim that
\[ \lambda_{q(n)} \in \left( \left( \mu_n^{5/4} - \delta \right)^{4/5}, \left( \mu_n^{5/4} + \delta \right)^{4/5} \right) \quad \text{for } n \geq n_0. \]

By (3.125) and (3.126), we have that
\[ |\lambda_{q(n)} - \mu_n| \leq C n^{-1/10} e^{-2n/5} \leq \delta \quad \text{for } n \geq n_0. \]

On the other hand, we have that
\[ \frac{(2n - 1)\pi}{4} + \max \left\{ 0, \frac{(n - 3)\pi}{8} \right\} < \mu_n^{5/4} - \delta \leq \mu_n^{5/4} + \delta \leq \frac{5n\pi}{8} \quad \text{for } n \geq n_0. \tag{3.127} \]

Using the inequality
\[ \sigma(-\sigma) \leq \xi - \eta \leq \sigma(\xi - \eta) \quad \text{for } 0 < \sigma < 1, \text{ and } \xi, \eta \geq 0, \]

and set
\[ A = \frac{\pi}{C} \left( \frac{\pi}{17} \right)^{5/4} \left( \frac{8}{5\pi} \right)^{1/5} \inf_{n \geq n_0} n^{-1/10} e^{2n/5} \geq 1, \]

we have that
\[ \left| \left( \mu_n^{5/4} \right)^{4/5} - \mu_n \right| \geq \frac{4 \delta}{5} \frac{1}{\left( \mu_n^{5/4} + \delta \right)^{1/5}} \geq \frac{4 \delta}{5} \left( \frac{8}{5\pi} \right)^{1/5} n^{-1/5} \]
\[ = \pi \left( \frac{\pi}{17} \right)^{5/4} \left( \frac{8}{5\pi} \right)^{1/5} n^{-1/5} \geq A \gamma_n n^{-1/10} e^{-2n/5} \]
\[ \geq A |\lambda_{q(n)} - \mu_n| \quad \text{for } n \geq n_0. \]

Hence, this establishes our claim that \( \lambda_{q(n)} \in \left( \left( \mu_n^{5/4} - \delta \right)^{4/5}, \left( \mu_n^{5/4} + \delta \right)^{4/5} \right) \) for \( n \geq n_0. \)
Next, we show that $\mu_{n+1}^{5/4} - \mu_n^{5/4} > 2\delta$ to conclude that the intervals $\left((\mu_n^{5/4} - \delta)^{4/5}, (\mu_n^{5/4} + \delta)^{4/5}\right)$ are mutually disjoint, and hence, $\lambda_{q(n)}$ for all $n \geq n_0$ are distinct. Thus, using the inequality
\[ \sigma(\xi - \eta)\xi^{\sigma - 1} \geq \xi^{\sigma} - \eta^{\sigma} \geq \sigma(\xi - \eta)\eta^{\sigma - 1} \quad \text{for } \sigma > 1, \text{ and } \xi > \eta \geq 0, \]
we have that
\[ \mu_{n+1}^{5/4} - \mu_n^{5/4} > 5/4(\mu_{n+1} - \mu_n)\mu_n^{1/4}. \quad (3.128) \]
We take $B = \inf_{n \geq n_0} (n + 1/2)^{-1/5} \mu_n^{1/4} \geq 1$ and recall from (3.60) that
\[ \mu_{n+1} - \mu_n \geq \frac{\pi}{2} \left( \frac{8}{15\pi} \right)^{1/5} \left( n + \frac{1}{2} \right)^{-1/5} \quad \text{for all } n \in \mathbb{N}. \quad (3.129) \]
Hence, inserting (3.129) into (3.128), we have that
\[ \mu_{n+1}^{5/4} - \mu_n^{5/4} > \frac{5}{8} \left( \frac{8}{15\pi} \right)^{1/5} \left( n + \frac{1}{2} \right)^{-1/5} \mu_n^{1/4} \geq \frac{5\pi}{8} \left( \frac{8}{15\pi} \right)^{1/5} B > 2\delta \quad \text{for all } n \geq n_0. \]
Furthermore, since $\lambda_{q(n)}$ is the closest eigenvalue to $\mu_n$ so that $\lambda_n < \lambda_{q(n)} \leq \mu_n$ for $n \geq n_0$, and given that $\lambda_{q(n)}^{5/4} < \mu_n^{5/4} + \delta$ for $n \geq n_0$, we conclude that $\lambda_n < \left(\mu_n^{5/4} + \delta\right)^{4/5}$ for $n \geq n_0$. Also, we have that
\[ \frac{5n\pi}{8} < \lambda_{n+1}^{5/4} \leq \frac{5(n+1)\pi}{8} \quad \text{for } n \in \mathbb{N}, \]
which together with (3.127) implies that $\left(\mu_n^{5/4} + \delta\right)^{4/5} < \lambda_{n+1}$ for $n \geq n_0$. Therefore, we have
\[ \lambda_n < \left(\mu_n^{5/4} + \delta\right)^{4/5} < \lambda_{n+1} \quad \text{for } n \geq n_0. \quad (3.130) \]
We further claim that there are at most $n_0 - 1$ eigenvalues not included in the above class. To establish this claim we proceed as follows. Recall from (3.64) that
\[ \sum_{n \geq 1} e^{-\lambda_{n} t} = \int_{0}^{\infty} e^{-\lambda t} \, dN(\lambda), \quad t > 0, \]
where $N(\lambda)$ is the spectral counting function associated with the non-local operator $H$. Let
\[ L := \{ l \in \mathbb{N} : l \neq q(n) \quad \text{for all } n \geq n_0 \}. \]
Note that $L$ is essentially the set of all mismatches $l \neq q(n)$ for $n \geq n_0$, and hence, its complement in $\mathbb{N}$ is exactly the set for which $q(n)$ matches $l \in \mathbb{N}$ for $n \geq n_0$. Using below $\lambda_{q(n)} < (\mu_n + \delta)^{4/5}$ and the simple fact that $e^{-\lambda t}$ is strictly monotone decreasing for every $\lambda > 0$ and any $t > 0$, we have that
\[ \sum_{l \in L} e^{-\lambda_{l} t} = \sum_{l \neq q(n)} e^{-\lambda_{l} t} - \sum_{n \geq n_0} e^{-\lambda_{q(n)} t} \]
\[ = \int_{0}^{\infty} e^{-\lambda t} \, dN(\lambda) - \sum_{n \geq n_0} e^{-\lambda_{q(n)} t} \]
\[ < \int_{0}^{\infty} e^{-\lambda t} \, dN(\lambda) - \sum_{n \geq n_0} e^{-\left(\mu_n^{5/4} + \delta\right)^{4/5} t}. \]

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But

\[ \int_{0}^{\infty} \frac{e^{-t}}{(\lambda_{n}^{5/4} + \delta)^{4/5}} \, d\lambda \leq \sum_{\mu_{n} \geq \mu_{n_{0}}} e^{-\left(\frac{5}{4}\lambda_{n} + \delta\right)^{4/5} t} \]

so that

\[ \sum_{l \in L} e^{-\lambda_{l} t} < \int_{0}^{\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5}} e^{-\lambda t} \, d\lambda. \]

Since \( N(\lambda) \) is monotone increasing on \((0, \left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5})\), we have

\[ \int_{0}^{\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5}} e^{-\lambda t} \, d\lambda \leq N \left(\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5}\right) \int_{0}^{\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5}} e^{-\lambda t} \, d\lambda \]

\[ = \frac{N \left(\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5}\right)}{t} \left(1 - e^{-\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5} t}\right) \quad (3.131) \]

Inserting \(3.132\) into \(3.131\), we obtain

\[ \sum_{l \in L} e^{-\lambda_{l} t} < N \left(\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5}\right). \]

But as \( t \searrow 0 \), the left-hand side converges to \(|L|\). It follows therefore that

\[ |L| < N \left(\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5}\right). \]

Using \(3.130\), it follows that \( N \left(\left(\frac{\mu_{n_{0}}}{4} + \delta\right)^{4/5}\right) = n \) for \( n \geq n_{0} \). Therefore, we have that \(|L| < n_{0}\), which establishes our claim.

Given that \( \lambda_{n} \leq (5n\pi/8)^{4/5} \) for all \( n \in \mathbb{N} \), then in particular, for \( l < n_{0} \), we have that \( \lambda_{l} \leq (5l\pi/8)^{4/5} \leq (5(n_{0} - 1)\pi/8)^{4/5} \). On the other hand, for \( n \geq n_{0} \), we have that \( \lambda_{q(n)} > \left(\frac{5}{4}\mu_{n_{0}} - \delta\right)^{4/5} > (5(n_{0} - 1)\pi/8)^{4/5} \). Thus, \( L \supseteq \{1, 2, 3, \ldots, n_{0} - 1\} \). However, we have that \(|L| \leq n_{0} - 1, \) which implies that \( L \subseteq \{1, 2, 3, \ldots, n_{0} - 1\} \). Hence, we must have that \( L = \{1, 2, 3, \ldots, n_{0} - 1\} \). We therefore conclude that \( q(n) = n \) for \( n \geq n_{0} \).

By direct calculation, we have that \( n_{0} = 2 \) implying that \( \lambda_{1} \) is the only eigenvalue excluded from the above eigenvalue set. But in particular, we have that \( \lambda_{1} \leq (5\pi/16)^{4/5} \) and given that \( \mu_{1} \geq (\pi/5)^{4/5} \), we obtain

\[ \lambda_{1} - \mu_{1} \leq \left(\frac{5\pi}{16}\right)^{4/5} - \left(\frac{\pi}{5}\right)^{4/5} < \frac{9}{100} \left(\frac{5}{\pi}\right)^{1/5} \approx 0.310282 < \frac{25/4}{5^{1/10}} \sqrt{\frac{\Gamma(4/5)}{\pi}} \gamma_{1} \approx 0.447194. \]

Therefore, we conclude that the result \((3.121)\) holds for all \( n \geq 1 \). □

In Table C.2 below, we compare numerical computations of the eigenvalues \( \lambda_{n} \) and approximations of these eigenvalues using the relation \((3.55)\) and also compare these numerical computations and \( \mu_{n} \).
3.7. FURTHER RESULTS: THE SMALL EFFECT

3.7.2. Small effect on the level of eigenfunctions

The next two results relate to the small effect arising from \( \widetilde{A_4}(y) \) at the level of eigenfunctions. In these results, we show that the eigenfunctions \( \psi_n(x) \) are well-approximated by the inverse Fourier transform of the dominating term \( A_4(|y| - \lambda_n) \) in (3.26) - (3.27) in its normalised form. First, we establish the result in \( L^2 \) sense, and then proceed with even stronger result in \( L^\infty \) sense.

Write \( \varphi_n(y) := C_n A_4(|y| - \lambda_n) \) where \( C_n := 1/\| A_4(\cdot | - \lambda_n) \|_2 \) and notice that \( \| \varphi_n \|_2 = 1 \) for all \( n \in \mathbb{N} \). We prove by Plancherel’s theorem the following result:

**Theorem 3.10.** For each \( n \in \mathbb{N} \), we have

\[
\| \psi_n - (\mathcal{F}^{-1} \varphi_n) \|_2 \leq \frac{2^{33/20} \Gamma^{1/10} \beta_n}{\pi^{13/10}}
\]

where \( \beta_n := n^{1/5} C_n \sqrt{\| c_{2,n} \|_{c_{1,n}}} \approx \pi n^{1/5} \lambda_n^{-1/8} e^{-\frac{3}{32} n} \approx \pi n^{1/10} e^{-\frac{3}{8} n} \).

**Proof.** Since \( \varphi_n \in L^2(\mathbb{R}) \), we have

\[
\varphi_n(y) = \sum_{l \geq 1} a_l \phi_l(y)
\]

with \( a_l := \langle \varphi_n, \phi_l \rangle \) and \( \sum_{l \geq 1} a_l^2 = \| \varphi_n \|_2^2 = 1 \). We now consider

\[
\| (L - \lambda_n) \varphi_n \|_2^2 = \sum_{l \geq 1} a_l^2 (\lambda_l - \lambda_n)^2 = \sum_{l \neq n} a_l^2 (\lambda_l - \lambda_n).\]

For \( l \neq n \), we have

\[
|\lambda_l - \lambda_n| \geq \lambda_{n+1} - \lambda_n \geq \frac{\pi}{2} \left( \frac{8}{15\pi} \right)^{1/5} n^{-1/5}.
\]

Therefore, from (3.134), we simply have that

\[
\sum_{l \neq n} a_l^2 \leq \frac{2}{\pi} \left( \frac{15\pi}{8} \right)^{2/5} n^{2/5} \| (L - \lambda_n) \varphi_n \|_2^2 - \sum_{l \neq n} a_l^2 \leq \frac{4}{\pi^2} \left( \frac{15\pi}{8} \right)^{2/5} n^{2/5} \| (L - \lambda_n) \varphi_n \|_2^2.
\]

Furthermore, from (3.133), we have \( \varphi_n(y) - a_n \phi_n(y) = \sum_{l \neq n} a_l \phi_l(y) \) so that using (3.135), we have

\[
\| \varphi_n - a_n \phi_n \|_2^2 = \sum_{l \neq n} a_l^2 \leq \frac{4}{\pi^2} \left( \frac{15\pi}{8} \right)^{2/5} C_{2,n} N^{2/5} \| (L - \lambda_n) A_4(| \cdot - \lambda_n) \|_2^2.
\]

By similar procedure adopted in the proof of (3.102), we can also easily show that

\[
\| (L - \lambda_n) A_4(| \cdot - \lambda_n) \|_2^2 \leq \frac{4 \sqrt{2} \Gamma \left( \frac{4}{3} \right) \sqrt{c_{2,n}}}{\pi^{1/5} c_{1,n}}.
\]

Hence, it follows that

\[
\| \varphi_n - a_n \phi_n \|_2 \leq \frac{2}{\pi} \left( \frac{15\pi}{8} \right)^{1/5} \sqrt{\frac{4 \sqrt{2} \Gamma \left( \frac{4}{3} \right)}{\pi^{1/5} c_{1,n}}} n^{1/5} C_n \sqrt{\| c_{2,n} \|_{c_{1,n}}}.
\]
Finally, we observe that \( \sum_{n \geq 1} a_n^2 = 1 \) implying that \( a_n \leq 1 \) for each \( n \in \mathbb{N} \). Therefore, we have that \( \| \varphi_n - \phi_n \|_2 \leq \| \varphi_n - a_n \phi_n \|_2 \), and hence, we complete the proof by recalling Plancherel’s theorem.  

Next, let \( \varphi_n(y) = c_{1,n} \, \text{Ai}_4(|y| - \lambda_n) \), we consider the difference between \( \psi_n(x) \) and the inverse Fourier transform of \( \varphi_n(y) \) in \( L^\infty \) sense. Thus, we prove the following statement:

**Theorem 3.11.** For each \( n \in \mathbb{N} \), we have

\[
\left\| \psi_n - \mathcal{F}^{-1} \varphi_n \right\|_\infty \leq \frac{2^{9/4}}{\sqrt{5\pi}} |c_{2,n}| \tag{3.137}
\]

where \( |c_{2,n}| \approx \sqrt{\frac{\pi}{2}} \lambda_n^{-1/8} e^{-\frac{1}{2} \lambda_n^{3/4}} \sim \sqrt{\frac{\pi}{2}} n^{-1/10} e^{-\frac{5}{2} n} \).

**Proof.** We recall that \( \phi_n(y) = c_{1,n} \, \text{Ai}_4(|y| - \lambda_n) + c_{2,n} \, \text{Ai}_4(|y| - \lambda_n) \) and with \( \varphi_n(y) := c_{1,n} \, \text{Ai}_4(|y| - \lambda_n) \), we have that

\[
\left| \psi_n(x) - \left( \mathcal{F}^{-1} \varphi_n \right)(x) \right| = \frac{1}{\sqrt{2\pi}} \left| \int_0^\infty e^{iy} \left[ \phi_n(y) - \varphi_n(y) \right] \, dy \right|
\]

\[
= \frac{|c_{2,n}|}{\sqrt{2\pi}} \left| \int_0^\infty e^{iy} \text{Ai}_4(|y| - \lambda_n) \, dy \right|
\]

\[
\leq |c_{2,n}| \sqrt{\frac{2}{\pi}} \int_0^\infty \left| \text{Ai}_4(y - \lambda_n) \right| \, dy. \tag{3.138}
\]

By similar procedure leading to (3.107), we also show that

\[
\left| \text{Ai}_4(y - \lambda_n) \right| \leq \frac{1}{\sqrt{2\pi}} |y - \lambda_n|^{-3/8} e^{-\frac{5}{4} (y - \lambda_n)^{5/4}}.
\]

Using this estimate in (3.138), we have that

\[
\int_0^\infty \left| \text{Ai}_4(y - \lambda_n) \right| \, dy \leq \frac{1}{\sqrt{2\pi}} \int_0^\infty |y - \lambda_n|^{-3/8} e^{-\frac{5}{4} (y - \lambda_n)^{5/4}} \, dy
\]

\[
= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \left[ \int_{\lambda_n-\epsilon}^{\lambda_n} e^{-\frac{2\sqrt{5}}{3} (\lambda_n-y)^{3/4}} \, dy + \int_{\lambda_n+\epsilon}^{\infty} e^{-\frac{2\sqrt{5}}{3} (y-\lambda_n)^{3/4}} \, dy \right]
\]

\[
\leq \frac{2}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \int_\epsilon^{\infty} z^{-3/8} e^{-\frac{5}{3} z^{5/4}} \, dz = \frac{2^{7/4}}{\sqrt{5}}. \tag{3.139}
\]

The result (3.137) then follows by the combination of (3.138) and (3.139).  

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Expressions and properties of $\tilde{A}_i(y)$, $G_3(y)$, and $G_4(y)$

A.1. Constructing expressions for $\tilde{A}_i(y)$, $G_3(y)$, and $G_4(y)$

Here we show how we have obtained the other independent solutions (i.e., $\tilde{A}_i(y)$, $G_3(y)$, and $G_4(y)$) given in (3.25). First, we derive expression for $\tilde{A}_i(y)$. To obtain this expression, we use the series expansion for $A_i(y)$ given in (2.32) (for the specific case $k = 4$) expressed in terms of the generalised hypergeometric functions. Thus, we have that

$$A_i(y) = \sum_{p=0}^{3} A_{i}^{(p)}(0) \frac{y^p}{p!} U_{4,p}(y),$$

(A-1)

where $A_{i}^{(p)}(0)$ is calculated directly from (2.21) (for $k = 4$), and

$$U_{4,0}(y) := \,_3F_3 \left( \begin{array}{c} 2, \ 3, \ 4 \\ 5, \ 5, \ 5 \end{array} \right| -\frac{y^5}{5^4} \right) ; \quad U_{4,1}(y) := \,_3F_3 \left( \begin{array}{c} 3, \ 4, \ 6 \\ 5, \ 5, \ 5 \end{array} \right| -\frac{y^5}{5^4} \right),$$

$$U_{4,2}(y) := \,_3F_3 \left( \begin{array}{c} 4, \ 6, \ 7 \\ 5, \ 5, \ 5 \end{array} \right| -\frac{y^5}{5^4} \right) ; \quad U_{4,3}(y) := \,_3F_3 \left( \begin{array}{c} 6, \ 7, \ 8 \\ 5, \ 5, \ 5 \end{array} \right| -\frac{y^5}{5^4} \right).$$

Furthermore, we have that

$$A_{i4}(y e^{2\pi i/5}) = \sum_{p=0}^{3} C_p e^{2\pi p i/5} y^p U_{4,p}(y); \quad A_{i4}(y e^{-2\pi i/5}) = \sum_{p=0}^{3} C_p e^{-2\pi p i/5} y^p U_{4,p}(y);$$

(A-2)

$$A_{i4}(y e^{4\pi i/5}) = \sum_{p=0}^{3} C_p e^{4\pi p i/5} y^p U_{4,p}(y); \quad A_{i4}(y e^{-4\pi i/5}) = \sum_{p=0}^{3} C_p e^{-4\pi p i/5} y^p U_{4,p}(y),$$

(A-3)

where $C_p := A_{i4}^{(p)}(0)/p!$. Let

$$A_1(y) := e^{-2\pi i/5} A_{i4}(y e^{-2\pi i/5}); \quad A_2(y) := e^{-4\pi i/5} A_{i4}(y e^{-4\pi i/5});$$

$$A_3(y) := e^{4\pi i/5} A_{i4}(y e^{4\pi i/5}); \quad A_4(y) := e^{2\pi i/5} A_{i4}(y e^{2\pi i/5}).$$

(A-4)

We observe that the sum

$$A_4(y) + A_1(y) := e^{2\pi i/5} A_{i4}(y e^{2\pi i/5}) + e^{-2\pi i/5} A_{i4}(y e^{-2\pi i/5})$$

and the difference

$$A_4(y) - A_1(y) := e^{2\pi i/5} A_{i4}(y e^{2\pi i/5}) - e^{-2\pi i/5} A_{i4}(y e^{-2\pi i/5}),$$
are purely real and purely imaginary, respectively, implying that \( A_1(y) \) and \( A_4(y) \) form a conjugate pair (similarly, \( A_2(y) \) and \( A_3(y) \) form another conjugate pair). Hence, for some real constants \( a_1, b_1 \), we consider

\[
(a_1 + i b_1) A_4(y) + (a_1 - i b_1) A_1(y) = 2 \sum_{p=0}^{3} \left[ a_1 \cos \left( \frac{2\pi(p+1)}{5} \right) - b_1 \sin \left( \frac{2\pi(p+1)}{5} \right) \right] C_p y^p U_{4, p}(y); \tag{A-6}
\]

where the expression in (A-6) has been obtained using (A-4) - (A-5) together with (A-3). Similarly, using (A-4) - (A-5) together with (A-2), we obtain

\[
(a_2 + i b_2) A_3(y) + (a_2 - i b_2) A_2(y) = 2 \sum_{p=0}^{3} \left[ a_2 \cos \left( \frac{4\pi(p+1)}{5} \right) - b_2 \sin \left( \frac{4\pi(p+1)}{5} \right) \right] C_p y^p U_{4, p}(y) \tag{A-7}
\]

for some real constants \( a_2, b_2 \). We now write the combination of (A-6) and (A-7) as follows:

\[
(a_1 + i b_1) A_4(y) + (a_1 - i b_1) A_1(y) + (a_2 + i b_2) A_3(y) + (a_2 - i b_2) A_2(y)
= 2 \sum_{p=0}^{3} \left[ a_1 \cos \left( \frac{2\pi(p+1)}{5} \right) - b_1 \sin \left( \frac{2\pi(p+1)}{5} \right) \right] C_p y^p U_{4, p}(y)
+ a_2 \cos \left( \frac{4\pi(p+1)}{5} \right) - b_2 \sin \left( \frac{4\pi(p+1)}{5} \right) \right] C_p y^p U_{4, p}(y). \tag{A-8}
\]

Furthermore, we observe that by setting \( a_1 = a_2 = -1 \) and \( b_1 = b_2 = 0 \) in (A-8), we obtain the expression for \( A_{i4}(y) \) given in (3.22) and/or (A-1). Instead, we set \( a_1 = a_2 = 0 \) and \( b_1 = -b_2 = \beta \neq 0 \) to obtain the required expression. Note that the value of \( \beta \) is not of any particular importance; hence, without loss of generality, one can simply set \( \beta = 1 \). Thus, the function we seek is given by the sum

\[
i[A_3(y) - A_2(y)] - i[A_4(y) - A_1(y)] = 2 \sum_{p=0}^{3} \sin \left( \frac{2\pi(p+1)}{5} \right) - \sin \left( \frac{4\pi(p+1)}{5} \right) \right] C_p y^p U_{4, p}(y).
\]

Let us denote this function by \( \tilde{A}_{i4}(y) \). Therefore, we have that

\[
\tilde{A}_{i4}(y) := i \sum_{r=1}^{4} (-1)^{(p+1)} A_r(y), \tag{A-9}
\]

which is the second independent solution to (3.19), and its expression in terms of the hypergeometric functions \( U_{4, p}(y) \) is given by

\[
\tilde{A}_{i4}(y) = \sum_{p=0}^{3} \tilde{C}_p y^p U_{4, p}(y), \tag{A-10}
\]

where

\[
\tilde{C}_p := 2 \left[ \sin \left( \frac{2\pi(p+1)}{5} \right) - \sin \left( \frac{4\pi(p+1)}{5} \right) \right] \frac{A_{i4}^{(p)}(0)}{p!}.
\]
A.2. AN INTEGRAL REPRESENTATION OF $\tilde{A}_i(y)$

Next, we obtain expressions for $G_3(y)$ and $G_4(y)$. These expressions are obtained with the help of (3.22) and (A-9). First, we observe from (3.22) and (A-9) that

$$A_i(y) = (A_1(y) A_2(y) A_3(y) A_4(y)) \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad \tilde{A}_i(y) = (A_1(y) A_2(y) A_3(y) A_4(y)) \begin{pmatrix} i \\ -i \\ i \\ -i \end{pmatrix},$$

respectively. Moreover,

$$\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} i \\ -i \\ i \\ -i \end{pmatrix} \quad \text{(A-11)}$$

are orthogonal vectors in $\mathbb{C}^4$. Therefore, for a complete set of independent solutions to (3.19), we construct an orthogonal basis for $\mathbb{C}^4$ using (A-11) and bearing in mind that the solution $\chi(y)$ of (3.19) is a real-valued function since the equation (3.19) is defined for $y \in \mathbb{R}$ (with $y > -\lambda$). Thus, we obtain

$$\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} i \\ -i \\ i \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ i \\ -i \\ i \end{bmatrix} \quad \text{(A-12)}$$

Therefore, the general solution to (3.19) is given by

$$\chi(y) = (c_1 c_2 c_3 c_4) \begin{pmatrix} -1 & -1 & -1 & -1 \\ i & -i & i & -i \\ 1 & -1 & -1 & 1 \\ -i & -i & i & i \end{pmatrix} \begin{pmatrix} A_1(y) \\ A_2(y) \\ A_3(y) \\ A_4(y) \end{pmatrix} = c_1 A_i(y) + c_2 \tilde{A}_i(y) + c_3 G_3(y) + c_4 G_4(y) \quad \text{(A-13)}$$

with

$$G_3(y) = A_1(y) + A_4(y) - (A_2(y) + A_3(y))$$

and

$$G_4(y) = i[A_4(y) - A_1(y)] + i[A_3(y) - A_2(y)].$$

We observe from graphical plots that $G_3(y)$ and $G_4(y)$ are divergent as $y \to +\infty$.

A.2. An integral representation of $\tilde{A}_i(y)$

Here we derive an integral representation for $\tilde{A}_i(y)$. To proceed, we observe that it follows directly from (A-9) that

$$\tilde{A}_i(y) = -2 \Re e \left( i \left[ A_2(y) + A_4(y) \right] \right), \quad \text{(A-14)}$$

with

$$A_2(y) = e^{-\frac{4\pi i}{5}} A_i \left( e^{-\frac{4\pi i}{5}} y \right) = \frac{e^{-4\pi i/5}}{2\pi i} \int_{\infty e^{-4\pi i/5}}^{\infty e^{4\pi i/5}} e^{-e^{-4\pi i/5} y t - t^5/5} dt \quad \text{(A-15)}$$
and

\[ A_4(y) = e^{2\pi i/5} \widetilde{A}_4 \left( e^{2\pi i/5} y \right) = \frac{e^{-4\pi i/5}}{2\pi i} \int_{0}^{\infty} e^{-e^{4\pi i/5} y t - t^5/5} \, dt. \]  

(A-16)

We make the change of variables \( u = t e^{-4\pi i/5} \) in (A-15) to obtain

\[ A_2(y) = \frac{1}{2\pi i} \int_{0}^{\infty} e^{-yu - u^5/5} \, du. \]  

(A-17)

We also make the change of variables \( u = t e^{2\pi i/5} \) in (A-16) to obtain

\[ A_4(y) = \frac{1}{2\pi i} \int_{0}^{\infty} e^{-yu - u^5/5} \, du. \]  

(A-18)

The paths of integration in (A-17) and (A-18) are shown in Figure A.1 below. These are indicated by \( C_2 \) and \( C_4 \), respectively.

![Figure A.1. The complex u-plane](image-url)

With the help of Cauchy’s theorem, we can replace \( C_2 \) by the two contours indicated by \( C_2' \) and \( C_2'' \) in Figure A.1 above. Similarly, we replace \( C_4 \) by the contours \( C_4' \) and \( C_4'' \). Thus, we have

\[ A_2(y) = \frac{1}{2\pi i} \left[ \int_{0}^{\infty} - \int_{0}^{\infty} \right] e^{-yu - u^5/5} \, du \]  

(A-19)
and
\[
A_4(y) = \frac{1}{2\pi i} \left[ \int_0^{\omega e^{-4\pi i/5}} - \int_0^{\omega e^{-2\pi i/5}} \right] e^{-yu^5/5} du, \tag{A-20}
\]
respectively. Furthermore, we combine (A-19) and (A-20) to obtain
\[
A_2(y) + A_4(y) = \frac{1}{2\pi i} \left[ \int_0^\infty + \int_0^{\omega e^{-4\pi i/5}} - \int_0^{\omega e^{-2\pi i/5}} \right] e^{-yu^5/5} du
\]
\[
= \frac{1}{2\pi i} \left[ \int_0^\infty + \int_0^{\omega e^{-4\pi i/5}} - 2 \int_0^{\omega e^{-2\pi i/5}} \right] e^{-yu^5/5} du. \tag{A-21}
\]
For \(y \in \mathbb{R}\), we can deform contours \(G_2 : 0 \to \omega e^{-4\pi i/5}, G_3 : \omega e^{-2\pi i/5} \to \omega e^{2\pi i/5}, \) and \(G_4 : 0 \to \omega e^{-2\pi i/5}\) continuously into the negative imaginary semi-axis, the full imaginary axis, and the positive real semi-axis, respectively. Thus, we obtain
\[
A_2(y) + A_4(y) = \frac{1}{2\pi i} \left[ - \int_0^\infty + \int_{-i\infty}^{i\infty} - \int_{-i\infty}^{i\infty} \right] e^{-yu^5/5} du
\]
\[
= -\frac{1}{2\pi i} \int_0^\infty e^{-yu^5/5} du - \frac{1}{2\pi} \int_0^\infty e^{i\left(\frac{u^5}{5} + yu\right)} du - \frac{1}{\pi} \int_0^\infty \cos \left(\frac{u^5}{5} + yu\right) du. \tag{A-22}
\]
Therefore, we multiply (A-22) by \(-2i\) and equate real parts using (A-14) to obtain the desired result. Hence, we have that
\[
\tilde{A}_4(y) = \frac{1}{\pi} \int_0^\infty e^{-y\left(\frac{t^5}{5} - \sin \left(\frac{t^5}{5} + y\right)\right)} dt, \quad y \in \mathbb{R}. \tag{A-23}
\]

![Figure A.2. Components of \(\tilde{A}_4(y)\)](image1)

![Figure A.3. Plots of \(A_4(y)\) (in red) and \(\tilde{A}_4(y)\) (in blue)](image2)

In Figure A.2 above, we show plots of \(\tilde{A}_4(y)\) and the two contributing integrals appearing in (A-23). The plots show that \(\tilde{A}_4(y)\) is mostly dominated by the integral \(\frac{1}{\pi} \int_0^\infty e^{-y\left(\frac{t^5}{5} - \sin \left(\frac{t^5}{5} + y\right)\right)} dt\). In fact, we
observe that the second integral containing \( \sin \left( \frac{y}{2} + yt \right) \) is strictly positive, convex, and has an exponential decay for \( y > 0 \) but oscillates for \( y < 0 \). However, the contribution of this integral to \( \widetilde{\text{Ai}}_4(y) \) on the negative real semi-axis is insignificantly small due to the exponential growth exhibited by the first integral containing \( e^{-y\tau - \frac{\pi}{2}} \). In a nutshell, whenever its is convenient, one can write

\[
\widetilde{\text{Ai}}_4(y) \approx \frac{1}{\pi} \int_0^\infty e^{-y\tau - \frac{\pi}{2}} \, d\tau, \quad y \in \mathbb{R}
\]

We have also plotted \( \text{Ai}_4(y) \) and \( \widetilde{\text{Ai}}_4(y) \) together; this is displayed in Figure A.3 above.

### A.3. Asymptotic Formulae for \( \text{Ai}_4(y) \) and \( \widetilde{\text{Ai}}_4(y) \) for \( y < 0 \)

Here we obtain asymptotic relations for \( \text{Ai}_4(y) \) and \( \widetilde{\text{Ai}}_4(y) \) when \( y < 0 \), which appeared in the proof of Corollary 3.2. We consider

\[
\text{Ai}_4(-y) = \frac{1}{2\pi i} \int_G e^{uy-u^5/5} \, du, \quad y > 0,
\]

where \( G \) is some contour in the complex \( u \)-plane. By setting \( u = y^{1/5}v \) and \( z = y^{2/5} \), we have

\[
I(z) = \frac{1}{2\pi} \int_G e^{-zh(v)} \, dv, \quad z > 0, \quad h(v) = \frac{v^5}{5} - v.
\]

We employ the method of steepest descent on \( I(z) \). Our aim is to replace the contour \( G \) by suitable paths of steepest descent associated with saddle points of \( h(v) \). By differentiation, \( h'(v) = 0 \) gives the following saddle points: \( v_1 = 1, v_2 = i, v_3 = -1, \) and \( v_4 = -i \). From [7, Table 1], the directions of steepest descent \( \theta_{q,p} \) associated with the saddle points \( v_q \) are given by

\[
\theta_{q,p} = -\frac{\theta_q}{2} + (2p + 1)\frac{\pi}{2}, \quad \text{for each } q = 1, 2, 3, 4 \text{ and } p = 0, 1,
\]

where \( \theta_q = \arg(h''(v_q)) \); in particular, we have \( \arg(h''(1)) = (\pi, 0) \) and \( \arg(h''(\pm i)) = (\pi/2, 3\pi/2) \). Set

\[
v - v_q = |v - v_q| e^{i\theta_q o_p}.
\]

Then, by change of variables \( \tau = h(v) - h(v_q) \), we consider

\[
\int_{v_q}^\infty e^{-zh(v)} \, dv = e^{-zh(v_q)} \int_{v_q}^\infty e^{-z(h(v) - h(v_q))} \frac{dv}{d\tau} \, d\tau
\]

\[
= e^{-zh(v_q)} \int_0^\infty e^{-\tau \tau} G(\tau) \, d\tau
\]

where \( G(\tau) = dv/d\tau \). Furthermore, since \( h'(v_q) = 0 \), we have that

\[
\tau = |h(v) - h(v_q)| = \frac{1}{2} |h''(v)| (v - v_q)^2 \left( 1 + O(|v - v_q|) \right).
\]
We have introduced the absolute value for convenience. By reversion and the use of (A-27), we have
\[ v - v_q = \tau^{1/2} \left( \frac{2}{|h''(v_q)|} \right)^{1/2} e^{i\theta_{q,p}} \left( 1 + O(\tau^{1/2}) \right). \]
We can differentiate this expression to obtain
\[ G(\tau) = \frac{1}{2} \tau^{-1/2} \left( \frac{2}{|h''(v_q)|} \right)^{1/2} e^{i\theta_{q,p}} \left( 1 + O(\tau^{1/2}) \right). \]
Inserting this expression into (A-28), we obtain
\[
\int_0^\infty e^{-z\tau} G(\tau) \, d\tau = \frac{1}{2} \left( \frac{2}{|h''(v_q)|} \right)^{1/2} e^{i\theta_{q,p}} \int_0^\infty e^{-z\tau} \tau^{-1/2} \left( 1 + O(\tau^{1/2}) \right) \, d\tau. \tag{A-29}
\]
By change of variables \( w = z\tau \), we have
\[
\int_0^\infty e^{-z\tau} \tau^{-1/2} \, d\tau = \sqrt{\frac{\pi}{z}},
\]
and
\[
\int_0^\infty e^{-z\tau} \tau^{-1/2} O(\tau^{1/2}) \, d\tau = O \left( \int_0^\infty e^{-z\tau} \, d\tau \right) = O \left( \frac{1}{z} \right)
\]
so that by combining (A-28) and (A-29), we obtain
\[
\int_{v_q}^\infty e^{-yh(v)} \, dv = \frac{1}{2i} \left( \frac{1}{|h''(v_q)|} \right)^{1/2} y^{-3/8} e^{-y^{5/4}h(v_q)+i\theta_{q,p}} \left( 1 + O \left( \frac{1}{y^{5/8}} \right) \right) \tag{A-30}
\]
Let \( D_{q,p} \) denote the paths of steepest descent with directions \( \theta_{q,p} \) associated with the saddle points \( v_q \). In (A-25), we replace \( G \) with \( D_{q,p} \), substitute \( z = y^{5/4} \), and write
\[
I_{q,p}(y) = \frac{y^{1/4}}{2\pi i} \int_{D_{q,p}} e^{-y^{5/4}h(v)} \, dv = \frac{y^{1/4}}{2\pi i} \int_{v_q}^\infty e^{-y^{5/4}h(v)} \, dv
\]
\[
= \frac{1}{2i} \left( \frac{1}{|h''(v_q)|} \right)^{1/2} y^{-3/8} e^{-y^{5/4}h(v_q)+i\theta_{q,p}} \left( 1 + O \left( \frac{1}{y^{5/8}} \right) \right)
\]
for each \( q = 1, 2, 3 \) and \( p = 0, 1 \).

Now, we recall from (3.22) and (A-14) that
\[
\text{Ai}_q(-y) = -\sum_{r=1}^4 A_r(-y) = -2\Re e \left[ A_2(-y) + A_4(-y) \right] \tag{A-31}
\]
and
\[
\widetilde{\text{Ai}}_q(-y) = -2\Re e \left( i \left[ A_2(-y) + A_4(-y) \right] \right) \tag{A-32}
\] where
\[
A_2(-y) = \frac{1}{2\pi i} \int_{-\infty}^\infty e^{yu-u^2/5} \, du, \quad A_4(-y) = \frac{1}{2\pi i} \int_{-\infty}^\infty e^{yu-4u^2/5} \, du.
\]
In terms of \( I_{q,p}(y) \) we obtain
\[
A_2(-y) = I_{1,1}(y) - I_{1,0}(y), \quad A_4(-y) = I_{4,0}(y) - I_{4,1}(y) + I_{3,1}(y).
\]
A.3. ASYMPTOTIC FORMULAE FOR $A_i(Y)$ AND $\tilde{A}_i(Y)$ FOR $Y < 0$

Hence, using (A-30) with $\theta_{q,p}$ given in (A-26), $h(v) = v^5/5 - v$, and $h'' = 4v^3$, we obtain

$$A_2(-y) = -\frac{1}{2i\sqrt{2\pi}} e^{\frac{4}{5}y^{5/4}} \left(1 + O\left(\frac{1}{y^{5/8}}\right)\right)$$

(A-33)

and

$$A_4(-y) = \frac{1}{2i\sqrt{2\pi}} y^{-3/8} \left[\cos\left(\frac{4}{5}y^{5/4} + \frac{\pi}{4}\right) - i \sin\left(\frac{4}{5}y^{5/4} + \frac{\pi}{4}\right)\right] \left(1 + O\left(\frac{1}{y^{5/8}}\right)\right)$$

(A-34)

By inserting (A-33) and (A-34) into (A-31), we simply have that

$$A_i(\tilde{A}_4(-y)) = \frac{1}{\sqrt{2\pi}} y^{-3/8} \sin\left(\frac{4}{5}y^{5/4} + \frac{\pi}{4}\right) \left(1 + O\left(\frac{1}{y^{5/8}}\right)\right).$$

(A-35)

Similarly, we insert (A-33) and (A-34) into (A-32), and simplify further to obtain

$$\tilde{A}_i(\tilde{A}_4(-y)) = \frac{1}{\sqrt{2\pi}} y^{-3/8} e^{\frac{4}{5}y^{5/4}} \left(1 + O\left(\frac{1}{y^{5/8}}\right)\right).$$

(A-36)

Finally, given $0 < y_0 \leq 1/2$, we observe that the asymptotic relations (A-35) and (A-36) particularly hold for every $y > y_0$. 

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APPENDIX B

Bell Polynomials

Here we recall some basic facts from combinatorics — we especially discuss briefly the Bell polynomials. Details of this appendix section have been used to derive explicit expressions for the various coefficients of the asymptotic expansions obtained for $A_i$ and $A'_i$, and also in computing the coefficients of the asymptotic expansions of their zeroes.

Consider the formal (power) double series expansion:

$$
\Psi(\zeta, t) := \exp \left( \zeta \sum_{m \geq 1} \frac{x^m}{m!} \right) = \sum_{r \geq 0} \frac{\zeta^r}{r!} \left( \sum_{m \geq 1} \frac{x^m}{m!} \right)^r = \sum_{r \geq 0} \frac{\zeta^r}{r!} \left\{ \sum_{\tau \geq 0} \frac{r!}{\prod_{j=1}^{\tau_1+\cdots+\tau_r} j!} \left( \prod_{j=1}^{\tau_1+\cdots+\tau_r} \tau_j \right)^r \right\}
$$

where $\tau_j \in \mathbb{Z}$ for all $j \in \mathbb{N}$. Therefore, we define as follows:

**Definition B.1** ((Exponential) Incomplete/partial and Complete Bell Polynomials). The (exponential) incomplete/partial Bell polynomials are the polynomials $B_{n,r} \equiv B_{n,r}(x_1, x_2, \ldots, x_{n-r+1})$ in an infinite number of variables $x_1, x_2, \cdots$ obtained as the coefficients of $\frac{\zeta^r}{n!}$ in the formal series expansion of $\Psi(\zeta, t)$, and are defined as:

$$
B_{n,r}(x_1, x_2, \cdots, x_{n-r+1}) = \sum_{\tau_1 + 2\tau_2 + \cdots + r\tau_r = n} \frac{n!}{\prod_{j=1}^{\tau_1+\cdots+\tau_r} j!} \left( \prod_{j=1}^{\tau_1+\cdots+\tau_r} \frac{x_j}{j!} \right)^{\tau_j}, \quad (B-2)
$$

where the sum is taken over all different non-negative integer solutions $(\tau_1, \tau_2, \cdots)$ of the equation $\tau_1 + 2\tau_2 + 3\tau_3 + \cdots = n$ subject to the following constraint: $\tau_1 + \tau_2 + \tau_3 + \cdots = r$. The (exponential) complete Bell polynomial is given by:

$$
Y_n := \sum_{1 \leq r < n} B_{n,r} \quad \text{with} \quad Y_0 := 1;
$$

it is obtained by simply setting $\zeta = 1$ in the formal series expansion of $\Psi(\zeta, t)$. 


Remark B.1. The partial Bell polynomials are homogeneous of degree \( r \) and of weight \( n \); they have integral coefficients, which follows from the fact that the expression \( \frac{n!}{\prod_{j=1}^{r} \tau_j! (j^!)} \) appearing in the notation of Definition B.1 represents the number of partitions of the set \( \{1, 2, \cdots, n\} \) into \( \tau_1 \) 1-part, \( \tau_2 \) 2-parts, \( \cdots \), as \( \tau_1 + 2\tau_2 + \cdots = n \), or into \( r \) blocks since \( \tau_1 + \tau_2 + \cdots = r \) (when every ‘empty part’ associated with any \( \tau_j = 0 \) is omitted and the numbering of equal parts have also been removed).

Lemma B.1. The following are some particular values of the partial Bell polynomials \( \mathcal{B}_{n,r} \):

1. \( \mathcal{B}_{n,r}(1, 1, 1, \cdots) = S(n, r) \) (Stirling number of the second kind);
2. \( \mathcal{B}_{n,r}(0!, 1!, 2!, \cdots) = |s(n, r)| = s(n, r) \) (signless Stirling number of the first kind);
3. \( \mathcal{B}_{n,r}(1, 2, 3, \cdots) = \binom{n}{r} r^{n-r} \);
4. \( \mathcal{B}_{n,r}(1!, 2!, 3!, \cdots) = \binom{n-1}{r-1} \frac{n!}{r!} \).

For the sake of simplicity, we chose to work with the ordinary partial Bell polynomials, which is obtained by considering the algebraic expansion of the formal (power) double series expansion

\[
\Phi(\zeta, t) = e^\zeta \sum_{m=1}^{\infty} x_m t^m = \sum_{r \geq 0} \frac{\zeta^r}{r!} \left( \sum_{m \geq 1} x_m t^m \right)^r.
\]

Furthermore, we have that

\[
\frac{1}{r!} \left( \sum_{m \geq 1} x_m t^m \right)^r = \sum_{n \geq r} \mathcal{B}_{n,r}(x_1, x_2, \cdots x_{n-r+1}) t^r \quad \text{for each } r \geq 0.
\]

Hence, we have the following:

Definition B.2 ((Ordinary) Incomplete/Partial Bell Polynomials). The (ordinary) partial Bell polynomials are the polynomials \( \mathcal{B}_{n,r} \equiv \mathcal{B}_{n,r}(x_1, x_2, \cdots x_{n-r+1}) \) in an infinite number of variables \( x_1, x_2, \cdots \) obtained as the coefficients of \( \zeta^r t^r \) in the formal series expansion of \( \Phi(\zeta, t) \), and are defined as:

\[
\mathcal{B}_{n,r}(x_1, x_2, \cdots x_{n-r+1}) = \sum_{j=1}^{n-r+1} \left( \prod_{j=1}^{r} \frac{1}{\tau_j!} x_j^{\tau_j} \right).
\]

where the sum is taken over all different non-negative integer solutions \( (\tau_1, \tau_2, \cdots) \) of the equation, \( \tau_1 + 2\tau_2 + 3\tau_3 + \cdots = n \) subject to the constraint \( \tau_1 + \tau_2 + \tau_3 + \cdots = r \).

The two partial Bell polynomials are related by

\[
\mathcal{B}_{n,r}(y_1, y_2, \cdots) = \frac{1}{n!} \mathcal{B}_{n,r}(1! y_1, 2! y_2, \cdots).
\]

With this relationship, it is easy to deduce results for the ordinary partial Bell polynomial from the results of the exponential Bell polynomials. In Wojdylo [83, pp. 260-261], a set of upper and lower
bounds were obtained for the ordinary partial Bell polynomials in terms of the Stirling number of the second kind, $S(n, r)$. 

APPENDIX C

Numerical Outputs

Table C.1. Numerical computations of eigenvalues $\lambda_n$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\lambda_{2j-1}$</th>
<th>$\Phi_2(-\lambda_{2j-1})$</th>
<th>$\lambda_{2j}$</th>
<th>$\Phi_1(-\lambda_{2j})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.97842695</td>
<td>$4.71 \times 10^{-4}$</td>
<td>2.35819511</td>
<td>$4.29 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>3.55762902</td>
<td>$3.72 \times 10^{-4}$</td>
<td>4.67603019</td>
<td>$7.69 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>5.70790501</td>
<td>$-5.43 \times 10^{-5}$</td>
<td>6.71126508</td>
<td>$-4.28 \times 10^{-6}$</td>
</tr>
<tr>
<td>4</td>
<td>7.66488539</td>
<td>$1.06 \times 10^{-5}$</td>
<td>8.60054100</td>
<td>$-4.71 \times 10^{-7}$</td>
</tr>
<tr>
<td>5</td>
<td>9.50196086</td>
<td>$1.62 \times 10^{-6}$</td>
<td>10.39051635</td>
<td>$-2.91 \times 10^{-8}$</td>
</tr>
<tr>
<td>6</td>
<td>11.25329529</td>
<td>$2.57 \times 10^{-7}$</td>
<td>12.10615459</td>
<td>$8.98 \times 10^{-10}$</td>
</tr>
<tr>
<td>7</td>
<td>12.93851429</td>
<td>$1.15 \times 10^{-7}$</td>
<td>13.76285910</td>
<td>$3.48 \times 10^{-9}$</td>
</tr>
<tr>
<td>8</td>
<td>14.57028767</td>
<td>$9.86 \times 10^{-8}$</td>
<td>15.37102717</td>
<td>$3.80 \times 10^{-9}$</td>
</tr>
<tr>
<td>9</td>
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<td>$9.62 \times 10^{-8}$</td>
<td>16.93811720</td>
<td>$3.82 \times 10^{-9}$</td>
</tr>
<tr>
<td>10</td>
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<td>$9.54 \times 10^{-8}$</td>
<td>18.46972152</td>
<td>$3.80 \times 10^{-9}$</td>
</tr>
<tr>
<td>11</td>
<td>19.22214645</td>
<td>$9.47 \times 10^{-8}$</td>
<td>19.97017707</td>
<td>$3.78 \times 10^{-9}$</td>
</tr>
<tr>
<td>12</td>
<td>20.70852054</td>
<td>$9.41 \times 10^{-8}$</td>
<td>21.44293784</td>
<td>$3.75 \times 10^{-9}$</td>
</tr>
<tr>
<td>13</td>
<td>22.16864714</td>
<td>$9.35 \times 10^{-8}$</td>
<td>22.89081441</td>
<td>$3.73 \times 10^{-9}$</td>
</tr>
<tr>
<td>14</td>
<td>23.60508543</td>
<td>$9.28 \times 10^{-8}$</td>
<td>24.31613481</td>
<td>$3.70 \times 10^{-9}$</td>
</tr>
<tr>
<td>15</td>
<td>25.01997094</td>
<td>$9.22 \times 10^{-8}$</td>
<td>25.72085641</td>
<td>$3.68 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

In Table C.1 above, numerical estimates for the first thirty eigenvalues are computed. These estimates have been obtained by computing the negative real zeroes of $\Phi_1(y)$ and $\Phi_2(y)$ as defined by (3.29), using the series expansions for $Ai_4(y)$ and $\tilde{Ai}_4(y)$ given in (A-1). Accuracy checks have also been carried out by plugging back these estimates into $\Phi_1$ and $\Phi_2$ using the integral representations of $Ai_4(y)$ and $\tilde{Ai}_4(y)$ this time. Note that we have restricted the upper limit of these integral representations to 5; this is to allow for efficient computation but also due to the capacity of the machine, which is very limited in our case.

Next, in Table C.2 below, we compare numerical computations of the eigenvalues $\lambda_n$ and approximations $\lambda_n^{\text{approx}}$ of these eigenvalues using the relation (3.55):

$$\lambda_n^{\text{approx}} = \left[ \frac{5(2n - 1)\pi}{16} \right]^{4/5}.$$
We also used Mathematica 9 to compute numerically the negative real zeroes of $\text{Ai}_d(y)$ and $\text{Ai}'_d(y)$ using the integral representation (3.20) directly. These zeroes denoted by $\mu_n$ according to (3.100) have also been compared with $\lambda_n$ for each $n \in \mathbb{N}$ (refer to columns 4 and 6).

| $n$ | $\lambda_n$ | $\lambda_n^{\text{approx}}$ | $\mu_n$ | $|\lambda_n - \lambda_n^{\text{approx}}|$ | $|\lambda_n - \mu_n|$ |
|-----|-------------|----------------------------|--------|---------------------------------|----------------|
| 1   | 0.97842695  | 0.98537132                 | 0.69054021 | $6.9 \times 10^{-3}$ | $2.9 \times 10^{-1}$ |
| 2   | 2.35819511  | 2.37299553                 | 2.47328205 | $1.5 \times 10^{-2}$ | $1.2 \times 10^{-1}$ |
| 3   | 3.55762902  | 3.57088545                 | 3.56708317 | $1.3 \times 10^{-2}$ | $9.5 \times 10^{-3}$ |
| 4   | 4.67603019  | 4.67388850                 | 4.70001675 | $2.1 \times 10^{-3}$ | $2.4 \times 10^{-2}$ |
| 5   | 5.70790501  | 5.71470640                 | 5.70805278 | $6.8 \times 10^{-3}$ | $1.5 \times 10^{-4}$ |
| 6   | 6.71126508  | 6.70986934                 | 6.72601722 | $1.4 \times 10^{-3}$ | $1.5 \times 10^{-2}$ |
| 7   | 7.66488539  | 7.66028006                 | 7.66503104 | $4.4 \times 10^{-3}$ | $1.5 \times 10^{-4}$ |
| 8   | 8.60054100  | 8.59949449                 | 8.61061536 | $1.0 \times 10^{-3}$ | $1.0 \times 10^{-2}$ |
| 9   | 9.50196086  | 9.50515170                 | 9.50201767 | $3.2 \times 10^{-3}$ | $5.7 \times 10^{-5}$ |
| 10  | 10.39051635 | 10.38969494                | 10.39807977 | $8.2 \times 10^{-4}$ | $7.6 \times 10^{-3}$ |
| 11  | 11.25329529 | 11.25577366                | 11.25332447 | $2.5 \times 10^{-3}$ | $2.9 \times 10^{-5}$ |
| 12  | 12.10615459 | 12.10548473                | 12.11215512 | $6.7 \times 10^{-4}$ | $6.0 \times 10^{-3}$ |
| 13  | 12.93851429 | 12.94052578                | 12.93853100 | $2.0 \times 10^{-4}$ | $1.7 \times 10^{-5}$ |
| 14  | 13.76285910 | 13.76229703                | 13.76780156 | $5.6 \times 10^{-4}$ | $4.9 \times 10^{-3}$ |
| 15  | 14.57028767 | 14.57197152                | 14.57029807 | $1.7 \times 10^{-3}$ | $1.0 \times 10^{-5}$ |
| 20  | 18.46972152 | 18.46934956                | 18.47289163 | $3.7 \times 10^{-4}$ | $3.2 \times 10^{-3}$ |
| 25  | 22.16864714 | 22.16954498                | 22.16864908 | $9.0 \times 10^{-4}$ | $1.9 \times 10^{-6}$ |
| 30  | 25.72085641 | 25.72062543                | 25.72278120 | $2.3 \times 10^{-4}$ | $1.9 \times 10^{-3}$ |
| 35  | 29.15213166 | 29.15277216                | 29.15213231 | $6.0 \times 10^{-4}$ | $6.5 \times 10^{-7}$ |
| 40  | 32.48656438 | 32.48640008                | 32.48791889 | $1.6 \times 10^{-4}$ | $1.4 \times 10^{-3}$ |
| 45  | 35.73605338 | 35.73649218                | 35.73605367 | $4.4 \times 10^{-4}$ | $2.9 \times 10^{-7}$ |
| 50  | 38.91432285 | 38.91419682                | 38.91535535 | $1.3 \times 10^{-4}$ | $1.0 \times 10^{-3}$ |
| 55  | 42.02788726 | 42.02823131                | 42.02788741 | $3.4 \times 10^{-4}$ | $1.5 \times 10^{-7}$ |
| 60  | 45.08566312 | 45.08556169                | 45.08649070 | $1.0 \times 10^{-4}$ | $8.3 \times 10^{-4}$ |
### Bibliography