A qualitative approach to the existence of random periodic solutions

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Additional Information:

- A Doctoral Thesis. Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University.

Metadata Record: https://dspace.lboro.ac.uk/2134/17355

Publisher: © K. O. Uda

Rights: This work is made available according to the conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) licence. Full details of this licence are available at: https://creativecommons.org/licenses/by-nc-nd/4.0/

Please cite the published version.
A Qualitative Approach to the Existence of Random Periodic Solutions

By

Kenneth Ogbonnaya Uda

A Doctoral Thesis
Submitted in partial fulfilment of the requirements for the award of
Doctor of Philosophy of Loughborough University

Department of Mathematical Sciences

September, 2014
© by K. O. Uda 2014
To My Mother
Abstract

In this thesis, we study the existence of random periodic solutions of random dynamical systems (RDS) by geometric and topological approach. We employed an extension of ergodic theory to random setting to prove that a random invariant set with some kind of dissipative structure, can be expressed as union of random periodic curves. We extensively characterize the dissipative structure by random invariant measures and Lyapunov exponents. For stochastic flows induced by stochastic differential equations (SDEs), we studied the dissipative structure by two point motion of the SDE and prove the existence exponential stable random periodic solutions.

**Keywords:** adjusted random variable, double skew product, two point motion, random invariant curves, random periodic curves, random periodic solutions, random semiuniform ergodic theorem.
Contents

Abstract ii

Acknowledgement v

Introduction 1

1 Foundations of Random Dynamical Systems 8
   1.1 Ergodic Dynamical Systems ........................................... 8
   1.2 Random dynamical systems .......................................... 12
   1.3 Perfection of a crude cocycle .................................... 15
   1.4 Generation of random dynamical systems ....................... 16
      1.4.1 Product of random mappings ................................ 16
      1.4.2 Random differential equations ............................... 18
      1.4.3 RDS and SDEs governed by semimartingale helices ...... 20
      1.4.4 RDS from classical stochastic differential equations .... 28

2 Random Ergodic Theory 31
   2.1 Random invariant probability measures ......................... 31
   2.2 Extremality and ergodicity of random invariant measures .... 44
   2.3 Lyapunov exponent and its extremal ............................ 46
   2.4 Furstenberg-Hasminskii formula for Lyapunov exponents .... 48

3 Random Invariant Periodic Curves 50
   3.1 Motivation and problem formulation .............................. 50
   3.2 Random semiuniform ergodic theorem ............................ 51
   3.3 Random continuous invariant graphs .............................. 62
3.4 Random periodic curves ................................................. 71

4 Random Periodic Solutions of Stochastic flows via Two Point Motion 79
  4.1 Bound of top Lyapunov exponent of stochastic flows ................. 79
  4.2 Exponentially stable non-trivial random stationary solutions for SDEs ............................................... 82
  4.3 Random periodic solutions for SDEs .................................. 90

Appendix 99
  A Selected results and facts from topology ............................... 99
  B Fréchet spaces ............................................................ 101
  C Stochastic calculus for RDS ............................................ 103
  D Proof of Theorem 1.2.2 .................................................. 105

Bibliography 108
Acknowledgement

I express my greatest gratitude to Professor Huaizhong Zhao, my supervisor who introduced me to random dynamical systems. He offered me invaluable supervision, despite most of my unpromising ideas, he paid undivided attention to them and offered tremendous advice and encouragement.

I would like to thank Dr. Chunrong Feng, who is always keen to attend to my questions. I thank all the support staff of Mathematical Sciences Department, Loughborough University for their invaluable help.

My special thanks to the Stochastic Analysis Seminar group at Loughborough University and East Midlands Stochastic Seminar group UK, for the opportunity to learn this subject among them.

My greatest thanks to the Government of Ebonyi State, Nigeria, for making it possible for me to undertake this programme. I also thank Chief Martin Nwancho Elechi for his financial support.

I thank my ever loving friends Louis Omenyi, Simon Baker, Igwe Obasi, Greg Roddick and Tomas Lazasukas for their encouragement and for providing necessary distraction when it is needed.

Finally, I express my special gratitude to my family for their love, encouragement and support.

Kenneth O. Uda
Introduction

Dynamical systems theory studies the changes over time that occur in physical systems. The mathematical models used to describe the solar system, the weather, the flow of water in a pipe, the dissolving of sugar in a cup of coffee, the stock market, the formation of traffic jams, the swinging of a clock pendulum are examples of dynamical systems. The mathematics behind the concept of dynamical system is founded on the fact that physical systems are subjected to certain laws. These laws are given implicitly by a relation that determines the state of a system for all future terms by the knowledge of the present state. Dynamical system is therefore, a pair \((X, \varphi)\), where \(X\) is the space of states and \(\varphi\) is a fixed rule that describes the future states following the present state. The concept of dynamical system is derived from the abstraction and generalization of differential equations and difference equations. This abstraction and idea of ergodicity could be traced back to 1927 and 1931 respectively to the American mathematician George D. Birkhoff (1884 - 1944) in his works [9], [10]. Qualitative theory of dynamical systems, employs geometric and topological point of view to understand the behaviour of solutions of systems. Qualitative theory explores the behaviour of dynamical systems without the knowledge of closed form solutions of the systems. This fascinating aspect of dynamical systems, is mainly inspired by the works of Henri Poincaré (1854 - 1912) and Aleksandr M. Lyapunov (1857 - 1918) and has hugely contributed to the development of the dynamical systems theory.

When considering physical systems, we cannot avoid influences or disturbances from environment or other sources. For examples, ecological and evolutionary systems involve interactions that are constantly subject to environmental and demographic fluctuations, financial market models are greatly influenced by some noise factors (insider information, uncoordinated market impulse, irrational choice, insider trade, etc). To be able to cope with these unavoidable influences or disturbances, we let randomness be embedded in the model. Random dynamical system is a dynamical system with an element of randomness. The idea of random dynamical system was discussed in
1945 by Ulam and Von Nuemann [88] and few years later by Kakutani [42] and continued in the
1970s in the framework of ergodic theory. The discovery by Elworthy, Meyer, Baxendale, Bismut,
Ikeda, Kunita, Watanabe and others ([7], [11], [16], [32], [39], [47], [48]) that stochastic di-
ferential equations (SDEs) induce stochastic flows gave substantial push to the subject and towards late
1980s, it became clear that the techniques from dynamical systems and probability theory could
produce the theory of random dynamical systems.

Random dynamical systems was extensively developed by Arnold [4] and his "Bremen group"
based on the work of Kunita [47], [48] and others on two-parameter stochastic flows generated by
stochastic differential equations. We mainly have three classes of random dynamical systems.

1. **Product of random maps** [4]: Let \( (\Omega, \mathcal{F}, \mathbb{P}, \theta) \) be an ergodic dynamical system, where
\( \theta : \Omega \to \Omega \) preserves the probability measure \( \mathbb{P} \) and let \( (X, \mathcal{B}) \) be a measurable space. Let
\( \psi : \Omega \times X \to X \) be a measurably invertible function, we can define the corresponding random
dynamical system \( \varphi : \mathbb{Z} \times \Omega \times X \to X \) by

\[
\varphi(n, \omega) := \begin{cases} 
\psi(\theta^{n-1} \omega) \circ \cdots \circ \psi(\omega), & n \geq 1 \\
\text{id}_X, & n = 0 \\
\psi(\theta^n \omega)^{-1} \circ \cdots \circ \psi(\theta^{-1} \omega)^{-1}, & n \leq -1.
\end{cases}
\] (0.0.1)

The random dynamical system \( \varphi \) is said to be generated by the measurable mapping \( \psi \).
Conversely, every discrete time random dynamical system has the form (0.0.1), known as
**product of random mappings**.

2. **Random differential equations** [4]: Let \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \) be an ergodic dynamical system,
where the ergodic flow \( (\theta_t)_{t \in \mathbb{R}} : \Omega \to \Omega \) preserves the probability measure \( \mathbb{P} \). Let \( f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n \) be a measurable function such that for almost all \( \omega \in \Omega \), the function \( (t, x) \mapsto f(\theta_t \omega, x) \)
is locally Lipschitz in \( x \), integrable in \( t \) and

\[
\|f(\omega, x)\| \leq \eta(\omega) \|x\| + \rho(\omega),
\]

where \( t \mapsto \eta(\theta_t \omega) \) and \( t \mapsto \rho(\theta_t \omega) \) are locally integrable. The random differential equation

\[ x' = f(\theta_t \omega, x) \]

uniquely generates a continuous random dynamical system \( \varphi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n \) satisfying

\[
\varphi(t, \omega)x = x + \int_0^t f(\theta_s \omega, \varphi(s, \omega)x)ds.
\]
3. **Stochastic evolution equations** \[4, 47, 48, 57\]: The classical Itô stochastic differential equation

\[
dX_t = f_0(X_t)dt + \sum_{k=1}^{m} f_k(X_t)dB_t^k
\]

where \(f_0, f_1, \cdots, f_m\) are smooth vector fields, and \((B_t)_{t \geq 0}\) is \(m\)-dimensional Brownian motion generates uniquely (up to indistinguishability) smooth random dynamical system \(\varphi\) over the filtered dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) describing the Brownian motion. This random dynamical system \(\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n\) almost surely satisfies the integral equation

\[
\varphi(t, \omega)x = x + \int_0^t f_0(\varphi(s, \omega)x)ds + \sum_{k=1}^{m} \int_0^t f_k(\varphi(s, \omega)x)dB_t^k, \quad t > 0.
\]

Following the formulation of the random dynamical systems, it is of great significant in applications and to the development of theory of random dynamical systems to consider its long time behaviour. There are two main issues that motivate the long time behaviour of a mathematical model with theoretical and practical consequences.

- The first one is to understand where the orbits (collection of solutions) converge to, in the long run.
- The second and equally important one is to ascertain whether the limiting behaviour is still essentially the same after small changes to the evolution rule.

Intuitively, the limiting behaviour of dynamical system is captured by the concept of stationary and periodic solutions. For a dynamical system \(\varphi : \mathbb{T} \times X \to X\), over \(t \in \mathbb{T}\), a stationary solution is a point \(x \in X\), such that

\[
\varphi_t(x) = x, \quad \text{for all } t \in \mathbb{T}.
\]

(0.0.2)

And a periodic solution is a periodic function \(u : \mathbb{T} \to X\) with period \(\tau \neq 0\), such that

\[
u(t + \tau) = u(t), \quad \text{and} \quad \varphi_t(u(s)) = u(t + s), \quad \text{for all } t, s \in \mathbb{T}.
\]

(0.0.3)

To understand and give the existence of such solutions have attracted vast interest in theory and applications. Periodic solutions have been crucial in the qualitative theory of dynamical systems and its systematic consideration was initiated by Poincaré in his work \[66\]. Periodic solutions have been studied for many fascinating physical problems, examples, van der Pol equations \[89\], Lienard equations \[51\], etc. However, once noise is added, the dynamics start to depend on both time and
the noise path, so the above definition of steady state (stationary and periodic) solutions may not exist for randomly perturbed systems.

The long time behaviour of systems become more interesting and difficult when we include noise in the systems, which in numerous applications are unavoidable. The long time behaviour of random dynamical systems is relatively new area of mathematical research and has seen a tremendous progress in last three decades. As in the dynamical systems setting, random stationary solutions are central in the long time behaviour of random dynamical systems. For a random dynamical system \( \varphi : T \times \Omega \times X \to X \), over a metric dynamical system \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T}) \), a random stationary solution is an \( \mathcal{F} \)-measurable random variable \( Y : \Omega \to X \) such that

\[
\varphi_t(\omega, Y(\omega)) = Y(\theta_t \omega), \quad \text{for all } t \in T, \quad \mathbb{P} - \text{almost surely}. \tag{0.0.4}
\]

The notion of random stationary solution of a random dynamical system is a natural extension of fixed point solution of the deterministic system. It is a "one force, one solution" setting that describes the pathwise invariance of the stationary solution over time along the dynamic \( \theta \) and the pathwise limit of the random dynamical system. The study of stability once the stationary solution is known were motivated by the pioneeer work of Has’minskii in 1969 (translated to English in 1980) \[38\], other contributions were made by Kushner \[50\], Pinksy \[65\], Mao \[55\] and others. Finding such solutions for stochastic evolution equations is one of the basic problems in stochastic analysis and has been studied recently by Schmalfuss \[78, 79\], Caraballo, Kloeden and Schmalfuss \[15\], E, Khanin, Mazel and Sinai \[31\], Sinai \[81, 82\], Zhang and Zhao \[91\]. Stable and unstable manifold was also recently considered by Duan, Lu and Schmalfuss \[29\], Mohammed, Zhang and Zhao \[59\] and others.

Analogous to the periodic solutions of dynamical system, the notion of random periodic solutions play similar role to random dynamical systems. In physical world around us (e.g. biology, chemical reactions, climatic dynamics, finance, etc), we encounter many phenomena which repeat after certain interval of time. Due to the unavoidable random influences, the phenomena may be best described by random periodic solutions rather than periodic solutions. For example, the maximum daily temperature in any particular region is a random process, however, it certainly has periodic nature driven by divine clock due to the rotation of the earth around the sun. There have been few attempts in physics to study random perturbation of limit cycle for some time \[49, 50, 92\]. One of the challenges that hinders real progress was lack of a rigorous mathematical definition of random periodic solution and appropriate mathematical tools. For a random path with some periodic

---

4
property, it is unclear what a reasonable mathematical relation between the random position \( \psi(t, \omega) \) at time \( t \) and \( \psi(t+\tau, \omega) \) at time \( t+\tau \) after a period \( \tau \) should be. However, as \( \psi(t, \omega) \) is a true path, so it is not necessarily true that \( \psi(t, \omega) = \psi(t+\tau, \omega) \). To require that \( \psi(t+\tau, \omega) \) is in the neighbourhood of \( \psi(t, \omega) \) by considering a small noise perturbation was worthwhile attempt. However, this approach does not apply to many stochastic differential equations and also lack rigour, and the scope of application is limited. Recently, in the works of Zhao and Zheng \[95\], Feng, Zhao and Zhou \[33\], Feng and Zhao \[34\], it has been observed that for fixed \( t \), \( (\psi(t+k\tau, \omega))_{k \in \mathbb{Z}} \) should be a random stationary solution of the discrete RDS \( \varphi(k\tau, \omega) \). This then led to the rigorous definition of random periodicity \( \psi(t+\tau, \omega) = \psi(t, \theta_t \omega) \). For a random dynamical system \( \varphi \) over a metric dynamical systems \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})\), a random periodic solution is an \( \mathcal{F} \)-measurable function \( \psi : \mathbb{T} \times \Omega \to X \) of period \( \tau \) such that

\[
\psi(t+\tau, \omega) = \psi(t, \omega) \quad \text{and} \quad \varphi_t(\omega, \psi(s, \omega)) = \psi(t+s, \theta_t \omega), \quad \text{for all} \ t, s \in \mathbb{T}.
\]  

(0.0.5)

The study of random periodic solution is more fascinating and difficult than the deterministic periodic solution. The extra essential difficulty is from the fact that trajectory (solution path) of the random dynamical systems starting at a point on the periodic curve does not follow the periodic curve, but moves from one periodic curve to another one corresponding to different \( \omega \). If one considers family of trajectories starting from different points on the closed curve \( \psi(., \omega) \), then the whole family of trajectories at time \( t \in \mathbb{T} \) will lie on a closed curve corresponding to \( \theta_t \omega \). There are few numerical evidence in the literature suggesting the existence of random periodic curves (examples; Stochastic Van der pol oscillator \[20\], Stochastic Goodwin-Lotka- Volterra model \[61\], Stochastic speculative financial model \[19\], Stochastic climate dynamics \[18\]).

Our interest in this thesis is to investigate the existence of random periodic solutions using qualitative (geomeric and topological) approach. It is subtle if not impossible to represent the solutions of stochastic systems in a closed form. So qualitative approach becomes a natural and or feasible point of view in the investigation of some important features of random dynamical systems. Our results are based on some geometric and topological invariant structures of RDS and it is mainly based on the extension of ergodic theorem to the random setting. The idea is to present results directly accessible to the theory of random bifurcation (dynamical bifurcation), which has numerous problems yet to be explored. In random D-bifurcation, random invariant structures such as random invariant sets, random invariant measures and Lyapunov exponents are mostly the major objects of investigation. The basic assumptions in the results here, are on these
invariant structures and there are quite natural for important systems.

Recently, Zhao and Zheng [95] used geometric approach to prove the existence of random periodic solutions on the cylinder $S^1 \times \mathbb{R}^d$, to the best of our knowledge, this is the first result on the geometric approach to the existence random periodic solutions. It is assumed in [95] that there is a random invariant compact set which consists of Lipschitz continuous graph and prove that the graph is random periodic. The result in chapter 3 is closely related to this work of Zhao and Zheng [95], but we did not assume that the random invariant compact set consists of Lipschitz graph. We employed an extension ergodic theory to the random setting to prove that the random invariant compact set consists of random periodic graph. In fact, we gave some conditions on the bound of top Lyapunov exponent of the RDS in the neighbourhood of the random invariant compact set to achieve this result. This random periodic graph is continuous (Theorem 3.3.10) and we are not sure if it is Lipschitz continuous, this would be investigated in the future work. As mentioned in the previous paragraph, the conditions here are consistent with that of random D-bifurcation ([4], [19], [76], [77] and [96]). We anticipate that further systematic study of our result would be applied in the Hopf bifurcation theory of random dynamical systems.

Analytic aspect of the existence of random periodic solutions of stochastic (partial) differential equations was extensively investigated by Feng, Zhou and Zhao [33] and Feng and Zhao [34]. Their results are based on infinite horizon stochastic integral equation and Wiener-Sobolev compact embedding argument. In fact, one of the basic assumptions in their works was some boundedness conditions on the vector fields associated with the stochastic (partial) differential equations. Our results in chapter 4, employed Lyapunov function technique to characterize this boundedness conditions (dissipativity of the stochastic flow) to prove the existence of unique stable random periodic solution. Advantage of our results is that the conditions are quite natural to some applicable stochastic differential equations in finite dimension. It has been known in few works (for example; [1], [55], [68]) that the technique of Lyapunov function could be used to provide a bound to the top Lyapunov exponent of stochastic differential equation. Thus, our results establish a connection between geometric and analytic approach to the existence of random periodic solution.

Finally, let us outline the structure of this thesis. In chapter 1, we provide some foundational aspects of random dynamical systems. First, we briefly introduce notion of global and local random dynamical systems. Perfection technique is as well introduced, which is actually the notion required to make the cocycle a random dynamical systems. In the remaining part of this chapter, we discuss
generation of random dynamical systems through stochastic/random difference equations, random differential equation and stochastic differential equations.

In chapter 2, we deal with some geometric aspect of random dynamical systems which serve as major tools to most of the results in this thesis. We explore the notion of random invariant measures for random dynamical systems. Next, we discuss the ergodicity and extremality of random invariant measure. We recall the celebrated multiplicative ergodic theory and based on the extremality of random invariant measures, we have the realisation of the top Lyapunov exponent. Finally, the idea of Furstenberg-Hasińinskii formula is briefly discussed for markov random dynamical system.

Random semiuniform ergodic theorem is introduced in the first part of chapter 3, the aspect of this fascinating result basic assumptions required to prove the existence random periodic random curves. We used the idea of double skew product dynamical systems to prove the existence of random periodic solutions of random dynamical systems in the cylinder $S^1 \times \mathbb{R}^d$.

In chapter 4, we used the notion of two point generator of stochastic differential equation to prove the existence of exponential stable random periodic solutions. The idea is not far from our results in chapter 3, as the assumptions require some bounds on the top Lyapunov exponents which is one of the basic assumptions in chapter 3. The advantage of our results here is that the assumptions seem natural and feasible in many important applications.
Chapter 1

Foundations of Random Dynamical Systems

This chapter is concerned with the basis of random dynamical systems, paying particular attention to perfection and generation problems. For comprehensive discussion of the theory and applications of random dynamical systems, we refer to the monograph by Arnold [4]. In what follows in this thesis, we will be concerned with probability space by which we mean a triplet \((\Omega, \mathcal{F}, P)\), where \(\Omega\) is a nonempty set, \(\mathcal{F}\) is a \(\sigma\)-algebra of the collection of sets in \(\Omega\), and \(P\) is a nonnegative measure on \(\mathcal{F}\) with \(P(\Omega) = 1\). The time \(T\) always stands for the following semigroups or groups:

- \(T = \mathbb{R}\): Two-sided continuous time,
- \(T = \mathbb{R}^+ := \{t \in \mathbb{R} : t \geq 0\}\) (sometimes \(T = \mathbb{R}^- := -\mathbb{R}^+\)): One-sided continuous time,
- \(T = \mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}\): Two-sided discrete time,
- \(T = \mathbb{Z}^+ := \{0, 1, 2, \cdots\}\) (sometimes \(T = \mathbb{Z}^- := -\mathbb{Z}^+\) or \(T = \mathbb{N} := \{1, 2, \cdots\}\)): One-sided discrete time.

We shall endow \(T\) with its Borel \(\sigma\)-algebra \(B(T)\).

1.1 Ergodic Dynamical Systems

A collection \((\theta_t)_{t \in T}\) of self mappings of \((\Omega, \mathcal{F})\) is called a measurable dynamical system with time \(T\) if it satisfies the following three conditions:
(a) $(\omega, t) \mapsto \theta_t \omega$ is measurable,

(b) $\theta_0 = \text{id}_\Omega = \text{identity on } \Omega$ (if $0 \in \mathbb{T}$),

(c) flow property: $\theta_{t+s} = \theta_t \circ \theta_s$, for all $s, t \in \mathbb{T}$.

Consider two measure spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$.

- A measurable mappings $\varphi : \Omega_1 \to \Omega_2$ such that
  \[ \mathbb{P}_1(\varphi^{-1}(A)) = \mathbb{P}_2(A), \quad \text{for all } A \in \mathcal{F}_2 \]

  is called a **homomorphism** of the measure spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$.

- A homomorphism of $(\Omega, \mathcal{F}, \mathbb{P})$ to itself is called an **endomorphism** ($\varphi : \Omega \to \Omega$, such that $\mathbb{P}(\varphi^{-1}(A)) = \mathbb{P}(A)$, for all $A \in \mathcal{F}$). In this case, we say that $\varphi$ preserves that the measure $\mathbb{P}$.

- A measurable dynamical system $\theta$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for which each $\theta_t$ is measure preserving (an endomorphism) is called **metric** (measure theoretic) dynamical system. The metric dynamical system will be denoted in this work by $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$ or by $\theta$ for short.

- A function $f$ is said to be invariant with respect to the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$, if $f(\theta_t \omega) = f(\omega)$ for all $t \in \mathbb{T}$, $\omega \in \Omega$.

- A set $A$ is said to be invariant with respect to $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$, if the indicator function $\mathbb{I}_A$ is invariant with respect to $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$ (that is, $\theta^{-1} A = A$).

An invariant set $A$ of a measurable dynamical system consists of the whole orbits or trajectories (that is, $(\theta_t \omega)_{t \in \mathbb{T}} \subset A$, if $\omega \in A$) and a set $A$ is **forward invariant** with respect to $\theta$, if $\theta^{-1} A \subset A$. One can easily check that a family of measurable invariant sets with respect to $\theta$ forms a sub $\sigma$-algebra $\{ A \in \mathcal{F} : \theta^{-1} A = A \} =: \mathcal{I} \subset \mathcal{F}$.

**Definition 1.1.1 (Ergodic dynamical system)** A metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$ is called **ergodic**, if $\mathbb{P}(A) \in \{0, 1\}$, for all $A \in \mathcal{I}$. 

---

\[ \text{Loughborough University Doctoral Thesis} \]
Examples of ergodic dynamical systems:

(i) **Circle map**: Let $\Omega = [0, 1), \mathcal{F} = \mathcal{B}([0, 1))$ equipped with the Lebesgue measure $\lambda$, define for $\alpha \in T = [0, 1)$
\[
\theta_\alpha : [0, 1) \to [0, 1), \quad \omega \mapsto \theta_\alpha \omega = \omega + \alpha \mod 1.
\]
We know that Lebesgue measure is translational invariant; that is
\[
\lambda(\theta_\alpha^{-1} I) = \lambda(I),
\]
for all interval $I \subset [0, 1)$. Therefore the collection $\mathcal{M} = \{ A \in \mathcal{F} : \lambda(\theta_\alpha^{-1} A) = \lambda(A) \}$ contains the algebra of finite disjoint union of intervals. We can check easily that $\mathcal{M}$ is a monotone class, and by monotone class theorem we deduce that $\mathcal{M}$ contains Borel measurable set (see Royden [72], Rudin [73]). Of course $\mathcal{M}$ also contains null sets. So, $\mathcal{M} = \mathcal{F}$ and thus,
\[
\lambda(\theta_\alpha^{-1} A) = \lambda(A),
\]
for all $A \in \mathcal{F}$.

We verify that $\theta_\alpha$ is ergodic whenever $\alpha \notin \mathbb{Q}$. Suppose $A$ is an invariant set, that is; $A \in \mathcal{I}$, and set $f = \mathbb{I}_A$, the indicator function. If we expand $f$ in Fourier series, then
\[
f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi in}.
\]
Since $A$ is invariant, we get that $f = f \circ \theta_\alpha$. The Fourier transform of $f \circ \theta_\alpha$ is
\[
f \circ \theta_\alpha = \sum_{n \in \mathbb{Z}} e^{2\pi in\alpha} \hat{f}(n) e^{2\pi in}.
\]
Equating coefficients, we have that
\[
\hat{f}(n) = \hat{f}(n) e^{2\pi in\alpha}.
\]
Thus, it is either that $\hat{f}(n) = 0$ or $e^{2\pi in\alpha} = 1$, and since $\alpha \notin \mathbb{Q}$, we get
\[
\hat{f}(n) = 0, \quad n \neq 0.
\]
Therefore,
\[
f = \hat{f}(0) = \int_{\Omega} \mathbb{I}_A(x) e^{-it\alpha x} dx = \lambda(A), \quad a.e.
\]
this implies that $\mathbb{I}_A = \lambda(A), \quad a.e.$

Hence, $\lambda(A) \in \{0, 1\}$ which shows the ergodicity of $\theta_\alpha$. 


(ii) **Torus map:** Let $\Omega$ be $n$-dimensional torus. Assume that its points are written as $\omega = (\omega_1, \cdots, \omega_n)$ with $\omega_i \in [0, 1)$. Let $\mathcal{F} = B(\text{Tor}^n)$ and $\mathbb{P}$ is the Lebesgue measure on $\text{Tor}^n$. We define the mapping

$$\theta_t \omega = (\omega_1 + ta_1 \mod 1, \cdots, \omega_n + ta_n \mod 1),$$

for a given $a = (a_1, \cdots, a_n)$. Then $(\theta_t)_{t \in \mathbb{R}}$ is a metric dynamical system and if $a_1, \cdots, a_n$ are rationally independent, we have that $(\theta_t)_{t \in \mathbb{R}}$ is ergodic.

(iii) **Wiener shift operator:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Wiener space, define the shift operator $\theta_t : \Omega \rightarrow \Omega$, by

$$\theta_t \omega(.) = \omega(t + .) - \omega(t).$$

Then $(\theta_t)_{t \in \mathbb{R}}$ is ergodic.

For this, let $\mathbb{P}$ be the Wiener measure, we know that Wiener measure is time shift invariant, we have

$$\mathbb{P}(\theta_t^{-1}A) = \mathbb{P}(A), \quad A \in \mathcal{F}.$$ 

Hence,

$$\theta_t \mathbb{P} = \mathbb{P}.$$ 

So, $(\theta_t)_{t \in \mathbb{R}}$ is measure preserving.

Let $B \in \mathcal{F}$ and $A \in \mathcal{I} = \{A \in \mathcal{F}; \theta_t(A) = A\}$.

$$\mathbb{P}(\theta_t^{-1}(A) \cap B) = \mathbb{P}(A \cap B), \quad \text{for all} \quad t \in \mathbb{R}.$$ 

Now, using the independent increments property of Brownian motion, we have that

$$\mathbb{P}(\theta_t^{-1}(A) \cap B) = \mathbb{P}(A) \mathbb{P}(B)$$

Since $B \in \mathcal{F}$ is arbitrary, we take $B = A$. So, that

$$\mathbb{P}(A) = [\mathbb{P}(A)]^2$$

this means that $\mathbb{P}(A) \in \{0, 1\}$. Thus, $(\theta_t)_{t \in \mathbb{R}}$ is ergodic.
1.2 Random dynamical systems

Definition 1.2.1 (Random dynamical system (RDS) [4]) A measurable RDS on a measurable space \((X, \mathcal{B})\) over a metric dynamical system \(\theta = (\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{T}})\) with time \(\mathbb{T}\) is a mapping

\[
\varphi : \mathbb{T} \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x) \equiv \varphi(t, \omega)x
\]

with the following properties:

- measurability: \(\varphi\) is \(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}\)-measurable.

- cocycle property: The mapping \(\varphi(t, \omega) \equiv \varphi(t, \omega, \cdot) : X \to X\), form a cocycle, that is

\[
\varphi(0, \omega) = \text{Id}_X, \quad \text{for all } \omega \in \Omega, \quad \text{if } 0 \in \mathbb{T}, \quad (1.2.1)
\]

\[
\varphi(t + s, \omega) = \varphi(t, \theta_s \omega \circ \varphi(s, \omega), \quad \text{for all } s, t \in \mathbb{T}, \quad \omega \in \Omega \quad (1.2.2)
\]

If \(X\) is a topological space and in addition, for each \(\omega \in \Omega\), the mapping

\[
\varphi(\omega) \equiv \varphi(\cdot, \omega, \cdot) : \mathbb{T} \times X \to X, \quad (t, x) \mapsto \varphi(t, \omega)x
\]

is continuous, we call the RDS a **continuous RDS**.

If \(X\) is a \(d\)-dimensional \(C^k\) manifold and in addition to topological (continuous) RDS, the mapping \(\varphi(t, \omega) : X \to X, \quad x \mapsto \varphi(t, \omega)x\) is \(k\)-times continuous differentiable and all derivatives are continuous with respect to \((t, x)\), we call it a **smooth RDS** or \(C^k\) RDS \((1 \leq k \leq \infty)\).

Theorem 1.2.2 (Basic properties of random dynamical systems with two-sided time [4])

Suppose that \(\mathbb{T} = \mathbb{R}\) or \(\mathbb{T} = \mathbb{Z}\).

(i) Let \(\varphi\) be a measurable RDS on a measurable space \((X, \mathcal{B})\) over \(\theta\). Then for all \((t, \omega) \in \mathbb{T} \times \Omega\), \(\varphi(t, \omega)\) is a bimeasurable bijection of \((X, \mathcal{B})\) and

\[
\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega), \quad \text{for all } (t, \omega) \in \mathbb{T} \times \Omega, \quad (1.2.3)
\]

or

\[
\varphi(-t, \omega) = \varphi(t, \theta_{-t} \omega)^{-1}, \quad \text{for all } (t, \omega) \in \mathbb{T} \times \Omega. \quad (1.2.4)
\]

Moreover, the mapping \((t, \omega, x) \mapsto \varphi(t, \omega)^{-1}x\) is measurable.
(ii) If \( \varphi \) is a continuous RDS on a topological space \( X \). Then for all \( (t, \omega) \in T \times \Omega \), we have that \( \varphi(t, \omega) \) is a homeomorphism, if

1. \( T = \mathbb{Z} \) or,
2. \( T = \mathbb{R} \) and \( X \) is a topological manifold, or
3. \( T = \mathbb{R} \) and \( X \) is a compact Hausdorff space.

Then \( (t, \omega) \mapsto \varphi(t, \omega)^{-1}x \) is continuous for all \( \omega \in \Omega \).

(iii) If \( \varphi \) is a \( C^k \) RDS on a manifold \( X \). Then for all \( (t, \omega) \in T \times \Omega \), \( \varphi(t, \omega) \) is a \( C^k \) diffeomorphism. Moreover, \( (t, x) \mapsto \varphi(t, \omega)^{-1}x \) is \( C^k \) with respect to \( x \) for \( \omega \in \Omega \).

**Remark 1.2.3**  
(1) Let \( \varphi \) be a continuous RDS with \( T = \mathbb{R} \), if \( X \) is not locally Euclidean or compact Hausdorff space, we cannot conclude in general that \( (t, x) \mapsto \varphi(t, \omega)^{-1}x = \varphi(-t, \theta_t \omega)x \) is continuous. This is due to the appearance of this measurable operator \( \theta_t \) in the formula

\[
\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega),
\]

(see the proof of Theorem 1.2.2 in appendix D on why we require locally Euclidean or compact Hausdorff space). So, we could always replace the continuity condition by continuity with respect to \( x \); for all \( (t, \omega) \in T \times \Omega \), and still conclude that \( \varphi(t, \omega) \) is a homeomorphism. In fact, continuity with respect to \( x \) is sufficient for the results in this work. But the reason for imposing continuity with respect to \( (t, x) \) in the definition is that we automatically obtain such RDS when solving random differential equations (RDEs) or stochastic differential equations (SDEs).

(2) Let the RDS \( \varphi \) be given. Define

\[
(\omega, x) \mapsto (\theta_t \omega, \varphi(t, \omega)x) =: \Theta_t(\omega, x), \quad t \in T.
\]

Now,

\[
\Theta_{t+s}(\omega, x) = (\theta_{t+s} \omega, \varphi(t + s, \omega)x) \\
= (\theta_t \circ \theta_s \omega, \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)x) \\
= \Theta_t(\theta_s \omega, \varphi(s, \omega)x) \\
= \Theta_t \circ \Theta_s(\omega, x), \quad \text{for all } (\omega, x) \in \Omega \times X,
\]
so that, $\Theta_{t+s} = \Theta_t \circ \Theta_s$.

Thus, $\Theta_t : \Omega \times X \to \Omega \times X$ is a measurable dynamical system, this is called the skew product of the metric dynamical system $\theta$ and the cocycle $\varphi(t, \omega)$ on $X$.

- $\varphi$ is a random dynamical system
- $\Theta(t)(x, \omega) = (\theta(t)\omega, \varphi(t, \omega)x)$ is a flow on the bundle

Figure 1.1: Random dynamical system as an action on a bundle

1. Evidently, we can use the cocycle property and the invariance of $\theta$ to deduce that a measurable RDS $\varphi$ has stationary increments.

2. Let $\phi(t, \omega)$ be a mapping satisfying the requirements for RDS except that instead of the usual cocycle property, we have

$$\phi(t + s, \omega) = \phi(t, \omega) \circ \phi(s, \theta_s \omega)$$

called the backward cocycle property. For two-sided time $T = \mathbb{R}$ or $\mathbb{Z}$, $\varphi(t, \omega)$ is a cocycle over $\theta$ if and only if $\phi(t, \omega) := \varphi(t, \omega)^{-1}$ is a backward cocycle over $\theta$. 
1.3 Perfection of a crude cocycle

We say that a certain function possess property $\mathcal{H}$ almost surely if the set of point for which this property is not satisfied is of probability zero. It is a tradition in probability theory to ignore events which occur with zero probability. For example, if a stochastic process has continuous paths with probability one (almost surely), then it is refered to as a continuous process and we do not care if it actually has discontinuous paths on some events of zero probability.

There is no reason the cocycle property (1.2.2) should hold for all $\omega \in \Omega$, it is enough that the cocycle property hold outside a set of measure zero. Precisely,

$$\varphi_{t+s}(\omega) = \varphi_t(\theta_s \omega) \circ \varphi_s(\omega), \quad \text{for all } t,s \in \mathbb{T}, \; \omega \notin N_{s,t}(\omega)$$

where $N_{s,t}(\omega)$ is a $\mathbb{P}$-null set generally depending on $s$ and $t$. If one could choose the $\mathbb{P}$-null set to be independent of $s$ and $t$, then we say that $\varphi$ is a perfect cocycle. If the cocycle $\varphi$ has right or left continuous sample path, the nulls set could be choosen to be independent of $t$ (e.g. $N_s = \bigcup \{N_{s,t} : t \in \mathbb{Q}\}$) which is always the case in this work. We shall only consider case when the null sets only depends on $s$. Perfection is a technique of finding another cocycle $\hat{\varphi}$ which is indistinguishable from the original cocycle $\varphi$ for which the cocycle property holds identically. It is crucial to mention that it is for perfect cocycle that one could define a skew product dynamical system on $\Omega \times X$, it is basically near impossible to carry on long time behaviour analysis on a crude cocycle. Thus, the results in this thesis are only possible for perfect cocycles.

To perfect a cocycle $\varphi$, we need quite sophisticated changes on the null sets $N_s$ to avoid contradictions in our analysis, this is because the cocycle property involve those $\omega$ at which we have to change $\varphi$ with those where it is already correctly define. Example [4]: suppose $\varphi$ is an almost perfect cocycle; that is, the cocycle property holds outside a null set $N \in \mathcal{F}$ ($N$ independent of $s$ and $t$). The first idea for constructing a perfect cocycle would be to put

$$\hat{\varphi}(t,\omega) = \begin{cases} \varphi(t,\omega) & \text{for all } t \in \mathbb{T}, \; \omega \notin N, \\ \text{id}_X & \text{for all } t \in \mathbb{T}, \; \omega \in N. \end{cases}$$

Assume for some $\omega \in N$, there is $s \in \mathbb{T}$ for which $\hat{\omega} = \theta_s \omega \notin N$. If $\hat{\varphi}$ were perfect cocycle for all $t \in \mathbb{T}$, one could have $\text{id}_X = \varphi(t,\theta_s \omega) \circ \text{id}_X$, thus $\varphi(t,\hat{\omega}) = \text{id}_X$ for all $t \in \mathbb{T}$, which is a contradiction.

Lucky enough, perfection problem has a satisfactory solution for many applicable problems and especially the problems considered in this work.
Theorem 1.3.1 (Perfection of a very crude cocycle [4], [I]) Let $T = \mathbb{Z}$ or $\mathbb{N}$ and $\mathbb{R}$ or $\mathbb{R}^+$. 

1. Let $\varphi$ be a very crude measurable/continuous/smooth cocycle over $\theta$ with discrete time $T$. Then there exists a measurable/continuous/smooth perfect cocycle $\hat{\varphi}$ over $\theta$ which is indistinguishable from $\varphi$, that is; there is a $\mathbb{P}$-null set $N \subset \mathcal{F}$ such that
$$\{\omega : \hat{\varphi}(t, \omega) \neq \varphi(t, \omega) \text{ for some } t \in T\} \subset N.$$

2. Let $\varphi$ be a very crude cocycle over $\theta$ on the Hausdorff topological space $X$. Then $\varphi$ can be perfected to a continuous cocycle $\hat{\varphi}$ which is indistinguishable from $\varphi$.

3. If $\varphi$ is a smooth cocycle over $\theta$ on a differentiable Manifold $X$. Then $\varphi$ can be perfected to a smooth cocycle $\hat{\varphi}$ which is indistinguishable from $\varphi$.

Remark 1.3.2 Perfection could as well be done for a cocycle with $T$ taken to be a Hausdorff topological group (see [4], [I]), but the above result is enough to what follows in subsequent sections and chapters.

1.4 Generation of random dynamical systems

Generation is a systematic discussion of the following objects:

- Random difference equations,
- Random differential equations,
- Stochastic (partial) differential equations

and their solutions from the dynamical systems point of view.

1.4.1 Product of random mappings

Consider the initial value problem for a stochastic difference equation
$$x_{n+1} = \psi(\theta^n \omega)x_n, \quad n \in \mathbb{N}, \quad x_0 = x \in X. \quad (1.4.1)$$

Let $\varphi(1, \omega) = \psi(\omega)$, the iterated random map $\varphi(n, \omega)$ defined by
$$\varphi(n, \omega) = \begin{cases} \psi(\theta^{n-1} \omega) \circ \cdots \circ \psi(\omega), & n \geq 1, \\ \text{id}_X, & n = 0, \end{cases} \quad (1.4.2)$$
forms a cocycle on $X$.

Important class of RDS arise when $\psi$ takes values in some sub-semigroup $S$ of all semigroup of self-mappings of $X = \mathbb{R}^d$.

(i) **Linear RDS: Product of random matrices:** Let $S = \mathbb{R}^{d \times d}$ be the semigroup of all $d \times d$ matrices, with matrix multiplication as composition. In this case, the mapping $\varphi(n, \omega)$ has the form

$$\varphi(n, \omega) = A_{n-1}(\omega) \cdots A_0(\omega), \quad A_k(\omega) := A(\theta^k \omega),$$

where $A : \Omega \to \mathbb{R}^{d \times d}$ is measurable.

(ii) **Affine discrete RDS:** Let $S$ be the semigroup of all affine mappings of $\mathbb{R}^d$, here $\psi(\omega)x = A(\omega)x + b(\omega)$, with $A : \Omega \to \mathbb{R}^{d \times d}$ and $b : \Omega \to \mathbb{R}^d$ are measurable.

For two sided discrete time $T = \mathbb{Z}$, if $\psi^{-1}$ exists, the stochastic difference equation

$$x_{n+1} = \psi(\theta^n \omega)x_n, \quad n \in \mathbb{Z}, \quad x_0 = x \in X,$$

(1.4.4)

can be solved forwards and backwards in time. Let $\varphi(1, \omega) = \psi(\omega)$ and $\varphi(-1, \omega) = \varphi(1, \theta^{-1} \omega) = \psi(\theta^{-1} \omega)^{-1}$, the random mapping $\varphi(n, \omega)$ given by

$$\varphi(n, \omega) = \begin{cases} 
\psi(\theta^{n-1} \omega) \circ \cdots \circ \psi(\omega), & n \geq 1, \\
\text{id}_X, & n = 0, \\
\psi(\theta^n \omega) \circ \cdots \circ \psi(\theta^{-1} \omega)^{-1}, & n \leq -1
\end{cases}$$

(1.4.5)
is a cocycle on $X$.

The important case $X = \mathbb{R}^d$ and $\psi \in GL(d, \mathbb{R})$ is the case of random matrices. In this situation, the RDS $\varphi(n, \omega)$ can also be viewed as a random walk in the group of $GL(d, \mathbb{R})$.

(1) **Affine RDS, two sided discrete time:** Let $\psi(\omega)x = A(\omega)x + b(\omega)$, we have

$$\varphi(1, \omega)x = A(\omega)x + b(\omega), \quad \varphi(-1, \omega)x = A(\theta^{-1} \omega)^{-1}(x - b(\theta^{-1} \omega))$$

and by induction, we have the random mapping

$$\varphi(n, \omega)x = \begin{cases} 
\Phi(n, \omega) \left( x + \sum_{j=0}^{n-1} \Phi(j+1, \omega)^{-1} b(\theta^{j} \omega) \right), & n \geq 1, \\
x, & n = 0, \\
\Phi(n, \omega) \left( x - \sum_{j=n}^{-1} \Phi(j+1, \omega)^{-1} b(\theta^{j} \omega) \right), & n \leq -1
\end{cases}$$

(1.4.6)

where $\Phi$ is the linear cocycle generated by $A$.  

17
(2) **Semilinear discrete RDS:** Let

$$\psi(\omega)x := A(\omega)x + F(\omega, x), \quad (\omega, x) \in \Omega \times \mathbb{R}^d,$$

be measurable, and $A(\omega) \in GL(d, \mathbb{R})$. Then $\psi$ generate an RDS if, and only if $x \mapsto h(\omega, x) := (I + A(\omega)^{-1}F(\omega, .))x$ is (jointly) measurably invertible.

### 1.4.2 Random differential equations

Let $\mathbb{T} = \mathbb{R}$ and $X = \mathbb{R}^d$. Then the random differential equation

$$\frac{dx}{dt} = f(x, \theta t \omega), \quad x(0) = x_0,$$

(1.4.8)

where $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ is measurable, induces a map $\varphi(t, \omega) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $x(t, \omega) := \varphi(t, \omega)x$ solves the random differential equation (1.4.8) in the sense that there is a correspondence given by the random integral equation

$$\varphi(t, \omega)x = x + \int_0^t f(\varphi(r, \omega)x, \theta r \omega)dr.$$

(1.4.9)

It is well known (see [4], [1], [22], et al) that under some boundedness condition on $f$, the maps $x \mapsto \varphi(t, \omega)x$ for a continuous RDS and all RDS which are absolutely continuous with respect to $t$ arise in this way. In fact, we have the following well known generation theorem.

**Theorem 1.4.1 (RDS from random differential equations [1, 4, 22])** Let $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ be measurable, put $f_\omega(t, x) := f(x, \theta t \omega)$.

- If $f_\omega \in L_{loc}(C^{0,1}_b, \mathbb{R})$ for all $\omega \in \Omega$, the equation (1.4.8) uniquely generates a continuous RDS $\varphi$ over $\theta$.

- If $f_\omega \in L_{loc}(C^{k,0}_b, \mathbb{R})$ for $k \geq 1$, for all $\omega \in \Omega$ then equation (1.4.8) uniquely generates a $C^k$ RDS $\varphi$ over $\theta$, and the Jacobian of $\varphi(t, \omega)$ at $x$,

$$D_x\varphi(t, \omega, x) := \left( \frac{\partial(\varphi(t, \omega)x)_i}{\partial x_j} \right),$$

is a linear (matrix) cocycle over the skew product $\Theta_t(\omega, x) = (\theta t \omega, \varphi(t, \omega)x)$ and uniquely solves the variational equation

$$D_x\varphi(t, \omega, x) = I + \int_0^t D_x f(\varphi(s, \omega)x, \theta s \omega)D_x\varphi(s, \omega, x)ds.$$
Moreover, we have Liouville’s equation
\[ \det D_x \varphi(t, \omega, x) = \exp \int_0^t (\text{trace} Df)(\varphi(s, \omega)x, \theta_s \omega) ds. \]

- Conversely, if \( \varphi \) is a continuous/smooth RDS over \( \theta \) such that \( t \mapsto \varphi(t, \omega)x \) is absolutely continuous for all \((\omega, x)\), then there exists a measurable function \( f(x, \omega) \) such that the random integral equation \[ (1.4.9) \] holds.

A special case of equation \((1.4.8)\) is the random differential equation
\[ \frac{dx}{dt} = f(x, \eta), \quad x(0) = x_0 \quad (1.4.10) \]
where \((\eta_t)_{t \in \mathbb{R}}\) is an Ornstein-Uhlenbeck process on \( \mathbb{R}^m \), that is the solution of the stochastic differential equation
\[ d\eta_t = -\lambda \eta_t dt + \sigma dW_t, \]
where \((W_t)_{t \geq 0}\) is a Brownian motion. The RDS \( \varphi(t, \omega) \) is a suitable mathematical model for many natural systems which are subjected to real noise (i.e. noise with non-vanishing correlation time). The process \((\eta_t)_{t \in \mathbb{R}}\) modelling the noise (fluctuation from the environment) is Markovian and ergodic (such noise is known as colored noise). The base dynamical system \((\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, \mathbb{P})\) is defined in the following canonical way: \( \Omega = C(\mathbb{R}, \mathbb{R}^m) \) the space of realisation of \( \eta_t \), \( \mathcal{F} \) is the product \( \sigma \)-algebra of the Borel \( \sigma \)-algebra of \( \mathbb{R}^m \), \( \mathbb{P} \) is the measure induced by \( \eta_t \), and \( \theta_t \) is the classical shift operator \( \theta_t \omega(\cdot) = \omega(t + \cdot) \).

Simple and important example of RDS generated by random differential equations are:

(a) **Linear random differential equation:** Let \( A : \Omega \to \mathbb{R}^{d \times d} \) be measurable such that \( A \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) and let \( f_\omega(t, x) = A(\theta_t \omega)x \), such that \( f_\omega \in L^{\text{loc}}(\mathbb{R}, C^\infty) \mathbb{P} - a.s. \).

Then \( \frac{dx}{dt} = A(\theta_t \omega)x \) generates a unique \( C^\infty \) RDS
\[ \Phi(t, \omega) = I + \int_0^t A(\theta_s \omega)\Phi(s, \omega)ds \]
and
\[ \det \Phi(t, \omega) = \exp \int_0^t \text{trace} A(\theta_s \omega) ds. \]

Also, \( \Phi(t, \omega) \circ \Phi(t, \omega)^{-1} = I \), yield
\[ \Phi(t, \omega)^{-1} = I - \int_0^t \Phi(s, \omega)^{-1} A(\theta_s \omega)ds. \]
(b) **Affine random differential equation:** The random equation

\[
\frac{dx}{dt} = A(\theta_t\omega)x + b(\theta_t\omega), \quad A, b \in L^1(\Omega, \mathcal{F}, \mathbb{P}),
\]

generates a unique $C^\infty$ RDS. By variation of constants formula we have

\[
\varphi(t, \omega)x = \Phi(t, \omega)x + \int_0^t \Phi(t, \omega) \circ \Phi(u, \omega)^{-1} b(\theta_u\omega)du
\]

\[
= \Phi(t, \omega)x + \int_0^t \Phi(t - u, \theta_u\omega) b(\theta_u\omega)du,
\]

where $\Phi$ is the matrix cocycle generated by $\frac{dx}{dt} = A(\theta_t\omega)x$.

(c) **Random differential equation with polynomial right-hand side:** Let $a_j \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for all $j$ and $N \geq 2$, then random differential equations

\[
\frac{dx}{dt} = \sum_{j=0}^N a_j(\theta_t\omega)x^j
\]

uniquely generates a local $C^\infty$ RDS in $\mathbb{R}$.

### 1.4.3 RDS and SDEs governed by semimartingale helices

Here we consider a large class of RDS which are just continuous but not absolutely continuous, in fact nowhere differentiable, and locally not of bounded variation with respect to time $t$. Precisely, integral equations defined in terms of Riemann or Stieltjes is not appropriate for this kind of generator. The generator here is stochastic differential equations (SDEs), which contain stochastic integrals. Stochastic integral is one of the central objects of stochastic analysis which make sense only as limits in probability and not $\omega$-wise limits. In particular, the result employed in the generation of RDS from random differential equations in the previous subsection 1.4.2 cannot be utilized here.

The class of RDS which have an SDE as generator consist of those RDS $\varphi$ which have additional statistical property of $t \mapsto \varphi(t, \omega)x$ being a semimartingale for each fixed $x$. The SDE will be driven by a semimartingale with stationary increments (semimartingale helix). We wish to briefly discuss a one-to-one correspondence between semimartingale helices and semimartingale cocycles. We do this by recalling some results extensively established by Arnold [4] and Kunita [48].

**Definition 1.4.2** ([2], [4]) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose $\mathcal{F}_s^t$, $s, t \in \mathbb{R}$, $s \leq t$, be a two-parameter family of sub-$\sigma$-algebras of $\mathcal{F}$ such that
i. $\mathcal{F}_s^t \subset \mathcal{F}_u^v$, for $u \leq s \leq t \leq v$,

ii. $\bigcap_{u \geq t} \mathcal{F}_u^s =: \mathcal{F}_s^t = \bigcap_{u \leq s} \mathcal{F}_u^t$, for $s \leq t$,

iii. $\mathcal{F}_s^t$ contains all sets of $\mathcal{F}$ with probability zero ($\mathbb{P}$-null sets of $\mathcal{F}$).

Then $\mathcal{F}_s^t$, $s \leq t$, is called a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathcal{F}_s^t$, $s \leq t$ be a filtration on a $(\Omega, \mathcal{F}, \mathbb{P})$. A metric dynamical system $\theta$ is called filtered dynamical systems if

$$\theta_{-r}(\mathcal{F}_s^t) = \mathcal{F}_{s+r}^{t+r}, \quad r \in \mathbb{R}, \quad -\infty < s \leq t < \infty.$$ 

- Let $\varphi$ be an RDS on a polish space $(X, \mathcal{B})$ with two sided time $T = \mathbb{R}$. Define

$$\mathcal{F}^- := \sigma(\varphi(-t,.)x : t \geq 0, \ x \in X),$$

$$\mathcal{F}^+ := \sigma(\varphi(t,.)x : t \geq 0, \ x \in X),$$

$\mathcal{F}^-$ and $\mathcal{F}^+$ are sub-$\sigma$-algebras of $\mathcal{F}$ (representing the "past" and "future" of $\varphi$). By measurability of $\theta$, we have

$$\theta_{-r}(\mathcal{F}^-) \subset \mathcal{F}^-, \quad \text{for all } r \leq 0, \quad \theta_{-r}(\mathcal{F}^+) \subset \mathcal{F}^+, \quad \text{for all } r \geq 0.$$ 

Take

$$\mathcal{F}^t := \theta_{-t}(\mathcal{F}^-), \quad \mathcal{F}_s := \theta_{-s}(\mathcal{F}^+), \quad \mathcal{F}_s^t = \mathcal{F}_s \cap \mathcal{F}^t, \quad s \leq t.$$ 

It could be shown that $\mathcal{F}^t$ is increasing in $t$ and $\mathcal{F}_s$ is decreasing in $s$. Further, $\mathcal{F}^t$ and $\mathcal{F}_s$ contain $\mathbb{P}$-null sets of $\mathcal{F}$, so is $\mathcal{F}_s^t$. With some work, we could show that $\mathcal{F}_s^t$ satisfies definition 1.4.2. By definition

$$\theta_{-r}(\mathcal{F}_s^t) = \theta_{-r}(\mathcal{F}_s) \bigcap \theta_{-r}(\mathcal{F}^t)$$

$$= \theta_{-r} \circ \theta_{-s}(\mathcal{F}^+) \bigcap \theta_{-r} \circ \theta_{-t}(\mathcal{F}^-)$$

$$= \mathcal{F}_{s+r}^{t+r}, \quad \forall r \in \mathbb{R}, \ s \leq t.$$ 

- Let $C_0(\mathbb{R}, \mathbb{R}^d)$ be a metric space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^d$ which vanishes at zero, endowed with the metric of uniform convergence $\rho$. Let $\mathcal{F}$ be Borel $\sigma$-algebra of this

$$\rho(f,g) = \left\{ \frac{\sup_{t \in [a,b]} |f(t)-g(t)|^2}{1+\sup_{t \in [a,b]} |f(t)-g(t)|^2} \right\}^{1/2}$$
space and let $\mathbb{P}$ be a Wiener measure. We know from section 1.1 that $\mathbb{P}$ is ergodic with respect to the "Wiener shift"

$$\theta_t\omega(.) = \omega(., + t) - \omega(t).$$

Set $\mathcal{F}^t_s := \sigma(\omega(u) - \omega(v) : s \leq u, v \leq t) \cup \mathcal{N}$, where $\mathcal{N}$ is $\mathbb{P}$-null set of $\mathcal{F}$. Then

1. $\mathcal{F}^t_s$ is a filtration on $(C_0(\mathbb{R}, \mathbb{R}^d), \mathcal{F}, \mathbb{P})$ and
2. $\theta$ is a filtered dynamical System.

A real valued stochastic process $X$ defined on $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ is a semimartingale if it can be decomposed as

$$X_t = M_t + A_t$$

where $M$ is a local martingale and $A$ is a càdlàg adapted process of local bounded variation.

An $\mathbb{R}^d$ process $X = (X^1, X^2, \cdots, X^d)$ is a semimartingale if each of its component $X^i$ is a semimartingale.

- Adapted and continuously differentiable processes are finite variation processes and hence semimartingales.
- Brownian motion is a Semimartingale.
- All càdlàg martingales, submartingales and supermartingales are semimartingales.
- Itô’s processes $X$ such that

$$dX = \mu dt + \sigma dW_t$$

are a semimartingales, where $\mu$ and $\sigma$ are adapted processes, $W$ is a Brownian motion.
- Every Lévy process is a semimartingale.
- Fractional Brownian motion $W^H$ with Hurst parameter $H \neq \frac{1}{2}$ is not a semimartingale, even though it is continuous and adapted process.

Let $\theta$ be a metric dynamical and $(H, \ast)$ a group, $F : \mathbb{R} \times \Omega \to H$ is called a (perfect) helix if

$$F(t + s, \omega) = F(t, \theta_s \omega) \ast F(s, \omega) \quad \text{for all} \quad s, t \in \mathbb{R}, \quad \omega \in \Omega. \quad (1.4.11)$$

As usual $F$ is called a (very) crude helix if for all $t, s \in \mathbb{R}$, equation (1.4.11) holds up to $\mathbb{P}$-null set which may depend on $s$ and or $t$. In what follows, we shall take $(H, \ast)$ to be $(\mathbb{R}^d, +)$ and study
processes which are both Semimartingales and helices at the same time. Given a filtered dynamical system \( \theta \), an \((\mathbb{R}^d, +)\)-valued helix \( F \) is called (Forward resp., backward resp.) Semimartingale helix if

\[
F_s(t, \omega) := F(t, \omega) - F(s, \omega)
\]

is and \( \mathcal{F}_s^t \)- (forward resp., backward resp.) semimartingale.

The next proposition shows that there is a standard way in which a semimartingale with stationary increments can be improved to be a semimartingale helix over a filtered dynamical system.

**Proposition 1.4.3 ([4])** Let \( E \) be a locally compact Hausdorff space with a countable base, let \( C(E, \mathbb{R}_d) \) be space of continuous functions from \( E \) into \( \mathbb{R}_d \) endowed with (metrizable) topology of uniform convergence on compact sets, and let \( C_0(\mathbb{R}, C(E, \mathbb{R}_d)) \) be the space of continuous functions on \( \mathbb{R} \) that vanishes at zero, also equipped with the topology of uniform convergence on compact sets. Let \( \mathcal{F}_s^t \), \( s \leq t \) be a filtration on the complete probability space \((\bar{\Omega}, \mathcal{F}, \bar{\mathbb{P}})\).

Assume \( \bar{F} : \mathbb{R} \times E \times \bar{\Omega} \to \mathbb{R}_d, (t, x, \bar{\omega}) \mapsto \bar{F}(t, x, \bar{\omega}) \) is jointly continuous in \((t, x)\) for all \( \bar{\omega} \in \bar{\Omega} \), \( \bar{F}(0, x, \bar{\omega}) = 0 \) for all \( x \in E, \bar{\omega} \in \bar{\Omega} \), and that

\[
\bar{F}_s(t, x, .) := \bar{F}(t, x, .) - \bar{F}(s, x, .)
\]

is an \( \mathcal{F}_s^t \)- (forward resp., backward resp.) semimartingale for all \( x \in E \).

Suppose that \( \bar{F} \) has stationary increments in the sense that the that the law of

\[
\{ \bar{F}(t + h, x, .) - \bar{F}(t, x, .), x \in E, h \in \mathbb{R} \}
\]
on \( C_0(\mathbb{R}, C(E, \mathbb{R}_d)) \) does not depend on \( t \in \mathbb{R} \).

Then there exists a filtered dynamical system \((\Omega, \mathcal{F}, (\mathcal{F}_s^t)_{s \leq t}, \mathbb{P})\) with (forward resp., backward resp.) semimartingale helices \( F(., x, .), x \in E \), which are also jointly continuous in \((t, x)\) for all \( \omega \in \Omega \) such that the laws of \( \bar{F} \) and \( F \) on \( C_0(\mathbb{R}, C(E, \mathbb{R}_d)) \) are the same.

---

2\( F \) is called

- an \( \mathcal{F}_s^t \)-forward semimartingale if \((F_s(s + t, \omega))_{t \geq 0}\) is an \((\mathcal{F}_s^{s+t})_{t \geq 0}\)-semimartingale,
- an \( \mathcal{F}_s^t \)-backward semimartingale if \((F_s(s - t, \omega))_{t \geq 0}\) is an \((\mathcal{F}_s^{s-t})_{t \geq 0}\)-semimartingale,
- an \( \mathcal{F}_s^t \)-semimartingale if it is both forward and backward semimartingale.
It will be necessary to give a characterization that will connect semimartingale helix to the usual one-parameter semimartingale known to us from stochastic analysis (usual sense).

**Proposition 1.4.4 ([4])** Given a filtered dynamical system \( \theta \). Let \( F \) be an \( \mathbb{R}^d \)-valued forward (backward resp.) semimartingale helix. Let

\[
F(t) = M^+(t) + B^+(t), \quad t \geq 0 \quad (M^-(t) + B^-(t), \quad t \leq 0 \text{ resp.})
\]

be the canonical decomposition of the \((\mathcal{F}_t^t)_{t \geq 0} \quad ((\mathcal{F}_t^0)_{t \leq 0} \text{ resp.})\)-semimartingale \( F|_{[0,\infty)} \quad (F|_{(-\infty,0]} \text{ resp.})\). Then there exists

- a strictly increasing continuous real-valued \( \mathcal{F}_t^t \)-adapted process \( A^+(t), t \geq 0 \) with \( A^+(0,\omega) = 0 \) \((\mathcal{F}_t^0 \)-adapted process \( A^-(t), t \leq 0 \) with \( A^-(0,\omega) = 0 \) resp.),

- an \( \mathcal{F}_t^t \)-predictable \((\mathcal{F}_t^0 \text{-backward predictable})\) \( \mathbb{R}^d \)-valued process \( b^+(t), t \geq 0 \) \((b^-(t), t \leq 0 \text{ resp.})\) and

- an \( \mathcal{F}_t^t \)-predictable \((\mathcal{F}_t^0 \text{-backward predictable})\) process \( a^+(t), t \geq 0 \) \((a^-(t), t \leq 0 \text{ resp.})\) taking values in \( \mathcal{S}_d \) (the set of non-negative definite \( d \times d \) matrices),

such that for all \( t \geq 0 \) \((t \leq 0 \text{ resp.})\), \( \omega \in \Omega \) we have

\[
B^+(t, \omega) = \int_0^t b^+(s, \omega) dA^+(s, \omega), \\
B^-(t, \omega) = \int_0^t b^-(s, \omega) dA^-(s, \omega), \\
Q_{ij}^+(t, \omega) = \langle M^+_i(., \omega), M^+_j(., \omega) \rangle_t = \int_0^t a^+_{ij}(s, \omega) dA^+(s, \omega), \\
Q_{ij}^-(t, \omega) = \langle M^-_i(., \omega), M^-_j(., \omega) \rangle_t = \int_0^t a^-_{ij}(s, \omega) dA^-(s, \omega).
\]

**Definition 1.4.5 (Local characteristics of semimartingale helix [4])** Let \( \theta \) be a filtered dynamical system, let \( F \) be an \( \mathbb{R}^d \)-valued (forward resp., backward resp.) semimartingale helix. Then the quantities \((a^+, b^+, A^+) \quad ((a^-, b^-, A^-) \text{ resp.})\) are called the **forward (backward resp.) local characteristics** of \( F \).

For semimartingale helix \( F \), we write

\[
M(t) + B(t) := \begin{cases} 
M^+(t) + B^+(t), & t \geq 0 \\
M^-(t) + B^-(t), & t \leq 0 
\end{cases}
\]
and then

\[ F(t) = M(t) + B(t), \quad t \in \mathbb{R} \]

with

\[
(a(t), b(t), A(t)) = \begin{cases} 
(a^+(t), b^+(t), A^+(t)), & t \geq 0 \\
(a^-(t), b^-(t), A^-(t)), & t \leq 0 
\end{cases}.
\]

The triple \((a, b, A)\) on all of \(\mathbb{R}\) is called the local characteristics of \(F\) which satisfies for all \(t \in \mathbb{R}\) and \(\omega \in \Omega\)

\[
B(t, \omega) = \int_0^t b(s, \omega) dA(s, \omega)
\]

\[
Q_{ij}(t, \omega) = \langle M_i(\cdot, \omega), M_j(\cdot, \omega) \rangle_t = \int_0^t a_{ij}(s, \omega) dA(s, \omega).
\]

**Definition 1.4.6 (Semimartingale helix with spatial parameter [4])** Let \(\theta\) be a filtered dynamical system, \(m \in \mathbb{Z}^+, 0 \leq \delta \leq 1\). Assume that for each \(x \in \mathbb{R}^d\), \(F(t, \omega, x)\) is an \(\mathbb{R}^d\)-valued semimartingale helix and let its canonical decomposition be given by

\[
F(t, \omega, x) = M(t, \omega, x) + B(t, \omega, x), \quad t \in \mathbb{R}
\]

Then \(F\) is called \(C^{m, \delta}_b\)-semimartingale helix, if for \(\mathbb{P}\)-almost all \(\omega \in \Omega\):

- \(M(t, \omega, \cdot) \in C^{m, \delta}_b\) and \(B(t, \omega, \cdot) \in C^{m, \delta}_b\) for all \(t \in \mathbb{R}\),

- for \(|\alpha| \leq m\), the spatial derivative \(D^\alpha_x M(t, \omega, x)\) is continuous with respect to \((t, x)\) and for each \(x\) a local martingale and \(D^\alpha_x B(t, \omega, x)\) is continuous with respect to \((t, x)\) and for each \(x\) is of locally bounded variation in \(t\).

**Uniformly controlled local characteristics ([2], [4])**

We assume that for the semimartingale helix \(F(t, \omega, x)\) there exists a continuous adapted increasing process \(A(t, \omega)\) with \(A(0, \omega) = 0\) such that

- there is a measurable function \(b(t, \omega, x)\) which is a predictable process for every \(x\), such that for all \(x \in \mathbb{R}^d, \omega \in \Omega\)

\[
B(t, \omega, x) = \int_0^t b(s, \omega, x) dA(s, \omega),
\]
there exists a measurable function \( a(t, \omega, x, y) \) which for every \( x, y \in \mathbb{R}^d \) is predictable process, such that for all \( x, y \in \mathbb{R}^d, t \in \mathbb{R}, \omega \in \Omega \)

\[
Q(t, \omega, x, y) = \langle M(t, \omega, x), M(t, \omega, y) \rangle = \int_0^t a(s, \omega, x, y) dA(s, \omega).
\]

The triple \((a, b, A)\) is called the local characteristics of \( F \). We will say that \( F \) is uniformly controlled by \( A \).

**Definition 1.4.7 (Semimartingale cocycles)** \([2, 4]\) Let \( \varphi \) be a cocycle over a filtered dynamical system \( \theta \) for which

\[
G_s(t, \omega, x) = \varphi(t, \omega) \circ \varphi(s, \omega)^{-1} x - x
\]

is a \( C^{m,\delta} \)-semimartingale. Then \( \varphi \) is called a \( C^{m,\delta} \)-semimartingale cocycle.

Define \( G(t, \omega, x) := G_0(t, \omega, x) = \varphi(t, \omega)x - x \). The following lemma is a direct deduction from the definition 1.4.7.

**Lemma 1.4.8** Given a metric dynamical system \( \theta \). The following statements are equivalent:

(i) \( \varphi \) is a cocycle over \( \theta \).

(ii) \( G_s(t, \omega, x) = G(t - s, \theta_s \omega, x) \).

(iii) \( G \) is a helix over the skew product \( \Theta = (\theta, \varphi) \), that is,

\[
G(t + s, \omega, x) = G(t, \theta_s \omega, \varphi(s, \omega)x) + G(s, \omega, x).
\]

**Theorem 1.4.9 (Global RDS from Stratonovich SDEs)** \([2, 4, 48]\) Given a filtered dynamical system \( \theta \). Let \( F \) be \( C^{m,\delta}_b \)-semimartingale helix over \( \theta \), where \( m \geq 1 \) and \( \delta \geq 0 \). Suppose that \( F \) has local characteristics \((a_F, b_F, A_F)\) such that \( a_F \in L_{loc}(\mathbb{R}, dA_F, C^{m+1,\delta}_b) \), \( b_F \in L_{loc}(\mathbb{R}, dA_F, C^{m,\delta}_b) \) and \( c_F \in L_{loc}(\mathbb{R}, dA_F, C^{m,\delta}_b) \), where \( c_F \) is the Stratonovich-Itô correction term given below. Then there exists a unique (up to indistinguishability) global \( C^m \)-RDS \( \varphi \) over \( \theta \) which for any \( \varepsilon < \delta \) is a \( C^{m,\varepsilon} \)-semimartingale cocycle which solves the SDE

\[
dx_t = F(x_t, dt)
\]

definition is in appendix C,
(this holds for fixed $t$ and $x$ outside $\mathbb{P}$-null set $N_{t,x}$). The semimartingale cocycle $\varphi$ has the local characteristics

$$A_\varphi = A_F,$$

$$a_\varphi(t, \omega, x, y) = a_F(t, \omega, \varphi(t, \omega)x, \varphi(t, \omega)y)$$

and

$$b_\varphi(t, \omega, x) = b_F(t, \omega, \varphi(t, \omega)x) + c_F(t, \omega, \varphi(t, \omega)x)$$

where

$$c_F(t, \omega, x) := \frac{1}{2} \sum_{j=1}^{d} \frac{\partial a_j}{\partial x_j}(t, \omega, x, y)|_{y=x}.$$  

We now state the converse of the above theorem.

**Theorem 1.4.10 (Stratonovich SDE from RDS [2], [4], [48] )** Given a filtered DS $\theta$. Let $\varphi$ be a $C^m$ RDS over $\theta$ such that

$$G_s(t, \omega, x) := \varphi(t, \omega) \circ \varphi(s, \omega)^{-1}x - x$$

$$= M_\varphi(t - s, \theta_s \omega, x) + B_\varphi(t - s, \theta_s \omega, x)$$

is a $C^{m, \delta}$-semimartingale with local characteristics $(a_\varphi, b_\varphi, A_\varphi)$ satisfying $a_\varphi \in L_{loc}(\mathbb{R}, dA_\varphi, \tilde{C}^{m, \delta})$, $b_\varphi \in L_{loc}(\mathbb{R}, dA_\varphi, C^{m, \delta})$ for some $m \geq 3$ and $\delta \geq 0$. Then there exists a unique (up to distinguishability) stochastic process $F(t, \omega, x)$ which for some $\varepsilon < \delta$ is a $C_{m-1, \varepsilon}$-semimartingale helix, such that $\varphi$ is generated by $F$, that is, $\varphi$ and $F$ satisfy

$$\varphi(t, \omega)x - x = \begin{cases} \int_{0}^{t} F(\varphi(s, \omega)x, \circ d^+s), & t \geq 0, \\ -\int_{t}^{0} F(\varphi(s, \omega)x, d^-s), & t \leq 0. \end{cases}$$

The local characteristics of $F$ are obtained from those of $\varphi$ as follows

$$A_F = A_\varphi,$$

$$a_F(t, \omega, x, y) = a_\varphi(t, \omega, \varphi(t, \omega)^{-1}x, \varphi(t, \omega)^{-1}y),$$

$$b_F(t, \omega, x) = b_\varphi(t, \omega, \varphi(t, \omega)^{-1}x) - d_\varphi(t, \omega, x)$$

where

$$d_\varphi(t, \omega, x) := \frac{1}{2} \sum_{j=1}^{d} \frac{\partial a_j}{\partial x_j}(t, \omega, \varphi(t, \omega)^{-1}x, \varphi(t, \omega)^{-1}y)|_{y=x}.$$
In sum, we have seen that the following classes of objects are basically the same:

1. Solutions of the SDE \( dx_t = F(x_t, \theta_t) dt \) given by a semimartingale helix over \( \theta \).

2. Semimartingale cocycles over \( \theta \).

It is well known that the technique discussed so far is the most general one for the generation of RDS, as we know that the semimartingales are the most general class of reasonable Itô stochastic integrators.

1.4.4 RDS from classical stochastic differential equations

Here we wish to make the discussions in the previous section little bit concrete by considering classical SDEs, that is SDEs driven by White noise (loosely speaking; the derivative of a Wiener process). We shall take our semimartingale helix \( F(t, \omega, x) \) to be

\[
F(t, \omega, x) := f_0(x) t + \sum_{j=1}^{n} f_j(x) B_j(t, \omega)
\]

Suppose that \( f_0 \in C_b^{m, \delta}, f_1, f_2, \ldots, f_n \in C_b^{m+1, \delta}, m \geq 1, \delta > 0 \), then \( F(t, \omega, x) \) define above is a \( C_b^{m, \delta} \)-semimartingale helix with the local characteristics \( (a, b, A) \) given by the following

\[
A(t, \omega) \equiv t, \quad a(t, x, y) = \sum_{j=1}^{n} f_j(x) f_j^T(y), \quad b(x) = f_0(x),
\]

and Stratonovich-Itô correction term given by

\[
c(t, x) = \frac{1}{2} \sum_{j=1}^{d} \sum_{i=1}^{n} f_j^i(x) \frac{\partial}{\partial x_i} f_j(x).
\]

By carefully looking at the definition of \( L_{loc}(\mathbb{R}, dA, C_b^{m, \delta}) \) with \( dA \equiv dt \) it is straightforward to see that \( b \in L_{loc}(\mathbb{R}, dt, C_b^{m, \delta}) \) if and only if, \( b = f_0 \in C_b^{m, \delta} \). Similarly, \( a \in L_{loc}(\mathbb{R}, dt, C_b^{m+1, \delta}) \), if and only if \( a \in C_b^{m+1, \delta} \) this is guaranteed by the supposition that \( f_1, f_2, \ldots, f_n \in C_b^{m+1, \delta} \). In fact, we have the following theorem.

**Theorem 1.4.11 (RDS from classic Stratonovich SDE [2], [4], [47], [48])** Let \( f_0 \in C_b^{m, \delta}, f_1, f_2, \ldots, f_n \in C_b^{m+1, \delta} \) and \( \sum_{j=1}^{n} \sum_{i=1}^{d} f_j^i \frac{\partial}{\partial x_i} f_j \in C_b^{m, \delta} \) and \( \delta > 0 \). Then:

i. The classical Stratonovich SDE

\[
\begin{align*}
\text{d}X_t &= f_0(X_t) \text{d}t + \sum_{j=1}^{n} f_j(x) \circ \text{d}W_t^j, \quad t \in \mathbb{R}
\end{align*}
\]
generates a unique (up to indistinguishability) a \( C^m \)-RDS \( \varphi \) over the filtered DS \( \theta \) describing the Brownian motion. For any \( \varepsilon \in (0, \delta) \), \( \varphi \) is a \( C^{m,\varepsilon} \)-semimartingale cocycle and \( (t, x) \mapsto \varphi(t, \omega)x \) belongs to \( C^{\beta,m,\varepsilon}_b \) for \( \beta < \frac{1}{2} \) and \( \varepsilon < \delta \).

ii. The RDS \( \varphi \) has stationary independent increments, in the sense that for all \( t_0 \leq t_1 \leq \cdots \leq t_n \) the random variables

\[
\varphi(t_1) \circ \varphi(t_0)^{-1}, \varphi(t_2) \circ \varphi(t_1)^{-1}, \cdots, \varphi(t_n) \circ \varphi(t_{n-1})^{-1}
\]

are independent, and the distribution of \( \varphi(t + h) \circ \varphi(t)^{-1} \) is independent of \( t \).

iii. If \( D\varphi(t, \omega)x \) denote the Jacobian of \( \varphi(t, \omega) \) at \( x \) then the pair \( (\varphi, D\varphi) \) is a \( C^{m-1} \)-RDS uniquely generated by \ref{1.4.12} together with

\[
dY_t = Df_0(X_t)Y_t dt + \sum_{j=1}^{n} Df_j(X_t)Y_t \circ dW^j_t, \quad t \in \mathbb{R}.
\]

Hence, \( D\varphi \) uniquely solves the variational Stratonovich SDE on \( \mathbb{R} \),

\[
D\varphi(t, \omega)x = I + \int_0^t Df_0(\varphi(s, \omega)x)D\varphi(s, \omega)x ds + \sum_{j=1}^{n} \int_0^t Df_j(\varphi(s, \omega)x)D\varphi(s, \omega)x ds dW^j_s, \quad t \in \mathbb{R},
\]

and thus a matrix cocycle over \( \Theta = (\theta, \varphi) \).

Finally, the determinant \( \det D\varphi(t, \omega)x \) satisfies Liouville’s equation on \( \mathbb{R} \),

\[
\det D\varphi(t, \omega)x = \exp \left( \int_0^t \text{trace} Df_0(\varphi(s, \omega)x) ds + \sum_{j=1}^{n} \int_0^t \text{trace} Df_j(\varphi(t, \omega)x) \circ dW^j_s \right), \quad t \in \mathbb{R},
\]

and hence a scalar cocycle over \( \Theta \).

The rigorous proof of above theorem is easily accessible through the classical lecture note by Kunita \cite{Kunita}.

**Example 1.4.12 (Affine Stratonovich SDE \cite{Kunita})** Consider on \( \mathbb{R} \) the affine SDE

\[
dX_t = (A_0X_t + b_0) dt + \sum_{j=1}^{n} (A_jX_t + b_j) \circ dW^j_t, \quad A_0, A_1, \cdots A_n \in \mathbb{R}^{d \times d}, \quad b_0, b_1, \cdots, b_n \in \mathbb{R}^d.
\]

Since \( f_j(x) = A_jx + b_j \in C_b^\infty \), \( j = 0, 1, \cdots, n \) and \( (\sum_{j=1}^{n} A_j^2)x \in C_b^\infty \), it generates uniquely a global \( C^\infty \)-RDS \( \varphi \), which consists of affine mappings given by the variation of constants formula

\[
\varphi(t, \omega)x = \Gamma(t, \omega) \left( x + \int_0^t \Gamma(s, \omega)^{-1}b_0 ds + \sum_{j=1}^{n} \int_0^t \Gamma(s, \omega)^{-1}b_j \circ dW^j_s \right),
\]
where $\Gamma$ is the fundamental matrix of the corresponding linear SDE

$$dX_t = A_0 X_t dt + \sum_{j=1}^{n} A_j X_t \circ dW^j_t$$

which is a linear RDS over $\theta$. 
Chapter 2

Random Ergodic Theory

In this chapter, we shall discuss some of these geometric structures relevant to what follows in the subsequent chapters.

2.1 Random invariant probability measures

A random probability measure is a notion for specifying a probability measure on the space of probability. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, let \( X \) be a polish space equipped with the metric \( d \) and let \( (\Omega \times X, \mathcal{F} \otimes B) \) be a measurable space.

\[
\pi_\Omega : \Omega \times X \to \Omega, \quad \pi_\Omega(\omega, x) = \omega
\]

and

\[
\pi_X : \Omega \times X \to X, \quad \pi_X(\omega, x) = x
\]

are measurable cannonical projections.

Definition 2.1.1 (Random probabilty measures [27]) A map \( \mu : \Omega \times \mathcal{B} \to [0, 1], (\omega, B) \mapsto \mu_\omega(B) \) satisfying

1. for every \( B \in \mathcal{B}, \omega \mapsto \mu_\omega(B) \) is \( \mathcal{F} \)-measurable and

2. for \( \mathbb{P} - a.e \omega \in \Omega, B \mapsto \mu_\omega(B) \) is a probability measure,

is called a random probability measure on \( X \) and it is denoted by \( \omega \mapsto \mu_\omega \).
Some specific examples of random probability are known by several names such as transition probabilities or probability kernels. Random probability measures are usually characterize using the following proposition.

**Proposition 2.1.2** ([27]) Suppose that $\omega \mapsto \mu_\omega$ is a random probability measure. Then

(i) for all measurable function $h : \Omega \times X \to \mathbb{R}$ with $h$ bounded or nonnegative, the map

$$\omega \mapsto \int_X h(\omega, x) \mu_\omega(dx)$$

is measurable,

(ii) the assignment

$$A \mapsto \int_\Omega \int_X 1_A(\omega, x) \mu_\omega(dx) \mathbb{P}(d\omega), \quad A \in \mathcal{F} \otimes \mathcal{B}$$

defines a probability measure on $\Omega \times X$, which is denoted by $\mu$ and the marginal of $\mu$ on $\Omega$ is $\mathbb{P}$.

**Definition 2.1.3** (Invariant measure for RDS [4],[27]) Given a measurable RDS $\varphi$ over $\theta$, a probability measure $\mu$ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ is said to be invariant with respect to $\varphi$ or $\varphi$-invariant, if it satisfies

- $\Theta_t \mu = \mu$, for all $t \in T$,
- $\pi_\Omega \mu = \mathbb{P}$.

Define the sets

$$\mathcal{P}_\mathbb{P}(\Omega \times X) := \{\mu \text{ probability on } (\Omega \times X, \mathcal{F} \otimes \mathcal{B}), \text{ with } \pi_\Omega \mu = \mathbb{P} \text{ on } (\Omega, \mathcal{F}, \mathbb{P})\}$$

and

$$\mathcal{I}_\mathbb{P}(\varphi) := \{\mu \in \mathcal{P}_\mathbb{P}(\Omega \times X) : \mu \text{ is } \varphi\text{-invariant}\}.$$ 

The sets $\mathcal{P}_\mathbb{P}(\Omega \times X)$ and $\mathcal{I}_\mathbb{P}(\varphi)$ are convex.

**Remark 2.1.4** (i) For two-sided time, it is enough to require the first condition of Definition 2.1.3 only for $t > 0$, since $\Theta_{-t} \mu = \Theta_{-t} \circ \Theta_t \mu = \mu$.

(ii) For discrete time, the first condition of Definition 2.1.3 follows, if $\Theta_1 \mu = \mu$.

Suppose $\mu \in \mathcal{P}_\mathbb{P}(\Omega \times X)$, the function $\mu(.) : \Omega \times X \to [0,1]$ is called a factorization (or disintegration or sample measure) if:
it is a random probability measure on \( X \) and

for all \( A \in \mathcal{F} \otimes \mathcal{B} \),

\[
\mu(A) = \int_\Omega \int_X \mathbb{1}_A(\omega, x) \mu_\omega(dx) \mathbb{P}(d\omega).
\]

We denote it by

\[
\mu(d\omega, dx) = \mu_\omega(dx) \mathbb{P}(d\omega)
\]

We could write equation (2.1.1) as

\[
\mu(A) = \int_\Omega \mu_\omega(A_\omega) \mathbb{P}(d\omega),
\]

where \( A_\omega := \{x \in X : (\omega, x) \in A\} \subset X \) is \( \omega \)-section of \( A \).

**Proposition 2.1.5 (Existence and uniqueness of factorization [4], [27])** Let \( \mu \in \mathcal{P}_\mathcal{P}(\Omega \times X) \) and suppose

(i) \( \mathcal{B} \) is countably generated,

(ii) the marginal \( \pi_\Omega \mu \) on \((X, \mathcal{B})\) can be compactly approximated.

Then a factorization of \( \mu \) exists and \( \mathbb{P} - a.s. \) unique.

**Definition 2.1.6 (Conditional expectation of random probability measure)** Let \( G \subset \mathcal{F} \) be a sub-\( \sigma \)-algebra and \( \mu \in \mathcal{P}_\mathcal{P}(\Omega \times X) \). If the factorization of \( \mu \) exists and almost surely unique. Then \( \mu(.) : G \times \mathcal{B} \rightarrow [0, 1] \) with respect to the probability measure \( \mathbb{P}|_G \) restricted on \( G \), is called the conditional expectation of \( \mu \) with respect to \( G \), denoted by \( \omega \mapsto \mathbb{E}(\mu|G)_\omega \) and that

\[
\mathbb{E}(\mu|G)_\omega(B) = \mathbb{E}(\mu(B)|G)_\omega, \quad \mathbb{P} - a.s., \quad \text{for all } B \in \mathcal{B}.
\]

The philosophy is that noise is something given to us and not in our disposal, we require that the marginal \( \pi_\Omega \mu = \mathbb{P} \), so, it becomes reasonable to express the invariance of the measure \( \mu \) with respect to the RDS \( \psi \) in terms its factorization (if the factorization exists).

**Lemma 2.1.7 (Factorization of image measure [4])** Let \( \psi \) be a measurable RDS on a polish space \( X \) and let \( \mu \in \mathcal{P}_\mathcal{P}(\Omega \times X) \). If \( \theta \) is measurably invertible, then the factorization of \( \Theta_\psi \mu \) is given by

\[
(\Theta_\psi \mu)_\omega = \varphi(t, \theta^{-1}_t \omega) \mu_{\theta^{-1}_t \omega} = \varphi(-t, \omega) \mu_{\theta_t^{-1} \omega}, \quad \mathbb{P} - a.s.,
\]

where the second equality holds for two-sided time.
**Theorem 2.1.8 (Invariance in terms of factorization [4])** Let $\varphi$ be a measurable RDS on a polish space $(X, \mathcal{B})$ and let $\mu \in \mathcal{P}_\mathbb{P}(\Omega \times X)$. Then

1. $\mu \in \mathcal{I}_\mathbb{P}(\varphi)$ if and only if, for all $t \in \mathbb{T}$,
   \[ \mathbb{E}(\varphi(t,.)\mu|_{\theta_t^{-1}\mathcal{F}}) = \mu_{\theta_t}\omega, \quad \mathbb{P} - \text{a.s.} \]

2. If $\theta$ is measurable invertible, then $\theta_t^{-1}\mathcal{F} = \mathcal{F}$ for all $t \in \mathbb{T}$ and $\mu \in \mathcal{I}_\mathbb{P}(\varphi)$ if and only if, for all $t \in \mathbb{T}$
   \[ \varphi(t,\omega)\mu_{\omega} = \mu_{\theta_t}\omega, \quad \mathbb{P} - \text{a.s.} \]

**Remark 2.1.9**

(i) If the property
   \[ \varphi(t,\omega)\mu_{\omega} = \mu_{\theta_t}\omega, \quad \mathbb{P} - \text{a.s.}, \tag{2.1.4} \]
   is satisfied, then $\mu_{\omega}$ is called equivariant with respect to $\varphi$ over $\theta$.

(ii) Equation (2.1.4) is sufficient for
   \[ \mathbb{E}(\varphi(t,.)\mu|_{\theta_t^{-1}\mathcal{F}}) = \mu_{\theta_t}\omega, \quad \mathbb{P} - \text{a.s} \tag{2.1.5} \]
   and thus for the invariance of $\mu$ also in the case where $\theta$ is not necessarily invertible.

(iii) Let $\mu \in \mathcal{I}_\mathbb{P}(\varphi)$ with $\rho = \pi_X\mu = \mathbb{E}(\mu)$ on $X$. Then equation (2.1.5) and invariance of $\mathbb{P}$ over $\theta$, gives us that
   \[ \rho_t := \mathbb{E}(\varphi(t,.)\mu) = \rho, \quad \forall t \in \mathbb{T}. \tag{2.1.6} \]

Up to this moment we know that any probability measure $\mu \in \mathcal{P}_\mathbb{P}(\Omega \times X)$ has $\mathbb{P} - \text{a.s}$ unique factorization
   \[ \mu(d\omega, dx) = \mu_{\omega}(dx)\mathbb{P}(d\omega) \]
and $\mu$ can be identified with this factorization. We shall recall a lemma that will be crucial in what follows in this section.

**Lemma 2.1.10 ([17], [27])** Suppose that $X$ is a separable metric space, $(\Omega, \mathcal{F})$ is a measurable space and $Y$ is another metric space. Assume that $f: \Omega \times X \to Y$ satisfies the following

- $\omega \mapsto f(\omega, x)$ is measurable for all $x \in X$ and
- $x \mapsto f(\omega, x)$ is continuous for all $\omega \in \Omega$. 

34
Then \( f \) is jointly measurable.

**Proof:** Let \( D \) be a countable dense set in \( X \). Given \( n \in \mathbb{N} \), choose a subset \( D_n \subset D \) and a partition \( \mathcal{V}_n \) of \( X \), consisting of measurable subsets, such that \( \mathcal{V}_n(x) \subset B(x, \frac{1}{n}) \) for all \( x \in D_n \), where \( \mathcal{V}(x) \) denotes the element of the partition \( \mathcal{V} \) containing \( x \).

Put

\[
f_n(\omega, x) = \sum_{y \in D_n} f(\omega, y)\mathbb{1}_{\mathcal{V}_n(y)}(x),
\]

this summation is just an abbreviating notation.

If \( G \subset Y \) is Borel set, then one can show that

\[
f_n^{-1}(G) = \bigcup_{y \in D_n} f(\omega, y)^{-1}(G) \times \mathcal{V}_n(y),
\]

it follows that \( f_n \) is jointly measurable.

On the other hand, for \((\omega, x) \in \Omega \times X\)

\[
d(f(\omega, x), f_n(\omega, x)) \leq \sup_{\{y: d(x, y) < \frac{1}{n}\}} d(f(\omega, x), f(\omega, y))
\]

hence,

\[
f(\omega, x) = \lim_{n \to \infty} f_n(\omega, x), \quad \forall (\omega, x) \in \Omega \times X.
\]

Let \( C_b(X) \) be the space of real-valued bounded continuous functions with 'sup norm' \( \|f\|_b := \sup_{x \in X} |f(x)| \). By Lemma 2.1.10, we can say that a function \( f: \Omega \to C_b(X) \) is measurable, if \((\omega, x) \mapsto f(\omega, x)\) is measurable.

\[
\mathbb{L}_b^1(\Omega, C_b(X)) = \{ f: \Omega \to C_b(X) \text{ measurable}, \|f\| = \int_{\Omega} \|f(\omega, \cdot)\|_b d\mu < \infty \}
\]

where, as usual \( f \) and \( g \) are identified if \( \|f - g\| = 0 \).

**Remark 2.1.11** For each \( f \in \mathbb{L}_b^1(\Omega, C_b(X)) \) and \( \mu \in \mathcal{P}(\Omega \times X) \) we have \( f \in L^1(\mu) \) and

\[
\mu(f) = \int_{\Omega \times X} f d\mu
\]

with

\[
|\mu(f)| \leq \|f\|.
\]

Further,

- \( f \mapsto \mu(f) \) is linear for each \( \mu \),
- \( \mu \mapsto \mu(f) \) is affine for each \( f \) (because of the convexity of \( \mu \mapsto \mu(f) \)).
Definition 2.1.12 (Narrow topology or topology of weak convergence [4], [27]) The smallest topology in \( P_\mathcal{P}(\Omega \times X) \) which makes \( \mu \mapsto \mu(f) \) continuous for each \( f \in L^1_\mathcal{P}(\Omega, C_b(X)) \) is called the **Narrow topology or topology of weak convergence** on \( P_\mathcal{P}(\Omega \times X) \).

A neighbourhood basis for the narrow topology in \( \nu \in P_\mathcal{P}(\Omega \times X) \) is given by

\[
U_{f_1, \ldots, f_n; \delta}(\nu) = \{ \mu \in P_\mathcal{P}(\Omega \times X); |\mu(f_k) - \nu(f_k)| < \delta, \quad k = 1, \ldots, n \}
\]

where \( n \in \mathbb{N}, \quad f_1, \ldots, f_n \in L^1_\mathcal{P}(\Omega, C_b(X)) \) and \( \delta > 0 \).

A net \( (\mu^\alpha)_{\alpha \in \Delta} \) converges in this topology to \( \mu \) if \( \mu^\alpha(f) \to \mu(f) \) for \( f \in L^1_\mathcal{P}(\Omega, C_b(X)) \).

Assume that \( P_\mathcal{P}(\Omega \times X) \) and \( P(X) \) are equipped with their respective topologies of weak convergence. Then the projection \( \pi_X : P_\mathcal{P}(\Omega \times X) \to P(X) \) define by assigning to each \( \mu \in P_\mathcal{P}(\Omega \times X) \) its marginal \( \rho = \pi_X(\mu) = \mathbb{E}(\mu) \) on \( X \) is continuous.

Now, we consider random invariant measures on random sets; here shall discuss all those random measures \( \mu \in I_\mathcal{P}(\varphi) \) for which

\[
\mu(A) = 1, \quad \text{for some } A \in \mathcal{F} \otimes \mathcal{B},
\]

equivalently

\[
\mu_\omega(A_\omega) = 1, \quad \mathbb{P} - a.s.
\]

We shall be concerned with the set valued maps

\[
A : \Omega \to 2^X.
\]

A set valued map is uniquely determined by its graph

\[
\text{graph}(A) = \{ (\omega, x); x \in A_\omega \} \subset \Omega \times X.
\]

Definition 2.1.13 (Random closed and open set [4], [27]) A set valued map \( A : \Omega \to 2^X \) taking values as a closed subset of a polish space \( X \), is said to be measurable, if for each \( x \in X \), the map \( \omega \mapsto d(x, A_\omega) \) is measurable. In this case, \( A \) is called a **random closed** set.

A set valued map \( \omega \mapsto U_\omega \) is said to be a **random open** set if its complement is a random closed set.

Random sets are referred in some books as measurable multifunctions.

---

\(^1\) A net \( (\mu^\alpha)_{\alpha \in \Delta} \) is a generalized sequence, where \( \Delta \) is a directed set.
Proposition 2.1.14 (Random selection [4],[17],[27]) The set valued map $K$ is a random closed set if and only if there exists a sequence $(k_n)_{n \in \mathbb{N}}$ of measurable maps $k_n : \Omega \to X$ such that

$$K(\omega) = \text{closure}\{k_n(\omega) : n \in \mathbb{N}\} \quad \text{for all } \omega \in \Omega.$$ 

In particular, if $K$ is a random closed set, then there exists a measurable selection, that is, a measurable map $k : \Omega \to X$ such that $k(\omega) \in K(\omega)$ for all $\omega \in \Omega$.

Definition 2.1.15 (Invariant sets of RDS [4],[27] ) Let $\varphi$ be a measurable RDS and $A \subset \Omega \times X$.

(a) $A$ is called forward invariant, if for $t > 0$

$$A_\omega \subset \varphi(t, \omega)^{-1}A_{\theta_t \omega}, \quad \mathbb{P} - a.s.$$ 

(b) $A$ is called invariant, if for all $t \in \mathbb{T}$

$$A_\omega = \varphi(t, \omega)^{-1}A_{\theta_t \omega}, \quad \mathbb{P} - a.s,$$

for two-sided time, this is equivalent to

$$\varphi(t, \omega)A_\omega = A_{\theta_t \omega}, \quad \mathbb{P} - a.s.$$ 

Remark 2.1.16 Strictly (forward) invariant if there is no exceptional set.

Definition 2.1.17 (Support of random measure [4],[27]) A probability measure $\mu \in \mathcal{P}_p(\Omega \times X)$ is said to be supported by a measurable set $\omega \mapsto A_\omega$, if

$$\mu(A) = 1,$$

where $A = \{(\omega, x) ; x \in A_\omega\}$ is the graph of the mapping $\omega \mapsto A_\omega$. Equivalently, if

$$\mu_\omega(A_\omega) = 1, \quad \mathbb{P} - a.s.$$ 

Lemma 2.1.18 ([4],[27]) Let $(\mu^\alpha)_{\alpha \in \Delta}$ be a net in $\mathcal{P}_p(\Omega \times X)$ converging to $\mu$ in the narrow topology. Then for each random closed set $A$

$$\limsup_{\alpha} \mu^\alpha(A) \leq \mu(A)$$

and for each random open set $U$

$$\liminf_{\alpha} \mu^\alpha(U) \geq \mu(U).$$
**Proof.** Let $A$ be a random closed set. Then $\omega \mapsto d(x, A_\omega)$ is measurable for all $x$ and clearly $x \mapsto d(x, A_\omega)$ is continuous for all $\omega$. Hence

$$f_n(\omega, x) := (1 - nd(x, A_\omega))^+ \in L^1(\Omega, C_b(X))$$

with $f_n \geq \mathbb{1}_A$ and $f_n \downarrow \mathbb{1}_A$.

Recall: for a net $(x^\alpha)_{\alpha \in \Delta}$, where $\Delta$ is a directed set,

- we say that $(x^\alpha)_{\alpha \in \Delta}$ converges to $x$ and write

$$\lim_{\alpha \in \Delta} x^\alpha = x,$$

if and only if for every neighbourhood $U$ of $x$, $(x^\alpha)_{\alpha \in \Delta}$ is eventually in $U$.

- For limit superior, we put

$$\limsup_{\alpha \in \Delta} x^\alpha = \limsup_{\alpha \in \Delta} \sup_{\beta \geq \alpha} x^\beta = \inf_{\alpha \in \Delta} \sup_{\beta \geq \alpha} x^\beta.$$

Now for each fixed $n$

$$\limsup_{\alpha} \mu^\alpha(A) \leq \lim_{\alpha} \mu^\alpha(f_n) = \mu(f_n).$$

Consequently,

$$\limsup_{\alpha} \mu^\alpha(A) \leq \inf_{n} \mu(f_n) = \mu(A).$$

And the second inequality follows by applying the first assertion on the complement of $U$. □

**Corollary 2.1.19 ([4])** Let $A$ be a random closed set.

(i) The set $\Gamma := \{ \mu \in \mathcal{P}(\Omega \times X) : \mu_\omega(A_\omega) = 1, \ P - a.s \}$ is convex and closed.

(ii) Let $\varphi$ be a continuous RDS and $A$ be a random closed set. Then the set of $\varphi$-invariant measures supported by $A$, denoted by

$$M_\varphi(A) := \mathcal{I}_\varphi(\varphi) \cap \Gamma = \{ \mu \in \mathcal{I}_\varphi(\varphi) : \mu_\omega(A_\omega) = 1 \}$$

is convex and closed.

**Proof.**

(i) If $\mu^\alpha \to \mu$ and $\mu^\alpha(A) = 1$, then by the previous lemma (Lemma 2.1.18) $\mu(A) = 1$.

(ii) $M_\varphi(A)$ is the intersection of two convex and closed sets. □
Proposition 2.1.20 ([4]) Let $\varphi$ be a continuous RDS and $\mu \in \mathcal{I}_{\varphi}(\varphi)$. Then the random closed $A_\omega := \text{supp}\mu_\omega$ is

(i) forward invariant if $\theta$ is invertible.

(ii) invariant if time is two-sided.

Proof. For any continuous function $f : X \to X$ and any Borel measure $\mu$,

$$f(\text{supp}\mu) \subset \text{supp}(f\mu).$$

(i) For invertible $\theta$, $\mu$ is $\varphi$-invariant if, and only if $\varphi(t, \omega)\mu_\omega = \mu_{\theta t \omega}, \ \mathbb{P} - a.s.$

$$\varphi(t, \omega)A_\omega = \varphi(t, \omega)\text{supp}\mu_\omega$$

$$\subset \text{supp}(\varphi(t, \omega)\mu_\omega)$$

$$= \text{supp}(\mu_{\theta t \omega})$$

$$= A_{\theta t \omega}. $$

Thus, $\varphi(t, \omega)A_\omega \subset A_{\theta t \omega}, \ \mathbb{P} - a.s.$

(ii) For a homeomorphism $f$, $f(\text{supp}\mu) = \text{supp}(f\mu)$. We know that when time is two sided, the RDS $\varphi(t, \omega)$ is a homeomorphism for each $(t, \omega) \in \mathbb{T} \times \Omega$, and thus we have

$$\varphi(t, \omega)A_\omega = \varphi(t, \omega)\text{supp}\mu_\omega$$

$$= \text{supp}(\varphi(t, \omega)\mu_\omega)$$

$$= \text{supp}(\mu_{\theta t \omega})$$

$$= A_{\theta t \omega}. $$

Definition 2.1.21 (Tight measures) Let $\mathcal{P}(X)$ be a space of probability measures on $X$, a set $G \subset \mathcal{P}(X)$ is said to be tight if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that

$$\rho(K_\varepsilon) \geq 1 - \varepsilon, \ \text{for all} \ \rho \in G.$$

Definition 2.1.22 (Tight random measures) A subset $\Gamma \subset \mathcal{P}(\Omega \times X)$ is said to be tight, if $\pi_X(\Gamma) \subset \mathcal{P}(X)$ is tight in the sense of definition 2.1.21. Thus, $\Gamma$ is tight if for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset X$ such that

$$\mu(\Omega \times K_\varepsilon) \geq 1 - \varepsilon, \ \text{for all} \ \mu \in \Gamma.$$
Alternatively, \( \Gamma \subset \mathcal{P}(\Omega \times X) \) is tight, if for every \( \varepsilon > 0 \), there exists a compact set \( \omega \mapsto K_{\varepsilon}(\omega) \), such that for every \( \mu \in \Gamma \),

\[
\mu(K_{\varepsilon}) = \int_{\Omega} \mu_{\omega}(K_{\varepsilon}(\omega))d\mathbb{P}(\omega) \geq 1 - \varepsilon.
\]

This means that \( \Gamma \subset \mathcal{P}(\Omega \times X) \) is tight, if and only if, its projection to the set of deterministic measure \( \pi_X(\Gamma) \subset \mathcal{P}(X) \) is tight. Roughly speaking, tight subset \( \Gamma \subset \mathcal{P}(\Omega \times X) \), gives us the collection of random probability measures that converges in the narrow topology.

**Theorem 2.1.23 (Prohorov Theorem for random measures)** \([4],[27]\) A subset \( \Gamma \subset \mathcal{P}(\Omega \times X) \) is tight if, and only if, it is relatively compact with respect to the narrow topology. In this case it is relatively sequentially compact (i.e., if \( (\mu^n)_{n\in\mathbb{N}} \) is a sequence in \( \Gamma \), then there exists a convergent subsequence \( (\mu^{n_k})_{k\in\mathbb{N}}) \).

**Lemma 2.1.24** \([4],[27]\) Let \( X \) be a metric space and \( \mu, \nu \in \mathcal{P}(\Omega \times X) \). Then \( \mu = \nu \) if, and only if, \( \mu(f) = \nu(f) \) for all \( f \in \mathbb{L}^1_{\mathbb{P}}(\Omega, C_b(X)) \)

**Proof:** This direction \( \Rightarrow \) is obvious.

For the other direction, it is enough to show that

\[
\mu(A \times F) = \nu(A \times F), \quad \forall A \in \mathcal{F} \text{ and all closed } F.
\]

Let

\[
f_n(\omega, x) = \mathbb{1}_A(\omega)g_n(x) \in \mathbb{L}^1_{\mathbb{P}}(\Omega, C_b(X))
\]

with \( g_n \downarrow \mathbb{1}_F \). But \( g_n(x) = (1 - nd(x, F))^+ \) is such a sequence, since \( d(x, F) = 0 \) if, and only if, \( x \in F \). \( \square \)

**Proposition 2.1.25** \([4]\) Let \( \varphi \) be a continuous RDS on a Polish space \( X \). Then

1. The mappings \( f \mapsto \Theta_t f, \quad t \in \mathbb{T} \), are commuting family of continuous linear mappings of \( \mathbb{L}^1_{\mathbb{P}}(\Omega, C_b(X)) \) to itself (isometries for two-sided time).

2. The mappings \( \mu \mapsto \Theta_t \mu, \quad t \in \mathbb{T} \) are commuting family of affine mappings of \( \mathcal{P}(\Omega \times X) \) to itself, which are continuous with respect to the Narrow topology.

**Proof:** The collection \( (\Theta_t)_{t\in\mathbb{T}} \) is commuting, since

\[
\Theta_t \circ \Theta_s = \Theta_{t+s} = \Theta_{s+t} = \Theta_s \circ \Theta_t \quad \forall t \in \mathbb{T}.
\]
1. For fixed \( t \), \( \Theta_t f(\omega, x) = f(\theta_t \omega, \varphi(t, \omega)x) \) is measurable and for fixed \( \omega \) continuous with respect to \( x \). But \( \| \Theta_t f(\omega, \cdot) \|_b \leq \| f(\theta_t \omega, \cdot) \|_b \) (which is equality for two-sided time, since \( \varphi(t, \omega) \) is bijective).

\[
\| \Theta_t f \| \leq \| f \|
\]

with equality for two-sided time. Showing that \( \Theta_t \) is bounded and linearity is obvious. Hence, \( f \mapsto \Theta_t f \) is continuous.

2. \( \Theta_t \) is affine, since by definition \( \Theta_t (\alpha \mu + (1 - \alpha) \nu) = \alpha \Theta_t \mu + (1 - \alpha) \Theta_t \nu \) for all \( \alpha \in [0, 1] \). We prove that \( \Theta_t \) is continuous. Let \( (\mu^\alpha)_{\alpha \in \Delta} \) be a net, such that \( \mu^\alpha \to \mu \), i.e., \( \mu^\alpha(f) \to \mu(f) \), for all \( f \in L^1_P(\Omega, C_b(X)) \). Then

\[
\Theta_t \mu^\alpha(f) = \mu^\alpha(\Theta_t f) \to \mu(\Theta_t f) = \Theta_t \mu(f)
\]

and \( \Theta_t f \in L^1_P(\Omega, C_b(X)) \) by part 1. \( \square \)

**Corollary 2.1.26** ([4]) Let \( \varphi \) be a continuous RDS on a polish space \( X \). Then \( I_P(\varphi) \) is a (possibly empty) convex and closed subset of \( P P(\Omega \times X) \).

**Theorem 2.1.27** (Krylov-Bogolyubov procedure for continuous RDS [4], [27]) Let \( \varphi \) be a continuous RDS on a polish space \( X \), and that \( \emptyset \neq \Gamma \subset P_P(\Omega \times X) \) is closed, tight, and convex set of random measures such that \( \Theta_t \Gamma \subset \Gamma \) for all \( t > 0 \). Let \( (\nu^N)_{N \in \mathbb{N}} \) be an arbitrary sequence in \( \Gamma \) and define the sequence of random measures \((\mu_N)_{N \in \mathbb{N}}\) by the scheme

\[
\mu_N = \begin{cases} 
\frac{1}{N} \sum_{n=0}^{N-1} \Theta_n \nu^N(t), & \text{T discrete}, \\
\frac{1}{T} \int_0^T \Theta_t \nu^N(t)dt, & \text{T continuous.}
\end{cases} \tag{2.1.7}
\]

(Similarly for \( N < 0 \) if \( T \) is two-sided). The sequence \((\mu_N)_{N \in \mathbb{N}}\) has a convergent subsequence and every convergent subsequence of \((\mu_N)_{N \in \mathbb{N}}\) converges in \( I_P(\varphi) \).

**Proof.** We shall consider only the continuous time case.

We are given that \( \Gamma \) is tight, so by Prohorov theorem for random measures (Theorem 2.1.23), \( \Gamma \) is sequentially compact. Using the fact that \( \Gamma \) is convex and invariant with respect to \( \Theta_t \), we have that \( \mu_N \in \Gamma, N \in \mathbb{N} \), and thus, \((\mu_N)_{N \in \mathbb{N}}\) has a convergent subsequence. Let \((\mu_{N_k})_{k \in \mathbb{N}}\) be the subsequence and let its limit be \( \mu \). We want to show that \( \mu \) is invariant with respect to \( \Theta_t \).
It is enough to prove that $\Theta_t \mu(f) = \mu(f)$ for all $f \in L^1_\Theta(\Omega, C_b(X))$ with $0 \leq f \leq 1$. Let $\Theta_t f(\omega, x) := f(\theta(t, \omega), \varphi(t, \omega)x)$ and then for any $f \in L^1_\Theta(\Omega, C_b(X))$

$$|\Theta_t \mu(f) - \mu(f)| = \left| \int_{\Omega \times X} \Theta_t f \, d\mu - \int_{\Omega \times X} f \, d\mu \right|$$

$$= \lim_{k \to \infty} \left| \int_{\Omega} \frac{1}{N_k} \int_{X} \left( \int_0^{N_k} (\Theta_{t+s} f(\omega, x) - \Theta_s f(\omega, x)) \, ds \right) d\nu^N_k(x) \, d\mathbb{P}(\omega) \right|$$

$$\leq \lim_{k \to \infty} \int_{\Omega} \frac{1}{N_k} \left( \int_{X} \Theta_s f(\omega, x) \, d\nu^N_k(x) \right) ds - \int_0^{N_k} \left( \int_{X} \Theta_s f(\omega, x) \, d\nu^N_k(x) \right) ds \, d\mathbb{P}(\omega)$$

$$\leq \lim_{k \to \infty} \frac{2t}{N_k} \int_{\Omega} \sup_{x \in X} |\Theta_s f(\omega, x)| ds + \int_0^{t} \sup_{x \in X} |\Theta_s f(\omega, x)| ds \, d\mathbb{P}(\omega)$$

$$= \lim_{k \to \infty} \frac{2t}{N_k} = 0,$$

where we have used $\sup_{x \in X} |\Theta_s f(\omega, x)| \leq 1$ for all $s \in \mathbb{T}$. Consequently, $\mu$ is invariant with respect to $\Theta_t$ and thus, $\mu \in \mathcal{I}_\Theta(\varphi)$.

\[\square\]

**Remark 2.1.28**  
(i) If in the random Krylov-Bogolyubov procedure, $\theta$ is measurably invertible (in particular, if time is two-sided) then factorized form of the random measure $\mu_N$ in equation (2.1.7) is

$$\mu_{N, \omega} = \left\{ \begin{array}{l}
\frac{1}{N} \sum_{n=0}^{N-1} \varphi(n, \theta^{-n}\omega) \nu^N_{\theta^{-n}\omega}, \\
\frac{1}{N} \int_0^N \varphi(t, \theta^{-1}\omega) \nu^N_{\theta^{-1}\omega} \, dt
\end{array} \right\}$$

and for two-sided time, we have

$$\mu_{N, \omega} = \left\{ \begin{array}{l}
\frac{1}{N} \sum_{n=0}^{N-1} \varphi(-n, \omega) \nu^N_{\theta^{-n}\omega}, \\
\frac{1}{N} \int_0^N \varphi(-t, \omega) \nu^N_{\theta^{-t}\omega} \, dt.
\end{array} \right\}$$

For two-sided time, Krylov-Bogolyubov procedure, builds up a random measure $\mu_{N, \omega}$ in the fibre $\{\omega\} \times X$ (i.e at $t = 0$) by first transporting for each fixed $t$, the measure $\nu^N_{\theta^{-t}\omega}$ from the fibre $\{\theta^{-t}\omega\} \times X$, by means of the mapping $\varphi(-t, \omega)$ to the measure $\varphi(-t, \omega)^{-1} \nu^N_{\theta^{-t}\omega}$ in the fibre $\{\omega\} \times X$, and then by averaging all those measures in $\{\omega\} \times X$ with respect to $t \in [0, N]$.

(ii) In particular, take $\nu_\omega = \delta_{x(\omega)}$, $x : \Omega \to X$ is a random variable, then $\mu_{N, \omega}(B)$ is the proportion of time from $[0, N]$ which the orbit $t \mapsto \varphi(-t, \omega)^{-1} x(\theta^{-t}\omega) \in \{\omega\} \times X$ spends in the subset $B \in \mathcal{B}$. 
(iii) The Krylov-Bogolyubov procedure remains valid, if \( t_N : \Omega \to \mathbb{T} \) is an increasing sequence of random times such that \( \limsup \mathbb{E}(t_N^{-1}) = 0 \), for \( N \to \infty \). In this situation, the scheme becomes

\[
\mu_N = \begin{cases} 
  t_N^{-1} \sum_{n=0}^{[t_N]-1} \Theta_n \nu^N(\cdot), & \mathbb{T} \text{ discrete}, \\
  t_N^{-1} \int_0^{t_N} \Theta_t \nu^N(\cdot) dt, & \mathbb{T} \text{ continuous.}
\end{cases}
\tag{2.1.8}
\]

(iv) The conditions of Krylov-Bogolyubov procedure are satisfied, in particular, for \( \Gamma = \mathcal{P}(\Omega \times X) \) in case the state space \( X \) is compact. However, for many interesting and relevant RDS the assumption of a compact state space is rather restrictive. In this case, \( \mathcal{P}(\Omega \times X) \) is not tight, and verification of tightness of a given set \( \Gamma \) is not completely trivial. Fortunately, in this thesis \( \Gamma = \mathcal{M}(K) \) the set of all random measures supported by an invariant random compact set \( K \), tightness can be verified (in fact, see the proof of Theorem 2.1.29).

**Theorem 2.1.29** ([25], [27]) Let \( \varphi \) be a continuous RDS, and suppose \( \omega \mapsto K(\omega) \) is a measurable compact forward \( \varphi \)-invariant set. Then the set \( \mathcal{M}(K) \) is nonempty.

**Proof.** Put \( \Gamma = \{ \mu \in \mathcal{P}(\Omega \times X); \mu_\omega(K(\omega)) = 1, \; \mathbb{P} - a.s. \} \). Then

- \( \Gamma \neq \emptyset \), since by Proposition 2.1.14, there is a measurable selection \( \omega \mapsto k(\omega) \in K(\omega) \), so \( \delta_{k(\omega)}(K(\omega)) = 1, \; \mathbb{P} - a.a. \omega \), whence \( \omega \mapsto \delta_{k(\omega)} \in \Gamma \).

- \( \Gamma \) is tight since

\[
\mu(K) = \int_\Omega \mu_\omega(K(\omega)) d\mathbb{P}(\omega) = 1, \; \text{for all } \mu \in \Gamma \text{ (see definition 2.1.22)},
\]

so by Prohorov theorem for random measures (Theorem 2.1.23), it is sequentially compact.

- \( \Gamma \) is closed and convex (already known from corollary 2.1.19).

- By forward invariance of \( K \) and the fact that \( \mu \mapsto \Theta_t \mu, \; t \in \mathbb{T} \) is commuting and affine (Proposition 2.1.25) we apply Markov-Kakutani fixed point Theorem (Theorem A.8 in the appendix) to have that \( \Theta_t \Gamma \subset \Gamma \), for \( t \geq 0 \).

By Krylov-Bogolyubov procedure (Theorem 2.1.27), the proof is complete. \( \Box \)
2.2 Extremality and ergodicity of random invariant measures

Let $\Phi : \Omega \times X \to \mathbb{R}$ be a function with the following property:

(i) for all $x \in X$, $\omega \mapsto \Phi(\omega, x)$ is measurable,

(ii) for all $\omega \in \Omega$, $x \mapsto \Phi(\omega, x)$ is continuous and bounded,

(iii) $\omega \mapsto \sup\{|\Phi(\omega, x)| : x \in X\}$ is $\mathbb{P}$-integrable.

If $\Phi$ and $\Psi$ are functions satisfying (i)-(iii) then the set

\[ \{ \omega : \Phi(\omega, \cdot) = \Psi(\omega, \cdot) \} \]

is measurable, provided $X$ is separable. Identify $\Phi$ and $\Psi$, if $\mathbb{P}\{\Phi(\omega, \cdot) \neq \Psi(\omega, \cdot)\} = 0$.

**Definition 2.2.1 (Random continuous function)** A random continuous function is (the equivalent class of) a function $\Phi$ satisfying (i)-(iii) above.

In fact, the set of all random continuous functions is the linear space $L^1_\mathbb{P}(\Omega, C_b(X))$, which we have encountered in the previous section.

**Definition 2.2.2 (Extremal point)** An extremal point of a convex set $C$ is a point $x \in C$ such that if $x = \alpha y + (1 - \alpha)z$, with $y, z \in C$, $\alpha \in [0, 1]$, then $y = x$ and/or $z = x$.

In other words, an extremal point is a point that is not an interior point of any line segment lying entirely in $C$.

**Lemma 2.2.3 ([27])** 1. Any ergodic invariant measure for $\varphi$ is an extremal point of the convex set $\mathcal{I}_\mathbb{P}(\varphi)$.

If $(\theta, \mathbb{P})$ is ergodic, then any extremal point of the set $\mathcal{I}_\mathbb{P}(\varphi)$ is ergodic.

2. Suppose that $\omega \mapsto K(\omega)$ is a random closed set. If $(\theta, \mathbb{P})$ is ergodic, then any extremal point of $\mathcal{M}_\mathbb{P}(K)$ is an ergodic invariant measure for $\varphi$.

Suppose that $\varphi$ is an RDS over an ergodic base dynamical system $(\theta, \mathbb{P})$, if $\omega \mapsto K(\omega)$ is a $\varphi$-invariant random compact set, then the convex set $\mathcal{M}_\mathbb{P}(K)$ is compact, and its extremal points are ergodic by the second part of Lemma [2.2.3]. Next, since for every random continuous function $f$, $\mu \mapsto \mu(f)$ is continuous, the set $\{\mu(f) : \mu \in \mathcal{M}_\mathbb{P}(K)\} \subset \mathbb{R}$ is the compact interval

\[ \left[ \min \left\{ \mu(f) : \mu \in \mathcal{M}_\mathbb{P}(K) \right\}, \max \left\{ \mu(f) : \mu \in \mathcal{M}_\mathbb{P}(K) \right\} \right]. \]  

(2.2.1)
So, for every random continuous function $f$, there are ergodic measures $\mu_*, \mu^* \in \mathcal{M}_P(K)$ such that

$$\int_{X \times \Omega} f d\mu_* = \min \left\{ \int_{X \times \Omega} f d\mu : \mu \in \mathcal{M}_P(K) \right\},$$

$$\int_{X \times \Omega} f d\mu^* = \max \left\{ \int_{X \times \Omega} f d\mu : \mu \in \mathcal{M}_P(K) \right\}.$$  

We note that $\mu_*$ and $\mu^*$ depends on $f$, and need not be unique.

**Proposition 2.2.4 (The supremum of time means is realised by an ergodic measure [27])**

Suppose that $\varphi$ is an RDS over an ergodic base dynamical system $(\theta, \mathcal{P})$, and that $\omega \mapsto K(\omega)$ is a $\varphi$-invariant compact set. Let $f : X \times \Omega \to \mathbb{R}$ be measurable, with $x \mapsto f(x, \omega)$ continuous on $K(\omega)$ $\mathcal{P}$-almost surely, and

$$\omega \mapsto \sup_{x \in K(\omega)} f^+(x, \omega)$$

is integrable with respect to $\mathcal{P}$.

Then the following limit exists $\mathcal{P}$-almost surely, and satisfies

$$\lim_{t \to \infty} \frac{1}{t} \left( \sup_{x \in K(\omega)} \int_0^t f \circ \Theta_s(x, \omega) ds \right) = \max \left\{ \int_{X \times \Omega} f d\mu : \mu \in \mathcal{M}_P(K) \right\}. \quad (2.2.3)$$

In particular, there exists an ergodic $\varphi$-invariant measure $\mu$, depending on $f$, such that

$$\lim_{t \to \infty} \frac{1}{t} \left( \sup_{x \in K(\omega)} \int_0^t f \circ \Theta_s(x, \omega) ds \right) = \int_{X \times \Omega} f(x, \omega) d\mu(x) d\mathcal{P}(\omega), \quad \mathcal{P} - \text{a.s.} \quad (2.2.4)$$

**Remark 2.2.5** It follows that random continuous functions satisfy the conditions of Proposition 2.2.4, we obtain that for every random continuous function and for every $\varphi$-invariant compact set $\omega \mapsto K(\omega)$, there exists an ergodic invariant measure $\mu$, supported by $K$, such that (2.2.3) is satisfied. But it is crucial to keep in mind that in general $\mu$ will depend on the function $f$ under consideration.

A measurable function $\Phi_n : X \to \mathbb{R}$ is subadditive with respect to the skew product $\Theta$ if

$$\Phi_{n+m} \leq \Phi_n \circ \Theta^n + \Phi_m, \quad \text{for all } n, m \in \mathbb{N}. \quad (2.2.5)$$

**Corollary 2.2.6 ([14])** There exits an ergodic measure $\nu \in \mathcal{M}_P(K)$, such that for a sequence of subadditive random continuous functions $(\Phi_n)_{n \in \mathbb{N}}$ we have

$$\lim_{n \to \infty} \frac{1}{n} \int_{X \times \Omega} \Phi_n(\omega, x) d\nu(\omega) d\mathcal{P}(\omega) = \lim_{n \to \infty} \frac{1}{n} \max_{x \in K(\omega)} \Phi_n(\omega, x). \quad (2.2.6)$$
2.3 Lyapunov exponent and its extremal

Now, we recall the celebrated **Multiplicative Ergodic Theorem** which is one of the major tools used in long time behaviour analysis in the theory of random dynamical systems.

**Theorem 2.3.1 (Multiplicative Ergodic Theorem (MET))** Let \( \varphi \) be a \( C^1 \) RDS on a Riemannian Manifold \( X \), with the random measure \( \mu \in \mathcal{I}_P(\varphi) \). Suppose

\[
(\omega, x) \mapsto \sup_{0 \leq t \leq 1} \log^+ \| (D_x\varphi(t, \omega, x))^{\pm} \| \in L^1(\Omega \times X, \mu) \quad ((\omega, x) \mapsto \log^+ \| (D_x\varphi(1, \omega, x))^{\pm} \|) \in L^1(\Omega \times X, \mu)
\]

Then there is \( (\theta, \varphi) =: \Theta \)-forward invariant set \( \Gamma \subset \Omega \times X \) with \( \mu(\Gamma) = 1 \), such that the following hold true

- The operator defined by
  \[
  \Lambda(\omega, x) := \lim_{t \to \infty} \left[ (D_x\varphi(t, \omega, x))^s(D_x\varphi(t, \omega, x)) \right]^{1/t} \text{ exists.}
  \]

- Let \( e^{\lambda^1(\omega, x)} < \cdots < e^{\lambda^s(\omega, x)} \) be the eigenvalues of the operator \( \Lambda(\omega, x) \), where \( s = s(\omega, x) \), the \( \lambda^r(\omega, x) \) are real, and \( \lambda^1(\omega, x) \) can be \(-\infty\), and \( U^1(\omega, x), \cdots, U^s(\omega, x) \) the corresponding eigenspaces. Let \( d_r(\omega, x) = \dim U^r(\omega, x) \). The functions \( \omega \mapsto \lambda^r(\omega, x) \), \( \omega \mapsto s(\omega, x) \) and \( \omega \mapsto d_r(\omega, x) \) are \( \Theta \)-invariant.

Put \( V^0(\omega, x) = \{0\} \) and \( V^r(\omega, x) = U^1(\omega, x) \oplus \cdots \oplus U^r(\omega, x) \) for \( r = 1, \cdots, s \).

Then for \( u \in V^r(\omega, x) \setminus V^{(r-1)}(\omega, x) \), \( 1 \leq r \leq s \),

\[
\lim_{t \to \infty} \frac{1}{t} \log \| D_x\varphi(t, \omega, u) \| = \lambda^r(\omega, x). \tag{2.3.1}
\]

**Remark 2.3.2** Let \( d = \dim X \), and \( d = d_1 + \cdots + d_s(\omega) \), we could see that MET assigns \( d \) numbers of \( \lambda \) to every random measure \( \mu \in \mathcal{I}_P(\varphi) \).

**Definition 2.3.3 (Lyapunov exponent and spectrum)** The numbers \( \lambda^1, \cdots, \lambda^s \) are called Lyapunov exponents of \( D_x\varphi \) and the numbers \( d_1, \cdots, d_s \) are their multiplicities, respectively. The collection \( \{(\lambda^1, d_1), \cdots, (\lambda^s, d_s)\} \) is called Lyapunov spectrum.

**Corollary 2.3.4** \([24]\) Let \( \varphi \) be an RDS on a Riemannian Manifold \( X \), \( d = \dim X \) and let \( \mu \in \mathcal{I}_P(\varphi) \) such that for some \( t_0 > 0 \)

\[
(\omega, x) \mapsto \sup_{0 < t \leq t_0} \left\{ \log^+ \| D_x\varphi(t, \omega, x) \| + \log^+ \| (D_x\varphi(t, \omega, x))^{-1} \| \right\} \in L^1(\Omega \times X, \mu)
\]
Then there is a measurable set $\Gamma \subset \Omega \times X$ with $\mu(\Gamma) = 1$, such that for all $(\omega, x) \in \Gamma$

$$\gamma^+_\mu := \lim_{t \to \infty} \frac{1}{t} \log \|D_x \varphi(t, \omega, x)\|,$$

$$\gamma^-_\mu := -\lim_{t \to \infty} \frac{1}{t} \log \|(D_x \varphi(t, \omega, x))^{-1}\| \quad \text{and}$$

$$\gamma^\Sigma_\mu := \lim_{t \to \infty} \frac{1}{t} \log \|\det D_x \varphi(t, \omega, x)\|$$

exist.

Note that $\gamma^\Sigma_\mu$ is the sum of the Lyapunov exponent associated with $\mu$, whereas $\gamma^+_\mu$ and $\gamma^-_\mu$ are the largest and smallest Lyapunov exponent associated with $\mu$ respectively; so that

$$\gamma^-_\mu \leq \frac{1}{d} \gamma^\Sigma_\mu \leq \gamma^+_\mu, \quad \mu - a.s. \quad (2.3.2)$$

The three functions $(\omega, x) \mapsto \gamma^+_\mu(\omega, x)$, $\gamma^-_\mu(\omega, x)$, $\gamma^\Sigma_\mu(\omega, x)$ are invariant with respect to the skew product $\Theta$ and thus, constant if $\mu$ is ergodic.

**Proposition 2.3.5** [24] Let $\varphi$ be an RDS on a compact Riemannian manifold $X$. Put

$$\lambda(\mu, \varphi) := \sup \left\{ \int \gamma^+_\mu \, d\mu : \mu \in \mathcal{I}_\varphi(\varphi) \right\}$$

$$\bar{\lambda}(\mu, \varphi) := \inf \left\{ \int \gamma^-_\mu \, d\mu : \mu \in \mathcal{I}_\varphi(\varphi) \right\}$$

If either $\lambda(\mu, \varphi) < 0$ or $\bar{\lambda}(\mu, \varphi) > 0$, then $\mathbb{P} - a.s. \mu_\omega$ is convex combination of finitely many Dirac measures with equal weights.

**Remark 2.3.6** The numbers $\lambda(\mu, \varphi)$ and $\bar{\lambda}(\mu, \varphi)$ are maximal and minimal Lyapunov exponent respectively, of the RDS $\varphi$. On compact manifolds, extremal exponents are realised, that is, there exists an ergodic invariant measure $\mu$ such that

$$\gamma^+_\mu = \lim_{t \to \infty} \frac{1}{t} \log \|D_x \varphi(t, \omega, x)\| = \lambda(\varphi, \mu). \quad (2.3.3)$$

And there another ergodic invariant measure for $\bar{\lambda}(\varphi, \mu)$. In particular, for every $\varphi$-forward invariant random compact set $K$, there exists $\mu \in \mathcal{M}_\varphi(K)$ such that $\gamma^+_\mu = \lambda(\varphi, \mu)$ (thanks to Corollary 2.2.6). However, this invariant ergodic measure is not unique in general.
2.4 Furstenberg-Hasminskii formula for Lyapunov exponents

In this section, we highlight on how to determine maximal (top) Lyapunov exponent for stochastic differential equation. Let $\Phi(t, \omega) \in GL(d, \mathbb{R})$ be linear cocycle over the metric dynamical system $\theta$, generated by the linear SDE

$$dV = A_0 V dt + \sum_{j=1}^{m} A_j V \circ dW_t^j, \quad V_0 = v \in \mathbb{R}^d, \quad (2.4.1)$$

where $A_0, A_1, \ldots, A_m$ are $d \times d$ matrices.

If we write (2.4.1) in polar coordinates, $s = \frac{v}{\|v\|} \in S^{d-1}$ and $r = \|v\| \in (0, \infty)$ in $\mathbb{R}^d \setminus \{0\}$, we obtain the SDE for the angular part

$$dS_t = h_0(S_t) dt + \sum_{j=1}^{m} h_j(S_t) \circ dW_t^j, \quad S_0 = s \in S^{d-1}, \quad (2.4.2)$$

the fields $h_j$ are the projections of the linear vector fields $A_j v$ onto $S^{d-1}$ given by

$$h_j(s) := A_j s - q_j(s)s, \quad q_j(s) := \langle A_j s, s \rangle.$$

While the radial part satisfies the SDE

$$dR_t = q_0(S_t) R_t dt + \sum_{j=1}^{m} q_j(S_t) R_t \circ dW_t, \quad R_0 = r \in (0, \infty). \quad (2.4.3)$$

Given a random dynamical system $\Phi$ with two-sided time $t \in \mathbb{R}$. Let

$$\mathcal{L} = A_0 + \frac{1}{2} \sum_{j=1}^{m} A_j A_j^*$$

be the generator of the Markov family $X_t^\omega = \Phi(t, \cdot, x)$ on $\mathbb{R}^+$. There is a one-to-one correspondence between the stationary measures $\rho$ and those invariant measures which are measurable with respect to the past $\mathcal{F}_{-\infty} := \sigma\{W_u - W_v : u, v \leq 0\} \subset \mathcal{F}$ of the Wiener process (see [2], [3], [4], [26] and few other references).

The correspondence is defined as follows: If $\mu_\omega$ is invariant and $\mathcal{F}_{-\infty}$-measurable, then

$$\rho = \mathbb{E}\mu_\omega$$

is a stationary measure.

Conversely, if $\rho$ is a stationary measure, then

$$\mu_\omega := \lim_{t \to \infty} \Phi(t, \theta^{-1} \omega) \rho$$

is invariant and $\mathcal{F}_{-\infty}$-measurable.
Theorem 2.4.1 (Top Lyapunov exponent equal to Furstenberg-Khasminskii number [3],[4])

Suppose that the Hörmander condition

\[ \dim L(s) = d - 1, \quad \text{for all } s \in S^{d-1} \]  

(2.4.4)
is satisfied, where \( L \) denotes the Lie algebra of vector fields on \( S^{d-1} \) generated by the vector fields \( h_0, h_1, \cdots, h_m \) of the angular SDE (2.4.2). Then

(i) The angular SDE (2.4.2) admit a unique stationary measure \( \rho \). The stationary measure \( \rho \) has a \( C^\infty \) density.

(ii) The top Lyapunov exponent of the linear RDS \( \lambda_1 \) is equal to the Furstenberg-Khasminskii number,

\[ \lambda_1 = \int_{S^{d-1}} Q(s) \rho(ds), \]  

(2.4.5)

where \( Q(s) = q_0(s) + \frac{1}{2} \sum_{j=1}^{m} \kappa_{A_j}(s) \) and \( \kappa_A(s) = \langle (A + A^*) s, As \rangle - 2 \langle s, As \rangle^2 \).

(iii) For any \( v \neq 0 \),

\[ \lambda(\omega, v) = \lambda_1, \quad \mathbb{P} - a.s. \]  

(2.4.6)

Remark 2.4.2 The Hörmander condition (2.4.4) (also know as hypoellipticity condition) ensures the smoothness of the density \( \rho \) associated with solutions of the stochastic differential equations. It turns out that under the Hörmander condition (2.4.4), the set \( \text{supp} \rho \) is unique and has nonempty interior (see chapter 6 of Arnold’s book [4] for more details). In fact, for markovian random dynamical systems, we see that assumption C in section 3.3 of chapter 3 is valid for all random measures with support in some random invariant compact set.
Chapter 3

Random Invariant Periodic Curves

3.1 Motivation and problem formulation

Consider a dynamical system on $\mathbb{R}^d$, $d \geq 2$ induced by the evolution equation

$$\frac{du}{dt} = Au + f(u).$$

(3.1.1)

Assume that (3.1.1) has a periodic solution of period $\tau$, $v : \mathbb{R} \to \mathbb{R}^d$ such that $v(t + \tau) = v(t)$ for all $t \in \mathbb{R}$. We wish to investigate if the following stochastic differential equation which is formally the random perturbation of equation (3.1.1),

$$du = (Au + f(u))dt + g(u)dW_t,$$

(3.1.2)

has periodic solutions. Although, there have been an attempt by few researchers to investigate this problem, but majority of the work we know have been when the function $g$ vanishes at the periodic solution of the deterministic equation (3.1.1) this kind of stochastic solution is refered to as trivial periodic solution (68). In fact, when there is such a periodic solution $v$ such that $g$ vanishes, stochastic equation (3.1.2) becomes a control problem. It is more interesting to consider nontrivial case, in this situation we do not have periodic solution in the sense $v(t + \tau) = v(t)$. We expect formally, a solution that will depend on the Wiener process $W_t$, random periodic solutions in the sense of Zhao and Zheng (95) become a reasonable notion here and we shall be interested in investigating such notion.

It is not quite direct to prove the existence of random periodic solutions of the stochastic differential equation (3.1.2), this is because the periodic solutions if they exist would be as a result
of the nonlinearity of its coefficients. What is normally done is to transform equation (3.1.2) to the stochastic differential equation with periodic coefficients. Now let us note the following observation by Feng and Zhao [34]

$$u(t) = X(t) + v(t),$$

where the stochastic process $X$ satisfies the stochastic differential equation

$$dX = (AX + F(t, X))dt + G(t, X)dW_t,$$

(3.1.3)

where $F(t, X) = f(X + v(t)) - f(v(t))$ and $G(t, X) = g(X + v(t))$. So that the functions $F$ and $G$ are time periodic. In our discussion we consider the problem (3.1.3) in autonomous form on $\mathbb{R}^{d+1}$ namely;

$$\begin{cases}
    dX(r) = AX(r(t))dt + F(r(t), X(t))dr + G(r(t), X(t))dW_t, \\
    dr(t) = dt, \\
    X(0) = x, \quad r(0) = t_0.
\end{cases}$$

(3.1.4)

Due to the $\tau$ periodicity of the functions $F$ and $G$, we can identify each point $(t, x) \in \mathbb{R}^{d+1}$ by $(t + \tau, x)$, hence equation (3.1.4) is in fact, time homogeneous equation on $I_\tau \times \mathbb{R}^d$, where $I_\tau$ is the closed interval with end point identified, i.e., $I_\tau$ is a circle $S^1$. Technically, here we have $s + t \mod \tau$, i.e., $k \in \mathbb{N}$

$$s + t - k\tau = \tau\left(\frac{s + t}{\tau} - k\right) = \tau\left(\frac{s + t}{\tau} \mod 1\right).$$

In this chapter, we consider a random dynamical system over the base dynamical system $(\Omega \times S^1, \mathcal{F} \otimes \mathcal{B}(S^1), \theta \times \eta)$, where $\eta$ is defined on the circle $S^1$ and $\theta$ is the usual Wiener shift.

### 3.2 Random semiuniform ergodic theorem

We would like to recall two of the most important theorems in ergodic theory: Birkhoff’s ergodic theorem and Kingman’s subadditive ergodic theorem, which can be found in various versions in most of the excellent books on ergodic theory, e.g. Ash and Doléans-Dade [6], Cornfield, Formin and Sinai [23], Krengel [46], Mañé [54], Peterson [64], Rudolph [74], Sinai [80], Walters [91].

**Theorem 3.2.1 (Birkhoff’s ergodic theorem)** Let $\theta$ be a measure preserving transformation on the measure space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ \theta^j = \mathbb{E}(f|\mathcal{F}),$$

(3.2.1)
where \( \mathcal{I} = \{ A \in \mathcal{F} : A = \theta^{-1}A \} \). Alternatively, \( \mathcal{I} \) is the \( \sigma \)-algebra of generated by all \( \theta \)-invariant functions.

**Theorem 3.2.2 (King’s subadditive ergodic theorem)** Let \( \theta \) be a measure preserving transformation on the measure space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( (g_n)_{n \in \mathbb{N}} \) be a sequence of \( \mathbb{P} \)-integrable functions satisfying

\[
  g_{n+m}(\omega) \leq g_n(\omega) + g_m(\theta^n \omega) \quad \text{(i.e. } g_n \text{ is subadditive with respect to } \theta). \tag{3.2.2}
\]

Then with measure one, we have

\[
  \lim_{n \to \infty} \frac{g_n(\omega)}{n} = g(\omega) \geq -\infty, \tag{3.2.3}
\]

where \( g(\omega) \) is \( \theta \)-invariant function.

Given \( \mu \in \mathcal{I}_\mathbb{P}(\varphi) \) and a sequence of random continuous functions \( (\Phi_n)_{n \in \mathbb{N}} \) subadditive with respect to \( \Theta \). We define

\[
  \mu(\Phi_n) := \int_{\Omega \times X} \Phi_n(\omega, x) d\mu(\omega, x) d\mathbb{P}(\omega), \tag{3.2.4}
\]

using the fact that \( (\Phi_n)_{n \in \mathbb{N}} \) is subadditive with respect to the skew product \( \Theta \) and invariance of the measure \( \mu \), have that

\[
  \mu(\Phi_{n+m}) \leq \mu(\Phi_n) + \mu(\Phi_m) \quad \text{(3.2.5)}
\]

provided both sides are well defined, such that the subadditivity limit

\[
  \hat{\Phi}_\mu := \lim_{n \to \infty} \frac{1}{n} \mu(\Phi_n) \quad \text{(3.2.6)}
\]

exists.

Given a random continuous function \( \Phi : \Omega \times X \to \mathbb{R} \) and a random compact set \( K \), we define

\[
  \Phi^K(\omega) := \max\{ \Phi(\omega, x) : x \in K(\omega) \}. \tag{3.2.7}
\]

**Lemma 3.2.3 ([41])** \( \Phi^K : \Omega \to \mathbb{R} \) is measurable and there is a measurable function \( b : \Omega \to X \) such that \( b(\omega) \in K(\omega) \) and \( \Phi^K(\omega) = \Phi(\omega, b(\omega)) \) for all \( \omega \in \Omega \).
Proof. Given the random compact set \( K \), random selection theorem ensures that there is a sequence \( \{ a_k(\omega) \}_{k \in \mathbb{N}} \) of measurable maps \( a_k : \Omega \to X \) such that \( K(\omega) = \text{closure}\{ a_k(\omega); k \in \mathbb{N} \} \) for all \( \omega \in \Omega \). And then using the fact that \( \Phi \) is random continuous, we have

\[
\Phi^K(\omega) = \max\{ \Phi(\omega, x); x \in K(\omega) \} = \sup\{ \Phi(\omega, a_k(\omega)); k \in \mathbb{N} \}
\]

is measurable. For \( l \in \mathbb{N} \), let \( k_l(\omega) = \min\{ k \in \mathbb{N}; \Phi(\omega, a_k(\omega)) > \Phi^K(\omega) - l^{-1} \} \). The functions \( k_l : \Omega \to \mathbb{N} \) is measurable. Therefore,

\[
A(\omega) = \bigcap_{j \in \mathbb{N}} \bigcup_{l > j} \{ a_{k_l(\omega)}(\omega) \} \subset \{ x \in K(\omega); \Phi(\omega, x) = \Phi^K(\omega) \}
\]

is a random compact set and by random selection theorem, there is a measurable selection \( b : \Omega \to X \) such that \( b(\omega) \in A(\omega) \subset K(\omega) \), for all \( \omega \in \Omega \). □

Lemma 3.2.4 ([14]) The function \( \mu \mapsto \hat{\Phi}_\mu \) is upper semi-continuous from \( \mathcal{I}_P(\varphi) \) to \( \mathbb{R} \).

Proof. This could be found in the reference [14] and it relies mostly on the subadditivity of \( (\Phi_n)_{n \in \mathbb{N}} \). Also, the upper semi-continuity is with respect to the narrow topology (2.1.12).

Lemma 3.2.5 ([41], [14])

(a) If the sequence \( (\Phi_n)_{n \in \mathbb{N}} \) of random continuous functions is subadditive and \( K \) is forward invariant random compact set, then the sequence \( (\Phi^K_n)_{n \in \mathbb{N}} \) is subadditive.

(b) If

\[
\hat{\Phi}^K := \inf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E}(\Phi^K_n) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(\Phi^K_n)
\]

exists, then we have

\[
\hat{\Phi}^K = \sup\{ \hat{\Phi}_\mu; \mu \in \mathcal{M}_P(K) \}; \quad (3.2.8)
\]

and the supremum is attained at some \( \mu^* \in \mathcal{M}_P(K) \).

Proof. We only prove (b) as (a) is obvious. Take \( b_n \) to be measurable selections such that \( \Phi^K_n(\omega) = \Phi_n(\omega, b_n(\omega)) \).

Define measures \( \mu_n \in \mathcal{M}_P(K) \) through their disintegrations

\[
\mu_{n, \omega} = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\varphi_i(\theta^{-i}\omega, b_n(\theta^{-i}\omega))}, \quad (3.2.9)
\]
where $\delta_x$ denotes the Dirac measure at the point $x \in X$. As $b_n(\omega) \in K(\omega)$ for each $\omega \in \Omega$ and all measures $\mu_n$ are supported by the forward invariant random compact set $K$. Then, by random Krylov-Bogolyubov theorem, there is a subsequence $(\mu_n)_{n \in \mathbb{N}}$ converging in the narrow topology to some $\mu^* \in \mathcal{M}_P(K)$.

Recall: If $\Phi_n$ is subadditive with respect to $\Theta$ and if $1 \leq l \leq m$, we have that

$$\Phi_{ml}(\omega, x) \leq \sum_{j=0}^{l-1} \Phi_m(\Theta^{jm}(\omega, x)). \quad (3.2.10)$$

Fix $m \in \mathbb{N}$ and suppose $n_k > m$. For each $i$, $1 \leq i \leq m$ there exists a unique choice of integers $c(i, k) \geq 0$ and $0 \leq r(i, k) < m$, such that $n_k = i + c(i, k)m + r(i, k)$. Now by subadditivity of $\Phi$, we have the estimate

$$\Phi^K_{n_k}(\omega) = \Phi_{n_k}(\omega, b_{n_k}) \leq \Phi_i(\omega, b_{n_k}(\omega)) + \sum_{j=0}^{c(i,k)-1} \Phi_m \circ \Theta^{i+jm}(\omega, b_{n_k}(\omega))$$

$$+ \Phi_r(i,k) \circ \Theta^{i+c(i,k)m}(\omega, b_{n_k}(\omega))$$

summing over $1 \leq i \leq m$, we have

$$m\Phi^K_{n_k}(\omega) \leq \sum_{i=1}^{m} \Phi_i(\omega, b_{n_k}(\omega)) + \sum_{i=1}^{m} \sum_{j=0}^{c(i,k)-1} \Phi_m \circ \Theta^{i+jm}(\omega, b_{n_k}(\omega))$$

$$+ \sum_{i=1}^{m} \Phi_r(i,k) \circ \Theta^{i+c(i,k)m}(\omega, b_{n_k}(\omega))$$

using the definition of $c(i, k)$ and $r(i, k)$, we have

$$m\Phi^K_{n_k}(\omega) \leq \sum_{i=1}^{m} \Phi_i(\omega, b_{n_k}(\omega)) + \sum_{i=1}^{n_k-m} \Phi_m \circ \Theta^i(\omega, b_{n_k}(\omega)) + \sum_{i=1}^{m} \Phi_r(i,k) \circ \Theta^{i+c(i,k)m}(\omega, b_{n_k}(\omega))$$

$$\leq \sum_{i=1}^{m} \Phi_i(\omega, b_{n_k}(\omega)) + \sum_{i=1}^{n_k-1} \Phi_m \circ \Theta^i(\omega, b_{n_k}(\omega)) + \sum_{i=1}^{m} \Phi_r(i,k) \circ \Theta^{i+c(i,k)m}(\omega, b_{n_k}(\omega))$$

$$- \Phi_m(\omega, b_{n_k}(\omega)) - \sum_{i=n_k-m+1}^{n_k-1} \Phi_m \circ \Theta^i(\omega, b_{n_k}(\omega)).$$

If we divide both sides of the estimate by $mn_k$, we have

$$\frac{1}{n_k} \Phi^K_{n_k}(\omega) \leq \frac{1}{mn_k} \sum_{i=0}^{n_k-1} \Phi_m \circ \Theta^i(\omega, b_{n_k}(\omega)) + \frac{1}{mn_k} \left( \sum_{i=1}^{m} \Phi_r(i,k) \circ \Theta^{i+c(i,k)m}(\omega, b_{n_k}(\omega)) \right.$$

$$+ \sum_{i=1}^{m} \Phi_i(\omega, b_{n_k}(\omega)) - \Phi_m(\omega, b_{n_k}(\omega)) - \sum_{i=n_k-m+1}^{n_k-1} \Phi_m \circ \Theta^i(\omega, b_{n_k}(\omega)) \Big). \quad (3.2.11)$$

54
Since $|\Phi^K_{n_k}| \in L^1(\mathbb{P})$ and $b_{n_k}(\omega) \in K(\omega)$, by Kingman’s subadditive Ergodic theorem (Theorem 3.2.2), we have that

$$\lim_{k \to \infty} \frac{1}{mn_k} \left( \sum_{i=1}^{m} \int_{\Omega} \Phi_{r(i,k)} \circ \Theta^{i_{+}(i,k)m}(\omega, b_{n_k}(\omega))d\mathbb{P}(\omega) + \sum_{i=1}^{m} \int_{\Omega} \Phi_{i}(\omega, b_{n_k}(\omega))d\mathbb{P}(\omega) \right) - \int_{\Omega} \Phi_{m}(\omega, b_{n_k}(\omega))d\mathbb{P}(\omega) = 0.$$  \hspace{1cm} (3.2.12)

So that,

$$\lim_{k \to \infty} \frac{1}{n_k} \int_{\Omega} \Phi_{K_{n_k}}(\omega)d\mathbb{P}(\omega) \leq \lim_{k \to \infty} \frac{1}{mn_k} \sum_{i=0}^{n_k-1} \int_{\Omega} \Phi_{m} \circ \Theta^{i}(\omega, b_{n_k}(\omega))d\mathbb{P}(\omega).$$  \hspace{1cm} (3.3.13)

We have already seen that the measure $\mu_n$ has a convergent subsequence $\mu_{n_k}$, let $\mu^* \in \mathcal{M}(K)$ be the limit, we have

$$\mu^*(\frac{1}{m} \Phi_{m}) = \lim_{k \to \infty} \frac{1}{m} \int_{\Omega \times X} \Phi_{m}d\mu_{n_k}$$

$$= \lim_{k \to \infty} \frac{1}{mn_k} \sum_{i=0}^{n_k-1} \int_{\Omega} \Phi_{m} \circ \Theta^{i}(\theta^{-i}\omega, b_{n_k}(\theta^{-i}\omega))d\mathbb{P}(\omega)$$

$$= \lim_{k \to \infty} \frac{1}{mn_k} \sum_{i=0}^{n_k-1} \int_{\Omega} \Phi_{m} \Theta^{i}(\omega, b_{n_k}(\omega))d\mathbb{P}(\omega) \quad (\text{\(\theta\) is \(\mathbb{P}\)-preserving})$$

$$\geq \lim_{k \to \infty} \frac{1}{n_k} \int_{\Omega} \Phi_{K_{n_k}}(\omega)d\mathbb{P}(\omega) \quad \text{(using estimate \(3.2.13\))}$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \mathbb{E}(\Phi^K_{n_k}).$$

Since this holds for all $m \in \mathbb{N}$, we have

$$\Phi_{\mu^*} = \inf_{m \in \mathbb{N}} \mu^*(\frac{1}{m} \Phi_{m}) \geq \lim_{m \to \infty} \frac{1}{m} \mathbb{E}(\Phi^K_{m}).$$

It remains to show that

$$\Phi_{\mu^*} \leq \lim_{m \to \infty} \frac{1}{m} \mathbb{E}(\Phi^K_{m}).$$  \hspace{1cm} (3.14.14)

In this case, Kingman’s subadditive ergodic theorem (Theorem 3.2.2) could be stated as: for every $\mu \in \mathcal{I}_{\mathbb{P}}(\varphi)$, there exists a measurable set $A \in \mathcal{F} \otimes \mathcal{B}(X)$ with $\mu(A) = 1$ and $\Phi \in L^1(\mu)$ such that

$$\lim_{m \to \infty} \frac{\Phi_{m}(\omega, x)}{m} = \bar{\Phi}(\omega, x) \quad \text{and}$$

$$\lim_{m \to \infty} \frac{1}{m} \int_{\Omega \times X} \Phi_{m}(\omega, x)d\mu_{\omega}(x)d\mathbb{P}(\omega) = \int_{\Omega \times X} \Phi(\omega, x)d\mu_{\omega}(x)d\mathbb{P}(\omega).$$
Since, \( \Phi_m(\omega, x) \leq \max\{\Phi_m(\omega, x) : x \in X\} \), we have that
\[
\lim_{m \to \infty} \frac{1}{m} \int_{\Omega \times X} \Phi_m(\omega, x) d\mu(x) d\mathbb{P}(\omega) \leq \lim_{m \to \infty} \frac{1}{m} \int_{\Omega} \max\{\Phi_m(\omega, x) : x \in X\} d\mathbb{P}(\omega).
\]
Hence,
\[
\sup\{\hat{\Phi}_\mu : \mu \in \mathcal{M}_\mathbb{P}(K)\} \leq \lim_{m \to \infty} \frac{1}{m} \mathbb{E}(\Phi^K_m).
\]

Recall a standard result from optimisation: Let \( f \) be a real valued function on a complete metric space \( X \). Suppose \( C \) is a compact subset of \( X \) and \( f \) upper semi-continuous on \( X \). Then there exist \( x^* \in C \) such that
\[
f(x^*) = \sup\{f(x) : x \in C\}.
\]
The function \( \mu \mapsto \hat{\Phi}_\mu \) is upper semi-continuous on \( \mathcal{I}_\mathbb{P}(\varphi) \) (Lemma 3.2.4) and \( \mathcal{M}_\mathbb{P}(K) \) is a compact subset of \( \mathcal{I}_\mathbb{P}(\varphi) \) (see section 2.1); so \( \mu \mapsto \hat{\Phi}_\mu \) attains its maximum on the compact set \( \mathcal{M}_\mathbb{P}(K) \).
That is, there exists \( \mu^* \in \mathcal{M}_\mathbb{P}(K) \) such that
\[
\hat{\Phi}_{\mu^*} = \sup\{\hat{\Phi}_\mu : \mu \in \mathcal{M}_\mathbb{P}(K)\}.
\]
We have,
\[
\hat{\Phi}_{\mu^*} \leq \lim_{m \to \infty} \frac{1}{m} \mathbb{E}(\Phi^K_m). \quad \square
\]

**Definition 3.2.6 (Tempered, slowly varying and adjusted random variable [4])**

1. A random variable \( R : \Omega \to \mathbb{R}^+ \) is **tempered** with respect to \( \theta \), if it satisfies
\[
\lim_{t \to \pm \infty} \frac{1}{t} \log R(\theta t) = 0, \quad \mathbb{P} - \text{a.s.}
\]
(3.2.15)

2. A random variable \( D : \Omega \to \mathbb{R}^+ \) is **\( \varepsilon \)-slowly varying** with respect to \( \theta \) if \( \mathbb{P} - \text{a.s.} \)
\[
e^{-\varepsilon|t|}D(\omega) \leq D(\theta t) \leq e^{\varepsilon|t|}D(\omega), \quad \text{for all } t \in \mathbb{T}.
\]
(3.2.16)

3. A random variable \( C : \Omega \to \mathbb{R} \) is **adjusted** to \( \theta \), if it satisfies
\[
\lim_{|t| \to \infty} \frac{1}{|t|} C(\theta t) = 0, \quad \mathbb{P} - \text{a.s.}
\]
(3.2.17)

**Proposition 3.2.7 ([4], [41])**

(i) If \( D_\varepsilon \) is \( \varepsilon \)-slowly varying for some \( \varepsilon \geq 0 \) then it is tempered.

(ii) \( C \) is adjusted to \( \theta \) if and only \( e^C \) is tempered with respect to \( \theta \).
(iii) If $R$ is tempered and $t \mapsto \log R(\theta_{i}\omega)$ is continuous $\mathbb{P}$-almost surely, then for any $\varepsilon > 0$ there is an $\varepsilon$-slow varying random variable $D_{\varepsilon}$ for which
\[
\frac{1}{D_{\varepsilon}(\omega)} \leq R(\omega) \leq D_{\varepsilon}(\omega).
\] (3.2.18)

(iv) Suppose $C : \Omega \to \mathbb{R}$ is measurable and $C \circ \theta - C$ has a $\mathbb{P}$-integrable minorant. Then $C$ is adjusted to $\theta$.

(v) Suppose that the random variable $C : \Omega \to \mathbb{R}$ is measurable and $C \circ \theta - C$ has a $\mathbb{P}$-integrable lower bound, then by Birkhoff Ergodic Theorem, one can show that $C$ is adjusted to $\theta$.

**Theorem 3.2.8 (Random semiuniform ergodic theorem [41])** Let $\Theta : \Omega \times X \to \Omega \times X$ be a skew product dynamical system with the ergodic base $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Suppose that $K$ is a forward invariant random compact set and that $(\Phi_{n})_{n \in \mathbb{N}}$ is a subadditive sequence of random continuous function with $|\Phi_{n}|^{K} \in L^{1}(\mathbb{P})$ for all $n \in \mathbb{N}$. Further, suppose that $\lambda \in \mathbb{R}$ is such that $\hat{\Phi}_{\mu} < \lambda$ for all $\mu \in \mathcal{M}_{\mathbb{P}}(K)$.

Then, there exists $\lambda' < \lambda$ and an adjusted random variable $C : \Omega \to \mathbb{R}$ such that
\[
\Phi_{n}(\omega, x) \leq C(\omega) + n\lambda', \quad \text{for all } n \in \mathbb{N}, \quad \mathbb{P} - a.e. \omega \in \Omega \quad \text{and all } x \in K(\omega). \tag{3.2.19}
\]

In particular, for $\delta \in (0, \lambda - \lambda')$ there is $N(\omega) \in \mathbb{N}$ such that
\[
\frac{1}{n} \Phi_{n}(\omega, x) \leq \lambda - \delta, \quad \text{for } n \geq N(\omega) \quad \text{and } x \in K(\omega). \tag{3.2.20}
\]

**Proof.** Suppose $\lambda > \hat{\Phi}_{\mu}$ for any $\mu \in \mathcal{M}_{\mathbb{P}}(K)$, then $\lambda > \hat{\Phi}_{K}$ (Lemma 3.2.5). And we can choose $\lambda' \in (\hat{\Phi}_{K}, \lambda)$. By Kingman’s subadditive Ergodic theorem (Theorem 3.2.2), we have
\[
\lim_{n \to \infty} \frac{1}{n} \Phi_{n}^{K}(\omega) = \hat{\Phi}_{K}, \quad \mathbb{P} - a.s.
\]
If $\Phi_{0}^{K} = 0$, then the random variable $C(\omega) = \sup_{n \geq 0}(-\lambda' n + \Phi_{n}^{K}(\omega))$ is nonnegative and $\mathbb{P} - a.s.$ finite. And we see that
\[
-n\lambda' + \Phi_{n}^{K}(\omega) \leq \sup_{n \geq 0}(-\lambda' n + \Phi_{n}^{K}(\omega)) = C(\omega),
\]
which implies that
\[
\Phi_{n}(\omega, x) \leq C(\omega) + n\lambda', \quad \text{for all } n \in \mathbb{N}, \quad \mathbb{P} - a.e. \omega \in \Omega \quad \text{for all } x \in K(\omega).
\]
It remains to show that $C$ is adjusted to $\theta$. If $C(\omega) = 0$, then

$$C(\theta \omega) - C(\omega) \geq 0. \quad (3.2.21)$$

Otherwise, using the fact that $\Phi^K_n$ is subadditive, we have

$$- \lambda' n + \Phi^K_n(\omega) \leq \left( - \lambda'(n-1) + \Phi^K_{n-1}(\theta \omega) \right) - \lambda' + \Phi^K_1(\omega) \leq C(\theta \omega) - \lambda' + \Phi^K_1(\omega),$$

for all $n \geq 1$, we have

$$C(\omega) \leq C(\theta \omega) - \lambda' + \Phi^K_1(\omega). \quad (3.2.22)$$

Combining estimates (3.2.21) and (3.2.22), we have

$$C(\theta \omega) - C(\omega) \geq \min\{0, \lambda' - \Phi^K_1(\omega)\}.$$

That is to say that, $C \circ \theta - C$ has a $\mathbb{P}$-integrable, minorant, thus, $C$ is adjusted to $\theta$. □

The following nontrivial facts from ergodic theory would be valuable in what follows.

**Proposition 3.2.9** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta : \Omega \to \Omega$ be a measure $\mathbb{P}$-preserving transformation. Then the following are equivalent:

1. $\theta$ is ergodic.
2. If $B \in \mathcal{F}$ with $\mathbb{P}(\theta^{-1} B \Delta B) = 0$, then $\mathbb{P}(B) = 0$, or 1.
3. If $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, then $\mathbb{P}(\bigcup_{k=1}^{\infty} \theta^{-k} A) = 1$.

**Proof.** We prove the direction we would require in this work: (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

Recall: A $\mathbb{P}$-preserving transformation $\theta$ is said to be ergodic if for every $A \in \mathcal{F}$, satisfying $\theta^{-1} A = A$, we have $\mathbb{P}(A) = 0$ or 1.

(1)$\Rightarrow$(2) Let $B \in \mathcal{F}$ be such that $\mathbb{P}(\theta^{-1} B \Delta B) = 0$. We define a measurable set $C$ with $C = \theta^{-1} C$ and $\mathbb{P}(C \Delta B) = 0$. Let

$$C = \{ \omega \in \Omega : \theta^k \omega \in B \text{ infinitely often} \} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \theta^{-n} B$$
Then $\theta^{-1}C = C$, hence by (1) we have that $\mathbb{P}(C) = 0$ or 1.

Furthermore,

$$
\mathbb{P}(C \Delta B) = \mathbb{P}\left( \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \theta^{-n}B \cap B^c \right) + \mathbb{P}\left( \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \theta^{-n}B^c \cap B \right).
$$

$$
= \mathbb{P}\left( \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \theta^{-n}B \cap B^c \right) + \mathbb{P}\left( \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \theta^{-n}B^c \cap B \right).
$$

$$
\leq \mathbb{P}\left( \bigcup_{n=1}^{\infty} \theta^{-n}B \cap B^c \right) + \mathbb{P}\left( \bigcup_{n=1}^{\infty} \theta^{-n}B^c \cap B \right)
$$

$$
\leq \sum_{n=1}^{\infty} \mathbb{P}(\theta^{-n}B \Delta B).
$$

It remains to show that $\mathbb{P}(\theta^{-n}B \Delta B) = 0$, we shall achieve this by induction.

For $n = 1$, $\mathbb{P}(\theta^{-1}B \Delta B) = 0$, by assumption. Assume it holds for $n = l$, $l \in \mathbb{N}$, that is,

$$
\mathbb{P}(\theta^{-l}B \Delta B) = 0.
$$

For $n = l + 1$, we have that

$$
\mathbb{P}(\theta^{-l-1}B \Delta B) \leq \mathbb{P}(\theta^{-l}B \Delta \theta^{-1}B) + \mathbb{P}(\theta^{-1}B \Delta B)
$$

$$
= 0.
$$

Hence, $\mathbb{P}(\theta^{-n}B \Delta B) = 0$, for $n \geq 1$ and it follows that $\mathbb{P}(C \Delta B) = 0$; which implies that $\mathbb{P}(C) = \mathbb{P}(B)$. Therefore, $\mathbb{P}(B) = 0$ or 1.

$(2) \Rightarrow (3)$ Let $\mathbb{P}(A) > 0$, and let $B = \bigcup_{k=1}^{\infty} \theta^{-k}A$, then $\theta^{-1}B \subset B$. Using the fact that $\theta$ is $\mathbb{P}$-preserving, we have that $\mathbb{P}(B) > 0$ and

$$
\mathbb{P}(\theta^{-1}B \Delta B) = \mathbb{P}(B) - \mathbb{P}(\theta^{-1}B) = 0.
$$

Thus, by (2), we conclude that $\mathbb{P}(B) = 1$. □

**Theorem 3.2.10 (Variant of random semiuniform ergodic theorem [41])** In the situation of Theorem 3.2.8 there exist $\lambda' < \lambda$ and $k_0 \geq 1$ such that for all $k \geq k_0$ there are adjusted random variable $\hat{C}_k : \Omega \rightarrow [0, \infty)$ and an ergodic component $\Omega_k$ of $\theta^k$ with $\mathbb{P}(\Omega_k) \geq \frac{1}{k}$ such that

$$
\Phi_k(\omega, x) \leq \hat{C}_k(\theta^k\omega) - \hat{C}_k(\omega) + k\lambda', \quad \text{for } \mathbb{P} - a.e. \omega \in \Omega_k \text{ and all } x \in K(\omega). \quad (3.2.23)
$$

The random variables $\hat{C}_k$ can also be chosen to take values in $(-\infty, 0]$, furthermore,

---

1An ergodic component of $\theta^k$ is a $\theta^k$-invariant set $\Omega_k$ of positive measure such that $\theta^k|_{\Omega_k}$ is ergodic.
(a) If \((\Phi_n)_{n \in \mathbb{N}}\) is additive\(^2\) then \(k_0 = 1\), so that \((3.2.23)\) holds for \(k = 1\) and \(\mathbb{P} - a.e. \omega \in \Omega\);

(b) If \(\theta\) totally is ergodic\(^3\) then \((3.2.23)\) holds for \(\mathbb{P} - a.e. \omega \in \Omega\).

**Proof.** As \(\hat{\Phi}^k < \lambda\), there is \(k_0 \in \mathbb{N}\) such that \(\mathbb{E}_{\mathbb{P}}[\Phi^K_k] = \int_{\Omega} \Phi^K_k \, d\mathbb{P} < \lambda'k\), for some \(\lambda' < \lambda\) and all \(k \geq k_0\). Fix any such \(k\), then by Birkhoff’s ergodic theorem (Theorem 3.2.1) we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \Phi^K_k (\theta^{-jk}\omega) = \mathbb{E}_{\mathbb{P}}[\Phi^K_k | \mathcal{I}_k], \quad \text{for } \mathbb{P} - a.e. \omega
\]

(3.2.24)

where \(\mathbb{E}_{\mathbb{P}}[\cdot | \mathcal{I}]\) denotes conditional expectation w.r.t. the \(\sigma\)-algebra \(\mathcal{I} \subset \mathcal{F}\) of all \(\theta^k\)-invariant sets.

Since \(\theta\) is ergodic, all sets of positive measure in \(\mathcal{I}_k\) have measures at least \(\frac{1}{k}\). As \(\int_{\Omega} \mathbb{E}_{\mathbb{P}}[\Phi^K_k | \mathcal{I}_k] d\mathbb{P} = \mathbb{E}_{\mathbb{P}}[\Phi^K_k] \leq \lambda'k\), this means that there is an ergodic component of \(\theta^k\)

\[
\Omega_k := \left\{ \omega : \forall j \geq 1 \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \Phi^K_k (\theta^{-jk}\omega) = \mathbb{E}_{\mathbb{P}}[\Phi^K_k | \mathcal{I}_k] \right\},
\]

such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \Phi^K_k (\theta^{-jk}\omega) \leq \lambda'k \quad \text{for } \mathbb{P} - a.e. \omega \in \Omega_k.
\]

Hence,

\[
0 \leq \hat{C}_k(\omega) = \sup_{n \geq 0} (-\lambda'nk + \sum_{j=1}^{n} \Phi^K_k (\theta^{-jk}\omega)) < \infty, \quad \text{for } \mathbb{P} - a.e. \omega \in \Omega_k.
\]

Let \(\omega \in \Omega_k\), we have that

\[
\hat{C}_k(\theta^k \omega) = \sup_{n \geq 0} (-\lambda'nk + \sum_{j=0}^{n-1} \Phi^K_k (\theta^{-jk}\omega))
\]

\[
\geq \sup_{n \geq 1} (-\lambda'nk + \sum_{j=0}^{n-1} \Phi^K_k (\theta^{-jk}\omega))
\]

\[
= \sup_{n \geq 1} (-\lambda'(n-1)k + \sum_{j=1}^{n-1} \Phi^K_k (\theta^{-jk}\omega)) - \lambda'k + \Phi^K_k (\omega).
\]

\[
= \hat{C}_k(\omega) - \lambda'k + \Phi^K_k (\omega).
\]

This implies that \(\hat{C}_k \circ \theta^k - \hat{C}_k\) has an integrable minorant \(-\lambda'k + \Phi^K_k\), it follows that \(\hat{C}_k \circ \theta^k - \hat{C}_k \in L^1(\mathbb{P})\) and \(\int (\hat{C}_k \circ \theta^k - \hat{C}_k) d\mathbb{P} = 0\). Which implies that \(\hat{C}_k\) is adjusted to \(\theta^k\). In order to show that

\(^2\)That is, \(\Phi_{n+m}(\omega) = \Phi_n(\Phi_m(\Theta^m))\), for all \(m, n \in \mathbb{N}\).

\(^3\)That is, \(\theta^k\) is ergodic for all \(k \in \mathbb{N}\).
it is also adjusted to \( \theta \), let \( l \in \{0, \cdots, k - 1\} \). Then by Birkhoff’s Ergodic Theorem we have that

\[
0 \leq \lim_{n \to \infty} \frac{1}{nk + l} \dot{C}_k(\theta^{nk+l}\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{k} (\dot{C}_k \circ \theta^k - \dot{C}_k)(\theta^{jk}(\theta^l\omega))
\]

\[
= \frac{1}{k} \mathbb{E}_{\mathbb{P}}[(\dot{C}_k \circ \theta^k) - \dot{C}_k](\theta^1\omega) \quad \text{for } \mathbb{P} - a.e.
\]

As \( \int \mathbb{E}_{\mathbb{P}}[\dot{C}_k \circ \theta^k - \dot{C}_k|\mathcal{I}_k] \, d\mathbb{P} = \int (\dot{C}_k \circ \theta^k - \dot{C}_k) \, d\mathbb{P} = 0 \), it follows that the limit is actually equal to zero for \( \mathbb{P} - a.e. \) \( \omega \). Hence \( \dot{C}_k \) is adjusted to \( \theta \). The case \( n \to -\infty \) follows similarly.

The random variable \( \dot{C}_k \) constructed above is nonnegative. To modify the proof for nonpositive random variable, we take

\[
\dot{C}_k(\omega) = - \sup_{n \geq 0} \left( -\lambda'nk + \sum_{j=0}^{n-1} \Phi^k_k(\theta^{jk}\omega) \right).
\]

As usual, one can show that there is an ergodic component \( \Omega_k \) of \( \theta^k \) such that \( -\infty \leq \dot{C}_k(\omega) \leq 0 \) for \( \mathbb{P} - a.e. \) \( \omega \in \Omega_k \); and

\[
\dot{C}_k(\theta^k\omega) = \inf_{n \geq 0} \left( \lambda'nk - \sum_{j=1}^{n} \Phi^k_k(\theta^{jk}\omega) \right)
\]

\[
= \inf_{n \geq 0} \left( \lambda'(n + 1)k - \sum_{j=0}^{n} \Phi^k_k(\theta^{jk}\omega) \right) - \lambda'k + \Phi^k_k(\omega)
\]

\[
\geq \dot{C}_k(\omega) - \lambda'k + \Phi^k_k(\omega).
\]

So, \( \dot{C}_k \circ \theta^k - \dot{C}_k \) has the integrable minorant \(-\lambda'k + \Phi^k_k(\omega)\), and it can easily be deduced that \( \dot{C}_k \) is adjusted to \( \theta \). □

**Remark 3.2.11** Now let \( \varphi \) be a \( C^1 \) random dynamical system and define a random continuous function by \( \Phi(t, \omega, x) := \log \| D\varphi(t, \omega, x) \| \), by cocycle property and chain rule of differentiation we have

\[
\log \| D\varphi(t + s, x) \| = \log \| D\varphi(t, \theta^s\omega, \varphi(s, \omega, x))D\varphi(s, \omega, x) \| \\
\quad \leq \log \| D\varphi(t, \theta^s\omega, \varphi(s, \omega, x)) \| + \log \| D\varphi(s, \omega, x) \|.
\]

So we have

\[
\Phi(t + s, \omega, x) \leq \Phi(s, \omega, x) + \Phi(t, \theta^s\omega, \varphi(s, \omega, x)),
\]

that is to say that the random continuous function \( \Phi : \mathbb{R} \times \Omega \times X \to \mathbb{R} \) is subadditive with respect to \( \Theta = (\theta, \varphi) \).
3.3 Random continuous invariant graphs

In the theory of nonautonomous dynamical systems, it is possible to have more than one base
dynamics; this is possible when there are more than one driving systems with different structures
(example; topological and measurable structures) present in the system.

The vast majority of literature in the theory of nonautonomous dynamical systems have been
extensively based on the case when the base dynamics are endowed with topological structures. We
want to consider nonautonomous dynamical system with ergodic and topological base dynamics.

Definition 3.3.1 Let \( \varphi : T \times \Omega \times E \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) be an RDS over the ergodic dynamical system
\((\Omega, \theta)\) and let \( \eta : T \times \Omega \times E \rightarrow E \) be a topological dynamical system, where \( E \) is a compact metric
space. The product map \( H : T \times \Omega \times E \times \mathbb{R}^d \rightarrow \Omega \times E \times \mathbb{R}^d \) given by
\[
\Omega \times E \times \mathbb{R}^d \ni (\omega, s, x) \mapsto H(t, \omega, s, x) := (\theta_t \omega, \eta_t(\omega, s), \varphi_t(\omega, s, x))
\]
is called a double skew product dynamical system.

Remark 3.3.2 It is crucial to note that, it is possible to define a random dynamical system \( \varphi \) over
the product base dynamic \( \theta \times \eta \), since

- \( \Omega \ni \omega \mapsto \theta_t \omega \) is measurable for for all \( t \in T \) and
- \( E \ni s \mapsto \eta_t(\omega, s) \) is continuous for all \((t, \omega) \in T \times \Omega\);

that is to say that \((\Omega \times E, \theta \times \eta)\) is a metric dynamical system (measure theoretical dynamical sys-
tem). Technically, metric dynamical system (not necessarily ergodic) is the base dynamic required
to define random dynamical system, we always assume ergodic base dynamic in order to have a
simplified discussion. Indeed, we have the following cocycle property
\[
\varphi_{t+r}(\omega, s, x) = \varphi_t(\theta_r \omega, \eta_r(\omega, s), \varphi_r(\omega, s, x))
\]

In what follows, we shall give some conditions on the product \( \theta \times \eta \), which seems natural and
compensate for the loss of ergodicity of base dynamical system in our discussion.

Definition 3.3.3 The product metric dynamical system \( \theta \times \eta \) is minimal, if each forward \( \theta \times \eta\)-
invariant subset \( A \subset \Omega \times E \) obeys the following dichotomy:

- either \( A(\omega) = E \) for \( \mathbb{P} - a.e. \omega \in \Omega \),
or $A(\omega) = \emptyset$ for $\mathbb{P} - a.e. \omega \in \Omega$.

The main idea in this section is to be able to demonstrate that random invariant set with certain structures consist of finite number of continuous random curves using random selection theorem (Proposition 2.1.14) and some topological techniques. By defining $\eta$ as $S^1 \ni s \mapsto \eta_t(\omega, s) := s + t \mod 1$, $(t, \omega) \in T \times \Omega$, in the next section (section 3.4), we are able to show that the continuous random graph is indeed periodic in appropriate sense (random periodic). The analysis in this section, could be looked at, as that of dynamical systems with random and deterministic forcing. Our major tools here will be some aspect of semiuniform ergodic theorem and pullback technique.

We have demonstrated that random invariant measure supports some random compact sets (example random attractor). That is to say that we can refer from time to time the regularity of some random compact sets in terms of the signs of the extremal Lyapunov exponents of the RDS. In a nutshell, semiuniform ergodic theorem, would employ Lyapunov exponents to give us the regularity of the random invariant graph.

We state the following results by Crauel and Flandoli [25] to illustrate possible candidate for such random compact set in the Assumptions A below. However, we are very much aware that connectedness is a bit too strong for the existence of random periodic curves, we included such assumption for completeness as we shall not refer to this in the prove of our main result in the next section.

**Lemma 3.3.4 (Facts about random sets [4], [41], [17])**

1. Let $A$ be a random set, then $\text{int}(A)$ is a random open set.

2. If $\theta \times \eta$ is a measurable homeomorphism and $A$ is a random compact set, then $(\theta \times \eta)(A)$ is a random compact set with fibres $\eta(\theta^{-1} \omega, A(\theta^{-1} \omega))$.

3. If $(A_n)_{n \in \mathbb{N}}$ are random compact sets, then $\cap_{n \in \mathbb{N}} A_n$ is a random compact set with fibres $\cap_{n \in \mathbb{N}} A_n(\omega)$.

4. If $A$ is random open or closed set, then $\pi_1(A) = \{ \omega \in \Omega : A(\omega) \neq \emptyset \}$ is $\mathcal{F}$-measurable.

**Lemma 3.3.5 ([25])** Suppose $A$ is a non-connected compact subset of a metric space $X$.

1. There exists $\alpha_0$ such that for $\alpha \leq \alpha_0$, the $\alpha$-neighbourhood of $U_\alpha(A) = \{ y \in X : d(y, A) < \alpha \}$ of $A$ is the disjoint union of two open sets.
(2) If $\alpha_0$ is as in (1), then
\[
\inf \{d(S, A) : A \subset S, \text{and } S \text{ is connected} \} \geq \alpha_0.
\]

**Proposition 3.3.6 (Connectedness of random attractor [25], [13])** Let $\varphi$ be an RDS on a connected space $X$, suppose that $\varphi$ has a global random attractor $K$, then $\mathbb{P}$-almost surely $K$ is connected.

**Proof.** The prove of this result is from the work of Crauel and Flandoli [25].

Since $K$ is random invariant, that is $\varphi_t(\omega)K(\omega) = K(\theta_t\omega)$, it is either $K$ is connected $\mathbb{P}$-almost surely, or $K$ is not connected $\mathbb{P}$-almost surely.

Suppose that $K$ is not connected $\mathbb{P}$-almost surely; we know from the above lemma that there is a number $\omega \mapsto \alpha_0(\omega)$ such that for $\alpha(\omega) \leq \alpha_0(\omega)$ the $\alpha$-neighbourhood of $K : U_\alpha(K) = \{x \in X : d(x, K) < \alpha\}$ is the disjoint union of two open nonempty sets. Pick a bounded connected subset $B$ such that
\[
\mathbb{P}\{K(\omega) \subset B\} \geq 1 - \varepsilon, \quad \text{for } 0 < \varepsilon \leq \frac{1}{2},
\]
(such choice of $B$ is possible since $X$ is connected and for arbitrary $x \in X$ the map $\omega \mapsto d(x, K(\omega))$ takes real values, so choosing $B$ as a ball around $x$ with sufficiently large radius will suffice.)

Then $\mathbb{P}\{K(\omega) \subset \varphi_t(\theta_{-t}\omega)B\} = \mathbb{P}\{K(\omega) \subset B\} \geq 1 - \varepsilon$ and since $\varphi_t(\theta_{-t}\omega)B$ is connected, we have by the above lemma (Lemma 3.3.5) that
\[
\mathbb{P}\{d(\varphi_t(\theta_{-t}\omega)B, K(\omega)) \geq \alpha_0(\omega)\} \geq 1 - \varepsilon
\]
for all $t \geq 0$.

On the other hand, since $B$ is a bounded subset and $K$ is a global random attractor, we have that the sequence
\[
d(\varphi_t(\theta_{-t}\omega)B, K(\omega))
\]
converges to zero $\mathbb{P}$-almost surely, so there exists $T$ such that
\[
\mathbb{P}\{d(\varphi_T(\theta_{-T}\omega)B, K(\omega)) \leq \frac{1}{2}\alpha_0(\omega)\} \geq 1 - \varepsilon,
\]
which gives a contradiction; thus, $K$ must be connected $\mathbb{P}$-almost surely. \qed
Assumptions

Here we give assumptions closely related to assumptions given by Zheng and Zhao [95] on their work on random periodic solutions. Consider a double skew product structure $H : \Omega \times E \times \mathbb{R}^d \to \Omega \times E \times \mathbb{R}^d$

\[
H(\omega, s, x) = (\theta \omega, \eta(\omega, s), \varphi(\omega, s, x))
\]

- the map $\eta(\omega, \cdot)$ is a homeomorphism,
- $(s \mapsto x \mapsto \varphi(\omega, s, x)$ is continuous and differentiable in $x$, and $D_x\varphi(\omega, s, x)$ continuous in $(s, x)$ for all $\omega \in \Omega$,

It follows that the mapping $\Omega \times E \ni (\omega, s) \mapsto (\theta \omega, \eta(\omega, s))$ is a random homeomorphism.

The Lyapunov exponent of $H$-invariant measure $\mu$ is defined by

\[
\lambda(\mu, \varphi) = \mu(\hat{\Phi}) = \inf_{n \in \mathbb{N}} \frac{1}{n} \mu(\Phi_n)
\]

where the subadditive random continuous function $\Phi_n$ is given by

\[
\Phi_n(\omega, s, x) := \log \|D_x\varphi^n(\omega, s, x)\|.
\]

We know from section [2.3] that for a forward invariant random compact set $K$, there exists $\mu^* \in \mathcal{M}_P(K)$, such that

\[
\lambda(\mu^*, \varphi) = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times E \times \mathbb{R}^d} \log \|D_x\varphi^n(\omega, s, x)\| d\mu^*(s, x) dP(\omega) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \max_{x \in K(\omega)} \log \|D_x\varphi^n(\omega, s, x)\| \right)
\]

Assumption A

A1: Assume that there exists an invariant random compact set $K$ and that the the product metric dynamical system $\theta \times \eta$ is minimal on $\Omega \times E$.

A2: Assume there is a random invariant compact set $K$ such that $K(\omega)$ is connected for $P-a.e. \omega \in \Omega$.

Assumption B

B1: The family $((s, x) \mapsto \log \|D_x\varphi^k(\omega, s, x)\|)_{\omega \in \Omega}$ is equicontinuous.

B2: Given $\varepsilon > 0$, there exists $r > 0$ such that, for all $k \in \mathbb{T}$

\[
\sup\{\log \|D_x\varphi^k(\omega, s, x)\| : \omega \in \Omega, (s, x) \in B_r(K(\omega))\} \leq \sup\{\log \|D_x\varphi^k(\omega, s, x)\| : \omega \in \Omega, (s, x) \in K(\omega)\} + \varepsilon.
\]
Assumption C: Assume that the Lyapunov exponent $\lambda(\mu, \varphi)$ satisfies

$$\lambda(\mu, \varphi) < 0, \quad \text{for all measures } \mu \in \mathcal{M}_p(K).$$

Next, for $\varepsilon \in (0,r]$, define $N_\varepsilon(\omega, s)$ to be the smallest number of open balls $B_{\varepsilon,\mathbb{C}(\omega)}(x) \subset \mathbb{R}^d$ centred at points $x \in K(\omega, s)$ that are required to cover $K(\omega, s)$.

Lemma 3.3.7 ([41])

1. The set $K$ is a random compact set over the base $(\Omega \times \mathbb{E}, \mathcal{F} \otimes \mathcal{B}(\mathbb{E}))$, and $\varphi^k(\omega, s, K(\omega, s)) = K(\theta \times \eta)^k(\omega, s)$ for $P - \text{a.e. } \omega \in \Omega$ and $s \in \mathbb{E}$.

2. The functions $N_\varepsilon$ are $\mathcal{F} \otimes \mathcal{B}(\mathbb{E})$-measurable.

3. For each $\omega \in \Omega$, the function $N_\varepsilon(\omega, \cdot) : \mathbb{E} \to \mathbb{N}$ is upper semicontinuous.

Proof.

1. For each $(\omega, s) \in \Omega \times \mathbb{E}$, the set $K(\omega, s) = \{x \in \mathbb{R}^d : (s, x) \in K(\omega)\}$ is compact, as it is a section of a compact set $K(\omega)$.

Denote the metric on $\mathbb{E}$ by $\rho$ and define, for each $n > 0$ a metric $d_n$ by

$$d_n((s, x), (\hat{s}, \hat{x})) = \|x - \hat{x}\| + n\rho(s, \hat{s}).$$

Each $d_n$ generates a product topology on $\mathbb{E} \times \mathbb{R}^d$. Further, $\omega \mapsto d_n((s, x), K(\omega))$ is measurable for all $(s, x) \in \mathbb{E} \times \mathbb{R}^d$ and $s \mapsto d_n((s, x), K(\omega))$ is continuous for $\omega \in \Omega$ and $x \in \mathbb{R}^d$. Which implies that $(\omega, s) \mapsto d_n((s, x), K(\omega))$ is measurable.

One can check that $d_n((s, x), K(\omega))$ is independent of $n$, hence $(\omega, s) \mapsto \sup_n d_n((s, x), K(\omega)) = d(x, K(\omega, s))$ is measurable, so that $K(\omega, s)$ is indeed a random compact set.

We know that $x \in K(\omega, s)$ if and only if $(s, x) \in K(\omega)$, so that $(\eta(\omega, s), \varphi(\omega, s, x)) \in K(\theta \omega)$ if and only if $\varphi(\omega, s, x) \in K((\theta \times \eta)(\omega, s)) = K(\theta \omega, \eta(\omega, s))$.

2. Now, the set $K$ is a random compact set over the base dynamical system $(\Omega \times \mathbb{E}, \mathcal{F} \otimes \mathcal{B}(\mathbb{E}))$, so by random selection theorem, there is a sequence of measurable maps $a_k : \Omega \times \mathbb{E} \to \mathbb{R}^d$ such that $K(\omega, s) = \text{closure}\{a_k(\omega, s) : k \in \mathbb{N}\}$ for all $(\omega, s) \in \Omega \times \mathbb{E}$.

Let $\varepsilon > 0$, for $n \in \mathbb{N}$ denote by $\mathcal{L}_n$ the family of subsets of $\mathbb{N}$ with $n$ elements. Then the sets

$$V_n := \bigcup_{\mathcal{L} \in \mathcal{L}_n} \bigcap_{k \in \mathbb{N}} \bigcup_{l \in \mathcal{L}} \{ (\omega, s) \in \Omega \times \mathbb{E} : \|a_k(\omega, s) - a_l(\omega, s)\| < \varepsilon \}$$

are $\mathcal{F} \otimes \mathcal{B}(\mathbb{E})$-measurable, and $N_\varepsilon(\omega, s) \leq n$ if and only if $(\omega, s) \in V_n$. 

66
(3) It is enough to verify that for $\omega \in \Omega$, for each $\alpha \in \mathbb{R}$, the set $U(\omega) := \{ s \in E : N_\varepsilon(\omega, s) < \alpha \}$ is open subset of $E$.

We know that $K(\omega, s)$ is a compact set, by random selection (Theorem 2.1.14), there exists $a_k : \Omega \times E \to \mathbb{R}^d$ such that $K(\omega, s) = \text{closure}\{a_k(\omega, s) : k \in \mathbb{N}\}$.

Also, $K(\omega) = \{(s, x) \in \Omega \times \mathbb{R}^d : x \in K(\omega, s)\}$; so that for any $y \in K(\omega)$ there exists $k \in \mathbb{N}$ such that $\|(s, a_k(\omega, s)) - y\| < \frac{\delta}{2}$.

Consider the system of subsets of $E$

$$B(\omega) = \bigcup_{k \in \mathbb{N}} \{ s \in E : (s, a_k(\omega, s)) \in B_\varepsilon(y) \}.$$  

The function $s \mapsto a_k(\omega, s)$ is continuous by Proposition A.4, so it follows that the function $(s, a_k(\omega, s))$ is also continuous, hence the system of subsets $B(\omega)$ consist of preimage of open sets under continuous function. Therefore, $B(\omega)$ is a system of open subsets of $E$.

Finally, for each $\alpha \in \mathbb{R}$, $N_\varepsilon(\omega, s) < \alpha$ if and only if $s \in B(\omega)$. Thus, $U(\omega)$ is open. $\square$

**Proposition 3.3.8** Let $K$ be an invariant random compact set such that Assumptions B and C hold true. Then there exist $c > 0$, $\delta > 0$ and $r > 0$ such for all $n \in \mathbb{T}$ and almost all $\omega \in \Omega$

$$\|D_x \varphi^k(\omega, s, x)\| \leq c e^{-\delta k}, \quad \text{for all } (s, x) \in B_r(K(\omega)). \quad (3.3.5)$$

**Proof.** By Theorem 3.2.10 there are $\lambda' < 0$, $k \geq 1$, ergodic component $\Omega_k$ of $\theta^k$ and an adjusted random variable $\hat{C} : \Omega \to (-\infty, 0]$ such that

$$\Phi_k(\omega, s, x) \leq \hat{C}(\theta^k \omega) - \hat{C}(\omega) + \lambda' k, \quad \text{for } \mathbb{P} - a.e. \omega \in \Omega_k \text{ and all } (s, x) \in K(\omega). \quad (3.3.6)$$

So that, by assumption B2, there are $\lambda'' < 0$ and $r > 0$ such that

$$\log \|D_x \varphi^k(\omega, s, x)\| \leq \hat{C}(\theta^k \omega) - \hat{C}(\omega) + \lambda'' k, \quad \text{for } \mathbb{P} - a.e. \omega \in \Omega_k \text{ and all } (s, x) \in B_r(K(\omega)). \quad (3.3.7)$$

In particular, for $\delta \in (0, -\lambda'')$, there is $N \in \mathbb{T}$ such that

$$\frac{1}{k} \Phi_k(\omega, s, x) \leq -\delta, \quad k \geq N \quad \text{for all } (s, x) \in B_r(K(\omega)), \quad (3.3.8)$$

$$\|D_x \varphi^k(\omega, s, x)\| \leq e^{-\delta k}, \quad k \geq N, \quad (s, x) \in B_r(K(\omega)).$$

Now, let $\tilde{c} = \max_{k \geq 1} \sup \{e^{\delta k} \|D_x \varphi^k(\omega, s, x)\| \}$ and then $\|D_x \varphi^k(\omega, s, x)\| \leq \tilde{c} e^{-k \delta}$, $k \leq N$, for all $(s, x) \in B_r(K(\omega))$. So that if we take $c = \max\{\tilde{c}, 1\}$ we have that for all $k \in \mathbb{T}$

$$\|D_x \varphi^k(\omega, s, x)\| \leq ce^{-k \delta}, \quad \forall (s, x) \in B_r(K(\omega)).$$
Proposition 3.3.9 ([41]) Let $H$ be a double skew product and $K$ be a random invariant compact set such that assumptions B1 or B2 and C hold true. Then there are $n \in \mathbb{N}$, a random variable $c : \Omega \to \mathbb{R}^+$ and a non-empty open forward $\theta \times \eta$-invariant set $A$ such that for $\mathbb{P} - a.e. \omega \in \Omega$

- $\#K(\omega, s) = n$ for all $s \in A(\omega)$,
- $\sup \{ \#K(\omega, s) : s \in \mathbb{E} \} < \infty$ and
- for all $s \in \mathbb{E}$, any two different points $y, y' \in K(\omega, s)$ have distance at least $c(\omega)$.

Proof. There exist $\lambda' < 0, k \geq 1$, an ergodic component $\Omega_k$ of $\theta^k$ and an adjusted random variable $\hat{\Phi}_k(\omega, s, x) \leq \hat{\Phi}(\theta^k \omega) - \hat{\Phi}(\omega) + \lambda' k$, for $\mathbb{P}$-a.e. $\omega$ and all $(s, x) \in K(\omega)$. (3.3.9)

Hence in view of assumption B2, there are $r > 0$ and $\gamma > 0$ such that

$$\log \|D_x \varphi^k(\omega, s, x)\| \leq \hat{\Phi}(\theta^k \omega) - \hat{\Phi}(\omega) - \gamma k,$$

for $\mathbb{P}$-a.e. $\omega \in \Omega_k$ and all $(s, x) \in B_r(K(\omega))$. (3.3.10)

Now fix $\varepsilon \in (0, r], \omega \in \Omega_k$ and $s \in \mathbb{E}$, and denote $N = N_\varepsilon(\omega, s)$. There are $x_1, \ldots, x_N \in K(\omega, s)$ such that $K(\omega, s) \subset \bigcup_{i=1}^N B_{e \varepsilon \hat{\Phi}(\omega)}(x_i)$. As $\varphi^k(\omega, s, K(\omega, s)) = K((\theta \times \eta)^k)(\omega, s)$ it follows that

$$K((\theta \times \eta)^k(\omega, s) \subset \bigcup_{i=1}^N \varphi^k(B_{e \varepsilon \hat{\Phi}(\omega)}(x_i)) \subset \bigcup_{i=1}^N B_{e^{-\gamma} e \varepsilon \hat{\Phi}(\theta^k \omega)}(\varphi^k(\omega, s, x_i)),$$

with points $\varphi^k(\omega, s, x_i) \in K((\theta \times \eta)^k(\omega, s))$.

Hence,

$$N_\varepsilon((\theta \times \eta)^k(\omega, s)) \leq N_{e^{-\gamma} \varepsilon}(((\theta \times \eta)^k(\omega, s)) \leq N_\varepsilon(\omega, s).$$ (3.3.11)

Consider the restricted system $(\theta \times \eta)^k|_{\Omega_k \times \mathbb{E}}$ and denote the normalized probability measure $\mathbb{P}|_{\Omega_k}$ by $\mathbb{P}_k$.

By lemma 3.3.7 (2) and (3), there is a subset $\tilde{\Omega}_k \subset \Omega_k$ of full measure such that the random sets $U_{\varepsilon, \alpha} = \{(\omega, s) \in \Omega_k \times \mathbb{E} : N_\varepsilon(\omega, s) < \alpha\}$ are open for all $\alpha \in \mathbb{R}$ and $\varepsilon = e^{-p\eta r}$, with $p \in \mathbb{N}$. For measurability purposes we restrict to these countable many values of $\varepsilon$ from now on. Let

$$n_\varepsilon(\omega) = \min \{ \alpha \in \mathbb{N} : U_{\varepsilon, \alpha}(\omega) \neq \emptyset \}$$

and the measurability of $n_\varepsilon$ follows from lemma 3.3.4 (4). Due to (3.3.11) we have

$$n_\varepsilon(\theta^k \omega) \leq n_\varepsilon(\omega) \quad \text{for } \mathbb{P}_k \text{-a.e. } \omega \in \Omega_k,$$

68
and thus, ergodicity of \((\theta^k, \mathbb{P}_k)\) implies that all \(n_x\) are constant \(\mathbb{P}_k - a.e.\) We denote these constants by \(n_x\) as well. By the first inequality in \(3.3.11\) we have \(n_x \leq n_{\varepsilon-n_x}\). But second inequality of \(3.3.11\) gives \(n_{\varepsilon-n_x} \leq n_x\) for all \(\varepsilon \in (0, r]\); so all \(n_x\) coincide. Denote their common value by \(n\)

Using \(3.3.11\) again, we see that the random open set \(U_{r,n}\) is \((\theta \times \eta)^k\)-invariant and we have \(U_{r,n} = U_{r,n}\) for all \(\varepsilon\). Similarly, for each integer \(m > n\) the set \(U_{r,m}\) is a non-empty \((\theta \times \eta)^k\)-invariant open random set and \(U_{r,m} = U_{r,m}\) for all \(\varepsilon\).

Next, we show that \(K(\omega), \omega \in \Omega_k\) intersects each fibre \(\{s\} \times K(\omega, s)\) in a finite number of points (this means that the cardinality of \(K(\omega, s)\) denoted by \(#K(\omega, s)\) = \(n\)). As \(K(\omega)\) is compact for fixed \(\omega\), there exists \((s_1, x_1), \ldots, (s_n, x_n)\) such that

\[
K(\omega) \subset \bigcup_{i=1}^{n} B_{r/2c(\omega)}(s_i, x_i). \tag{3.3.12}
\]

We will show that for any \(s \in U_{r,n}(\omega)\) the cardinality of \(K(\omega, s)\) is at most \(n\). Suppose not, there exists \(\hat{s} \in U_{r,n}(\omega)\) with \(#K(\omega, \hat{s}) > n\). Choose \(n+1\) distinct points \(x_1, \ldots, x_{n+1} \in K(\omega, \hat{s})\) and let

\[
\min_{i \neq j} \|x_i - x_j\|.
\]

Furthermore, for fixed \(k \in \mathbb{N}\) such that \(r e^{\hat{C}(\omega)} \alpha^k < a\), for some \(\alpha \in (0, 1)\) and choose, for each \(i = 1, \ldots, n+1\), some \(y_i \in (\varphi^k(\omega, \eta^{-k}(\omega, \hat{s})))^{-1} \{x_i\} \in K(\omega)\) (such \(y_i\) exists due the fact that \(K(\omega)\) is invariant), due to \(3.3.12\) there exists \(l \in \{1, \ldots, n\}\) and \(i, j \in \{1, \ldots, n+1\}\) such that \(y_i, y_j \in B_{r/2c(\omega)}(x_l)\). Hence the distance between two points is less that \(r e^{\hat{C}(\omega)}\), hence from Proposition \(3.3.8\) we conclude that

\[
\|x_i - x_j\| = \|\varphi^k(\omega, \eta^{-k}(\omega, \hat{s}), y_i) - \varphi^k(\omega, \eta^{-k}(\omega, \hat{s}), y_j)\| \\
\leq \alpha^k r e^{\hat{C}(\omega)} < a,
\]

contracting the definition of \(a\). Thus, for \(\mathbb{P}_k - a.e.\ \omega \in \Omega_k\), the following holds:

- \(#K(\omega, s) = n(\omega)\) for all \(s \in U_{r,n}(\omega)\),
- \(\sup\{#K(\omega, s) : s \in \mathbb{E}\} < \infty\) and
- \(d(x, x') \geq c(\omega) := r e^{\hat{C}(\omega)}\) for all \(s \in \mathbb{E}\) and any two different points \(x, x' \in K(\omega, s)\).

Let \(A_k = U_{r,n} = \{\omega, s) \in \Omega_k \times \mathbb{E} : N_r(\omega, s) < n\}.\) If \(k = 1, A = A_k\) satisfies the assertions of the proposition. Otherwise, we let

\[
A = \bigcup_{i=0}^{k-1}(\theta \times \eta)^i(A_k),
\]

69
then as $\bigcup_{i=0}^{k-1} \theta^i(\Omega_k) = \Omega$ up to set of $P$-measure zero, and the map $\eta^i(\omega)$ are homeomorphisms, and

$$\varphi^i(\omega, s, K(\omega, s)) = K((\theta \times \eta)^i(\omega, s)),$$

the above conclusions carry over to $P$ – a.e. $\omega \in \Omega$ as follows:

Observe that for ergodic component $\Omega_k$ of $\theta^k$, we have that

$$\bigcup_{i=0}^{k-1} \theta^i(\Omega_k) = \Omega \text{ up to set of } P\text{-measure zero.}$$

By Theorem 3.2.8 and assumption B2, there exist $r > 0, \lambda' < 0, i \in \{0, 1, \ldots, k - 1\}$ and an adjusted random variable $C$ such that if we let $\hat{\Omega} := \theta^i \Omega$, we have

$$\log \|D_x \varphi^k(\theta^{-i} \omega, s, x)\| \leq C(\theta^{k-i} \omega) - C(\theta^{-i} \omega) + \lambda' k, \quad \text{for all } k \in \mathbb{N}, P - a.e. \omega \in \hat{\Omega}, (s, x) \in B_r(K(\theta^{-i} \omega)).$$  \hspace{1cm} (3.3.13)

Following same argument as above, we have that for $\hat{P} - a.e. \omega \in \theta^i \Omega_k$

$$\#K(\theta^{-i} \omega, \eta^i(\theta^{-i} \omega, s)) = n(\theta^{-i} \omega) \quad \text{for all } s \in A_k(\theta^{-i} \omega).$$  \hspace{1cm} (3.3.14)

As $\eta^i(\theta)$ and $\varphi^i(\omega)$ are homeomorphisms, hence one-to-one and using the invariance of $K(\omega, s)$ with respect to $\varphi^i(\omega, s)$, we have that for $P_k - a.e. \omega \in \Omega_k$

$$\#\varphi^i(\theta^{-i} \omega, \eta^i(\theta^{-i} \omega, s)) = \#K(\omega, s);$$  \hspace{1cm} (3.3.15)

so that for all $P - a.e. \omega \in \Omega$

$$\#K(\omega, s) = n(\omega), \quad \text{for all } s \in A(\omega),$$  \hspace{1cm} (3.3.16)

where $A = \bigcup_{i=0}^{k-1}(\theta \times \eta)^i(A_k)$.

**Theorem 3.3.10** (P1) Let $H$ be a double skew product and $K$ be a random compact $H$-invariant set such that assumptions A1, B1 or B2 and C hold true. Then there are $n \in \mathbb{N}$ and a random variable $c : \Omega \rightarrow \mathbb{R}^+$ such that, for $P - a.e. \omega \in \Omega$

- $\#K(\omega, s) = n$ for all $s \in E$,
- the map $s \mapsto K(\omega, s)$ from $E$ to $K(\mathbb{R}^d)$ is continuous and
- for all $s \in E$, any two different points $y, y' \in K(\omega, s)$ have distance at least $c(\omega)$.
If we replace assumption A1 with A2, then for $P$-a.e. $\omega \in \Omega$, the set $K(\omega)$ consists of a single continuous graph.

**Proof.** The idea here is apply proposition 3.3.9, we note that one the highlights of the proof of the proposition is the construction of a non-empty forward $\theta \times \eta$-invariant random set $A \subset \Omega \times E$. It follows that $A^c$ is a backward $\theta \times \eta$-invariant random set and $A^c \neq \emptyset$ for $P$-a.e. $\omega \in \Omega$. Using the assumption that $\theta \times \eta$ is a minimal homeomorphism, we have that $A^c = \emptyset$ for $P$-a.e. $\omega \in \Omega$ and hence $A(\omega) = E$ for $P$-a.e. $\omega \in \Omega$.

First and second assertions of proposition 3.3.9 together with random compactness of $K$ implies that $E \ni s \mapsto \phi_i(\omega, s) \in K(\omega, s)$, $i = 1, 2, \ldots, n(\omega)$, note that we have applied random selection theorem (Proposition 2.1.14) here. And then as $K(\omega, s)$ is the $s$-section of $K(\omega)$, we have that $K(\omega) = \text{graph} \phi_i i = 1, 2, \ldots, n(\omega)$.

It remains to show that $E \ni s \mapsto \phi_i(\omega, s)$ are continuous, Recall that for a compact metric space $E$ a function define on $E$ is continuous on if and only if its graph is compact (see proposition A.4 in the appendix).

**If we replace assumption A1 with A2:** By the second assertion of proposition 3.3.9 together with connectedness of $K(\omega)$, we have that there is a subset $\hat{\Omega} \subset \Omega$ of full measure such that $K(\omega, s) \subset \mathbb{R}^d$ consists of a single point for all $(\omega, s) \in \hat{\Omega} \times E$. As $K(\omega) \subset E \times \mathbb{R}^d$ is compact, then application of proposition A.4 shows that $K(\omega)$ must be the graph of a continuous function $\phi(\cdot, s) : E \to \mathbb{R}^d$. As $\{\phi(\omega, s)\} = K(\omega, s)$ is the only possible selection of $K$, $\phi$ is measurable (Proposition 2.1.14). \(\square\)

### 3.4 Random periodic curves

In this section, we shall consider a double skew product such that $E = S^1$ with

$$H_t(\omega, s, x) = (\theta_t(\omega, s + t \mod 1, \varphi_t(\omega, s, x)) \text{ (}\omega, s, x\text{) } \in \Omega \times E \times \mathbb{R}^d. \quad (3.4.1)$$

It is easy to verify that such $H$ is indeed a dynamical system on $\Omega \times S^1 \times \mathbb{R}^d$. The extra assumption on the random dynamical system $\varphi$ is that $x \mapsto \varphi_t(\omega, s, x)$ is at least $C^1$ for all $(\omega, s) \in \Omega \times S^1$. We aim to obtain a result similar to Theorem 3.3.10 such that $K(\omega)$ consists finite number of random periodic curves.

**Definition 3.4.1** ([95]) Let $\phi(\omega) : \mathbb{R} \to \mathbb{R}^d$ be a continuous periodic function with period $\tau \in \mathbb{N}$ for each $\omega \in \Omega$. Define $L^\omega := \text{graph}(\phi(\omega)) = \{(s \mod 1, \phi(\omega, s)) : s \in \mathbb{R}\}$. If $L^\omega$ is invariant with
respect to the RDS $\varphi$ on $S^1 \times \mathbb{R}^d$, that is, $\varphi_t(\omega)L^\omega = L^{\theta_t \omega}$ and there is a minimum $T > 0$ (maximum $T < 0$) such that for any $s \in [0, \tau)$

$$
\varphi_t(\omega, s \mod 1, \phi(\omega, s)) = (s \mod 1, \phi(\theta_t \omega, s))
$$

for almost all $\omega \in \Omega$, then it is said that $\varphi$ has random periodic solution of period $T$ with random periodic curve $L^\omega$ of winding number $\tau$.

We know from the previous section that $K(\omega)$ consist of graph of continuous function, that is, there is a random continuous function $\phi(\omega, \cdot)$ such that

$$
\{ (s \mod 1, \phi(\omega, s)) : s \in S^1 \} \subset K(\omega).
$$

We define $(\varepsilon, \delta)$-neighbourhood of $(s \mod 1, x) \in K(\omega)$ as follows

$$
B(s \mod 1, x \mod 1, \delta \mod 1, \varepsilon) = \{ (\hat{s} \mod 1, \hat{x}) : |s - \hat{s}| \leq \delta, ||\phi(\omega, s) - \hat{x}|| \leq \varepsilon \},
$$

and for any $s \in S^1$, $B_s(\delta, \varepsilon) = \bigcup_{(s, x) \in K(\omega)} B(s, x, \delta, \varepsilon)$, so that $B_\varepsilon(K(\omega)) = \bigcup_{s \in S^1} B_s(\delta, \varepsilon)$.

Now define $\hat{\theta} \omega := \theta_{t_1} \omega$, where $t_1$ is the time the particle in $S^1$ rotate a full circle. With this $\hat{\theta}$, we reduce our continuous time dynamical system $H$ to discrete time dynamical system $\hat{H}$. We note that the system $\hat{\theta} : \mathbb{Z} \times \Omega \rightarrow \Omega$ is $\mathbb{P}$-preserving such that for any $n, m \in \mathbb{Z}$ one gets

$$
H_{t_1}^{n+m}(\omega, s, x) = \hat{H}^{n+m}(\omega, s, x) = \hat{H}^n \circ \hat{H}^m(\omega, s, x),
$$

for $(s, x) \in S^1 \times \mathbb{R}^d$ and $\mathbb{P}$-almost all $\omega \in \Omega$. Let us denote our reference random dynamical system on $S^1 \times \mathbb{R}^d$ by

$$
G(\omega, s, x) := (h(s), \hat{\phi}(\omega, s, x))
$$

where $h(s) := s + t_1 \mod 1$, so that for any $n, m \in \mathbb{N}$, we have

$$
G^{m+n}(\omega, s, x) = G^n(\hat{\theta}^m \omega) \circ G^m(\omega, s, x) \quad \mathbb{P} - a.s.
$$

We wish to employ pullback technique to show that the random invariant compact set $K(\omega)$ indeed consist of finite number of continuous periodic curves. Indeed, we want to prove the following result.
**Theorem 3.4.2 (Main result)** Under assumptions A1 or A2, B1 or B2 and C, there exist \( n(\omega) \in \mathbb{N} \) continuous periodic functions \( \phi_1(\omega), \ldots, \phi_{n(\omega)}(\omega) \) with periods \( \tau_1(\omega), \ldots, \tau_{n(\omega)}(\omega) \in \mathbb{N} \) respectively such that

\[
K(\omega) = \bigcup_{i=1}^{n(\omega)} L_i^\omega,
\]

where \( L_i^\omega = \text{graph}(\phi_i(\omega)) = \{(s, \phi_i(\omega, s)) : s \in [0, \tau_i(\omega))\}, i = 1, \ldots, n(\omega). \)

**Remark 3.4.3**

1. In fact, we only need to verify that the continuous functions \( \phi(\omega) \) from Theorem 3.3.10 is indeed random periodic, given that \( \mathbb{E} = \mathbb{S}^1 \ni s \mapsto \eta_t(\omega, s) = s + t \mod 1. \)

2. If we suppose that A2 hold, then Theorem 3.4.2 would give that \( K(\omega) \) consists of a single graph of random periodic function, that is, \( n(\omega) = 1. \) So, it is more interesting to provide the proof with Assumption A1.

We start by covering the invariant set \( K(\omega) \) by boxes within which we have semiuniform contraction. Rather than considering a complete covering of the whole set \( K(\omega), \) we concentrate on a strip \( D_{[s^*, s^*+\delta]}(\omega) = \{\{s\} \times K(\omega, s) : s \in [s^* - \delta, s^* + \delta]\}. \) For any \( s \in \mathbb{S}^1 \) define

\[
D_s(\omega) := \{s\} \times K(\omega, s)
\]

and for any \( (s, x) \in D_s(\omega), \) let \( U(s, x, \delta, \varepsilon) \) be the interior of \( B(s, x, \delta, \varepsilon). \) Then for any \( s^* \in \mathbb{S}^1, \)

\[
\{U(s^*, x, \delta, \varepsilon) : (s^*, x) \in D_{s^*}(\omega)\}
\]

is an open covering of \( D_{s^*}(\omega). \) By compactness of \( D_{s^*}(\omega), \) a finite subcover, \( U(s^*, x_{1\omega}, \delta, \varepsilon), \ldots, U(s^*, x_{p(\omega)}, \delta, \varepsilon), \) could be found. Define

\[
U^\omega(s^*, \delta, \varepsilon) = \bigcup_{i=1}^{p(\omega)} U(s^*, x_{i\omega}, \delta, \varepsilon),
\]

\[
B^\omega(s^*, \delta, \varepsilon) = \bigcup_{i=1}^{p(\omega)} B(s^*, x_{i\omega}, \delta, \varepsilon).
\]

It is easy to see that \( B^\omega(s^*, \delta, \varepsilon) \) is the closure of \( U^\omega(s^*, \delta, \varepsilon) \) and that \( D_{s^*}(\omega) \subset B^\omega(s^*, \delta, \varepsilon). \)

We merge overlap boxes \( B(s^*, x^i, \delta, \varepsilon) \) (if any) and work with the connected components of \( B^\omega(s^*, \delta, \varepsilon) \) which we will denote by \( B_1^\omega(s^*, \delta, \varepsilon), B_2^\omega(s^*, \delta, \varepsilon), \ldots, B_{d(\omega)}^\omega(s^*, \delta, \varepsilon) \) and let the minimal distance between any two of them be \( \beta^\omega > 0. \)

**Proposition 3.4.4** Under the assumption A1, B1 or B2 and C, \( K(\omega) \) is a union of a finite number of continuous periodic curves.

73
Proof. Let $\delta$ be independent of $s \in S^1$ and let $M \in \mathbb{N}$ such that $\frac{1}{M} \leq \delta$. Define $s_m = \frac{m}{M}$, $m = 1, 2, \cdots, M$. Then $\{(s_{m-1}, s_{m+1}) : m = 1, 2, \cdots, M\}$ (in which $s_{M+1} = s_1, s_0 = s_M$) covers $S^1$. By Theorem 3.3.10 we know that $D_{[s_{m-1}, s_{m+1}]}(\omega)$ contains finite number of continuous curves and denote their number by $d(\omega, m)$. We claim that $d(\omega, m)$ is independent of $m$ and the continuous curves on $D_{[s_{m-1}, s_{m+1}]}(\omega)$ can be extended to $\mathbb{S}^1$.

- $d(\omega, m)$ is independent of $m$ : The interval $[s_m, s_{m+1}]$ is contained in the interval $[s_{m-1}, s_{m+1}]$ and thus the strip $D_{[s_m, s_{m+1}]}(\omega)$ contains $d(\omega, m)$ curves. On the other hand, the interval $[s_m, s_{m+1}]$ is contained in the interval $[s_m, s_{m+2}]$ and the strip $D_{[s_m, s_{m+2}]}(\omega)$ contains $d(\omega, m+1)$ curves, it follows that the strip $D_{[s_m, s_{m+1}]}(\omega)$ contains $d(\omega, m+1)$ curves. Hence, $d(\omega, m) = d(\omega, m+1)$ for all $m \in \mathbb{N}$, so $d(\omega, m)$ must be independent of $m$ and we denote it by $d(\omega)$.

- Extension of curves on $D_{[s_{m-1}, s_{m+1}]}(\omega)$ to $\mathbb{S}^1$ : Denote the $d(\omega)$ curves on each strip $D_{[s_{m-1}, s_{m+1}]}(\omega)$ by $\phi_{m, 1}(\omega), \cdots, \phi_{m, d(\omega)}(\omega)$. Since $d(\omega)$ is independent of $m$, each $\phi_{m,i}$ can be extended to the whole $S^1$. More precisely, let us move to the universal cover $\mathbb{R}$ (as used in the definition of $L^\omega$) and $s_m = \frac{m}{M}$ for all $m \in \mathbb{N}$ and lift the function $\phi_{m,i}(\omega)$ to the cover $\mathbb{R}$ by $\hat{\phi}_{m,i}(\omega, s) = \phi_{m \text{ mod } M, i}(\omega, s \text{ mod } 1)$ for $s \in [s_{m-1}, s_{m+1}]$.

Now start by defining $\hat{\phi}_i(\omega) := \hat{\phi}_{0,i}(\omega)$ on $[s_{-1}, s_1]$, this agrees with some $\phi_{1,j}(\omega)$ for unique $j \in \{1, \cdots, d(\omega)\}$ on $[s_0, s_1]$. So defining $\hat{\phi}_i(\omega) = \phi_{1,j}(\omega)$, continue in this way to define $\hat{\phi}_i(\omega)$ uniquely for $s \geq 0$. Similarly, $\hat{\phi}_i$ agrees with some $\phi_{1,j}(\omega)$ on $[s_{-1}, s_0]$, we continue to the right to define $\hat{\phi}_i(\omega)$ for $s \leq 0$. Hence

$$D_s(\omega) = \{(s \mod 1, \phi_i(\omega, s)) : i \in \{1, \cdots, d(\omega)\}\}$$

The continuous curves on $D_{[s_{m-1}, s_{m+1}]}(\omega)$ extend to $S^1$ and we have the following random return map

$$G^l(\theta^{-l}\omega) : D_s(\theta^{-l}\omega) \to D_s(\omega)$$

which $D_s(\theta^{-l}\omega), D_s(\omega), l \in \mathbb{N}$, are finite sets containing $d(\omega)$ elements:

$$D_s(\theta^{-l}\omega) = \{(s \mod 1, \phi_i(\theta^{-l}\omega, s)) : i = 1, 2, \cdots, d(\theta^{-l}\omega)\},$$

$$D_s(\omega) = \{(s \mod 1, \phi_i(\omega, s)) : i = 1, 2, \cdots, d(\omega)\}.$$
for a fixed \( s \in \mathbb{R} \).

Also, we can verify that, for \( l \in \mathbb{N} \)

\[
G(\theta^{-l}\omega)D_s(\theta^{-l}\omega) = D_{s+d(\omega)}(\omega) = D_s(\omega).
\]

By the finiteness of \( D_s(\omega) \) and by continuity of \( \phi_i(\omega) \), we have

\[
\phi_i(\omega, s + 1) = \phi_i(\omega, s)
\]

\[
\phi_i(\omega, s + 2) = \phi_i(\omega, s)
\]

\[
\vdots
\]

\[
\phi_i(\omega, s + d(\omega)) = \phi_i(d(\omega), \omega)
\]

- If exact one of \( i_1, i_2, \ldots, i_d(\omega) \) is equal to \( i \); say \( i_{\tau_i(\omega)} = i \). Then

\[
\phi_i(\omega, s + \tau_i(\omega)) = \phi_i(\omega, s),
\]

for any \( s \in \mathbb{R} \). So \( \phi_i(\omega) \) is a periodic function of period \( \tau_i(\omega) \).

- If more than one of \( i_1, i_2, \ldots, i_d(\omega) \) is equal to \( i \). Denote \( \tau_i(\omega) \) the smallest number \( j \) such that \( i_{\tilde{\tau}_i(\omega)} = i \). Then

\[
\phi_i(\omega, s + \tau_i(\omega)) = \phi_i(\omega, s)
\]

\[
\phi_i(\omega, s + \tilde{\tau}_i(\omega)) = \phi_i(\omega, s)
\]

But

\[
\phi_i(\omega, s + \tilde{\tau}_i(\omega)) = \phi_i(\omega, s + \tilde{\tau}_i(\omega) - \tau_i(\omega) + \tau_i(\omega))
\]

\[
= \phi_i(\omega, s + \tilde{\tau}_i(\omega) - \tau_i(\omega))
\]

\[
= \ldots
\]

\[
= \phi_i(\omega, s + \tilde{\tau}_i(\omega) - k\tau_i(\omega)),
\]

where \( k \) is the smallest integer such that \( \tilde{\tau}_i(\omega) - (k + 1)\tau_i(\omega) \leq 0 \). Then by definition of \( \tau_i(\omega) \),

\[
\tilde{\tau}_i(\omega) - k\tau_i(\omega) = \tau_i(\omega),
\]

so

\[
\tilde{\tau}_i(\omega) = (k + 1)\tau_i(\omega).
\]

Therefore \( \phi_i(\omega) \) is a periodic function of period \( \tau_i(\omega) \).
• If none of \(i_2, i_3, \ldots, i_d(\omega)\) is equal to \(i\). In this case, at least two of \(i_1, i_2, \ldots, i_d(\omega)\) must be equal. Say \(\tau_2(\omega) > \tau_1(\omega)\) are such integers such that \(i_{\tau_1(\omega)} = i_{\tau_2(\omega)}\) with smallest difference \(\tau_2(\omega) - \tau_1(\omega)\). Then

\[
\phi_i(\omega, s + \tau_1(\omega)) = \phi_i(\omega, s + \tau_2(\omega)).
\]

Denote \(s + \tau_1(\omega)\) by \(s\), then

\[
\phi_i(\omega, s) = \phi_i(\omega, s + \tau_2(\omega) - \tau_1(\omega)), \quad \forall s \in \mathbb{R}.
\]

Same as in the second case, we can see for all other possible \(\tilde{\tau}_2(\omega)\) and \(\tilde{\tau}_1(\omega)\), \(\tilde{\tau}_2(\omega) > \tilde{\tau}_1(\omega)\) and \(i_{\tilde{\tau}_2(\omega)} = i_{\tilde{\tau}_1(\omega)}\), \(\tilde{\tau}_2(\omega) - \tilde{\tau}_1(\omega)\) must be an integer multiple of \(\tau_2(\omega) - \tau_1(\omega)\). So, \(\phi_i(\omega)\) is a periodic function of period \(\tau_2(\omega) - \tau_1(\omega)\). □

Now for any \((s, x) \in B_s(K(\omega))\), for \(l \in \mathbb{N}\) denote

\[
h_1(s) = h_l(s),
\]

\[
\varphi_1(\omega, s, x) = \varphi^l(\omega, s, x) = \varphi^{l-1}(\omega, h_l^{-1}(s), \varphi^{l-1}(\omega, s, x)),
\]

\[
H_1(\omega, s, x, i) := (h_1(s), \varphi_1(\omega, s, x)).
\]

We know that there are finite number of continuous periodic functions \(\phi_1(\omega), \ldots, \phi_n(\omega)\) on \(\mathbb{R}\). Denote their periods by \(\tau_1(\omega), \ldots, \tau_n(\omega)\) respectively. So that

\[
K(\omega) = L_1^\omega \cup \cdots \cup L_n^\omega,
\]

where

\[
L_i^\omega = \text{graph}(\phi_i(\omega)) = \{(s \mod 1, \phi_i(\omega, s)) : s \in [0, \tau_i(\omega))\},
\]

and from the proof of proposition 3.4.4 \(\tau_1 + \cdots + \tau_n = d(\omega)\). But

\[
H_1(\hat{\theta}^{-l}_1, K(\hat{\theta}^{-l}_i)) = K(\omega).
\]

So

\[
H_1(\hat{\theta}^{-l}_1, L_1^\hat{\theta}^{-l}_1) \cup \cdots \cup H_1(\hat{\theta}^{-l}_i, L_i^\hat{\theta}^{-l}_i) = L_1^\omega \cup \cdots \cup L_n^\omega. \tag{3.4.5}
\]

Since \(L_i^\hat{\theta}^{-l}_1\) is a closed curve and \(H_1(\hat{\theta}^{-l}_i)\) is a continuous map, one can get that \(H_1(\hat{\theta}^{-l}_1, L_i^\hat{\theta}^{-l}_i)\) is a closed curve. Moreover, since \(H_1\) is a homeomorphism, so

\[
H_1(\hat{\theta}^{-l}_1, L_i^\hat{\theta}^{-l}_1) \cap H_1(\hat{\theta}^{-l}_1, L_j^\hat{\theta}^{-l}_j) = \emptyset, \quad \text{when } i \neq j. \tag{3.4.6}
\]

76
Therefore the left hand side of (3.4.5) is indeed a union of \( n(\hat{\theta}^{-l}\omega) \) distinct closed curves and the right hand side of (3.4.5) is a union of \( n(\omega) \) distinct closed curves. Hence for any \( i \in \{1, 2, \cdots, n(\hat{\theta}^{-l}\omega)\} \), there is a unique \( j \in \{1, 2, \cdots, n(\omega)\} \) such that

\[
H_1(\hat{\theta}^{-l}\omega, L_i^{\hat{\theta}^{-l}\omega}) = L_j^\omega. \tag{3.4.7}
\]

Similarly, for any \( t \in \mathbb{R} \),

\[
K(\theta_{-t}\omega) = L_1^{\theta_{-t}\omega} \cup \cdots \cup L_n^{\theta_{-t}\omega}, \tag{3.4.8}
\]

and

\[
G(t, \theta_{-t}\omega, L_1^{\theta_{-t}\omega}) \cup \cdots \cup G(t, \theta_{-t}\omega, L_n^{\theta_{-t}\omega}) = L_1 \cup \cdots \cup L_n^\omega, \tag{3.4.9}
\]

and here \( G : \mathbb{R} \times \Omega \times S^1 \times \mathbb{R}^d \to S^1 \times \mathbb{R}^d \) is defined by

\[
G(t, \omega, s, x) = (s + t \mod 1, \varphi_t(\omega, s, x)).
\]

Without confusion, we can re-order \( i's \) and denote the unique \( j \) by \( i \), so that for each \( \omega \), we have

for any \( t \in \mathbb{R} \),

\[
G(t, \theta_{-t}\omega, L_i^{\theta_{-t}\omega}) = L_i^\omega \tag{3.4.10}
\]

**Lemma 3.4.5** For any \( t \in \mathbb{R} \), \( \tau_i(\theta_{-t}\omega) = \tau_i(\omega) \) for any \( i = 1, 2, \cdots, n(\omega) \).

**Proof.**

- Consider when \( t = kt_1, k \in \mathbb{N} \) (as indicated earlier in this section, \( t_1 \) is the time the particle in \( S^1 \) rotate a full circle) and note that for any \( s \in \{0, 1, \cdots, \tau_i^{\theta_{-t}\omega}\} \),

\[
\pi_{\mathbb{S}^1} G(t, \theta_{-t}\omega, (s, \phi_i(\theta_{-t}\omega, s))) = s + k. \tag{3.4.11}
\]

So for \( t = kt_1 \), from (3.4.10) and (3.4.11), it turns out that

\[
\pi_{\mathbb{S}^1} G(t, \theta_{-t}\omega, 0, \phi_i(\omega, 0)) = 0 + k,
\]

\[
\pi_{\mathbb{S}^1} G(t, \theta_{-t}\omega, \tau_i(\theta_{-t}\omega), \phi_i(\theta_{-t}\omega, \tau_i(\theta_{-t}\omega))) = \tau_i(\theta_{-t}\omega)
\]

and

\[
\pi_{\mathbb{S}^1} G(t, \theta_{-t}\omega, \tau_i(\theta_{-t}\omega), \phi_i(\theta_{-t}\omega, \tau_i(\theta_{-t}\omega))) - \pi_{\mathbb{S}^1} G(t, \theta_{-t}\omega, 0, \phi_i(\omega, 0)) = \tau_i(\omega)
\]

we have that, \( \tau_i(\omega) = \tau_i(\theta_{-t}\omega) \).
Consider also, the case when \( t \in (kt_1, (k + 1)t_1) \), \( k \in \mathbb{N} \) and for any \( s \in \{0, 1, 2, \ldots, \tau_i^{\omega - t} \} \),

\[
\pi_{\mathbb{S}_1} G(t, \theta_{-t} \omega, s, \phi_i(\omega, s)) \in (s + k, s + k + 1),
\]

since \( \phi_i(\theta_{-t} \omega, .) \) is periodic with period \( \tau_i(\theta_{-t} \omega) \) we have

\[
\pi_{\mathbb{S}_1} G(t, \theta_{-t} \omega, \tau_i(\theta_{-t} \omega), \phi_i(\theta_{-t} \omega)) = \tau_i(\theta_{-t} \omega) + k
\]

\[
= \tau_i(\theta_{-t} \omega) + \pi_{\mathbb{S}_1} G(t, \theta_{-t} \omega, 0, \phi_i(\theta_{-t} \omega, 0)).
\]

And for each \((t, \omega) \in \mathbb{R} \times \Omega\), using the fact that the curve \( L_i^\omega \) is invariant under the homeomorphism \( G(t, \omega) : \mathbb{S}^1 \times \mathbb{R}^d \to \mathbb{S}^1 \times \mathbb{R}^d \), to get

\[
\tau_i(\omega) = \pi_{\mathbb{S}_1} G(t, \theta_{-t} \omega, \tau_i(\theta_{-t} \omega), \phi_i(\theta_{-t} \omega)) - \pi_{\mathbb{S}_1} G(t, \theta_{-t} \omega, 0, \phi_i(\theta_{-t} \omega, 0)) = \tau_i(\theta_{-t} \omega).
\]

Finally, let \( \phi \) represent any \( \phi_i \) and \( \tau(\omega) \) represent any \( \tau_i(\omega) \), we already know that \( \tau(\theta_{-t} \omega) = \tau(\omega) \) for any \( t \in \mathbb{R} \) and define \( \hat{t} = kt_1 \).

Then for any \( s \in [0, \tau(\omega)) \)

\[
G(\hat{t}, \theta_{-t} \omega, s, \phi(\theta_{-t} \omega, s)) = (s \mod 1, \phi(\omega, s)) := \hat{L}^\omega.
\]

Therefore from (3.4.12) and the cocycle property of \( \varphi \), we have that for any \( s \in [0, \tau(\omega)) \)

\[
G(\hat{t} + t, \theta_{-t} \omega, \hat{L}^{\theta_{-t} \omega}) = G(t, \theta_{-t} \omega, G(-\hat{t}, \theta_{-t} \omega, \hat{L}^{\theta_{-t} \omega})) = G(t, \theta_{-t} \omega, \hat{L}^{\theta_{-t} \omega}).
\]

This gives for any \( s \in [0, \tau(\omega)) \)

\[
G(t + \hat{t}, \omega, s, \phi(\omega, s)) = G(t, \theta_{\omega}, s, \phi(\theta_{\omega}, s)))
\]

for any \( t \leq 0 \). Which implies that \( G \) has a periodic curve with period \( \hat{t} \) and winding number \( \tau \) and there are \( n \) such \( \phi \). That is to say \( G \) has \( n \) random periodic solutions. \( \square \)
Chapter 4

Random Periodic Solutions of
Stochastic flows via Two Point Motion

4.1 Bound of top Lyapunov exponent of stochastic flows

The pioneering work of Has’minskii in his book [38] championed the stability theory of stochastic
differential equations. Has’minskii systematically adopted the concept Lyapunov function $V$, for
the SDE case. The flavour in his concept is the fact that the average growth $V$ along the trajectory
$X(t, x)$ is expressed by

$$\mathcal{L}V(x) = \lim_{t \downarrow 0} \frac{E[V(X(t, x)) - V(x)]}{t}.$$ 

For some suitable $V \in C^2$, we can use Itô’s formula to write $\mathcal{L}$ as

$$\mathcal{L}V(x) = \sum_{i=1}^{d} b^i(x) \frac{\partial V(x)}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x, x) \frac{\partial^2 V(x)}{\partial x^i \partial x^j}.$$ 

Using this operator $\mathcal{L}$, Has’minskii established the stability of trivial solution of stochastic differential
equations. This operator $\mathcal{L}$ is well known as one-point generator of stochastic differential equations.
The stability theory for trajectory $\varphi(t)x_0$ of a deterministic dynamical system investigates the time
evolution of the distance $|\varphi(t)x - \varphi(t)x_0|$ and near by trajectory $\varphi(t)x$, that is, for which $|x_0 - x|$ is
small. Stochastic stability due to Has’minskii is based on the information provided by the operator $\mathcal{L}$, hence principally unable to consider this problem, as the joint distribution of $X(t, x_0)$ and $X(t, x)$ has to be investigated which cannot be obtained from $\mathcal{L}$, unless $X(t, x_0) = x_0$ (trivial or
deterministic stationary solution). So the theorem of stability due to Has’minskii is a one-point
problem and thus cannot be used to account for the stability of non-trivial solution, this critical observation was first made by Arnold in his paper [3].

In the study of stochastic flows, Kunita introduced $n$-point generator of stochastic flows given by the following partial differential operator

$$
L^n V(\bar{x}) = \sum_{i=1}^{d} \sum_{v=1}^{n} b^i(x_v) \frac{\partial V(\bar{x})}{\partial x^i_v} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{v,u=1}^{n} a^{ij}(x_v, x_u) \frac{\partial^2 V(\bar{x})}{\partial x^i_v \partial x^j_u},
$$

where $\bar{x} = (x^1, x^2, \ldots, x^n)$ and $x^1, x^2, \ldots, x^n \in \mathbb{R}^d$, in other words $\bar{x} \in \mathbb{R}^{nd}$.

Let’s consider the two-point generator of stochastic flows;

$$
L^2 V(\bar{x}) = \sum_{i=1}^{d} \sum_{v=1}^{2} b^i(x_v) \frac{\partial V(\bar{x})}{\partial x^i_v} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{v_1,v_2=1}^{2} a^{ij}(x_{v_1}, x_{v_2}) \frac{\partial^2 V(\bar{x})}{\partial x^i_{v_1} \partial x^j_{v_2}}, \quad (4.1.1)
$$

where $\bar{x} = (x_1, x_2)$, $x_1, x_2 \in \mathbb{R}^d$.

Our main concern in this section is to give an idea how general assumptions in section 3.3 are related to the assumptions in the result we presented in this chapter. We would be able to verify assumptions similar to assumptions B and C in chapter 3 and assumption A can be achieved by construction and or deduction from the result here. There are special estimates for stochastic flows of diffeomorphisms that guarantees the assumption B in section 3.3. Consider the stochastic flows $\varphi$ induced by the stochastic system

$$
dX_t = F(X_t, dt), \quad (4.1.2)
$$

where $F$ is a spatial semimartingale (see section 1.4.3). For integer $k \in \mathbb{Z}$ and $\delta > 0$, we denote $B^{k,\delta}_{ub}$ to be the class of semimartingale with local characteristics $(a(., t), b(., t), A(\cdot))$ such that $\|a(., t)\|_{k,\delta}$ and $\|b(., t)\|_{k,\delta}$ is bounded (see section B in the Appendix for the definition of these semi-norms).

**Proposition 4.1.1** ([40], [47], [48]) A. Suppose $F \in B^{0,1}_{ub}$ and fix $T > 0$. Then there exists $C \geq 0$ such that for all $x, y \in \mathbb{R}^d$ and all $p \geq 1$, we have:

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|\varphi_t(x) - \varphi_t(y)\|^p \leq \exp(Cp^2) \|x - y\|^p, \quad (4.1.3)
$$

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|\varphi_t(0)\|^p \leq \exp(Cp^2). \quad (4.1.4)
$$

B. Assume that $F \in B^{k,1}_{ub}$ for some $k \in \mathbb{N}$. Then for all $T > 0$ there exists $C \geq 0$ such that for
all \( 1 \leq |\alpha| \leq k \), \( p \geq 1 \) and \( x, y \in \mathbb{R}^d \) we have:

\[
E \sup_{0 \leq t \leq T} \| D^\alpha \varphi_t(x) - D^\alpha \varphi_t(y) \|^p \leq \exp(Cp^2)(\| x - y \|^p \wedge 1),
\]

(4.1.5)

\[
E \sup_{0 \leq t \leq T} \| D^\alpha \varphi_t(x) \|^p \leq \exp(Cp^2).
\]

(4.1.6)

In fact, estimates similar to those in assumption B in section 3.3 would be derived whenever \( C \geq 0 \) is independent of \( T > 0 \). With little more work we can achieve this using the assumptions in the results in section 4.3 we will not derive this in this work.

Top Lyapunov exponent was central in the result obtained in chapter 3, in this section we want to estimate its bound using the method of Lyapunov function. The idea of using Lyapunov function to estimate a bound for top Lyapunov exponent was from stability analysis of stochastic flows. In the past three decades Mao has done many works in both stochastic differential equations and stochastic functional differential equations ([55]). Mao employed his results to study the bounds of the Lyapunov exponents of stochastic flows. The results due to Mao extended Hasminskii’s idea in [38], as mentioned earlier one investigates the difference between two trajectories of a stochastic differential equations, where one of the trajectories is a zero solution. In many cases, especially for stochastic systems zero solution many not exist, so it is of signifance to investigate a nontrivial random fixed point (to be reviewed in the next section), and thus the rate of infinitesimal separation of these two nontrivial trajectories.

To make connection between assumptions B and C in section 3.3 and the assumptions that we going apply in the results in section 4.3 of this chapter, let us define \( X := L^p(\Omega, \mathcal{F}_0, \mathbb{P}) \), let \( x, y \in X \) with \( x \neq y \) and if there are postive numbers \( C > 0 \), \( \lambda > 0 \) such that for \( p \geq 1 \) and with some assumptions on the operator \( L^2 \) (this assumptions will be made clear in the subsquent sections) we have that

\[
E \| \varphi(t, \omega, x) - \varphi(t, \omega, y) \|^p \leq Ce^{-\lambda t}E\| x - y \|^p.
\]

(4.1.7)

In order to estimate the bound for the top Lyapunov of the RDS \( \varphi \), let us suppose that \( D_x \varphi \) exists and then, we rewrite the inequality (4.1.7) as

\[
\frac{\| \varphi(t, \omega, x) - \varphi(t, \omega, y) \|_X}{\| x - y \|_X} \leq C^\frac{1}{p} \exp(-\frac{\lambda t}{p}).
\]

Then, as \( y \to x \), we have that

\[
\| D_x \varphi(t, \omega, x) \|_X \leq C^\frac{1}{p} \exp(-\frac{\lambda t}{p}),
\]

81
consequently we have that
\[
\limsup_{t \to \infty} \frac{1}{t} \log \|D_x \varphi(t, \omega, x)\|_X \leq -\frac{\lambda}{p}.
\]

In what follows, the dissipative assumption on stochastic flows will be on the sign of the bound of the top Lyapunov exponent \( \lambda \).

### 4.2 Exponentially stable non-trivial random stationary solutions for SDEs

In this section, we wish to present random stationary solution for an RDS generated by stochastic differential equations in finite dimensional space. This will serve as motivation to discussing theory of random periodic solutions.

In the usual setting, let \( \varphi \) be an RDS on a topological space \( X \) over a filtered dynamical system \( \theta \), a measurable random variable \( S : \Omega \to X \) is called a random stationary solution if

\[
\varphi(t, \omega, S(\omega)) = S(\theta_t \omega), \quad \text{almost surely.} \tag{4.2.1}
\]

In this case, we say that the RDS \( \varphi(t, \omega) \) has a stationary trajectory.

1. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a Wiener space, that is \( \Omega = C_0(\mathbb{R}; \mathbb{R}^d) \), \( \mathcal{F} := \mathcal{B}(\Omega) \) and \( \mathbb{P} \) is Wiener measure and consider the SDE

\[
d\varphi(t) = f_0(\varphi(t))dt + \sum_{k=1}^{m} f_k(\varphi(t))dW^k_t,
\]

where the vector fields \( f_0, f_1, \ldots, f_m : \mathbb{R}^d \to \mathbb{R}^d \) are in \( C^k_b \) for \( k \geq 1, \delta \in (0,1) \). As usual take \( \theta \) to be Wiener shift. Suppose \( f_0(x_0) = f_k(x_0) = 0 \) for some fixed \( x_0 \in \mathbb{R}^d \), then \( S(\omega) = x_0 \) for all \( \omega \in \Omega \) is a random stationary solution for the SDE.

2. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be one dimensional Wiener space, consider one-dimensional affine SDE

\[
d\varphi(t) = \alpha \varphi(t)dt + dW_t,
\]

where \( \alpha > 0 \) is fixed. The random variable \( S(\omega) \) given by

\[
S(\omega) := -\int_{-\infty}^{0} e^{-\alpha s}dW_s,
\]

is a random stationary solution of the above affine SDE.
(3) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be one-dimensional Wiener space, consider the following affine Stratonovich SDE
\[
dX_t = (\lambda - \alpha X_t) dt + 2\sigma X_t \circ dW_t,
\]
for some constants \(\lambda, \alpha, \sigma\). This SDE generates the following RDS represented via variation of constants formula by
\[
\varphi(t, \omega, x) = \exp(-\alpha t + 2\sigma W_t(\omega))(x + \lambda \int_0^t \exp(\alpha \tau - 2\sigma W_\tau(\omega)) d\tau).
\]
Then the random variable \(S(\omega)\) defined by
\[
S(\omega) := \lambda \int_{-\infty}^0 \exp(\alpha t - 2\sigma W_t(\omega)) dt,
\]
has a measurable version which is a random stationary solution of the RDS.

(4) Now let \((\Omega, \mathcal{F}, \mathbb{P})\) be two-dimensional Wiener space; that is \(\Omega := C_0(\mathbb{R}; \mathbb{R}^2)\), consider two-dimensional affine SDE
\[
d\varphi(t) = A \varphi(t) dt + G dW_t,
\]
where \(A\) is a fixed hyperbolic \(2 \times 2\) diagonal matrix,
\[
A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_2 < 0 < \lambda_1,
\]
and \(G\) is a \(2 \times 2\) constant matrix
\[
G = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\]
Here \(W = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}\) where \(W_t^1, W_t^2\) are one-dimensional Brownian motions.
Set \(S := \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}\), where
\[
S_1 := -a_{11} \int_0^\infty e^{-\lambda_1 s} dW_s^1 - a_{12} \int_0^\infty e^{-\lambda_1 s} dW_s^2
\]
and
\[
S_2 := a_{21} \int_{-\infty}^0 e^{-\lambda_2 s} dW_s^1 + a_{22} \int_{-\infty}^0 e^{-\lambda_2 s} dW_s^2.
\]
Then \(S\) has a measurable version which is a random stationary solution of our SDE.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be $d$-dimensional Wiener space; that is $\Omega := C_0(\mathbb{R}; \mathbb{R}^d)$, consider $d$-dimensional affine SDE
\[ dX_t = (A_0X_t + b_0)dt + \sum_{j=1}^{m}(A_jX_t + b_j)dW^j_t, \]
where $A_0, A_j \in \mathbb{R}^{d \times d}$ and $b_0, b_j \in \mathbb{R}^d, j = 0, 1, 2, \cdots m$. This SDE generate the following RDS $\varphi$ in $\mathbb{R}^d$ which can be represented via variation of constants formula by
\[ \varphi(t, \omega, x) = \Phi(t, \omega)x + \int_0^t (\Phi(s, \omega)^{-1}b_0ds + \sum_{j=1}^{m}\int_0^t \Phi(s, \omega)^{-1}b_j \circ dW^j_s(\omega)), \]
where $\Phi$ is a fundamental matrix of the following stochastic equation
\[ dV_t = A_0V_t dt + \sum_{j=1}^{m}A_jV_t \circ dW^j_t. \]
Suppose that
\[ \lambda := \lim_{t \to \infty} \frac{1}{t} \log \| \Phi(t, \omega) \| < 0. \]
Then the random variable represented by
\[ S(\omega) := \int_{-\infty}^{0} \Phi(t, \omega)^{-1}b_0dt + \sum_{j=1}^{m}\int_{-\infty}^{0} \Phi(t, \omega)^{-1}b_j \circ dW^j_t(\omega), \]
has a measurable version which is a random stationary solution of the affine SDE.

The study of existence of random stationary solutions is a recent development in stochastic analysis and generalises the concept of fixed point solution in deterministic dynamical systems. This type of solutions are vital in the stability of nontrivial solutions and the existence of random invariant manifolds of stochastic differential equations. The existence of exponentially stable random stationary was proved by Caraballo, Kloeden and Schmalfuss [15, 78] using the concept of Lyapunov function. We shall review the result due to Schmalfuss [78] and adopted the same approach to investigate the existence of stable random periodic solutions of stochastic differential equations. But before then, we would like introduce some basic concepts and results that will be used in what follows in this section and the subsequent section .

**Theorem 4.2.1 (Exponential local martingale inequality [55] )** Let $M = (M_t)_{t \geq 0}$ be a continuous local martingale. Then for any positive constants $\tau, \gamma, \delta$, we have
\[ \mathbb{P} \left\{ \omega : \sup_{t \leq \tau} (M_t - \frac{\gamma}{2}(M)^2_t) > \delta \right\} \leq \exp(-\gamma\delta) \]
Corollary 4.2.2 ([55]) Let \((M_t)_{t \geq 0}\) be a continuous real-valued local martingale vanishing at \(t = 0\). Let \((\tau_k)_{k \geq 1}\) and \((\gamma_k)_{k \geq 0}\) be two sequences of positive numbers with \(\tau_k \to \infty\). Let \(g(t)\) be a positive increasing function on \(\mathbb{R}^+\) such that
\[
\sum_{k=1}^{\infty} g(k)^{-\theta} < \infty, \quad \text{for some } \theta > 1.
\]
Then for almost all \(\omega \in \Omega\) there is a random integer \(k_0(\omega)\) such that for all \(k \geq k_0(\omega)\)
\[
M_t \leq \frac{\gamma_k}{2} (M)_t + \frac{\theta}{\gamma_k} \log(g(k)) \text{ on } 0 \leq t \leq \tau_k.
\]
Proof. Now, we use the exponential local martingale inequality (Theorem 4.2.1), to get that
\[
P\left\{ \omega : \sup_{0 \leq t \leq \tau_k} \left[ M_t - \frac{\gamma_k}{2} (M)_t \right] > \frac{\theta}{\gamma_k} \log(g(k)) \right\} \leq g(k)^{-\theta}, \quad \text{for all } k \geq 1.
\]
Hence, by Borel-Cantelli lemma, for almost all \(\omega \in \Omega\) there is \(k_0(\omega)\) such that for all \(k \geq k_0(\omega)\)
\[
\sup_{0 \leq t \leq \tau_k} \left[ M_t - \frac{\gamma_k}{2} (M)_t \right] \leq \frac{\theta}{\gamma_k} \log(g(k)).
\]
That is,
\[
M_t \leq \frac{\gamma_k}{2} (M)_t + \frac{\theta}{\gamma_k} \log(g(k)), \quad \text{on } 0 \leq t \leq \tau_k, \quad \text{as required.} \quad \Box
\]
Consider the stochastic differential equation of Itô type
\[
dX_t = f_0(X_t)dt + \sum_{j=1}^{m} f_j(X_t)dW^j_t, \quad (4.2.2)
\]
where \(f_0, f_1, f_2, \ldots, f_m\) are vector fields in \(\mathbb{R}^d\) such that \(f_0 \in C_b^{k,\delta}, f_1, \ldots, f_m \in C_b^{k+1,\delta}\) for \(k \in \mathbb{N}, \delta \in (0,1)\). We know from section 1.4.4 precisely from Theorem 1.4.11 that if
\[
\sum_{j=1}^{m} \sum_{i=1}^{d} f_j^i \frac{\partial}{\partial x_i} f_j \in C_b^{k,\delta},
\]
then the SDE (4.2.2) generates a smooth RDS \(\varphi\) over a filtered dynamical system \(\theta\) (Wiener shift).
Now we reconsider the two point generator
\[
\mathcal{L}^2V(\bar{x}) = \sum_{i=1}^{d} \sum_{v=1}^{2} b^i(x_v) \frac{\partial V(\bar{x})}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{v_1,v_2=1}^{2} a^{ij}(x_{v_1}, x_{v_2}) \frac{\partial^2 V(\bar{x})}{\partial x^i \partial x^j}. \quad (4.2.3)
\]
In particular, when $V(x, y) = V(x - y)$ which will turn to be the case we shall consider in what follows, the two point generator $L^2$ simplifies to

$$L^2 V(x - y) = V_x(x - y)(f_0(x) - f_0(y)) + \frac{1}{2} \text{trace} \left( (g(x) - g(y))^T V_{xx}(x - y)(g(x) - g(y)) \right),$$

(4.2.4)

where $V_x = \left( \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d} \right)$, $V_{xx} = \left( \frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{d \times d}$ and $g(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ We now recall some of the results due to Schmalfuss and give some idea on how to apply the results on random perturbation theory.

Theorem 4.2.3 (Exponentially attracting non-trivial random stationary solution [78])

Let $\varphi$ be a continuous RDS over a filtered dynamical system $\theta$ generated by the SDE (4.2.2). Let $V$ be a $C^2$ function such that $V(0) = 0$ and

$$L^2 V(x - y) \leq -\lambda V(x - y), \quad |x|^p \leq V(x), \quad \text{for } x, y \in \mathbb{R}^d,$$

(4.2.5)

where $\lambda, p$ are postive constants. In addition, assume that $V(\varphi(s, \theta_s \omega, x) - x)$, $s \in [0, 1]$ is $(\theta_s)_{s \in \mathbb{Z}}$-tempered. Then there a unique (up to indistinguishability) $\mathcal{F}_0^\infty$-measurable random variable $S$ such that for $t \in \mathbb{R}^+$

$$S(\theta_t \omega) = \varphi(t, \omega, S(\omega)), \quad \text{almost surely.}$$

The process $t \mapsto \varphi(t, \omega, S(\omega))$, which solves (4.2.2) is predictable. For any $\mathcal{F}_0^\infty$-random variable $X$ we have that

$$|\varphi(t, \omega, X(\omega)) - S(\theta_t \omega)| \to 0 \quad \text{as } t \to \infty \quad \text{almost surely,}$$

exponentially fast.

The random variable $S$ in the above theorem is some sort of pre-random fixed point. The definition of this random variable depends on an initial condition $x$. The random variables $S = S_x$ constructed for different initial conditions are only almost surely identical. Under additional conditions, it is possible to avoid such dependence with respect to the construction of $x$. In addition, this random variable will be tempered.

Theorem 4.2.4 (Exponentially attracting random fixed point [78]) Suppose that $\varphi$ is a continuous RDS over a filtered dynamical system $\theta$ generated by the SDE (4.2.2) with two-point generator $L^2$. Let $V$ be a $C^2$ function such that $V(0) = 0$ and

$$L^2 V(x - y) \leq -\lambda V(x - y), \quad |x|^p \leq V(x) \leq c|x|^p, \quad \text{for } x, y \in \mathbb{R}^d,$$

86
where $\lambda, p$ and $c$ are positive constants with $p > 2d$ and $c \geq 1$. Then there exists a unique random fixed point $S$ attracting compact random sets exponentially fast.

**Examples 4.2.5 ([78])** (1) For the SDE (4.2.2), suppose that

$$ (f_0(x) - f_0(y), x - y) \leq -c_0|x - y|^2, \quad |f_k(x) - f_k(y)| \leq L|x - y|, \quad \text{for } x, y \in \mathbb{R}^d, $$

$k = 1, 2, \ldots, m, c_0 > 0, L > 0$. Suppose further that

$$ 2c_0 > mL^2(p - 1) \quad \text{and} \quad p > 2d. $$

Then there exists an exponentially stable random stationary solution $S$ for the SDE and this random variable $S$ is a random fixed point which attract random compact sets exponentially fast.

Take $V(x) = |x|^p$ for some $p \geq 1$, let’s compute $\mathcal{L}^2 V(x - y)$ with the information we have from our SDE above

\[
\frac{\partial V(z)}{\partial z^i} = p z^i \left( \sum_{n=1}^{d} (z^n)^2 \right)^{\frac{p}{2} - 1} = p z^i |z|^{p-2}.
\]

\[
\frac{\partial^2 V(z)}{\partial z^{i_1} \partial z^{i_2}} = 2 \left( \frac{p}{2} - 1 \right) z^{i_1} z^{i_2} |z|^{p-4} + \delta_{i_1, i_2} p |z|^{p-2},
\]

where $\delta_{i_1, i_2}$ is the Kronnecker symbol. Since we know that

\[
a^{i_1, i_2}(x_1, x_2) = \sum_{k=1}^{m} f^{i_1}_k(x_1) f^{i_2}_k(x_2),
\]

we obtain

\[
\sum_{j_1, j_2=1}^{2} (-1)^{j_1+j_2} \sum_{i_1, i_2=1}^{d} (x_1^{i_1} - x_2^{i_2}) a^{i_1, i_2}(x_1, x_2) (x_1^{i_1} - x_2^{i_2})
\]

\[
= \sum_{k=1}^{m} \sum_{i_1, i_2=1}^{d} (f^{i_1}_k(x_1) - f^{i_1}_k(x_2)) (f^{i_2}_k(x_1) - f^{i_2}_k(x_2)) (x_1^{i_1} - x_2^{i_1}) (x_1^{i_2} - x_2^{i_2})
\]

\[
\leq mL^2 |x_1 - x_2|^4,
\]

\[
\sum_{k=1}^{m} \sum_{i=1}^{d} (f^{i}_k(x_1) - f^{i}_k(x_2)) (f^{i}_k(x_1) - f^{i}_k(x_2)) \leq mL^2 |x_1 - x_2|^2.
\]
Now,
\[
\mathcal{L}^2 V(x_1 - x_2) = \sum_{i=1}^{d} (f_0^i(x_1) - f_0^i(x_2)) p(x_1^i - x_2^i)|x_1 - x_2|^{p-2} \\
+ \frac{1}{2} \sum_{i_1,i_2=1}^{d} \left( \sum_{k=1}^{m} (x_1^{i_1} - x_2^{i_2}) (f_k^{i_1}(x_1) - f_k^{i_1}(x_2)) (f_k^{i_2}(x_1) - f_k^{i_2}(x_2)) (x_1^{i_2} - x_2^{i_2}) \right) \times \\
\{2\left(\frac{p}{2} - 1\right)p|x_1 - x_2|^{p-4} \} \\
+ \left( \sum_{k=1}^{m} (f_k^{i_1}(x_1) - f_k^{i_1}(x_2)) (f_k^{i_2}(x_1) - f_k^{i_2}(x_2)) \delta_{i_1,i_2} |x_1 - x_2|^{p-2} \right) \right] \\
\leq -pc_0|x_1 - x_2|^p + p\frac{mL^2}{2}(p-1)|x_1 - x_2|^p \\
= -\lambda V(x_1 - x_2), \text{ for all } x_1, x_2 \in \mathbb{R}^d,
\]

where \( \lambda = p(c_0 - \frac{mL^2}{2}(p-1)) \). From the assumption that \( 2c_0 > mL^2(p-1) \), it follows that \( \lambda > 0 \), thus, we have the inequality (4.2.5):
\[
\mathcal{L}^2 V(x - y) \leq -\lambda V(x - y), \text{ for all } x, y \in \mathbb{R}^d.
\]

Since the coefficients \( f_0, f_1, \cdots, f_m \) are Lipschitz continuous, we have that \( \varphi(t, \omega, X) \in L^p \) if \( X \in L^p \). Thus, the assumption of Theorem 4.2.3 are satisfied. In particular, we have that \( E[V(\varphi(t, \omega, x))] < \infty \) implying that the temperedness assumption is fulfilled. Hence there is an exponentially attracting random stationary solution \( S(\omega) \). In addition, if \( p > 2d \), then from Theorem 4.2.4 we have that this random stationary solution can be represented by a random fixed point attracting random compact sets exponentially fast.

(2) Random perturbation of linear ordinary differential equation: Let \( A \) be a positive definite matrix in \( \mathbb{R}^{d \times d} \):
\[
(-Ax, x) \geq c_0 \|x\|^2, \quad c_0 > 0.
\]

From stability theory of ODE, the system
\[
\frac{dx}{dt} = Ax, \quad x(0) = x_0 \in \mathbb{R}^d.
\]

has exponentially stable steady state 0. If we perturb this system by the noise \( f_1(x)\frac{dW_1}{dt} \) where \( f_1 \) is Lipschitz with the constant \( L > 0 \). Then the perturbed system has a random stationary solution \( S \) if
\[
c_0 - \frac{L^2}{2} > 0.
\]
Furthermore, if
\[ c_0 - \frac{L^2}{2}(p - 1), \quad \text{and} \quad 2p > d. \]
This random variable \( S \) is a random fixed point which attract compact sets exponentially fast.

(3) **Random perturbation of semilinear ordinary differential equation:** Let \( f_0 \) be a smooth vector field in \( \mathbb{R}^d \) such that
\[ (f_0(x) - f_0(y), x - y) \leq -c_0|x - y|^2, \quad c_0 > 0. \]
if the system of ODE
\[ \frac{dx}{dt} = f_0(x), \quad x(0) = x_0 \in \mathbb{R}^d, \]
has a deterministic steady state, then it is exponentially stable. We perturb this system by the noise \( f_1(x)\frac{dW}{dt} \), where \( f_1 \) is Lipschitz continuous with the constant \( L > 0 \). Then the perturbed system has a random stationary solution \( S \) if
\[ c_0 - L^2 > 0. \]
This random variable \( S \) is a random fixed point which attract random compact sets exponentially fast if
\[ c_0 - L^2(p - 1) > 0, \quad \text{and} \quad p > 2d. \]

From the above examples we could say that exponentially stable deterministic steady state is persistent under small influence of noise. The above examples will be a great source of motivation to tackling similar problem for random periodic solutions of time homogeneous SDE.

**Remark 4.2.6** The conditions
\[ (f_0(x) - f_0(y), x - y) \leq -c_0|x - y|^2 \quad \text{and} \quad |f_k(x) - f_k(y)| \leq L|x - y|, \quad \text{for} \quad x, y \in \mathbb{R}^d, \quad (4.2.6) \]
k = 1, 2, \cdots, m, c_0 > 0, L > 0, are know as **one-sided Lipschitz and Lipschitz** conditions respectively. The conditions \( 4.2.7 \) will be replaced with local one-sided Lipschitz and local Lipschitz conditions respectively. That is to say that there exist \( c_r > 0 \) and \( L_r > 0 \) such that
\[ (f_0(x) - f_0(y), x - y) \leq -c_r|x - y|^2 \quad \text{and} \quad |f_k(x) - f_k(y)| \leq L_r|x - y|, \quad \text{for} \quad x, y \in \mathbb{R}^d \quad \text{with} \quad |x| \leq r, |y| \leq r. \quad (4.2.7) \]
In this situation, we suppose further that
\[ 2c_r > mL^2(p - 1) \quad \text{and} \quad p > 2d, \text{for } x \neq y \text{ with } |x| \vee |y| \leq r. \] (4.2.8)

To arrive at the conclusion of Theorem 4.2.4 in addition to (4.2.8) we would require further boundedness assumptions:
\[ \limsup_{t \to \infty} \frac{1}{t} \int_0^t \mu(\varphi(s, \omega, x)) ds \leq M \quad \text{a.s.} \] (4.2.9)
for some \( \mu \in \mathcal{K} \) and a constant \( M > 0 \) such that for all \( x \in \mathbb{R}^d \). Where \( \mathcal{K} \) is the class function given by
\[ \mathcal{K} := \{ \mu : \mathbb{R}^d \to \mathbb{R}^+ : \lim_{|x| \to \infty} \mu(x) = \infty \}. \]

### 4.3 Random periodic solutions for SDEs

**Definition 4.3.1 (Random periodic solution for stochastic flow [33], [34], [95])** A random periodic solution of period \( \tau \) of a stochastic flow \( X : \Delta \times \Omega \to \mathbb{R}^d \) is an \( \mathcal{F} \)-measurable function \( \phi : T \times \Omega \to \mathbb{R}^d \) such that
\[ \phi(t + \tau, \omega) = \phi(t, \theta_{\tau} \omega) \quad \text{and} \quad X(t + s, t, \omega, \phi(t, \omega)) = \phi(t + s, \omega), \]
for any \( s, t \in \mathbb{T} \) and \( \omega \in \Omega \), where \( \Delta := \{(t, s) \in \mathbb{T}^2; s \leq t\} \).

**Remark 4.3.2** Suppose that \( X(t, s, \omega, x) \) is a temporary homogeneous Brownian (flow with stationary increments). This kind of flow corresponds to the case when our SDE is time homogeneous (that is equation (4.2.2)). In this case, we can write
\[ X(t, s, \omega, x) = X(t - s, 0, \theta_s \omega, x), \]
we define \( \varphi(t, \omega, x) := X(t, 0, \omega, x) \)

Let \( \phi(t, \omega) \) be a random periodic solution with period \( \tau \), define \( \eta(t, \omega) := \phi(t, \theta_{-t} \omega) \) so that,
\[ \eta(t + \tau, \omega) = \phi(t + \tau, \theta_{-t - \tau} \omega) \]
\[ = \phi(t, \theta_{\tau} \circ \theta_{-t - \tau} \omega) \]
\[ = \eta(t, \omega), \]
and using the fact that \( \phi(s, \omega) \) is a random periodic solution and that the flow \( X(t, s, \omega, x) \) is a temporary Brownian flow, we have that

\[
\eta(t + s, \theta t \omega) = \phi(t + s, \theta s \omega) \\
= X(t + s, s, \theta s \omega, \phi(s, \theta s \omega)) \\
= X(t, 0, \omega, \eta(s, \omega)) \\
= \varphi(t, \omega, \eta(s, \omega)).
\]

Hence, we have that

\[
\eta(t + \tau, \omega) = \eta(t, \omega) \quad \text{and} \quad \varphi(t, \omega, \eta(s, \omega)) = \eta(t + s, \theta t \omega),
\]

(4.3.1)
corresponding to the definition of random periodic solution we have in the introduction (equation (0.0.5)).

**Example 4.3.3** Consider the following stochastic differential equation

\[
\begin{cases}
   dX = -\alpha(t)Xdt + dW_t \\
   X(t_0) = x_0 \in \mathbb{R}
\end{cases} \quad t \geq t_0 \geq 0,
\]

(4.3.2)
where \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function. Evidently the SDE (4.3.2) has no zero solution and hence will not follow the usual result of studying the stability via one point generator.

However, suppose there is \( \tau > 0 \) such that \( \alpha(t + \tau) = \alpha(t) \) and

\[
\int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u) du} ds < \infty, \quad \text{for} \quad 0 \leq t \leq \tau.
\]

The random variable \( \phi(t, \omega) \) defined by

\[
\phi(t, \omega) = \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u) du} dW_s(\omega)
\]
is a random periodic solution of the stochastic flow generated \( X(t, t_0, \omega, x_0) \) generated by the SDE (4.3.2) defined by

\[
X(t, t_0, \omega, x_0) = x_0 e^{-\int_{t_0}^{t} \alpha(u) du} + \int_{t_0}^{t} e^{-\int_{s}^{t} \alpha(u) du} dW_s(\omega),
\]

Indeed, by suitable change of variable, we have that

\[
\phi(t, \theta t \omega) = \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u) du} dW_{s+\tau}(\omega)
\]

\[
= \int_{-\infty}^{t+\tau} e^{-\int_{s}^{t+\tau} \alpha(u) du} dW_s
\]

\[
= \phi(t + \tau, \omega),
\]

91
and

\[ X(t + s, t, \omega, \phi(t, \omega)) = e^{-f_{t+s}^{t} \alpha(u)du} \int_{-\infty}^{t} e^{-f_{\tau}^{t} \alpha(u)du} dW_{\tau}(\omega) + \int_{t}^{t+s} e^{-f_{\tau}^{t+s} \alpha(u)du} dW_{\tau}(\omega) = \int_{-\infty}^{t+s} e^{-f_{\tau}^{t+s} \alpha(u)du} dW_{\tau}(\omega) = \phi(t + s, \omega). \]

Consider an autonomous stochastic differential equations

\[
\begin{cases}
    dX = f_0(X)dt + \sum_{k=1}^{m} f_k(X)dW^k_t, & t \geq t_0. \\
    X(t_0) = x \in \mathbb{R}^d
\end{cases}
\] (4.3.3)

Let \( y(t) \) be a periodic solution of the following ordinary differential equation with period \( \tau \)

\[
\frac{dy}{dt} = f_0(y). \tag{4.3.4}
\]

The recent work by Feng and Zhao [34] introduced the following transformation defined by \( X(t, \omega) = y(t) + Z(t, \omega) \), where \( Z \) satisfies the stochastic differential equation

\[
dZ = [f_0(y(t) + Z) - f_0(y(t))] dt + \sum_{k=1}^{m} f_k(y(t) + Z)dW^k_t. \tag{4.3.5}
\]

Suppose that \( \phi(t, \omega) \) is a random periodic solution of the SDE (4.3.5), define

\[
\tilde{X}(t, \omega) = y(t) + \phi(t, \omega).
\]

Then

\[
d\tilde{X} = f_0(y) dt + [f_0(y + \phi) - f_0(y)] dt + \sum_{k=1}^{m} f_k(y + \phi)dW^k_t = f_0(\tilde{X}) dt + \sum_{k=1}^{m} f_k(\tilde{X})dW^k_t.
\]

And \( \tilde{X}(t + \tau, \omega) = y(t + \tau) + \phi(t + \tau, \omega) = y(t) + \phi(t, \theta_\tau \omega) = \tilde{X}(t, \theta_\tau \omega) \).

Our investigation will be to find a random periodic solution for SDE (4.3.5). The following theorem gives us the existence stable random periodic solution for the some highly nonlinear SDE with periodic coefficients given that the coefficients satisfy some regularities conditions.
In the following, we consider time inhomogeneous stochastic differential equations in $\mathbb{R}^d$

\[
\begin{cases}
    dX = f_0(t, X)dt + \sum_{k=1}^{m} f_k(t, X)dW^k_t \\
    X(t_0) = x \in \mathbb{R}^d,
\end{cases}
\]  

$t \geq t_0$.  

(4.3.6)

Where assume that $f_0, f_1, f_2, \cdots, f_m : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous and Lipschitz with respect to $x \in \mathbb{R}^d$. Existence and uniqueness theorem for such SDE tells us that there is a unique (up to indistinguishability) predictable solution $X(t, t_0, \omega, x)$ that solves the SDE for an $\mathcal{F}^{t_0}_{-\infty}$-mesurable random variable $x$. Here the two point generator has additional term and given by

\[
\mathcal{L}^2 V(t, \bar{x}) = \frac{\partial V(t, \bar{x})}{\partial t} + \sum_{i=1}^{d} \sum_{v=1}^{2} b^i(t, x_v) \frac{\partial V(t, \bar{x})}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{v_1,v_2=1}^{2} a^{ij}(t, x_{v_1}, x_{v_2}) \frac{\partial^2 V(t, \bar{x})}{\partial x^i \partial x^j},
\]

where $\bar{x} = (x^1, x^2)$ and $x^1, x^2 \in \mathbb{R}^d$. Here

\[
b(t, x) = f_0(t, x) \tag{4.3.7}
\]

and

\[
a(t, x, y) = \sum_{k=1}^{m} f_k(t, x)f^T_k(t, y) \tag{4.3.8}
\]

In particular, considering the distance between two solutions starting from different initial values, the two point generator simplifies to

\[
\mathcal{L}^2 V(t, x - y) = V_t(t, x - y) + V_x(t, x - y) \left(f_0(t, x) - f_0(t, y)\right)
\]

\[
+ \frac{1}{2} \text{trace} \left((g(t, x) - g(t, y))^T V_{xx}(t, x - y)(g(t, x) - g(t, y))\right), \tag{4.3.9}
\]

where

\[
g(t, x) = (f_1(t, x), f_2(t, x), \cdots, f_m(t, x)).
\]

**Theorem 4.3.4 (Existence of random periodic solutions for stochastic flows)** Suppose that $f_0, f_1, f_2, \cdots, f_m$ are periodic in $t$ with period $\tau > 0$. Let $V \in C^{1,2}(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^+)$ such that $V(t, 0) = 0$, and

\[
\mathcal{L}^2 V(t, x - y) \leq -\lambda V(t, x - y), \quad |x|^p \leq V(t, x), \quad \text{for all } t \in \mathbb{R}, \quad x, y \in \mathbb{R}^d, \tag{4.3.10}
\]

for some $\lambda > 0, p \geq 1$. Suppose further, that

\[
\mathbb{E} \sup_{t > t_0} \ln V(t_0, X(t, t_0, \omega, x) - x) < \infty.
\]
Then there exists an $F_{-\infty}$-measurable random variable $S(t, \omega)$ such that

$$X(t + \tau, t, \omega, S(t, \omega)) = S(t + \tau, \omega) = S(t, \theta_\tau \omega), \quad \mathbb{P} \text{- almost surely.}$$

**Proof.** We wish to show that the sequence $\{X(t, t - n\tau, \omega, x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in a space of continuous function $C(\mathbb{R}^+, \mathbb{R}^d)$. We recall that a sequence $(f_n)_{n \in \mathbb{N}} \subset C(\mathbb{R}^+, \mathbb{R}^d)$ is a Cauchy sequence if and only if

$$\sup_{t \in \mathbb{R}^+} |f_n(t) - f_N(t)| \to 0, \quad \text{as} \quad n, N \to \infty.$$ 

For this, we shall first modify the initial value by a random variable in such a way that $X(t, t_0, \omega, x) \neq X(t, t_0, \omega, y)$ for some $y \in \mathbb{R}^d$. This modification is necessary to avoid difficulties in the definiteness of some integrals and logarithms. We can disregard this modification at the end, using the fact $X(t, t_0, \omega, x)$ is a homeomorphism (see [47]) so that $X(t, t_0, \omega, x) = X(t, t_0, \omega, y)$ if and only if $x = y$. (See [78] for time homogeneous SDE, where such modification was made explicitly).

Let $X^x(t) := X(t, t_0, \omega, x)$, $X^y(t) := X(t, t_0, \omega, y)$, we apply Itô’s formula on $\ln(V(t, X^x(t) - X^y(t)))$ to get

$$\ln V(t, X^x(t) - X^y(t)) = \ln V(t_0, x - y) + \int_{t_0}^t \frac{C^2 V(s, X^x(s) - X^y(s))}{V(s, X^x(s) - X^y(s))} ds$$

$$+ N(t, \omega) - \frac{1}{2} q(t, \omega),$$

where

$$q(t, \omega) = \int_{t_0}^t \left( \mathcal{H}V(s, X^x(s), X^y(s)) \right)^2 ds,$$

$$N(t, \omega) = \int_{t_0}^t \left( \mathcal{H}V(s, X^x(s), X^y(s)) \right) dW_s$$

and

$$\mathcal{H}V(t, x, y) := \frac{V_x(t, x - y)(g(t, x) - g(t, y))}{V(t, x - y)}.$$ 

Here $W_t = (W^1_t, \ldots, W^n_t)$ and we observe that $q(t, \omega)$ is a quadratic variation of $N(t, \omega)$ and as a consequence of corollary 4.2.2 we have

$$\mathbb{P} \left\{ \omega \in \Omega : \sup_{t \in [t_0, t_0 + k]} \left( N(t, \omega) - \frac{1}{2} q(t, \omega) \right) > 2 \ln k \right\} \leq \frac{1}{k^2}.$$ 

An application of Borel-Cantelli lemma yields that for almost all $\omega \in \Omega$, there exists a random integer $k_0 := k(\omega)$ such that for any $k \geq k_0$

$$\sup_{t_0 \leq t \leq t_0 + k} \left( N(t, \omega) - \frac{1}{2} q(t, \omega) \right) \leq 2 \ln k.$$ 

94
In particular,

\[
\frac{1}{t} \left( N(t, \omega) - \frac{1}{2} q(t, \omega) \right) \leq \frac{2 \ln k}{t_0 + k - 1}, \quad \text{for } t_0 + k - 1 \leq t \leq t_0 + k. \quad (4.3.11)
\]

Applying the assumptions on V and on the two point generator, we have

\[
\frac{1}{k-1} \sup_{k-1 \leq t \leq k} |X^x(t) - X^y(t)| \leq \frac{1}{p(k-1)} \sup_{k-1 \leq t \leq k} \ln V(t, X^x(t) - X^y(t)) \leq \frac{1}{p(k-1)} \ln V(t_0, x - y) + \frac{1}{p(k-1)} \sup_{k-1 \leq t \leq k} \int_{t_0}^{t} \frac{L^2 V(s, X^x(s) - X^y(s))}{V(s, X^x(s) - X^y(s))} ds \\
+ \frac{1}{p(k-1)} \sup_{k-1 \leq t \leq k} \left( N(t, \omega) - \frac{1}{2} q(t, \omega) \right) \leq \frac{1}{p(k-1)} \ln V(t_0, x - y) + \frac{1}{p(k-1)} \sup_{k-1 \leq t \leq k} \left( -\lambda (t - t_0) + \frac{2 \ln k}{p(t_0 + k - 1)} \right) \leq \frac{1}{p(k-1)} \ln V(t_0, x - y) - \frac{\lambda (k - 1 - t_0)}{p(k-1)} + \frac{2 \ln k}{p(t_0 + k - 1)}
\]

Hence, there exists a finite random variable \( \beta \), such that for \( \varepsilon > 0 \) and \( k \) large enough, we have that

\[
\frac{\ln |X^x(t) - X^y(t)|}{t - t_0} \leq \beta - \frac{\lambda}{p} + \varepsilon, \quad \text{for all } t > t_0, \quad (4.3.12)
\]

that is,

\[
|X(t, t_0, \omega, x) - X(t, t_0, \omega, x)| \leq \exp((\beta - \frac{\lambda}{p} + \varepsilon)(t - t_0)), \quad \text{for all } t \geq t_0. \quad (4.3.13)
\]

Let \( n \geq N \), for \( 0 < \varepsilon < \frac{1}{p} \) and by flow property we have that

\[
\frac{1}{n^\tau} \ln |X(t, t - n \tau, \omega, x) - X(t, t - N \tau, \omega, x)| \\
= \frac{1}{n^\tau} \ln |X(t, t - n \tau, \omega, x) - X(t, t - n \tau, \omega, X(t - n \tau, t - N \tau, \omega, x))| \\
\leq \frac{1}{n^\tau} \ln V(t - n \tau, x - X(t - n \tau, t - N \tau, \omega, x)) - \frac{\lambda}{p} + \varepsilon.
\]

From the fact that \( \mathbb{E}\sup_{t \geq t_0} \log V(t_0, X(t, t_0, \omega, x) - x) < \infty \), we have that for almost all \( \omega \in \Omega \), there exist a finite random variable \( \gamma(\omega) \) such that

\[
\sup_{t \geq 0} |X(t, t - n \tau, \omega, x) - X(t, t - N \tau, \omega, x)| \leq \gamma \exp(-\frac{\lambda n \tau}{p}), \quad \text{almost surely.} \quad (4.3.14)
\]

So, the sequence \( \{X(t, t - n \tau, \omega, x)\}_{n \in \mathbb{N}} \) is a Cauchy sequence. Let \( S(t, \omega) \) be the limit of the sequence \( \{X(t, t - n \tau, \omega, x)\}_{n \in \mathbb{N}} \).
Next,

\[ X(t + \tau, t, \omega, \lim_{n \to \infty} X(t, t - n\tau, \omega, x)) = \lim_{n \to \infty} X(t + \tau, t - n\tau, \omega, x) = S(t + \tau, \omega), \quad \text{for any } \tau > 0. \]

Using the fact that \( f_0, f_1, f_2, \cdots, f_m \) are time periodic and denoted their period by \( \tau \), we have that

\[
X(t, t - n\tau, \theta \tau, \omega, x) = x + \int_{t-n\tau}^{t} f_0(r, X(r, r - n\tau, \theta \tau, \omega, x))dr \\
+ \int_{t-n\tau}^{t} \sum_{k=1}^{m} f_k(r, X(r, r - n\tau, \theta \tau, \omega, x))dW_k^r(\omega)
\]

\[
= x + \int_{t-\tau}^{t+\tau} f_0(r, X(r, r - \tau - n\tau, \theta \tau, \omega, x))dr \\
+ \int_{t-\tau}^{t+\tau} \sum_{k=1}^{m} f_k(r, X(r, r - \tau - n\tau, \theta \tau, \omega, x))dW_k^r(\omega),
\]

and

\[
X(t + \tau, t + \tau - n\tau, \omega, x) = x + \int_{t+\tau-n\tau}^{t+\tau} f_0(r, X(r, r - n\tau, \omega, x))dr \\
+ \int_{t+\tau-n\tau}^{t+\tau} \sum_{k=1}^{m} f_k(r, X(r, r - n\tau, \omega, x))dW_k^r(\omega).
\]

By uniqueness of solution of SDE and the fact that \( \mathbb{P} \) is \( \theta \)-invariant, we have that

\[ X(t, t - n\tau, \theta \tau, \omega, x) = X(t + \tau, t + \tau - n\tau, \omega, x), \quad \mathbb{P} - \text{almost surely}, \]

so that

\[ S(t + \tau, \omega) = S(t, \theta \tau, \omega), \quad \mathbb{P} - \text{almost surely}. \]

Now, from the inequality \[ (4.3.13) \], we see that for any \( \mathcal{F}_t^{\omega} \)-measurable random variable \( X(t, \omega) \)

\[
\lim_{k \to \infty} \sup_{t \in [k-1, k]} |X(t, r, \omega, S(t, \omega)) - X(t, r, \omega, X(t, \omega))| = 0,
\]

exponentially fast almost surely. \( \square \)

**Corollary 4.3.5** Suppose the coefficients of the SDE \[ (4.3.6) \] are completely continuous and periodic in \( t \) with period \( \tau \) such that

\[
(f_0(t, x) - f_0(t, y), x - y) \leq -\beta|x - y|^2, \quad |f_k(t, x) - f_k(t, y)| \leq L|x - y|, \quad \text{for all } x, y \in \mathbb{R}^d, t \in \mathbb{R},
\]

\[ (4.3.15) \]

\[ k = 1, 2, \cdots, m, \text{ for some } \beta > 0, L > 0. \] Let \( V(t, x) = |x|^p \), for some \( p \geq 1 \) such that

\[ 2\beta - mL^2(p - 1) > 0. \]

Then SDE \[ (4.3.6) \] has a random periodic solution.
Then there exists an

for all \( \lambda \)  

Theorem 4.3.7 flexible and applicable condition.

It follows that, SDE (4.3.3) has a random periodic solution

above conditions in the same theorem, then the SDE (4.3.5) has a random periodic solution SDE (4.3.3) generates an RDS and if we can find a Lyapunov function

Remark 4.3.6 a measurable random variable

If \( X_0(\omega) \in L^p \) then solution \( X(t, t_0, \omega, X_0) \in L^p \) and since \( V \in C^2(\mathbb{R} \times \mathbb{R}^d - \{0\}; \mathbb{R}^+) \) we have that \( E[V(t_0, X(t, t_0, \omega, x))] < \infty \) satisfying the temperedness assumption of theorem 4.3.4. Thus, there a measurable random variable \( S(t, \omega) \) such that

\[ X(t + \tau, t, \omega, S(t, \omega)) = S(t + \tau, \omega) = S(t, \theta_x \omega), \quad \text{for all } t \in \mathbb{R}. \]

\[ \square \]

Remark 4.3.6 In the case of random dynamical systems take

\[ f_0(t, x) = f_0(y(t) + x) - f_0(y(t)), \quad f_k(t, x) = f_k(y(t) + x), \]

where \( g(t) \) is a periodic solution of the differential equation \( \frac{d}{dt} f_0(y(t)) = f_0(y) \). Since the SDE (4.3.3) generates an RDS and if we can find a Lyapunov function \( V \in C(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^+) \) with above conditions in the same theorem, then the SDE (4.3.5) has a random periodic solution \( S(t, \omega) \).

It follows that, SDE (4.3.3) has a random periodic solution \( y(t) + S(t, \omega) \).

We can relax the condition \( L^2 V(t, \omega) \leq -\lambda V(t, \omega) \) for some positive constant \( \lambda \) by more flexible and applicable condition.

\[ L^2 V(t, t - y) \leq -\lambda V(t, \omega) \]

\[ \text{for all } t \in \mathbb{R}, \text{ for some } c > 0 \text{ and } p \geq 1. \]

Suppose further that

\[ \mathbb{E} \sup_{t > t_0} \ln V(t_0, X(t, t_0, \omega, x) - x) < \infty. \]

Then there exists an \( \mathcal{F}_{-\infty}^t \)-measurable random variable \( S(t, \omega) \) such that

\[ X(t + \tau, t, \omega S(t, \omega)) = S(t + \tau, \omega) = S(t, \theta_x \omega), \quad \mathbb{P} - \text{a.s.} \]
Proof. It follows the same argument as Theorem 4.3.4, we present the prove breifly. For \( x \neq y \), let \( X^x(t) := X(t, t_0, \omega, x) \) and \( X^y(t) := X(t, t_0, \omega, y) \) applying Itô formula, we have

\[
\ln V(r, X^x(t) - X^y(t)) = \ln V(t_0, x - y) + \int_{t_0}^{t} \frac{\mathcal{L}^2 V(s, X^x(s) - X^y(s))}{V(s, X^x(s) - X^y(s))} ds + N(t, \omega) - \frac{1}{2} q(t, \omega),
\]

where \( N(t, \omega) \) is a local martingale with \( N(0, \omega) = 0 \) and \( q(t, \omega) \) is the quadratic variation of \( N(t, \omega) \), and there are defined in the proof of Theorem 4.3.4. By exponential Martingale inequality, we have

\[
P\{\omega \in \Omega : \sup_{t \in [t_0, t_0 + k]} (N(t, \omega) - \frac{1}{2} q(t, \omega)) > 2 \ln k\} \leq \frac{1}{k^2}.
\]

We apply Borel-Cantelli lemma to deduce that

\[
\frac{1}{t_0 + k - 1} \sup_{t \in [t_0 + k - 1, t_0 + k]} \ln |X^x(t) - X^y(t)| \leq \frac{1}{p(k - 1)} \ln V(t_0, x - y) + \sup_{r \in [k - 1, k]} \frac{1}{p(k - 1)} \int_{t_0}^{t} \lambda(s) ds + 2 \frac{\ln k}{p(k - 1)},
\]

and

\[
\leq \frac{1}{p(k - 1)} \ln V(t_0, x - y) + \frac{\alpha(k - 1)}{p(t_0 + k)} + 2 \frac{\ln k}{p(t_0 + k - 1)}, \quad \mathbb{P} - a.s.
\]

Hence, there is \( \Omega_0 \) of full measure such that for any \( \omega \in \Omega_0 \) and \( 0 < \varepsilon < -\frac{\alpha}{p} \), such that for \( k \) large enough, we have that

\[
|X^x(t) - X^y(t)| < e^{(\frac{\alpha}{p} + \varepsilon)k} \tag{4.3.18}
\]

It then follows from the proof of Theorem 4.3.4 that \( \{X(t, t - n \tau, x)\}_{n \in \mathbb{N}} \) is a Cauchy sequence and the existence of random periodic solutions follows. \( \square \)
Appendix

A  Selected results and facts from topology

Theorem A.1 (Continuous image of a compact set) Let $X$ be a compact topological space and $Y$ be a topological space. Let $f : X \to Y$, be continuous. Then, the set $f(X)$ is a compact subset of $Y$.

Proof: The collection $\{f^{-1}(A) ; A \in \mathcal{A}\}$ is a covering $X$; these sets are open in $X$, since $f$ is continuous. Hence, there is a finite subcover

$$f^{-1}(A_1), \ldots, f^{-1}(A_n).$$

Then the sets $A_1, \ldots, A_n$ cover $f(X)$. □

Lemma A.2 Every compact subspace of a Hausdorff space is closed

Proof. Let $A$ be a compact subspace of the Hausdorff space $X$. We shall prove that $X \setminus A$ is open. Let $x_0 \in X \setminus A$. We have to show that there is a neighbourhood of $x_0$ that is disjoint from $A$. For each $y \in A$, let us choose disjoint neighbourhoods $U_{x_0}$ and $V_y$ of points $x_0$ and $y$ (using Hausdorff condition). The collection $\{V_y ; y \in A\}$ is a covering of $A$ by open sets open in $X$, therefore $V_{y_1}, \ldots, V_{y_n}$ cover $A$. The open set $V = V_{y_1} \cup \cdots \cup V_{y_n}$ contains $A$ and it is disjoint from the open set $U = U_{x_0} \cap \cdots \cap U_{x_0}$. For if $z \in V$, then $z \in V_{y_i}$, for some $i$, hence $z \notin U_{x_0}$. We have that $U$ is a neighbourhood of $x_0$ disjoint from $A$. □
**Theorem A.3** Let $f : X \to Y$ be a bijective continuous map. If $X$ is compact and $Y$ is Hausdorff, then $f$ is a homeomorphism.

**Proof.** If $A$ is closed in $X$, then $A$ is compact so $f(A)$ is compact. Since $Y$ is Hausdorff, then $f(A)$ is closed. □

**Proposition A.4** Let $X$ and $Y$ be metric spaces. Let $f : X \to Y$ be any map, define its graph by

$$G_f = \{ (x, y) \in X \times Y : y = f(x) \} \subset X \times Y.$$  

1. If $f$ is continuous, the $G_f$ is a closed subset of $X \times Y$.
2. If $X$ is compact, the map $f$ is continuous if and only if, its graph $G_f$ is compact.

**Proof.**

1. Suppose that $G_f \ni (x_n, y_n) \to (x, y)$ as $n \to \infty$. This implies that $y_n = f(x_n)$ by definition of $G_f$ and $x_n \to x$, $f(x_n) \to y$ as $n \to \infty$.

   However, $f$ is assumed to be continuous, so $f(x_n) \to f(x)$ as $n \to \infty$ and by uniqueness of limit, we have that $y = f(x)$ and hence $(x, y) \in G_f$.

2. The set $X$ is compact, implies for $x_n \in X$ there exists $x_{n_k} \to x$ as $k \to \infty$.

   Assume that $f$ is continuous, we have that $f(x_{n_k}) \to f(x)$ as $k \to \infty$. Let $(x_n, y_n) \in G_f$, we have that $y_n = f(x_n)$, hence there exists $(x_{n_k}, y_{n_k}) \to (x, f(x)) \in G_f$, since $G_f$ is closed.

   Assume that $G_f$ is compact. Let $C \subset Y$ be closed and suppose $x_n \in f^{-1}(C)$ and $x_n \to x \in X$ as $n \to \infty$. Then $(x_n, f(x_n)) \in G_f$ has a convergent subsequence $(x_{n_k}, f(x_{n_k}))$ with limit $(x, y) \in G_f$ (since $G_f$ is closed), so $y = f(x) \in Y$ and since $f(x_{n_k}) \to y$ in $Y$, it implies that $y = f(x) \in C$. Thus, $x \in f^{-1}(C)$ which is therefore closed and hence $f$ is continuous.

**Theorem A.5 (Principle of invariance of domain)** Let $M$ and $N$ be topological Manifolds without boundary, let $f : M \to N$ be continuous and locally injective, then $f$ is a homeomorphism.

**Definition A.6 (Lower and upper semicontinuous functions)** Let $X$ be a topological space, a function $f : X \to \mathbb{R}$ is continuous at $x_0 \in X$, if for all $\alpha \in \mathbb{R}$ such that $f(x_0) > \alpha$, there exists a neighbourhood $U$ of $x_0$, such that

$$f(x) > \alpha \quad \forall x \in U$$

100
and it is upper semicontinuous at $x_0$, if for all $\alpha \in \mathbb{R}$ such that $f(x_0) < \alpha$, there exists a neighbourhood $U$ of $x_0$, such that

$$f(x) < \alpha, \quad \forall x \in U.$$  

The function is said to lower (resp. upper) semicontinuous, if it lower (resp. upper) semicontinuous at all point $x \in X$, that is for all $\alpha \in \mathbb{R}$ such that $f(x) > \alpha$ (resp. $f(x) < \alpha$) there exists $U \in \mathcal{N}(x)$, $x \in X$, such that

$$f(y) > \alpha \quad (\text{resp. } f(y) < \alpha), \quad \forall y \in U.$$  

The below Theorem is always helpful in recognising a lower (resp. upper) semicontinuous functions.

**Theorem A.7** Let $X$ be a topological space, a function $f : X \to \mathbb{R}$ is lower (resp. upper) semicontinuous iff, $f^{-1}((\alpha, \infty])$ (resp. $f^{-1}([-\infty, \alpha))$) is an open subset of $X$.

Let $K$ be a nonempty convex subset of Hausdorff vector space $X$. A map $g : K \to K$ is called **affine** if

$$g((1 - \lambda)x + \lambda y) = (1 - \lambda)g(x) + \lambda g(y) \quad (A.1)$$

for all $0 \leq \lambda \leq 1$ and $x, y \in K$.

**Theorem A.8 (Markov-Kakutani fixed point theorem)** Let $K$ be a nonempty convex subset of Hausdorff topological vector space $X$. Let $\mathcal{G}$ be a set of continuous affine maps $g : K \to K$. Suppose that all elements of $\mathcal{G}$ commute, that is, $g_1 \circ g_2 = g_2 \circ g_1$ for all $g_1, g_2 \in \mathcal{G}$. Then there exists a point in $K$ which is fixed by all elements of $\mathcal{G}$.

**B Frechét spaces**

Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ be a multi index of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Define

$$D^\alpha f = D^\alpha_x f := \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^d)^{\alpha_d}}.$$  

Let $m, R$ be non-negative integers, $\delta \in (0, 1]$ and $p > 1$. Let $D$ be a domain of $\mathbb{R}^d$, $K$ be a compact subset of $D$ and $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$.  

101
Consider the following Semi-norms:

\[ \|f\|_{m,K} = \sup_{x \in K} \frac{|f(x)|}{1 + |x|} + \sum_{1 \leq |\alpha| \leq m} \sup_{x \in K} |D^\alpha f(x)|, \]

\[ \|f\|_{m,R} = \|f\|_{m:B_R}, \]

\[ \|f\|_m = \|f\|_{m:D}, \]

\[ \|f\|_{m,p,R} = \left( \sum_{|\alpha|} \int_{B_R} |D^\alpha f(x)|^p dx \right)^\frac{1}{p} \]

\[ \|f\|_{m+\delta,K} = \|f\|_{m,K} + \sup_{|\alpha|=m} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\delta}, \]

\[ \|f\|_{m+\delta} = \|f\|_{m+\delta,D}, \]

\[ \|g\|_{m,K} = \sup_{x,y \in K} \frac{|g(x,y)|}{1 + |x|} + \sum_{1 \leq |\alpha| \leq m} \sup_{x,y \in K} |D_x^\alpha D_y^\alpha g(x,y)|, \]

\[ \|g\|_{m,K} = \sup_{x,y \in K} \sup_{x \neq y} \frac{|g(x,y) - g(x',y')|}{|x - x'|^\delta |y - y'|^\delta}, \]

\[ \|g\|_{m+\delta,K} = \|g\|_{m,K} + \sum_{|\alpha|=m} \|D_x^\alpha D_y^\alpha g\|_{m,K}, \]

\[ \|F\|_{m,R} = \|F\|_{m,\infty;R} = \sup_{t \in [0,T]} \|F(t)\|_{m,R}, \]

\[ \|F\|_{m,p,R} = \sup_{t \in [0,T]} \|F(t)\|_{m,p,R}. \]

The following function spaces endowed respectively with the above Semi-norms are Frechet spaces\(^1\).

Let \( C^m = C^{m,0} = \{ f(x), x \in D; m\text{-times continuously differentiable} \} \).

The space of \( m\)-times locally continuously differentiable functions with all derivatives locally \( \delta\)-Hölder continuous, endowed the semi-norm \( \|f\|_{m+\delta,K} \) is a Frechet space and it is given by

\[ C^{m,\delta} = \{ f \in C^m; \|f\|_{m+\delta,K} < \infty \text{ for any compact subset } K \text{ of } D \}, \]

Let \( C^m_b = \{ f \in C^m; \|f\|_m < \infty \} \).

The space of \( m\)-times continuously differentiable with all derivatives \( \delta\)-Hölder continuous, endowed with the semi-norm \( \|f\|_{m+\delta} \) is a Frechet space is given by

\[ C^{m,\delta}_b = \{ f \in C^{m,\delta}; \|f\|_{m+\delta} < \infty \}. \]

\(^1\)Frechet spaces are generalization of Banach spaces, they are locally convex spaces which are complete with countable family of Semi-norms.
And when we have functions of two variables \( g(x, y) \) we have the following Frechet spaces, similar to the above descriptions:

\[
\tilde{C}^m = \{ g(x, y), x, y \in D, m\text{-times continuously differentiable with respect to } x \text{ and } y \},
\]

\[
\tilde{C}^{m, \delta} = \{ g \in \tilde{C}^m; \| g \|_{\tilde{m} + \delta, K} < \infty \text{ for any compact subset } K \text{ of } D \},
\]

\[
\tilde{C}_b^m = \{ g \in \tilde{C}^m; \| g \|_{\tilde{m}} < \infty \}, \quad \tilde{C}_b^{m, \delta} = \{ g \in \tilde{C}^{m, \delta}; \| g \|_{\tilde{m} + \delta} \}.
\]

Let \( m \in \mathbb{Z}^+, 0 \leq \delta \leq 1 \) and let \( A : \mathbb{R} \to \mathbb{R} \) be continuous increasing function with \( A(0) = 0 \). Define \( L_{loc}(\mathbb{R}, dA, C_b^{m, \delta}) \) to be the set of measurable function \( f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) for which

- \( f(., t) \in C_b^{m, \delta} \) for every \( t \in \mathbb{R} \) ("for \( A \)-almost \( t \) would be enough ),
- for every \( t \in \mathbb{R} \)
  \[
  | \int_0^t \| f(., s) \|_{m, \delta} dA(s) | < \infty.
  \]

(B.1)

With the seminorms \( | \int_0^t \| f(., s) \|_{m, \delta} dA(s) | \), \( L_{loc}(\mathbb{R}, dA, C_b^{m, \delta}) \) is a Fréchet space. In addition, we have the following continuous inclusions

\[
L_{loc}(\mathbb{R}, dA, C_b^{m, \delta}) \hookrightarrow L_{loc}(\mathbb{R}, dA, C_b^{m, \delta}) \hookrightarrow L_{loc}(\mathbb{R}, dA, C_b^{m-1, \epsilon})
\]

The spaces \( L_{loc} \) of \( C^{m, \delta} \), \( \tilde{C}_b m, \delta \) and \( \tilde{C}^{m, \delta} \) are similarly defined.

## C Stochastic calculus for RDS

**Definition C.1 (Stratonovich stochastic integrals with respect to semimartingale helix [4, 48])**

Let \( F(t, \omega, x) \) be \( C^{0,1} \)-semimartingale helix with local characteristics \( (a, b, A) \) such that \( a \in L_{loc}(\mathbb{R}, dA, \tilde{C}^{2, \delta}) \) for some \( \delta \geq 0 \) and \( b \in L_{loc}(\mathbb{R}, dA, C^{1,0}) \). Let \( f(s, t) \) be a semimartingale.

- **Forward Stratonovich stochastic integral:** Let \( s \leq t \), then

  \[
  I_s(t) = \int_s^t F(f(s, u), \circ d^+ u)
  \]

  \[
  := \lim_{h \to 0} \text{pr.} \sum_{k=0}^{n-1} \frac{1}{2} \{ F(f(s, t_{k+1}), t_{k+1}) + F(f(s, t_k), t_{k+1}) - F(f(s, t_{k+1}), t_k) - F(f(s, t_k), t_k) \}
  \]

where the limit in probability is taken over a sequence of partitions \([s, t]\) for which the mesh \( h := \max_{0 \leq k \leq n-1} (t_{k+1} - t_k) \to 0 \). The limit exists and is a forward semimartingale.
• Backward Stratonovich stochastic integral: Let \( t \leq s \)

\[
I_s(t) = \int_t^s F(f(s, u), \circ d^- u) \\
:= \lim_{h \to 0} \text{in pr.} \sum_{k=0}^{n-1} \frac{1}{2} \{ F(f(s, t_{k+1}), t_{k+1}) + F(f(s, t_k), t_{k+1}) - F(f(s, t_{k+1}), t_k) - F(f(s, t_k), t_k) \}
\]

where the limit in probability is taken over a sequence of partitions of \([t, s]\) for which \( h \to 0 \).

This limit exists and is a backward semimartingale.

It is necessary to notice that there no relation whatsoever between the forward and backward stochastic integrals. Combining both cases in one, we conclude that \((I_s(t))_{s,t \in \mathbb{R}}\) is a semimartingale.

The helix property of \( F \) allows us to reduce one end point of the integral to zero; given in the below proposition.

**Proposition C.2**  \( [4] \)

- Let \( t \geq 0 \), then

\[
\int_t^{s+t} F(f(s, u), \circ d^+ u) = \int_0^t \theta_s F(f(0, u), \circ d^+ u), \quad \mathbb{P} \text{- a.s.,}
\]

where \( \theta_s F(t, \omega, x) = F(t, \theta_s \omega, x) \). In case \( f(s, t + s, \omega) = f(0, t, \theta_s \omega) \), for all \( t \geq 0 \) and \( \omega \notin N_s \) we have

\[
\int_t^{t+s} F(f(s, u), \circ d^+ u) = \int_0^t \theta_s F(f(0, u), \circ d^+ u), \quad \mathbb{P} \text{- a.s.}
\]

- Let \( t \leq 0 \), then

\[
\int_{t+s}^s F(f(s, u), \circ d^- u) = \int_t^0 \theta_s F(f(0, u), \circ d^- u), \quad \mathbb{P} \text{- a.s.}
\]

In case \( f(s, s + t, \omega) = f(0, t, \theta_s \omega) \) for all \( t \leq 0 \) and \( \omega \notin N_s \) we have

\[
\int_{t+s}^s F(f(s, u), \circ d^- u) = \int_t^0 \theta_s F(f(0, u), \circ d^- u), \quad \mathbb{P} \text{- a.s.}
\]

**Definition C.3** (Itô stochastic integrals with respect to semimartingale helix \([2], [4]\)) Let \( F(t, \omega, x) \) be a \( C^{0,1} \)-semimartingale helix and let \( f(s, t) \) be a semimartingale.

- **Forward Itô stochastic integral**: Let \( s \leq t \), then

\[
I_s(t) = \int_s^t F(f(s, u), d^+ u) := \lim_{h \to 0} \text{in pr.} \sum_{k=0}^{n-1} \{ F(f(s, t_k), t_k) - F(f(s, t_{k+1}), t_{k+1}) \},
\]

where the limit in probability is taken over a sequence of partitions \([s, t]\) for which the mesh

\[
h := \max_{0 \leq k \leq n-1} (t_{k+1} - t_k) \to 0.
\]

The limit exists and is a forward semimartingale.

104
• **Backward Itô stochastic integral**: Let \( t \leq s \), then

\[
I_s(t) = \int_s^t F(f(s,u), d^-u) := \lim_{h \to 0} \text{in pr.} \sum_{k=0}^{n-1} \{ F(f(s, t_k), t_{k+1}) - F(f(s, t_k), t_k) \},
\]

where the limit in probability is taken over a sequence of partitions of \([t, s]\) for which \( h \to 0 \).

This limit exists and is a backward semimartingale.

**Remark C.4** Note that Itô stochastic integrals are also define for less regular \( F \) and \( f \), but for our purpose in this discussion we gave the above definition.

Stratonovich and Itô stochastic integrals have some relation and this is the subject of the below Theorem.

**Theorem C.5** ([4] [47], [48]) Let \( F(t, \omega, x) \) be \( C^{0,1} \)-semimartingale helix with local characteristics \((a, b, A)\) such that \( a \in L_{loc}(\mathbb{R}, dA, C^{2,\delta}) \) for some \( \delta \geq 0 \) and \( b \in L_{loc}(\mathbb{R}, dA, C^{1,0}) \). Suppose \( f(s,t) \) is a semimartingale, then

\[
\int_s^t F(f(s,u), d^+u) = \int_s^t F(f(s,u), d^-u) + \frac{1}{2} \sum_{j=1}^d \left\langle \int_s^t \frac{\partial F}{\partial x}(f(s,u), d^+u), f^j(s,t) \right\rangle.
\]

**D Proof of Theorem 1.2.2**

Here, we present a comprehensive proof of Theorem 1.2.2 (basic properties of RDS with two sided time \( T = \mathbb{R} \) or \( T = \mathbb{Z} \)), the presentation would explain some of the implicit assumptions we have employed when we have RDS with two sided time in this thesis. We restate the theorem for easy access to the reader.

**Theorem D.1** (Basic properties of random dynamical systems [4])

(i) Let \( \varphi \) be a measurable RDS on a measurable space \((X, \mathcal{B})\) over \( \theta \) and let \( T = \mathbb{R} \) or \( T = \mathbb{Z} \). Then for all \((t, \omega) \in T \times \Omega, \varphi(t, \omega)\) is bimeasurable bijection of \((X, \mathcal{B})\) and

\[
\varphi(t, \omega)^{-1} = \varphi(-t, \theta t \omega), \quad \text{for all } (t, \omega) \in T \times \Omega, \tag{D.1}
\]

or

\[
\varphi(-t, \omega) = \varphi(t, \theta_{-t} \omega)^{-1}, \quad \text{for all } (t, \omega) \in T \times \Omega. \tag{D.2}
\]

Moreover, the mapping \((t, \omega, x) \mapsto \varphi(t, \omega)^{-1}x\) is measurable.
(ii) If $\varphi$ is a continuous RDS on a topological space $X$. Then for all $(t, \omega) \in T \times \Omega$, we have that $\varphi(t, \omega)$ is a homeomorphism, if

1. $T = \mathbb{Z}$ or,
2. $T = \mathbb{R}$ and $X$ is a topological manifold, or
3. $T = \mathbb{R}$ and $X$ is a compact Hausdorff space.

Then $(t, \omega) \mapsto \varphi(t, \omega)^{-1}x$ is continuous for all $\omega \in \Omega$.

(iii) If $\varphi$ is a $C^k$ RDS on a manifold $X$. Then for all $(t, \omega) \in T \times \Omega$, $\varphi(t, \omega)$ is a $C^k$ diffeomorphism. Moreover, $(t, x) \mapsto \varphi(t, \omega)^{-1}x$ is $C^k$ with respect to $x$ for $\omega \in \Omega$.

**Proof.**

(i) First we have to show that $\varphi(t, \omega)$ is a bijection of $X$ onto $X$. Let $x, y \in X$, such that

$$\varphi(t, \omega)x = \varphi(t, \omega)y,$$

for all $(t, \omega) \in T \times \Omega$,

using the cocycle property

$$\varphi(-t, \theta t \omega) \circ \varphi(t, \omega)x = \varphi(-t, \theta t \omega) \circ \varphi(t, \omega)y,$$

for all $(t, \omega) \in T \times \Omega$,

so,

$$x = y,$$

which shows that $\varphi(t, \omega)$ is injective.

Next, we know that

$$\varphi(t, \omega)X \subset X.$$

It is enough to show that $X \subset \varphi(t, \omega)X$

$$\varphi(t, \omega)X \subset X,$$

for all $(t, \omega) \in T \times \Omega$.

In particular,

$$\varphi(-t, \omega)X \subset X,$$

$$\varphi(t, \theta -t \omega) \circ \varphi(t, \omega)X \subset \varphi(t, \theta -t \omega)X.$$

Which implies that

$$X \subset \varphi(t, \omega)X,$$

for all $(t, \omega) \in T \times \Omega$,
hence $\varphi(t, \omega)X = X$.

So, $\varphi(t, \omega) : X \to X$ is a bijection.

Next,

$$\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \quad \text{for all } (t, \omega) \in \mathbb{T} \times \Omega,$$

$t = -s$, one gets

$$\text{Id}_X = \varphi(-s, \theta_s \omega) \circ \varphi(s, \omega) \quad \text{(D.3)}$$

$s = -t$, we have

$$\text{Id}_X = \varphi(t, \theta_{-t} \hat{\omega}) \circ \varphi(-t, \hat{\omega}),$$

let $\theta_{t\omega} = \hat{\omega} \Rightarrow \omega = \theta_{-t} \hat{\omega}$, so that

$$\text{Id}_X = \varphi(t, \omega) \circ \varphi(-t, \theta_{t\omega}) \quad \text{(D.4)}$$

If $s = t$,

$$\text{Id}_X = \varphi(-t, \theta_{t\omega}) \circ \varphi(t, \omega)$$

and

$$\text{Id}_X = \varphi(t, \omega) \circ \varphi(-t, \theta_{t\omega})$$

which implies that

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta_{t\omega}).$$

The measurability of $(t, \omega, x) \mapsto \varphi(t, \omega)^{-1}x$, follows from the fact that $\varphi(t, \omega)^{-1}x = \varphi(-t, \theta_{t\omega})x$ is a composition of the measurable maps $(-t, \theta_{t\omega}, x)$ and $\varphi(t, \omega)x$.

(ii) We already know from (i) that

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta_{t\omega}),$$

from definition of continuous RDS, $\varphi(-t, \theta_{t\omega})$ is continuous, hence $\varphi(t, \omega) \in \text{Homeo}(X)$.

1. Obvious

2. $(t, x) \mapsto (t, \varphi(t, \omega)x)$ is continuous by definition and bijection by part (i).

   We appeal to the principle of invariance of domain (Theorem A.6), that a continuous bijection $f : \mathbb{R} \times X \to \mathbb{R} \times X$ is a homeomorphism. So, the inverse map $(t, x) \mapsto (t, \varphi(t, \omega)^{-1}x)$ is continuous. In particular, $(t, x) \mapsto \varphi(t, \omega)^{-1}x$ is continuous.
3. A continuous bijection of a compact space into a Hausdorff space is a homeomorphism, thanks to Theorem A.3. We apply this on the continuous bijective map $f(\omega): K \times X \to K \times X$, where $K$ compact subset of $\mathbb{R}$ and $f(\omega)(t, x) = (t, \varphi(t, \omega)x)$. Since $X$ is a compact Hausdorff space, then $f(\omega)$ is a homeomorphism and it follows that

$$(t, x) \mapsto \varphi(t, \omega)^{-1}x$$ is continuous.

(iii) From the formula in part (i) $\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega)$ we have that

$$\varphi(t, \omega) \in \text{Diff}^k(X),$$

since $\varphi(-t, \theta_t \omega)x$ is $k$-times continuously differentiable with respect to $x$, by definition of smooth RDS.

Next, we know that the derivative of a diffeomorphism is nonsingular, and given by the formula,

$$D\varphi(t, \omega)^{-1}x = (D\varphi(t, \omega)y|_{y=\varphi(t, \omega)^{-1}x})^{-1}.$$ 

Hence, $(t, x) \mapsto D\varphi(t, \omega)^{-1}x$ is continuous, because

- $(t, x) \mapsto D\varphi(t, \omega)^{-1}x$ is continuous by assumption,
- $(t, x) \mapsto \varphi(t, \omega)^{-1}x$ is continuous by part (ii).

Similar argument will also be applied for higher order derivatives. \(\Box\)
Bibliography


