Bi-Hamiltonian systems and Jordan-Kronecker invariants of finite-dimensional Lie algebras

Jordan-Kronecker Invariants of finite-dimensional Lie algebras and bi-Hamiltonian systems

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Outline

- Motivation and Generalised Argument Shift Conjecture
- Jordan-Kronecker decomposition theorem
- Jordan-Kronecker invariants of a finite-dimensional Lie algebra
- General facts about JK invariants
- Realisation Problem
- Examples

Motivation

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- It is a very natural idea.
  A Lie algebra is defined by its structure tensor c_{ij}^k. Too complicated!
  Classical approach: Take $\mathrm{ad}_\xi = \sum_i c_{ij}^k \xi^i$ for generic $\xi \in \mathfrak{g}$ and then study the properties of this operator.
  Another option: Take a form $\mathcal{A}_x = \sum_k c_{ij}^k x_k$ for generic $x \in \mathfrak{g}^*$.
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- This approach seems to be useful for the study of arbitrary Lie algebras, not necessarily semisimple. It would be nice to have universal tools.
- Finally, the main reason for us is the generalised "argument shift conjecture". The "algebra of shifts" is one of those indeed universal constructions which should be studied in detail.

Historical remarks

1. The first and main ingredient is the argument shift method suggested by A. Mischenko and A. Fomenko in 1976 as a generalisation of S. Manakov construction.
2. In 1988, I. Gelfand and I. Zakharevich observed a very important relationship between compatible Poisson brackets and Jordan–Kronecker decomposition theorem and used it to study the so-called micro-Kronecker pencils and their applications.
3. Jordan–Kronecker decomposition for a pencil of skew-symmetric forms (C.Jordan, L.Kronecker, F.Gantmacher, G.Gurevich, R.Thompson, . . .).
4. Transition from algebraic canonical forms of pencils in linear algebra to normal forms of compatible Poisson brackets was done by F.-J. Turiel (series of papers since 1989).
5. The idea of JK invariants was conceptualised in the framework of an informal Moscow–Loughborough research seminar for PhD students (A. Izosimov, P. Zhang, A. Konjaev, A. Vorontsov, I. Kozlov, D. Dowell).
6. This talk is based on our joint paper with Pumei Zhang.

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$$\{f,g\}(x) = \mathcal{A}_x(df(x), dg(x)) = \sum \left(c_{ij}^k x_k\right) \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.$$

The second is:

$$\{f,g\}_a(x) = \mathcal{A}_a(df(x), dg(x)) = \sum \left(c_{ij}^k a_k\right) \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}, \quad a \in \mathfrak{g}^* \text{ (fixed)}.$$

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Consider the following family of functions on $\mathfrak{g}^*$:

$$\mathcal{F}_a = \{f(x + \lambda a) \mid f \in \mathrm{I}(\mathfrak{g}), \lambda \in \mathbb{R}\}.$$

Theorem (Mischenko, Fomenko)

1) *$\mathcal{F}_a$ is commutative w.r.t. both $\{\ ,\ \}$ and $\{\ ,\ \}_a$.*
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Mischenko-Fomenko conjecture.
For any $\mathfrak{g}$, there is a complete commutative family $\mathcal{G} \subset P(\mathfrak{g})$ of polynomials in involution.

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Generalised Argument Shift Conjecture.
For any $\mathfrak{g}$, there is a complete commutative family $\mathcal{G}_a \subset P(\mathfrak{g})$ of polynomials in bi-involution.

Jordan-Kronecker decomposition

This theorem gives a classification of skew-symmetric bilinear forms $\mathcal{A}, \mathcal{B}$ on a finite-dimensional vector space by reducing them simultaneously to an elegant block-diagonal form.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be two skew-symmetric bilinear forms on a complex vector space V. Then by an appropriate choice of a basis, their matrices can be simultaneously reduced to the following canonical block-diagonal form:

$$\mathcal{A} \mapsto \begin{pmatrix} \mathcal{A}_1 & & & \\ & \mathcal{A}_2 & & \\ & & \ddots & \\ & & & \mathcal{A}_k \end{pmatrix}, \qquad \mathcal{B} \mapsto \begin{pmatrix} \mathcal{B}_1 & & & \\ & \mathcal{B}_2 & & \\ & & \ddots & \\ & & & \mathcal{B}_k \end{pmatrix}$$

where the pairs of the corresponding blocks $\mathcal{A}_i$ and $\mathcal{B}_i$ can be of the following three types:

Jordan-Kronecker decomposition

	$\mathcal{A}_i$	$\mathcal{B}_i$
Jordan block ($\lambda_i \in \mathbb{C}$)	$\begin{pmatrix} & J(\lambda_i) \\ -J^\top(\lambda_i) & \end{pmatrix}$	$\begin{pmatrix} & -\mathrm{Id} \\ \mathrm{Id} & \end{pmatrix}$
Jordan block ($\lambda_i = \infty$)	$\begin{pmatrix} & -\mathrm{Id} \\ \mathrm{Id} & \end{pmatrix}$	$\begin{pmatrix} & J(0) \\ -J^\top(0) & \end{pmatrix}$
Kronecker block	$\left(\begin{array}{c\|c} & \begin{matrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \end{matrix} & \end{array}\right)$	$\left(\begin{array}{c\|c} & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} \\ \hline \begin{matrix} 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \end{matrix} & \end{array}\right)$

Some comments

- The characteristic number λ_i plays the same role as an "eigenvalue" in the case of linear operators (recursion operators). More precisely, λ_i are those numbers for which the rank of $\mathcal{A}_\lambda = \mathcal{A} + \lambda\mathcal{B}$ for $\lambda = \lambda_i$ is not maximal.
- If $\mu \neq \lambda_i$, then $\mathcal{A}_\mu = \mathcal{A} + \mu\mathcal{B}$ is called regular (in the pencil $\mathcal{P} = \{\mathcal{A}_\lambda\}$).
- The sizes of Kronecker blocks are odd $2k_i - 1$, the sizes of Jordan blocks are even $2j_m$. The numbers k_i and j_m are called Kronecker and Jordan indices of the pencil.
- The set of characteristic numbers λ_i of the pencil $\mathcal{P}$ will be denoted by Λ.
- The Jordan–Kronecker decomposition theorem implies the existence of a large subspace which is isotropic simultaneously for all forms from the pencil $\mathcal{P}$ (bi-Lagrangian subspace).

Theorem

For every pencil $\mathcal{P} = \{\mathcal{A}_\lambda\}$, there is a bi-Lagrangian subspace $L \subset V$. This means that L is isotropic with respect to all $\mathcal{A}_\mu \in \mathcal{P}$ and is maximal isotropic for all regular forms $\mathcal{A}_\lambda \in \mathcal{P}$. In particular, $\dim L = \frac{1}{2}(\dim V + \operatorname{corank} \mathcal{P})$.

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Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual space. We say that $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$ is a generic pair if the type of the Jordan-Kronecker decomposition of $\mathcal{A}_x = \sum_k c_{ij}^k x_k$ and $\mathcal{A}_a = \sum_k c_{ij}^k a_k$ is the same for all points in the neighborhoods of (x, a).

Definition

The type of the Jordan-Kronecker canonical form for the pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ for a generic pair $(x, a) \in \mathfrak{g}^*$, is called the Jordan-Kronecker invariant of $\mathfrak{g}$.

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In particular, a Lie algebra $\mathfrak{g}$ is

- of Kronecker type,
- of Jordan (symplectic) type,
- of mixed type,

if the Jordan–Kronecker decomposition of the (generic) pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ consists of

- only Kronecker blocks,
- only Jordan blocks
- both Jordan and Kronecker blocks.

The Kronecker and Jordan indices of a generic pencil $\{\mathcal{A}_x + \lambda \mathcal{A}_a\}$ are said to be the Kronecker and Jordan indices of $\mathfrak{g}$.

Prerequisites: basic notions

- Finite-dimensional Lie algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$
- Adjoint and coadjoint representations of the Lie group G and Lie algebra $\mathfrak{g}$: $\quad \mathrm{Ad}_X \xi = X\xi X^{-1}, \quad \langle \mathrm{Ad}^*_X a, \xi \rangle = \langle a, \mathrm{Ad}^{-1}_X \xi \rangle$
  Similarly: $\quad \mathrm{ad}_\xi \eta = [\xi, \eta], \quad \langle \mathrm{ad}^*_\xi a, \eta \rangle = \langle a, -[\xi, \eta] \rangle.$
- Lie-Poisson bracket on $\mathfrak{g}^*$:
$$\{f, g\}(x) = \langle x, [df(x), dg(x)] \rangle$$
- Symplectic leaves of the Lie–Poisson structure $=$ Ad^*-orbits; Casimir functions $=$ invariants of the coadjoint representation.
- Annihilator of $a \in \mathfrak{g}^*$:
$$\mathrm{Ann}\, a = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi a = 0\}$$
- Index of $\mathfrak{g}$ is the codimension of generic Ad^*-orbits:
$$\mathrm{ind}\, \mathfrak{g} = \min_{x \in \mathfrak{g}^*} \dim \mathrm{Ann}\, x$$
  If $\mathrm{ind}\, \mathfrak{g} = 0$, then the Lie algebra $\mathfrak{g}$ is called Frobenius.
- Singular set
$$\mathrm{Sing} = \{y \in \mathfrak{g}^* \mid \dim \mathrm{Ann}\,(x) > \mathrm{ind}\, \mathfrak{g}\}$$

Kronecker case

Theorem

The following properties of a Lie algebra $\mathfrak{g}$ are equivalent

1. $\mathfrak{g}$ *is of Kronecker type, i. e. the Jordan–Kronecker decomposition of the (generic) pencil* $\mathcal{A}_x + \lambda \mathcal{A}_a$ *consists of Kronecker blocks only,*
2. $\operatorname{codim} \mathrm{Sing} \geq 2$, *where*

$$\mathrm{Sing} = \{y \in \mathfrak{g}^* \mid \dim \mathrm{Ann}\, y > \mathrm{ind}\, \mathfrak{g}\} \subset \mathfrak{g}^*$$

   is the singular subset of $\mathfrak{g}^*$,

3. *the algebra of shifts* $\mathcal{F}_a$ *is complete.*

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Example

Let $\mathfrak{g}$ be semisimple. Then $\mathfrak{g}$ is of Kronecker type and the Kronecker indices $k_1, \ldots, k_s$, $s = \mathrm{ind}\, \mathfrak{g}$, are exactly the degrees of the basic Casimirs $f_1, \ldots, f_s$ (invariants of the adjoint representation).
This property holds for many other classes of Lie algebras, e. g., $e(n) = so(n) + \mathbb{R}^n$.

Jordan case

Theorem

The following properties of a Lie algebra $\mathfrak{g}$ are equivalent

1. *$\mathfrak{g}$ is of Jordan type, i.e. the Jordan–Kronecker decomposition of the generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ consists of Jordan blocks only,*
2. *a generic form $\mathcal{A}_x$ is non-degenerate, i.e., $\mathrm{ind}\, \mathfrak{g} = 0$ and $\mathfrak{g}$ is a Frobenius Lie algebra,*
3. *$\mathcal{F}_a$ is trivial, i.e., $\mathcal{F}_a = \mathbb{C}$.*

General case

Proposition

1) The number of Kronecker blocks in the JK decomposition is equal to the index of $\mathfrak{g}$.
2) The number of trivial Kronecker blocks is greater or equal to the dimension of the center of $\mathfrak{g}$.
3) The number of independent functions in the family of shifts $\mathcal{F}_a$ (i.e., $\mathrm{tr.deg.}\,\mathcal{F}_a$) is equal to $\sum k_i$, where $k_1, \ldots, k_s$, $s = \mathrm{ind}\,\mathfrak{g}$ are the Kronecker indices of $\mathfrak{g}$.

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Theorem (A. Vorontsov, 2011)

Let $f_1(x), f_2(x), \ldots, f_s(x) \in P(\mathfrak{g})$ be algebraically independent polynomial Ad^*-invariants of $\mathfrak{g}$, $s = \mathrm{ind}\,\mathfrak{g}$, and $m_1 \leq m_2 \leq \cdots \leq m_s$ be their degrees, $m_i = \deg f_i$. Then

$$m_i \geq k_i,$$

where $k_1 \leq k_2 \leq \cdots \leq k_s$ are Kronecker indices of the Lie algebra $\mathfrak{g}$.

In the semisimple case (but not only!): $m_i = k_i$.

Characteristic numbers and singular set

- The characteristic numbers λ_i of the Lie algebra $\mathfrak{g}^*$ are, by definition, the characteristic numbers for a generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$, $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$.
- The characteristic numbers $\lambda_i = \lambda_i(x, a)$ are defined by the following algebraic condition $x + \lambda_i a \in \mathrm{Sing}$.
- The characteristic numbers exist if and only if $\operatorname{codim} \mathrm{Sing} = 1$.
- The codimension one component of the singular set Sing is defined by one polynomial equation $f(x) = 0$, where f, the characteristic polynomial, is the greatest common divisor of all polynomials of the form $\mathrm{Pf}\, C_{i_1,\dots,i_m}$, where $C_{i_1,\dots,i_m}$ is a diagonal submatrix of $\mathcal{A}_x = \left(c_{ij}^k x_k\right)$, $m = \dim \mathfrak{g} - \mathrm{ind}\, \mathfrak{g}$.
- This polynomial may be reducible and may contain multiple factors:

$$f(x) = \left(f_1(x)\right)^{s_1} \cdot \ldots \cdot \left(f_k(x)\right)^{s_k}$$

We can also consider the reduced polynomial

$$f_{\mathrm{red}}(x) = f_1(x) \cdot \ldots \cdot f_k(x)$$

which still define the singular set Sing.

Characteristic numbers and singular set

- $\deg f(x)$ and $\deg f_{red}(x)$ are called the algebraic and geometric degree of the singular set Sing.
- The characteristic numbers of $\mathfrak{g}$ are the roots of $f(x+\lambda a)$ (or, equivalently, of $f_{\mathrm{red}}(x+\lambda a)$).
- The number of distinct characteristic numbers of $\mathfrak{g}$ is equal to $\deg f_{\mathrm{red}}$.
- Let $f(x) = \left(f_1(x)\right)^{s_1} \cdot \ldots \cdot \left(f_k(x)\right)^{s_k}$ be the decomposition of the characteristic polynomial into irreducible components. Then s_i are multiplicities of characteristic numbers.
- The symmetric polynomials of characteristic numbers $\lambda_1, \cdots, \lambda_m$ are rational functions of x and a. Moreover, if $a \in \mathfrak{g}^*$ is fixed, then they are polynomial in x.
- $\lambda_i(x, a)$ are in bi-involutions and commute with the algebra of shifts $\mathcal{F}_a$.

Frobenius Lie algebras

Let $\mathfrak{g}$ be Frobenius, i.e., $\operatorname{ind}\mathfrak{g} = 0$. Then Sing is defined by one polynomial, namely, $\mathrm{Pf}(\mathcal{A}_x) = \sqrt{\det\left(c_{ij}^k x_k\right)}$. The degree of this polynomial is $\frac{1}{2}\dim\mathfrak{g}$.

Theorem

Let $\mathfrak{g}$ be a Frobenius Lie algebra, and the (geometric) degree of $\mathrm{Sing} \subset \mathfrak{g}^*$ be equal to $k = \frac{1}{2}\dim\mathfrak{g}$.
Then a generic pencil $\mathcal{A}_x + \lambda\mathcal{A}_a$ is diagonalisable (i.e. has no Jordan blocks of size greater than 2×2), all characteristic numbers are distinct, and the coefficients of the "characteristic polynomial" $\mathrm{p}(\lambda) = \mathrm{Pf}\,\mathcal{A}_{x+\lambda a}$ form a complete family of polynomials in bi-involution.

Mixed case

Consider the polynomial $\mathrm{f}_{\mathrm{red}}(x)$, substitute $x + \lambda a$ and consider it as a polynomial in λ:

$$\mathrm{p}(\lambda) = \mathrm{f}_{\mathrm{red}}(x + \lambda a) = g_0(x) + \lambda g_1(x) + \lambda^2 g_2(x) + \cdots + \lambda^m g_m(x).$$

The homogeneous polynomials $g_0(x), \ldots, g_m(x)$ are obviously the symmetric polynomials of characteristic numbers. So they are in bi-involution and, moreover, they are in bi-involution with the algebra of shifts $\mathcal{F}_a$.

Combining the collection of g_k's with the algebra of shifts $\mathcal{F}_a$, we obtain an extended algebra of functions in bi-involution $\mathcal{G}_a$.

Question. Is $\mathcal{G}_a$ complete?

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Theorem (A.Isosimov)

The algebra $\mathcal{G}_a$ is complete if and only if, the annihilator of a generic singular element is isomorphic to $\mathfrak{a}_2 \oplus \mathrm{centre}$, where $\mathfrak{a}_2$ is the two-dimensional non-Abelian Lie algebra.

Realisation problem

Let $\mathcal{P} = \{\mathcal{A} + \lambda\mathcal{B}\}$ be an arbitrary pencil of skew-symmetric bi-linear forms.

Question. Can $\mathcal{P}$ can be realised as a generic pencil $\mathcal{A}_{x+\lambda a}$ for a suitable Lie algebra $\mathfrak{g}$?

Observation. The JK invariants of a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ can naturally be obtained from those of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by "summation". In particular, the set of characteristic numbers for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ can be understood as the *disjoint* union of the corresponding sets for $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Thus, first it is natural to study the realisation problem for the following simplest cases:

- a single Kronecker block,
- a single λ-block which consists of several Jordan blocks.

Examples of such Lie algebras were constructed by I. Kozlov.

Single Kronecker block

This case can be realised by the Lie algebra $\mathfrak{g}$ with the basis $e_1, \dots, e_k$, $f_1, \dots, f_{k+1}$ and commutation relations:

$$[e_l, f_i] = f_i, \quad [e_i, f_{i+1}] = -f_{i+1}, \qquad i = 1, \dots, k.$$

This Lie algebra admits the following matrix representation

$$\begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \in \mathrm{gl}(k+2, \mathbb{C}),$$

where A denotes the matrix $\mathrm{diag}(a_1, a_2 - a_1, a_3 - a_2, \dots, a_k - a_{k-1}, -a_k)$, i.e., an arbitrary diagonal matrix with zero trace, and b is a column of length $k+1$ with arbitrary entries.

The index of $\mathfrak{g}$ equals 1.

The singular set Sing consists of several connected components each of which has codimension 2 and is defined by two linear equations $f_i = 0, f_j = 0,\ i \neq j$.

The Casimir function of the Lie-Poisson bracket on $\mathfrak{g}^*$ is $f_1 f_2 \cdot \ldots \cdot f_{k+1}$.

Single λ-block

$$\mathfrak{g} = \left\{ \begin{pmatrix} a_0 & x_1 & x_2 & \dots & x_m & b_0 \\ & A_1 & 0 & \dots & 0 & y_1 \\ & & A_2 & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & y_{m-1} \\ & & & & A_m & y_m \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \right\}.$$

Here x_k is an arbitrary row of length n_k, y_k is an arbitrary column of length n_k, and A_k is the $n_k \times n_k$-matrix of the form:

$$A_k = \begin{pmatrix} a_0 & x_k^1 & x_k^2 & \dots & x_k^{n_k-2} & x_k^{n_k-1} \\ & a_0 & x_k^1 & \ddots & & x_k^{n_k-2} \\ & & a_0 & \ddots & & \vdots \\ & & & \ddots & x_k^1 & x_k^2 \\ & & & & a_0 & x_k^1 \\ & & & & & a_0 \end{pmatrix}$$

This Lie algebra is Frobenius, its singular set $\mathrm{Sing} \subset \mathfrak{g}^*$ is defined by the linear equation $b_0^* = 0$. Let $n_1 = \max_{k=1,\dots,m} n_k$. Then $\mathfrak{g}$ is of pure Jordan type and the Jordan indices are $n_1 + 1, n_2, \dots, n_m$.

One restriction

The Jordan indices can be arbitrary with the only restriction. Namely, the largest Jordan block is unique. This restriction is unavoidable (F.-J.Turiel). In particular, there is no Frobenius Lie algebra with diagonalisable λ-blocks if the multiplicity of λ is greater than 1.

Corollary

If a characteristic number of a Frobenius Lie algebra $\mathfrak{g}$ has non-trivial multiplicity, then $\mathrm{ind}\,\mathrm{Ann}\, y > \mathrm{ind}\,\mathfrak{g}$ for any generic singular $y \in \mathrm{Sing}$.

However this restriction disappears if we allow Kronecker blocks. The simplest example which illustrates this phenomenon is the Heisenberg algebra with the basis e_i, f_i, h $(i = 1, \dots, n)$ and relations $[e_i, f_j] = \delta_{ij} h$. A generic pencil $\mathcal{A}_{x+\lambda a}$ consists of one trivial Kronecker block and n Jordan 2×2 blocks with the same characteristic number $\lambda(x, a) = -\frac{\langle h,x \rangle}{\langle h,a \rangle}$.

Elashvili conjecture

Let $\operatorname{Ann} a = \{\xi \in \mathfrak{g} \mid \operatorname{ad}^*_\xi a = 0\}$ be the stationary subalgebra of $a \in \mathfrak{g}^*$ with respect to the coadjoint representation. The following estimate is well-known:

$$\operatorname{ind} \operatorname{Ann} a \geq \operatorname{ind} \mathfrak{g}$$

Elashvili conjecture: if $\mathfrak{g}$ is semisimple, then $\operatorname{ind} \operatorname{Ann} a = \operatorname{ind} \mathfrak{g}$ for all $a \in \mathfrak{g} = \mathfrak{g}^*$.

Interpretation in terms of Jordan-Kronecker decomposition:

Proposition

Let $a \in \mathfrak{g}^*$ be fixed and $x \in \mathfrak{g}^*$ is generic in the sense that the type of the Jordan-Kronecker decomposition of $\mathcal{A}_x$ and $\mathcal{A}_a$ does not change in a certain neighborhood of x. Then

$$\operatorname{ind} \operatorname{Ann} a = \operatorname{ind} \mathfrak{g}$$

if and only if the Jordan–Kronecker decomposition does not contain any non-trivial Jordan blocks, i.e., the Jordan part is diagonalisable. Otherwise, i.e. if there are non-trivial Jordan blocks, we have strong inequality:

$$\operatorname{ind} \operatorname{Ann} a > \operatorname{ind} \mathfrak{g}$$

Examples

- Semisimple Lie algebras:
  Pure Kronecker case, the Kronecker indices k_i coincide with the degrees of m_i of basis Casimirs:
  $$k_i = m_i.$$
  For classical series these numbers m_i are:
  - A_n: $2, 3, 4, \ldots, n+1$;
  - B_n: $2, 4, 6, \ldots, 2n$;
  - C_n: $2, 4, 6, \ldots, 2n$;
  - D_n: $2, 4, 6, \ldots, 2n-2$ and n.
- Some semidirect sums.
  - $e(n) = so(n) + \mathbb{R}^n$:
    JK invariants are the same as those for $so(n+1)$.
  - $\mathfrak{g} = sl(n) + \mathbb{R}^n$:
    Pure Kronecker type with one single Kronecker block, i.e., $k_1 = \frac{1}{2}(\dim \mathfrak{g} + 1)$.
  - $\mathfrak{k} +_\rho V$ with $\mathfrak{k}$ simple and ρ irreducible:
    Pure Kronecker type
    (F. Knop, P. Littelmann, B. Priwitzer, AB)
- Lie algebras of low dimension ≤ 5. The complete list of JK invariants is obtained by P. Zhang.

Examples

- Affine Lie algebra $\mathfrak{aff}(n) = gl(n) + \mathbb{R}^n$. The Pfaffian $\mathrm{Pf}(\mathcal{A}_x) = \sqrt{\det\left(c_{ij}^k x_k\right)}$ is irreducible and we can apply one of above theorems. The pencil $\mathcal{A}_x + \lambda\mathcal{A}_a$ is of Jordan type, diagonalisable, with $\frac{1}{2}\dim g$ distinct characteristic number. In other words, the Jordan indices of $\mathfrak{g}$ are

$$\underbrace{1, 1, \dots, 1}_{k}, \quad k = \frac{1}{2}(n^2 + n) = \frac{1}{2}\dim \mathfrak{aff}(n).$$

- Another interesting example $\mathfrak{g} = gl(n) + \mathbb{R}^{n^2}$, where $\mathbb{R}^{n^2}$ is realised as the space of $n \times n$-matrices, and the action of $gl(n)$ on it is left multiplication. The matrix realisation of $\mathfrak{g}$ is: $\begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix}$.
  This Lie algebra is Frobenius, the singular set is defined by $\det C = 0$. Hence, we have n distinct characteristic numbers, the multiplicity of each of them is $2n$, and the Jordan indices are

$$\underbrace{1, 1, \dots, 1}_{n-2}, 2$$

Triangular Lie algebra

Let $\mathfrak{t}_n$ be the Lie algebra of upper triangular $n \times n$ matrices.

The description of Jordan-Kronecker invariants for $\mathfrak{t}_n$ easily follows from the results by A.Arkhangelskii.

If n is even, then $\mathfrak{t}_n$ is of mixed type. The coadjoint invariants are rational functions $f_k = \dfrac{P_k}{Q_k}$, $k = 1, \dots, \frac{n}{2}$ with $\deg P_k = k$ and $\deg Q_k = k - 1$. The Kronecker indices are exactly $\deg P_k + \deg Q_k$, namely

$$1, 3, 5, \dots, n-1.$$

The singular set $\mathrm{Sing} \subset \mathfrak{t}_n^*$ is defined by an irreducible polynomial f of degree $\frac{n}{2}$. Therefore, $\mathfrak{t}_n$ possesses $\frac{n}{2}$ distinct characteristic numbers, each of multiplicity one. In particular, the Jordan part of a generic pencil $\mathcal{A}_{x+\lambda a}$ is diagonalisable and Jordan indices are $1, \dots, 1$ ($\frac{n}{2}$ times).

If n is odd, then $\mathfrak{t}_n$ is of Kronecker type and the Kronecker indices are $1, 3, 5, \dots, n$.

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Important: For all these Lie algebras, the Generalised Argument Shift Conjecture holds.