Bi-Hamiltonian systems and Jordan-Kronecker invariants of finite-dimensional Lie algebras

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Jordan-Kronecker Invariants of finite-dimensional Lie algebras and bi-Hamiltonian systems

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Finite Dimensional Integrable Systems, CIRM, 15–19 July, 2013
Outline

▶ Motivation and Generalised Argument Shift Conjecture

▶ Jordan-Kronecker decomposition theorem

▶ Jordan-Kronecker invariants of a finite-dimensional Lie algebra

▶ General facts about JK invariants

▶ Realisation Problem

▶ Examples
Motivation

It is a very natural idea. A Lie algebra is defined by its structure tensor $c_{ij}$. Too complicated!

Classical approach: Take $\text{ad}_\xi = \sum_i c_{ij} \xi^i$ for generic $\xi \in g$ and then study the properties of this operator.

Another option: Take a form $A_x = \sum_k c_{kj} x^k$ for generic $x \in g^*$. Unfortunately, the only invariant is the rank of $A_x$.

But... non-trivial invariants appear if we consider a pair of forms $A_x$ and $A_a$, $x, a \in g^*$. Can we get anything interesting?

▶ Many classical facts become more transparent and some new results can be derived.

▶ This approach seems to be useful for the study of arbitrary Lie algebras, not necessarily semisimple. It would be nice to have universal tools.

▶ Finally, the main reason for us is the generalised “argument shift conjecture”. The “algebra of shifts” is one of those indeed universal constructions which should be studied in detail.
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Historical remarks

1. The first and main ingredient is the argument shift method suggested by A. Mischenko and A. Fomenko in 1976 as a generalisation of S. Manakov construction.

2. In 1988, I. Gelfand and I. Zakharevich observed a very important relationship between compatible Poisson brackets and Jordan–Kronecker decomposition theorem and used it to study the so-called micro-Kronecker pencils and their applications.


4. Transition from algebraic canonical forms of pencils in linear algebra to normal forms of compatible Poisson brackets was done by F.-J. Turiel (series of papers since 1989).

5. The idea of JK invariants was conceptualised in the framework of an informal Moscow–Loughborough research seminar for PhD students (A. Izosimov, P. Zhang, A. Konjaev, A. Vorontsov, I. Kozlov, D. Dowell).

6. This talk is based on our joint paper with Pumei Zhang.
Conjecture

In the two forms $A^x A^a$, one can easily recognise two famous compatible Poisson structures on the dual space $g^*$ of a Lie algebra $g$.

The first is the standard Lie-Poisson bracket:

$$\{ f, g \}(x) = A^x(df(x), dg(x)) = \sum (c_{kij} x_k) \partial f / \partial x_i \partial g / \partial x_j.$$ 

The second is:

$$\{ f, g \}^a(x) = A^a(df(x), dg(x)) = \sum (c_{kij}^a a_k) \partial f / \partial x_i \partial g / \partial x_j, \quad a \in g^*(\text{fixed}).$$

Consider the following family of functions on $g^*$:

$$F^a = \{ f(x + \lambda a) | f \in I(g), \lambda \in \mathbb{R} \}.$$ 

Theorem (Mischenko, Fomenko)

1) $F^a$ is commutative w.r.t. both $\{ , \}$ and $\{ , \}^a$.

2) If $g$ is semisimple, then $F^a$ is complete.

Mischenko-Fomenko conjecture.

For any $g$, there is a complete commutative family $G \subset P(g)$ of polynomials in involution.
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$$\{f, g\}_a(x) = A_a(df(x), dg(x)) = \sum (c_{ij}^k a_k) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad a \in g^* \text{ (fixed).}$$
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**Mischenko-Fomenko conjecture.**

For any $\mathfrak{g}$, there is a complete commutative family $\mathcal{G} \subset P(\mathfrak{g})$ of polynomials in involution.
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Generalised Argument Shift Conjecture.

For any $g$, there is a complete commutative family $\mathcal{G}_a \subset P(g)$ of polynomials in bi-involution.
Jordan-Kronecker decomposition

This theorem gives a classification of skew-symmetric bilinear forms $\mathcal{A}, \mathcal{B}$ on a finite-dimensional vector space by reducing them simultaneously to an elegant block-diagonal form.

**Theorem**

Let $\mathcal{A}$ and $\mathcal{B}$ be two skew-symmetric bilinear forms on a complex vector space $V$. Then by an appropriate choice of a basis, their matrices can be simultaneously reduced to the following canonical block-diagonal form:

$$
\mathcal{A} \mapsto \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \\ & & & A_k \end{pmatrix}, \quad \mathcal{B} \mapsto \begin{pmatrix} B_1 & & \\ & B_2 & \\ & & \ddots \\ & & & B_k \end{pmatrix}
$$

where the pairs of the corresponding blocks $A_i$ and $B_i$ can be of the following three types:
Jordan-Kronecker decomposition

\[
\begin{align*}
\mathcal{A}_i &= \begin{pmatrix} J(\lambda_i) \\ -J^\top(\lambda_i) \end{pmatrix} \\
\mathcal{B}_i &= \begin{pmatrix} -\text{Id} \\ \text{Id} \\ -J^\top(0) \end{pmatrix}
\end{align*}
\]

Jordan block \((\lambda_i \in \mathbb{C})\)

Jordan block \((\lambda_i = \infty)\)

Kronecker block
The characteristic number $\lambda_i$ plays the same role as an “eigenvalue” in the case of linear operators (recursion operators). More precisely, $\lambda_i$ are those numbers for which the rank of $A_\lambda = A + \lambda B$ for $\lambda = \lambda_i$ is not maximal.

If $\mu \neq \lambda_i$, then $A_\mu = A + \mu B$ is called regular (in the pencil $P = \{A_\lambda\}$).

The sizes of Kronecker blocks are odd $2k_i - 1$, the sizes of Jordan blocks are even $2j_m$. The numbers $k_i$ and $j_m$ are called Kronecker and Jordan indices of the pencil.

The set of characteristic numbers $\lambda_i$ of the pencil $P$ will be denoted by $\Lambda$.

The Jordan–Kronecker decomposition theorem implies the existence of a large subspace which is isotropic simultaneously for all forms from the pencil $P$ (bi-Lagrangian subspace).

**Theorem**

For every pencil $P = \{A_\lambda\}$, there is a bi-Lagrangian subspace $L \subset V$. This means that $L$ is isotropic with respect to all $A_\mu \in P$ and is maximal isotropic for all regular forms $A_\lambda \in P$. In particular, $\dim L = \frac{1}{2}(\dim V + \text{corank } P)$. 
Definition of Jordan-Kronecker invariants

Let $g$ be a Lie algebra and $g^\ast$ its dual space. We say that $(x, a) \in g^\ast \times g^\ast$ is a generic pair if the type of the Jordan-Kronecker decomposition of $Ax = \sum k c_{ij} x_k$ and $Aa = \sum k c_{ij} a_k$ is the same for all points in the neighborhoods of $(x, a)$.

Definition

The type of the Jordan-Kronecker canonical form for the pencil $Ax + \lambda Aa$ for a generic pair $(x, a) \in g^\ast$, is called the Jordan-Kronecker invariant of $g$.

In particular, a Lie algebra $g$ is

- of Kronecker type,
- of Jordan (symplectic) type,
- of mixed type,

if the Jordan–Kronecker decomposition of the (generic) pencil $Ax + \lambda Aa$ consists of

- only Kronecker blocks,
- only Jordan blocks,
- both Jordan and Kronecker blocks.

The Kronecker and Jordan indices of a generic pencil $\{Ax + \lambda Aa\}$ are said to be the Kronecker and Jordan indices of $g$. 
Definition of Jordan-Kronecker invariants

Let \( \mathfrak{g} \) be a Lie algebra and \( \mathfrak{g}^* \) its dual space. We say that \( (x, a) \in \mathfrak{g}^* \times \mathfrak{g}^* \) is a **generic pair** if the type of the Jordan-Kronecker decomposition of \( A_x = \sum_k c_{ij}^k x_k \) and \( A_a = \sum_k c_{ij}^k a_k \) is the same for all points in the neighborhoods of \( (x, a) \).

**Definition**

The **type of the Jordan-Kronecker canonical form** for the pencil \( A_x + \lambda A_a \) for a generic pair \( (x, a) \in \mathfrak{g}^* \), is called the **Jordan-Kronecker invariant** of \( \mathfrak{g} \).
Definition of Jordan-Kronecker invariants

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual space. We say that $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$ is a generic pair if the type of the Jordan-Kronecker decomposition of $A_x = \sum_k c_{ij}^k x_k$ and $A_a = \sum_k c_{ij}^k a_k$ is the same for all points in the neighborhoods of $(x, a)$.

Definition

The type of the Jordan-Kronecker canonical form for the pencil $A_x + \lambda A_a$ for a generic pair $(x, a) \in \mathfrak{g}^*$, is called the Jordan-Kronecker invariant of $\mathfrak{g}$. In particular, a Lie algebra $\mathfrak{g}$ is

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The Kronecker and Jordan indices of a generic pencil $\{A_x + \lambda A_a\}$ are said to be the Kronecker and Jordan indices of $\mathfrak{g}$. 
Prerequisites: basic notions

- Finite-dimensional Lie algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$
- Adjoint and coadjoint representations of the Lie group $G$ and Lie algebra $\mathfrak{g}$:
  \[ \text{Ad}_X \xi = X \xi X^{-1}, \quad \langle \text{Ad}^*_X a, \xi \rangle = \langle a, \text{Ad}^{-1}_X \xi \rangle \]
  Similarly:
  \[ \text{ad}_\xi \eta = [\xi, \eta], \quad \langle \text{ad}^*_\xi a, \eta \rangle = \langle a, -[\xi, \eta] \rangle. \]
- Lie-Poisson bracket on $\mathfrak{g}^*$:
  \[ \{f, g\}(x) = \langle x, [df(x), dg(x)] \rangle \]
- Symplectic leaves of the Lie–Poisson structure = $\text{Ad}^*$-orbits; Casimir functions = invariants of the coadjoint representation.
- Annihilator of $a \in \mathfrak{g}^*$:
  \[ \text{Ann} a = \{ \xi \in \mathfrak{g} \mid \text{ad}^*_\xi a = 0 \} \]
- Index of $\mathfrak{g}$ is the codimension of generic $\text{Ad}^*$-orbits:
  \[ \text{ind} \mathfrak{g} = \min_{x \in \mathfrak{g}^*} \dim \text{Ann} x \]
  If $\text{ind} \mathfrak{g} = 0$, then the Lie algebra $\mathfrak{g}$ is called Frobenius.
- Singular set
  \[ \text{Sing} = \{ y \in \mathfrak{g}^* \mid \dim \text{Ann}(x) > \text{ind} \mathfrak{g} \} \]
Theorem

The following properties of a Lie algebra $\mathfrak{g}$ are equivalent

1. $\mathfrak{g}$ is of Kronecker type, i.e. the Jordan–Kronecker decomposition of the (generic) pencil $A_x + \lambda A_a$ consists of Kronecker blocks only,
2. $\text{codim} \text{Sing} \geq 2$, where

$$\text{Sing} = \{y \in \mathfrak{g}^* \mid \dim \text{Ann} y > \text{ind } \mathfrak{g}\} \subset \mathfrak{g}^*$$

is the singular subset of $\mathfrak{g}^*$, 

3. the algebra of shifts $F_a$ is complete.

Example

Let $\mathfrak{g}$ be semisimple. Then $\mathfrak{g}$ is of Kronecker type and the Kronecker indices $k_1, \ldots, k_s$, $s = \text{ind } \mathfrak{g}$, are exactly the degrees of the basic Casimirs $f_1, \ldots, f_s$ (invariants of the adjoint representation). This property holds for many other classes of Lie algebras, e.g., $e(n) = \mathfrak{so}(n) + \mathbb{R}n$. 
Kronecker case

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Let $\mathfrak{g}$ be semisimple. Then $\mathfrak{g}$ is of Kronecker type and the Kronecker indices $k_1, \ldots, k_s, s = \text{ind} \mathfrak{g}$, are exactly the degrees of the basic Casimirs $f_1, \ldots, f_s$ (invariants of the adjoint representation).

This property holds for many other classes of Lie algebras, e.g., $e(n) = so(n) + \mathbb{R}^n$. 

Theorem

The following properties of a Lie algebra $\mathfrak{g}$ are equivalent

1. $\mathfrak{g}$ is of Jordan type, i.e. the Jordan–Kronecker decomposition of the generic pencil $A_x + \lambda A_a$ consists of Jordan blocks only,

2. a generic form $A_x$ is non-degenerate, i.e., $\text{ind} \, \mathfrak{g} = 0$ and $\mathfrak{g}$ is a Frobenius Lie algebra,

3. $\mathcal{F}_a$ is trivial, i.e., $\mathcal{F}_a = \mathbb{C}$.  

General case

Proposition

1) The number of Kronecker blocks in the JK decomposition is equal to the index of $g$.
2) The number of trivial Kronecker blocks is greater or equal to the dimension of the center of $g$.
3) The number of independent functions in the family of shifts $\mathcal{F}_a$ (i.e., tr.deg. $\mathcal{F}_a$) is equal to $\sum k_i$, where $k_1, \ldots, k_s$, $s = \text{ind } g$ are the Kronecker indices of $g$. 

Theorem (A. Vorontsov, 2011)

Let $f_1(x), f_2(x), \ldots, f_s(x) \in \mathcal{P}(g)$ be algebraically independent polynomial Ad$^*$-invariants of $g$, $s = \text{ind } g$, and $m_1 \leq m_2 \leq \cdots \leq m_s$ be their degrees, $m_i = \text{deg } f_i$. Then $m_i \geq k_i$, where $k_1 \leq k_2 \leq \cdots \leq k_s$ are Kronecker indices of the Lie algebra $g$. 

In the semisimple case (but not only!): $m_i = k_i$. 

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$$m_i \geq k_i,$$

where $k_1 \leq k_2 \leq \cdots \leq k_s$ are Kronecker indices of the Lie algebra $g$.

In the semisimple case (but not only!): $m_i = k_i$. 
The characteristic numbers $\lambda_i$ of the Lie algebra $\mathfrak{g}^*$ are, by definition, the characteristic numbers for a generic pencil $A_x + \lambda A_a$, $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$.

The characteristic numbers $\lambda_i = \lambda_i(x, a)$ are defined by the following algebraic condition $x + \lambda_i a \in \text{Sing}$.

The characteristic numbers exist if and only if $\text{codim Sing} = 1$.

The codimension one component of the singular set $\text{Sing}$ is defined by one polynomial equation $f(x) = 0$, where $f$, the characteristic polynomial, is the greatest common divisor of all polynomials of the form $\text{Pf} C_{i_1, \ldots, i_m}$, where $C_{i_1, \ldots, i_m}$ is a diagonal submatrix of $A_x = (c_{ij}^k x_k)$, $m = \dim \mathfrak{g} - \text{ind } \mathfrak{g}$.

This polynomial may be reducible and may contain multiple factors:

$$f(x) = (f_1(x))^{s_1} \cdot \ldots \cdot (f_k(x))^{s_k}$$

We can also consider the reduced polynomial

$$f_{\text{red}}(x) = f_1(x) \cdot \ldots \cdot f_k(x)$$

which still define the singular set $\text{Sing}$. 
Characteristic numbers and singular set

- \(\text{deg } f(x)\) and \(\text{deg } f_{\text{red}}(x)\) are called the algebraic and geometric degree of the singular set \(\text{Sing}\).
- The characteristic numbers of \(g\) are the roots of \(f(x + \lambda a)\) (or, equivalently, of \(f_{\text{red}}(x + \lambda a)\)).
- The number of distinct characteristic numbers of \(g\) is equal to \(\text{deg } f_{\text{red}}\).
- Let \(f(x) = (f_1(x))^{s_1} \cdot \ldots \cdot (f_k(x))^{s_k}\) be the decomposition of the characteristic polynomial into irreducible components. Then \(s_i\) are multiplicities of characteristic numbers.
- The symmetric polynomials of characteristic numbers \(\lambda_1, \ldots, \lambda_m\) are rational functions of \(x\) and \(a\). Moreover, if \(a \in g^*\) is fixed, then they are polynomial in \(x\).
- \(\lambda_i(x, a)\) are in bi-involutions and commute with the algebra of shifts \(F_a\).
Let $\mathfrak{g}$ be Frobenius, i.e., $\text{ind} \mathfrak{g} = 0$. Then $\text{Sing}$ is defined by one polynomial, namely, $\text{Pf}(A_x) = \sqrt{\det \left( c_{ij}^k x_k \right)}$. The degree of this polynomial is $\frac{1}{2} \dim \mathfrak{g}$.

**Theorem**

Let $\mathfrak{g}$ be a Frobenius Lie algebra, and the (geometric) degree of $\text{Sing} \subset \mathfrak{g}^*$ be equal to $k = \frac{1}{2} \dim \mathfrak{g}$.

Then a generic pencil $A_x + \lambda A_a$ is diagonalisable (i.e. has no Jordan blocks of size greater than $2 \times 2$), all characteristic numbers are distinct, and the coefficients of the “characteristic polynomial” $p(\lambda) = \text{Pf} A_x+\lambda A_a$ form a complete family of polynomials in bi-involution.
Consider the polynomial $f_{\text{red}}(x)$, substitute $x + \lambda a$ and consider it as a polynomial in $\lambda$:

$$p(\lambda) = f_{\text{red}}(x + \lambda a) = g_0(x) + \lambda g_1(x) + \lambda^2 g_2(x) + \cdots + \lambda^m g_m(x).$$

The homogeneous polynomials $g_0(x), \ldots, g_m(x)$ are obviously the symmetric polynomials of characteristic numbers. So they are in bi-involution and, moreover, they are in bi-involution with the algebra of shifts $\mathcal{F}_a$.
Combining the collection of $g_k$’s with the algebra of shifts $\mathcal{F}_a$, we obtain an extended algebra of functions in bi-involution $\mathcal{G}_a$.

**Question.** Is $\mathcal{G}_a$ complete?
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**Question.** Is $\mathcal{G}_a$ complete?

**Theorem (A.Isosimov)**

*The algebra $\mathcal{G}_a$ is complete if and only if, the annihilator of a generic singular element is isomorphic to $\mathfrak{a}_2 \oplus \text{centre}$, where $\mathfrak{a}_2$ is the two-dimensional non-Abelian Lie algebra.*
Realisation problem

Let $\mathcal{P} = \{A + \lambda B\}$ be an arbitrary pencil of skew-symmetric bi-linear forms.

**Question.** Can $\mathcal{P}$ can be realised as a generic pencil $A_x + \lambda a$ for a suitable Lie algebra $\mathfrak{g}$?

**Observation.** The JK invariants of a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ can naturally be obtained from those of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by “summation”. In particular, the set of characteristic numbers for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ can be understood as the *disjoint* union of the corresponding sets for $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Thus, first it is natural to study the realisation problem for the following simplest cases:

- a single Kronecker block,
- a single $\lambda$-block which consists of several Jordan blocks.

Examples of such Lie algebras were constructed by I. Kozlov.
Single Kronecker block

This case can be realised by the Lie algebra \( \mathfrak{g} \) with the basis \( e_1, \ldots, e_k, f_1, \ldots, f_{k+1} \) and commutation relations:

\[
[e_i, f_i] = f_i, \quad [e_i, f_{i+1}] = -f_{i+1}, \quad i = 1, \ldots, k.
\]

This Lie algebra admits the following matrix representation

\[
\begin{pmatrix}
A & b \\
0 & 0
\end{pmatrix} \in \mathfrak{gl}(k + 2, \mathbb{C}),
\]

where \( A \) denotes the matrix \( \text{diag}(a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, -a_k) \), i.e., an arbitrary diagonal matrix with zero trace, and \( b \) is a column of length \( k + 1 \) with arbitrary entries.

The index of \( \mathfrak{g} \) equals 1.

The singular set \( \text{Sing} \) consists of several connected components each of which has codimension 2 and is defined by two linear equations \( f_i = 0, f_j = 0, \ i \neq j \).

The Casimir function of the Lie-Poisson bracket on \( \mathfrak{g}^* \) is \( f_1 f_2 \cdot \ldots \cdot f_{k+1} \).
Single $\lambda$-block

\[
g = \begin{cases} 
\begin{pmatrix} 
a_0 & x_1 & x_2 & \ldots & x_m & b_0 \\
A_1 & 0 & \ldots & 0 & y_1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & 0 & y_{m-1} & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 
\end{pmatrix} 
\end{cases} 
\]

Here $x_k$ is an arbitrary row of length $n_k$, $y_k$ is an arbitrary column of length $n_k$, and $A_k$ is the $n_k \times n_k$-matrix of the form:

\[
A_k = \begin{pmatrix} 
a_0 & x_k^1 & x_k^2 & \ldots & x_k^{n_k-2} & x_k^{n_k-1} \\
a_0 & x_k^1 & \ddots & & x_k^{n_k-2} \\
a_0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & x_k^1 & x_k^2 \\
a_0 & x_k^1 & x_k^2 & \ldots & a_0 \\
a_0 & x_k^1 & \ldots & \ddots & \ddots \\
\end{pmatrix} 
\]

This Lie algebra is Frobenius, its singular set $\text{Sing} \subset g^*$ is defined by the linear equation $b_0^* = 0$. Let $n_1 = \max_{k=1,\ldots,m} n_k$. Then $g$ is of pure Jordan type and the Jordan indices are $n_1 + 1, n_2, \ldots, n_m$. 
The Jordan indices can be arbitrary with the only restriction. Namely, the largest Jordan block is unique. This restriction is unavoidable (F.-J.Turiel). In particular, there is no Frobenius Lie algebra with diagonalisable $\lambda$-blocks if the multiplicity of $\lambda$ is greater than 1.

**Corollary**

*If a characteristic number of a Frobenius Lie algebra $\mathfrak{g}$ has non-trivial multiplicity, then $\text{ind } \text{Ann } y > \text{ind } \mathfrak{g}$ for any generic singular $y \in \text{Sing}$.***

However this restriction disappears if we allow Kronecker blocks. The simplest example which illustrates this phenomenon is the Heisenberg algebra with the basis $e_i, f_i, h$ ($i = 1, \ldots, n$) and relations $[e_i, f_j] = \delta_{ij} h$. A generic pencil $A_{x+\lambda} a$ consists of one trivial Kronecker block and $n$ Jordan $2 \times 2$ blocks with the same characteristic number $\lambda(x, a) = -\frac{\langle h, x \rangle}{\langle h, a \rangle}$. 
Elashvili conjecture

Let $\text{Ann } a = \{\xi \in g \mid \text{ad}^* \xi a = 0\}$ be the stationary subalgebra of $a \in g^*$ with respect to the coadjoint representation. The following estimate is well-known:

$$\text{ind } \text{Ann } a \geq \text{ind } g$$

Elashvili conjecture: if $g$ is semisimple, then $\text{ind } \text{Ann } a = \text{ind } g$ for all $a \in g = g^*$.

Interpretation in terms of Jordan-Kronecker decomposition:

**Proposition**

Let $a \in g^*$ be fixed and $x \in g^*$ is generic in the sense that the type of the Jordan-Kronecker decomposition of $A_x$ and $A_a$ does not change in a certain neighborhood of $x$. Then

$$\text{ind } \text{Ann } a = \text{ind } g$$

if and only if the Jordan–Kronecker decomposition does not contain any non-trivial Jordan blocks, i.e., the Jordan part is diagonalisable. Otherwise, i.e. if there are non-trivial Jordan blocks, we have strong inequality:

$$\text{ind } \text{Ann } a > \text{ind } g$$
Examples

- Semisimple Lie algebras:
  Pure Kronecker case, the Kronecker indices $k_i$ coincide with the degrees of $m_i$ of basis Casimirs:
  
  $$k_i = m_i.$$ 

For classical series these numbers $m_i$ are:

- $A_n$: $2, 3, 4, \ldots, n + 1$;
- $B_n$: $2, 4, 6, \ldots, 2n$;
- $C_n$: $2, 4, 6, \ldots, 2n$;
- $D_n$: $2, 4, 6, \ldots, 2n - 2$ and $n$.

- Some semidirect sums.
  - $e(n) = so(n) + \mathbb{R}^n$:
    JK invariants are the same as those for $so(n + 1)$.
  - $g = sl(n) + \mathbb{R}^n$:
    Pure Kronecker type with one single Kronecker block, i.e.,
    $k_1 = \frac{1}{2}(\dim g + 1)$.
  - $\mathfrak{t} + \rho \mathfrak{V}$ with $\mathfrak{t}$ simple and $\rho$ irreducible:
    Pure Kronecker type
    (F. Knop, P. Littelmann, B. Priwitzer, AB)

- Lie algebras of low dimension $\leq 5$. The complete list of JK invariants is obtained by P. Zhang.
Examples

- Affine Lie algebra $\text{aff}(n) = \text{gl}(n) + \mathbb{R}^n$. The Pfaffian $\text{Pf}(A_x) = \sqrt{\det \left(c_{ij}^k x_k\right)}$ is irreducible and we can apply one of above theorems. The pencil $A_x + \lambda A_a$ is of Jordan type, diagonalisable, with $\frac{1}{2} \dim g$ distinct characteristic number. In other words, the Jordan indices of $g$ are

$$\underbrace{1, 1, \ldots, 1}_k, \quad k = \frac{1}{2} (n^2 + n) = \frac{1}{2} \dim \text{aff}(n).$$

- Another interesting example $g = \text{gl}(n) + \mathbb{R}^{n^2}$, where $\mathbb{R}^{n^2}$ is realised as the space of $n \times n$-matrices, and the action of $\text{gl}(n)$ on it is left multiplication. The matrix realisation of $g$ is: $\begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix}$. This Lie algebra is Frobenius, the singular set is defined by $\det C = 0$. Hence, we have $n$ distinct characteristic numbers, the multiplicity of each of them is $2n$, and the Jordan indices are

$$\underbrace{1, 1, \ldots, 1}_n, \underbrace{2}_{n - 2}.$$
Let $t_n$ be the Lie algebra of upper triangular $n \times n$ matrices.

The description of Jordan-Kronecker invariants for $t_n$ easily follows from the results by A.Arkhangelskii.

If $n$ is even, then $t_n$ is of mixed type. The coadjoint invariants are rational functions $f_k = \frac{P_k}{Q_k}, k = 1, \ldots, \frac{n}{2}$ with $\deg P_k = k$ and $\deg Q_k = k - 1$. The Kronecker indices are exactly $\deg P_k + \deg Q_k$, namely

$$1, 3, 5, \ldots, n - 1.$$ 

The singular set $\text{Sing} \subset t_n^*$ is defined by an irreducible polynomial $f$ of degree $\frac{n}{2}$. Therefore, $t_n$ possesses $\frac{n}{2}$ distinct characteristic numbers, each of multiplicity one. In particular, the Jordan part of a generic pencil $A_{x+\lambda}a$ is diagonalisable and Jordan indices are $1, \ldots, 1$ ($\frac{n}{2}$ times).

If $n$ is odd, then $t_n$ is of Kronecker type and the Kronecker indices are $1, 3, 5, \ldots, n$. Important: For all these Lie algebras, the Generalised Argument Shift Conjecture holds.
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