Conceptualising a university mathematics teaching practice in an activity theory perspective

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In this article I present a theorisation of a university mathematics teaching practice, based on a research study into the teaching of linear algebra in a first year mathematics undergraduate course. The research was largely qualitative and consisted of data collected in interviews with the lecturer and in observations of his lectures. Using Leontiev's (1981) activity theory framework I categorised the teaching of linear algebra on three levels: activity-motive, actions-goals and operations-conditions. Each level of analysis provided insights into the lecturer's teaching approach, his motivation, his intentions and his strategies in relation to his teaching. I developed a model of the teaching process that relates goals as expressed by the lecturer in interviews to the strategies that he designed for his teaching.

The research presented in this article is a qualitative study of one lecturer's teaching of mathematics at a UK university. This involved the teaching and learning of linear algebra, a mathematical topic that is commonly taught to students in their first year. The study explored the lecturer's teaching practice through observations of his lectures, as well as insights into his perspectives on teaching and learning. The lecturer, a research mathematician, expressed his views, his aims and his intentions in research meetings with two mathematics education researchers. Meetings took the form of conversations and discussions with the lecturer and provided insights into his mathematical and didactical thinking and planning in relation to his teaching. Observations of his lectures provided insights into his day-to-day teaching of students. Both meetings and observations provided the data for analyses which, when combined with a theoretical framework, led to a characterisation of the teaching practice in terms of the lecturer's intention and strategies.

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There is a growing body of research into students' learning of mathematics. This provides a wealth of insights and findings for researchers in the field, while research into teaching and teaching approaches that are based on empirical studies are, in comparison, far fewer (see Speer, Smith & Horvath, 2010). Thus, this article is a contribution to a still relatively under-researched area, that of the nature of teaching mathematics at university level. Such research depends on access to university teachers, often research mathematicians, who are willing to collaborate and engage with researchers in mathematics education.

Before participating in the research, the lecturer in this study had taught the linear algebra module once before at this university to a large cohort of undergraduate mathematics students and had decided, before agreeing to take part in the research, to take a new and completely different approach in his teaching of the module.

Data from the meetings and lecture observations, mainly in the form of audio-recordings, were first transcribed and then analysed using a grounded approach involving coding, categorisations, interpretation and theorisation.

Literature informing the study

The last decade has seen an increase in research into the teaching and learning of mathematics at university level. The majority of this research is into the teaching and learning of calculus, into areas such as limit of a function, and proofs and proving (for example, Hemmi, 2010; Weber, 2004). However, there has also been a steady increase in the number of studies in the area of linear algebra, a topic that is new to UK students when they arrive at university. This is similar, for example, for students entering higher education in Sweden, but unlike the situation in the US where some linear algebra topics are introduced at college (pre-university) level.

Research into the teaching and learning of linear algebra has been conducted over the past 30 years (for example, Carlson, 1993; Dorier, Robert, Robinet & Rogalski, 2000a; Hannah, Stewart & Thomas, 2013; Harel, 1989; Hillel, 2000; Pedersen, 2007; Sierpinska, 2005; Stewart, 2009; Stewart & Thomas, 2009; Thomas, 2012). Most researchers agree that students find linear algebra difficult (see for example Hillel & Sierpinska, 1994). The three most commonly quoted areas of students' difficulties are: the overwhelming number of new concepts and definitions in a first year linear algebra course, the high level of abstraction required in mastering the linear algebra concepts, and the lack of connection with students' prior knowledge and experience, particularly with mathematics at school level (see for example, Dorier, 2000). Dorier and Sierpinska (2001) provide a comprehensive review of, and reasons for students' difficulties. However, research findings and suggestions for improvement in terms of curriculum or teaching approach have varied. Early in the 1990s researchers mostly considered curricular issues and changes to the syllabus for a linear algebra course (Carlson, Johnson, Lay & Porter, 1993, for example). Dorier (2000) working in collaboration with several colleagues focussed on both content and teaching methods and consolidated results from a five year study. Dorier, Robert, Robinet and Rogalski (2000b) focussed on developing teaching methods aimed at overcoming cognitive difficulties through the use of a ”meta-lever”. The meta-lever was described as a deliberate (often oral) ”meta” intervention by the teacher that would lead students to reflect on a mathematical problem posed. In a more recent development Rasmussen, Wawro and Zandieh (2015) have been working with two different theoretical perspectives in order to research teaching alongside learning within the topic area of linear algebra.

Many publications in relation to higher education mathematics, including in linear algebra, arose from mathematicians reflecting on their own teaching and developing new teaching methods for university mathematics courses. These can be seen as professional publications rather than research publications (see Treffert-Thomas & Jaworski, 2015). Uhlig (2002, 2003) for example, advocated a matrix-based approach to teaching linear algebra that arose from his own experience of teaching students and from mathematical considerations. Uhl (1999) abandoned lectures altogether and taught mathematics (not linear algebra) in a more student-centred approach. He changed his teaching to laboratory-style tutorials based on a desire for closer interaction with his students. Love, Hodge, Grandgenett and Swift (2014) studied students' learning in a ”flipped” linear algebra classroom which incorporated technological aids such as pre-recorded screencasts that students watched prior to a lesson. The aim was to free up time in lessons for more active learning and problem solving. Nanes (2014) discussed his adaptation of team-based learning in a linear algebra classroom and the positive outcomes in terms of students' attitudes and achievement. In the Nordic context Pedersen (2007) used Brousseau's theory of didactic situations to develop teaching in relation to eigenvectors and eigenvalues, a central theme in linear algebra, through the use of modelling with technology. Others (Millet, 2001; Pritchard, 2010; Wu, 1999) have retained and advocated a traditional university lecturing style as an effective means for teaching a large student cohort in advanced mathematics. This approach is often criticised (for example, Alsina, 2001) and taken as the point of departure for developing a new approach.

Studies based on practitioners' reflections are extremely valuable in terms of informing mathematical practice. They differ from empirical studies and are often, but not always published in professional journals. Research into teaching and teaching practices, according to recognised and formal research methodologies exist but are rarer in comparison (for example, Jaworski, 2002; Nardi, Jaworski & Hegedus, 2005). They involve an analysis of empirical data collected in accordance with expressed research aims and objectives. For example, Weber (2004) analysed a mathematics professor's traditional lecturing style in the context of real analysis in the US, and Bergsten (2007) provided insights into lecturing and the lecture format in the Swedish context, in Calculus. Barnard and Morgan (1996) presented a case study of a single lecture that investigated the teacher's consciously expressed intentions and his practice while Ioannou and Nardi (2009) considered the teaching of abstract algebra in a tutorial setting. In Sweden Viirman (2014) conducted an in-depth study of a Calculus teaching practice adopting a commognitive theoretical framework (based on Sfard, 2008). Petropoulou, Potari and Zachariades (2011) conducted a similar study (based on lecture observations and interviews with the lecturer) at a Greek university, also in Calculus. The latter are, or are based on doctoral dissertations. To date, apart from the study that I report here, I have not found an in-depth study of a linear algebra teaching practice, nor one using an activity theory perspective.

Design of the study

The aim of this study was to explore university mathematics teaching from a lecturer's perspective. The research was qualitative and interpretive, and followed the principles of a naturalistic inquiry. As a researcher I took a sociocultural perspective where teaching and learning is viewed as embedded in the cultural and social context in which the teaching occurs. I was seeking insights into, and an understanding of the lecturer's teaching practice and posed the question "What does it mean to teach linear algebra at university?" This question led the research and the design of the study. I combined a grounded theory approach with an activity theory analysis to pursue this question. Taking a grounded approach to data collection and data analyses my aim was to allow issues to emerge as the research developed. Using Leontiev's activity theory framework provided the analytical structure towards a theorisation of the practice. A set of more refined research questions was formulated as a result of these analyses. These were:

1. What are the strategies used by the lecturer in his teaching of linear algebra?
2. How and why does the lecturer use these strategies? What are his intentions for student learning?
3. How can we make sense of the complexity of factors that influence and contribute to the lecturer's teaching practice?

The research findings presented in this article are the result of data collected in interviews with the lecturer and through observations of his lectures. The interviews were informal and often took the form of conversations with the lecturer who was encouraged to talk freely about any issues that he wished in relation to his teaching of the linear algebra module. The interviews were audio-recorded and took place in meetings between the lecturer and two mathematics education researchers.

In addition, all lectures and tutorials held by the lecturer in his teaching of the linear algebra module were audio-recorded. The recordings captured what the lecturer said to his students in the lecture (while teaching) and were complemented by field notes taken by the researchers sitting in the lectures among students. Both methods of data collection were aimed at capturing the context of teaching and the actual setting.

This research was an in-depth study of teaching that took account of emerging issues. It included seeking a collaborative way of engaging with the lecturer in order to discuss issues and initial findings. Thus data transcripts and records of initial analyses were made available to the lecturer. The aim was to be transparent and open in the research process and to involve the lecturer closely with the research as a whole.

Methodological and theoretical perspectives

In this research I took a sociocultural perspective rooted in the work of Lev Vygotsky (1978, 1981) to investigate the teaching of linear algebra at university. A central tenet in Vygotskian theory is mediation, an appropriate concept to apply in a teaching and learning context. After an initial analysis of data from a grounded theory (Glaser & Strauss, 1967) perspective I used Leontiev's (1981) work on activity theory as an analytical tool to deepen the analyses, make sense of the data, and answer my research questions.

In a Vygotskian perspective, all human actions are mediated by tools and signs (e.g. Vygotsky, 1978, 1981; see also Daniels, 2008; Wertsch, 1991). Material tools include physical objects such as textbooks while psychological tools relate to symbolic systems (signs, numbers, notation and language). Both are cultural tools. They are developed, preserved and passed on in any particular culture. Learning and mental development are seen as situated in the individual's social and cultural context and as developing within social interactions. For my study I viewed teaching as a mediational process involving the teacher, the learner and the mathematical knowledge of linear algebra. I investigated the lecturer's teaching approach and his didactical thinking and planning of his teaching within the university context in which they took place. The mediational means consisted of the lecturer's actions (his strategies) alongside material and psychological tools (teaching resources, course notes, modes of talking to students) that the lecturer designed and applied in his teaching of undergraduate students.

Leontiev (1978, 1981) furthered Vygotsky's work on mediation by defining human activity as central to all human mental functioning. That is, through engaging in activity, including practical and social activity, the individual acquires knowledge about the world. Leontiev structured human activity on three levels. At the top level the subject's *activity* is fundamentally related to its *motive*. "There can be no activity without a motive" (Leontiev, 1981, p. 59), the driving force for activity. At the second level *actions* are related to the *goals* that they are designed to achieve. Activity cannot exist without actions which are determined by their goal-directedness. Actions in turn consist of operations, the "how it can be done". At the third level the *operations* are related to the *conditions* (or *constraints*) that influence or limit an activity. Leontiev's three levels of activity provide a hierarchical structure that I applied in analysing and categorising my data. The elements at each level relate to each other in different ways which results in three *qualitatively different* levels of analysis.

The following diagram by Goodchild (1997, p. 28) shows Leontiev's formulation of activity (figure 1).

As an activity unfolds, there can be considerable movement between levels, e.g. an action can become an operation and vice versa. In addition, as Leontiev wrote, an action can have more than one goal or be associated with a set of goals that follow each other (Leontiev, p. 65). Similarly, more than one action may be taken in pursuit of a goal.

Leontiev's structural elements of activity provided an analytical tool that furthered an initial data analysis based on a grounded theory approach

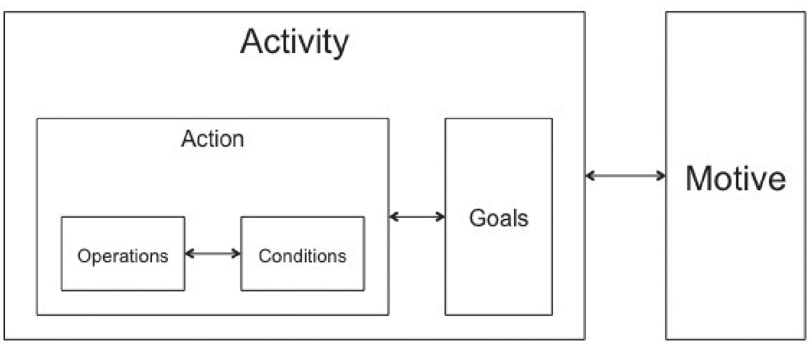

Figure 1. *The Structural Elements of Activity (Goodchild, 1997)*

(Glaser, 1992; Glaser & Strauss, 1967). The grounded theory analysis resulted in a multitude of codes and categories. Applying Leontiev's three levels crystallised and condensed these codes and categories within an activity theoretical framework and explained the teaching practice. Activity theory thus acted as an analytical tool and as theoretical lens (Simon, 2009) that oriented the researcher within a sociocultural perspective on teaching and learning.

The university, the lecturer and the linear algebra course

This study into a university mathematics teaching practice took place at a UK university with a strong tradition in engineering and design technology. The focus of the study was a compulsory linear algebra course (called a *module*) for first year undergraduate students (approximately 240) enrolled on a single or joint honours mathematics degree programme. The research presented refers to the teaching of linear algebra in Semester 1, the autumn term at the university and spanning twelve weeks. At the start of Semester 2, another lecturer took over the module for a further twelve weeks. Both semesters were designed to complement each other and both lecturers contributed towards setting the final examination which students took at the end of the academic year, in June.

At the time of the study the lecturer of the module in Semester 1, a research mathematician in his mid-thirties, was in his second year at the university and in teaching this module. In research meetings he talked enthusiastically about his subject and about his teaching.

As a result of school reforms in the UK in the 1980's linear algebra is not commonly taught at school level in the UK[1]. The topics covered in this particular module included an introduction to the core concepts of vector spaces and subspaces, linear independence and bases, and eigenvalues and eigenvectors. The lecturer in the study had taught the linear algebra module in the year preceding the study using a traditional university lecturing style. This was based on exposition of the main theory and the presentation of definitions and theorems prior to any examples and exercises for students. He had re-designed the module adopting a more *inductive* or *bottom-up* approach to his teaching after observing his students struggling with the content of linear algebra in the previous year. In this new approach he decided to present examples and ask students to work on the examples in lectures by themselves (or in collaboration with other students) before he introduced relevant definitions or theorems. The approach was aimed at engaging students with the mathematics so that they might develop mathematical ideas and concepts in relation to the example. In Semester 1 all linear algebra concepts were introduced in the vector space $\mathbb{R}^n$ and in Semester 2 the same concepts were revisited in the context of abstract vector spaces.

To inform his teaching the lecturer in the study had accessed some professional literature including the work of Frank Uhlig (2002, 2003), a research mathematician who published reflections on his own teaching of linear algebra. Uhlig (2003) advocated a matrix-based approach that included the teacher presenting examples and posing guiding questions. The lecturer had consulted the literature of his own accord and prior to taking part in the research study. He had not previously accessed mathematics education literature per se and did not contribute to the analyses and interpretations that I present in this article and that I have developed elsewhere (Thomas, 2012).

The analytical process

Data were analysed through several coding cycles within a grounded approach (Glaser, 1992; Glaser & Strauss, 1967) that generated a multitude of codes. As the research focus narrowed, the research questions were re-formulated to focus on the lecturer's aims, strategies and intentions. I began to re-code and amalgamate previously coded sentences or paragraphs into more overarching codes. As this process developed, the need arose for some structure due to the complexity of the emerging codes and categories. Thus, introducing Leontiev's (1981) activity theory framework provided a methodological and theoretical basis from which the analysis took shape and led to a theorisation of the teaching practice. A brief summary of the data collected and analysed for this article:

Meetings data: These are the audio-recordings of the interviews with the lecturer. In total there were 20 meetings lasting between half an hour to one hour. Meetings were labelled M1 to M20 which I used for referencing quotations.

Lecture observation data: These are the audio-recordings of the lecturer teaching his students in lectures and tutorials. There were 32 lectures/tutorials lasting fifty minutes each and labelled L1 to L32 for referencing purposes.

With reference to the research questions, any statements by the lecturer that expressed his aims, his intentions or his strategies in respect of his teaching were of particular interest for this research. On the following page (top) is an example of a typical coded paragraph of a research meeting with the lecturer.

48:18 I think that's one of the big steps that students need to take from A-level mathematics to degree level mathematics. There is, one thing is to be able to do a calculation that you have been shown how to do, and the other thing is to learn a set of concepts with which you can work and which you can potentially apply to problems that are not precisely what you have been shown. And actually, I had a discussion on that with my small group tutorial the other day … I took the opportunity there to explain to that small group that, ideally, at the university level they should be able to understand the ideas that go into the way the calculation works, and then be able to adapt that to whatever problems they are faced with. But of course that is not something that you can just tell the students and then they know it. That's a new way of looking at what it means to do mathematics and that will take a long, long time to sink in. (M2, 48:18)	*Focus*: learning outcomes *Focus*: computational vs. conceptual *Intent*: conceptual understanding *Intent*: conceptual understanding *Aim*: competent problem-solving *Strategy*: telling students *Aim*: change student view *Strategy*: more time

The aim was to code whole sentences or short paragraphs. I decided that more than one code could be assigned to any section. Each code relates a theme such as ”intent” (for lecturer intention), ”strategy” (for lecturer strategy), ”aim” (for a more overarching intention), ”focus” (for content discussed), to name a few. I also used ”memo-ing” to keep track of ideas and current thinking when assigning codes.

The lecturer expressed many different intentions and strategies. The codes associated with these formed the basis of further analyses applying Leontiev's activity theory framework. Overlaying the data with Leontiev's structural elements of activity-motive, actions-goals and conditions-operations, resulted in relating the theoretical concepts of actions and goals, for example, with intentions and strategies in analysis. Moving the analysis forward was accomplished by identifying the hierarchical nature of the goals. The extract below is one of several that led me to make a connection between goals.

And also you should have had a look at your problem sheet where I put down a couple of problems asking you to do something with these terms. And the purpose of these problems and the purpose of the new problems for this week is that you get some experience in working with these objects and these concepts and you get a feel for how they fit together and how they work. (L15, 05:23)	*Strategy*: telling explicitly *Intent*: hands-on experience *Intent*: get a feel *Intent*: engage conceptually

The intentions labelled above as ”hands-on experience”, ”get a feel” and ”engage conceptually” in analysis were interpreted within Leontiev's framework as the goals of engagement, intuitive understanding and conceptual understanding. Giving a name to each goal I stayed close to the origin of the expression, as captured in the data transcripts. Glaser (1992) refers to this type of naming a code as a "conceptual name". There is extensive research in mathematics education around the notions of mathematical intuition and/or procedural and conceptual understanding (e.g. Fischbein, 1987; Harel, 1997 in the context of linear algebra). I wish to add that the goals formulated in the coming sections are not directly related to this literature but a result of labelling codes by adopting the lecturer's own words and expressions. The strategy "telling explicitly" in the excerpt above was interpreted as the action of verbalising intentions, that is, the lecturer made his intention for students' learning explicit to his students in the lecture. As I will show later this strategy spanned all goals and is the only one that did.

Both statements in the meetings with the lecturer and statements that the lecturer made in lectures to his students while teaching contributed to the theorisation of the teaching. As a result I developed a multi-stage action-goal model of teaching that was very coherent in respect of the data held.

Conceptualising the teaching practice

The activity-motive level of analysis

Leontiev (1981) defined three elements, or levels of activity: activity-motive, actions-goals and operations-conditions. He wrote that activity is characterised by its motive, and that the motive may be recognised by its energising function for activity (see Leontiev, 1981, p. 60).

In this research study the lecturer stated in meetings with the researchers that he wanted to change the way that students viewed mathematics. He wanted to make students think about a mathematical topic and mathematical problem solving "the way a mature mathematician would" (cited in meeting M8, 18:18), and for students to "grow into the community of professional mathematicians" (M5, 55:20). For the lecturer this involved changing students' view of the nature of mathematics and how mathematicians engaged with mathematics. When the term "enculturation" was suggested in a meeting by one of the researchers, the lecturer seized upon it saying,

> I like that word. I probably wouldn't have, no, I *certainly* [2] wouldn't have come up with that word. But I like it because that's exactly what I'm after. I'm hoping students are going to change the way they think about mathematics as they change [from school] to university. ... And so, to that extent that's exactly what I'm hoping for. (M5, 56:18)

Thus the lecturer stated explicitly that he wanted to change students' way of thinking. He re-designed his teaching of the linear algebra module and re-structured the course materials in order to teach the concepts of linear algebra in an inductive style. This consisted of "presenting an example first, followed by a definition or theorem". In Thomas (2012) I referred to this approach as EAG where the initials stand for "Example – Argument – Generalisation", and describe the process of:

we introduce an *Example*,

we make an *Argument* on the example, and then

we *Generalise* to an observation (i.e. a definition or theorem).

In brief, a linear algebra concept such as linear independence was introduced with an example that asked the student to determine whether any of the vectors stated could be written as a linear combination of the other vectors. If this was not possible, then the set of vectors was linearly independent. Only when the solution was obtained, did the lecturer introduce the formal definition of linear independence and how to apply the definition to subsequent examples. (See Jaworski, Treffert-Thomas & Bartsch, 2009, 2011; Thomas, 2012; Treffert-Thomas, 2013a for more details of how this approach was applied.) His aim was to engage students more conceptually with the material and to encourage a different way of engaging with the mathematics through exploration. To aid this approach he produced course notes that contained blank areas following the examples which provided students with spaces to enter the solutions in the lecture.

Thus, in the meetings, the lecturer stated explicitly that he wanted to change students' way of thinking and to enculturate them into mathematical practice. This was an overarching goal and the motive in activity theoretical terms. A desire to change students' views of mathematics and mathematics learning energised the lecturer's activity, that of teaching linear algebra inductively. The lecturer's overt realisation of the term in meeting M5 (above) meant that he became conscious of his motivation and the overall aims of his teaching. This was a direct result of the lecturer's participation in the research study.

The lecturer's inductive approach to teaching linear algebra is culturally and socially rooted in the lecturer's own life and work experiences with colleagues and students.

The action-goal level of analysis

In activity theory terms, actions realise and give form to activity. They are goal-directed and determined by their goal orientation. Leontiev (1981, p. 59) wrote, "We call a process an action when it is subordinated to the idea of achieving a result, i.e., a process that is subordinated to a conscious goal". Goals were identified mainly in the analyses of statements that the lecturer made in meetings, that is from the meetings data, while actions were identified also from more contextual factors, that is from both meetings data and lecture observation data.

Goal 1. Engagement with mathematics

In research meetings the lecturer talked about wanting his students to ”engage” and ”get some hands-on experience” (M12, 13:47), to ”get a feel” (L15, 05:12), by ”playing with the concepts” (of linear algebra) (M9, 00:26). For example, in a research meeting he said,

> I'm not thinking so much about the class test, but maybe having some more hands-on experience with what linear independence is about would have made it easier to go over the lecture on rank-nullity. (M9, 04:37)

The lecturer expressed the goal of engagement with mathematics also directly to his students in the lectures. For example, in Week 6 of his teaching of the linear algebra course he said,

> And also you should have had a look at your problem sheet where I put down a couple of problems asking you to do something with these terms. And the purpose of these problems and the purpose of the new problems for this week is that *you get some experience in working with these objects* and these concepts and you get a feel for how they fit together and how they work. (L15, 05:23)

The lecturer presented examples in lectures, as well as exercises set in tutorials, or on problem sheets (where problem sheets were designed to be completed by students outside of lectures). He allocated time in lectures for students to try finding a solution, either by themselves or working with a neighbour. In a research meeting, the lecturer said that he wanted to ”encourage students and challenge students to do that” [to engage more conceptually]. He said,

> On the other hand, also that's very difficult, and ... that's why I am trying to find ways to encourage students to get engaged with the material in that way. And that's also one of the main reasons why I give them, to some extent, exploratory questions, examples to do in the lecture themselves, before I show them the first example of how things work. And as we have seen there are quite a few students who take that up, and who do it well, ... (M15, 46:42)

The lecturer stated his intentions (that is, his goals) explicitly also directly to his students in lecture (see quotation L15, 05:23 above which related to a lecture).

In taking account of the analysis I related four actions to the goal of engagement, that is mental engagement with the mathematics and physical engagement by writing solutions and discussing with peers. I referenced the four actions as presenting examples, providing course notes with gaps, making breaks in the lecture and verbalising intentions.

There were also instances where the lecturer related one goal to another, possibly higher level goal. For example, the lecturer said to his students in a lecture,

> [...] and the purpose of the new problems for this week is that you get some experience in working with these objects ... and you get a feel for how they fit together and how they work. (L15, 05:23)

With this comment the goal of engagement with mathematics ("working with these objects") indicated a necessary step towards attaining the next or higher level goal of an intuitive understanding (to "get a feel"). A second example that served to highlight this relationship between the two goals relates to a research meeting (M18, 25:18) towards the end of the study. The researcher posed the question whether conceptual understanding could not also come through repeatedly practicing procedures. The researcher asked, "If you do it lots and lots of times, won't the concept come to you?" . To which the lecturer replied,

> Perhaps. Probably not automatically, but one of the assumptions, or one of the hopes of the way I've taught this semester, is of course that it will, and that's why I put in a lot of different examples that show phenomena in a different way, and I phrased essentially the same question in different ways at different points. I think, it won't come automatically, but if it is to come it can only come from working with the objects. (M18, 25:18)

Through engaging with the mathematics the lecturer envisaged that students progressed towards a conceptual understanding of linear algebra concepts. The lecturer considered this progression as not necessarily immediate but as taking in intermediate steps in the form of informal and intuitive ways of understanding.

Goal 2. Intuitive understanding

This goal is closely linked to the previous one and was hinted at in the quotations above. In research meetings the lecturer said that he wanted students to "get a feel" (M2, 44:06) or "develop a feel" (M8, 06:56) for the concepts in linear algebra, and ”to make summaries of that sort” (M2, 18:29), meaning summaries of an informal or intuitive nature. In the context of solving linear equation systems using Gaussian elimination the lecturer said,

> But the reason why I'm asking them actually to solve things by hand is so that *they get a feel* of what can happen, and so that *they get a feel* for how the algorithm works, and what the different cases are that can happen. Ultimately we're going to diagnose the solvability, inconsistency, unique solvability or number of free parameters of a linear equation system from the echelon form. And we will see that those are all possibilities. (M2, 44:06)

In order to produce mathematical problems that students could try without direct help in the lecture, the lecturer formulated all linear algebra concepts in the context of the vector space $\mathbb{R}^n$, that is the space of all n-component column vectors, a relatively ”simple” vector space. Using vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, for example, would be sufficient to describe a problem spatially in two or three dimensions, respectively. The lecturer said,

> So that's … what I'm aiming for is to talk about these linear combinations, and linear independence, these crucial concepts, in the context of column vectors where most people feel comfortable they can calculate with them. … That's why I asked them today, and I ask them again on the problem sheet, ”Write this vector as a linear combination of the other vectors. Can it be done?”, because I'm hoping that students … will develop a feel for what it means that one vector is a linear combination of others, and that vector is not. And also, that's also why I'm putting the emphasis on, where I can, putting the emphasis on really what we're talking about … (M8, 07:34)

Reducing the complexity of the problem by focussing on a simple vector space, $\mathbb{R}^n$, the lecturer wanted to direct students' attention away from the computational and towards the more conceptual aspects of linear algebra. The vector space $\mathbb{R}^n$ affected the formation of the course notes and the language used (in the course notes and in lectures) - language that was ”simpler” than would have been the case had the course notes been formulated in the context of a more abstract vector space. The lecturer said,

> My hope is to get students acquainted with the basic concepts … in the context of column vectors rather than abstract vector spaces where hopefully they feel reasonably at home. (M1, 56:07)

The lecturer made his goal of intuitive understanding also explicit in lecture by telling his students of his goal as indicated by a quotation (L15, 05:23) in the previous section. I referenced three actions with this goal: presenting mathematical problems, formulating in the vector space $\mathbb{R}^n$, and verbalising intentions. Identifying not only the actions associated with each goal but also a dependence and hierarchy of goals in the analytic process represented a crucial step in conceptualising the teaching.

Goal 3. Acquisition of mathematical language

In research meetings the lecturer talked about students needing to "learn the language of linear algebra" (M12, 02:03; M6, 12:22; M7, 01:51), to "get fluent in that notation" (M5, 04:08), and to be "able to read a definition" (M5, 27:15). For example, in one of the meetings he said,

> Yes, of course [learning the language is] important because [the students are] supposed to start reading mathematics on their own. And our students are very slow at that, and many probably even when they graduate, couldn't take up a mathematics book and read it. But that's why I'm putting a lot of emphasis on that language. (M12, 02:03)

The lecturer was referring to the terminology of linear algebra, the formulation of concepts in terms of definitions and theorems and general mathematical notation since "that's really what you need in order to get hold of the concepts" (M8, 26:44). For example, students needed to understand the notions of subspace, spanning set and linear independence in terms of their formal definitions. That the formal language of linear algebra presents great difficulties for students is known from research literature. Dorier (1998, 2000) pointedly wrote about "the obstacle of formalism" while Dorier and Sierpinska (2001) recalled Hillel in saying that "students had the feeling of landing on a new planet and were not able to find their way in this new world" (p. 259). In a research meeting the lecturer referred to the formalism of mathematics as "where the power of mathematics came from" (M15, 44:28). He emphasised the importance of mathematical language when addressing his students in lectures. He said,

> [Subspace is] an important new idea and also an important new word that you need to learn to use. ... And you will, as we go on through this chapter, you will find a lot of new words that I need to introduce to you, ... and it's very important that you learn how to use these words properly and to speak that language because this is the language in which we will be able to formulate observations, theorems that are much more general than we observe here. (L12, 43:25)

The lecturer designed course material for students' use, that is, the mathematical content of coursework (which was assessed) and of problem sheets (which were homework tasks) that encouraged and required students to practice and use correct mathematical language in developing their solutions to the linear algebra problems posed.

Goal 4. Conceptual understanding

The term conceptual understanding (but not necessarily a unified meaning) is known to most who teach. The lecturer in my study frequently talked about encouraging students to engage more conceptually with the material. He acknowledged that students found conceptual work difficult but that it was desirable and necessary that students worked conceptually during their undergraduate studies. He said,

> That's the fundamental problem that we have to deal with here, to find that balance between ... *challenging students to get involved conceptually* with the material and, on the other hand, asking too much of them. And ... I mean, to some extent our students come in and they probably never have been asked to do conceptual work on mathematics before. And so that's not only a problem with that conceptual material being hard, it's also a problem of what does a student expect work on mathematics to be like. (M15, 53:03)

In lectures, when addressing his students, the lecturer referred to "focussing on the ideas" in linear algebra by drawing attention to underlying principles, and he stressed that "the challenge is in understanding what the language means" (L13, 20:00). He particularly articulated the link between learning the language of linear algebra in order to gain a conceptual understanding of the topics. For example, in a research meeting he said,

> And ... this is one of the reasons why I'm putting so much emphasis on the language of linear algebra here because that's really what you need in order to get hold of the concepts. And ... I'm hoping that this is going to help students get there, think about these formulae.
> (M8, 26:44)

As with all previous goals the lecturer communicated the goal of conceptual understanding directly also to his students in lectures. He devised mathematical content (coursework and problem sheets) for students to work on in order to encourage a more conceptual view of linear algebra.

Goal 5. Mathematical competence

The lecturer frequently talked about skills training that focussed on computations and algorithms as insufficient in enabling students to deal with mathematical problem solving in a variety of situations and contexts. He described mathematical competence in terms of being able to apply mathematical knowledge in situations that are wholly unfamiliar.

> I think that's one of the big steps that students need to take from A-level mathematics to degree level mathematics. There is, one thing is, to be able to do a calculation that you have been shown how to do, and the other thing is to learn a set of concepts with which you can work and which you can potentially apply to problems that are not precisely what you have been shown ... And so, I took the opportunity there to explain to that small group that, ideally, at the university level they should be able to understand the ideas that go into the way the calculation works, and then be able to adapt that to whatever problems they are faced with. (M2, 48:18)

The lecturer stated that to become mathematically competent students needed a sufficiently deep, that is, conceptual understanding of the mathematics. This was an overarching goal at this, the second level and action-goal level of analysis. The lecturer communicated this goal directly to his students in lectures, the only action associated with this goal.

Consolidation of the action-goal level analyses

I identified five goals from analysing data through coding, categorising and interpreting, and by applying Leontiev's framework. In summary, the goal of engagement with mathematics related to engagement as physical and mental participation in lecture. The goal of an intuitive understanding was supported by the lecturer's use of informal reasoning about the concepts in linear algebra, in particular in the use of examples in an inductive approach to teaching. The goal of the acquisition of mathematical language was associated with students acquiring the language of linear algebra, its body of definitions and theorems, and its notation. The goal of conceptual understanding related to the ability to make connections between concepts, rather than relying on a purely computational view for solving a mathematical problem, say. The goal of mathematical competence was associated with students becoming competent in applying rules and procedures to applications that appear unfamiliar. This applied in particular to real-life problem solving, and to students being able, in those situations, to adapt methods and procedures if the need arose.

With each goal I identified and associated a number of actions. In addition, determining how goals related to one another opened up a hierarchical structure of intermediate goals in pursuit of a more over-arching goal. Leontiev wrote, that "an activity is usually carried out by some aggregate of actions subordinated to <u>partial goals</u>, which can be distinguished from the <u>overall goal</u>" (1981, p. 61).

To bring together the analyses I developed an action-goal model of the teaching (see figure 2). The goals are arranged on the horizontal axis

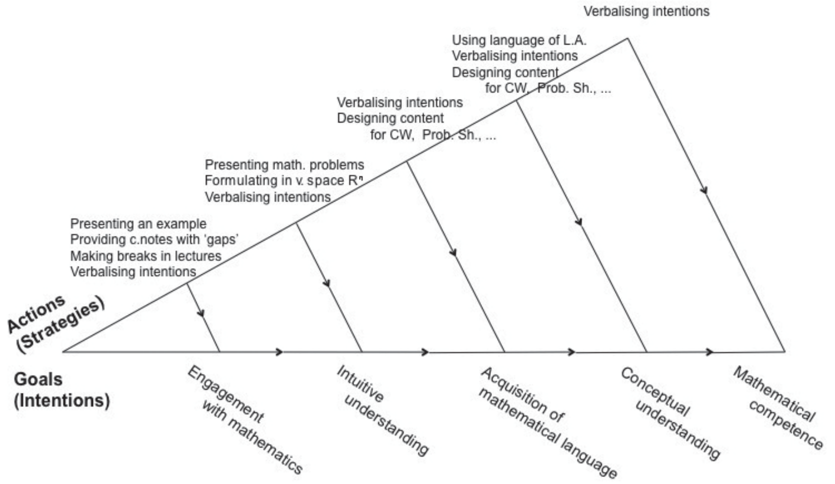

Figure 2. *Modelling the teaching process*

forming a sequential order and the actions are arranged on the diagonal line.

This model summarises the lecturer's intentions and strategies (the actions and goals) in an inductive approach to teaching. In this model an intuitive understanding came before a more formal understanding of the mathematics, and an intuitive understanding relied on students engaging with mathematics. In the approach taken by the lecturer this meant students engaging in the lecture and with the material (the examples, the mathematical problems, etc.) presented in the lecture. Thus these first two goals and their associated actions were crucial in developing students' learning and understanding so that, in turn, the higher goals could be attained. The lecturer, in talking about his didactical thinking and planning, drew particular attention to these first two goals. Linking these two goals, one depending on the other, exemplifies Leontiev's assertion, that "we must keep in mind that any kind of well-developed activity presupposes the attainment of a series of concrete goals, some of which are rigidly ordered" (1981, p. 61).

In my research I considered teaching a conscious and well–structured activity with the lecturer having concrete goals for his students' learning. Intermediate goals were acquiring language and conceptual understanding. Both contributed to the overarching goal of becoming mathematically competent.

The operation-condition level of analysis

I now consider Leontiev's third level of activity. In activity-theoretical terms operations are the processes that need to, or can be performed in order to carry out an action. Operations are dependent on and often limited by (external) conditions.

The lecturer was teaching mathematics to a large cohort of undergraduate students in the format of a university lecture. Both mathematics as a subject and university lecturing as a teaching style bring with them certain traditions, assumptions and expectations by university staff and students alike. Apart from teaching the lecturer spent some of his time engaged in research in his field and performing administrative duties for his department.

The condition for teaching was a mathematics (i.e. linear algebra) lecture. The lecturer prepared mathematical examples for lectures, problem sheets for students to complete as homework (outside of lectures), and various assessments including a final examination. He delivered the lecture in person, as it was not, for example an on-line lecture. Lecturing involved addressing his students from the front of the lecture theatre and presenting mathematics, both orally and in writing. Talking to his students frequently took the form of monologues (or unidirectional talk). Writing, for the most part, involved writing solutions to examples on the overhead projector since course materials were pre-printed for students as lecture hand-outs.

In summary, I identified four operations in data analyses: maintaining a physical presence, talking, writing and providing mathematical content. This completes the structural analysis of the data.

Concluding remarks

This research was a systematic, in-depth study of university mathematics teaching. It was part of a wider study that included research into the student viewpoint and a theoretical analysis of three linear algebra concepts (see Thomas, 2012; Treffert-Thomas, 2013b). Working closely with a mathematician who designed and taught a linear algebra course I documented and conceptualised a university teaching practice in an activity theory framework. I used Leontiev's (1981) structural elements of activity, that are activity-motive, actions-goals and operations-conditions alongside a traditional grounded theory approach in analysis and developed a multi-stage model of the teaching process. The model explains university teaching (in the context of linear algebra) in terms of the lecturer's intentions and strategies. The model arose from analysing data in relation to the lecturer's views on teaching, his didactical thinking and planning and his face-to-face teaching of students.

Although the goals in the model were identified within linear algebra teaching they have a universal character and generalise to other areas of mathematics. For example, developing conceptual understandings and becoming mathematically competent are not linear algebra specific goals. Some of the strategies that the lecturer devised in the context of linear algebra are expected to generalise while others may need to be adapted.

The model arose from a methodological choice of combining grounded theory and activity theory in analysis. This offers a research methodology for investigating teaching practices at all levels of education and will be of interest to researchers in these fields. In my study research methodology and theory became interlinked. Hence activity theory, as one of the socio-cultural theories of learning, was an analytical lens in research (Simon, 2009) as well as providing a theoretical framework.

The study as a whole provided insights into the lecturer's rationale for developing an inductive approach to his teaching. As a result of this research I learnt of one mathematician's motivations and beliefs, in particular his desire for changing students' view of the nature of mathematics by changing how students *learnt* mathematics. The creative process of mathematics has been documented in the literature by mathematicians (for example, Hadamard, 1945, 1973) as well as by researchers in mathematics education (Burton, 2004; Nardi, 2008). Holton (2005) described mathematicians' ways of working as finding a problem, "followed by an intense period of experimentation where lots of examples are found" (p. 305), in order "to try and get some feeling" (p. 305) as to whether a conjecture may be true or not. The lecturer's design resonates with this description.

No attempt is made to present the lecturer's teaching design as a panacea for overcoming students' difficulties with linear algebra which are well known (see Dorier & Sierpinska, 2001, for a review). However, it presents an alternative way of thinking about designing an undergraduate mathematics course that can provide a different learning experience for students.

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Notes

1 In the UK there are school leavers who may have met certain linear algebra topics during their advanced studies at school. These students took an additional advanced mathematics A-level and may have met topics such as

matrices and transformations. However, the way that the topic is offered at university with an emphasis on definitions and proofs and proving extends greatly what would have been expected of pupils at school. In this sense linear algebra is new to most UK students when they enter university.

2 As part of qualitative data analysis words in italics indicate an emphasis placed on these words by the researcher during interpretation. Emphasised words were often the result of a perceived change in the lecturer's intonation.

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