A Symbolic Dance: the Interplay between Movement, Notation and Mathematics on a Journey towards Solving Equations

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This paper analyses the use of the software Grid Algebra\(^1\) with a mixed ability class of twenty-one 9-10 year old students who worked with complex formal notation involving all four arithmetic operations. Unlike many other models to support learning, Grid Algebra has formal notation ever present and allows students to ‘look through’ that notation and interpret it either in terms of physical journeys on a grid, or in terms of mathematical operations. A dynamic fluidity was found between the formal notation, imagery of movements on a grid, and the process of mathematical operations. This fluidity is interpreted as a ‘dance’ between these three. The significant way in which this dynamic took place reflects the scaffolding and fading offered by the software which was crucial to the students’ fluency with formal notation well beyond what has been reported from students of that age.

This paper analyses a number of episodes from a teaching sequence of three lessons where a class of mixed ability students, of age 9-10 years old, learnt to interpret and use formal notation in tasks such as substituting and solving linear equations. The number of lessons was a constraint from the school but the focus was on what students might be able to achieve within this given timeframe. As such this paper cannot say anything about the students’ longer term understanding of algebra but indicates what is possible for students of that age to achieve within just three lessons. The confidence with which students of that age worked with complex formal notation was noticeable, especially given the limited timeframe. By formal notation I mean the conventional way in which a series of arithmetic operations is expressed. In particular this includes:

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\(^1\) Grid Algebra is available from the Association of Teachers of Mathematics: 
http://www.atm.org.uk/shop/products/sof071.html
• Writing division using a line with the divisor below the dividend, such as $\frac{16}{2}$ rather than $16 \div 2$;

• Writing an addition or subtraction sign at the same vertical level as the division line when it follows immediately after that division. For example $\frac{16}{2} + 3$ rather than $\frac{16 + 3}{2}$;

• Writing multiplication as $2(3+5)$ or $2x$, without the use of a multiplication sign.

The students had never met these conventions before and they had also rarely met expressions involving more than one or two operations. Also, the students had not used letters before in a mathematical context where they represented an unknown or variable.

Studies over the last ten years have shown that young students can work meaningfully with addition and subtraction situations involving letters (Blanton & Kaput, 2004; Brizuela & Schliemann, 2004; Carraher, Schliemann, & Brizuela, 2001; Moss & Beatty, 2006; Schliemann et al., 2003; Schmittau, 2005) but this paper shows that students of this age can also work in a meaningful way with complex notation involving all four arithmetic operations and indeed solve equations of complexity well beyond what is expected of students of that age. A key factor in this was how the innovative software approach allowed students to interpret formal notation both as a journey made from physical movements around a grid as well as a series of mathematical operations. The focus of this paper is on the interchange between interpreting formal notation in terms of a physical journey and interpreting it as a series of mathematical operations, whilst students were learning the conventions of that formal notation, the order of operations within a formal expression, and working towards solving linear equations.
Making journeys is something which all students already had prior experience in their daily lives and this helped make many of the tasks seem intuitive and supported their learning of working with formal notation. Other models which offer an image to support work with formal notation and solving equations, such as the use of a balance or cups and tiles (Caglayan & Olive, 2010), have the issue that formal notation is not present within the model itself and at some point the notation has to be introduced and sit alongside the model; then, at a later point the model needs to fade in order that students are left working independently with formal notation. The use of spreadsheets or specialised calculators for algebraic work also suffer from the issue that the notation used for the spreadsheet or calculator is different from the formal notation used elsewhere. This paper shows that the software, Grid Algebra, is uniquely different in that the formal notation is ever present from the outset and there is a fluidity of when students interpret that notation as journeys on a grid or as mathematical operations. This fluidity allows individual students to come in and out of different interpretations at different times whilst still being able to complete the same tasks alongside each other. It also allows a teacher to shift from stressing one interpretation to stressing another as and when appropriate without leaving the formal notation.

Elsewhere (Hewitt, 2012), I describe the ways in which these students became confident with interpreting and using formal notation and how a number of well documented difficulties students often experience in algebra (some of which are discussed below) can be avoided. This paper addresses the question of how the dynamic changed between interpreting notation as movements on a grid or as mathematical operations whilst students were learning that notation and using it for substitution tasks and solving linear equations. It shows that the fluidity in the way the notation was interpreted allowed students of varying abilities to engage with similar tasks and for them all to develop confidence working with complex formal notation which is usually considered to be well beyond their age.
BACKGROUND

During the 1980s and 1990s there were a number of studies which identified difficulties students had with learning algebra (for example Herscovics, 1989; Herscovics & Linchevski, 1994; MacGregor & Stacey, 1997). Many of these difficulties relate to understanding and working with formal notation. MacGregor and Stacey (1996, 1997) found that students do not find it easy to express relatively simple mathematical operations and relationships in formal notation. Students often call upon their understandings of the context within which the letter was introduced (Drouhard & Teppo, 2004) and sometimes these contexts, such as $a$ standing for *apples*, can provide difficulties for later work within algebra. A classic example is the well known ‘students and professors’ problem (Clement, Lochhead, & Monk, 1981) where someone is asked to write down in symbols the relationship of there being six times as many students as professors in a university. Here, it is not uncommon for someone to show that they see letters as labels rather than variables by writing $6s = p$.

Radford (2010) points out that the power of algebra involves the detachment of signs from the specific things they denote and to gain meaning from their relational connection with other signs. As Mercer (2000, p.67) says of words, they “gather meanings from ‘the company they keep’” and the same can be said for much mathematical notation as well. This level of abstraction is the power of algebra but at the same time it provides the cause of many difficulties students experience. Mason (1980) points out that a possible rush to the symbolic can cause future trouble unless there are images, for example, to fall back upon which capture a relationship. Mason also suggests that whether a symbol might be viewed as symbolic (as opposed to enactive or iconic) in Bruner’s (1966) sense, is highly relative. Wilensky (1991) suggests that the level of abstraction is not so much about the notation but about someone’s relationship with that object. As such formal notation does not *have* to feel abstract for students as it will depend upon their relationship with it and that will initially be related to the
context within which it is introduced. So, despite there being evidence that many students have difficulty with formal notation (as cited above) this does not imply that this has to be the case. This paper will show a way in which formal notation was introduced to young students who became confident with quite complex expressions which are never usually met by students of their age.

The ability to see expressions as both an object and a process is important in algebra. If students are always used to seeing an expression such as $2 + 4$ as an invitation to carry out a process and replace it with a single number $6$, then this causes difficulty when meeting an expression such as $x + 4$, where the calculation cannot be carried out. This can lead to students concatenating an expression down to as close to a single symbol as possible in an attempt to carry out the process (Booth, 1984; Lee & Messner, 2000). Students need to be able to accept $x + 4$ as a single object as well as a potential process which could be carried out if told the value of $x$. This desired flexibility Gray and Tall (1994) have called proceptual thinking.

Since 2000 there have been several studies showing that quite young students are capable of working with algebraic ideas and mathematical symbols (Blanton & Kaput, 2004; Carraher, et al., 2001; Schliemann, et al., 2003; Schmittau, 2005). These have been important in pointing the way to viewing algebra as something which is accessible to young students but none of these involve particularly complex notation. They invariably involve a letter and a few additions or subtractions. This study advances that work. I show primary aged students working with relatively complex formal notation, sometimes involving up to 11 operations, including combinations of all four arithmetic operations. The students developed appropriate conventions for order of operations as necessary to consolidate and operate on expressions-as-objects.
Radford (2009) argues the case that mathematical cognition is mediated by gestures and actions as well as conventional mathematical symbols. Indeed Cook et al. (2008) showed that learning was retained in a follow-up test significantly more by students who had been told to use gestures as well as speech when carrying out mathematical tasks, compared with those who were asked to speak but not use gestures. A key feature of this study is that an original approach to teaching algebra is used, with the software Grid Algebra, where carrying out physical movements is an integral part of how students became confident working with reasonably complex notation. These movements naturally led students to using gesturing within their discussions.

THE STUDY

The study took place within a primary school in the West Midlands, UK, which had lower results than the national average in the national Key Stage 1 and 2 tests (taken in Years 2 and 6 respectively) and a higher than average number of students with free school meals. A mixed ability class of 21 Year 5, 9-10 year old, students were taught three lessons over a three day period (one hour for the first lesson and one and a half hours for the remaining two lessons). The students were selected on the basis that all were included who wished to be part of the study and whose parents returned permission slips. Prior to the study the students had not met formal notation nor had they been introduced to letters in a mathematics context.

The lessons were taught by myself using the software Grid Algebra with most of the time spent working as a whole class using the software on an interactive whiteboard (during which time there were discussions in pairs about a task before working on this as a whole group). There were some pen and paper activities based upon whole class work and there were two half hour sessions in a computer room where individual and pair work took place on computer generated tasks from the software. Both these periods of time are included
within the time stated above for the lessons overall. This paper will only focus on a selection of the activities but below is an overview of what took place over the three lessons:

- Getting used to the structure of the grid
- Introducing notation
- Learning order of operations
- Introducing letters
- Substitution
- Multiplying out brackets and equivalence of expressions
- Inverse operations
- Solving linear equations

The teaching style involved mainly questioning and setting challenges (such as some mentioned above). The mathematics involved in the lesson was not explained to the students nor was any explanation offered regarding the visual appearance of the formal notation, such as use of a division line, brackets, etc. It should be noted that as well as teaching the students I was also the author of the software and as such this paper should be read with that in mind. It should also be noted that there was a novelty factor to this study. The students came from a mix of two classes and we were in a different room to either of their normal classrooms. The arrangement of chairs was also different from the usual; the chairs were positioned in rows facing an interactive whiteboard and no tables were used except for when students worked on related pencil and paper tasks. They were also being taught by someone different from their usual teacher, although I had been in the school on a number of occasions prior to this study and I was known to some of the students.
GRID ALGEBRA

Grid Algebra is based upon a grid of numbers with the first row being the one times table, the second row being the two times table, etc. Figure 1 shows the first two rows of the grid but the grid can be extended to have more rows. The grid can also start with higher numbers or indeed negative numbers. The software has a lot of functionality but I restrict the description to features which are relevant to this paper.

\[\begin{array}{cccccc}
  & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 2 & 4 & 6 & 8 & 10 \\
\end{array}\]

FIGURE 1  The first two rows of the grid.

A key feature of the software is that movements can be made between numbers with notation appearing which expresses the arithmetic relationship between those numbers. Addition is a movement to the right, subtraction to the left, multiplication down and division up. So the number 2 in row one can be dragged one cell to the right to produce $2 + 1$ or one cell to the left to produce $2 - 1$ (see Figure 2). The latter expression, for example, can be viewed as an object in its own right and can also be dragged. Figure 2 shows the expression $2 - 1$ being dragged from row one down to row three which involves multiplying by three. Also, in Figure 2, the number 4 in row two is dragged three cells to the right. Since this is in the two times table, moving three cells to the right will involve adding six. This expression has then been dragged up to row one (dividing by two) and then dragged one cell to the left to result in the expression $\frac{4 + 6}{2} - 1$. The software provides the formal notation for the operations/movements which have taken place.
In this way expressions written in formal notation can be built up through a series of movements on the grid. Note that not all cells have to have numbers in them. A single number can be placed in one cell of an otherwise empty grid and taken on a journey through a series of movements. Expressions within a cell can also be electronically ‘rubbed out’ using a rubber button on the toolbar. Letters can also be placed into a cell and taken on a journey to produce an expression involving that letter. If a letter is placed in an otherwise empty grid, there is choice about which number could be placed into that letter cell. Thus $n$ is a variable. If a number is placed in a cell then the grid becomes defined and $n$ takes on the status of an unknown.

Note that the arrows appearing in Figure 2 are for the purpose of clarity. They do not appear on the software, although later examples are offered where the route is marked by the software.

Sfard and Linchevski (1994, p. 211) have commented that an “equation requires suspension of actual calculations for the sake of static description of relationships between quantities” and although work with Grid Algebra in this study started with the carrying out of calculations, it soon shifted focus onto the operations between numbers in the grid. When this
happened, calculations were rarely carried out as the objects under consideration were the formal notational expressions of those relationships.

Grid Algebra does not attempt to make connections across different representations, such as Function Supposer (Schwartz & Yerushalmy, 1992), but instead makes the formal notation (whether involving letters or not) the focus of all the activity. Other common software used in UK schools for algebra, such as spreadsheets or Autograph (Hatsell, Butler, & Couzens, 2004), have notational forms which are constrained by the fact that expressions or functions are entered and displayed within a line, using only the slash symbol (/) for division. This means that the notation is different to the traditional pencil and paper notation and so there is still a job to be done for students to become familiar with this formal notation (Bills, Ainley, & Wilson, 2006). Grid Algebra provides traditional formal notation as a consequence of movements made on the grid and this notation is continually being interpreted and used by students whilst they are carrying out activities using the software. So, one educational focus of the software is to help students become familiar with formal notation. Grid Algebra also offers the image of a series of movements to see how an expression is made up of individual arithmetic operations. There is a one-to-one correspondence between movements made on the grid and mathematical operations appearing in the expression. The expression is thus both a process (series of movements) and an object (a label for a cell and an object which can be picked up and dragged to other cells). Students already have understandings about journeys through such activities as travelling to school or indeed movements they have made with their hands as part of daily life. For example, after moving the salt on a dinner table and being asked to return the salt afterwards, the same distance is moved but in the opposite direction. Likewise when reversing the journey to school, the route is covered in reverse order and in the opposite direction. These experiences will support the potential awareness in Grid Algebra of inverse mathematical
operations as they are associated with making inverse journeys and have similar attributes (change order of operations, with each operation being inversed, etc.). This, in turn, can assist the solving of equations. So there is a very different emphasis on what aspects of algebra might benefit from the use of different technological tools (Grid Algebra, spreadsheets, Autograph, Function Supposer, etc.).

THEORETICAL FRAMEWORKS

There are two frameworks which are relevant to this paper. The first involves the notion of blended space (Fauconnier & Turner, 2002) which I will introduce first. This looks at the dynamics between formal notation, movements on the grid and mathematical operations. The second concerns the well known idea of scaffolding (Wood, Bruner, & Ross, 1976) and will focus on the support Grid Algebra offers and the extent to which that support is prominent or has faded at different points during the three lessons.

I begin introducing the first framework by discussing Kaput et al. (2008) who used the metaphor of a windshield which offered an image for how one symbol system might be viewed in relation to another where both have a shared referential meaning.

For example, Caglayan and Olive (2010) discussed the way in which cups and tiles were used with students to help them solve equations. Each cup held the same unknown number of tiles inside. They related this to the windshield metaphor by considering that the conventional notation system is in the windshield plane D with the cups and tiles system in A (see Figure 3). This metaphor gives a sense of the way in which someone might look through one symbol system whilst considering the other. In their case the image of the cups was offered first and then the traditional notation was introduced whilst still talking about cups and tiles. In the case of Grid Algebra, the situation is a little different. The formal notation is provided by the software and is met at the same time as movements take place. When they look at the notation, they can view it either in terms of mathematical operations or the physical movements. For example, the expression $3(2-1)$ in Figure 2 can be viewed in terms of mathematical operations (subtract one and then multiply by three) or physical movements (move one cell to the left and then down to row three). Thus I offer a modification of Kaput et al.’s image in Figure 4.

FIGURE 4 My adaptation for Grid Algebra of the windshield metaphor.
Here, the students look through the formal notation and might interpret it as either movements on the grid or as mathematical operations. Fauconnier and Turner (2002) offer the notion of a blended space where two or more aspects are blended together to make something different which helps develop a new emergent structure. Edwards (2009) offers the example of a number line which brings knowledge of numbers and the imagery of a geometric line. These come together to form a number line which has properties not found in either of the others individually. In this study, the blended space involves two ideas. The first is the usual interpretation of formal notation in terms of mathematical operations. The second comes from the fact that formal notation is produced through movements around the grid. Thus, the world of movements merges with the world of mathematical operations, both being linked through the formal notation which appears in the grid (see Figure 4). Students already have some knowledge of mathematical operations, albeit not expressed in formal notation. They also have some implicit knowledge relating to movements through their life-experiences of doing such things as walking from one place to another and moving their arms as they pick up and move objects. So the formal notation which appears on the screen is more than just written notation on paper. Students can metaphorically look through the windshield of the notation space and have meanings relating to its position within the grid and the way in which it has been built up through movements. Hence interaction with this software is more than just meeting notation on paper and interpreting it in terms of mathematical operations. It is also different from the world of movements alone since the same movement in different rows will produce a difference in the notation shown (e.g. a movement two cells to the left in row two results in subtracting four, whereas in row one it would result in only subtracting two). So the blended space which results within Grid Algebra is a world which is different from both just describing mathematical operations and also just describing movements. Yet it combines the two by using movement to embody mathematical operations. By this I mean
that students were found to be making physical movements whilst talking about the mathematical operations, so part of the way in which they discussed mathematical operations was enacted through body movements. This sense of embodiment comes from the notion that human cognition is partly “bodily-grounded” (Núñez, Edwards, & Matos, 1999, p. 46).

The second framework concerns scaffolding. Yelland and Masters (2007) re-looked at the notion of scaffolding in the information age and considered three types of scaffolding: cognitive, technical and affective. With regard to what they called ‘technical scaffolding’, they considered that the design of a technological environment can offer “inbuilt constructs to facilitate understandings and problem solution” (p. 367). Sutherland and Balacheff (1999, p. 11) said that “in the case of algebra the support which pupils need is inextricably linked to the semiotic system of language”. Grid Algebra looks after the formal language of mathematical notation, in the sense that it provides the appropriate notation as a consequence of movements made on the grid. As seen later in this paper, the software also has an ‘expression calculator’ where a student can enter in a series of mathematical operations carried out on a number or letter, and the calculator will provide the appropriate formal expression for those operations. When introducing the notion of scaffolding, Wood et al. (1976, p. 90) talked about an “adult ‘controlling’ those elements of the task that are initially beyond the learner's capacity, thus permitting him [sic] to concentrate upon and complete only those elements that are within his range of competence”. In this case, it is not only an adult which is doing this controlling but also it is the software which is taking care of the way in which quite complex formal notation is written, whilst at the same time requiring students to engage in a meaningful way with that notation.

Several authors have talked about another key element with any scaffolding offered, and that is the temporary nature of that support (Collins, Brown, & Newman, 1989; Holton & Clarke, 2006; Mason, 2000; Yelland & Masters, 2007). It is important that the scaffolding is
withdrawn as the students become more competent. The issue for a teacher is when to withdraw that support. This is a difficult decision for a teacher: too early and students can lose whatever confidence they were developing; too late and the students become reliant upon the support and fail to develop their own confidence to act independently. So, the framework of blended space will form the basis of looking at the dynamic between notation, movement and mathematical operations. Since it is movement which is the basis of the visual support offered by the software, the way in which movement features in that dynamic will be an indicator of how much the scaffolding is prominent or how much it has faded.

**METHODOLOGY**

The three lessons involved three forms of activity: whole group work with the interactive whiteboard; pencil and paper tasks; and individual or paired work on computers in a computer room. The work with the whole class was video recorded using one camera facing the interactive whiteboard and a second camera pointing towards the students in order to capture their reactions and gestures along with helping to identify students as they contributed to the discussions. This second camera was not available, though, for the third lesson. Pencil and paper tasks, along with the computer room activities, were recorded using a roaming video camera. As such only individual students or pairs of students were captured at any one time. The decision as to whom to video record was made in the moment by the camera operator in consultation with the two researchers presents (a colleague, Kirsty Wilson, and myself). These decisions were made to capture a mix of students who were being successful and ones who appeared to be gaining less success. In addition to this, work on two computers was captured using Camtasia software which recorded internally on the computer the detail of what happened on the screen along with the conversations which took place at the time. Students’ written work was also collected in. This included worksheets along with personal student notebooks which were used occasionally during the whole class activities.
The video data were analysed using Studiocode software which allows coding to be carried out easily. The coding was initially based upon attention to when notation was being introduced and the language used during activities, particularly in relation to notation. An element of Grounded theory (Strauss & Corbin, 1990) was used where I began with certain sensitivities and these developed and extended over continual viewings of the video leading to new links and codings. This included an increased sensitivity to the visual aspects surrounding the use of the software and noting when the visual support of the software was a focus and when mathematical operations was a focus. It also included use of gestures by the students as they used their bodies to engage physically with the interpretation of expressions as movements on the grid.

Written work was analysed to gain a general sense of the level of success students had but was also used to analyse the process of gradual adoption of formal notation. This latter aspect was done by counting the overall numbers of students who wrote expressions in various forms of notation (whether ‘correct’ or ‘incorrect’) and also looking in detail at the use (both ‘correct’ and ‘incorrect’) of forms of notation used within each expression (such as whether the ‘÷’ sign was used, the positioning and length of a division line, whether a multiplication sign was used, etc.).

ANALYSIS OF LESSON EPISODES

In this paper I analyse six particular episodes from the three lessons and these are indicated within the overview of the lessons below. Note that Episode 3 contains an example of how attention began to shift from movements to mathematical notation (indicated as Episode 3a) and an example from later on in the lesson of that shift having taken place (Episode 3b):
• Getting used to the structure of the grid
• Introducing notation [Episode 1]
• Learning order of operations [Episode 2] [Episode 3a]
• Introducing letters [Episode 4]
• Substitution [Episode 3b]
• Multiplying out brackets and equivalence of expressions
• Inverse operations [Episode 5]
• Solving linear equations [Episode 6]

The dynamic between movement, notation and mathematics changed during the course of the lessons. Each lesson episode discussed below has been chosen to exemplify a particular dynamic between these three aspects and to explore how their relationship at the time influenced the learning situation. I start with students meeting new notation for the first time and end up with an episode related to a time when they were learning to solve linear equations.

**Episode 1: mathematics supporting learning of notation**

It is common in UK schools for division to be written using the ‘÷’ symbol for much of students’ early schooling and often well into secondary school. The Year 5 students in this study had only met the division line within the context of fractions, such as “three quarters” being written as \( \frac{3}{4} \). However, the division line is used as part of formal notation and so at some stage a shift is needed for students to read \( \frac{3}{4} \) as “three divided by four” as well as the already familiar “three quarters”. In the first of these three Year 5 lessons, division was introduced as the inverse of multiplication. Using Grid Algebra, the 5 in row one had just been moved down to the second row to show ‘5 x 2’ and I asked what the opposite of that
would be. The class replied that it would be dividing by two and that if the 8 from row 2 was moved back to row one it would say eight divided by two. This meant that the students knew that the movement would be division and so there was a sense that whatever notation appeared due to this particular movement of the 8, it would represent eight divided by two. When the movement was made and $\frac{8}{2}$ appeared (see Figure 5. Note the arrows indicate movements made but they were not visible within the software) this was a surprise for many students as they commented that $\frac{8}{2}$ was a fraction.

![Figure 5](image)

**FIGURE 5** The first viewing of how division was written.

The fact that there was a reaction from some students indicated that they did have the process of ‘eight divided by two’ in their minds when they saw the notation and this conflicted with the meaning they already had for this notation. Repeated examples of such movements resulted in students becoming used to this notation and they began adopting this in their notebooks. For example, in Figure 6, Pushpinder started off writing $\frac{6}{3}$ for when the number 6 in row two was dragged up to row one. This should have been $\frac{6}{2}$ but he put the 3 instead of the 2 as he may have been thinking this new division notation was expressing the resultant number, 3, rather than the division. The grid here might reinforce this initially since he could see that the 6 was being dragged up to the cell with 3 in it (see Figure 1). He
correctly wrote \( \frac{8}{2} \) for the next movement but in both these first two he also included the ‘÷’ notation as well as if reluctant to let go of what was familiar.

In a later example, the expression \( 8-4 \) was dragged up. Here it can be seen by looking carefully that he initially wrote \( \frac{8-4}{2} \) before extending the division line after seeing the resultant expression on the interactive whiteboard. He learnt from this and the next two divisions were written correctly, along with the multiplication which was carried out. So, this shows a development in adoption of formal notation as he gradually learnt from the examples carried out.

![Image](image.png)

FIGURE 6 An example of gradual adoption of conventional notation.

The pedagogic technique, of establishing the mathematical operation before the notation was revealed, continued with such expressions as \( \frac{8-4+6}{2} \) and \( \frac{8-4+6}{2}-1 \) so that
they had to figure out the mathematical operations first and could then place that awareness into the new notation when it appeared on the screen. A similar situation occurred when the notation for multiplying expressions, such as $9 + 2$, was introduced. It was established first that multiplication by two was going to take place before they saw the expression $2(9 + 2)$ appear. At no time did a student ask why the notation for multiplication was written how it was. It just seemed to be accepted by students as how the software wrote multiplication.

![Diagram](image)

**FIGURE 7** Mathematical operations informed the learning of notation.

The notation of the division line and brackets were known at some level for the students already; the division line within the context of a fraction and brackets within the context of language. For each, however, a new meaning for these symbols was required (‘divided by’ for the line which appears within a fraction and ‘multiplied by’ for brackets). This was done by establishing the mathematical meaning first (division or multiplication), ahead of the symbols appearing. This enabled mathematical meaning to be placed immediately into the notation when it appeared rather than the notation appearing and there being a period of time with students not ‘understanding’ what they saw. At this stage the
dynamic between movement, notation and mathematical operations (written as ‘mathematics’ for simplicity) might be represented as in Figure 7.

The early movements not only established interpretations of the new formal notation but also introduced movements as a way of creating that notation. This is particularly important as this effectively created the blended space. Thus the notation began to be known both as a way of expressing mathematical operations and also expressing movements carried out within the grid. This introduced the two spaces (mathematical operations and movements on the grid) which now had the potential to become blended together as they both shared the common notation which appeared in the cells.

Elsewhere (Hewitt, 2012), I have indicated that students gained increasing confidence with notation, particularly when they went to the computer room to work on computer generated tasks from the software, and there are examples of this below. The computer generated tasks involved them reading a given formal expression and trying to re-create that same expression through journeys on the grid. More is said about this in the next section.

**Episode 2: notation as something visual**

Formal notation had initially been met by students, as stated above, in the context of expressing relationships between numbers in the grid. Relatively quickly notation was used as part of a challenge to re-create expressions which had been made from movements around the grid. For example, in Figure 8 the number 15 had been moved around an otherwise empty grid as indicated by the arrows (these did not appear within the software) to produce the expression \( \frac{2(15+2)-6}{2} + 2 \). There would normally be intermediate expressions appearing on the grid after each individual move but these were electronically rubbed out to leave only the initial number 15 and the final expression. The task for the class was to try to re-create
this expression. This would require them to work out what the movements would be from their interpretation of the notation.

FIGURE 8 A journey made with the number 15.

This was worked on successfully by the class as a whole with one student making the movements on the interactive whiteboard (IWB) with the rest of the class calling out directions. The main difficulty did not concern deciding which movements to make. It was the physical control over the pressure of the pen on the IWB which led to the student not making a horizontal movement along two cells in one movement. By reducing the pen pressure halfway through such a move this resulted in it being recorded as two separate movements of one cell (15+1+1), rather than one movement of two cells (15+2). Although the rest of the students were watching one student make these movements, nearly all the students became very animated and involved in the process with many of them standing up and making gestures with their hands as to which direction the expression should be moved next. Following this example, pairs of students worked on similar computer generated tasks in a computer room where they were all physically making movements with the mouse in order to re-create the given expressions. Whilst working with the whole class I often asked the question “Can you see this (meaning the expression they had generated so far) in here (pointing to the final target expression)?” This was a useful way for students to check how they were doing so far as it was purely a visual matter. Could they see, within the final
expression, the arrangement of symbols they had created through their movements so far? They did not need to have understanding in terms of mathematical operations to determine this. They did not need to see brackets as indicating multiplication, only something which usually appeared when they moved downwards in the grid. Likewise the division line could represent moving upwards for them rather than division as such.

During these “re-create a journey” tasks there were occasional incorrect moves made on the grid before finding the correct series of movements. It is these incorrect movements which I will now focus on. One pair, Paulette and Sofia were trying to create the expression \(2(7+2)+2\) given the number 7 which appeared in one of the cells (see Figure 9 where the target expression is given at the bottom left of the screen). They successfully moved the initial number 7 two cells to the right to produce \(7+2\) but then thought that adding was next rather than multiplying. This resulted in them moving to the right to produce \(7+2+1\) (see Figure 9) and then dragging it down to get \(2(7+2+1)\). They realised this was not correct and so went back to \(7+2\) and dragged it two cells to the right to get \(7+2+2\) still thinking that addition was the next operation to be done. They dragged this down to get \(2(7+2+2)\). They looked at this and decided that this was not right either and realised they needed to drag \(7+2\) down first before moving to the right, eventually getting the target expression of \(2(7+2)+2\). The visual comparison between the notation produced as a consequence of movements, and the notation of the final expression, was enough to guide them to working out the correct order of movements. Up to this particular question, Paulette and Sofia had only got three out of seven questions correct within the time allocated for each question. Following this question they appeared far more confident and got seven out of the remaining nine questions correct within the time allocated, just missing the allocated time for one other. This improvement included the fact that all but two of these later questions
involved expressions with four operations rather than the three which appeared in
\[ 2(7 + 2) + 2. \]

FIGURE 9 Trying to create the expression \( 2(7 + 2) + 2 \).

Chris was working on re-creating \( 4 \left( \frac{12}{2} - 1 \right) + 12 \) and made incorrect movements and
adjusted them when he saw the notation was different from that in the final expression. He
also spoke as he was carrying out the tasks and occasionally said the incorrect operation as he
moved. For example, when he moved 12 to produce \( \frac{12}{2} \) he said *take away two* and not
*divided by two*. This highlighted for me the point that it was not *necessary* to have a correct
mathematical meaning for these visual symbols in order to carry out the task successfully. I
am not suggesting that these particular students did not have some mathematical meaning for
the symbols, especially since this was stressed before the division line and brackets first
appeared, but that if someone had missed the first part of the lesson, for example, they *could*
still develop meaning for the symbols in terms of movements alone.

I also note here that the above task for Paulette and Sofia had the start number already
present on the screen and through making movements from this start number and seeing the
expressions obtained, students seemed to develop an unstated (by any of the students or myself) rule that, given an expression such as $\frac{3(4+6)-12}{2}$, the start number would always be the left-most one within the inner parentheses (so ‘4’ in this case). Although never stated, I noted the rule in action as students would always started with the number or letter in this position when carrying out tasks. This is another aspect of the visual nature of how expressions were viewed.

In relation to solving an equation, Radford and Puig (2007, p. 156) say that “a definite shift of attention occurs: attention moves from the verbal meaning of signs to the shapes of the expressions that they constitute. This shift leads one to see the equation as an iconic, spatial object” [their emphasis]. They were thinking in terms of an equation as a diagram which visually indicated the relations between its parts. With Grid Algebra, visual positioning of parts of an expression comes from the particular route taken on the grid. So the relations between the parts of an expression can be thought of in terms of movements rather than operations. For example, if a student is trying to create $2(3+4)-2$ but ends up with $2(3+4-2)$, is it sufficient for them to note a visual difference in order for them to know they were not ‘correct’. The meaning here being partly gained from the relational connection between the signs (Radford, 2010) since in both these expressions there are exactly the same signs involved. Both the strongest and weakest students in the class were able to engage with these tasks since the blended space allowed some students to stress mathematical operations whilst other students focused on movements. This difference came out within the language used whilst carrying out the re-creating tasks. Some students used the language of mathematical operations (add, subtract, times, divide, etc) whilst others used the language of movements (up, down and across, etc).
This was the time when the ‘technical scaffolding’ (Yelland & Masters, 2007) offered by the software began to be significant. Since the software provided the correct formal notation for any particular movement, students were able to learn whether their movement was the correct one by comparing the notation they obtained with the target expression.

Figure 10 indicates that in this situation mathematical operations can take a back seat and the learning of order within the symbolic notation can be achieved without the need to interpret the notation in terms of mathematical operations. Instead the task can stay at the level of the visual image of movements. This offers a powerful pedagogic tool as students can engage with, and become comfortable with, relatively complex notation even if they are still a little unsure about the combination of mathematical operations the notation represents.

**FIGURE 10** The reflexive relationship between movement and notation helped provide feedback to students whilst they were learning order of operations.

**Episode 3: language supporting shift from movement to mathematics**

The mathematical notation appearing in the software can evoke a sense of movement or can evoke the carrying out of mathematical operations, yet the notation is the same
collection of symbols. With the windshield metaphor (Kaput, et al., 2008) the sense of movement and the carrying out of mathematical operations are both seen through the windshield of the formal notation. A shift between these two senses occurred during the second lesson where attention had previously been on movements on the grid and the visual nature of notation and then moved onto mathematical operations and the carrying out of arithmetic processes. Movement played a central role within the software and as such could become the focus of attention. For example, when students were involved in re-creating expressions tasks they often used words such as ‘up’, ‘across’ and ‘down’. However, in the second lesson students were asked to talk in groups to decide how they thought $2(23 + 2) - 6$ had been created. Their conversations were accompanied by occasional pointing to the interactive whiteboard and, more significantly, a lot of gesturing made by their arms and fingers as to which route should be taken. The talk was of mathematical operations whilst their bodies were physically involved in movements. This is an example of the blended space where both interpretations of the notation co-existed.

After their discussion one student, Abdul (one of the weakest in the class) was asked to describe what happened to the number 23 in the expression $2(23 + 2) - 6$. This was different from coming to the board and physically making the journey and required a more verbal response.

1. Abdul: You added two, you went down one, you take away six.
2. DH: Now just change... tell me something different than going down one.

Students began to be challenged when using directional words so that the mathematical operations were said instead. This began a gradual shift into notation being interpreted as mathematical operations. The next expression, as with the last, involved using
just two rows (the grid can be changed to have from one to six rows in view). Hence the number 2 appeared several times within this larger expression of

\[
2 \left( \frac{2(33 - 2) + 2}{2} + 3 \right) - 2 \right) - 4
\]

since multiplication and division were limited to moving between these two rows. When Julie talked through what was done to 33, I followed what she said by pointing to the appropriate place within the expression, thus stressing the mathematical operations rather than the physical journey round the grid. Fluency had been gained from the “re-create the journey” tasks and once that fluency had been gained, students could stress the mathematical space over the movement space. This allowed the sense of order in the notation to be brought into mathematical contexts. An example of such a context was substitution, which required mathematical interpretation of the notation.

FIGURE 11 Julie explaining a substitution task.
By the end of the second lesson, letters had been introduced. This was obviously significant and I will address how they were introduced in the next section. The students went into a computer room to work on computer generated tasks from the software. Julie was working on substitution tasks, one of which was to substitute $b = 8$ into $2\left(\frac{b+8}{2} - 2\right)$, as shown in Figure 11. The task was to drag the correct number (in this case, 12) from a number box situated below the grid, into the expression cell. Julie was typical of all the students as she now paid attention to the mathematical expression rather than the spatial positioning within the grid. If she attended to the grid she would have known that since the cell marked $b$ was 8, the cell with the expression was only the next cell to the right; meaning that this cell must be four more (as it was in row four), and so would be 12. So the visual image of the two cells being next to one another in the grid was ignored. Instead Julie focused on the mathematical operations as she spoke out loud each operation on her way to calculating the answer 12. The transcript in Table 1 is of Julie speaking and using interesting gestures as she worked on this task. Julie had her right hand holding the mouse and her left hand was pointing throughout this transcript. The pointing hand was held some distance in front of the screen so that she was pointing in space rather than at a particular object on the screen (except for the first occasion mentioned below). The transcript is presented with the speech in one column and the associated gestures in the other.
TABLE 1

Julie’s speech and gestures whilst she talks through her answer to a substitution task.

<table>
<thead>
<tr>
<th>Speech</th>
<th>Gestures</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$b$ equals eight.</td>
</tr>
<tr>
<td>2</td>
<td>So eight times [sic] eight equals sixteen.</td>
</tr>
<tr>
<td>3</td>
<td>Divide by two equals eight,</td>
</tr>
<tr>
<td>4</td>
<td>minus two…</td>
</tr>
<tr>
<td>5</td>
<td>equals six.</td>
</tr>
<tr>
<td>6</td>
<td>Times two equals twelve.</td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

The point of interest is that not only did Julie talk through all the operations, but her gesturing followed the visual positioning of the numbers and operations within the formal expression. By looking at the physical positioning of the numbers and operations within the expression $2\left(\frac{b+8}{2} - 2\right)$, Julie’s movements followed the operations in order. The $b+8$ has two ‘numbers’ (since she knew $b$ was a particular number), one to the right of the other, which is the finger movement she made in line 2 (I note the fact that she said “times” rather than “add” here, but she still carried out an addition). The dividing by two involves a division line and the number 2 below the previous addition, and I noted the mouse movement.
downwards when she said line 3. When she said “minus two” in line 4, she moved her pointing finger to the right and the position of that subtraction is to the right of the division. Lastly, when she said “times two” she moved her finger to the left a greater distance than her previous movement to the right (line 6). This fits in with the position, within the expression, of the multiplicative two in relation to the previous subtraction. So her focus, physically as well as mentally, was on the arrangement of symbols in the formal notation and not any movements within the grid.

Figure 12 shows the way in which language was the vehicle to relate the notation back from being about movements to being about mathematical operations. It can be noted that much of the time movement was not being made within the software. Students verbally described the order of operations or worked out numerical values of expressions in their head. As such the potential scaffolding within the software was not used.

![Figure 12](image-url)

**FIGURE 12** Use of language to shift interpretation of notation onto mathematical operations.
Episode 4: the introduction of letters

The introduction of letters is often very significant for students and also for those analysing students’ learning of algebra. Prior to the introduction of letters, students had been working on tasks which involved re-creating an expression, such as in Figure 9.

They had done that collectively and also individually in the computer room. By now they were quite proficient at doing such tasks. These involved being given a start number, a final expression, and having to re-create the journey to make that final expression. A key factor in the introduction of letters was that the focus whilst doing these tasks was with the operations or movements which needed to be made. The actual start number was irrelevant to the task. This was shown when students frequently did not mention the start number when describing what was done to create the expression. For example, just before a letter was introduced, the expression $\left( \frac{2(33-2)+2}{2} + 3 \right) - 2$ was created on the interactive whiteboard and the following was said:

1. DH: I’d like you to tell me what it is I did.
2. Julie: Minus two then...
3. DH: What did I start with by the way?
4. Julie: Thirty-three, minus two... [she continued to say several more operations in the correct order and DH pointed to the various parts of the equation after she said each operation]

Julie started talking about the operations which were carried out and needed specifically to be asked before she mentioned the start number. Whatever the start number was, it did not change the series of operations which were carried out and the series of
movements which needed to be made. As such there was a sense of generality. The
irrelevance of the start number allowed an opportunity to put in a letter at the start, as it did
not affect the activity since it was not significant for completing the task successfully. A
relevant metaphor might be that of a magician who deliberately does something to take the
audience’s attention away from what is significant for the trick. For example, they might do
something flamboyant with one hand whilst the other hand palms a card. In this case,
attention is on the operations and movements, the result of which means that attention is
away from the start number. This allows a letter to be introduced with relatively little
attention given to it. Having said this, the reality was that I gave more attention to it than I
had intended. This was due to the fact that I decided to ask a pupil for a letter rather than just
choosing one myself; and also the fact that the letter they chose was the letter ‘o’, which I felt
was unfortunate as it could be seen as the number zero. As a consequence I said far more than
I had intended to do. The transcript of the introduction was as follows:

1. DH: Right, by the way I’m just fed up with numbers so your favourite letter is?
2. Students: [several things said, including “Letter?”]
3. Student A: Oh [the letter ‘o’].
4. DH: An ‘o’. So here is an ‘o’ [drags letter ‘o’ into a cell]. Umm. There it is. By the
   way that is an ‘o’ and not a zero, it is not a number, it’s just a letter there. OK. Well, it
   might be a number later but it is the letter ‘o’, OK? [Quickly moves letter to the right
   on the grid to produce ‘o+4’] What have I done to it?
5. Students [in unison]: Added four.
6. DH: And...? [moving ‘o+4’ to produce $\frac{o+4}{2}$]
7. Students: Divided by two [one or two voices heard to say “Times”]. [DH moves
   expression again to create $\frac{o+4}{2} - 3$] Take away three. [DH moves it again to produce
\[2\left(\frac{o + 4}{2} - 3\right)\] Times two. [DH moves this again to produce \(2\left(\frac{o + 4}{2} - 3\right) + 4\)] Add four.

There was an initial reaction in line 2 with one student being heard to say “Letter?” indicating their surprise. In line 4 I was feeling a little flustered, wondering what to do with the fact that the letter ‘o’ had been chosen. All this was potentially confusing and put more of a spotlight onto the letter than I had intended. As a consequence I decided not to dwell more on the issue and started making movements. The focus of attention then moved onto the operations involved. This series of questions was followed immediately by asking for another letter and creating the expression \(\frac{2\left(\frac{r - 2}{2}\right) - 2}{2} + 2\). The class were then asked to talk in pairs to decide what had happened to the letter ‘r’. This time no-one reacted at all to a letter being chosen and all the conversation in pairs concerned the operations involved with no-one asking anything about a letter being present.

There was no real difference in the way a student needed to think to do this activity due to the fact that a letter ‘r’ was involved rather than a number. The focus remained on order of operations whether or not a letter is involved. After the initial surprise, the letter was ignored in a similar way to how the start number had been ignored.

Since the same activity was used before and after letters were introduced, the focus remained on the order of operations. I noted no change in the way notation was being interpreted in terms of movements or mathematical operations. As such it remained as in Figure 12, with the stress on mathematical operations.
Episode 5: re-appearance of visual support when introducing inverse operations

The students were confident with reading order of operations within an expression by the end of the second lesson. However, many students were not intuitively confident about how they might work out the inverse operations within a given expression. They did have intuitive notions about inverse journeys through their everyday life experiences and it proved helpful to allow the image of journeys to run alongside the learning of inverse mathematical operations. I start here by showing what happened when the image of journeys was not present and contrast it with the time when the visual image was introduced.

A short time after the beginning of the third lesson, I loaded a pre-prepared file which only had the single expression \( \frac{x - 4}{2} + 3 \) in a cell. Putting my hand to my head I announced that I had rubbed out where the \( x \) was and asked how we could find out where the letter was. One student was still focussed on the “re-create a journey” activities they had been doing previously and said the order of operations within the expression: “you took away four divided by two and added three”. Another two students spoke, both of whom were relatively able students within the class. One correctly identified which cell the letter would be in and the other gave an eloquent description of the process:

“But don’t you do the expression in reverse? The whole expression in reverse, so you do, you start the other way round as well. So you start with adding three, so you take away three from that box. Then you times by two because of the divide in the expression. Then you add four. And then you end up in the correct box.”

This was impressive but at the time I felt that many students had not followed what was said. There were many ‘blank’ faces in the room and everyone else was quiet. As a
consequence I decided to clear the grid and place the letter \( d \) into the top left cell and mark out a route I was going to take the letter on (see Figure 13).

![Figure 13](image13.png)

FIGURE 13 A route marked out on the grid: the letter \( d \) was placed in the cell marked 1.

After making the journey from the cell marked 1 to the cell marked 4, the expression \( 2(d + 2) + 4 \) appeared in the final cell. This could be seen if the interactive whiteboard pen was pressed in this final cell, as in Figure 14.

![Figure 14](image14.png)

FIGURE 14 The expression produced after the letter \( d \) was taken along that route.
Through the use of questioning, the students were able to express how they would get back to the original cell, marked 1. The following transcript was at the start of that questioning:

1. DH: OK. So if I wanted to get to find out where this was [pointing to the letter $d$ in cell marked 1], which we know where it is now don’t we? What do I have to do to this [pointing to the expression in cell marked 4]?
2. Student C: Do it the other way round.
3. DH: What do I have to do to that? What can I do? If I am here [pointing to cell marked 4], how can I get back to there [pointing to cell marked 1]?
4. Student D: Take away four.

The conversation allowed one student to engage with the visual route (line 2) and another student to engage with the mathematical operations (line 4). The blended space allowed both to make contributions towards getting back to the original cell. What was significant was that now 12 different students contributed to carrying out this task. Far more students were involved compared to when the route was not shown. In addition to the visual route marked on the grid, an ‘inverse’ button was pressed down in the toolbox within the software. This had the effect of cancelling the last operation within an expression if the inverse movement was made with that expression. For example, with the expression $2(d + 2) + 4$ (as in Figure 14), if a movement was made two cells to the left it would show $2(d + 2)$ and not $2(d + 2) + 4 - 4$ as the movement made cancelled out the plus four within the original expression. This added a visual feedback to support students deciding how to get back to the letter $d$. There were now two meanings for the goal of getting back to the letter $d$. The first was physically moving back to the cell which contained the letter $d$. The second was a gradual simplifying of the expression until there was only the letter $d$ left. The first
concerned a physical ‘inverse’ journey and the second concerned a successive cancellation of mathematical operations. The blended space enabled the imagery of inverse journeys to support the learning of inverse operations.

The blended space was also present in the next challenge. A student came up to the board and made a journey, starting with the letter \( s \) and ending up with the expression

\[
2\left(\frac{s+2}{2} - 2 - 1\right)
\]

in the final cell (Figure 15).

![FIGURE 15 A journey created by a student.](image)

The route taken was then marked on the grid as in Figure 16.

![FIGURE 16 The route taken by that journey is marked out.](image)

The route was marked in the order of the movements taken to create that particular expression. The task was to carry out the inverse journey, going from the final expression back to the letter. This inverse journey was emphasised through the gesture of a hand movement over the grid. This hand movement indicated the required route from the final
expression back to the cell with the single letter inside. This emphasised the visual space and allowed students to see what was needed visually.

The verbal interaction, however, focused on the mathematical operations. The verbal interchange following the hand movement was as follows:

1. DH: What am I going to do to go back to my letter?
2. Student E: Divided by two.
3. DH: OK and what would that get rid of here [points to the expression]? 
4. Student E: It will get rid of the dividing by two.
5. Student B: Times two.
6. DH: [Drags the expression upwards which gets rid of the multiplication] Oh! What did it get rid of?
7. Students: The brackets.
8. DH: OK and what is the brackets and the two [pointing to the brackets and the 2 in front of the brackets]? 
10. DH: So I got rid of the timsing by two, and does that make sense, that doing that [makes gesture of moving upwards] would get rid of timsing by two?
11. Students: Yes.
12. DH: What have I done by going up, by the way?
13. Students: Divide/Divide by
14. DH: Divide by...?
15. Students: Two.
16. DH: And is that ‘get rid of multiplying by two’, if I divide by two?
17. Students: Yes/Because it is the opposite.
18. DH: Absolutely. Absolutely. So what am I going to get rid of next [pointing to different parts of the remaining expression]?
19. Students: Take away/Take away one.
20. DH: OK I am going to get rid of the taking away one [points to the subtract one]. How do I get rid of take away one?
21. Students: Add one/Plus one.

The verbal interaction concerned the mathematical operations and attention was drawn to the mathematical notation within the expressions through the use of pointing as students were talking about the relevant mathematical operations (lines 8, 18 and 20). Thus the notation was the focus for talking about mathematical operations whilst there was a strong visual support in terms of the route marked out on the grid and the hand movement indicating the journey which needed to be taken. In these early stages of working on inverse operations several students were not sure of what needed to be done to arrive back at the letter. This was particularly true when the route was not marked in the first example above. However, when the route was marked and with help of my own arm movement gestures indicating the inverse route, the blended space allowed students to know what was going on in terms of movement even if they were not sure of what was going on in terms of mathematical operations. The language, though, stayed with mathematical operations even when students later came up to the board themselves to carry out further tasks of making the inverse journey back to the letter. As a consequence, over time, the language used by students was that of mathematical operations.

During this part of the lesson, it seemed that the new task of thinking about inverse mathematical operations within a formal expression was clear only to a very few of the students. This led to the visual being emphasised once more in the form of a route marked on the grid and the use of hand movements. The blended space then allowed some students to
gain meaning in terms of physical movements whilst the verbal interactions talked about the mathematical operations. Figure 17 shows the way in which both the notation of the mathematical operations and the physical movements supported the mathematics required to work out which operations needed to be carried out and in which order.

![Diagram](image)

**FIGURE 17** Learning of order of inverse operations was informed not only by the notation but also by the visual image of an inverse journey.

The scaffolding offered by the visual image of a journey was crucial. Instead of students becoming confused – as many had begun to be at the start when the route was not present – the image of a route enabled them to know visually what needed to be done. After all, they have all had experiences of reversing walks they have made, etc. A key feature of the software was that the same notation can be discussed verbally in terms of mathematical operations whilst being addressed visually in terms of a physical journey. Students’ confidence with the latter supported their learning of the former.
**Episode 6: Fading of visual support**

Soon after the examples above, students continued with tasks of trying to get back to the original letter. However, the visual support of the route was not put on the grid and neither did the students see the expression being formed in the first place. Instead, a pre-prepared file was loaded where the only thing in the grid was the final expression. Thus, they had few visual clues as to where they needed to go. Instead the work needed to be carried out just from the notation. The language when talking about the notation continued to be that of mathematical operations. Consequently the visual support was faded gradually.

It made a re-appearance when work began on solving an equation. This activity was similar to before except that a number was placed into the same cell as the final expression, thus creating an equation. Figure 18 shows the letter $d$ starting in the cell marked 1 and a window appears below the grid showing the contents of that cell (just the letter $d$). The letter was then taken on a journey to the cell marked 6, producing the expression $\frac{2(d + 2) - 6}{2} + 4$. The number 27 was then placed into the same cell to create the equation $27 = \frac{2(d + 2) - 6}{2} + 4$. Another window appeared below the grid showing the contents of this cell. I noted that on this occasion I just stated that I would make this expression equal to 27 as I dragged that number into the same cell as the expression. I did not dwell on the fact that there was choice as to what this expression could equal. As such, the idea of the expression being a function which could take different values was not stressed on this occasion although it had been earlier with other expressions.

The challenge for the students was to find a way of working out what number must be in the $d$ cell. This time, the number 27 was dragged back along the route to the $d$ cell to create a solution to the equation. It was dragged back along the whole route and not directly just
three cells to the left. So the inverse journey went along the route backwards through the cells marked, 6, 5, 4, 3, 2 and 1 to produce the solution \( \frac{2(27 - 4) + 6}{2} - 2 = d \).

\[
27 = \frac{2(27 - 4) + 6}{2} + 4
\]

**FIGURE 18**  The letter \( d \) is taken along the indicated route to produce an expression. Then the number 27 is entered into the same cell as the expression to create an equation.

As before, when something conceptually new was introduced the visual support became more prominent so that students could use mix of the visual movements and the mathematical operations to work on the task, in this case, of finding the value of the letter.

The next example (see Figure 19) had less support with the route no longer shown. Thus, it could no longer be done by following the visual image of the route.

**FIGURE 19**  The route has gone but the grid is still present as a support.
The support continued to be reduced with the next example having the grid hidden and an ‘Expression Calculator’ was used to enter in the calculation needed to be done to the number 49 to arrive back at the letter x (Figure 20). This meant the solution to the equation had to be entered as a series of operations rather than being created through movements on the grid. This gradual fading of support meant that students were getting closer to the situation of solving such equations on paper, where attention is purely on the mathematical notation and the mathematics of solving equations.

**FIGURE 20** The grid has now gone and only an ‘Expression Calculator’ is used to enter what needs to be done to 49 to get back to x.

In all only 17 minutes was spent on solving equations using Grid Algebra as this was towards the end of the third lesson. During this time, as the support was faded gradually, I made a point of asking each student in turn what operation should be carried out next and why. Every student in the group was able to state correctly the operation and give a reason for why. This was followed at the very end by students choosing one of two worksheets, one offered visual support by showing a picture of the route on a grid along with each equation to solve. The other offered no visual support at all and was just a set of equations to solve. 18 sheets were collected in (some students worked in pairs) of which 13 were the visual support sheets and five were the sheets without support. Overall 67.9% of questions answered across
the whole group were correct. When other questions, which had arithmetic errors but were correct algebraically, were counted, this increased to 72.6%. What was also of significance was that rarely was any working shown as they could solve these equations, ranging in complexity from \(2(t + 3) = 46\) to \(2\left(\frac{b-18}{3}+4\right)=128\), in their head. Where working was shown, all but one was written with correct formal notation, including the sheet written by the weakest student in the class.

Having given support to the students’ learning of mathematics and formal notation, it was desirable in the end for students to be able to solve equations without the need to get out a grid and start making journeys around the grid. The image of the grid was a pedagogic ‘scaffolding’ tool to offer support to help students learn confidence with formal notation and carry out certain algebraic tasks. As such, planning to have that support fade (Collins, et al., 1989; Seeley Brown, Collins, & Duguid, 1989) so that students can work directly and confidently with the symbols of mathematics, was as important as introducing that support in the first place. The gradual fading of the support offered by the grid, helped students work with quite complex expressions and equations with a fluency which is unusual for students of this age.
FIGURE 21 Many students were working directly with the algebraic notation when solving equations without making any noticeable reference to the visual imagery of the grid.

Figure 21 offers the image of the grid having done its job of assisting learning and then fading out so that students are left being able to work directly with formal notation in order to carry out mathematical tasks. This was successful for the students who solved equations using the sheet at the end of the lesson which did not have support. Of course, I do not know whether some students still used the imagery of the grid in their heads but I saw no evidence of this on the video captured. The gestures that Julie showed earlier when substituting showed her wrapped up in the visual arrangement of symbols within the notation and not the grid. With regard to the students who chose the sheet with visual support, it is not known how many of those students decided to ‘play safe’ and went with that sheet but would also have been able to be successful with the traditional equation solving sheet. There was little time left for students to work on these sheets before the lesson had to come to an end. What did happen prior to starting work on the sheets was that each student was able to articulate what operation needed to be carried out and why, whilst the Grid Algebra support was gradually faded out.
DISCUSSION

I discuss below three aspects of this study, the first is about the use of blended space, the second about the ‘technical scaffolding’ offered by the software, and the third is about the students’ learning of formal notation and solving equations.

A blended space was created through the close association between movements on the grid and mathematical operations. Students with a wide range of prior mathematics attainment were able to engage confidently with complex expressions and learn how to read order within a formal expression. This learning of order, and the inverse order required to solve equations, could be learned through attending to the meaning of movements just as much as the meaning of mathematical operations. This allowed their met-before (Tall, 2004) implicit knowledge of order and inverse within the context of movement to be utilised in their learning of a key mathematical idea. Access to this learning was thus accomplished without the need for significant prior mathematics knowledge. Once order was established, this was transferred to the context of mathematical operations through the use of language. The blended space meant that this could happen at different times for different students as the two spaces co-existed alongside each other. Indeed a student may shift between both spaces several times before becoming confident within the mathematical operations context. The blended space also allowed for notation to be spoken in terms of mathematical operations whilst visually seen as a journey. These two ways of interpreting formal notation could happen simultaneously, enabling students’ confidence with the visual world of movement to support learning in the verbal world of mathematical operations.

A symbolic ‘dance’ of the changing dynamics between movement, notation and mathematical operations was a key aspect of the learning which took place over the three lessons:
• **learning of notation**: meaning was given through prior establishment of mathematical relationships between numbers on the grid.

• **learning of the mathematical notion of order within formal notation**: this was informed by movement with students initially using the language of up, down, left and right. There was also feedback provided as a consequence of movement where students could see visually whether their sense of order was correct or not. Students could either learn order through attention to movement or they could learn through attention to the mathematical operations. The blended space allowed every student to engage successfully in these tasks irrespective of their confidence with interpreting formal notation in terms of mathematical operations.

• **use of order within substitution tasks**: substitution requires the notation to be viewed in the context of mathematical operations. So prior to this, language was used to shift the verbal expressions of movements onto the mathematical operations of add, subtract, multiply and divide.

• **learning of the mathematical notion of inverse operations**: this was informed by movement with the visual appearance of a route on the grid and the use of hand movements. Parallel with this visual support was the verbal expression of mathematical operations. Thus the blended space was once again present, this time with one space operating visually whilst simultaneously the other space was operating verbally.

• **learning to solve linear equations**: this was initially supported with the blended space of visual movements alongside verbal mathematical operations. However, there was gradual fading of the visual support to leave only formal notation. At the end of the three lessons, some students were still at a stage where the visual support was significant for them whilst others were able to solve relatively complex linear equations with no visual support at all.
These stages can be represented in the sequence, left to right, as shown in Figure 22 where each of these arrow diagrams are smaller versions of the ones shown throughout this paper. This gives a sense of the changing dynamic between movement, notation and mathematics across the three lessons.

The ‘technical scaffolding’ (Yelland & Masters, 2007) offered by the software allowed individual students to interpret notation in different ways at different times whilst still working on the same task as other students. It provided the formal notation associated with the movements made on the grid. Consequently, that formal notation was ever present and continuously ‘looked through’ rather than a teacher deciding when formal notation was to be introduced and hoping that students’ confidence would not be affected by the new formality. This is a significant feature as other models, such as the image of a balance, do not have the formal notation present within the model. The notation has to be introduced separately. The structure of the lessons also allowed the imagery of movement to be stressed at some times and allowed to fade at other times without any change in the formality of the notation which appeared on the screen and without any change in the fact that the formal notation was constantly at the centre of all activities. Thus the ‘dance’ encapsulated in the above diagrams was only possible because the software allowed that flexibility in what was being stressed at any point in time. This flexibility could be used both by a teacher in the pedagogic decisions made during lessons, and also by students who can shift in and out of different interpretations whilst still progressing forward with the challenges offered.

FIGURE 22  The symbolic ‘dance’ which occurred over the three lessons.
Regarding the learning of formal notation and the solving of equations, the software shifts attention onto operations rather than numbers. This focus on order and inverse stayed across the shift from expressions with only numbers (which might be regarded as an arithmetic topic within the curriculum) to expressions involving a letter (which might be regarded as an algebraic topic within the curriculum). Towards the end, students were able to work with complex formal expressions with confidence. Even those who got some questions wrong in the solving equations sheets were still showing an understanding of what those expressions meant in terms of mathematical operations and were writing expressions themselves in correct formal notation. Their fluency with such expressions was in contrast to studies which showed that many students of a much older age have difficulties with algebraic notation (MacGregor & Stacey, 1997). It supports the increasing evidence that relatively young students can understand and work with letters and notation (Blanton & Kaput, 2004; Carraher, et al., 2001; Schliemann, et al., 2003; Schmittau, 2005) whilst this study shows that young students can also work successfully with relatively complex formal notation involving all four arithmetic operations. Although this work might be considered an abstract topic, the students were very involved in the work throughout the three lessons, often clapping and cheering when a target expression was achieved, and having animated conversations about complex expressions. As Wilensky (1991) commented, abstraction is more about the relationship students have with notation, rather than it being a property of notation per se. The unique scaffolding offered by the software and the blended space were key to establishing this positive relationship students had with notation.
REFERENCES


