Approaches to teaching mathematics and their relation to students’ mathematical meaning making.

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This paper addresses theory and practice in the teaching of mathematics in university lectures and tutorials that is designed to promote students’ mathematical meaning-making. It draws on areas of theory and five research studies to characterise teaching, to illustrate how theory can be used to analyse teaching and to relate meaning making in teachers’ teaching and students’ mathematics. It begins to develop a body of research on teaching practice and its development at university level, identify key areas of knowledge and to present questions for further research.

INTRODUCTION

This paper is a theoretical/philosophical paper dealing with mathematics teaching at university level and associated meanings. It is linked closely to teaching practice through several studies that have sought to illuminate practice at this level. The focus is the relationship between teaching approaches, the meanings that students make of the mathematics taught, the ways in which teachers gain access to student meanings and ways in which teaching can focus on creation of meaning.

Kilpatrick, Hoyles and Skovsmose (2005) write “Teachers of mathematics must deal with questions of meaning, sense making, and communication if their students are to be proficient learners and users of mathematics” (p. vii). They ask the questions:

“How can meanings for teachers and didacticians be developed?”

“How can teachers best explore the meanings which students have constructed?”

Skovsmose writes further “… for students to ascribe meanings to concepts that have to be learned, it is essential to provide meaning to the educational situation in which the students are involved (p. 85, our italics). Artigue, Batanero & Kent (2007) suggest that learning at this level is seen as enculturation in advanced mathematical practices, while Ben-Zvi & Arcavi (2001) write that making meaning in mathematics is a process of “socialisation” into the culture and values of “doing mathematics” (p. 36). The studies to which we refer below address the above questions, taking into account the full sociocultural context of learning and teaching as far as is possible. In terms of what we mean by ‘teaching’, we follow Pring (2000 and 2004) who claims:

An action might be described as ‘teaching’ if, first, it aims to bring about learning, second, it takes account of where the learner is at, and, third it has regard for the nature of what has to be learnt. (2000, p. 23). For example, the teaching of a particular concept
in mathematics can be understood only within a broader picture of what it means to think mathematically, and its significance and value can be understood only within the wider evaluation of the mathematics programme. (Pring, 2000) and [not attending to this] is to accept a limited and impoverished understanding of teaching (2004, p. 22), (our italics).

Further, we seek to redress the deficiency expressed by Speer, Smith and Horvath (2010) who write “very little research has focused directly on teaching practice [at university level] – what teachers do and think daily, in class and out, as they perform their teaching work” (p. 99). We refer to five studies, below, which arise from an in-depth focus on specific examples of teaching practice with qualitative analyses which reveal aspects of teaching, capture the intentions and reflective thinking of the teacher and subject outcomes to critical scrutiny through rigorous analysis. They expand on existing theory and introduce new theory, illuminate the nature of teaching at this level and its relations to students’ meaning-making, raise and elaborate on issues that arise in the practice of teaching and offer insights that can be of relevance and significance more generally. In doing so, they begin to theorise this (relatively) new area of study into the university teaching of mathematics.

**STUDIES HIGHLIGHTING CHARACTERISTICS OF TEACHING**

We exemplify the practices, to which we refer, through references to research which is taking or has taken a sociocultural approach, seeking to address the full context of learning and teaching as far as is possible. This has involved the use of qualitative approaches to data collection including participant observation of teaching-learning events and interviews or conversations with the teacher in each case. Analysis is grounded and interpretative with care taken to justify interpretations in relation to the wider context of the events. These studies have drawn variously on existing theory and in some cases have developed theory through the research. We sketch briefly some of the significant findings in each case (with reference to publications which provide greater detail: some in earlier PME proceedings) and follow with a theoretical synthesis and questions for further research.

1. **Characterising pedagogy in mathematics small group tutorial teaching**

Nardi, Jaworski & Hegedus (2005) report a qualitative study in the UK of the teaching of six mathematicians in *small group tutorials* over one university term (8 weeks). Analyses led to a characterization of teaching approaches from the perspectives of the tutors. All tutors recognized students’ difficulties and dealt with them in differing ways, episodes from which were seen to fit into or between four pedagogic characterizations: *Naive and Dismissive; Intuitive and Questioning; Reflective and Analytic; and Confident and Articulate*; the whole being characterized as a *Spectrum of Pedagogical Awareness*. This spectrum offers a theoretical

<table>
<thead>
<tr>
<th>Naive and dismissive</th>
<th>Intuitive and questioning</th>
<th>Reflective and analytic</th>
<th>Confident and articulate</th>
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</table>

Spectrum of pedagogical awareness/development
perspective on the links between mathematics and pedagogy, and the knowledge of the teachers in working for the meaning-making of their students. In parallel, teaching episodes were analysed using a theoretical construct “The Teaching Triad” (Jaworski, 1994; 2003), consisting of 3 domains – Management of Learning (ML), Sensitivity to Students (SS) and Mathematical Challenge (MC). Findings showed largely that ML involved teachers in showing and explaining mathematics; SS involved ensuring the student was made aware of the correct mathematics; while MC left it up to the student to go away and make sense of the mathematics presented to them. There were of course exceptions: for example, one tutor frequently invited a student to go to the board and explain his or her solution to a given problem. He explained his own actions in relation to the student’s activity at the board:

I do promise to help; or will help . . . they actually know I’ll start them off. They won’t just be stood at the board and me twiddling my thumbs. I might, after a few seconds, like 30 seconds, or something like that, or perhaps even less if they’re looking panicky, I would suggest, er, “Well, OK, write down, what’s the first line? What’s it mean to say that?” (Jaworski, 2003, p. 89)

For example, the following text is recorded from a dialogue about group theory in which a student (S2) writes at the board and the Tutor (T) supports him:

4 S2 If you’ve got \( h \) and \( l \) in \( H \), then this one tells us that \( gh = kl \) (pause) for some \( l \) in \( H \) [S2 and tutor say this together] and same sort of thing for the twos, [he writes \( gh = kl \)]. (pause)

5 T … call it \( h' \) and \( l' \), [h-prime and l-prime] well, call it \( l' \), you might want the same \( h \) as … (Jaworski 2003, p.84)

This kind of teaching was seen to show SS in both affective and cognitive domains as the tutor helped the student to feel supported and encouraged in the given task. Moreover MC was seen in the tutor’s requirement for students to express their own mathematical meanings publicly. This kind of activity was characterized in the spectrum as “Reflective and Analytic”, whereas the more common forms of teaching activity, mentioned above, were seen to be largely in the domain of “Intuitive and Questioning”, with some at the level of “Naive and Dismissive” (Nardi et al, 2005).

2. Characterising teaching in mathematics lecturing

An ongoing study in Greece, of teaching in mathematics lectures in two University Mathematics Departments, involves observation of lectures and interviews with six lecturers who are research mathematicians. Analysis identifies a lecturer’s teaching goals, actions and characteristics, allowing insights into how the actual teaching practice in this context takes students’ needs into account. The Teaching Triad is used as an analytical frame to characterise teaching episodes and to interpret the identified scheme of teaching actions, goals and characteristics (e.g. Petropoulou et al, 2015). Sensitivity to Students and Mathematical Challenge are identified through lecturers’ main goals; are inherent in the nature of their teaching actions and are reflected in basic characteristics of their teaching. For example, one lecturer aims to
stimulate students affectively by drawing on students’ experiences and encouraging their engagement in the lecture through interaction, which is for him an important element of the learning process. He challenges students mathematically, encouraging mathematical investigation by posing open questions and making connections by exemplifying basic mathematical ideas in dialogue with students. For example,

L: Now, can you hypothesize, when a series may converge? [an open, and challenging question] You can base on the series S1 and S2.

\[ S1 = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1 \]

S: The base [of the \(x^k\)] in the first case [S1] is a positive number. [student meaning]

L: Not necessarily. If \(x\) is a negative number the sign [of \(x^k\)] changes. …

Dialogue allows the lecturer access to students’ meanings, to which he can then respond. Analyses are showing “An uneasy balance” between Sensitivity to Students and Mathematical Challenge in this teaching, reflected in a tension between lecturer’s intentions to include students’ thinking in the activity of a lecture while at the same time presenting mathematical meanings in a rigorous form. The ways in which this tension is addressed in teaching episodes can be seen to fit with differing positions on the Spectrum of Pedagogical Awareness.

3. Characterising one lecturer’s teaching of linear algebra

A UK study of the teaching of linear algebra involved a small community of inquiry of two mathematics educators and one mathematician (the lecturer) in which teaching in lectures was explored in depth and characterised. The focus was centrally on the thinking and actions of the lecturer with the three members engaging deeply with ideas from linear algebra, approaches to teaching and the engagement/meaning-making of students. The teacher talked extensively about the mathematics and the ways in which he would (and did) work with students on this mathematics, reflecting particularly after a lecture in which he had given time to students to work on a task and observed their work. For example, after a lecture on subspaces, he reflected:

At some points I realised I need to find different ways of phrasing the questions in order to make them more accessible. One example of that was the introductory example on subspaces, where I had asked the students to find solutions to a homogeneous equation system with unknown coefficient matrix, given that they know a couple of solutions that I’ve given them. That was one question where I saw quite clearly that some of the students found it very easy, and some of the students didn’t have the slightest idea even if they tried. (Thomas, 2012, p. 117)

Central to analysis was the lecturer’s articulation of teaching goals and his realization of the goals in his day to day teaching. The study showed episodes of teaching and talking about teaching in the realm of ‘reflective and analytic’ as the lecturer expressed his goals for teaching for the benefit of his co-researchers. We noted particularly two modes of reflection which we called ‘expository’ (in which the
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lecturer expressed the mathematical meanings he wanted students to make) and ‘didactic’ (in which he articulated his goals for teaching and the associated actions he would, and did, take in his lecturers) (Jaworski, Treffert-Thomas & Bartsch, 2009). The study provides important insights into the teaching of linear algebra. (Thomas, 2012.)

4. Knowledge in mathematics teaching in small group tutorials

An ongoing UK study of university teaching in small group tutorials is exploring tutors’ knowledge in teaching. Initially 26 tutorials were observed from 26 different tutors, including both mathematicians and mathematics educators. Subsequently tutorials of three tutors, over one semester, are being studied in depth. The focus of analysis is on teachers’ knowledge for teaching in mathematical, didactical and pedagogic domains, drawing on a grounded analytical approach and a range of theoretical positions in the literature (e.g., Teaching Triad; Knowledge Quartet – Rowland, Huckstep & Thwaites, 2005). One key concept emerging is tutors’ use of mathematical examples: dialogue between tutor and students provides insights into students’ meaning-making and tutors’ adaptation of teaching to students’ thinking as they see it (Mali, Biza and Jaworski, 2014).

The study seeks to identify characteristics of university mathematics teaching (an example of which is a tutor’s use of a generic set of examples for students’ meaning making of a mathematical concept) where a characteristic of teaching constitutes a pattern of teaching identified repeatedly in the data. The characteristics as a whole form an image of a tutors’ teaching practice and, in a finer layer of analysis, each one of them is distilled into tools and strategies for teaching. For example, in one tutorial, a generic set of monotonic functions on intervals was used for the students’ appreciation of the property that every monotonic function is injective. This set formed a mathematical object coded as ‘tool for teaching’, and included the functions

\[
\begin{align*}
  f(x) &= x^2 \text{ on } [0, +\infty]; \\
  f(x) &= \sin(x) \text{ on } [-\pi/2, \pi/2], [\pi/2, 3\pi/2]; \\
  f(x) &= \log_a(x) \text{ for } a>1, 0<a<1; \text{ and } f(x) = x.
\end{align*}
\]

A strategy for teaching is a process consisting of the mode of tool use and the associated decision making. In this tutorial, the tutor attempted to make the connection between a concrete set of examples and the abstract mathematical concept of monotonicity and its properties. The strategy here was the use of different mathematical representations to foster students’ meaning making of mathematics through making connections within mathematics. The layers of characterization, tools and strategies form the basis of a theoretical construct to capture knowledge in teaching (Mali et al, 2014). The particular tools and strategies used by the tutor comprise the tutor’s pedagogy; an example of a tool-strategy pair was given above.

5. Developing mathematics teaching to address students’ meaning making

In this study, one university tutor designed teaching approaches to focus on the mathematical meanings made by her students in a small-group tutorial, and adjusted the approaches to relate to meanings discerned. The students were first years, in a
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joint degree in Mathematics and Sport Science, and were relatively weak in mathematics. The study involved a small community of inquiry of two researchers, one being also the tutor; as the tutor designed teaching, worked with students and reflected on learning outcomes, her co-researcher gathered data and acted as a sounding board for tutor reflections. A fine-grained analysis was made of dialogue from the tutorials, using the Teaching Triad as a tool (Jaworski & Didis, 2014).

It was clear from the beginning of work with her five students that the students were unaccustomed to speaking their mathematics or explaining concepts, so it was hard to discern students’ meanings. Various approaches were seen to address these issues. For example, one tutorial was spent entirely on the definition of ‘limit’. After realising that students were unfamiliar with quantifiers, and unaccustomed to reading mathematics aloud, the tutor first established meanings of symbols and then requested students to read parts of the definition, sometimes independently and sometimes chanting as a group. These strategies led to each student’s being able to read the definition in a meaningful way which gave some insight into their understanding of the limit concept. However, at the end of the tutorial, one student expressed frustration with not having tackled any problems in the tutorial. The tutor learned from such observations and adapted her practice for future tutorials.

A question established during this study was “How can we foster student expression of mathematical meanings in relation to the teaching experienced?” We recognised that the tutor was a learner, developing teaching through a design/action/reflection approach. In terms of the spectrum she might be seen as shifting between ‘intuitive and questioning’ and ‘reflective and analytic’ as reflection and analysis enabled her to became more aware of the meanings and perceptions of her student.

DISCUSSION

The papers referenced in each case above provide details of research which explores and characterises the university teaching of mathematics in both lectures and small group tutorials. In each case the research has studied the ways in which the teachers (lecturers and tutors) constructed teaching to enable students to make sense of the mathematics, to make mathematical meanings. The studies have been interested in teachers’ thinking related to didactics and pedagogy – how they design tasks and use strategies in teaching, and how they interact with students in meaningful ways (or not). We see teachers identifying students’ difficulties and using strategies to deal with difficulty; using examples to aid students’ meaning-making; relating goals for student comprehension to strategic action in teaching sessions, and designing teaching actions to cope with tensions between sensitivity to students and the rigorous mathematics desired by the tutor.

Two theoretical constructs, particularly, have been used to make sense of teaching approaches: the spectrum of pedagogical awareness captures the degree to which teachers engage consciously with didactic and pedagogic issues and design teaching to engage students and enable them to create meaning; the teaching triad identifies
aspects of sensitivity towards students and associated mathematical challenge within an overall management of the learning environment. Where challenge can be seen as offered with due sensitivity, mathematical rigour might seem to be in question. This is an important issue to explore further. Relationships between the spectrum and the triad also require further exploration.

These studies are embedded in a sociocultural perspective, thus seeking to consider the holistic nature of the teaching/learning process rather than fragmented images of the tutor’s practices and students’ meaning. Recognition of goals and tools is related to Vygotskian theory, in particular the mediational triangle in which tools are seen to mediate the achievement of the object of activity, and goals are related to actions in activity (e.g., Leont’ev, 1979). For instance, Study 4 conceptualises tools for teaching – the generic set of examples is seen as a tool which mediates students’ understanding of the concept of injectivity. Analysis here is linking tool use, and the strategies teachers are seen to use, to knowledge in teaching, discerned through in-depth conversations with each tutor. In Study 3, goals are the teachers’ intentions for teaching, instantiated in his teaching actions in a lecture and theorised in terms of Leont’ev’s theory of activity (Leont’ev 1979; Thomas, 2012). In Study 5, teaching was developed to respond to students’ meaning making and perceived learning needs, and to support their meaning-making, thus demonstrating forms of sensitivity to students which recognises their particular context (e.g., sport-science students, rather than mainstream mathematics). These studies together start to define a field of research in which the following elements are significant:

1. Emergence of theoretical constructs: provides ways of categorising teaching and comparing teaching processes and events.
2. Different ways of characterising teaching (related to theoretical constructs): allow others to question their practices and develop knowledge in the field.
3. Methodological approaches and methods: allow in-depth study of teaching practice within a sociocultural frame with attention to details of context.
4. Recognition of teaching intentions, goals, and actions, and identification of tools in teaching, provides the beginnings of a ‘tool box’ (Nardi et al, 2005) for teaching which can be developed through scrutiny and critique.
5. Meaning making in teaching, related to students’ mathematical meanings: opens up ways of linking teaching with learning in very initial and tentative approaches which can be developed further through scrutiny and critique.

Finally we recognise key questions for further research:

I. In what ways can teaching be characterised so that university teachers can gain relevant insights to teaching processes and develop teaching?

II. How can theories of teaching be employed to aid the design and development of teaching?

III. How do/can meanings in teaching relate to students’ mathematical meanings to enable students to gain deeper conceptual meaning in mathematics?
References


