Mathematical modelling of nonlinear ring waves in a stratified fluid

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Mathematical Modelling of Nonlinear Ring Waves in a Stratified Fluid

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A Doctoral Thesis
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Abstract

Oceanic waves registered by satellite observations often have curvilinear fronts and propagate over various currents. In this thesis, we study long linear and weakly-nonlinear ring waves in a stratified fluid in the presence of a depth-dependent horizontal shear flow. It is shown that despite the clashing geometries of the waves and the shear flow, there exists a linear modal decomposition, which can be used to describe distortion of the wavefronts of surface and internal waves, and systematically derive a 2+1-dimensional cylindrical Korteweg-de Vries (cKdV) - type equation for the amplitudes of the waves. The general theory is applied to the case of the waves in a two-layer fluid with a piecewise-constant shear flow, with an emphasis on the effect of the shear flow on the geometry of the wavefronts. The distortion of the wavefronts is described by the singular solution (envelope of the general solution) of the nonlinear first order differential equation, constituting generalisation of the dispersion relation in this curvilinear geometry. There exists a striking difference in the shape of the wavefronts: the wavefront of the surface wave is elongated in the shear flow direction while the wavefront of the interfacial wave is squeezed in this direction. We solve the derived 2+1-dimensional cKdV-type equation numerically using a finite-difference scheme. The effects of nonlinearity and dispersion are studied by considering numerical results for surface and interfacial ring waves generated from a localised source with and without shear flow and the 2D dam break problem. In these examples, the linear and nonlinear surface waves are faster than interfacial waves, the wave height decreases faster at the surface, the shear flow leads to the wave height decreasing slower downstream and faster upstream, and the effect becomes more prominent as the shear flow strengthens.
Key words: ring waves, cylindrical Korteweg - de Vries (cKdV) - type equation, shear flow, two-layer model, singular solution, wavefront, numerical solution, nonlinearity, dispersion
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Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>KdV</td>
<td>Korteweg - de Vries</td>
</tr>
<tr>
<td>cKdV</td>
<td>Cylindrical Korteweg - de Vries</td>
</tr>
<tr>
<td>KP</td>
<td>Kadomtsev - Petviashvili</td>
</tr>
<tr>
<td>1-D</td>
<td>One Dimensional</td>
</tr>
<tr>
<td>2-D</td>
<td>Two Dimensional</td>
</tr>
<tr>
<td>KH instability</td>
<td>Kelvin - Helmholtz instability</td>
</tr>
<tr>
<td>IST</td>
<td>Inverse Scattering Transformation</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary Differential Equation</td>
</tr>
</tbody>
</table>
Contents

List of Figures viii

1 Introduction 1

2 Theoretical background 5
   2.1 Problem formulation 7
   2.2 Derivation of cKdV-type equation for waves in a uniform fluid over a shear flow 11
   2.3 Derivation of cKdV equation for internal waves in a stratified fluid without shear flow 16
   2.4 Concluding remarks 22

3 Ring waves in a stratified fluid over a shear flow 23
   3.1 Derivation of a cKdV-type equation for ring waves in a stratified fluid over a shear flow 24
   3.2 Reduction to the previously studied cases 31
   3.3 Two-layer example: dispersion relation and wavefronts 35
      3.3.1 Two-layer Model 35
      3.3.2 Dispersion relation and approximations 36
      3.3.3 Wavefronts of surface and interfacial ring waves 47
      3.3.4 Critical layer condition 51
      3.3.5 Coefficients of the derived cKdV-type equation 53
   3.4 Concluding remarks 56

4 Numerical method 58
   4.1 Implicit finite-difference method 59
## CONTENTS

4.2 Order of accuracy .......................................................... 63  
4.3 Results in physical coordinate frame ................................. 64  
4.4 Concluding remarks ....................................................... 67  

5 Concentric waves without shear flow — numerical results  
5.1 Linear waves in a two-layer model ................................... 70  
5.2 Nonlinear propagation of concentric waves ........................ 76  
5.2.1 Example 1: concentric wave generated from a localised source  
5.2.1.1 Numerical results for surface wave .......................... 80  
5.2.1.2 Numerical results for interfacial waves ..................... 81  
5.2.1.3 Comparison of surface and interfacial waves .............. 87  
5.2.2 Example 2: the 2D dam break concentric waves .............. 90  
5.2.2.1 Numerical results for surface wave .......................... 92  
5.2.2.2 Numerical results for interfacial waves ..................... 97  
5.3 Concluding remarks ....................................................... 104  

6 Ring waves on a shear flow—numerical results  
6.1 Model initial condition for ring waves on a shear flow .......... 106  
6.2 Numerical results for interfacial waves on a shear flow .......... 108  
6.3 Numerical results for surface waves on a shear flow .......... 113  
6.4 Concluding remarks ....................................................... 116  

7 Conclusions and future work  
7.1 Conclusions ............................................................... 117  
7.2 Future work .............................................................. 119  

Appendix A: MATLAB Code for solving the cKdV-type equation  
Appendix B: Simulation parameters of numerical results  
Bibliography
List of Figures

2.1 Schematic of the problem formulation. ........................................ 8

3.1 Two-layer model. ................................................................. 35

3.2 Function $k(\alpha)$ at $d = 0.5$: surface mode (left) and interfacial mode (right). 42

3.3 Function $k(\alpha)$ at $d = 0.6$: surface mode (left) and interfacial mode (right). 43

3.4 Function $\theta(\alpha)$ at $d = 0.5$: surface mode (left) and interfacial mode (right). 43

3.5 Function $\theta(\alpha)$ at $d = 0.6$: surface mode (left) and interfacial mode (right). 44

3.6 Function $k(\theta)$ at $d = 0.5$: surface mode (left) and interfacial mode (right). 44

3.7 Function $k(\theta)$ at $d = 0.6$: surface mode (left) and interfacial mode (right). 45

3.8 Function $k(\theta)$: rigid-lid approximation (3.47) (blue) and exact solution (3.48) (red) for internal waves; $d = 0.5$ and $U_1 - U_2 = 0.1$. 46

3.9 Function $k(\theta)$: rigid-lid approximation (3.47) (blue) and exact solution (3.48) (red) for internal waves; $d = 0.6$ and $U_1 - U_2 = 0.1$. 46

3.10 Wavefronts of surface ring waves described by $k(\theta) = 1$ when $d = 0.5$
for $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.5$ (green), $U_1 - U_2 = 1$ (blue), and
$U_1 - U_2 = 1.5$ (red). ............................................................ 47

3.11 Wavefronts of surface ring waves described by $k(\theta) = 1$ when $d = 0.6$
for $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.5$ (green), $U_1 - U_2 = 1$ (blue), and
$U_1 - U_2 = 1.5$ (red). ............................................................ 48

3.12 Wavefronts of interfacial ring waves described by $k(\theta) = 1$ when $d = 0.5$
for $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.1$ (green), $U_1 - U_2 = 0.15$ (blue), and
$U_1 - U_2 = 0.2$ (red). ............................................................ 48

3.13 Wavefronts of interfacial ring waves described by $k(\theta) = 1$ when $d = 0.6$
for $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.1$ (green), $U_1 - U_2 = 0.15$ (blue), and
$U_1 - U_2 = 0.2$ (red). ............................................................ 49

3.14 Wave speeds (3.52) as functions of the strength of the shear flow at $d = 0.5$. 50

3.15 Wave speeds (3.52) as functions of the strength of the shear flow at $d = 0.6$. 51
5.1 Localised initial condition .................................................. 76
5.2 Initial values in \((\xi, R)\) coordinates at \(R_0 = 0.1\) ................................ 77
5.3 Concentric surface waves generated from a localised source \((d = 0.5)\) ........................................ 78
5.4 Concentric surface waves generated from a localised source \((d = 0.6)\) ........................................ 79
5.5 Surface waves in the directions \(\theta = 0\) and \(\theta = \pi\) for \(d = 0.5\) when \(t = 0\)
(blue), \(t = 5\) (black), \(t = 10\) (red), \(t = 15\) (green) and \(t = 20\) (cyan). ........................................ 81
5.6 Surface waves in the directions \(\theta = 0\) and \(\theta = \pi\) for \(d = 0.6\) when \(t = 0\)
(blue), \(t = 5\) (black), \(t = 10\) (red), \(t = 15\) (green) and \(t = 20\) (cyan). ........................................ 82
5.7 The linear (blue) and nonlinear (red) surface waves at \(t = 20\) .............................. 82
5.8 Concentric interfacial waves generated from a localised source \((d = 0.5)\) ........................................ 83
5.9 Concentric interfacial waves generated from a localised source \((d = 0.6)\) ........................................ 84
5.10 Interfacial waves in the directions \(\theta = 0\) (downstream) and \(\theta = \pi\) (upstream) for \(d = 0.5\) when \(t = 0\)
(blue), \(t = 20\) (black), \(t = 40\) (red), \(t = 60\) (green) and \(t = 80\) (cyan). ........................................ 85
5.11 Interfacial waves in the directions \(\theta = 0\) (downstream) and \(\theta = \pi\) (upstream) for \(d = 0.6\) when \(t = 0\)
(blue), \(t = 20\) (black), \(t = 40\) (red), \(t = 60\) (green) and \(t = 80\) (cyan). ........................................ 86
5.12 The linear (blue) and nonlinear (red) interfacial waves at \(t = 80\) .............................. 87
5.13 The surface (blue) and interfacial (red) waves at \(t = 20\) .............................. 88
5.14 The surface (blue) and interfacial (red) waves at \(r = 15\) \((R = 0.3)\) .............................. 88
5.15 Initial condition for the 2D dam break problem ........................................ 90
5.16 Linear surface waves at \(y = 0\) for \(d = 0.5\) when \(t = 0\) (blue), \(t = 10\)
(black), \(t = 20\) (red), \(t = 30\) (green) and \(t = 40\) (cyan). ........................................ 91
5.17 Linear surface waves at \(y = 0\) for \(d = 0.6\) when \(t = 0\) (blue), \(t = 10\)
(black), \(t = 20\) (red), \(t = 30\) (green) and \(t = 40\) (cyan). ........................................ 91
5.18 Nonlinear surface waves in the directions \(\theta = 0\) and \(\theta = \pi\) for \(d = 0.5\)
when \(t = 0\) (blue), \(t = 10\) (black), \(t = 20\) (red), \(t = 30\) (green) and \(t = 40\)
(cyan). ........................................ 93
5.19 Nonlinear surface waves in the directions \(\theta = 0\) and \(\theta = \pi\) for \(d = 0.6\)
when \(t = 0\) (blue), \(t = 10\) (black), \(t = 20\) (red), \(t = 30\) (green) and \(t = 40\)
(cyan). ........................................ 94
5.20 The linear (blue) and nonlinear (red) surface waves at \(t = 30\) for \(Q = 40\)
when \(d = 0.5\). ........................................ 95
5.21 Linear interfacial waves at \(y = 0\) for \(d = 0.5\) when \(t = 0\) (blue), \(t = 50\)
(black), \(t = 100\) (red), \(t = 150\) (green) and \(t = 200\) (cyan). ........................................ 96
5.22 Nonlinear interfacial waves in the directions $\theta = 0$ and $\theta = \pi$ when $t = 0$ (blue), $t = 50$ (black), $t = 100$ (red), $t = 150$ (green) and $t = 200$ (cyan) when $d = 0.5$. ........................................... 98
5.23 The linear (blue) and nonlinear (red) interfacial waves at $t = 150$ for $Q = 40$ when $d = 0.5$. ........................................... 99
5.24 Linear interfacial waves in the directions $\theta = 0$ and $\theta = \pi$ for $d = 0.6$ when $t = 0$ (blue), $t = 50$ (black), $t = 100$ (red), $t = 150$ (green) and $t = 200$ (cyan). ........................................... 100
5.25 Nonlinear interfacial waves in the directions $\theta = 0$ and $\theta = \pi$ when $t = 0$ (blue), $t = 50$ (black), $t = 100$ (red), $t = 150$ (green) and $t = 200$ (cyan) when $d = 0.6$. ........................................... 101
5.26 The linear (blue) and nonlinear (red) interfacial waves at $t = 150$ for $Q = 40$ when $d = 0.6$ ........................................... 102

6.1 The initial condition along the shear flow $\theta = 0$ (blue) and against the shear flow $\theta = \pi$ (red) at $R_0 = 0.1$. ........................................... 106
6.2 Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.5$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan) for $U_1 - U_2 = 0.05$. ........................................... 108
6.3 Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.6$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan) for $U_1 - U_2 = 0.05$. ........................................... 109
6.4 Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.5$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan) for $U_1 - U_2 = 0.1$. ........................................... 109
6.5 Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.6$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan) for $U_1 - U_2 = 0.1$. ........................................... 110
6.6 Interfacial waves in the directions $\theta = 0$ (downstream) at $t = 80$ for $U_1 - U_2 = 0$ (red), 0.05 (black) and 0.1 (blue). ........................................... 111
6.7 Interfacial waves in the directions $\theta = \pi$ (upstream) at $t = 80$ for $U_1 - U_2 = 0.05$ (black) and 0.1 (blue). ........................................... 111
6.8 Surface waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.5$ when $t = 0$ (blue), $t = 5$ (black), $t = 10$ (red), $t = 15$ (green) and $t = 20$ (cyan) for $U_1 - U_2 = 0.1$. ........................................... 113
6.9 Surface waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.6$ when $t = 0$ (blue), $t = 5$ (black), $t = 10$ (red), $t = 15$ (green) and $t = 20$ (cyan) for $U_1 - U_2 = 0.1$. ........................................ 114

6.10 Surface waves in the directions $\theta = 0$ (downstream) at $t = 20$ for $U_1 - U_2 = 0$ (red) and 0.1 (blue). ......................................................... 115
Chapter 1

Introduction

The mathematical theory of water waves has been an intriguing and fast developing area in the past two centuries. Most of the recent research was devoted to the study of nonlinear waves including internal waves, supported by density contrast. The Korteweg-de Vries (KdV) equation is the canonical equation for the wave amplitude of long free surface waves with nonlinear and dispersive effects, which leads to the existence of the solitary wave. The solitary wave was first observed by J.S. Russell at the Edinburgh-Glasgow canal [75]. This solitary wave was studied by Boussinesq [15], Rayleigh [74] and finally became known as the KdV soliton after the works by Korteweg - de Vries [50; 83]. In recent years observations and records of internal waves became more frequent with technological improvements and internal waves have been intensively study as well [41]. Many images of coastal oceans internal waves can be found at http://www.internalwaveatlas.com. The classical KdV equation and its generalisations are successfully used to describe long weakly-nonlinear internal waves that are commonly observed in the oceans [10; 34; 35; 36; 37; 39; 59], as well as describing weakly-nonlinear shallow-water surface waves [1; 18]. The waves described by these models have plane or nearly-plane fronts and mainly plane solitons were investigated in experiments [20; 24; 68]. However, waves registered by satellite observations often
have curvilinear fronts. This motivates us to study the waves with curvilinear fronts.

In natural oceanic environments these waves often propagate over currents, for example, tides, wind-drift currents, river flows etc. There are many studies of the effects of shear flow on the waves. For example, the effect of a background shear flow on solitary waves [9], gravity waves in two dimensions [76] and three dimensions [69], internal wave fields [55; 67; 70; 82] and surface ship waves [28] were studied analytically and numerically. We aim to develop an asymptotic theory which could be used to model long ring waves in a stratified fluid over a shear flow.

The cylindrical (or concentric) Korteweg - de Vries (cKdV) equation is a universal weakly-nonlinear weakly-dispersive wave equation in cylindrical geometry. Originally derived in the context of ion-acoustic waves in plasma [60], it was later derived for the surface waves in a uniform fluid, first from Boussinesq equations [63], and then from the full Euler equations [42]. Versions of the equation were also derived for internal waves in a stratified fluid without shear flow [57], and surface waves in a uniform fluid with a shear flow [43]. The original equation is integrable [27], and there exists a useful map between cKdV and KP equations [42; 44; 48], while a generic shear flow leads to a non-integrable cKdV-type equation. The properties of this equation have been studied, including the asymptotic solutions [45] and numerical solutions [51; 61].

The thesis is organised as follows. In Chapter 2 we introduce the theoretical background for the derivation of the cKdV-type equations. We review and re-derive two versions of the cKdV-type equations from the full set of Euler equations for an inviscid, incompressible fluid with free surface and rigid bottom boundary conditions. The first version is the cKdV-type equation for surface waves in a uniform fluid over a shear flow, considered by Johnson [43]. The second version is the cKdV equation for internal waves in a stratified fluid without shear flow, derived by Lipovskii [57].

In Chapter 3, we develop an asymptotic method to derive the cKdV-type equation
for the surface and internal waves in the general case of a stratified fluid with a horizontal shear flow. This is a new version of a 2+1-dimensional cKdV-type equation. It can be simplified to a 1+1-dimensional cKdV-type equation when the shear flow is absent or the fluid is uniform. The two-layer model with a piecewise constant shear flow is used to illustrate the general theory in the case of surface and interfacial waves. The explicit description of wavefronts at the surface and the interface is given and illustrated for different strengths of the shear flows. We derive the explicit expressions for the coefficients of the derived equation at the end of this chapter.

In Chapter 4, we describe the numerical method, used to solve our derived equation numerically. This method is a development of the finite-difference scheme introduced by Feng and Mitsui [30]. Since our derived equation is written in the coordinate system of fast and slow variables, we also will discuss the map between the coordinate systems. The choice of boundary conditions and the domain of independent variables are also discussed.

In Chapter 5 and Chapter 6, we discuss the numerical solutions and the effect of nonlinearity on the surface and interfacial waves in the two-layer model. In Chapter 5, we consider the concentric waves without shear flow with two initial conditions. We use the 2D linear wave equations in this two-layer model to obtain the initial condition for our equation at a fixed radius. The first initial condition is provided by an explicit analytical solution of the 2D linear wave equation [25]. The second initial condition corresponds to a 2D version of the dam break problem, when we solve the linear wave equation numerically. We compare the linear and nonlinear wave solutions at the surface and the interface to illustrate the effect of nonlinearity. In the second example, we vary the initial wave amplitude to illustrate its effect on the subsequent dynamics in the physical variables. Some conclusions are drawn in the end.

In Chapter 6, the general ring waves with shear flow at surface and interface will be studied numerically. We use an artificial wave height equation, which has the same
properties with Dobrokhотов’s results [25] at near field, as initial condition of our derived equation. We will compare the surface and interfacial waves at different shear flows to study the effect of the nonlinearity and the shear flow.

The final chapter summarises the results throughout the thesis, along with some possible directions in which this work can be developed in the future study.
Chapter 2

Theoretical background

The classical cKdV equation,

\[ 2A_R + 3AA_\xi + \frac{1}{3}A_{\xi\xi\xi} + \frac{A}{R} = 0 \]  

(2.1)

is a universal weakly-nonlinear, weakly-dispersive wave equation in cylindrical geometry. Originally derived in the context of ion-acoustic waves in plasma [60], it was later derived for the surface waves in a homogeneous fluid, first from Boussinesq equations [63], and then from the full set of Euler equations [42]. We consider the derivations of two versions of the cKdV-type equations: cKdV-type equation for surface waves in a uniform fluid with a shear flow, which was derived by Johnson [43], and cKdV equation for internal waves in a stratified fluid without shear flow, which was derived by Lipovskii [57].

First, we formulate the problem in the general case of ring waves in a stratified fluid over a shear flow for an inviscid, incompressible fluid, which is considered in the Thesis. Then we reduce the problem to the two versions studied before and use the method of asymptotic multiple-scales expansions to rederive these two versions of the cKdV-type equations. We show that these two versions of the cKdV-type equation
have the 1 + 1-dimensional form

\[ \mu_1 A_R + \mu_2 A_{\xi} + \mu_3 A_{\xi\xi} + \mu_4 \frac{A}{R} = 0, \]

but with different coefficients \( \mu_i \) \((i = 1, 2, 3, 4)\).
2. THEORETICAL BACKGROUND

2.1 Problem formulation

We consider a ring wave propagating in an inviscid incompressible fluid, described by the full set of Euler equations and the equation of mass conservation (see, for example [35]):

\[ \rho (u_t + uu_x + vu_y + wu_z) + p_x = 0, \]  \hspace{1cm} (2.2)

\[ \rho (v_t + uv_x + vv_y + vw_z) + p_y = 0, \]  \hspace{1cm} (2.3)

\[ \rho (w_t + uw_x + vw_y + ww_z) + p_z + \rho g = 0, \]  \hspace{1cm} (2.4)

\[ \rho \tau + u \rho_x + v \rho_y + w \rho_z = 0, \]  \hspace{1cm} (2.5)

\[ u_x + v_y + w_z = 0, \]  \hspace{1cm} (2.6)

with the free surface and rigid bottom boundary conditions appropriate for the oceanic conditions:

\[ w = h_t + uh_x + vh_y \quad \text{at} \quad z = h(x, y, t), \]  \hspace{1cm} (2.7)

\[ p = p_a \quad \text{at} \quad z = h(x, y, t), \]  \hspace{1cm} (2.8)

\[ w = 0 \quad \text{at} \quad z = 0. \]  \hspace{1cm} (2.9)

Here, \( u, v, w \) are the velocity components in \( x, y, z \) directions respectively, \( p \) is the pressure, \( \rho \) is the density, \( g \) is the gravitational acceleration, \( z = h(x, y, t) \) is the free surface location (with \( z = 0 \) at the bottom), and \( p_a \) is the constant atmospheric pressure at the surface. In the basic state \( u_0 = u_0(z), \ v_0 = w_0 = 0, \ p_{0z} = -\rho_0 g, \ h = h_0 \). Here \( u_0(z) \) is a horizontal shear flow in the \( x \)-direction, and \( \rho_0 = \rho_0(z) \) is a stable background density stratification (see Figure 2.1). We assume that perturbations of the basic state are caused only by the propagating waves.

When considering the derivation of the equations for internal waves, it is convenient to use the vertical particle displacement \( \zeta \) as an additional dependent variable,
2. THEORETICAL BACKGROUND

which is defined by the equation

$$\zeta_t + uu_x + vv_y + wz = w,$$  \hspace{1cm} (2.10)

and satisfies the obvious surface boundary condition

$$\zeta = h - h_0 \quad \text{at} \quad z = h(x, y, t),$$  \hspace{1cm} (2.11)

where $h_0$ is the unperturbed depth of the fluid.

We aim to derive an equation for the wave heights of the long surface and internal waves. Thus, we use the following non-dimensional variables:

$$x \rightarrow \lambda x, \quad y \rightarrow \lambda y, \quad z \rightarrow h_0 z, \quad t \rightarrow \frac{\lambda}{c}t,$$

$$u \rightarrow c^* u, \quad v \rightarrow c^* v, \quad w \rightarrow \frac{h_0 c^*}{\lambda} w, \quad (\rho_0, \rho) \rightarrow \rho^*(\rho_0, \rho),$$

$$h \rightarrow h_0 + a\eta, \quad \zeta \rightarrow a\zeta, \quad p \rightarrow p_a + \int_{z}^{h_0} \rho^* \rho_0(s) g \, ds + \rho^* g h_0 p.$$  

Here, $\lambda$ is the wave length, $a$ is the wave height, which is defined as the maximum
displacement from the undisturbed flow. $\rho^*$ is the dimensional reference density of the fluid, while $\rho_0(z)$ is the non-dimensional function describing stratification in the basic state, and $\eta = \eta(x, y, t)$ is the non-dimensional free surface elevation, $c^*$ is the typical long-wave speed of surface or internal waves ($\sqrt{gh_0}$ or $h^*N^*$, respectively, where $N^*$ is a typical value of the buoyancy frequency, and $h^*$ is a typical depth of the stratified layer). In both cases non-dimensionalisation leads to the appearance of the same small parameters in the problem, the amplitude parameter $\varepsilon = a/h_0$ and the wavelength parameter $\delta = h_0/\lambda$. In the second case, a third small (Boussinesq) parameter will appear as well, but the Boussinesq approximation is not used in the subsequent derivation. Thus, it is natural to non-dimensionalise the general problem formulation, including both surface and internal waves, using the parameters of the faster surface waves, and measure the speeds of the internal waves as fractions of the surface wave speed, etc. However, if one is primarily interested in the study of internal waves, it is more natural to use the typical speed of the internal waves.

Introducing the cylindrical coordinate system moving at a constant speed $c$ [43] (the choice of $c$ will be discussed later and a natural choice is the flow speed at the bottom), considering deviations from the basic state (the same notations $u$ and $v$ have been used for the projections of the deviations of the speed on the new coordinate axis), and scaling the appropriate variables using the amplitude parameter $\varepsilon$,

\[
\begin{align*}
    x &\rightarrow ct + r \cos \theta, \quad y \rightarrow r \sin \theta, \quad z \rightarrow z, \quad t \rightarrow t, \\
    u &\rightarrow u_0(z) + \varepsilon(u \cos \theta - v \sin \theta), \quad v \rightarrow \varepsilon(u \sin \theta + v \cos \theta), \\
    w &\rightarrow \varepsilon w, \quad p \rightarrow \varepsilon p, \quad \rho \rightarrow \rho_0 + \varepsilon \rho,
\end{align*}
\]

we arrive at the following non-dimensional problem formulation in the moving cylin-
2. THEORETICAL BACKGROUND

spherical coordinate frame:

\[
\begin{align*}
\left( \rho_0 + \varepsilon \rho \right) \left[ u_t + \varepsilon \left( u u_r + \frac{v}{r} u_r + w u_z - \frac{v^2}{r} \right) + \left( (u_0 - c) u_r + u_{0z} w \right) \cos \theta \\
- (u_0 - c) (u_\theta - v) \frac{\sin \theta}{r} \right] + p_r &= 0, \quad (2.12) \\
\left( \rho_0 + \varepsilon \rho \right) \left[ v_t + \varepsilon \left( u v_r + \frac{v}{r} v_\theta + w v_z + \frac{w}{r} \right) + (u_0 - c) v_r \cos \theta \\
- \left( (u_0 - c) \frac{v_\theta}{r} + \frac{u}{r} \right) + u_{0z} w \right] \sin \theta + \frac{p_\theta}{r} &= 0, \quad (2.13) \\
\delta^2 \left( \rho_0 + \varepsilon \rho \right) \left[ w_t + \varepsilon \left( u w_r + \frac{v}{r} w_\theta + w w_z \right) + (u_0 - c) \left( w_r \cos \theta - \frac{w_\theta \sin \theta}{r} \right) \right] + p_z + \rho &= 0, \quad (2.14) \\
\rho_t + \varepsilon \left( u \rho_r + \frac{v}{r} \rho_\theta + w \rho_z \right) + (u_0 - c) \left( \rho_r \cos \theta - \rho_\theta \frac{\sin \theta}{r} \right) + \rho_{0z} w &= 0, \quad (2.15) \\
u_r + \frac{u}{r} + \frac{v_\theta}{r} + w_z &= 0, \quad (2.16) \\
w &= \eta_t + \varepsilon \left( u \eta_r + \frac{v}{r} \eta_\theta \right) + (u_0 - c) \left( \eta_r \cos \theta - \eta_\theta \frac{\sin \theta}{r} \right) \quad \text{at} \quad z = 1 + \varepsilon \eta, \quad (2.17) \\
\varepsilon \rho = \int_1^{1+\varepsilon \eta} \rho_0(s) ds \quad \text{at} \quad z = 1 + \varepsilon \eta, \quad (2.18) \\
w &= 0 \quad \text{at} \quad z = 0, \quad (2.19)
\end{align*}
\]

With the vertical particle displacement satisfying the following equation and boundary condition:

\[
\begin{align*}
\zeta_t + \varepsilon \left( u \zeta_r + \frac{v}{r} \zeta_\theta + w \zeta_z \right) + (u_0 - c) \left( \zeta_r \cos \theta - \zeta_\theta \frac{\sin \theta}{r} \right) &= w, \quad (2.20) \\
\zeta &= \eta \quad \text{at} \quad z = 1 + \varepsilon \eta. \quad (2.21)
\end{align*}
\]

For the sake of simplicity, in the subsequent derivation we impose the condition \( \delta^2 = \varepsilon \), although this is not the necessary condition. Indeed, variables can be scaled further to replace \( \delta^2 \) with \( \varepsilon \) in the equations [44]. So, the non-dimensional Euler equation and the equation of mass conservation for the most general case have been given. We will add the necessary assumptions to this set of equations (2.12)-(2.21) in the cases without stratification and without shear flow.
2. THEORETICAL BACKGROUND

2.2 Derivation of cKdV-type equation for waves in a uniform fluid over a shear flow

In this section, we review the case of surface waves in a uniform fluid, studied by Johnson [43; 44]. If there is no shear flow, the wave propagates at speed 1 in every direction $\theta$ and the wavefront is $r - t = \text{constant}$. But when there is a shear flow $u_0(z)$ present in $x$ direction, the wave must propagate at different speed in each direction $\theta$. A function $k(\theta)$ is introduced to describe the distortion of the wave front in the presence of a shear flow. Then, to leading order, the wavefront in direction $\theta$ at given time $t$, is described by the equation

$$H(r, \theta, t) = rk(\theta) - t = \text{constant}.$$  

We choose the following set of fast and slow variables

$$\xi = rk(\theta) - t, \quad R = \varepsilon rk(\theta), \quad \theta = \theta,$$

and write the equation (2.12)-(2.19) with the assumption that density is constant. We shall seek the asymptotic solution to these equations by expanding each dependent variable in the form

$$[u, v, w, p, \eta] = [u_1, v_1, w_1, p_1, \eta_1] + \varepsilon [u_2, v_2, w_2, p_2, \eta_2] + O(\varepsilon^2).$$

Here, $\varepsilon$ is a small parameter. In the derivation given by Johnson [43; 44], the vertical particle displacement $\zeta$ is not introduced because in the uniform fluid, only the surface elevation is interested in. Also, equation (2.15) is not present because the density is
2. THEORETICAL BACKGROUND

uniform. The leading order equations are

\[
F_{u_{1\xi}} + u_0 \xi \cos \theta + kp_{1\xi} = 0, \\
F_{v_{1\xi}} - u_0 \xi w_1 \sin \theta + k'p_{1\xi} = 0, \\
p_{1z} = 0, \\
k_{u_{1\xi}} + k'v_{1\xi} + w_1 = 0,
\]

with boundary conditions

\[
w_1 = 0, \quad \text{at } z = 0, \\
p_1 = \eta_1, \quad w_1 = F\eta_{1\xi}, \quad \text{at } z = 1,
\]

where \(F(z, \theta) = -1 + (u_0 - c)(k \cos \theta - k' \sin \theta).\)

One can solve this system of equations and write the dependent variables in terms of \(\eta_1\), which are

\[
u_1 = \left( u_0 \xi \cos \theta (k^2 + k'^2) \int_0^z \frac{1}{F^2} \, dz + \frac{k}{F} \right) \eta_1, \\
v_1 = \left( u_0 \xi \sin \theta (k^2 + k'^2) \int_0^z \frac{1}{F^2} \, dz - \frac{k'}{F} \right) \eta_1, \\
p_1 = \eta_1, \\
w_1 = \eta_{1\xi} (k^2 + k'^2) F \int_0^z \frac{1}{F^2} \, dz,
\]

where the function \(k(\theta)\) satisfies the generalised Burns condition

\[
(k^2 + k'^2) \int_0^1 \frac{1}{F^2} \, dz = 1, \quad (2.22)
\]

which can be reduced to the well known Burns condition [17] by letting \(k(\theta) = 1\) and \(\theta = 0\). The generalised Burns condition is a nonlinear first-order ordinary differential equation (ODE) for the function \(k(\theta)\). The function \(k(\theta)\) can be determined from
2. THEORETICAL BACKGROUND

this equation and we will choose the singular solution (the envelope of the general solution) to satisfy the assumption that the waves propagate outwards everywhere. The singular solution has the form \[ k(\theta) = a \cos(\theta) + b(a) \sin(\theta), \quad (2.23) \]
\[ 0 = \cos \theta + b'(a) \sin \theta, \quad (2.24) \]
\[ (a^2 + b^2(a)) \int_0^1 \frac{dz}{[1 - (u_0(z) - c)a]^2} = 1. \quad (2.25) \]

Indeed, the general solution (2.23) can not describe an outward propagating ring wave since \( k(\theta) \) can not be strictly positive.

At higher order, \( O(\varepsilon) \), we obtain the following set of equations:

\[ Fu_{2\xi} + u_{0\xi} \cos \theta w_2 + kp_{2\xi} + (F + 1)u_{1R} - (u_0 - c) \frac{\sin \theta k}{R} (u_{1\theta} - v_1) + (ku_1 + k'v_1)u_{1\xi} \]
\[ + u_{1\xi} w_1 + kp_{1R} = 0, \quad (2.26) \]
\[ Fv_{2\xi} - u_{0\xi} \sin \theta w_2 + k'p_{2\xi} + (F + 1)v_{1R} - (u_0 - c) \frac{\sin \theta k}{R} (v_{1\theta} + u_1) + (ku_1 + k'v_1)v_{1\xi} \]
\[ + v_{1\xi} w_1 + k'p_{1R} + \frac{k}{R} p_{1\theta} = 0, \quad (2.27) \]
\[ p_{2\xi} + Fw_{1\xi} = 0, \quad (2.28) \]
\[ ku_{2\xi} + k'v_{2\xi} + w_{2\xi} + ku_{1R} + k'v_{1R} + \frac{k}{R} (u_1 + v_{1\theta}) = 0, \quad (2.29) \]

with boundary conditions:

\[ w_2 = 0, \quad \text{at } z = 0, \quad (2.30) \]
\[ p_2 = \eta_2 - \eta_1 p_{1z} \quad \text{at } z = 1, \quad (2.31) \]
\[ w_2 = F\eta_{2\xi} + (F + 1)\eta_{1R} - (u_0 - c) \frac{k \sin \theta}{R} \eta_{1\theta} + (ku_1 + k'v_1)\eta_{1\xi} \]
\[ + u_{0\xi} \eta_1 \eta_{1\xi} (k \cos \theta - k' \sin \theta) - \eta_1 w_{1z} \quad \text{at } z = 1. \quad (2.32) \]
From (2.28) with boundary condition (2.31), one obtains

\[ p_2 = \eta_2 + (k^2 + k'^2)\eta_{1\xi \xi} \int_z^1 F(z_1, \theta)^2 \int_0^{z_1} \frac{1}{F(z_2, \theta)^2} \, dz_2 \, dz_1. \]

Substituting (2.26)×k+(2.27)×k' and \( p_2 \) into equation (2.29), we can obtain an equation involving only \( w_2, \eta_2 \) and \( \eta_1 \), which has the form

\[-F^2 \left( \frac{w_2}{F} \right)_z + (k^2 + k'^2)\eta_{2\xi} = M,\]

where

\[
M = \left( k^2 + k'^2 \right)^2 \int_0^1 F^2 \left( \int_0^z \frac{1}{F^2} \, dz_2 \right) \, dz_1 \eta_{1\xi \xi \xi \xi} \\
+ (k^2 + k'^2)^2 \left( \int_0^z \frac{1}{F^2} \, dz \right) \eta_{1R} \\
+ (k^2 + k'^2)^2 \left( \int_0^z \frac{1}{F^2} \, dz + (F_z - FF_{z\zeta})(\int_0^z \frac{1}{F^2} \, dz)^2 \right) \eta_{1\xi} \\
+ \left( k(k + k') - \frac{k'k'\sin \theta(k + k')}{k \cos \theta - k' \sin \theta} \frac{FF_z}{F} \int_0^z \frac{1}{F^2} \, dz \right) \\
- \frac{2 \sin^2 \theta(k^2 + k'^2)(k + k')}{(k \cos \theta - k' \sin \theta)^2} F F_z \int_0^z \frac{1}{F^2} \, dz \\
+ \frac{2kk'(k + k') \sin \theta F + 1}{k \cos \theta - k' \sin \theta} \\
+ \frac{k(k^2 + k'^2) \sin \theta (F + 1)}{(k \cos \theta - k' \sin \theta)^2} \\
+ \frac{2kk'(k + k') \sin \theta}{k \cos \theta - k' \sin \theta} \int_0^z \frac{1}{F^2} \, dz \\
+ \frac{2k(k^2 + k'^2)(k + k') \sin^2 \theta}{(k \cos \theta - k' \sin \theta)^2} F F_z \int_0^z \frac{1}{F^2} \, dz \\
+ \frac{k \sin \theta (k^2 + k'^2)}{k \cos \theta - k' \sin \theta} \frac{FF_z}{F} \int_0^z \frac{1}{F^2} \, dz \\
+ \frac{k \sin \theta (k^2 + k'^2)}{k \cos \theta - k' \sin \theta} \int_0^z \frac{1}{F^2} \, dz \frac{\eta_{1\theta}}{R}.\]

From the above equation with the boundary condition (2.30), the solution for \( w_2 \) has
the form

\[ w_2 = (k^2 + k^2)\eta_2 F \int_0^z \frac{1}{F^2} \, dz - F \int_0^z \frac{M}{F^2} \, dz. \]

So, at \( z = 1 \), from the boundary condition (2.32) and using the generalised Burns condition (2.22), one obtains

\[
\begin{align*}
  w_2 &= F\eta_{2\xi} - F \int_0^1 \frac{M}{F^2} \, dz \\
  &= F\eta_{2\xi} + Fz\eta_{1\xi} + (F + 1)\eta_{1R} - (u_0 - c) k \sin \theta R \eta_{10} + (ku_1 + k'v_1)\eta_{1\xi}.
\end{align*}
\]

Then the equation for the dominant behaviour of the free surface elevation \( \eta \) (i.e. \( \eta_1 \), and here \( A = \eta_1 \) to make the dependent variable consistent with the general cKdV type equation that will be derived in Chapter 3) can be written as

\[ \mu_1 A_R + \mu_2 AA_\xi + \mu_3 A_{\xi\xi\xi} + \mu_4 \frac{A}{R} + \mu_5 \frac{A}{R^3} = 0, \tag{2.33} \]

where the coefficients are given by the following formulae:

\[
\begin{align*}
  \mu_1 &= 2(k^2 + k^2)I_3, \tag{2.34} \\
  \mu_2 &= -3(k^2 + k^2)^2 I_4, \tag{2.35} \\
  \mu_3 &= -(k^2 + k^2)^2 \int_0^1 \frac{1}{F^2} \int_0^1 \int_0^{z_2} \int_0^{z_3} \frac{1}{F^2} \, dz_3 \, dz_2 \, dz_1, \tag{2.36} \\
  \mu_4 &= -\frac{k(k + k'')}{k \cos \theta - k' \sin \theta} \left((k \cos \theta + 3k' \sin \theta)I_2 + 4k' \sin \theta I_3\right) \\
  &\quad -\frac{3(k(k + k''))(k^2 + k^2) \sin^2 \theta}{(k \cos \theta - k' \sin \theta)^2} (I_2 + 2I_3 + I_4), \tag{2.37} \\
  \mu_5 &= \frac{2k}{k \cos \theta - k' \sin \theta} \left(k \sin \theta + k' \cos \theta\right)I_2 + (k^2 + k^2) \sin \theta I_3. \tag{2.38}
\end{align*}
\]

where

\[ I_n = \int_0^1 \frac{dz}{F^n}. \]
2.3 Derivation of cKdV equation for internal waves in a stratified fluid without shear flow

In this section, we review the derivation of the cKdV equation for internal waves in a stratified fluid without shear flow given by Lipovskii [57]. Internal waves in the absence of a shear flow are concentric. The solution of the linearised problem allows for a modal decomposition similar to the well known result in the Cartesian geometry [35]. Thus, the fast and slow variables are defined by

\[ \zeta = r - st, \quad R = \varepsilon r, \]

where \( s \) is the internal long wave speed. When we non-dimensionalise the variables, we choose the typical long wave speed \( c^* \) equal to the surface wave speed \( \sqrt{gh_0} \) and measure the speeds of the internal waves as fractions of the surface wave speed. If we choose \( c^* = h^* N^* \), where \( N^* \) is a typical value of the buoyancy frequency, and \( h^* \) is a typical depth of the stratified layer, then \( s = 1 \) here. We look for a solution of this problem in the form of asymptotic multiple-scales expansions of the form

\[ \zeta = \zeta_1 + \varepsilon \zeta_2 + O(\varepsilon^2), \]

and similar expansions for other dependent variables, where

\[ \zeta_1 = A(\xi, R)\phi(z). \]
When there is no shear flow, we assume \( u_0 = c = 0 \) in the set of equations (2.12)-(2.21), and the leading order of equations are

\[
\begin{align*}
-s\rho_0 u_{1\xi} + p_{1\xi} &= 0, \\
-s\rho_0 v_{1\xi} &= 0, \\
p_{1z} + \rho_1 &= 0, \\
-s\rho_{1\xi} + \rho_{0z} w_1 &= 0, \\
u_{1\xi} + w_{1z} &= 0, \\
-s\zeta_{1\xi} &= w_1,
\end{align*}
\]

with the boundary conditions given by

\[
\begin{align*}
w_1 &= 0, \quad \text{at } z = 0, \\
p_1 &= \rho_0 \eta_1 \quad w_1 = -s\eta_{1\xi} \quad \zeta_1 = \eta_1 \quad \text{at } z = 1.
\end{align*}
\]

The leading order solution can be written in terms of \( A(\xi, R) \) and \( \phi(z) \)

\[
\begin{align*}
u_1 &= sA\phi_z, \\
v_1 &= 0, \\
w_1 &= -sA_\xi \phi, \\
p_1 &= s^2 \rho_0 A\phi_z, \\
\rho_1 &= -\rho_{0z} A\phi, \\
\eta_1 &= A\phi, \quad \text{at } z = 1,
\end{align*}
\]
where the function \( \phi(z) \) satisfies the modal equations

\[
\begin{align*}
  s^2(\rho_0 \phi_z)_z - \rho_0 \phi_z &= 0, & \text{at } 0 \leq z \leq 1, \\
  \phi &= 0, & \text{at } z = 0, \\
  \phi &= s^2 \phi_z, & \text{at } z = 1.
\end{align*}
\] (2.39)

The modal equations have the same form as those in Cartesian coordinates [35]. The modal equations (2.39)-(2.41) are discussed in [33; 65] (see also references therein).

At higher order, \( O(\varepsilon) \), we obtain the following set of equations:

\[
\begin{align*}
  -s \rho_0 u_2 \xi + p_2 \xi + \rho_0 (u_1 u_1 \xi + w_1 u_1 z) - s \rho_1 u_1 \xi + p_{1R} &= 0, \\
  -s \rho_0 v_2 \xi + \rho_0 (u_1 v_1 \xi + w_1 v_1 z) - s \rho_1 v_1 \xi + \frac{p_{1\theta}}{R} &= 0, \\
  p_2z + p_2 - s \rho_0 w_1 \xi &= 0, \\
  \rho_0 w_2 - s \rho_2 \xi + u_1 p_1 \xi + w_1 p_1 z &= 0, \\
  u_2 \xi + w_2z + u_{1R} + \frac{v_{1\theta}}{R} + \frac{u_1}{R} &= 0, \\
  w_2 + s \zeta_2 \xi - u_1 \zeta_1 \xi - w_1 \zeta_1 z &= 0.
\end{align*}
\] (2.42)

and boundary conditions

\[
\begin{align*}
  w_2 &= 0, & \text{at } z = 0, \\
  p_2 &= \rho_0 \eta_2 - p_{1z} \eta_1 + \frac{1}{2} \rho_0 z \eta_1^2 & \text{at } z = 1, \\
  w_2 &= -s \eta_2 \xi + u_1 \eta_1 \xi - \eta_1 w_1 z & \text{at } z = 1, \\
  \zeta_2 &= \eta_2 - \eta_1 \zeta_1 z & \text{at } z = 1.
\end{align*}
\] (2.48)

Equation (2.47) yields

\[
  w_2 = -s \zeta_2 \xi - sAA_\xi \phi \phi_z + sAA_\xi \phi \phi_z = -s \zeta_2 \xi,
\] (2.52)
Substituting (2.42) into (2.46) to eliminate $u_1$, one obtains

$$p_{2z} = -sp_0w_2 - p_0s^2AA_x\dot{\phi}_z^2 + s^2\rho_0AA_x\dot{\phi}_z^2 - s^2\rho_0AA_x\ddot{\phi}_z - 2s^2\rho_0A_R\dot{\phi}_z - \frac{s^2\rho_0A\dot{\phi}_z}{R}.$$  
(2.53)

On the other hand, we can obtain the equation for $p_2$ and $w_2$ by eliminating $\phi$ from (2.44) and (2.45)

$$sp_{2z\xi} = -\rho_0z^2 - s^3\rho_0A_\xi\phi\dot{\phi} - s\rho_0zAA_x\phi^2.$$  
(2.54)

Equating the expressions for $p_{2z\xi}$ from (2.53) and (2.54) and substituting the expression for $w_2$ from (2.52), we can obtain the equation for $\zeta$ in the form

$$s^2(p_0\zeta_{2z\xi})_z - p_0\zeta_{2z} = M,$$  
(2.55)

where

$$M = -s^2\rho_0\phi A_\xi + s^2(\rho_0\phi^2)eA_\xi + s^2(\rho_0\phi^2)eA_\xi + s^2(\rho_0\phi^2)eA_\xi.$$  

From the expression for $w_2$ in equation (2.52) with the boundary condition (2.48), one obtains

$$\zeta_{2z} = 0 \quad \text{at } z = 0.$$  
(2.56)

From (2.49) and (2.51), one gets

$$p_2 = \rho_0\zeta + \rho_0A^2\phi\dot{z} - s^2A^2p_0\phi\dot{z} + \frac{1}{2}\rho_0A^2\phi^2,$$

which is used to eliminate $p_2$ in (2.53), then we have

$$s^2p_0\zeta_{2z\xi} - p_0\zeta_{2z} = N \quad \text{at } z = 1,$$  
(2.57)
where
\[ N = 2s^2 \rho_0 \phi_z A_R + s^2 \rho_0 \phi_z \frac{A}{R} + (3 \rho_0 s^2 \phi_z^2 - 2s^2 \rho_0 \phi_{z\xi}) A A_{\xi}. \]

The coefficients of \( N \) are simplified by using (2.41). We can integrate the equation for \( \zeta_2 \) (2.55) by parts twice
\[
\int_0^1 M \phi \, dz = \int_0^1 \left( s^2 (\rho_0 \zeta_{2\xi})_\xi - \rho_0 \zeta_{2\xi} \right) \phi \, dz
\]
\[
= s^2 \rho_0 \zeta_{2\xi} \phi |_{z=0}^{1} - s^2 \int_0^1 \rho_0 \zeta_{2\xi} \phi_z \, dz - \int_0^1 \rho_0 \zeta_{2\xi} \phi \, dz
\]
\[
= s^2 \rho_0 \zeta_{2\xi} \phi |_{z=0}^{1} - s^2 \phi_z \rho_0 \zeta_{2\xi} |_{z=0}^{1} + s^2 \int_0^1 \zeta_{2\xi} (\rho_0 \phi_z) \, dz - \int_0^1 \rho_0 \zeta_{2\xi} \phi \, dz.
\]

Substituting the modal equations (2.39)-(2.41) and boundary conditions for \( \zeta_2 \) (2.56),(2.57), the above equation can be reduced to
\[
\int_0^1 M \phi \, dz = s^2 \rho_0 \zeta_{2\xi} \phi |_{z=0}^{1} - s^2 \phi_z \rho_0 \zeta_{2\xi} |_{z=0}^{1} + s^2 \int_0^1 \zeta_{2\xi} (\rho_0 \phi_z) \, dz - \int_0^1 \rho_0 \zeta_{2\xi} \phi \, dz
\]
\[
= \left( s^2 \rho_0 \phi \zeta_{2\xi} - s^2 \rho_0 \phi_z \zeta_{2\xi} \right) |_{z=1}^{1} + \int_0^1 \rho_0 \phi_z \zeta_{2\xi} \, dz - \int_0^1 \rho_0 \phi \zeta_{2\xi} \, dz
\]
\[
= \left( (s^2 \rho_0 \phi \zeta_{2\xi} - \rho_0 \zeta_{2\xi}) \phi \right) |_{z=1}^{1}
\]
\[
= (N \phi) |_{z=1}^{1}. \]

The compatibility condition is given by
\[
\int_0^1 M \phi \, dz - (N \phi) |_{z=1}^{1} = 0.
\]

From the compatibility condition we can obtain the cKdV equation for the function in form \( A \)
\[
\mu_1 A_R + \mu_2 A A_\xi + \mu_3 A_{\xi\xi\xi} + \mu_4 \frac{A}{R} = 0,
\]
(2.58)
where the coefficients are given by

\[ \mu_1 = 2 \int_0^1 \rho_0 \phi_z^2 \, dz, \quad \mu_2 = 3 \int_0^1 \rho_0 \phi_z^3 \, dz, \]
\[ \mu_3 = \int_0^1 \rho_0 \phi^2 \, dz, \quad \mu_4 = \int_0^1 \rho_0 \phi_z^2 \, dz. \]
2. THEORETICAL BACKGROUND

2.4 Concluding remarks

In this chapter, we have first formulated the full set of Euler equations and equation of mass conservation in the general case, when ring waves propagate in a stratified fluid over a shear flow, with the free surface and rigid bottom boundary conditions. We non-dimensionalised the variables and mapped them into a moving cylindrical coordinate frame.

Then we used asymptotic multiple-scales expansions to rederive the cKdV-type equation in the two cases, which have been considered before, assuming that either the fluid is uniform, i.e. $\rho_0$ is constant, or the shear flow is not present, i.e. $u_0 = 0$. We obtained the cKdV-type wave amplitude equations (2.33) and (2.58), respectively, which coincide with the results obtained by Johnson [43; 44] and Lipovskii [57]. In the case of surface ring waves over a shear flow, the amplitude equation is $2 + 1$-dimensional cKdV-type equation. In the case of concentric waves over a stratified fluid, the amplitude equation is $1 + 1$-dimensional cKdV-type equation.
Chapter 3

Ring waves in a stratified fluid over a shear flow

In this chapter, we study the propagation of internal and surface ring waves in a stratified fluid over a prescribed shear flow, generalising the previous studies. We firstly derive a 2+1-dimensional cKdV-type model for the wave heights of surface and internal waves, by finding an appropriate linear modal decomposition (different from the known modal decomposition in Cartesian coordinates) and techniques from the asymptotic multiple-scales analysis. Then we use the derived 2+1-dimensional cKdV-type equation to recover the two 1+1-dimensional cKdV-type equations in the simplified cases, discussed in Chapter 2. We use a two-layer model with a piecewise constant shear flow to illustrate the general theory. We show how to determine the wave speed without shear flow, the distortion function for the wavefronts and the modal functions. We also obtain conditions guaranteeing that there are no critical levels, and calculate explicit expressions for the coefficients of the derived wave amplitude equations.
3.1 Derivation of a cKdV-type equation for ring waves in a stratified fluid over a shear flow

We have formulated the general problem in Chapter 2. It is given by the system of equations (2.12)-(2.21), including the full set of Euler equation, the equation of mass conservation and the equation for the vertical particle displacement with the free surface and rigid bottom boundary conditions. We choose the set of fast and slow variables similarly to the case of surface ring waves with shear flow

\[ \xi = rk(\theta) - st, \quad R = \varepsilon rk(\theta), \quad \theta = \theta, \]

where we have added the parameter \( s \), the wave speed in the absence of a shear flow. The function \( k(\theta) \) describes the distortion of the wavefront in the presence of a shear flow and has to be determined. To leading order, the wavefront at any fixed moment of time \( t \) is described by the equation

\[ h(r, \theta, t) = rk(\theta) - st = \text{constant}. \]

We seek the asymptotic solution to this problem by expanding the dependent variables in the form

\[ [u, v, w, p, \eta, \zeta] = [u_1, v_1, w_1, p_1, \eta_1, \zeta_1] + \varepsilon[u_2, v_2, w_2, p_2, \eta_2, \zeta_2] + O(\varepsilon^2), \]
where \( \varepsilon \) is a small parameter. At leading order the equations have the form

\[
\begin{align*}
\rho_0 (F_{u_{\xi}} + \cos \theta_0 w_1) + k p_{1\xi} &= 0, \\
\rho_0 (F_{v_{\xi}} - \sin \theta_0 w_1) + k' p_{1\xi} &= 0, \\
p_{1\xi} + \rho_1 &= 0, \\
F p_{1\xi} + \rho_0 w_1 &= 0, \\
k u_{1\xi} + k' v_{1\xi} + w_{1z} &= 0, \\
F \zeta_{1\xi} &= w_1,
\end{align*}
\]

with boundary conditions:

\[
\begin{align*}
w_1 &= 0 \quad \text{at } z = 0, \\
p_1 &= \rho_0 \eta_1 \quad \text{at } z = 1, \\
w_1 &= F \eta_{1\xi} \quad \text{at } z = 1, \\
\zeta_1 &= \eta_1 \quad \text{at } z = 1,
\end{align*}
\]

where \( F(\theta, z) = -s + (u_0 - c)(k \cos \theta - k' \sin \theta) \).

We anticipate that the solution of the linearised problem allows for a modal decomposition, similar to the well-known result in the Cartesian geometry and the case of internal waves without shear flow in cylindrical geometry \( \zeta_1 = A(\xi, R)\phi(z) \), but expect that it has more complicated structure for the ring waves over a shear flow because of the loss of the radial symmetry in the problem formulation. Therefore the modal function \( \phi \) and amplitude function \( A \) should depend on \( \theta \). The modal decomposition is defined as

\[
\zeta_1 = A(\xi, R, \theta)\phi(z, \theta). \tag{3.1}
\]

We can solve this set of leading order of equations and write the leading order of
dependent variables in terms of the modal function \( \phi(z, \theta) \) and the amplitude function \( A(\xi, R, \theta) \):

\[
\begin{align*}
 u_1 &= -A\phi u_0 z \cos \theta - \frac{kF}{k^2 + k' c^2} A\phi_z, \\
 v_1 &= A\phi u_0 z \sin \theta - \frac{k'F}{k^2 + k' c^2} A\phi_z, \\
 w_1 &= A\phi, \\
 p_1 &= \frac{\rho_0}{k^2 + k' c^2} AF^2 \phi_z, \\
 \rho_1 &= -\rho_0 A\phi, \\
 \eta_1 &= A\phi \quad \text{at} \quad z = 1.
\end{align*}
\]

Here the modal function \( \phi(z, \theta) \) satisfies the following modal equations:

\[
\begin{align*}
 \left( \frac{\rho_0 F^2}{k^2 + k' c^2} \phi_z \right)_z - \rho_0 c^2 \phi &= 0, \quad (3.8) \\
 \phi &= 0 \quad \text{at} \quad z = 0, \quad (3.9) \\
 \frac{F^2}{k^2 + k' c^2} \phi_z - \phi &= 0 \quad \text{at} \quad z = 1, \quad (3.10)
\end{align*}
\]

where we now have fixed the speed of the moving coordinate frame \( c \) to be equal to the speed of the shear flow at the bottom, \( c = u_0(0) \) (then, \( F = -s \neq 0 \) at \( z = 0 \), and the condition \( F\phi = 0 \) at \( z = 0 \) implies (3.9)). The values of the wave speed \( s \) in the absence of the shear flow, and the pair of functions \( \phi(z, \theta) \) and \( k(\theta) \) constitute solution of the modal equations (3.8) - (3.10). Note that the wave speed \( s \) and the function \( k(\theta) \) will be determined from some generalisation of the ‘dispersion relation’, and examples will be discussed in the subsequent section.
At $O(\varepsilon)$ we obtain the following set of equations:

\[ \rho_0 (F u_{\xi} + u_0 w_2 \cos \theta) + \rho_1 (F u_{\xi} + u_0 w_1 \cos \theta) + \rho_0 [(F + s)u_{1R} + (k u_1 + k' v_1)u_{1\xi} - (u_0 - c)(u_{1\theta} - v_1)k \frac{R}{\sin \theta + u_{1z}w_1} + k(p_{2\xi} + p_{1R}) = 0, \] (3.11)

\[ \rho_0 (F v_{2\xi} - u_0 w_2 \sin \theta) + \rho_1 (F v_{1\xi} - u_0 w_1 \sin \theta) + \rho_0 [(F + s)v_{1R} + (k u_1 + k' v_1)v_{1\xi} - (u_0 - c)(v_{1\theta} + u_1)k \frac{R}{\sin \theta} + v_{1z}w_1] + \frac{k' p_{2\xi} + p_{1R}}{R} = 0, \] (3.12)

\[ p_{2z} + p_2 + \rho_0 F w_{1\xi} = 0, \] (3.13)

\[ Fp_{2\xi} + \rho_0 w_2 + (F + s)p_{1R} - (u_0 - c)k \frac{R}{\rho_{1\theta}} \rho_{1\xi} \sin \theta + (k u_1 + k' v_1)p_{1\xi} + w_{1}p_{1z} = 0, \] (3.14)

\[ ku_{2\xi} + k' v_{2\xi} + w_{2z} + ku_{1R} + k \frac{v_{1\theta} + u_1}{1} + k' v_{1R} = 0, \] (3.15)

\[ Fz_{2\xi} - w_{2} + (F + s)z_{1R} - (u_0 - c)k \frac{R}{z_{1\theta}} z_{1\xi} \sin \theta + (k u_1 + k' v_1)z_{1\xi} + w_{1}z_{1z} = 0, \] (3.16)

and boundary conditions:

\[ w_2 = 0 \quad \text{at} \quad z = 0, \] (3.17)

\[ p_2 = \rho_0 \eta_2 + \frac{1}{2} \rho_0 \eta_1^2 - \eta_1 p_{1z} \quad \text{at} \quad z = 1, \] (3.18)

\[ w_2 = F \eta_{2\xi} + (F + s)\eta_{1R} - (u_0 - c)k \frac{R}{\eta_{1\theta}} \eta_{1\xi} \sin \theta + (k u_1 + k' v_1)\eta_{1\xi} + F_2 \eta_{1\xi} \eta_{1z} - \eta_1 w_{1z} \quad \text{at} \quad z = 1, \] (3.19)

\[ z_2 = \eta_2 - \eta_1 z_{1z} \quad \text{at} \quad z = 1. \] (3.20)

By substituting the leading order solutions (3.2)-(3.6), equation (3.16) yields

\[ w_2 = F z_{2\xi} + (F + s)A_{R\phi} - (u_0 - c)k \frac{R}{\sin \theta} (A_{\phi})_0 - F z_{2\phi}^2 A A_{\xi}. \] (3.21)

Next, we find $u_2\xi$ from (3.11) and $v_2\xi$ from (3.12) and substitute them into (3.15),
obtaining the following equation:

\[-(k^2 + k'^2)p_{2z} + \rho_0(Fw_{2z} - F_z w_2) = \rho_0 \left\{ -\frac{k}{R} F(v_{1\theta} + u_1) - (u_0 - c) \frac{k}{R} \sin \theta [k(u_{1\theta} - v_1) + k'
\right.

\[+ k'(v_{1\theta} + u_1)] + s(ku_{1R} + k'v_{1R}) + \frac{1}{2} [((ku_1 + k'v_1)^2]_x + (ku_1 + k'v_1)z w_1 \right\} - \rho_1(Fw_{1z} - F_z w_1) + (k^2 + k'^2)p_{1z} + \frac{kk'}{R} p_{1\theta}. \tag{3.22}\]

On the other hand, finding \( \rho_2 \) from (3.13) and substituting it into (3.14) we get

\[ F p_{2z\xi} - \rho_{2z} w_2 = \]

\[-\rho_0 k^2 w_{1\xi \xi} + (F + s) \rho_{1R} - (u_0 - c) \frac{k}{R} \rho_{1\theta} \sin \theta + (ku_1 + k'v_1) \rho_{1\xi} + \rho_{1z} w_1. \tag{3.23}\]

Equating the expressions for \( p_{2z\xi} \) from the equations (3.22) and (3.23), using (3.21) to exclude \( w_2 \) and substituting the leading order solution (3.2) - (3.7) and modal equation (3.8), we obtain the equation for \( \zeta_2 \) in the form

\[ \left( \frac{\rho_0 F^2}{k^2 + k'^2} \zeta_{2\xi \xi} \right)_z - \rho_{0z} \zeta_{2\xi} = M_2, \tag{3.24}\]

where

\[-(k^2 + k'^2)M_2 = 2s(\rho_0 F \phi_z)_z A R + (k^2 + k'^2) \rho_0 F^2 \phi A_{\xi \xi \xi} + \left\{ \rho_0 \left[ k(k + k') \left[ \frac{F^2}{k^2 + k'^2} - \left( \frac{2k'}{k^2 + k'^2} + (u_0 - c) \sin \theta \right)^2 \right] \phi_x \right. \right.

\[+ 2kF \left( \frac{k'}{k^2 + k'^2} + (u_0 - c) \sin \theta \right) \phi_x \right\}_z A \frac{\theta}{R} - \left\{ 2\rho_0 kF \left[ \frac{k'}{k^2 + k'^2} + (u_0 - c) \sin \theta \right] \phi_x \right\}_z A \frac{\theta}{R}. \]

Next, substituting (3.21) into the boundary condition (3.17) and recalling that \( \phi|_{z=0} = 0 \), we obtain \( F \zeta_{2\xi} = 0 \) at \( z = 0 \), implying that

\[ \zeta_{2\xi} = 0 \quad \text{at} \quad z = 0, \tag{3.25}\]
since \( F = -s \) at \( z = 0 \) by our choice of the constant \( c \). The boundary condition (3.20) implies
\[
\eta_2 = \zeta_2 + A^2 \phi \phi_z \quad \text{at} \quad z = 1. \tag{3.26}
\]
Substituting (3.26) into (3.18) we get
\[
p_2 = \rho_0 (\zeta_2 + A^2 \phi \phi_z) - \frac{A^2 \phi}{k^2 + k'^2} (\rho_0 F^2 \phi_z + 2 \rho_0 F F_z \phi_z + \rho_0 F^2 \phi_{zz}) + \frac{1}{2} \rho_0 A^2 \phi^2 \quad \text{at} \quad z = 1. \tag{3.27}
\]
Substituting (3.21) and leading order solutions (3.2)-(3.6) into the boundary condition (3.19), one obtains
\[
F \zeta_{2z} - 2 F \phi \phi_z \phi_z = F \eta_{2z} - 2 A \phi \phi (F \phi)_{zz},
\]
which implies the same boundary condition as (3.26).

Differentiating (3.27) with respect to \( \xi \), using (3.22) to eliminate \( p_{2z} \), and using (3.19) to exclude \( w_2 \) (all at \( z = 1 \)), we obtain
\[
\rho_0 \left[ \frac{F^2}{k^2 + k'^2} \zeta_{2z} - \zeta_{2z} \right] = N_2 \quad \text{at} \quad z = 1, \tag{3.28}
\]
where
\[
-(k^2 + k'^2) N_2 = 2 s \rho_0 F \phi_z A_R + [-3 \rho_0 F^2 \phi_z^2 + 2 \rho_0 F^2 \phi_{zz}] A A_{\xi} + 2 k F \left[ \frac{k' F}{k^2 + k'^2} + (u_0 - c) \sin \theta \right] \phi_z + 2 k F \left[ \frac{k' F}{k^2 + k'^2} + (u_0 - c) \sin \theta \right] \phi_{z \theta} A_R - 2 \rho_0 k F \left[ \frac{k' F}{k^2 + k'^2} + (u_0 - c) \sin \theta \right] \phi_z A \phi_{z \theta}.
\]
Thus, we obtain the non-homogeneous equation (3.24) for the function \( \zeta_{2z} \) with the boundary conditions (3.25), (3.28).

We can derive the compatibility condition from the problem for \( \zeta_2 \) by using inte-
gration by parts and modal equations (3.8)-(3.10):

\[
\int \left( \frac{\rho_0 F^2}{k^2 + k'^2} \xi \varphi - \rho_0 \xi \right) \varphi \, dz \\
= \left( \frac{\rho_0 F^2}{k^2 + k'^2} \xi \varphi \right)_{z=0}^{1} - \int_{0}^{1} \frac{\rho_0}{k^2 + k'^2} \varphi \xi \, dz \\
= \left( \rho_0 \left[ \frac{F^2}{k^2 + k'^2} \xi - \xi \right] \varphi \right)_{z=1}^{0},
\]

which implies

\[
\int_{0}^{1} M_2 \varphi \, dz - [N_2 \varphi]_{z=1} = 0.
\]

The compatibility condition yields the 2+1-dimensional evolution equation for the slowly varying amplitude of the ring wave in the form

\[
\mu_1 \frac{A_{xx}}{r} + \mu_2 A_{xx} + \mu_3 A_{xxx} + \frac{\mu_4 A}{R} + \frac{\mu_5 A_0}{R} = 0,
\]

where the coefficients are given in terms of solutions of the modal equations (3.8) - (3.10) by the following formulae:

\[
\mu_1 = 2s \int_{0}^{1} \rho_0 F \phi_z^2 \, dz, \tag{3.30}
\]

\[
\mu_2 = -3 \int_{0}^{1} \rho_0 F^2 \phi_z \, dz, \tag{3.31}
\]

\[
\mu_3 = -(k^2 + k'^2) \int_{0}^{1} \rho_0 F^2 F' \, dz, \tag{3.32}
\]

\[
\mu_4 = - \int_{0}^{1} \left\{ \frac{\rho_0 \phi_z^2 k(k + k''')}{(k^2 + k'^2)^2} \left( (k^2 - 3k'^2)F^2 - 4k'(k^2 + k'^2)(u_0 - c) \sin \theta F \\
- \sin^2 \theta (u_0 - c)^2(k^2 + k'^2)^2 \right) + \frac{2\rho_0 k}{k^2 + k'^2} F' \phi_z \phi_{zz} (k'F + (k^2 + k'^2)(u_0 - c) \sin \theta) \right\} \, dz, \tag{3.33}
\]

\[
\mu_5 = - \frac{2k}{k^2 + k'^2} \int_{0}^{1} \rho_0 F \phi_z^2 \left[ k'F + (u_0 - c)(k^2 + k'^2) \sin \theta \right] \, dz. \tag{3.34}
\]
3. RING WAVES IN A STRATIFIED FLUID OVER A SHEAR FLOW

3.2 Reduction to the previously studied cases

In this section, we revisit the two particular cases reviewed in Chapter 2, and obtain the relevant cKdV-type equations as reductions of our general model.

When considering the case of surface ring waves in a homogeneous fluid, \( \rho_0 \) is a constant and we normalise \( \phi \) by setting \( \phi = 1 \) at \( z = 1 \). In this case, the wave speed \( s \) in the absence of a shear flow, function \( k(\theta) \) and the modal function \( \phi \) can be easily found from the modal equations (3.8) - (3.10). Indeed, the modal function \( \phi \) is given by

\[
\phi = (k^2 + k'^2) \int_0^1 \frac{1}{F^2} \, dz.
\]

When there is no shear flow, \( k = 1, \ u_0(z) = c = 0 \), and

\[
\int_0^1 \frac{1}{s^2} \, dz = 1 \quad \Rightarrow \quad s^2 = 1.
\]

Thus, the wave speed of the outward propagating wave in the absence of a shear flow is equal to 1 and \( F = -1 + (u_0 - c)(k \cos \theta - k' \sin \theta) \). The function \( k(\theta) \) satisfies the generalised Burns condition:

\[
(k^2 + k'^2) \int_0^1 \frac{1}{F^2} \, dz = 1,
\]

which is the same as (2.22) but now it is obtained directly by substituting the modal function into the normalisation condition \( \phi = 1 \) at \( z = 1 \). Here the generalised Burns condition, the ODE for the function \( k(\theta) \) is the same, so the singular solution for \( k(\theta) \) is the same.
In Johnson’s result \cite{43, 44}, the coefficient $\mu_5$, in front of $A_0/R$ is
\begin{align*}
\mu_5 &= -\frac{2k}{k \cos \theta - k' \sin \theta} \int_0^1 \frac{k(k \sin \theta + k' \cos \theta)F + (k^2 + k'^2) \sin \theta}{F^3} \, dz \\
&= -2k \int_0^1 \left( \frac{k^2 \sin \theta + k'(k \cos \theta - k' \sin \theta) + k^2 \sin \theta}{(k \cos \theta - k' \sin \theta)F^2} + \frac{(u_0 - c)(k^2 + k'^2) \sin \theta}{(F + 1)F^3} \right) \, dz \\
&= -2k \int_0^1 \left( \frac{k'F}{F^2} + \frac{(u_0 - c)(k^2 + k'^2) \sin \theta}{(F + 1)F^3} \right) \, dz.
\end{align*}

Differentiating the generalised Burns condition (3.36), one obtains
\[2(k + k'') \int_0^1 \frac{k'F + (u_0 - c)(k^2 + k'^2) \sin \theta}{F^3} \, dz = 0.\]

Since $k + k'' \neq 0$ in the singular solution, we have
\[\mu_5 = -2k \int_0^1 \frac{k'F + (u_0 - c)(k^2 + k'^2) \sin \theta}{F^3} \, dz = 0.\]

Thus, the amplitude equation reduces to the form of the 1+1 - dimensional cKdV-type equation (i.e. $\mu_5 = 0$) for any shear flow, and not just for stationary and constant shear flows, as previously thought \cite{43, 44}.

Substituting the modal function (3.35) into the expressions for coefficients (3.30) - (3.34), the amplitude equation (3.29) can be written as
\[\tilde{\mu}_1 A_R + \tilde{\mu}_2 A_\xi + \tilde{\mu}_3 A_\xi \xi + \frac{\tilde{\mu}_4}{R} A = 0,\] (3.37)
and the expressions for the coefficients can be brought to the previously derived form:

\[ \tilde{\mu}_1 = 2(k^2 + k'^2)I_3, \]
\[ \tilde{\mu}_2 = -3(k^2 + k'^2)^2I_4, \]
\[ \tilde{\mu}_3 = -(k^2 + k'^2)^2 \int_0^1 \int_0^z \int_0^{z_1} \frac{F^2(z_1, \theta)}{F^2(z_2, \theta)} \, dz_2 \, dz_1 \, dz, \]
\[ \tilde{\mu}_4 = \frac{-k(k + k'')}{k \cos \theta - k' \sin \theta}((k \cos \theta + 3k' \sin \theta)I_2 + 4k' \sin \theta I_3) \]
\[ - \frac{3k(k + k'')(k^2 + k'^2) \sin^2 \theta}{(k \cos \theta - k' \sin \theta)^2}(I_2 + 2I_3 + I_4), \]

where \( I_n = \int_0^1 \frac{dz}{F^n}, \quad \tilde{\mu}_i = \frac{\mu_i}{k^2 + k'^2}, \quad i = 1, 2, 3, 4. \)

Here we confirm that the coefficient in front of \( A_\theta/R \) is 0 in the case of the homogeneous fluid and the formulae of coefficients coincide with Johnson’s result [43; 44].

When considering the case of no shear flow, \( u_0(z) = 0 \), the ring waves become concentric waves. Here, \( F = -s, k(\theta) = 1, \) and \( k'(\theta) = 0. \) Then, the derived equation (3.29) again reduces to the form of the 1+1-dimensional cKdV-type equation

\[ \tilde{\mu}_1 A_R + \tilde{\mu}_2 AA_\xi + \tilde{\mu}_3 A_{\xi\xi\xi} + \frac{\tilde{\mu}_4}{R} A = 0, \]

(3.38)

and expressions for the coefficients are given by

\[ \tilde{\mu}_1 = 2 \int_0^1 \rho_0 \phi^2 \, dz, \quad \tilde{\mu}_2 = 3 \int_0^1 \rho_0 \phi^3 \, dz, \]
\[ \tilde{\mu}_3 = \int_0^1 \rho_0 \phi^2 \, dz, \quad \tilde{\mu}_4 = \int_0^1 \rho_0 \phi^2 \, dz, \]

where \( \tilde{\mu}_i = \frac{\mu_i}{s^2}, \quad i = 1, 2, 3, 4, \)

where \( \phi = \phi(z) \) is the modal function, which does not depend on \( \theta \) any more, satisfying
the same modal equations as (2.39)-(2.41):

\[ s^2 (\rho_0 \phi) z - \rho_0 \phi = 0, \]
\[ \phi = 0 \quad \text{at} \quad z = 0, \]
\[ s^2 \phi_z - \phi = 0 \quad \text{at} \quad z = 1. \]

This reduction also agrees with the equation previously derived in Chapter 2 (2.33).

Thus, in both cases the derived amplitude equation (3.29) correctly reduces to the previously derived models. Importantly, in both of these previously studied cases the coefficient \( \mu_5 = 0 \). However, it is not equal to zero in the general case of the waves in a stratified fluid with a shear flow.
3.3 Two-layer example: dispersion relation and wavefronts

3.3.1 Two-layer Model

In order to clarify the general theory developed in the previous section and to illustrate the different effect of a shear flow on the wavefronts of surface and internal ring waves, here we discuss a simple piecewise-constant setting, frequently used in theoretical and laboratory studies of long internal and surface waves (Figure 3.1, see, for example, [2; 5; 6; 13; 19; 21; 22; 37; 56; 58; 66; 73; 80; 81] and references therein). In these theoretical and laboratory studies, the model is often chosen to yield explicit formulae, but is regarded as a reasonable abstraction for a background flow with smooth density and shear profiles across the interface, in the long wave approximation. It is also necessary to note that this background flow is subject to Kelvin-Helmholtz (K-H) instability arising as short waves, which are excluded in the long wave theory due to the large separation of scales (see [23; 26; 78] and references therein).

Here, in non-dimensional form, both the density of the fluid and the shear flow are...
piecewise-constant functions \((0 \leq z \leq 1)\):

\[
\begin{align*}
\rho_0 &= \rho_2 H(z) + (\rho_1 - \rho_2) H(z - d), \\
u_0 &= U_2 H(z) + (U_1 - U_2) H(z - d),
\end{align*}
\]

where \(d\) is the thickness of the lower layer and \(H(z)\) is the Heaviside step function.

In the rigid lid approximation, the condition for the linear stability of the long waves in a fluid of finite depth in nondimensional variables used in our paper is given by \([71; 72]\):

\[
(U_1 - U_2)^2 < \frac{(\rho_2 - \rho_1)(\rho_1 d + \rho_2(1 - d))}{\rho_1 \rho_2},
\]

(3.39)
see also \([12; 14; 53]\) and references therein. The free surface results show that long waves are stable both for small shears, as in the rigid lid case, but also for sufficiently large shears \([71; 72]\), see also \([7; 53]\).

Long interfacial waves are observed both in experiments and in natural environments (for example, see reviews by \([3; 37]\)), which means that in the situations when they are observed there exist some extra mechanisms which prevent the development of the K-H instability. Among such stabilising mechanisms not present in the simplified model are the continuity of the actual density and shear flow profiles, with a thin intermediate layer in between the two main layers, and surface tension (see \([14; 53; 78]\) and references therein). The two-layer model is an abstraction of the actual continuous density and shear flow profiles, suitable for the theoretical study of the long interfacial waves.

### 3.3.2 Dispersion relation and approximations

Solution to the modal equations \((3.8) - (3.10)\) in the two layers is given by the linear functions of \(z\) \((\phi_1\) is the modal function in the upper layer and \(\phi_2\) is the modal function}
3. RING WAVES IN A STRATIFIED FLUID OVER A SHEAR FLOW

in the lower layer):

\[
\phi_1 = \left( \frac{F_1^2}{k^2 + k'^2} + z - 1 \right) \Lambda, \quad \phi_2 = \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) \frac{\Lambda z}{d},
\]

where \(\Lambda\) is a constant and the continuity of \(\phi\) is satisfied, while the jump condition at the interface

\[
\frac{[\rho_0 F^2 \phi_2]}{k^2 + k'^2} = [\rho_0] \phi \quad \text{at} \quad z = d
\]

provides the ‘dispersion relation’,

\[
(\rho_2 - \rho_1)d(1 - d)(k^2 + k'^2)^2 - \rho_2(dF_1^2 + (1 - d)F_2^2)(k^2 + k'^2) + \rho_2 F_1^2 F_2^2 = 0,
\]

with \(F_1 = -s + (U_1 - U_2)(k \cos \theta - k' \sin \theta), \quad F_2 = -s\). This nonlinear first-order differential equation for the function \(k(\theta)\) is a further generalisation of both Burns and generalised Burns conditions [17, 43]. First, we find the wave speed \(s\) by letting \(U_1 - U_2 = 0\), while \(k = 1\). The dispersion relation takes the form

\[
\rho_2 s^4 - \rho_2 s^2 + (\rho_2 - \rho_1)d(1 - d) = 0.
\]

So the wave speed in the absence of the shear flow is given by

\[
s^2 = \frac{\rho_2 \pm \sqrt{\rho_2^2 - 4\rho_2(\rho_2 - \rho_1)d(1 - d)}}{2\rho_2} = \frac{1 \pm \sqrt{(2d - 1)^2 + 4\rho_1 / \rho_2 d(1 - d)}}{2},
\]

where the upper sign should be chosen for the faster surface mode, and the lower sign for the slower internal mode. For example, assuming \(\rho_1 = 1, \rho_2 = 1.2\), we obtain \(s_{\text{sur}} \approx 0.9780\) and \(s_{\text{int}} \approx 0.2087\) at \(d = 0.5\) and \(s_{\text{sur}} \approx 0.9789\) and \(s_{\text{int}} \approx 0.2043\) at \(d = 0.6\).

When the shear flow is present, equation (3.41) constitutes a nonlinear first-order
differential equation for the function \( k(\theta) \). We have

\[
k^2 + k'^2 = \frac{\rho_2 [dF_1^2 + (1 - d)F_2^2] \pm \sqrt{\Delta}}{2(\rho_2 - \rho_1)(1 - d)d},
\]

(3.43)

where

\[
\begin{align*}
\Delta &= \rho_2^2 (dF_1^2 + (1 - d)F_2^2)^2 - 4\rho_2(\rho_2 - \rho_1)d(1 - d)F_1^2F_2^2 \\
&= \rho_2^2 (dF_1^2 - (1 - d)F_2^2)^2 + 4\rho_1\rho_2d(1 - d)F_1^2F_2^2 \geq 0,
\end{align*}
\]

and the upper / lower signs correspond to the internal / surface modes, respectively.

The generalised Burns condition for the surface waves in a homogeneous fluid with this two-layer shear flow reads

\[
(k^2 + k'^2) \left( \frac{1 - d}{F_1^2} + \frac{d}{F_2^2} \right) = 1 \iff (dF_1^2 + (1 - d)F_2^2)(k^2 + k'^2) = F_1^2F_2^2,
\]

(3.44)

and can be recovered from our more general equation (3.41) in the limit \( \rho_2 - \rho_1 \to 0 \).

If the density contrast is small, i.e. \( \rho_2 - \rho_1 \ll \rho_1, \rho_2 \), one can obtain a simplified equation not only for the surface mode (see (3.44)), but also for the interfacial mode, by replacing the free surface condition (3.9) with the rigid lid approximation (see, for example [35]),

\[
\phi = 0 \quad \text{at} \quad z = 1.
\]

Then, the modal function \( \phi \) in the two layers is found to be

\[
\phi_1 = \Lambda(z - 1), \quad \phi_2 = \frac{d - 1}{d} \Lambda z,
\]

where \( \Lambda \) is a constant, and the jump condition at \( z = d \) again provides the ‘dispersion relation’:

\[
(\rho_2 - \rho_1)d(1 - d)(k^2 + k'^2) = \rho_1dF_1^2 + \rho_2(1 - d)F_2^2.
\]

(3.45)
In this case, the interfacial wave speed in the absence of the shear flow is given by

\[ s^2 = \frac{(\rho_2 - \rho_1)d(1 - d)}{\rho_1d + \rho_2(1 - d)}, \]

which is found by letting \( U_1 - U_2 = 0 \) and \( k = 1 \). The singular solution to the ODE (3.45) of \( k(\theta) \) has the form

\[
\begin{align*}
    k(\theta) &= a \cos \theta + b(a) \sin \theta, \\
    b'(a) &= -1/ \tan \theta, \\
    (\rho_2 - \rho_1)d(1 - d)(a^2 + b^2) &= \rho_1d(-s + (U_1 - U_2)a)^2 + \rho_2(1 - d)s^2.
\end{align*}
\]

We can find the function \( b(a) \) explicitly from the third equation in (3.46). Substituting \( b(a) \) into the second equation in (3.46), we obtain a quadric equation for \( a \), which can be solved explicitly in terms of \( \theta \). Substituting \( a \) and \( b \) into the first equation in (3.46) and letting the value of \( k(\theta) \) to be positive everywhere, we obtain the singular solution \( k(\theta) \) explicitly, in the following form:

\[
k(\theta) = \left\{ \begin{array}{ll}
    \sqrt{\frac{a^2(1 - a(U_1 - U_2))}{1 + a(U_1 - U_2)} + \frac{\cos^2 \theta}{\alpha(1 - a(U_1 - U_2))} - \frac{a \cos \theta}{\alpha(U_1 - U_2)}, & \text{if } \theta \in [0, \frac{\pi}{2}] \cup \left[ \frac{3\pi}{2}, 2\pi \right], \\
    -\sqrt{\frac{a^2(1 - a(U_1 - U_2))}{1 + a(U_1 - U_2)} + \frac{\cos^2 \theta}{\alpha(1 - a(U_1 - U_2))} - \frac{a \cos \theta}{\alpha(U_1 - U_2)}}, & \text{if } \theta \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right),
\end{array} \right.
\]

where \( \alpha = \frac{\rho_1(U_1 - U_2)}{(1 - d)(\rho_2 - \rho_1)} \). The square root sign, here and throughout the thesis, only chooses the positive sign branch. The solution (3.47) can be verified directly, by substitution into the equation (3.45).

Let us note that equations asymptotically equivalent to both approximate equations (3.44) and (3.45) can be formally obtained from the full dispersion relation (3.41) if we first solve it as a quadratic equation with respect to \( k^2 + k'^2 \), see (3.43), and then Taylor expand \( \sqrt{\Delta} \) in powers of the small parameter \( (\rho_2 - \rho_1)/\rho_2 \):

\[
k^2 + k'^2 = \frac{\rho_2[F_1^2 + (1 - d)F_2^2]}{2(\rho_2 - \rho_1)(1 - d)d} \left[ 1 \pm \left( \frac{(\rho_2 - \rho_1)d(1 - d)F_1^2F_2^2}{\rho_2[F_1^2 + (1 - d)F_2^2]^2} + \ldots \right) \right].
\]
The approximate equations (3.44) and (3.45) correspond to the lower and upper signs in the above equation, respectively.

The general solution of equation (3.41) can be found in the form similar to the general solution of the generalised Burns condition [43], allowing us then to find the necessary singular solution relevant to the ring waves in a stratified fluid in parametric form:

\[
\begin{align*}
   k(\theta) &= a \cos \theta + b(a) \sin \theta, \\
   b'(a) &= -1/ \tan \theta, \\
   a^2 + b^2 &= \frac{\rho_2[d(-s+a(U_1-U_2))^2+(1-d)s^2] \pm \sqrt{\Delta}}{2(\rho_2-\rho_1)d(1-d)},
\end{align*}
\]

where

\[
\Delta = \rho_2^2 \left[d(-s+a(U_1-U_2))^2+(1-d)s^2 \right] - 4\rho_2(\rho_2-\rho_1)d(1-d)s^2 \left[-s + a(U_1-U_2) \right]^2, 
\]

\[
= \rho_2^2 \left[d(-s+a(U_1-U_2))^2 - (1-d)s^2 \right] + 4\rho_1\rho_2d(1-d)s^2 \left[-s + a(U_1-U_2) \right]^2. 
\]

Therefore,

\[
0 \leq \sqrt{\Delta} \leq \rho_2 \left[d(-s+a(U_1-U_2))^2+(1-d)s^2 \right].
\]

The upper sign should be chosen in (3.48) for the interfacial mode and the lower sign for the surface mode, as previously discussed.

Let us denote

\[
a^2 + b^2 = \frac{\rho_2[d(-s+a(U_1-U_2))^2+(1-d)s^2] \pm \sqrt{\Delta}}{2(\rho_2-\rho_1)d(1-d)} = Q.
\]

Then the condition \( b^2 = Q - a^2 \geq 0 \) determines the domain of \( a \). However, the condition involves a quartic and may define several intervals. We require \( k(\theta) \) to be positive everywhere to describe the outward propagating ring wave. Therefore, one needs to choose the interval \([a_{\text{min}}, a_{\text{max}}]\) which contains \( a = 0 \), since \( a \) should take both positive and negative values (in particular, \( k(\theta) \) should be positive both at \( \theta = 0 \) and
Then,

\[ 2bb' = Q_a - 2a, \quad \Rightarrow b' = \frac{Q_a - 2a}{2b}, \quad \Rightarrow \tan \theta = -\frac{2b}{Q_a - 2a}. \]

Here, \( Q_a = \frac{\partial Q}{\partial a} \). Therefore, \( k(\theta) \) can be written in the form

\[ k(\theta) = a \cos \theta + b \sin \theta = \cos \theta(a + b \tan \theta) = \text{sign} \left( \frac{-2b \cos \theta}{\tan \theta} \right) \frac{aQ_a - 2Q}{\sqrt{(Q_a - 2a)^2 + 4b^2}}. \]

Since \( k(\theta) > 0 \), we obtain

\[ \text{sign}(k(\theta)) = -\text{sign} \left[ b(aQ_a - 2Q) \frac{\cos \theta}{\tan \theta} \right] = 1. \]

When there is no shear flow, \( aQ_a - 2Q = -2Q < 0 \). Therefore, it is natural to assume that this inequality will continue to hold in the case of the relatively weak shear flow considered here, and check that it is satisfied once the solution is constructed. Also, in the particular case of the small density gradient, \( \rho_2 - \rho_1 \ll \rho_1, \rho_2 \), this condition can be verified directly using the approximation (3.45) for the interfacial mode. Thus, we assume that \( aQ_a - 2Q < 0 \) in the interval \( [a_{\min}, a_{\max}] \). Then

\[ k(a) = -\frac{aQ_a - 2Q}{\sqrt{(Q_a - 2a)^2 + 4b^2}}. \]

and

\[ \text{sign}(b) = \text{sign} \left( \frac{\cos \theta}{\tan \theta} \right) = \text{sign} \left( \frac{\cos^2 \theta}{\sin \theta} \right) = \begin{cases} 1 & \text{if } \theta \in (0, \pi), \\ -1 & \text{if } \theta \in (\pi, 2\pi). \end{cases} \]

Therefore, if \( \theta \in (0, \pi) \), then

\[ b = \sqrt{Q - a^2}, \quad \tan \theta = -\frac{2\sqrt{Q - a^2}}{Q_a - 2a}. \]
3. RING WAVES IN A STRATIFIED FLUID OVER A SHEAR FLOW

(1) Surface mode: $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.5$ (green), $U_1 - U_2 = 1$ (blue), and $U_1 - U_2 = 1.5$ (red).

(2) Interfacial mode: $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.1$ (green), $U_1 - U_2 = 0.15$ (blue), and $U_1 - U_2 = 0.2$ (red).

Figure 3.2: Function $k(a)$ at $d = 0.5$: surface mode (left) and interfacial mode (right).

and we let

$$\theta = \begin{cases} 
\arctan\left(-\frac{2\sqrt{Q-a^2}}{Q_a - 2a}\right) & \text{if } Q_a - 2a < 0, \\
\arctan\left(-\frac{2\sqrt{Q-a^2}}{Q_a - 2a}\right) + \pi & \text{if } Q_a - 2a > 0.
\end{cases}$$

Similarly, if $\theta \in (\pi, 2\pi)$, then

$$b = -\sqrt{Q - a^2}, \quad \tan \theta = \frac{2\sqrt{Q-a^2}}{Q_a - 2a},$$

and we let

$$\theta = \begin{cases} 
\arctan\left(\frac{2\sqrt{Q-a^2}}{Q_a - 2a}\right) + \pi & \text{if } Q_a - 2a > 0, \\
\arctan\left(\frac{2\sqrt{Q-a^2}}{Q_a - 2a}\right) + 2\pi & \text{if } Q_a - 2a < 0.
\end{cases}$$

Thus, we obtained the required singular solution analytically, in parametric form. The functions $k(a)$, $\theta(a)$ and $k(\theta)$ for both surface and interfacial ring waves at $d = 0.5$ and $d = 0.6$ are shown in Figures 3.2-3.7. As before, we choose $\rho_1 = 1$, $\rho_2 = 1.2$ and two values of the lower layer depth $d = 0.5$ and $d = 0.6$ as examples, and consider several values of the strength of the shear flow. Here, the function $k$ cannot be conveniently written in the form $k = k(\theta)$. The equation for $\theta$ is a quartic equation. Therefore
(1) Surface mode: $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.5$ (green), $U_1 - U_2 = 1$ (blue), and $U_1 - U_2 = 1.5$ (red).

(2) Interfacial mode: $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.1$ (green), $U_1 - U_2 = 0.15$ (blue), and $U_1 - U_2 = 0.2$ (red).

Figure 3.3: Function $k(a)$ at $d = 0.6$: surface mode (left) and interfacial mode (right).

(1) Surface mode: $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.5$ (green), $U_1 - U_2 = 1$ (blue), and $U_1 - U_2 = 1.5$ (red).

(2) Interfacial mode: $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.1$ (green), $U_1 - U_2 = 0.15$ (blue), and $U_1 - U_2 = 0.2$ (red).

Figure 3.4: Function $\theta(a)$ at $d = 0.5$: surface mode (left) and interfacial mode (right).
3. RING WAVES IN A STRATIFIED FLUID OVER A SHEAR FLOW

Figure 3.5: Function $\theta(a)$ at $d = 0.6$: surface mode (left) and interfacial mode (right).

Figure 3.6: Function $k(\theta)$ at $d = 0.5$: surface mode (left) and interfacial mode (right).
(1) Surface mode: $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.5$ (green), $U_1 - U_2 = 1$ (blue), and $U_1 - U_2 = 1.5$ (red).
(2) Interfacial mode: $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.1$ (green), $U_1 - U_2 = 0.15$ (blue), and $U_1 - U_2 = 0.2$ (red).

Figure 3.7: Function $k(\theta)$ at $d = 0.6$: surface mode (left) and interfacial mode (right).

we define the functions $k$ and $\theta$ in terms of parameter $a$, and plot the function $k(\theta)$ numerically.

The approximate solution (3.47) for the internal waves is compared with the exact solution (3.48) in Figure 3.8 and Figure 3.9 at $d = 0.5$ and 0.6, respectively, when $U_1 - U_2 = 0.1$. It is shown that the simpler approximate solution is rather close to the exact solution, with the advantage that the function $k$ can be written explicitly as a function of $\theta$. 
3. RING WAVES IN A STRATIFIED FLUID OVER A SHEAR FLOW

Figure 3.8: Function $k(\theta)$: rigid-lid approximation (3.47) (blue) and exact solution (3.48) (red) for internal waves; $d = 0.5$ and $U_1 - U_2 = 0.1$.

Figure 3.9: Function $k(\theta)$: rigid-lid approximation (3.47) (blue) and exact solution (3.48) (red) for internal waves; $d = 0.6$ and $U_1 - U_2 = 0.1$. 
3. RING WAVES IN A STRATIFIED FLUID OVER A SHEAR FLOW

Figure 3.10: Wavefronts of surface ring waves described by $k(\theta)r = 1$ when $d = 0.5$ for $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.5$ (green), $U_1 - U_2 = 1$ (blue), and $U_1 - U_2 = 1.5$ (red).

3.3.3 Wavefronts of surface and interfacial ring waves

We note that to leading order, the waves propagate at the speed $s/k(\theta)$ in the direction $\theta$. As discussed before, the wavefronts are described by the equation $H(r, \theta, t) = rk(\theta) - st = \text{constant}$. Here, we choose $t = 1$ and constant $= 0$. We show the wavefronts for the surface mode and interfacial mode of the equation (3.41) respectively at $d = 0.5$ in Figures 3.10 and 3.12 and $d = 0.6$ in Figures 3.11 and 3.13, for $\rho_1 = 1, \rho_2 = 1.2$ and several values of the strength of the shear flow.

We see that the shear flow has significantly different effect on the surface and internal ring waves: the surface ring waves shown in Figure 3.10 and 3.11 are elongated in the direction of the shear flow, while the interfacial ring waves shown in Figure 3.12 and 3.13 are squeezed in the direction of the flow. There is small difference of wavefronts between $d = 0.5$ and $d = 0.6$. We also note that when the value of $U_1 - U_2$ is increased, there is a threshold after which the equation for $k(\theta)$ corresponding to interfacial waves does not have a solution, which indicates that the interfacial waves
Figure 3.11: Wavefronts of surface ring waves described by $k(\theta)r = 1$ when $d = 0.6$ for $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.5$ (green), $U_1 - U_2 = 1$ (blue), and $U_1 - U_2 = 1.5$ (red).

Figure 3.12: Wavefronts of interfacial ring waves described by $k(\theta)r = 1$ when $d = 0.5$ for $U_1 - U_2 = 0$ (black), $U_1 - U_2 = 0.1$ (green), $U_1 - U_2 = 0.15$ (blue), and $U_1 - U_2 = 0.2$ (red).
become unstable as the shear flow strengthens.

To understand why this happens, we consider the behaviour of \( k(\theta) \) around the angles \( \theta = 0, \pi \). Locally, in the area around the angles \( \theta = 0, \pi \), the ring waves can be treated as plane waves over a shear flow (propagating along or opposite the flow). The modal equation is given by [35]:

\[
\begin{align*}
\rho_0(C - u_0)^2 \phi_z z - \rho_\infty \phi &= 0, \\
(C - u_0)^2 \phi_z - \phi &= 0, \quad \text{at } z = 1, \\
\phi &= 0, \quad \text{at } z = 0,
\end{align*}
\]

where \( C \) is the wave speed. In the two-layer case, the dispersion relation takes the form

\[
d\rho_1(C - U_1)^2 - \rho_2(C - U_2)^2(d - 1 + (C - U_1)^2) = d(\rho_1 - \rho_2)(d - 1 + (C - U_1)^2).
\]
Substituting the coefficients $\rho_1 = 1, \rho_2 = 1.2$, and $d = 0.5$ into the general solution of this quartic equation, one obtains the following formula for the wave speed $C$:

$$C = \frac{U_1 + U_2}{2} \pm \frac{1}{6} \sqrt{9(U_1 - U_2)^2 + 3 \sqrt{72(U_1 - U_2)^2 + 30} + 18.}$$ (3.52)

Letting the Cartesian coordinate frame move at the speed $U_2$, we plot the four branches of solutions for the wave speed $C$ in Figure 3.14. At $d = 0.6$, the formula of the wave speed $C$ is more complicated. We plot the four branches of solutions for the wave speed $C$ in Figure 3.15. Here, the top and bottom curves in both figures show the speeds of surface waves, propagating along and opposite the shear flow, while the curves in between show the speeds of the slower moving internal waves. Both surface and internal waves can propagate when there is no shear flow ($U_1 - U_2 = 0$). When the strength of the shear flow is increased ($U_1 - U_2 > 0$), the difference between the speeds of the surface waves along and opposite the flow increases, while the similar difference for the internal waves decreases. This indicates that the wavefronts of the
surface waves become elongated in the direction of the flow, while the wavefronts of the internal waves are indeed squeezed in this direction. The graph also shows the onset of the K-H instability for the long waves at $U_1 - U_2 \approx 0.5$ and stabilisation for the values of the shear flow exceeding $U_1 - U_2 \approx 2$, in agreement with the results of [71]. We note that the stabilisation persists within the scope of the full equations of motion [53; 72].

### 3.3.4 Critical layer condition

The critical layer is a region in the neighbourhood of a line at which the local wave speed is equal to the shear flow speed, see [31; 46] and references therein. We only consider a relatively weak shear flow, when the critical layer does not appear, which we will justify next.

The wavefront at a fixed time $t$ can be described as

$$H(r, \theta, t) = rk(\theta) - st = \text{constant}.$$
The local wave speed in the direction 
\[
\frac{\nabla H}{|\nabla H|} = \frac{ke_r + k'e_{\theta}}{\sqrt{k^2 + k'^2}}
\]
is given by
\[
- \frac{H_t}{|\nabla H|} = \frac{s}{\sqrt{k^2 + k'^2}}
\]
(see [44]).

So the critical layer occurs when
\[
\frac{s}{\sqrt{k^2 + k'^2}} = (U_1 - U_2) \cos(\alpha + \theta), \quad \text{where } \cos \alpha = \frac{k}{\sqrt{k^2 + k'^2}}
\]
which is equivalent to the condition
\[
F_1 = -s + (U_1 - U_2)(k \cos \theta - k' \sin \theta) = 0,
\]
when the linear problem formulation fails. Note that $F_2 = -s \neq 0$.

We know that
\[
F_{1\theta} = -(U_1 - U_2)(k + k'') \sin \theta,
\]
where, without loss of generality, we assume that $U_1 - U_2 > 0$ and $k + k'' > 0$ on the selected singular solution ($k(\theta) > 0$). Then, $F_{1\theta} < 0$ if $\theta \in (0, \pi)$ and $F_{1\theta} > 0$ if $\theta \in (\pi, 2\pi)$, which implies that $F_1$ reaches its maximum value at $\theta = 0$. Therefore, to avoid the appearance of critical layers, we require that
\[
F_1 \leq F_{1\theta=0} = -s + (U_1 - U_2)k(0) < 0,
\]
which yields the following constraint on the strength of the shear flow:
\[
(U_1 - U_2)k(0) < s. \quad (3.53)
\]
We note that \( k(0) \) depends on \( U_1 - U_2 \). However, since \( s/k(\theta) \) represents the local wave speed in the direction of \( \theta \), we know that \( s/k(0) \geq s \), implying that \( k(0) \leq 1 \). Thus, the exact condition (3.53) can be replaced with a simplified estimate:

\[
U_1 - U_2 < s \leq \frac{s}{k(0)}.
\]

### 3.3.5 Coefficients of the derived cKdV-type equation

Finally, we will calculate the coefficients of the derived 2 + 1-dimensional amplitude equation (3.29) for both surface and interfacial ring waves in this two-layer case. In the surface case, we normalise \( \phi \) by setting \( \phi = 1 \) at \( z = 1 \). The constant \( \Lambda \) in the modal function (3.40) is

\[
\Lambda_s = \frac{k^2 + k'^2}{F_1^2}.
\]

Substituting the modal function into the formulae (3.30) - (3.34), the coefficients are given by

\[
\begin{align*}
\mu_1 &= \frac{2s(k^2 + k'^2)^2}{F_1^4} \left( (1 - d) \rho_1 F_1 + \frac{\rho_2 F_2}{d} \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) \right), \\
\mu_2 &= -\frac{3(k^2 + k'^2)^3}{F_1^6} \left( (1 - d) \rho_1 F_1 + \frac{\rho_2 F_2}{d^2} \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) \right), \\
\mu_3 &= \frac{(k^2 + k'^2)^3}{3F_1^4} \rho_1 F_2^2 \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) + \rho_2 F_2^2 \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right), \\
\mu_4 &= \frac{(1 - d) \rho_1 k(k + k'^2)(k^2 + k'^2)}{F_1^4} \left( F_1^2 + 4k' F_1 (U_1 - U_2) \sin \theta + 3(k^2 + k'^2)(U_1 - U_2)^2 \sin^2 \theta \right) \\
&\quad - \frac{\rho_2 (k + k'^2) k F_2^2}{d F_1^2} \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) \left( k^2 - 3k'^2 \right) \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) \\
&\quad + 4(d - 1) k(k' F_1 + (U_1 - U_2)(k^2 + k'^2) \sin \theta) \\
&\quad \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right), \\
\mu_5 &= \frac{-2k(k^2 + k'^2)}{F_1^4} \left( (1 - d) \rho_1 F_2 (k' F_1 + (U_1 - U_2)(k^2 + k'^2) \sin \theta) + \frac{\rho_2 k' F_2^2}{d} \left( \frac{F_1^2}{k^2 + k'^2} + d - 1 \right) \right),
\end{align*}
\]
where

\[
F_1 = -s + (U_1 - U_2)(k \cos \theta - k' \sin \theta),
\]

\[
F_2 = -s,
\]

\[
s^2 = \frac{1 + \sqrt{(2d-1)^2 + 4\rho_1/\rho_2d(1-d)}}{2},
\]

and the function \( k(\theta) \) is defined by formula (3.48) (lower sign).

In the interfacial case, we normalise \( \phi \) by setting \( \phi = 1 \) at \( z = d \). The constant \( \Lambda \) in the modal function (3.40) is

\[
\Lambda_i = \frac{k^2 + k'^2}{F_1^2 + (d-1)(k^2 + k'^2)}.
\]

Substituting the modal function into the formulae (3.30) - (3.34), the coefficients are given by

\[
\mu_1 = 2s \left( \frac{(1 - d)\rho_1 F_1 (k^2 + k'^2)^2}{(F_1^2 + (d-1)(k^2 + k'^2))^2} + \frac{\rho_2 F_2}{d} \right),
\]

\[
\mu_2 = -3 \left( \frac{(1 - d)\rho_1 F_1^2 (k^2 + k'^2)^3}{(F_1^2 + (d-1)(k^2 + k'^2))^3} + \frac{\rho_2 F_2^2}{d^2} \right),
\]

\[
\mu_3 = -\frac{\rho_1 F_1^2}{3(F_1^2 + (d-1)(k^2 + k'^2))^2} \left( F_1^6 - (F_1^2 + (d-1)(k^2 + k'^2))^3 \right) - \frac{1}{3} d \rho_2 F_2^2 (k^2 + k'^2),
\]

\[
\mu_4 = -\frac{(1 - d)\rho_1 k(k + k') (F_1^6 - 4k'(k^2 + k'^2)F_1^2 - 4k'(k^2 + k'^2)F_1(U_1 - U_2) \sin \theta - (U_1 - U_2)^2 (k^2 + k'^2)^2 \sin^2 \theta)}{(F_1^2 + (d-1)(k^2 + k'^2))^2}
\]

\[
- \frac{4(1 - d)\rho_1 k(k + k') F_1^2}{(F_1^2 + (d-1)(k^2 + k'^2))^3} (k' F_1 + (k^2 + k'^2)(U_1 - U_2) \sin \theta)^2 - \frac{\rho_2 k(k + k')(k^2 - 3k'^2) F_2^2}{d(k^2 + k'^2)^2},
\]

\[
\mu_5 = -\frac{2(1 - d)\rho_1 F_1 k(k^2 + k'^2)}{(F_1^2 + (d-1)(k^2 + k'^2))^2} (k' F_1 + (U_1 - U_2)(k^2 + k'^2) \sin \theta) - \frac{2k' \rho_2 F_2^2}{d(k^2 + k'^2)},
\]

(3.55)
where

\[ F_1 = -s + (U_1 - U_2)(k \cos \theta - k' \sin \theta), \]
\[ F_2 = -s, \]
\[ s^2 = \frac{1 - \sqrt{(2d - 1)^2 + 4\rho_1/\rho_2d(1 - d)}}{2}, \]

and the function \( k(\theta) \) is defined by the formula (3.48) (upper sign). One can also use the explicit formula (3.47), obtained in the rigid lid approximation.
3.4 Concluding remarks

From the formulation of the general fluid dynamic problem for the ring waves in a stratified fluid over a shear flow, we developed an asymptotic theory describing the propagation of long linear and weakly nonlinear ring waves. We found that there exists a linear modal decomposition, which has more complicated structure than the known modal decomposition in Cartesian geometry. Then, we derived the modal equation at the leading order and the 2+1-dimensional cKdV-type amplitude equation (3.29) at higher order, with coefficients given by (3.30)-(3.34). Assuming that either the fluid is homogenous or there is no shear flow, we recovered the two versions of 1+1-dimensional cKdV-type equation derived in Chapter 2 for the surface waves in a homogeneous fluid over a shear flow (2.33) and internal waves in a stratified fluid in the absence of a shear flow (2.58). The coefficient in front of $A \theta / R$ is 0 in these two particular cases.

After that, we used a two-layer model with a piecewise-constant shear flow as an example to illustrate our theory. We found the modal function and obtained the generalised dispersion relation, which we used to derive the wave speed when there is no shear flow, and to find the distortion function. The wavefronts of surface and interfacial ring waves were described in terms of two branches of the singular solution of the derived nonlinear first-order differential equation, constituting further generalisation of the Burns and generalised Burns conditions [17; 43]. Remarkably, the two branches of this singular solution could be described in parametric form, and an explicit analytical solution was developed for the wavefront of the interfacial mode in the case of the low density contrast. The constructed solutions have revealed the qualitatively different behaviour of the wavefronts of surface and interfacial waves propagating over the same shear flow. Indeed, while the wavefront of the surface ring wave is elongated in the direction of the flow, the wavefront of the interfacial wave is squeezed in this direction.
Finally, for the case of the two-layer model we also derived a constraint on the strength of the shear flow, which guarantees that there are no critical layers, and obtained explicit expressions for the coefficients of the derived amplitude equation in terms of the physical and geometrical parameters of the model, which provides a fully developed asymptotic theory for this case.
Chapter 4

Numerical method

The KdV and cKdV equations are integrable by the Inverse Scattering Transformation (IST) [32]. The general cKdV-type equation (3.29) derived in Chapter 3 is non-integrable. In this chapter, we introduce an implicit finite-difference method in order to find numerical solutions for the derived 2+1-dimensional cKdV-type equation. This method is a development of the numerical method given by Feng and Mitsui [30], used to solve KdV and KP equations numerically. We will show that this numerical method provides sufficiently accurate results.
4. NUMERICAL METHOD

4.1 Implicit finite-difference method

The derived 2 + 1 - dimensional cKdV-type equation (3.29) is written in the form

$$\mu_1 A_R + \mu_2 A R A_{\xi} + \mu_3 A_{\xi\xi\xi} + \mu_4 \frac{A}{R} + \mu_5 \frac{A_{\theta}}{R} = 0,$$

(4.1)

where $A = A(\xi, R, \theta)$ and $\mu_i = \mu_i(\theta)$, $i = 15$. The initial condition should be given by the function $A(\xi, R_0, \theta)$, i.e. the value of $A$ at fixed $R_0$. Suppose the region of interest of $\xi$ is $[\xi_{\text{min}}, \xi_{\text{max}}]$ (the interval of $\xi$ will be discussed later). $\theta$ is the angle variable, whose domain is $[0, 2\pi]$. We discretise domains of $\xi$ and $\theta$ into grids with equal spacings $\Delta \xi$ and $\Delta \theta$. The variable $R$ starts from $R_0$ and increases by $\Delta R$. We approximate the grid values $A(\xi_{\text{min}} + l\Delta \xi, R_n, m\Delta \theta)$ by $A^n_{l,m}$ where $R_n = R_0 + n\Delta R$, $l = 0, 1, 2, ...; L; m = 0, 1, 2, ...; M; n = 0, 1, 2, ..., N$ with $L = (\xi_{\text{max}} - \xi_{\text{min}})/\Delta \xi$ and $M = 2\pi/\Delta \theta - 1$ (because $\theta = 0$ and $\theta = 2\pi$ correspond to the same direction), and approximate the grid value coefficients $\mu_i(m\Delta \theta)$ by $\mu_{i,m}$. The initial condition is $A^0 = [A^0_{11}, A^0_{21}, \ldots, A^0_{1L}, A^0_{2L}, \ldots, A^0_{LM}]^T$. Thus, the central difference approximations of partial derivatives in (4.1) are

$$A^n_{\xi, l,m} = \frac{A^n_{l+1,m} - A^n_{l-1,m}}{2\Delta \xi} + O(\Delta \xi^2),$$

(4.2)

$$A^n_{\xi\xi\xi, l,m} = \frac{A^n_{l+2,m} - 2A^n_{l+1,m} + 2A^n_{l-1,m} - A^n_{l-2,m}}{2\Delta \xi^3} + O(\Delta \xi^2),$$

(4.3)

$$A^n_{\theta, l,m} = \frac{A^n_{l,m+1} - A^n_{l,m-1}}{2\Delta \theta} + O(\Delta \theta^2).$$

(4.4)

We use the notation $f = A^2$ and $\frac{1}{2} f_{\xi} = A A_{\xi}$. To obtain $A^{n+1}$ from $A^n$, a set of nonlinear algebraic equations has to be solved. We need to linearise the equations by using Taylor’s expansion of $f$ with respect to $R$:

$$f_{l,m}^{n+1} = f_{l,m}^n + \frac{\partial f}{\partial R}_{l,m}^n \Delta R + O(\Delta R^2) = f_{l,m}^n + D_{l,m}^n A_{l,m}^{n+1} + O(\Delta R^2),$$

(4.4)
where \( D^n_{l,m} = \frac{\partial f}{\partial A} \bigg|^{n}_{l,m} = 2A^n_{l,m} \) and \( \Delta A^{n+1}_{l,m} = A^{n+1}_{l,m} - A^n_{l,m} \). Then

\[
f^{n+1}_{l,m} + f^n_{l,m} \approx 2f^n_{l,m} + 2\Delta A^n_{l,m}(A^{n+1}_{l,m} - A^n_{l,m}) = 2A^n_{l,m}A^{n+1}_{l,m}.
\]

Through the central difference approximations, the equation (4.1) at \( R_n + \frac{1}{2}\Delta R \) can be written as

\[
\mu_{1,m} A^{n+1}_{l,m} - A^n_{l,m} \frac{\Delta A^{n+1}_{l,m}}{\Delta R} + \mu_{2,m} \frac{(A^n_{l,m} A^{n+1}_{l,m})_{\xi} + (A^{n+1}_{l,m} + A^n_{l,m})_{\xi\xi\xi}}{2} + \mu_{3,m} \frac{2A^n_{l,m}}{R_n + \frac{R_{n+1}}{2}} + \frac{\mu_{5,m}}{2} \frac{(A^n_{l,m})_{\xi} + (A^{n+1}_{l,m})_{\xi}}{R_n + \frac{R_{n+1}}{2}} = 0. \tag{4.5}
\]

Substituting (4.2)-(4.4) into the above equation yields the following linear system of equations:

\[
\begin{align*}
&\frac{\mu_{5,m}}{4R_{n+1}\Delta \theta} A^{n+1}_{l,m} + \frac{\mu_{3,m}}{4\Delta \xi} A^{n+1}_{l,m-1} + \frac{\mu_{3,m}}{4\Delta \xi} A^{n+1}_{l,m+1} + \frac{\mu_{2,m}}{4\Delta \xi} A^{n+1}_{l,m-2} + \frac{\mu_{3,m}}{2\Delta \xi^3} A^{n+1}_{l,m-1} - \frac{\mu_{2,m}}{2\Delta \xi^3} A^{n+1}_{l,m-2} = d^n_{l,m}, \tag{4.6}
\end{align*}
\]

where

\[
d^n_{l,m} = -\frac{\mu_{5,m}}{4R_n\Delta \theta} A^n_{l,m+1} + \frac{\mu_{5,m}}{4R_n\Delta \theta} A^n_{l,m-1} + \frac{\mu_{3,m}}{4\Delta \xi} A^n_{l,m-1} - \frac{\mu_{3,m}}{4\Delta \xi} A^n_{l,m+1} + \frac{\mu_{3,m}}{2\Delta \xi^3} A^n_{l,m-1} - \frac{\mu_{3,m}}{2\Delta \xi^3} A^n_{l,m+1} + \frac{\mu_{5,m}}{4\Delta \xi} A^n_{l,m-2} + \frac{\mu_{3,m}}{4\Delta \xi^3} A^n_{l,m-2}.
\]

The vector form of equation (4.6) can be expressed as

\[
T \cdot A^{n+1} = d^n, \tag{4.7}
\]
where

\[
A^{n+1} = \begin{bmatrix}
A_{11}^{n+1} \\
A_{21}^{n+1} \\
\vdots \\
A_{L1}^{n+1} \\
A_{12}^{n+1} \\
\vdots \\
A_{LM}^{n+1}
\end{bmatrix},
T = \begin{bmatrix}
a_{11,11} & a_{11,21} & \cdots & a_{11,L1} & a_{11,12} & \cdots & a_{11,LM} \\
a_{21,11} & a_{21,21} & \cdots & a_{21,L1} & a_{21,12} & \cdots & a_{21,LM} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{L1,11} & a_{L1,21} & \cdots & a_{L1,L1} & a_{L1,12} & \cdots & a_{L1,LM} \\
a_{12,11} & a_{12,21} & \cdots & a_{12,L1} & a_{12,12} & \cdots & a_{12,LM} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{LM,11} & a_{LM,21} & \cdots & a_{LM,L1} & a_{LM,12} & \cdots & a_{LM,LM}
\end{bmatrix},
\mathbf{d}^m = \begin{bmatrix}
d_{11}^n \\
d_{21}^n \\
\vdots \\
d_{L1}^n \\
d_{12}^n \\
\vdots \\
d_{LM}^n
\end{bmatrix}.
\]

At the boundary, \( \theta \in [0, 2\pi] \) is the angle variable, which is periodic. We have

\[
A_{l-1}^{n} = A(\xi, R, -\Delta \theta) = A(\xi, R, 2\pi - \Delta \theta) = A_{LM}^{n},
\]

\[
A_{l,M+1}^{n} = A(\xi, R, 2\pi) = A(\xi, R, 0) = A_{10}^{n},
\]

(4.8)

for every \( n \) and \( l \). We choose the domain of \( \xi \) to satisfy the condition that \( A_{l,m}^{n} \) tends to 0 at \( l = 0 \) and \( L \), i.e. at \( \xi = \xi_{\text{min}} \) and \( \xi_{\text{max}} \). Then the value of \( A_{l,m}^{n} \) outside of the interval \([\xi_{\text{min}}, \xi_{\text{max}}]\) is 0,

\[
A_{-2,m}^{n} = A_{-1,m}^{n} = A_{L+1,m}^{n} = A_{L+2,m}^{n} = 0.
\]

(4.9)

From equation (4.6), the terms in the coefficient matrix \( T \) are determined by the
following formulae

\[
\begin{align*}
    d_{l+1,m}^{n} &= \frac{\mu_{5,m}}{4R_{n+1}\Delta \theta'}, \\
    d_{l,m}^{n} &= \frac{\mu_{5,m}}{4R_{n+1}\Delta \theta'} \\
    d_{l+2,m}^{n} &= \frac{\mu_{3,m}}{4\Delta \xi^3}, \\
    d_{l+1,m}^{n} &= \frac{\mu_{2,m}}{4\Delta \xi}A_{l+1,m}^{n} - \frac{\mu_{3,m}}{2\Delta \xi'^3}, \\
    d_{l,m}^{n} &= \frac{\mu_{1,m}}{\Delta R} + \frac{\mu_{4,m}}{2R_{n+1}'}, \\
    d_{l-1,m}^{n} &= -\frac{\mu_{2,m}}{4\Delta \xi}A_{l+1,m}^{n} + \frac{\mu_{3,m}}{2\Delta \xi'^3}, \\
    d_{l-2,m}^{n} &= -\frac{\mu_{3,m}}{4\Delta \xi'^3}.
\end{align*}
\]

Note that the coefficient matrix at the boundary needs to be changed by using the boundary conditions (4.8) and (4.9). The coefficients \( \mu_i \) can be determined by the formulae (3.30)-(3.34). Thus, we can obtain \( A^{n+1} \) by

\[
A^{n+1} = T^{-1} \cdot d^n,
\]

at each step of \( R \), where \( T^{-1} \) is the inverse matrix of \( T \).
4. NUMERICAL METHOD

4.2 Order of accuracy

Substituting the central difference approximations (4.2)-(4.4) and Taylor’s expansion of \( A_{l,m}^{n+1} \) and \( A_{l,m}^{n+1}/R^{n+1} \):

\[
A_{l,m}^{n+1} = A_{l,m}^n + \Delta R(A_{l,m}^n)R + \frac{1}{2}((\Delta R)^2(A_{l,m}^n))_{RR} + O(\Delta R^3),
\]

\[
\frac{A_{l,m}^{n+1}}{R^{n+1}} = \frac{A_{l,m}^n}{R^n} + \Delta R\left(\frac{A_{l,m}^n}{R^n}\right)_R + O(\Delta R^2),
\]

into the difference equation (4.5), one obtains

\[
\mu_{1,m}(A_{l,m}^n)_{RR} + \frac{\mu_{2,m}}{2} \left(2f_{l,m}^n + \Delta(\frac{\partial f_{l,m}^n}{\partial \xi})_\xi\right) + \frac{\mu_{3,m}}{2} \left(2A_{l,m}^n + \Delta R(A_{l,m}^n)_{\xi\xi\xi}\right)
+ \frac{\mu_{4,m}}{2} \left(2\left(\frac{A_{l,m}^n}{R^n}\right)_R + \Delta R\left(\frac{A_{l,m}^n}{R^n}\right)_{\xi R}\right) + O(\Delta R^2 + \Delta \xi^2, \Delta R^2 + \Delta \theta^2)
= \mu_{1,m}(A_{l,m}^n)_{RR} + \mu_{2,m}(f_{l,m}^n) + \mu_{3,m}(A_{l,m}^n)_{\xi\xi\xi} + \mu_{4,m}\left(\frac{A_{l,m}^n}{R^n}\right)_R + \mu_{5,m}\left(\frac{A_{l,m}^n}{R^n}\right)_{\theta R}
+ \frac{1}{2} \Delta R \left(\mu_{1,m}(A_{l,m}^n)_{RR} + \mu_{2,m}(f_{l,m}^n) + \mu_{3,m}(A_{l,m}^n)_{\xi\xi\xi} + \mu_{4,m}\left(\frac{A_{l,m}^n}{R^n}\right)_R + \mu_{5,m}\left(\frac{A_{l,m}^n}{R^n}\right)_{\theta R}\right) + O(\Delta R^2 + \Delta \xi^2, \Delta R^2 + \Delta \theta^2).
\]

(4.10)

If \( A_{l,m}^n \) is the exact solution of the cKdV-type equation (4.1), we will have

\[
\mu_{1,m}(A_{l,m}^n)_{RR} + \mu_{2,m}(f_{l,m}^n) + \mu_{3,m}(A_{l,m}^n)_{\xi\xi\xi} + \mu_{4,m}\left(\frac{A_{l,m}^n}{R^n}\right)_R + \mu_{5,m}\left(\frac{A_{l,m}^n}{R^n}\right)_{\theta R} = 0.
\]

Substituting the above equation into (4.10), we can conclude that the truncation error of our linear system (4.6) is \( O(\Delta R^2 + \Delta \xi^2, \Delta R^2 + \Delta \theta^2) \).
4. NUMERICAL METHOD

4.3 Results in physical coordinate frame

In order to present numerical results in the fixed coordinate frame, we need to consider the transformations between the coordinate systems. In our derived equation (4.1), the wave amplitude $A$ depends on ($\xi, R, \theta$). But in the fixed coordinate frame, it should depend on the radius $r$, the time $t$ and the angle $\theta$. The relations between these two coordinate systems are

$$
\xi = r k(\theta) - st, \quad R = \varepsilon r k(\theta), \quad \theta = \theta,
$$

$$
\implies r = \frac{R}{\varepsilon k(\theta)}, \quad t = \frac{R}{\varepsilon s} - \frac{\xi}{s}, \quad \theta = \theta,
$$

where $\varepsilon$ is the amplitude parameter, $s$ is the wave speed in the absence of a shear flow and the function $k(\theta)$ is the distortion function. The parameter $s$ and the function $k(\theta)$ are defined by the density of the fluid in the basic state and the given shear flow (an example of the calculation of $s$ and $k(\theta)$ has been shown in Chapter 3). The initial condition for the derived equation (4.1) has the form $A(\xi, R_0, \theta)$. In a physical setting, one can put probes at fixed points to measure the wave amplitude at various depths for a certain time period, see for example Ramirez et al [73]. In our case, the initial condition can be obtained by putting probes at $r_0 = R_0/(\varepsilon k(\theta))$, determined by $R_0$ and $\theta$, and measuring the wave amplitude at time $t \in [0, t_1]$. We assume that waves are generated by a disturbance in the vicinity of the origin ($r < r_0$), i.e. for $r > r_0$, $A_{l,m}^n = 0$ when $l > (R_0/\varepsilon - \xi_{\min} + n\Delta R/\varepsilon)/\Delta \xi$. Here

$$
t_1 = \frac{R_0}{\varepsilon s} - \frac{\xi_{\min}}{s},
$$

which implies

$$
\xi_{\min} = \frac{R_0}{\varepsilon} - st_1.
$$
From the initial condition and the implicit finite-difference method, we can forecast the wave amplitude at \( r > r_0 \) at \( t \in [0, t_2] \). Here

\[
t_2 = \frac{rk(\theta) - \xi_{\text{min}}}{s} = t_1 + \frac{R - R_0}{\varepsilon s}.
\]

The lower boundary of \( t \) is equal to 0, implying

\[
\frac{r_0k(\theta) - \xi_{\text{max}}}{s} < \frac{r_{\text{max}}k(\theta) - \xi_{\text{max}}}{s} = 0,
\]

which implies

\[
\xi_{\text{max}} = \frac{R_{\text{max}}}{\varepsilon}.
\]

The value range of \( \xi \) is \( \xi \in [R_0/\varepsilon - st_1, R_{\text{max}}/\varepsilon] \). So when we obtain numerical results \( A^n, (n = 1, 2, \ldots, N) \) for our derived equation (4.1), we can plot the wave amplitude with respect to time \( t \) at a fixed point \((r, \theta)\). At a fixed point \((r, \theta)\), the values of \( m \) and \( n \) in \( A^n_{l,m} \) are given by

\[
m = \frac{\theta}{\Delta\theta}, \quad n = \frac{\varepsilon k(\theta)(r - r_0)}{\Delta R},
\]

and we change the value of \( l \) as a function of \( t \)

\[
l = \frac{rk(\theta) - \xi_{\text{min}} - st}{\Delta \xi}.
\]

Furthermore, we can also plot the wave at a fixed time \( t \) in the coordinates \((r, \theta)\). Note that only the wave amplitude in the area \( r > r_0 \) is forecasted using our model and we just use the linear solution in the area \( r < r_0 \) to show the shape of wave on the whole area \( r < R/(\varepsilon k(\theta)) \). At each point \((r, \theta)\)

\[
n = \frac{\varepsilon k(\theta)r - R_0}{\Delta R}, \quad m = \frac{\theta}{\Delta\theta}.
\]
Since $t$ is fixed, we have

$$\xi_{\text{min}} + l \Delta \xi = \frac{R_0 + n \Delta R}{\varepsilon} - st,$$

which implies

$$l = \frac{R_0 + \Delta R}{\varepsilon \Delta \xi} - \frac{st + \xi_{\text{min}}}{\Delta \xi},$$

where $l$ is a function of $n$. 
4. NUMERICAL METHOD

4.4 Concluding remarks

In this chapter, we described the implicit finite difference method used to simulate solutions of the derived equation (4.1). The scheme is given in a vector form

\[ A^{n+1} = T^{-1} \cdot d^n, \]

where \( T \) and \( d^n \) are determined by \( A^n \). The order of accuracy of this method for the derived 2+1-dimensional cKdV-type equation is \( O(\Delta R^2 + \Delta \xi^2, \Delta R^2 + \Delta \theta^2) \).

To present numerical results in the physical coordinate frame, we need to transform independent variables \((\xi, R, \theta)\) into \((r, \theta, t)\). We discussed the method to obtain the initial value and determine the interval of \( \xi \). In the end, we gave two ways to show our numerical results: in the first way, the wave amplitude changes with respect to time at fixed point \((r, \theta)\) and in the second way, the wave shape is shown at \((r, \theta)\) plane at a fixed time \( t \).
Chapter 5

Concentric waves without shear flow — numerical results

In Chapter 3, we derived the cKdV-type equation (3.29) to describe weakly-nonlinear ring waves in a stratified fluid over a depth-dependent horizontal shear flow. We considered a two-layer example to illustrate the theory. In Chapter 4, we introduced a numerical method which can be used to solve the derived cKdV-type equation with the initial condition given at \( R = R_0 \).

In this Chapter, we will study the effect of nonlinearity on the propagation of the surface and interfacial waves in the two-layer model without shear flow. We will first derive the 2D wave equations for surface and interfacial waves from the full set of Euler equations with free surface and rigid bottom boundary conditions in the two-layer model. Then, we solve the derived cKdV equation with two initial conditions to discuss the effect of nonlinearity. The first example is the nonlinear stage of the evolution of a linear concentric wave generated in a neighbourhood of a point, described analytically by Dobrokhotov et al [25] for the linear 2D wave equation. We use the explicit analytical solution at \( r_0 \) as the initial condition for our derived cKdV equation (3.29). Using the numerical method, we obtain the surface and interfacial
waves in the area \( r > r_0 \). The second example is a version of the 2D dam break problem, where the fluid height is greater in central area, and the dam is released at time \( t = 0 \). We solve the 2D linear wave equations numerically and use these solutions at \( r_0 \) as the initial conditions for our derived cKdV equation (3.29). Using the numerical method introduced in Chapter 4, we solve the cKdV equation numerically. By comparing the numerical results for the cKdV equation in these two examples with the linear solutions, we show the effects of nonlinearity and dispersion. We also discuss the effects of the depth of the lower layer and the initial wave amplitude.
5.1 Linear waves in a two-layer model

Let us consider the two-layer constant density model considered in Section 3.3, but here we assume that the shear flow is absent. In the non-dimensional coordinates, the basic density of the fluid is

\[ \rho_0 = \begin{cases} 
\rho_1 & \text{if } z \in (d, 1), \\
\rho_2 & \text{if } z \in (0, d), 
\end{cases} \]

where \( \rho_1 \) and \( \rho_2 \) are constants and \( d \) is the depth of the lower layer. In this case, one can derive two separate 2D linear wave equations for the surface and interfacial modes (for example [54]). The Euler equations in \( (x, y, z) \) coordinates are given by

\[ \rho_i \left( u_{(i)t} + u_{(i)}u_{(i)x} + v_{(i)}u_{(i)y} + w_{(i)}u_{(i)z} \right) + p_{(i)x} = 0, \quad (5.1) \]
\[ \rho_i \left( v_{(i)t} + u_{(i)}v_{(i)x} + v_{(i)}v_{(i)y} + w_{(i)}v_{(i)z} \right) + p_{(i)y} = 0, \quad (5.2) \]
\[ \varepsilon \rho_i \left( w_{(i)t} + u_{(i)}w_{(i)x} + v_{(i)}w_{(i)y} + w_{(i)}w_{(i)z} \right) + p_{(i)z} + p_{(i)} = 0, \quad (5.3) \]
\[ u_{(i)x} + v_{(i)y} + w_{(i)z} = 0, \quad (5.4) \]

where \( i = 1, 2 \). Here, \( u, v, w \) are the velocity components in \( x, y, z \) directions respectively and \( p \) is the pressure. The variables with the subscript \((1)\) belong to the upper layer \((z \in (d, 1))\) and the variables with the subscript \((2)\) belong to the lower layer \((z \in (0, d))\).

Using the asymptotic multiple-scales expansions of the form

\[ u_{(i)} = \varepsilon \tilde{u}_{(i)} + O(\varepsilon^2), \quad v_{(i)} = \varepsilon \tilde{v}_{(i)} + O(\varepsilon^2), \quad w_{(i)} = \varepsilon \tilde{w}_{(i)} + O(\varepsilon^2), \]
\[ p_{(i)} = p_{0(i)}(z) + \varepsilon \tilde{p}_{(i)} + O(\varepsilon^2), \quad \text{where } p_{0(i)z} = \rho_{(i)}. \]
\[ \rho_{(i)} = \rho_{0(i)} + \varepsilon \tilde{\rho}_{(i)} + O(\varepsilon^2), \]
to leading order equations (5.1) - (5.4) yield (ignore tilde hereafter)

\[ \rho \frac{\partial \eta}{\partial t} + \rho \frac{\partial \eta}{\partial x} = 0, \quad (5.5) \]
\[ \rho \frac{\partial \eta}{\partial t} + \rho \frac{\partial \eta}{\partial y} = 0, \quad (5.6) \]
\[ \frac{\partial \eta}{\partial z} = 0, \quad (5.7) \]
\[ u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} + w \frac{\partial \eta}{\partial z} = 0. \quad (5.8) \]

We apply the free surface and rigid bottom boundary conditions and let \( \varepsilon \eta(x, y, t) \) represent the surface wave amplitude and \( \varepsilon \zeta(x, y, t) \) represent the interfacial wave amplitude. The boundary and continuity conditions are

\[ w(1) = \eta_t, \quad p(1) = \rho_1 \eta, \quad \text{at } z = 1 + \varepsilon \eta, \]
\[ w(2) = 0, \quad \text{at } z = 0, \]
\[ w(1) = w(2) = \zeta_t, \quad p(1) = \zeta \rho_1, \quad p(2) = \zeta \rho_2, \quad \text{at } z = d + \varepsilon \zeta. \]

Thus to leading order,

\[ w(1) = \eta_t, \quad \text{at } z = 1, \quad (5.9) \]
\[ p(1) = \rho_1 \eta, \quad \text{at } z = 1, \quad (5.10) \]
\[ w(2) = 0, \quad \text{at } z = 0, \quad (5.11) \]
\[ w(1) = w(2) = \zeta_t, \quad \text{at } z = d, \quad (5.12) \]
\[ p(1) - p(2) = \zeta (\rho_1 - \rho_2), \quad \text{at } z = d. \quad (5.13) \]

From equations (5.5)-(5.8) and boundary conditions (5.9)-(5.11), one can obtain

\[ w(1)_t = (z - 1)(\eta_{xx} + \eta_{yy}) + \eta_{tt}, \]
\[ w(2)_t = \frac{\zeta}{\rho_2} (p(2)_{xx} + p(2)_{yy}), \quad \text{where } p(2) = p(2)(x, y, t). \]
From the continuity condition (5.13) at the interface $z = d$, one gets

$$p^{(2)} = \rho_1 \eta + (\rho_2 - \rho_1) \zeta. \quad (5.14)$$

Substituting (5.14) into (5.12), one obtains

$$\eta_t = \left( \frac{d(\rho_1 - \rho_2)}{\rho_2} + 1 \right) \left( \eta_{xx} + \eta_{yy} \right) + \frac{d(\rho_2 - \rho_1)}{\rho_2} (\zeta_{xx} + \zeta_{yy}), \quad (5.15)$$

$$\zeta_t = \frac{\rho_1 d}{\rho_2} (\eta_{xx} + \eta_{yy}) + \frac{(\rho_2 - \rho_1) d}{\rho_2} (\zeta_{xx} + \zeta_{yy}). \quad (5.16)$$

Now, let us consider the linear combinations of $\eta$ and $\zeta$:

$$\psi = \eta + b \zeta, \quad \text{where } b \text{ is a constant},$$

such that they satisfy the linear wave equation

$$\psi_{tt} - s^2 (\psi_{xx} + \psi_{yy}) = 0.$$

Using equations (5.15) and (5.16), one obtains

$$\psi_{tt} - s^2 (\psi_{xx} + \psi_{yy})$$

$$= \eta_{tt} + b \zeta_{tt} - s^2 (\eta_{xx} + \eta_{yy}) - b s^2 (\zeta_{xx} + \zeta_{yy})$$

$$= \left( \frac{d(\rho_1 - \rho_2)}{\rho_2} + 1 + \frac{b \rho_1 d}{\rho_2} - s^2 \right) (\eta_{xx} + \eta_{yy})$$

$$+ \left( \frac{d(\rho_2 - \rho_1)}{\rho_2} + \frac{b(\rho_2 - \rho_1) d}{\rho_2} - b s^2 \right) (\zeta_{xx} + \zeta_{yy})$$

$$= 0.$$
It gives us the system of equations

\[
\begin{align*}
\frac{d(\rho_1 - \rho_2)}{\rho_2} + 1 + \frac{b\rho_2 d}{\rho_2} - s^2 &= 0, \\
\frac{d(\rho_2 - \rho_1)}{\rho_2} + \frac{b(\rho_2 - \rho_1)d}{\rho_2} - bs^2 &= 0,
\end{align*}
\]

which has two solutions

\[
\begin{align*}
s_+^2 &= \frac{1}{2} + \frac{1}{2} \sqrt{(1 - 2d)^2 + \frac{4d(1-d)\rho_1}{\rho_2}} = \frac{1+\sqrt{D}}{2}, \\
b_1 &= \frac{-2d(\rho_1 - \rho_2) - \rho_2 + \sqrt{4d(1-d)\rho_2(\rho_1 - \rho_2) + \rho_2^2}}{2d\rho_1} = \frac{-2d(\rho_1 - \rho_2) - \rho_2 (1 - \sqrt{D})}{2d\rho_1}, \\
s_-^2 &= \frac{1}{2} - \frac{1}{2} \sqrt{(1 - 2d)^2 + \frac{4d(1-d)\rho_1}{\rho_2}} = \frac{1-\sqrt{D}}{2}, \\
b_2 &= \frac{-2d(\rho_1 - \rho_2) - \rho_2 - \sqrt{4d(1-d)\rho_2(\rho_1 - \rho_2) + \rho_2^2}}{2d\rho_1} = \frac{-2d(\rho_1 - \rho_2) - \rho_2 (1 + \sqrt{D})}{2d\rho_1},
\end{align*}
\]

where \( D = (1 - 2d)^2 + \frac{4d(1-d)\rho_1}{\rho_2} \).

The derived wave speeds \( s_+ \) and \( s_- \) coincide with the previously obtained wave speeds (3.42). The modal equations (3.49)-(3.51) in \((x, y, z)\) coordinate system in the two-layer case have the form

\[
\begin{align*}
s^2 \phi_{zz} &= 0 \quad \text{at } 0 < z < 1, \\
s^2 \phi_z - \phi &= 0 \quad \text{at } z = 1, \\
\phi &= 0 \quad \text{at } z = 0.
\end{align*}
\]

We obtain the modal function \( \phi \) in the form

\[
\phi = \begin{cases} 
\Lambda (s^2 - 1 + z) & \text{at } d < z < 1, \\
(s^2 - 1 + d) \frac{z}{d} & \text{at } 0 < z < d,
\end{cases}
\]

\( \Lambda \) is an integration constant.
where $\Lambda$ is a constant. For the surface mode, the wave speed $s^2 = s^2_+$ and we set $\phi_s = 1$ at $z = 1$, which implies $\Lambda = 1/s_+^2$. The modal function for the surface mode is

$$\phi_s = \begin{cases} \frac{s_+^2 - 1 + z}{s_+^2} & \text{at } d < z < 1, \\ \frac{(s_+^2 - 1 + d)z}{s_+^2 d} & \text{at } 0 < z < d. \end{cases}$$

In the interfacial mode, the wave speed $s^2 = s^-_2$ and we set $\phi_i = 1$ at $z = d$, which implies $\Lambda = 1/(s^-_2 - 1 + d)$. The modal function for the interfacial mode is

$$\phi_i = \begin{cases} \frac{s^-_2 - 1 + z}{s^-_2 - 1 + d} & \text{at } d < z < 1, \\ \frac{z}{d} & \text{at } 0 < z < d. \end{cases}$$

In our derived equation,

$$\zeta = (A_s \phi_s + A_i \phi_i)_{z=d}, \quad \eta = (A_s \phi_s + A_i \phi_i)_{z=1},$$

where $A_s$ denotes the wave amplitude of the surface wave at $z = 1$ and $A_i$ denotes the wave amplitude of the interfacial wave at $z = d$. Thus,

$$\psi_1 \quad \text{and} \quad \psi_i,$$

where

$$P_{1s} = 1 + \frac{2d(\rho_2 - \rho_1) - \rho_2(1 - \sqrt{D})}{2d\rho_1} \cdot \frac{s_+^2 - 1 + d}{s_+^2},$$

$$P_{1i} = \frac{s^-_2}{s^-_2 - 1 + d} + \frac{2d(\rho_2 - \rho_1) - \rho_2 + \rho_2 \sqrt{D}}{2d\rho_1} = 0.$$
and

\[(A_s)_{tt} - s^2_s ((A_s)_{xx} + (A_s)_{yy}) = \frac{\psi_{1tt} - s^2_s (\psi_{1xx} + \psi_{1yy})}{P_{ls}} = 0,\]

the surface mode amplitude \(A_s\) satisfies the 2D linear wave equation

\[(A_s)_{tt} = s^2_s ((A_s)_{xx} + (A_s)_{yy}).\]

Similarly, we can show that \(\psi_2 = P_{2i}A_i\), where

\[P_{2i} = \frac{(1 - 4d)\rho_2 \sqrt{D} + 2\rho_2 D}{2d \rho_1 (2d - 1) - 2d \rho_1 \sqrt{D}}.\]

So the interfacial mode amplitude \(A_i\) also satisfies another 2D linear wave equation

\[(A_i)_{tt} = s^2_\pm ((A_i)_{xx} + (A_i)_{yy}).\]

Therefore, both the surface wave height \(A_s\) and interfacial wave height \(A_i\) satisfy the 2D linear wave equations

\[A_{tt} - s^2 (A_{xx} + A_{yy}) = 0,\] (5.20)

where \(s = s_+\) for the surface wave, and \(s = s_-\) for the interfacial wave.
5.2 Nonlinear propagation of concentric waves

The linear surface and interfacial waves in the two-layer fluid model without shear flow satisfy the 2D linear wave equations. The subsequent nonlinear evolution of the long weakly-nonlinear waves is described by nonlinear cKdV equation. We consider two examples to show the combined effect of nonlinearity and dispersion on the propagation of the waves: the first example is the concentric wave generated from a localised source in the neighbourhood of a point and the second one is a version of the 2D dam break problem.

5.2.1 Example 1: concentric wave generated from a localised source

We are interested in the concentric waves, which are generated from a neighbourhood of a point. The linear problem has been studied by Dobrokhotov and Sekerzh-Zen’kovich [25] and they gave an explicit analytical solution to the 2D linear wave equation

\[ A_{tt} - s^2(A_{xx} + A_{yy}) = 0, \]
5. CONCENTRIC WAVES WITHOUT SHEAR FLOW — NUMERICAL RESULTS

Figure 5.2: Initial values in \((\xi, R)\) coordinates at \(R_0 = 0.1\) in the form

\[
A(x, y, t) = Q \Re \left( \frac{1 + ist/v}{\left( (1 + ist/v)^2 + (x^2 + y^2)/v^2 \right)^{3/2}} \right),
\]

Here \(x^2 + y^2\) can be replaced by \(r^2\) to map the solution from \((x, y, z)\) coordinates to \((r, \theta, z)\) coordinates, and the angle variable \(\theta\) is not presented in the solution because of radial symmetry. Then, the linear solution is

\[
A(r, t) = Q \Re \left( \frac{1 + ist/v}{\left( (1 + ist/v)^2 + r^2/v^2 \right)^{3/2}} \right),
\]

(5.21)

where \(s\) is the wave speed, \(Q\) is the wave height factor and \(v\) is an arbitrary positive constant. This solution is generated by a localised initial condition shown in Figure 5.1 (with \(Q = 5, v = 0.5\)).

To solve the derived nonlinear cKdV equation by the numerical method introduced in Chapter 4, we need to know the value of the wave amplitude at \(R_0 = \varepsilon r_0\). In this
example, we assume $\varepsilon = 0.02$ and take the initial condition (5.21) written in $(\xi, R)$ coordinates at $R = R_0$ is

$$A(\xi, R_0) = 5 \text{Re} \left( \frac{1 + 2i(50R_0 - \xi)}{((1 + 2i(50R_0 - \xi))^2 + (100R_0)^2)^{3/2}} \right).$$

We use one and the same initial condition for both surface and interfacial waves. This initial condition in $(\xi, R)$ coordinates at $R = R_0 = 0.1$ is shown in Figure 5.2.
5. CONCENTRIC WAVES WITHOUT SHEAR FLOW — NUMERICAL RESULTS

Figure 5.4: Concentric surface waves generated from a localised source ($d = 0.6$)
5. CONCENTRIC WAVES WITHOUT SHEAR FLOW — NUMERICAL RESULTS

5.2.1.1 Numerical results for surface wave

Let us substitute parameters $\rho_1 = 1$, $\rho_2 = 1.2$ and two values of the lower layer depth $d = 0.5$ and $d = 0.6$, which are the same parameters used in the discussion of the wavefronts in Chapter 3, into the wave speed formula (3.42) and the cKdV-type equation (3.29). At $d = 0.5$, we obtain the wave speed of surface wave $s_+ \approx 0.9780$ and the Cauchy problem

$$-2.0911 A_R - 3.1366 A_{\xi} - 0.3153 A_{\xi\xi\xi} - 1.0455 \frac{A}{R} = 0,$$

$$A(\xi, 0.1) = 5 \text{Re} \left( \frac{1 + 2i(5 - \xi)}{(1 + 2i(5 - \xi))^2 + 100)^{3/2}} \right),$$

and at $d = 0.6$, the wave speed of surface wave $s_+ \approx 0.9789$ and the Cauchy problem is

$$-2.1358 A_R - 3.2015 A_{\xi} - 0.3236 A_{\xi\xi\xi} - 1.0679 \frac{A}{R} = 0,$$

$$A(\xi, 0.1) = 5 \text{Re} \left( \frac{1 + 2i(5 - \xi)}{(1 + 2i(5 - \xi))^2 + 100)^{3/2}} \right).$$

Using the numerical method described in Chapter 4, the profiles of the surface wave at times $t = 5$, 10, 15, 20 are plotted in Figure 5.3 for $d = 0.5$ and Figure 5.4 for $d = 0.6$. We also plot the cross-section along directions $\theta = 0$ and $\theta = \pi$, in Figure 5.5 ($d = 0.5$) and Figure 5.6 ($d = 0.6$). The profiles of the surface waves are very similar when $d = 0.5$ and $d = 0.6$.

At $t = 20$ and $d = 0.5$, for example, the linear wave solution for the surface mode and the nonlinear wave solution are shown in Figure 5.7. We see that nonlinearity and dispersion, acting together, yield the generation of an oscillatory dispersive wave train.
5. CONCENTRIC WAVES WITHOUT SHEAR FLOW — NUMERICAL RESULTS

5.2.1.2 Numerical results for interfacial waves

Substituting the same parameters into the wave speed formula (3.42) and the cKdV-type equation (3.29), for $d = 0.5$, we obtain the wave speed of the interfacial wave $s_\pi \approx 0.2087$ and the Cauchy problem

$$-0.4182 A_R + 0.05988 A A_\xi - 0.01535 A_{\xi\xi\xi} - 0.2091 \frac{A}{R} = 0,$$

$$A(\xi,0.1) = 5 \text{ Re} \left( \frac{1 + 2i(5 - \xi)}{((1 + 2i(5 - \xi))^2 + 100)^{3/2}} \right),$$

Figure 5.5: Surface waves in the directions $\theta = 0$ and $\theta = \pi$ for $d = 0.5$ when $t = 0$ (blue), $t = 5$ (black), $t = 10$ (red), $t = 15$ (green) and $t = 20$ (cyan).
5. CONCENTRIC WAVES WITHOUT SHEAR FLOW — NUMERICAL RESULTS

Figure 5.6: Surface waves in the directions $\theta = 0$ and $\theta = \pi$ for $d = 0.6$ when $t = 0$ (blue), $t = 5$ (black), $t = 10$ (red), $t = 15$ (green) and $t = 20$ (cyan).

Figure 5.7: The linear (blue) and nonlinear (red) surface waves at $t = 20$. 
Figure 5.8: Concentric interfacial waves generated from a localised source ($d = 0.5$)
Figure 5.9: Concentric interfacial waves generated from a localised source ($d = 0.6$)
Figure 5.10: Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.5$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan).

and for $d = 0.6$, the wave speed of the interfacial wave $s_- \approx 0.2043$ and the Cauchy problem

$$-0.4272 A_R + 0.6719 A A_\xi - 0.0150 A \xi A_\xi - 0.2136 \frac{A}{R} = 0,$$

$$A(\xi, 0.1) = 5 \text{ Re} \left( \frac{1 + 2i(5 - \xi)}{(1 + 2i(5 - \xi))^2 + 100} \right).$$

The profiles of the interfacial wave at times $t = 20, 40, 60, 80$ are shown in Figure 5.8 ($d = 0.5$) and Figure 5.9 ($d = 0.6$). The cross-section along the direction $\theta = 0$ and $\theta = \pi$ is plotted in Figure 5.10 and Figure 5.11 for $d = 0.5$ and 0.6 respectively, to show the wave profiles. The profiles of interfacial wave are also very similar when $d = 0.5$ and $d = 0.6$

At $t = 80$ and $d = 0.5$, for example, the the linear wave and the numerical solution
Figure 5.11: Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.6$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan).
are shown in Figure 5.12, to illustrate the effect of nonlinearity and dispersion on the interfacial waves. For the chosen values of parameters, the effects of nonlinearity and dispersion are less pronounced than for the surface mode, although qualitative behaviour is similar.

5.2.1.3 Comparison of surface and interfacial waves

The surface and interfacial waves at the same time, for example $t = 20$ at $d = 0.5$, are compared in Figure 5.13. The surface wave propagates faster than the interfacial wave. When the two waves get to the same radius $r$, for example $r = 15$ at $d = 0.5$, we can compare the surface and interfacial waves as shown in Figure 5.14. For $r_0 = 5$ (initial condition), both waves have the same shape, but they are different afterwards. We can conclude that for chosen set of parameters the effect of nonlinearity is weaker on interfacial waves than on surface waves. The wave height of the interfacial wave decreases slower and its profile remains close to the linear wave profile within the
Figure 5.13: The surface (blue) and interfacial (red) waves at $t = 20$

Figure 5.14: The surface (blue) and interfacial (red) waves at $r = 15$ ($R = 0.3$)
scope of this model.
5.2.2 Example 2: the 2D dam break concentric waves

In the example 1, we used the waves generated from a localised source in the neighbourhood of a point. In this example, we will consider a version of the 2D dam break problem, in which the wave length is larger. In the same two-layer model, the fluid heights of both upper and lower layers are greater in the central area of the dam, which is released at time $t = 0$. To avoid numerical instability at the sharp boundaries, we use hyperbolic tangent functions to approximate the initial condition as

$$\tilde{A}(x, y, 0) = Q\left(\frac{\tanh\left(-\nu(x^2 + y^2 - \tilde{r}_0^2)\right)}{2} + \frac{1}{2}\right),$$

where $\tilde{r}_0$ is the location of the dam and $\nu$ and $Q$ are suitable positive constants. The initial condition is shown in Figure 5.15 (with $Q = 1$, $\nu = 0.15$, and $\tilde{r}_0 = 8$).
Figure 5.16: Linear surface waves at $y = 0$ for $d = 0.5$ when $t = 0$ (blue), $t = 10$ (black), $t = 20$ (red), $t = 30$ (green) and $t = 40$ (cyan).

Figure 5.17: Linear surface waves at $y = 0$ for $d = 0.6$ when $t = 0$ (blue), $t = 10$ (black), $t = 20$ (red), $t = 30$ (green) and $t = 40$ (cyan).
5.2.2.1 Numerical results for surface wave

Considering the same two-layer model as in example 1, the wave speed of the surface wave is 
\[ s_+ \approx 0.9780 \text{ for } d = 0.5 \text{ and } s_+ \approx 0.9789 \text{ for } d = 0.6. \]

The linear Cauchy problems, choosing the location of ‘dam’ at \( \tilde{r}_0 = 8 \),

\[
\dot{A}_{tt} - s_+^2 (\dot{A}_{xx} + \dot{A}_{yy}) = 0, \\
A(x, y, 0) = Q \left( \tanh \left( -0.15(x^2 + y^2 - 64) \right) \right) + \frac{1}{2},
\]

can be solved numerically. The cross-section diagrams at \( t = 0, 10, 20, 30, 40 \) at \( y = 0 \) are plotted in Figure 5.16 for \( d = 0.5 \) and Figure 5.17 for \( d = 0.6 \) choosing \( Q = 1 \). Note that, the wave height parameter \( Q \) can be chosen arbitrarily because these equations are linear equations. Then the numerical solutions of linear wave equations can be set as the initial condition of the cKdV equation. Here we assume \( \varepsilon = 0.02 \) and choose the initial point of nonlinear waves \( r_0 = 12 (R_0 = \varepsilon r_0 = 0.24) \). The coefficients of the cKdV equation are the same as those in example 1. The nonlinear Cauchy problems are

\[
-2.0911A_R - 3.1366AA_\xi - 0.3153A_{\xi\xi\xi} - 1.0455\frac{A}{R} = 0, \\
A(\xi, 0.24) = A(12, 0, 12 - \xi, 0.9780),
\]

for \( d = 0.5 \) and

\[
-2.1358A_R - 3.2015AA_\xi - 0.3236A_{\xi\xi\xi} - 1.0679\frac{A}{R} = 0, \\
A(\xi, 0.24) = A(12, 0, 12 - \xi, 0.9789),
\]

for \( d = 0.6 \).

By using the numerical method derived in Chapter 4, the cKdV equations can be solved. Here we choose different values of the wave height parameter \( Q \) to show
Figure 5.18: Nonlinear surface waves in the directions $\theta = 0$ and $\theta = \pi$ for $d = 0.5$ when $t = 0$ (blue), $t = 10$ (black), $t = 20$ (red), $t = 30$ (green) and $t = 40$ (cyan).
The amplitude of initial condition is $Q = 10$ (1) $Q = 20$ (2) $Q = 30$ (3) $Q = 40$ (4)

Figure 5.19: Nonlinear surface waves in the directions $\theta = 0$ and $\theta = \pi$ for $d = 0.6$ when $t = 0$ (blue), $t = 10$ (black), $t = 20$ (red), $t = 30$ (green) and $t = 40$ (cyan).
the more pronounced effect of nonlinearity at a fixed value of the radius, as the wave height factors increase. The cross-section diagrams along the directions $\theta = 0$ and $\theta = \pi$ at a series of times with several values of $Q$ are shown in Figure 5.18 ($d = 0.5$) and Figure 5.19 ($d = 0.6$). We see the formation of dispersive shock waves, similar to the waves described by the KdV equation (see [38; 40] and references therein). The 1D dam break problem for a two-layer fluid was recently considered by Esler and Pearce [29] (see also [22] and [64]). The profiles of surface waves for $d = 0.5$ and $d = 0.6$ are similar. At time $t = 30$, $Q = 40$ and $d = 0.5$ for example, the linear wave solution and the nonlinear wave solution are shown in the same Figure 5.20. In this figure, we can see that the linear wave is propagating slower than the nonlinear wave and the nonlinearity and dispersion result in the formation of two dispersive shock waves.
Figure 5.21: Linear interfacial waves at $y = 0$ for $d = 0.5$ when $t = 0$ (blue), $t = 50$ (black), $t = 100$ (red), $t = 150$ (green) and $t = 200$ (cyan).
5.2.2.2 Numerical results for interfacial waves

Considering the same two-layer example, the wave speed of interfacial wave is \( s_\approx 0.2087 \). The linear Cauchy problem, also choosing the location of ‘dam’ at \( r_0 = 8 \),

\[
\begin{align*}
\tilde{A}_H - 0.2087^2 (\tilde{A}_{xx} + \tilde{A}_{yy}) &= 0, \\
\tilde{A}(x, y, 0) &= Q \left( \tanh \left( \frac{-0.15(x^2 + y^2 - 64)}{2} \right) + \frac{1}{2} \right),
\end{align*}
\]

can be solved numerically. The cross-section diagrams at \( t = 0, 50, 100, 150, 200 \) at \( y = 0 \), is shown in Figure 5.21 for \( Q = 1 \). Similarly, we use the numerical solution of linear wave equation as the initial condition for the cKdV equation and choose the same parameters \( \epsilon = 0.02, r_0 = 12 \) and \( R_0 = 0.24 \). The nonlinear Cauchy problem is

\[
-0.4182 A_R + 0.05988 A A_\xi - 0.01535 A_{\xi\xi\xi} - 0.2091 \frac{A}{R} = 0, \\
A(\xi, 0.24) = \tilde{A}(12, 0, \frac{12 - \xi}{0.2087}).
\]

In Figure 5.22, the cross-section diagrams along the directions \( \theta = 0 \) and \( \theta = \pi \) of interfacial waves are shown at \( t = 0, 50, 100, 150, 200 \) and \( Q = 10, 20, 30, 40 \). In Figure 5.23 the linear and nonlinear wave solutions are shown for \( t = 150 \) and \( Q = 40 \).

We can see for this set of parameters that the effect of nonlinearity is much weaker for the interfacial waves than for the surface waves. This is because the coefficients \( \mu_2 \) of quadratic term is smaller than other coefficients. This phenomenon is also observed in plane waves because the coefficient of the quadratic term tends to 0 in the two-layer case when \( d \approx 0.5 \). This problem has been considered by Knickerbocker and Newell [49]. Let us choose \( d = 0.6 \) to show the stronger effect of the lower layer depth \( d \).
5. CONCENTRIC WAVES WITHOUT SHEAR FLOW — NUMERICAL RESULTS

Figure 5.22: Nonlinear interfacial waves in the directions $\theta = 0$ and $\theta = \pi$ when $t = 0$ (blue), $t = 50$ (black), $t = 100$ (red), $t = 150$ (green) and $t = 200$ (cyan) when $d = 0.5$. 
Figure 5.23: The linear (blue) and nonlinear (red) interfacial waves at $t = 150$ for $Q = 40$ when $d = 0.5$. 
Figure 5.24: Linear interfacial waves in the directions $\theta = 0$ and $\theta = \pi$ for $d = 0.6$ when $t = 0$ (blue), $t = 50$ (black), $t = 100$ (red), $t = 150$ (green) and $t = 200$ (cyan).
The amplitude of initial condition is $Q = 10$.

The amplitude of initial condition is $Q = 20$.

The amplitude of initial condition is $Q = 30$.

The amplitude of initial condition is $Q = 40$.

Figure 5.25: Nonlinear interfacial waves in the directions $\theta = 0$ and $\theta = \pi$ when $t = 0$ (blue), $t = 50$ (black), $t = 100$ (red), $t = 150$ (green) and $t = 200$ (cyan) when $d = 0.6$. 
Figure 5.26: The linear (blue) and nonlinear (red) interfacial waves at $t = 150$ for $Q = 40$ when $d = 0.6$
In this case, the wave speed is \( s_\pm = 0.2043 \). The linear Cauchy problem is

\[
\ddot{A} - 0.2043^2 (\ddot{A}_{xx} + \ddot{A}_{yy}) = 0, \\
\dot{A}(x, y, 0) = Q \left( \tanh \left( -0.15 (x^2 + y^2 - 64) \right) + \frac{1}{2} \right),
\]

and the numerical solutions for this linear Cauchy problem on directions 0 and \( \pi \) at times \( t = 0, 50, 100, 150, 200 \) are shown in Figure 5.24. The nonlinear Cauchy problem becomes

\[
-0.4272 \dot{A} + 0.6719 A \dot{A} - 0.0150 A_{\xi\xi\xi} - 0.2136 \frac{A}{R} = 0, \\
A(\xi, 0.24) = \dot{A}(12, 0, \frac{12 - \xi}{0.2043}).
\]

The value of \( \mu_2 \) in this case is larger than other coefficients. The cross-section diagram along the directions \( \theta = 0 \) and \( \pi \) are shown in Figure 5.25 at times \( t = 0, 50, 100, 150, 200 \) for \( Q = 10, 20, 30, 40 \). In this case, the effect of nonlinearity can be seen more clearly. Figure 5.26 shows the linear and nonlinear waves at time \( t = 150 \) for \( Q = 40 \). We can conclude that strong and well developed dispersive shock waves are formed at the interface.
5.3 Concluding remarks

In this chapter, we first reviewed the derivation of the 2D linear wave equations for a two-layer fluid without shear flow. These equations are used to describe the initial evolution of the waves for a given initial condition. At higher order, the waves are described by the nonlinear cKdV equation. To study the effect of nonlinearity, we chose two examples: concentric waves generated from a localised source in the neighbourhood of a point and the 2D dam break waves.

In the first example, the explicit linear wave solution is available. We solved the cKdV equation with this explicit linear wave solution used at \( r = r_0 \) numerically. From the numerical solution for surface and interfacial waves, we can conclude that:

1. both linear and nonlinear surface waves propagate faster than interfacial waves;
2. the nonlinear wave amplitude decreases faster for the surface wave than for the interfacial wave, both for fixed time interval or a fixed distance from the source;
3. nonlinearity and dispersive action together yield the development of oscillatory wave trains.

In the second example, we firstly solved the 2D linear wave equation with the dam break initial condition. Then the cKdV equation with this numerical linear wave solution at \( r_0 \) was solved. The wave length of the dam break wave was larger than in the first example. In this example, we also considered the influence of the wave amplitude and the depth of the lower layer. In addition to the conclusions obtained in example 1, we can conclude that:

1. the increase of the initial wave amplitude can strengthen the effect of nonlinearity at a fixed distance from the source;
2. the nonlinearity is weak when \( d = 0.5 \) in a two-layer fluid in the cKdV equation;
3. the nonlinearity and dispersion yield the development of dispersive ring shock waves.
Chapter 6

Ring waves on a shear flow—numerical results

In Chapter 5, we obtained the numerical solutions for the concentric waves, when there is no shear flow. In this chapter, we will illustrate the effect of the shear flow on the ring waves.

We will first introduce a model initial condition, given by a distorted solution of the 2D linear wave equation [25], where the wave amplitude depends on a direction. Then we numerically solve the Cauchy problem for the cKdV-type equation (3.29) with this initial condition. Using the numerical method introduced in Chapter 4, we obtain the numerical solution for surface and interfacial ring waves with the shear flow. We then discuss the qualitative effect of the shear flow.
6.1 Model initial condition for ring waves on a shear flow

In a two-layer fluid, the wave amplitude does not satisfy the 2D linear wave equation any more when the shear flow is present. Recently, Arkhipov et al [4] researched long nonlinear ring waves on the interface of a two-layer fluid with a piecewise-constant shear flow using a coupled system of equations [25]. In this chapter, we solve a qualitatively similar problem using the following model initial condition. At $R = R_0 = 0.1$ (where $R_0 = \varepsilon r_0(\theta)k(\theta)$ with $k(\theta)$ described in Chapter 3), we define:

$$A(\xi, R_0, \theta) = 5\left(4 - \frac{3|\pi - \theta|}{\pi}\right)\text{Re}\left(\frac{1 + 2i(50R_0 - \xi)}{(1 + 2i(50R_0 - \xi))^2 + (100R_0)^2}^{3/2}\right),$$

which is the analytical solution to the 2D linear wave equation used in the previous chapter where now the amplitude depends on the direction. This model condition is shown in Figure 6.1 when $\theta = 0$ and $\pi$, $R_0 = 0.1$. Here, the wave height is four times...
higher in the direction against shear flow ($\theta = \pi$) than that in the direction along the shear flow. This model initial condition mimics the properties of the waves in [4]: the wave is lower along the shear flow and higher against the shear flow. This initial condition is not continuous at $r = 0$. However, we are only interested in the initial condition at $r = r_0(\theta)$ in the time period $t \in [0, t_1]$ and the wave amplitude at $r = 0$ is not used. We would like to compare the qualitative features of our model problem with the character of solutions in [25]. In this section, we choose the same parameters as in example 1 of Chapter 5. The densities of the two layers are $\rho_1 = 1$, $\rho_2 = 1.2$ and $\varepsilon = 0.02$. Two values of the depth of the lower layer are considered: $d = 0.5$ and $d = 0.6$.

At $d = 0.5$, the surface and interfacial wave speeds without shear flow are $s_+ \approx 0.9780$ and $s_- \approx 0.2087$ and they are $s_+ \approx 0.9789$ and $s_- \approx 0.2043$ at $d = 0.6$
6. RING WAVES ON A SHEAR FLOW—NUMERICAL RESULTS

Figure 6.2: Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.5$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan) for $U_1 - U_2 = 0.05$.

6.2 Numerical results for interfacial waves on a shear flow

We derived the 2+1-dimensional cKdV-type equation (3.29) and given formulae (3.55) for the coefficients of interfacial waves in Chapter 3. The coefficients depend on $\theta$. We solve this equation numerically using the method described in Chapter 4. When the shear flow is absent, the effects of nonlinearity and dispersion have been studied in Chapter 5 the example 1. Here we plot the cross-section diagrams in downstream ($\theta = 0$) and upstream ($\theta = \pi$) directions for $U_1 - U_2 = 0.05$ (Figure 6.2 for $d = 0.5$ and Figure 6.3 for $d = 0.6$) and $U_1 - U_2 = 0.1$ (Figure 6.4 for $d = 0.5$ and Figure 6.5 for $d = 0.6$).

We can see that the wave height of interfacial waves decreases faster upstream. Indeed, for $d = 0.5$ when $t = 80$, the wave height in the upstream direction is 0.7392 (3.70% of initial wave height) when $U_1 - U_2 = 0.05$ and 0.6158 (3.08% of initial wave height) when $U_1 - U_2 = 0.1$, while in the downstream direction it is 0.2188 (4.38% of
Figure 6.3: Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.6$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan) for $U_1 - U_2 = 0.05$.

Figure 6.4: Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.5$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan) for $U_1 - U_2 = 0.1$. 
6. RING WAVES ON A SHEAR FLOW—NUMERICAL RESULTS

Figure 6.5: Interfacial waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.6$ when $t = 0$ (blue), $t = 20$ (black), $t = 40$ (red), $t = 60$ (green) and $t = 80$ (cyan) for $U_1 - U_2 = 0.1$.

initial wave height) when $U_1 - U_2 = 0.05$ and 0.2242 (4.5% of initial wave height) when $U_1 - U_2 = 0.1$. For $d = 0.6$ when $t = 80$, the wave height in the upstream direction is 0.6146 (3.07% of initial wave height) when $U_1 - U_2 = 0.05$ and 0.5289 (2.64% of initial wave height) when $U_1 - U_2 = 0.1$, while in the downstream direction it is 0.2179 (4.36% of initial wave height) when $U_1 - U_2 = 0.05$ and 0.2240 (4.48% of initial wave height) when $U_1 - U_2 = 0.1$. In Arkhipov et al [4], interfacial waves along the shear flow have smaller wave height and decrease slower. Thus there is a qualitative agreement.

We plot the interfacial waves along the shear flow at $t = 80$ and $d = 0.5$ for the shear flow $U_1 - U_2 = 0$, 0.05, and 0.1 in Figure 6.6 and against the shear flow for the shear flow $U_1 - U_2 = 0.05$ and 0.1 in Figure 6.7. We can see the shear flow increases the wave speed downstream and decreases the wave speed upstream. We also can see that the speed changes more in the upstream rather than the downstream direction. This coincides with the result in Chapter 3 that the interfacial ring waves squeeze the
Figure 6.6: Interfacial waves in the directions $\theta = 0$ (downstream) at $t = 80$ for $U_1 - U_2 = 0$ (red), 0.05 (black) and 0.1 (blue).

Figure 6.7: Interfacial waves in the directions $\theta = \pi$ (upstream) at $t = 80$ for $U_1 - U_2 = 0.05$ (black) and 0.1 (blue).
wavefront in the direction of the flow. We also observe that the wave height decreases slower downstream and faster upstream, as the shear flow strengthens.
6.3 Numerical results for surface waves on a shear flow

The coefficients of the 2+1-dimensional cKdV-type equation (3.29) are given in (3.54). Using the same model initial condition, we solve this Cauchy problem numerically. We plot the cross-section diagrams in the downstream ($\theta = 0$) and upstream ($\theta = \pi$) directions for $U_1 - U_2 = 0.1$ (Figure 6.8 when $d = 0.5$ and Figure 6.9 when $d = 0.6$). In these figures, the wave height also decreases faster upstream ($0.5765, 2.88\%$ of initial wave height for $d = 0.5$ and $0.5260, 2.63\%$ of initial wave height for $d = 0.6$ when $t = 20$) than downstream ($0.1555, 3.11\%$ of initial wave height for $d = 0.5$ and $0.1423, 2.85\%$ of initial wave height for $d = 0.6$ when $t = 20$). We plot the surface waves along the shear flow at $t = 20$ and $d = 0.5$ when the shear flow $U_1 - U_2 = 0$ and $0.1$ (Figure 6.10). The effect of the same shear flow on the surface waves is weaker than its effect on the interfacial waves (see Figure 6.6), because the shear flow speed $U_1 - U_2 = 0.1$ is
Figure 6.9: Surface waves in the directions $\theta = 0$ (downstream) and $\theta = \pi$ (upstream) for $d = 0.6$ when $t = 0$ (blue), $t = 5$ (black), $t = 10$ (red), $t = 15$ (green) and $t = 20$ (cyan) for $U_1 - U_2 = 0.1$. 
much smaller than the surface wave speed without shear flow. Qualitatively, the shear flow has similar effect on the surface waves as on the interfacial waves. The important difference is that the shear flow elongates the wavefronts of the surface waves.

Figure 6.10: Surface waves in the directions $\theta = 0$ (downstream) at $t = 20$ for $U_1 - U_2 = 0$ (red) and 0.1 (blue).
6.4 Concluding remarks

In this Chapter, we introduced a model initial condition with the wave height being lower in the direction along the shear flow. Continuing the study in the example 1 of Chapter 5, we used the numerical solution of the 2+1-dimensional cKdV-type equation with this model initial condition to show the effect of the shear flow on the surface and interfacial ring waves. From the plots of the numerical solutions, we can conclude that:

(1) the wave height decreases slower downstream than upstream for both surface and interfacial waves;

(2) the shear flow increases the wave speed downstream and decreases the wave speed upstream for both surface and interfacial waves, but it squeezes the wavefronts of the interfacial waves and elongates the wavefronts of the surface waves in the direction of the shear flow;

(3) the same shear flow affects the interfacial waves stronger than the surface waves.
Chapter 7

Conclusions and future work

7.1 Conclusions

The main result of this thesis is the derivation and application of a new 2+1-dimensional cKdV-type equation (3.29) to model the surface and internal ring waves in a stratified fluid with a shear flow.

The developed linear formulation provides, in particular, a description of the distortion of the shape of the wavefronts of the ring waves by the shear flow, which has been illustrated by considering the classical setting of a two-layer fluid with a piecewise-constant shear flow. The wavefronts of surface and interfacial ring waves were described in terms of two branches of the singular solution of the derived nonlinear first-order differential equation, constituting further generalisation of the well-known Burns and generalised Burns conditions [17; 43]. Remarkably, the two branches of this singular solution could be described in parametric form, and an explicit analytical solution was developed for the wavefront of the interfacial mode in the case of the low density contrast. The constructed solutions have revealed the qualitatively different behaviour of the wavefronts of surface and interfacial waves propagating over the same shear flow. Indeed, while the wavefront of the surface ring wave is
7. CONCLUSIONS AND FUTURE WORK

elongated in the direction of the flow, the wavefront of the interfacial wave is squeezed in this direction.

The derived 2+1-dimensional cKdV-type equation constitutes generalisation of the previously derived 1+1-dimensional equation for the surface waves in a homogeneous fluid over a shear flow [43] and internal waves in a stratified fluid in the absence of a shear flow [57]. Strictly speaking, Johnson has derived a 2+1-dimensional model [43], but as a by-product of our study we have shown that the complicated formula for one of the coefficients of his equation will yield zero coefficient for any shear flow, which then reduces the equation to a 1+1-dimensional model. For the case of the two-layer model we also derived a constraint on the strength of the shear flow, which guarantees that there are no critical layers, and obtained explicit expressions for the coefficients of the derived amplitude equation in terms of the physical and geometrical parameters of the model, which provides a fully developed asymptotic theory for this case.

The finite-difference scheme, developed from the scheme given by Feng and Mitsui [30] for KdV and KP equations, has been used in order to solve the derived 2+1-dimensional cKdV-type equation numerically. We used the waves generated from a localised source without and with shear flow and 2D cylindrical dam break problem as examples in order to study the effects of nonlinearity and dispersion. The following observations were made:

(1) The propagation speed is faster at the surface than the interface for both linear and nonlinear waves. It is affected by the shear flow: increases downstream and decreases upstream. Quantitative effect is different: the shear flow squeezes the wavefronts of the interfacial waves and elongates the wavefronts of the surface waves in the direction of the shear flow.

(2) The wave height decreases in every direction, and it decreases faster at the surface than at the interface through the same time or the same distance. The shear flow led to the wave height decreasing slower downstream and faster upstream for
both surface and interfacial waves. This effect becomes more prominent as the shear flow increases.

(3) The combined action of nonlinearity and dispersion has led to the emergence of either oscillatory wave trains or dispersive shock waves.

7.2 Future work

In this thesis, we used a two-layer model as an example to illustrate the key constructions of the theory in Chapter 3 and give numerical solutions in Chapters 5 and 6. In this model, fluid has discontinuous stratification. But in the real oceanic problems, density is a continuous function of depth and the changing rate is large at pycnocline layer, whose thickness is assumed to be zero in the two-layer model. Future work will include applications to the cases with continuous stratification. Moreover, the shear flow was also discontinuous in our example, when present. Considering continuous shear flows is another reasonable extension of the work presented in this Thesis.

In Chapter 6, we used a model initial condition in order to study the effect of the shear flow on the ring waves generated from a localised source. In future studies, we would like to consider the generation of the linear wave in the near field with shear flow, which will provide a realistic initial condition.

A motivation for our study is to model the oceanic waves with curvilinear fronts, registered by satellites. It would be interesting to use our model with the realistic oceanic data and compare our results with the observations of the waves in the oceans.

In additional, similar equations can be derived and studied in other physical contexts.
Appendix A: MATLAB Code for solving the cKdV-type equation

Here we give an example of the code used to solve the the $2 + 1$ - dimensional cKdV -type equation: surface ring waves with a model initial condition. The following code is used to plot Figure 6.8 in Chapter 6.

***************surfacering.m***************

cic;
clear;

mu=0.5;
c=0.9779751861;

Xmin= -15;  % Rmin/eps - c*tmax  0.1/0.02- 0.9779751861*20
Xmax= 28;   % (Rmax/eps)-c*tmin(==0) :0.56/0.02
dx= 0.05;
L= (Xmax-Xmin)/dx +1;

eps=0.02;

amin=-1.049;
amax=0.9476;
APPENDIX A: MATLAB CODE FOR SOLVING THE CKDV-TYPE EQUATION

M=18;
da=(amax-amin)/(M-1);

Rmin= 0.1;
N= 231; %Rmax=0.56 (0.56-0.1)/dR+1
dR= 0.002;

xi=zeros(1,L);
R=zeros(1,N);
aa=zeros(1,M);
theta=theta=zeros(1,M);

z=zeros(N,L*M);

cof=zeros(L*M,L*M);
nu=zeros(L*M,1);
zzer=zeros(L*M,1);
zzz=zeros(L*M,1);

for m=1:M
    aa(m)=amin+(m-1)*da;
    thetatheta(m)=theta(aa(m));
end

for i=1:L  % Initial value z(i,j), given the value of xi(i), a(j) and theta(j);
    xi(i) = Xmin+(i-1)*dx;
    for m=1:M
        if Rmin/eps/c-xi(i)/c<0
            z(1,i+(m-1)*L)=0;
        else
            z(1,i+(m-1)*L) = initial(Rmin/eps/kk(aa(m)),Rmin/eps/c-xi(i)/c,mu,c);
        end
    end
end
end

R(1)=Rmin;
for k=1:(N-1) % time interval
    R(k+1) = Rmin+k*dR;
    for ik=1:L*M % solve the linear equations, given RHS
        m= floor((ik-1)/L)+1;
        l= mod((ik-1),L)+1;
        if (m==1)
            if (l==1)
                nu(ik)=-mu5(aa(m))/(4*R(k)*da)*z(k,ik+L)
                +mu5(aa(m))/(4*R(k)*da)*z(k,ik+L)
                -mu3(aa(m))/(4*dx^3)*z(k,ik+2)
                +mu3(aa(m))/(2*dx^3)*z(k,ik+1)
                +(mu1(aa(m))/dR-mu4(aa(m))/(2*R(k)))*z(k,ik);
            elseif (l==2)
                nu(ik)=-mu5(aa(m))/(4*R(k)*da)*z(k,ik+L)
                +mu5(aa(m))/(4*R(k)*da)*z(k,ik+L)
                -mu3(aa(m))/(4*dx^3)*z(k,ik+2)
                +mu3(aa(m))/(2*dx^3)*z(k,ik+1)
                + (mu1(aa(m))/dR-mu4(aa(m))/(2*R(k)))*z(k,ik)
                -mu3(aa(m))/(2*dx^3)*z(k,ik-1);
            elseif (l==L-1)
                nu(ik)=-mu5(aa(m))/(4*R(k)*da)*z(k,ik+L)
                +mu5(aa(m))/(4*R(k)*da)*z(k,ik+L)
                +mu3(aa(m))/(2*dx^3)*z(k,ik+1)+ (mu1(aa(m))/dR)
                -mu4(aa(m))/(2*R(k)))*z(k,ik)
                -mu3(aa(m))/(2*dx^3)*z(k,ik-1)
                +mu3(aa(m))/(4*dx^3)*z(k,ik-2);
            elseif (l==L)
                nu(ik)=-mu5(aa(m))/(4*R(k)*da)*z(k,ik+L)
                +mu5(aa(m))/(4*R(k)*da)*z(k,ik+L)
                +mu3(aa(m))/(2*dx^3)*z(k,ik+1)
                +(mu1(aa(m))/dR-mu4(aa(m))/(2*R(k)))*z(k,ik)
                -mu3(aa(m))/(2*dx^3)*z(k,ik-1)
                +mu3(aa(m))/(4*dx^3)*z(k,ik-2);
        end
    end
end
\begin{align*}
&+\left(\frac{\mu_1(aa(m))}{dR} - \frac{\mu_4(aa(m))}{2R(k)}\right) z(k, ik) \\
&-\frac{\mu_3(aa(m))}{2dx^3} z(k, ik-1) \\
&+\frac{\mu_3(aa(m))}{4dx^3} z(k, ik-2); \\
\text{else} \\
&\nu(ik) = -\frac{\mu_5(aa(m))}{4R(k) da} z(k, ik+L) \\
&+\frac{\mu_5(aa(m))}{4R(k) da} z(k, ik+L) \\
&-\frac{\mu_3(aa(m))}{4dx^3} z(k, ik+2) \\
&+\frac{\mu_3(aa(m))}{2dx^3} z(k, ik+1) \\
&+\left(\frac{\mu_1(aa(m))}{dR} - \frac{\mu_4(aa(m))}{2R(k)}\right) z(k, ik) \\
&-\frac{\mu_3(aa(m))}{4dx^3} z(k, ik-1) \\
&+\frac{\mu_3(aa(m))}{2dx^3} z(k, ik-2); \\
\text{end} \\
\text{elseif} (m==M) \\
\text{if} (l==1) \\
&\nu(ik) = -\frac{\mu_5(aa(m))}{4R(k) da} z(k, ik-L) \\
&+\frac{\mu_5(aa(m))}{4R(k) da} z(k, ik-L) \\
&-\frac{\mu_3(aa(m))}{4dx^3} z(k, ik+2) \\
&+\frac{\mu_3(aa(m))}{2dx^3} z(k, ik+1) \\
&+\left(\frac{\mu_1(aa(m))}{dR} - \frac{\mu_4(aa(m))}{2R(k)}\right) z(k, ik) \\
&-\frac{\mu_3(aa(m))}{4dx^3} z(k, ik-1); \\
\text{elseif} (l==2) \\
&\nu(ik) = -\frac{\mu_5(aa(m))}{4R(k) da} z(k, ik-L) \\
&+\frac{\mu_5(aa(m))}{4R(k) da} z(k, ik-L) \\
&-\frac{\mu_3(aa(m))}{4dx^3} z(k, ik+2) \\
&+\frac{\mu_3(aa(m))}{2dx^3} z(k, ik+1) \\
&+\left(\frac{\mu_1(aa(m))}{dR} - \frac{\mu_4(aa(m))}{2R(k)}\right) z(k, ik) \\
&-\frac{\mu_3(aa(m))}{2dx^3} z(k, ik-1); \\
\text{elseif} (l==L-1) \\
&\nu(ik) = -\frac{\mu_5(aa(m))}{4R(k) da} z(k, ik-L) \\
&+\frac{\mu_5(aa(m))}{4R(k) da} z(k, ik-L) \\
&+\frac{\mu_3(aa(m))}{2dx^3} z(k, ik+1) \\
&+\left(\frac{\mu_1(aa(m))}{dR} - \frac{\mu_4(aa(m))}{2R(k)}\right) z(k, ik) \\
&-\frac{\mu_3(aa(m))}{2dx^3} z(k, ik-1) \\
&+\frac{\mu_3(aa(m))}{4dx^3} z(k, ik-2); 
\end{align*}
elseif \( l = L \)

\[
\nu(ik) = -\frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} + \frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} + \frac{\mu_1(aa(m))}{dR - \mu_4(aa(m))} \cdot \frac{(2 \cdot R(k))}{2 \cdot dx^3} \cdot z(k, ik) - \frac{\mu_3(aa(m))}{2 \cdot dx^3} \cdot z(k, ik-1) + \frac{\mu_3(aa(m))}{dR - \mu_4(aa(m))} \cdot \frac{(2 \cdot R(k))}{2 \cdot dx^3} \cdot z(k, ik-2);
\]

else

\[
\nu(ik) = -\frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} + \frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} - \frac{\mu_3(aa(m))}{4 \cdot dx^3} \cdot z(k, ik-1) + \frac{\mu_3(aa(m))}{dx^3} \cdot z(k, ik)}{2 \cdot dx^3} \cdot z(k, ik+2) + \frac{\mu_3(aa(m))}{2 \cdot dx^3} \cdot z(k, ik+1) + \frac{\mu_1(aa(m))}{dR - \mu_4(aa(m))} \cdot \frac{(2 \cdot R(k))}{2 \cdot dx^3} \cdot z(k, ik) - \frac{\mu_3(aa(m))}{2 \cdot dx^3} \cdot z(k, ik-1) + \frac{\mu_3(aa(m))}{dx^3} \cdot z(k, ik-2)}{4 \cdot dx^3} \cdot z(k, ik-2);
\]

end

else

if \( l = 1 \)

\[
\nu(ik) = -\frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} + \frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} - \frac{\mu_3(aa(m))}{4 \cdot dx^3} \cdot z(k, ik-1) + \frac{\mu_3(aa(m))}{2 \cdot dx^3} \cdot z(k, ik+1) + \frac{\mu_1(aa(m))}{dR - \mu_4(aa(m))} \cdot \frac{(2 \cdot R(k))}{2 \cdot dx^3} \cdot z(k, ik) - \frac{\mu_3(aa(m))}{2 \cdot dx^3} \cdot z(k, ik-1) + \frac{\mu_3(aa(m))}{dx^3} \cdot z(k, ik)}{4 \cdot dx^3} \cdot z(k, ik-2);
\]

elseif \( l = 2 \)

\[
\nu(ik) = -\frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} + \frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} - \frac{\mu_3(aa(m))}{4 \cdot dx^3} \cdot z(k, ik+2) + \frac{\mu_3(aa(m))}{2 \cdot dx^3} \cdot z(k, ik+1) + \frac{\mu_1(aa(m))}{dR - \mu_4(aa(m))} \cdot \frac{(2 \cdot R(k))}{2 \cdot dx^3} \cdot z(k, ik) - \frac{\mu_3(aa(m))}{2 \cdot dx^3} \cdot z(k, ik-1) + \frac{\mu_3(aa(m))}{dx^3} \cdot z(k, ik)}{4 \cdot dx^3} \cdot z(k, ik-2);
\]

else \( l = L-1 \)

\[
\nu(ik) = -\frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} + \frac{\mu_5(aa(m))}{(4 \cdot R(k) \cdot da) \cdot z(k, ik-L)} + \frac{\mu_3(aa(m))}{2 \cdot dx^3} \cdot z(k, ik+1)
\]
\[ \begin{align*}
+ \left( \frac{\mu_1(aa(m))}{dR} - \frac{\mu_4(aa(m))}{2R(k)} \right) \ast z(k, ik) \\
- \mu_3(aa(m)) \left( \frac{2}{dx^3} \right) \ast z(k, ik-1) \\
+ \mu_3(aa(m)) \left( \frac{4}{dx^3} \right) \ast z(k, ik-2); \\
\end{align*} \]

\[ \begin{align*}
e\text{ elseif } (l==L) \\
nu(ik) &= -\frac{\mu_5(aa(m))}{(4 \ast R(k) \ast da)} \ast z(k, ik+L) \\
+ \frac{\mu_5(aa(m))}{(4 \ast R(k) \ast da)} \ast z(k, ik-L) \\
+ \left( \frac{\mu_1(aa(m))}{dR} - \frac{\mu_4(aa(m))}{2R(k)} \right) \ast z(k, ik) \\
- \mu_3(aa(m)) \left( \frac{2}{dx^3} \right) \ast z(k, ik-1) \\
+ \mu_3(aa(m)) \left( \frac{4}{dx^3} \right) \ast z(k, ik-2); \\
\end{align*} \]

\[ \begin{align*}
e\text{ else} \\
nu(ik) &= -\frac{\mu_5(aa(m))}{(4 \ast R(k) \ast da)} \ast z(k, ik+L) \\
+ \frac{\mu_5(aa(m))}{(4 \ast R(k) \ast da)} \ast z(k, ik-L) \\
- \mu_3(aa(m)) \left( \frac{4}{dx^3} \right) \ast z(k, ik+2) \\
+ \mu_3(aa(m)) \left( \frac{2}{dx^3} \right) \ast z(k, ik+1) \\
+ \left( \frac{\mu_1(aa(m))}{dR} - \frac{\mu_4(aa(m))}{2R(k)} \right) \ast z(k, ik) \\
- \mu_3(aa(m)) \left( \frac{4}{dx^3} \right) \ast z(k, ik-1) \\
+ \mu_3(aa(m)) \left( \frac{4}{dx^3} \right) \ast z(k, ik-2); \\
\end{align*} \]

\[ \begin{align*}
e\text{ end} \\
\text{end} \\
\text{end} \\
\end{align*} \]

\[ \begin{align*}
\text{for } ii=1:L \ast M \\
m &= \text{ floor}((ii-1)/L)+1; \\
l &= \text{ mod}((ii-1),L)+1; \\
\text{if } (m==1) \\
\text{ if } (l==1) \\
\text{ cof}(ii,ii+L) &= \frac{\mu_5(aa(m))}{(4 \ast R(k+1) \ast da)}; \\
\text{ cof}(ii,ii+L) &= -\frac{\mu_5(aa(m))}{(4 \ast R(k+1) \ast da)}; \\
\text{ cof}(ii,ii+2) &= \mu_3(aa(m)) \left( \frac{4}{dx^3} \right); \\
\text{ cof}(ii,ii+1) &= \mu_2(aa(m)) \left( \frac{4}{dx} \right) \ast z(k, ii+1) - \mu_3(aa(m)) \left( \frac{2}{dx^3} \right); \\
\text{ cof}(ii,ii) &= \mu_1(aa(m)) \left( \frac{dR}{dR} + \mu_4(aa(m)) \right) \left( \frac{2}{R(k+1)} \right); \\
\text{ elseif } (l==2) \\
\text{ cof}(ii,ii+L) &= \frac{\mu_5(aa(m))}{(4 \ast R(k+1) \ast da)}; \\
\end{align*} \]
cof(ii,ii+L) = -mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+2) = mu3(aa(m))/(4*dx^3);
cof(ii,ii+1) = mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
cof(ii,ii) = mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
cof(ii,ii-1) = -mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
cof(ii,ii-2) = -mu3(aa(m))/(4*dx^3);

elseif (l==L-1)
cof(ii,ii+L) = mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+L) = -mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+1) = mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
cof(ii,ii) = mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
cof(ii,ii-1) = -mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
cof(ii,ii-2) = -mu3(aa(m))/(4*dx^3);

elseif (l==L)
cof(ii,ii+L) = mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+L) = -mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+2) = mu3(aa(m))/(4*dx^3);
cof(ii,ii+1) = mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
cof(ii,ii) = mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
cof(ii,ii-1) = -mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
cof(ii,ii-2) = -mu3(aa(m))/(4*dx^3);

else
cof(ii,ii+L) = mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+L) = -mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+2) = mu3(aa(m))/(4*dx^3);
cof(ii,ii+1) = mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
cof(ii,ii) = mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
cof(ii,ii-1) = -mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
cof(ii,ii-2) = -mu3(aa(m))/(4*dx^3);
end

elseif (m==M)

if (l==1)
cof(ii,ii-L) = mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii-L) = -mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+2) = mu3(aa(m))/(4*dx^3);
cof(ii,ii+1) = mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);

else

end

elseif (m==M)

if (l==1)
elseif (l==2)
    cof(ii,ii-L) = mu5(aa(m))/(4*R(k+1)*da);
    cof(ii,ii-L) = -mu5(aa(m))/(4*R(k+1)*da);
    cof(ii,ii+2) = mu3(aa(m))/(4*dx^3);
    cof(ii,ii+1) = mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
    cof(ii,ii) = mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
    cof(ii,ii-1) = -mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
elseif (l==L-1)
    cof(ii,ii-L) = mu5(aa(m))/(4*R(k+1)*da);
    cof(ii,ii-L) = -mu5(aa(m))/(4*R(k+1)*da);
    cof(ii,ii) = mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
    cof(ii,ii-1) = -mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
    cof(ii,ii-2) = -mu3(aa(m))/(4*dx^3);
elseif (l==L)
    cof(ii,ii-L) = mu5(aa(m))/(4*R(k+1)*da);
    cof(ii,ii-L) = -mu5(aa(m))/(4*R(k+1)*da);
    cof(ii,ii) = mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
    cof(ii,ii-1) = -mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
    cof(ii,ii-2) = -mu3(aa(m))/(4*dx^3);
else
    cof(ii,ii-L) = mu5(aa(m))/(4*R(k+1)*da);
    cof(ii,ii-L) = -mu5(aa(m))/(4*R(k+1)*da);
    cof(ii,ii+2) = mu3(aa(m))/(4*dx^3);
    cof(ii,ii+1) = mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
    cof(ii,ii) = mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
    cof(ii,ii-1) = -mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
    cof(ii,ii-2) = -mu3(aa(m))/(4*dx^3);
end

else
    if (l==1)
cof(ii,ii+L)=mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii-L)=-mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+2)=mu3(aa(m))/(4*dx^3);
cof(ii,ii+1)=mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
cof(ii,ii)=mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));

elseif (l==2)
cof(ii,ii+L)=mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii-L)=-mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+1)=mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
cof(ii,ii)=mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
cof(ii,ii-1)=-mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
cof(ii,ii-2)=-mu3(aa(m))/(4*dx^3);

elseif (l==L-1)
cof(ii,ii+L)=mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii-L)=-mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+1)=mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
cof(ii,ii)=mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
cof(ii,ii-1)=-mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
cof(ii,ii-2)=-mu3(aa(m))/(4*dx^3);

elseif (l==L)
cof(ii,ii+L)=mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii-L)=-mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii)=mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
cof(ii,ii-1)=-mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
cof(ii,ii-2)=-mu3(aa(m))/(4*dx^3);

else
cof(ii,ii+L)=mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii-L)=-mu5(aa(m))/(4*R(k+1)*da);
cof(ii,ii+2)=mu3(aa(m))/(4*dx^3);
cof(ii,ii+1)=mu2(aa(m))/(4*dx)*z(k,ii+1)-mu3(aa(m))/(2*dx^3);
cof(ii,ii)=mu1(aa(m))/dR+mu4(aa(m))/(2*R(k+1));
cof(ii,ii-1)=-mu2(aa(m))/(4*dx)*z(k,ii-1)+mu3(aa(m))/(2*dx^3);
cof(ii,ii-2)=-mu3(aa(m))/(4*dx^3);
end
end
end
zz=coef\nu;
for i=1:L*M
    z(k+1,i)=zz(i);
end
for t=0:5:20
    tt=t/5+1;
    for i=N+1:2*N-1
        ll(i,tt)=round((R(i-N+1)/eps-t*c-Xmin)/dx)+1;
    end
end
x1=zeros(4*N-3);
y1=zeros(4*N-3);
y2=zeros(4*N-3);
y3=zeros(4*N-3);
y4=zeros(4*N-3);
y5=zeros(4*N-3);
for i=1:N
    rr(i)=Rmin/(eps*(N-1))*(i-1);
    if i==1
        x1(2*N-1)=rr(i);
        y1(2*N-1)=initial(rr(i),0,mu,c);
        y2(2*N-1)=initial(rr(i),5,mu,c);
        y3(2*N-1)=initial(rr(i),10,mu,c);
        y4(2*N-1)=initial(rr(i),15,mu,c);
        y5(2*N-1)=initial(rr(i),20,mu,c);
    else
        x1(2*N-2+i)=rr(i)/kk(amax);
APPENDIX A: MATLAB CODE FOR SOLVING THE CKDV-TYPE EQUATION

```matlab
% APPENDIX A: MATLAB CODE FOR SOLVING THE CKDV-TYPE EQUATION

x1(2*N-i) = -rr(i)/kk(amin);
y1(2*N-2+i) = initial(rr(i)/kk(amin), 0, mu, c);
y1(2*N-i) = initial(rr(i)/kk(amin), 5, mu, c);
y2(2*N-2+i) = initial(rr(i)/kk(amax), 0, mu, c);
y2(2*N-i) = initial(rr(i)/kk(amax), 5, mu, c);
y3(2*N-2+i) = initial(rr(i)/kk(amax), 10, mu, c);
y3(2*N-i) = initial(rr(i)/kk(amin), 10, mu, c);
y4(2*N-2+i) = initial(rr(i)/kk(amax), 15, mu, c);
y4(2*N-i) = initial(rr(i)/kk(amin), 15, mu, c);
y5(2*N-2+i) = initial(rr(i)/kk(amax), 20, mu, c);
y5(2*N-i) = initial(rr(i)/kk(amin), 20, mu, c);
end
end
for i = N+1:2*N-1
    rr(i) = R(i-N+1)/eps;
x1(2*N-2+i) = rr(i)/kk(amax);
x1(2*N-i) = -rr(i)/kk(amin);
y1(2*N-2+i) = z(i-N+1, ll(i,1)+(M-1)*L);
y1(2*N-i) = z(i-N+1, ll(i,1));
y2(2*N-2+i) = z(i-N+1, ll(i,2)+(M-1)*L);
y2(2*N-i) = z(i-N+1, ll(i,2));
y3(2*N-2+i) = z(i-N+1, ll(i,3)+(M-1)*L);
y3(2*N-i) = z(i-N+1, ll(i,3));
y4(2*N-2+i) = z(i-N+1, ll(i,4)+(M-1)*L);
y4(2*N-i) = z(i-N+1, ll(i,4));
y5(2*N-2+i) = z(i-N+1, ll(i,5)+(M-1)*L);
y5(2*N-i) = z(i-N+1, ll(i,5));
end

figure
plot(x1, y1, 'b-', x1, y2, 'k-', x1, y3, 'r-', x1, y4, 'g-', x1, y5, 'c-', 'LineWidth', 2)
axis([-24 24 -0.3 0.7])
```
APPENDIX A: MATLAB CODE FOR SOLVING THE CKDV-TYPE EQUATION

```matlab
xlabel('x')
ylabel('A')
set(get(gca, 'XLabel'), 'FontSize', 18, 'Vertical', 'top');
set(get(gca, 'YLabel'), 'FontSize', 18, 'Vertical', 'bottom');
set(gca, 'linewidth', 2);
set(findobj('FontSize', 10), 'FontSize', 14);

*****************initial.m***********************
function vinitial = initial(r, t, mu, s, th)
vinitial = 5 * (4 - 3 * abs(pi - th) / pi) * real((1 + t * s * i / mu) / ((1 + t * s * i / mu)^2 + (r / mu)^2)^(1/2));

*****************kk.m***********************
function vk = kk(a)
vk = -(a * Qa(a) - 2 * Q(a)) / sqrt((Qa(a) - 2 * a)^2 + 4 * Q(a) - 4 * a^2);

*****************kp.m***********************
function vkp = kp(a)
vkp = -cos(theta(a)) * (a * tan(theta(a)) - sqrt(Q(a) - a^2));

*****************kpkp.m***********************
function vkpkp = kpkp(a)
vkpkp = cos(theta(a)) / thetaa(a) * (2 * sqrt(Q(a) - a^2) / (Qa(a) - 2 * a) + (Qa(a) - 2 * a) / (2 * sqrt(Q(a) - a^2)));

*****************mu1.m***********************
function vmu1 = mu1(a)
vmu1 = 2 * 0.9779751861 * Q(a)^2 / (-0.9779751861 + 0.1 * a)^4 * (0.5 * (-0.9779751861 + 0.1 * a)^2 + 1.2 * (-0.9779751861) / 0.5 * ((-0.9779751861 + 0.1 * a)^2 / Q(a) - 0.5)^2);

*****************mu2.m***********************
function vmu2 = mu2(a)
vmu2 = -3 * Q(a)^3 / (-0.9779751861 + 0.1 * a)^6 * (0.5 * (-0.9779751861 + 0.1 * a)^2 + 1.2 * 0.9779751861^2 / 0.25 * ((-0.9779751861 + 0.1 * a)^2 / Q(a) - 0.5)^3);

*****************mu3.m***********************
```
function vmu3=mu3(a)
vmu3=-Q(a)^3/(3*(-0.9779751861+0.1*a)^4)*((-0.9779751861+0.1*a)^2/Q(a)-0.5)^3
+1.2*(-0.9779751861+0.1*a)^2/Q(a)-0.5)^2);

function vmu4=mu4(a)
vmu4=-0.5*kk(a)*kpkp(a)*Q(a)/(-0.9779751861+0.1*a)^4*(-0.9779751861+0.1*a)^2
+4*kp(a)*((-0.9779751861+0.1*a)^2/Q(a)+3*Q(a)*0.1^2*(sin(theta(a)))/(-0.9779751861+0.1*a)^2)
-1.2*kpkp(a)*kk(a)*0.9779751861^2/0.5*(-0.9779751861+0.1*a)^4)
* ((-0.9779751861+0.1*a)^2/Q(a)-0.5)^2)*((kk(a)^2-3*kp(a)^2)/(-0.9779751861+0.1*a)^2)
+0.1*Q(a)*sin(theta(a)))/(-0.9779751861+0.1*a));

function vmu5=mu5(a)
vmu5=-2*kk(a)*Q(a)/(-0.9779751861+0.1*a)^4*(-0.9779751861+0.1*a)
+(kp(a)*((-0.9779751861+0.1*a)+0.1*Q(a)*sin(theta(a))))^2
+1.2*Q(a)*sin(theta(a)))/(-0.9779751861+0.1*a));

function vQ=Q(a) %Q(a)
vQ=6*(-0.9779751861+0.1*a)^2+5.738612788
-10*sqrt(1.44*(-0.9779751861+0.1*a)^2-0.4782177323)^2
+1.147722558*(-0.9779751861+0.1*a)^2);

function vQa=Qa(a) %diff(Q(a),a)
vQa=-1.173570223+0.12*a+5.000000000*(-(2.88*(0.5*(-0.9779751861+0.1*a)^2-0.4782177323)
+1.147722558*(-0.9779751861+0.1*a)^2));

function vQaa=Qaa(a) %diff(Q(a),a^2)

function vQaa=Qaa(a) %diff(Q(a),a^2)
\[vQaa = 0.12 + 2.5 \times (2.88 \times (0.5 \times (-0.9779751861 + 0.1 \times a)^2 - 0.4782177323) \times (-0.09779751860 + 0.01 \times a)^2
\]
\[-0.224488364 + 0.0229545116 \times a)^2 / (1.44 \times (0.5 \times (-0.9779751861 + 0.1 \times a)^2)^2 + (0.09779751861 + 0.1 \times a)^2)^{(3/2)}
\]
\[-0.09779751860 + 0.01 \times a)^2 - 0.2244888364 + 0.0229545116 \times a)^2 / (1.44 \times (0.5 \times (-0.9779751861 + 0.1 \times a)^2 - 0.4782177323) \times (-0.9779751861 + 0.1 \times a)^2)^{(3/2)}
\]

\[v\theta = \begin{cases} \arctan \left( -2 \sqrt{Q(a) - a^2} / (Qa(a) - 2 \times a) \right) & \text{if } Qa(a) - 2 \times a < 0 \\ \arctan \left( -2 \sqrt{Q(a) - a^2} / (Qa(a) - 2 \times a) \right) + \pi & \text{else} \end{cases}
\]

\[v\theta a = -\left( (Qa(a) - 2 \times a)^2 - 2 \times (Qaa(a) - 2) \times (Q(a) - a^2) / (4 \times (Q(a) - a^2)
\]
\[+ (Qa(a) - 2 \times a)^2) \times \sqrt{Q(a) - a^2} \right) ;
\]
Appendix B: Simulation parameters of numerical results

Section 5.2.1.1

\[\begin{align*}
\varepsilon_{\min} &= -20, \quad \varepsilon_{\max} = 28, \quad \Delta \varepsilon = 0.025, \quad L = \frac{\varepsilon_{\max} - \varepsilon_{\min}}{\Delta \varepsilon} + 1; \\
R_{\min} &= 0.1, \quad R_{\max} = 0.56, \quad \Delta R = 0.001, \quad N = 461.
\end{align*}\]

Section 5.2.2

\[\begin{align*}
\varepsilon_{\min} &= -30, \quad \varepsilon_{\max} = 15, \quad \Delta \varepsilon = 0.025, \quad L = \frac{\varepsilon_{\max} - \varepsilon_{\min}}{\Delta \varepsilon} + 1; \\
R_{\min} &= 0.24, \quad R_{\max} = 1.04, \quad \Delta R = 0.001, \quad N = 801.
\end{align*}\]

Section 6.2

\[\begin{align*}
\varepsilon_{\min} &= -15, \quad \varepsilon_{\max} = 28, \quad \Delta \varepsilon = 0.05, \quad L = \frac{\varepsilon_{\max} - \varepsilon_{\min}}{\Delta \varepsilon} + 1; \\
R_{\min} &= 0.1, \quad R_{\max} = 0.56, \quad \Delta R = 0.002, \quad N = 231; \\
\Delta t &= \frac{a_{\max} - a_{\min}}{M - 1}, \quad M = 18.
\end{align*}\]
Section 6.3

\( \xi_{\text{min}} = -15, \quad \xi_{\text{max}} = 28, \quad \Delta \xi = 0.05, \quad L = \frac{\xi_{\text{max}} - \xi_{\text{min}}}{\Delta \xi} + 1; \)

\( R_{\text{min}} = 0.1, \quad R_{\text{max}} = 0.56, \quad \Delta R = 0.002, \quad N = 231; \)

\( \Delta t = \frac{a_{\text{max}} - a_{\text{min}}}{M - 1}, \quad M = 18. \)
Bibliography


