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Post-local buckling-driven delamination in bilayer composite beams

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Abstract

Analytical theories are developed for post-local buckling-driven delamination in bilayer composite beams. The total energy release rate (ERR) is obtained more accurately by including an axial strain energy contribution from the intact part of the beam and by developing a more accurate expression for the post-buckling mode shape than that in the work by Chai et al. (1981) and Hutchinson and Suo (1992). The total ERR is partitioned by using partition theories based on the Euler beam, Timoshenko beam and 2D-elasticity theories. Independent experimental tests by Kutlu and Chang (1995) show that, in general, the analytical partitions based on the Euler beam theory predicts the propagation behaviour very well and much better than the partitions based on the Timoshenko beam and 2D-elasticity theories.

Keywords: Composite materials, Delamination propagation, Mixed-mode partition, Post-local buckling
1. Introduction

Interface delamination in layered materials is often driven by buckling and post-buckling. Some examples include the delamination of laminated composite beams, plates and shells under in-plane compression, and the surface spalling of thermal and environmental barrier coatings. This topic has attracted the attention of many researchers for decades. Ref. [1] gives a recent review.

Although post-buckling-driven delamination generally occurs as mixed-mode fracture with all three opening, shearing and tearing actions (i.e. mode I, II and III), post-buckling-driven one-dimensional (1D) delamination has received more attention because it is simpler, still captures the essential mechanics, and also serves as a 'stepping stone' towards the study of general mixed-mode delamination. The term ‘1D delamination’ means that a delamination propagates in one direction with mode I opening and mode II shearing action only. Some examples of 1D
delamination include through-width delamination in beams, and blisters in laminated composite plates and shells.

The focus of the present work is 1D post-local buckling-driven delamination. A detailed definition of this will be given in the next section. Key tasks in studying 1D post-local buckling-driven delamination include: (1) determining the critical buckling strain and the post-buckling deformation, (2) calculating the post-local buckling total energy release rate (ERR) \( G \), (3) partitioning the total ERR \( G \) into its individual mode I and II ERR components, \( G_I \) and \( G_{II} \), which govern the propagation of mixed-mode delamination, and (4) predicting the delamination propagation behaviour.

Analytical, numerical and experimental approaches are all commonly used for this kind of study. Some representative analytical studies, numerical studies and experimental studies are given in Refs. [2,3], [4-10] and [8,9] respectively. Ref. [2] is regarded as a pioneering and instrumental study. It gives full analytical developments for calculating the total ERR \( G \) for cases of thin-film, thick-column and general post-local buckling-driven delamination in laminated beam-like plates by using Euler beam theory. No partition of the total ERR \( G \) into its individual mode I and II ERR components, \( G_I \) and \( G_{II} \), is attempted in Ref. [2]. Ref. [3] gives analytical calculations for both the total ERR \( G \) and its components, \( G_I \) and \( G_{II} \), for the case of thin-film post-local buckling-driven delamination. The partition is based on 2D elasticity theory [3]. The numerical studies in Refs. [4-7] are developed by using layer wise plate/shell theory. The studies in Refs. [8-10] are based on 2D elasticity and the study in Ref. [10] also uses the 3D finite element method. The virtual crack closure technique is used to calculate the ERRs in Refs. [4,5,8-10] and the cohesive zone model is used in Refs. [6,7].

The present work aims to develop an improved analytical method to complete the four key tasks stated above, based on the work in Refs. [2,3] and [11-20]. The structure of the paper is as follows: the analytical development is given in Section 2, and in Section 3, the numerical verification and experimental validation are reported. Finally, conclusions are given in Section 4.

2. Analytical development [2,3,11-20]

Fig. 1 shows a post-locally buckled bilayer composite beam. The Young’s moduli of the upper and lower layers are \( E_1 \) and \( E_2 \) respectively, and the corresponding thicknesses are \( h_1 \) and \( h_2 \) with \( h_2 >> h_1 \). The beam has a total length \( L \) and a width \( b \) with a central through-width interfacial delamination of length \( a \). The delamination tips are labelled ‘B’. The beam is clamped at both ends and is under uniform end-shortening compression. The local buckling, as
shown, divides the beam into three parts, namely, the locally-buckled part labelled ‘1’, the
substrate part labelled ‘2’ and the intact parts labelled ‘3’. The following development assumes
that the whole process of buckling, post-buckling and delamination propagation is localised in
the upper layer, that is, the bending action in both parts 2 and 3 is negligible.

2.1. Deformation, internal forces and bending moments

The uniform end-shortening compression is represented by a strain $\varepsilon_0$, defined as $\varepsilon_0 = u_0 / L$
with $u_0$ being the end-shortening displacement and $L$ being the total length of the beam. The
compressive axial strains of the neutral surfaces of each the three parts of the beam are
represented by $\varepsilon_i$ (with $i=1,2,3$). Similarly, $N_i$ and $M_i(x_i)$ represent the axial forces and
bending moments respectively in each part, where $x_i$ is the axial axis on each neutral surface.
The directions of the axes of the three parts together are shown in Fig. 1 where only their
directions are indicated. The axial forces $N_i$ can be expressed as

$$N_i = -E_i A_{ie} \varepsilon_i$$

where the effective cross-sectional areas $A_{ie}$ are given by

$$A_{ie} = bh_i, \quad A_{2e} = bh_i \eta \gamma, \quad A_{3e} = bh_i (1 + \eta \gamma)$$

and $\eta = E_2 / E_1$ and $\gamma = h_2 / h_1$, which are the modulus and thickness ratios respectively. Before
the local buckling of part 1, $\varepsilon_i = \varepsilon_0$, $N_i = -E_i A_{ie} \varepsilon_0$ and $M_i(x_i) = 0$, that is, all three parts are
under constant uniform axial compressive strain $\varepsilon_0$ and there is no bending.

After the local buckling of part 1, part 1 is under both axial compression and bending action
while parts 2 and 3 are still assumed to be under axial compression only without bending action.
The axial strain $\varepsilon_i$ is assumed to remain constant at the critical local-buckling strain $\varepsilon_c$
throughout [2, 3], that is,

$$\varepsilon_i = \varepsilon_c$$

The axial strain $\varepsilon_2$ can be expressed by using the axial equilibrium condition, $N_1 + N_2 = N_3$,
giving

$$\varepsilon_2 = \varepsilon_3 + \frac{\varepsilon_3 - \varepsilon_c}{\eta \gamma}$$

from which it is obvious that $\varepsilon_2 > \varepsilon_3$. Also the axial strain $\varepsilon_3$ should be smaller than the end-
shortening strain $\varepsilon_0$ after local buckling, that is, $\varepsilon_3 < \varepsilon_0$. From these two observations, it is
reasonable to assume that the following is a good approximation:
\[ \varepsilon_2 = \varepsilon_0 \]  

Then Eq. (4) gives

\[ \varepsilon_3 = \frac{\varepsilon_c + \eta \gamma \varepsilon_0}{1 + \eta \gamma} \]  

In order to determine the critical local buckling strain \( \varepsilon_c \) and bending moment \( M_i(x_i) \) accurately, it is essential to find an accurate post-locally buckled mode shape. Here, it is assumed to be

\[ V_i(x_i) = \frac{A}{2} \left[ \cos \left( \frac{2\alpha \pi x_i}{a} \right) - \cos(\alpha \pi) \right] \]  

where \( \alpha \) is the correction factor for the quality of the clamped end condition at the crack tip. In Refs. [2,3], the value of \( \alpha \) is taken as 1. The critical local-buckling strain \( \varepsilon_c \) can be determined by considering the free-body diagram of a symmetrical half of the buckled upper layer shown in Fig. 2.

Horizontal equilibrium combined with Eqs. (1) and (3) gives \( N_{10} = N_{1B} = -E_1 A \varepsilon_c \) and bending moment equilibrium gives \( M_{1B} = M_{10} - N_{10} V_{10} \), which together give \( M_{10} = M_{1B} - E_1 A \varepsilon_c V_{10} \). Classical beam theory and Eq. (7) give \( M_{10} = -4E_1 I_1 V_{10}(\alpha \pi/a)^2 - 2E_1 I_1(\alpha \pi/a)^2 \cos(\alpha \pi) \). Therefore the critical local-buckling strain \( \varepsilon_c \) is obtained as

\[ \varepsilon_c = \frac{(\alpha \pi)^2}{3} \left( \frac{h_i}{a} \right)^2 \]  

The value of the correction factor \( \alpha \) for the problem under consideration can be determined either from numerical simulations or from experimental tests. More details about the value of \( \alpha \) will be given in Section 3 which deals with the experimental validation. The amplitude \( A \) is now determined by using the following assumption, where \((1 - \varepsilon_c)a/2\) represents half-length of part 1 at the instant of local buckling, \((1 - \varepsilon_0)a/2\) represents the half-length of part 2 during post-local buckling, and \( ds \) represents the differential arc length of part 1’s buckled mode shape:

\[ \frac{(1 - \varepsilon_c)a}{2} = \int ds = \int_{0}^{1(1 - \varepsilon_c)^{1/2}} \sqrt{1 + \left( \frac{dV_i}{dx_i} \right)^2} \, dx_i \]  

Note that this assumption implies that the curved half-length of the buckled part 1 remains constant at \((1 - \varepsilon_c)a/2\) during post-buckling. In order to determine the amplitude \( A \) accurately,
particularly in the deep post-buckling region, a third-order series expansion based on \((dV_1/dx_1)^2\) is used to expand the integrand on the right-hand-side of Eq. (9), which results in the following:

\[
(\varepsilon_0 - \varepsilon_c)A = \int_0^{(1-\varepsilon_c)/2} \left[ \left( \frac{dV_1}{dx_1} \right)^2 - \frac{1}{4} \left( \frac{dV_1}{dx_1} \right)^4 + \frac{1}{8} \left( \frac{dV_1}{dx_1} \right)^6 \right] dx_1 \]

(10)

Let \(\varepsilon_a = \varepsilon_0 - \varepsilon_c\), which represents the additional end-shortening strain beyond the critical buckling, and approximate the upper limit on the integration as \((1-\varepsilon_0)a/2 \approx a/2\).

\[
\varepsilon_a A \approx \int_0^{a/2} \left[ \left( \frac{dV_1}{dx_1} \right)^2 - \frac{1}{4} \left( \frac{dV_1}{dx_1} \right)^4 + \frac{1}{8} \left( \frac{dV_1}{dx_1} \right)^6 \right] dx_1
\]

(11)

Substituting Eq. (7) into Eq. (11) and evaluating the integration gives

\[
C_6 \bar{A}^6 + C_4 \bar{A}^4 + C_2 \bar{A}^2 - \varepsilon_a = 0
\]

(12)

where

\[
\bar{A} = \frac{A \alpha \pi}{2a}
\]

(13)

\[
C_2 = 1 - \sin \left( 2\alpha \pi \right) / 2\alpha \pi
\]

(14)

\[
C_4 = -\frac{3}{4} + \sin \left( 2\alpha \pi \right) / 4 \sin \left( 4\alpha \pi \right) / 4 \alpha \pi
\]

(15)

\[
C_6 = \frac{5}{4} - \frac{15}{8} \sin \left( 2\alpha \pi \right) / 2\alpha \pi + \frac{3}{4} \sin \left( 4\alpha \pi \right) / 4 \alpha \pi - \frac{1}{8} \sin \left( 6\alpha \pi \right) / 6\alpha \pi
\]

(16)

Since \(\alpha\) is typically close to 1, the harmonic terms can be neglected as a further approximation. The polynomial in Eq. (12) can then be solved, which gives the amplitude \(A\) as

\[
A = \frac{2 \sqrt{15} a}{15 \alpha \pi} \left[ \frac{\varepsilon^{1/3} \left( \varepsilon^{2/3} + 3 \varepsilon^{1/3} \right) - 51}{\varepsilon^{1/3}} \right]
\]

(17)

where

\[
\epsilon = 30 \sqrt{3} \left( 675 \varepsilon_a^2 - 243 \varepsilon_a + 71 + 1350 \varepsilon_a \varepsilon - 243 \right)
\]

(18)

The bending moment at the delamination tip \(B\) is then obtained by using Eqs. (7), (8), and (17),

\[
M_{1B} = \frac{\sqrt{3} E h_i c_a \sqrt{\varepsilon_a} \varepsilon_a}{3}
\]

(19)

where

\[
c_a = -\frac{\sqrt{15}}{15 \sqrt{\varepsilon_a}} \left[ \frac{\varepsilon^{1/3} \left( \varepsilon^{2/3} + 3 \varepsilon^{1/3} \right) - 51}{\varepsilon^{1/3}} \right] \cos(\alpha \pi)
\]

(20)
2.2. Strain energy and total energy release rate

By using the internal bending moment in part 1 and the internal axial forces in parts 1, 2 and 3, but neglecting the internal bending moments in parts 2 and 3, the strain energy \( U \) in one half of the symmetrical post-buckled beam is

\[
U = \frac{1}{2} \left[ E_1 A_e c_e^2 \int_0^{y/2} E_1 I_1 \left( \frac{d^2 V}{dx_1^2} \right)^2 dx_1 + E_1 A_{e_0} c_0^2 \int_0^{y/2} E_1 I_1 \left( \frac{d^2 V}{dx_1^2} \right)^2 dx_1 + E_1 A_{3e} c_3^2 \frac{L-a}{2} \right]
\]

\[
= \frac{1}{2} \left[ \frac{N_1^2}{E_1 A_1} a + \int_0^{y/2} \frac{M_1^2}{E_1 I_1} dx_1 + \frac{N_2^2}{E_1 A_{2e}} a + \frac{N_3^2}{E_1 A_{3e}} \frac{L-a}{2} \right]
\]

These assumptions are consistent with Section 2.1. The total ERR \( G \) is then calculated as

\[
G = \frac{1}{2bE_1} \left[ \frac{M_{1b}^2}{I_1} + \frac{N_{1b}^2}{A_1} + \frac{N_{2b}^2}{A_{2e}} - \frac{N_{3b}^2}{A_{3e}} \right]
\]

It is worth noting that ERR represents the strain energy density difference or ‘pressure’ across the delaminated and intact parts. Since uniform axial compression results in no strain energy density difference, it does not produce any ERR. Therefore, an effective axial force \( N_{1be} \) is defined as

\[
N_{1be} = -E_1 A_1 (\varepsilon_1 - \varepsilon_2) = E_1 b h_1 \varepsilon_a
\]

The total ERR \( G \) in Eq. (22) then becomes

\[
G = \frac{1}{2bE_1} \left[ \frac{M_{1b}^2}{I_1} + \left( \frac{1}{A_1} - \frac{1}{A_{3e}} \right) N_{1be}^2 \right] = \frac{6}{E_1 b^2 h_1^2} \left( \frac{M_{1b}^2 + \frac{\lambda h_1^2}{12} N_{1be}^2}{12} \right)
\]

where \( \lambda = \eta \gamma / (1 + \eta \gamma) \). Substituting \( M_{1b} \) from Eq. (19) and \( N_{1be} \) from Eq. (23) into Eq. (24) gives

\[
G = \frac{1}{2} E_1 h_1 \varepsilon_a \left( 4c_a^2 \varepsilon_c + \lambda \varepsilon_a \right)
\]

Note that when \( \lambda = 1, \alpha = 1 \) and \( c_a = 1 \) Eq. (19) becomes the same as that in Refs. [2,3].

2.3. Partitions of energy release rate

2.3.1. Euler beam partition

From the authors’ previous work [14-17,20], the Euler beam partition of the total ERR \( G \) in Eq. (24) can be written as

\[
G_{IE} = c_{IE} \left( M_{1b} - \frac{N_{1be}}{\beta} \right) \left( M_{1b} - \frac{N_{1be}}{\beta'} \right)
\]

(26)
where \((\theta, \beta)\) and \((\theta', \beta')\) are the two sets of orthogonal pure modes. The \(\theta\) and \(\beta'\) pure modes correspond to zero relative shearing displacement and zero relative opening displacement respectively just ahead of the crack tip [14-17,20]. Using the beam mechanics in Section 2.1 in conjunction with these conditions, and then the orthogonality condition [14-17,20] through the ERR in Eq. (24) to obtain the orthogonal \(\theta'\) and \(\beta\) pure modes, gives the following:

\[
(\theta, \beta) = \left( \frac{6}{h_i}, \frac{2}{\lambda h_i} \right)
\]

\[
(\theta', \beta') = (0, \infty)
\]

Note that the zero value of \(\theta'\) results from the approximate nature of the total ERR \(G\) in Eq. (24) and is due to neglecting the bending action in parts 2 and 3 of the bilayer beam. This does not prevent from the mode II ERR \(G_{IE}\) from being obtained as it is readily obtained as \(G - G_{IE}\) when the mode I ERR \(G_{IE}\) is known. The coefficient \(c_{IE}\) in Eq. (26) is calculated by using Eqs. (24) and (26) together, and noting that \(G = G_{IE}\) when \(M_{1B} = 1\) and \(N_{1Be} = \theta\), giving

\[
c_{IE} = \frac{6}{E_i h_i^2 h_i^3} \left( 1 + \frac{\lambda h_i^2}{12} \theta^2 \right) \left\{ \left[ 1 - \frac{\theta}{\beta} \right] \left[ 1 - \frac{\theta}{\beta'} \right] \right\} = \frac{6}{E_i h_i^2 h_i^3}
\]

Now the ERR partitions, \(G_{IE}\) and \(G_{IE}\), are known in terms of the delamination tip bending moment \(M_{1B}\) in Eq. (19) and the effective axial force \(N_{1Be}\) in Eq. (23). For the sake of convenience, they are also given below in terms of the critical buckling strain \(\varepsilon_c\) and the additional end-shortening strain \(\varepsilon_a\).

\[
G_{IE} = E_i h_i c_a \sqrt{\varepsilon_c} \varepsilon_a \left( 2 c_a \sqrt{\varepsilon_c} - \sqrt{3} \lambda \sqrt{\varepsilon_a} \right)
\]

\[
G_{IE} = E_i h_i \lambda c_a^{3/2} \left( \sqrt{3} c_a \sqrt{\varepsilon_c} + 1/2 \sqrt{\varepsilon_a} \right)
\]

Note that when \(N_{1Be} > \beta M_{1B}\) or \(\sqrt{\varepsilon_a} > (2 c_a \sqrt{\varepsilon_c}) / (\sqrt{3} \lambda)\), the crack tip normal stress becomes compressive, and so \(G_{IE}\) is taken to be zero with \(G_{IE} = G\).

2.3.2. Timoshenko beam partition

From the authors’ previous work [14-17,20], the Timoshenko beam partition of the total ERR \(G\) in Eq. (24) can be written as
\[ G_{IT} = c_{IT} \left( M_{1B} - \frac{N_{1Be}}{\beta} \right)^2 \]  
\[ G_{II} = c_{II} \left( M_{1B} - \frac{N_{1Be}}{\theta} \right)^2 \]

where

\[ c_{IT} = \frac{6}{E_i b^2 h_i^3} \left( 1 + \frac{\lambda h_i^2}{12} \beta^2 \right) \left( 1 - \frac{\theta}{\beta} \right)^{-2} = \frac{6}{(1 + 3\lambda) E_i b^2 h_i^3} \]  
\[ c_{II} = \frac{6}{E_i b^2 h_i^3} \left( 1 + \frac{\lambda h_i^2}{12} \beta^2 \right) \left( 1 - \frac{\beta}{\theta} \right)^{-2} = \frac{6}{(1 + 1/(3\lambda)) E_i b^2 h_i^3} \]

In terms of the critical buckling strain \( \varepsilon_c \) and the additional end-shortening strain \( \varepsilon_a \), they become

\[ G_{IT} = \frac{1}{2(1 + 3\lambda)} E_i h_i \varepsilon_a \left( 2 \varepsilon_a \sqrt{\varepsilon_c} - \sqrt{3\lambda} \sqrt{\varepsilon_a} \right)^2 \]  
\[ G_{II} = \frac{\lambda}{2(1 + 3\lambda)} E_i h_i \varepsilon_a \left( 2 \sqrt{3} \varepsilon_a \sqrt{\varepsilon_c} + \sqrt{\varepsilon_a} \right)^2 \]

Again note that when \( N_{1Be} > \beta M_{1B} \) or \( \sqrt{\varepsilon_a} > \left( 2 \varepsilon_a \sqrt{\varepsilon_c} \right) / \left( \sqrt{3\lambda} \right) \), the crack tip normal stress becomes compressive, and so \( G_{IT} \) is taken to be zero with \( G_{II} = G \).

### 2.3.3. 2D elasticity partition

In general, if there is a material mismatch across the interface and Young’s modulus ratio \( \eta = E_2 / E_1 \) is not equal to 1, then the 2D-elasticity-based partition of ERR is crack extension size-dependent ERR due to the complex stress intensity factor [21]. It has been one most challenging fracture mechanics problems to obtain analytical solutions for the ERR partition and the stress intensity factors. Recently Harvey et al. [22,23] have solved this problem by using a novel and powerful methodology. It is expected, however, that the effect of material mismatch across the delamination is not significant in this study as the local deformation in the upper layer dominates the fracture. Therefore the 2D-elasticity-based partition theory in Refs. [3,19] for homogeneous beams with no material mismatch across the interface is used instead. The total ERR \( G \) in Eq. (24) can be partitioned as

\[ G_{12D} = c_{12D} \left( M_{1B} - \frac{N_{1Be}}{\beta_{2D}} \right)^2 \]  
\[ G_{II2D} = c_{II2D} \left( M_{1B} - \frac{N_{1Be}}{\theta_{2D}} \right)^2 \]
where

\[
(\theta_{2D}, \beta_{2D}) = \left( -\frac{2.697}{h_i}, \frac{4.450}{\lambda h_i} \right)
\]  

(41)

\[
c_{12D} = \frac{6}{E_i b^2 h_i^3} \left( 1 + \frac{\lambda h_i^2}{12} \theta_{2D}^2 \right) \left( 1 - \frac{\theta_{2D}}{\beta_{2D}} \right)^2 = \frac{4.450^2 \left( 12 + 2.697^2 \lambda \right)}{12 (4.450 + 2.697 \lambda)^2} \frac{6}{E_i b^2 h_i^3}
\]  

(42)

\[
c_{II2D} = \frac{6}{E_i b^2 h_i^3} \left( 1 + \frac{\lambda h_i^2}{12} \beta_{2D}^2 \right) \left( 1 - \frac{\beta_{2D}}{\theta_{2D}} \right)^2 = \frac{2.697^2 \left( 12 \lambda + 4.450^2 \lambda \right)}{12 (4.450 + 2.697 \lambda)^2} \frac{6}{E_i b^2 h_i^3}
\]  

(43)

In terms of the critical buckling strain \( \varepsilon_c \) and the additional end-shortening strain \( \varepsilon_a \), they become

\[
G_{12D} = \frac{12 + 2.697^2 \lambda}{6 (4.450 + 2.697 \lambda)} E_i h_i \varepsilon_a \left( 4.450 \varepsilon_a \sqrt{\varepsilon_c} - \sqrt{3} \lambda \sqrt{\varepsilon_a} \right)^2
\]  

(44)

\[
G_{II2D} = \frac{(12 \lambda + 4.450^2 \lambda)}{6 (4.450 + 2.697 \lambda)^2} E_i h_i \varepsilon_a \left( 2.697 \varepsilon_a \sqrt{\varepsilon_c} + \sqrt{3} \lambda \sqrt{\varepsilon_a} \right)^2
\]  

(45)

Again note that when \( N_{Ile} > \beta_{2D} M_{1b} \) or \( \varepsilon_a > \left( 4.450 \varepsilon_a \sqrt{\varepsilon_c} \right) / (\sqrt{3} \lambda) \), the crack tip normal stress becomes compressive, and so the \( G_{12D} \) is taken to be zero and \( G_{II2D} = G \).

2.4. Crack propagation and stability

In general the propagation criterion can be expressed in the form

\[
f(G_I, G_{II}, G_K, G_{IIc}) = 0
\]  

(46)

where \( G_K \) and \( G_{IIc} \) are the respective critical mode I and II ERRs. The form of Eq. (46) is not unique but is crack interface-dependent and is determined from experimental testing for a given interface. At the instant when Eq. (46) is met, two scenarios could occur. One is unstable crack propagation in which the crack continues to advance without increasing end-shortening. The other is the stable crack propagation in which the crack stops propagating unless further end-shortening is applied. Mathematically, these two scenarios can be expressed as

\[
\frac{\partial f}{\partial \alpha} = \begin{cases}  
\geq 0 & \text{unstable} \\
\leq 0 & \text{stable}
\end{cases}
\]  

(47)

Alternatively, the stability of crack propagation can be checked by finding the value of \( f \) at the critical end-shortening strain for propagation at the initial delamination length and then at a slightly increased delamination length. An increasing value of \( f \) indicates unstable propagation.
3. Numerical verification and experimental validation

This section aims to examine the capability of the analytical development in Section 2 for predicting the propagation behaviour of post-local bucking-driven delamination by making comparisons with independent numerical [4,5] and experimental data [8,9]. The quantities of interest are the critical propagation end-shortening strain, the ERR partitions during propagation and the propagation stability. Two composite beams [8,9] are studied, which both contain a single through-width delamination, and which are subjected to uniform end-shortening displacement at the clamped ends, as shown in Fig. 1. The composite beams are made from T300/976 graphite/epoxy plies and have a total length $L$ equal to 50.8 mm, and a width $b$ equal to 5.08 mm. Table 1 gives more details of the two cases. The double slashes “//” denote the location of the delaminated interface. All plies have equal thickness. The ply longitudinal modulus $E_i$ is 139.3 GPa. The critical ERR for mode I $G_{Ic}$ is 87.6 N/m and for mode II $G_{IIc}$ is equal to 315.2 N/m. Experimental studies in Refs. [8,9] suggest that the material has a linear failure criterion, that is, Eq. (46) takes the form

$$f(G_I, G_{II}, G_{Ic}, G_{IIc}) = \frac{G_I}{G_{Ic}} + \frac{G_{II}}{G_{IIc}} - 1 = 0$$

(48)

which will be used in the following studies. For these two cases, an empirical formula for the critical buckling strain correction factor $\alpha$ in Eq. (8) is obtained by using finite element method simulations and is given by

$$\alpha^2 = 11.738 \left( \frac{h}{a} \right)^2 - 3.6654 \left( \frac{h}{a} \right) + 0.9755$$

(49)

3.1. Comparison of total ERR $G$ in Eq. (25) with independent numerical results [5]

Accurate calculation of total ERR $G$ is a crucial pre-requisite step towards the accurate prediction of propagation behaviour. The following exercise aims to examine the accuracy of the total ERR $G$ given by Eq. (25) and the solutions in Refs. [2,3] by comparing them against independent numerical results in Ref. [5]. Tables 2 and 3 record the comparisons for Case 1 and Case 2 respectively. In general, good agreement is observed between the present solutions and the numerical results in Ref. [5] for both cases. The solutions from Refs. [2,3] have reasonable agreement for Case 1 and very poor agreement for Case 2.
3.2. Comparison of delamination propagation behaviour with independent experimental results [9]

It is well known that fracture toughness depends on fracture mode partition. The validity of a particular mixed-mode partition theory can only be validated against experimental tests. Thorough and comprehensive experimental test data from several independent research groups [24-29] shows [17,20] that Wang and Harvey’s Euler beam partition theory gives the most accurate prediction of mixed-mode fracture toughness. The exercise in this section aims to establish whether this partition theory also governs the propagation of mixed-mode delamination driven by post-local buckling.

Case 1 is considered first. Table 4 and Fig. 3 record the delamination propagation behaviour predicted by the three partition theories described in Section 2. The symbol \( f \) in Table 4 represents the propagation criterion in Eq. (48) with \( f < 0 \) indicating no propagation and \( f = 0 \) indicating stable propagation. Note that the bold values of the end-shortening strains \( \varepsilon_0 \) in Table 4 are those that are discussed here. Both the Euler and Timoshenko beam partition theories predict an initial mixed-mode delamination followed by a pure-mode-II delamination, with delamination propagation beginning in the pure-mode-II region at an end-shortening strain of \( \varepsilon_0 = 2.76 \times 10^{-3} \) and reaching the clamped ends at an end-shortening strain of \( \varepsilon_0 = 2.92 \times 10^{-3} \). Although the propagation is stable, it takes only \( 0.17 \times 10^{-3} \) of extra end-shortening strain (or 0.0085 mm of end-shortening displacement) to extend the delamination by 12.7 mm. This might suggest an observation of unstable propagation in experimental tests. The 2D elasticity partition theory predicts a mixed-mode delamination which begins to propagate at an end-shortening strain of \( \varepsilon_0 = 2.52 \times 10^{-3} \) and reaches the clamped ends at an end-shortening strain of \( \varepsilon_0 = 2.91 \times 10^{-3} \). It takes an extra end-shortening strain of \( 0.39 \times 10^{-3} \) (or 0.0020 mm of end-shortening displacement) to extend the delamination by the same 12.7 mm, which is much larger than the \( 0.17 \times 10^{-3} \) of extra end-shortening strain predicted by the Euler and Timoshenko partition theories. This might suggest an observation of stable propagation in experimental tests. The propagation behaviour is also shown graphically in Fig. 3 as delamination length \( a \) vs. end-shortening strain \( \varepsilon_0 \). The two beam partition theories predict a much steeper growth rate than the 2D elasticity partition theory does. It is seen that the predictions from the two beam partition theories are considerably different from that of the 2D elasticity partition theory.
Experimental test data in Ref. [9] are used next to assess the accuracy of each partition theory. The tests record the history of the compression force per unit width $F$ against the upper surface mid-span axial strain $\varepsilon_{sc}$. The compression force per unit width is calculated analytically as

$$F = \left( N_1 + N_2 \right)/b = E_i h_i \left( \varepsilon_c + \eta \varepsilon_0 \right)$$  \hspace{1cm} (50)$$

and the upper surface mid-span axial strain is calculated analytically as

$$\varepsilon_{sc} = -\frac{h_i}{2} \left( \frac{d^2 V_1}{dx_1^2} \right)_{x_1=0} - \varepsilon_c = \frac{h_i A \alpha^2 \pi^2}{a^2} - \varepsilon_c$$  \hspace{1cm} (51)$$

Fig. 4 compares the three partition theories with the test results [9]. The following points are noted: (1) the analytical critical local-buckling compression force is much smaller than the experimental one. One possible reason for this is the sticking of the specimen’s sub-laminates through the Teflon film—inserted to create the initial delamination—during manufacturing, thus increasing the buckling load [9]. Note that both the analytical and experimental results display bifurcation-type local buckling, which appears as the first sharp corner in the figure. (2) By cross-comparing with the results in Table 4, the two beam partition theories predict pure-mode-II propagation, beginning at the second sharp corner and ending at the third one, which corresponds to the complete delamination. During the delamination propagation process, the compression force does not change very much, which equates to an almost-unstable propagation. On the other hand, the 2D elasticity partition theory predicts mixed-mode propagation, starting smoothly and ending at about the same point predicted by the two beam prediction theories. During the delamination propagation, the compression force does change significantly, which equates to a stable propagation. (3) The experimental results [9] do show an almost-unstable propagation and both the initial- and end-propagation compression forces agree very well with the predictions of the two beam partition theories. (4) The significant discrepancy between the analytical and experimental critical local-buckling compression forces results in a significant difference between the predicted and experimental loading curves. This needs to be investigated in order to examine the partition theories more thoroughly.

In the following, an approximate expression for the critical local-buckling end-shortening strain $\varepsilon_{ce}$ is derived where the subscript $e$ indicates that it is based on experimental results. Similar to in Eq. (8), $\varepsilon_{ce}$ is written as

$$\varepsilon_{ce} = \frac{\left( \frac{\alpha_c \pi}{3} \right)^2}{\left( \frac{h_i}{a} \right)^2} = \left( \frac{\alpha_c}{\alpha} \right)^2 \varepsilon_c$$  \hspace{1cm} (52)$$
where the correction factor $\alpha_e$ needs to be determined based on experimental results. It is perhaps the case that, in general, the ratio $\alpha_e/\alpha$ varies with the ratio $h_1/a$; however, $\alpha_e/\alpha$ is assumed here to be constant at its value at the initial-buckling delamination length due to lack of experimental results for other crack lengths. The accuracy of this assumption will be examined shortly. It is now only required to determine the value of $\alpha_e$ at the point of initial buckling. From Fig. 4, two approximate critical local-buckling end-shortening strains $\varepsilon_{ce}$ are found from the upper-surface mid-span axial strain and the compression force at the bifurcation point of the experimental results: (1) since $\varepsilon_i = \varepsilon_0$ before the local buckling of part 1, at this location $\varepsilon_{ce} = \varepsilon_{ce} = 0.748 \times 10^{-3}$. (2) Before the local buckling of part 1, $\varepsilon_z = \varepsilon_0$ also, giving $F = E_i h_i \varepsilon_0 (1 + \eta \gamma)$ or $\varepsilon_{ce} = F/[E_i h_i (1 + \eta \gamma)] = 0.903 \times 10^{-3}$ at this location. By averaging these two values, an approximate critical local-buckling end-shortening strain is obtained as $\varepsilon_{ce} = 0.825 \times 10^{-3}$. Therefore the value of $\alpha_e$ at the critical local-buckling point is determined from Eq. (52) to be $\alpha_e = 1.163$ and the ratio $\alpha_e/\alpha = 1.207$. The critical local-buckling strain $\varepsilon_{ce}$ at any delamination length is then calculated from Eq. (52) as $\varepsilon_{ce} = 1.207^2 \varepsilon_e$.

Fig. 5 compares the test results [9] with the three partition theories, which now use the critical local-buckling end-shortening strain $\varepsilon_{ce}$ based on experimental results. The two beam partition theories predict the propagation behaviour very well and much better than the 2D elasticity partition theory does. The delamination propagation is indeed the pure-mode-II propagation predicted by the two beam partition theories. It is now clear that the 2D elasticity partition theory does not provide the right partition for predicting the propagation behaviour of buckling-driven delamination for Case 1. The question of which beam partition theory provides the right partitions when the propagation is not pure mode II, however, still needs to be answered. Case 2 is considered next to answer this question.

Case 2 is now considered in the same manner. Table 5 and Fig. 6 record the delamination propagation behaviour predicted by the three partition theories. Note that the bold values of the end-shortening strains $\varepsilon_0$ in Table 5 are those that are discussed here. All three partition theories predict an initial mixed-mode delamination after the local buckling of the upper layer at $\varepsilon_e = 2.073 \times 10^{-3}$, followed by unstable mixed-mode delamination propagation and then stable propagation. The Euler beam partition theory predicts mode-I-dominated unstable propagation occurring at an end-shortening strain of $\varepsilon_0 = 2.46 \times 10^{-3}$, during which the delamination extends to a total length of 28.99 mm. Then the delamination propagates stably as mode-II-dominated to
a total length of 29.67 mm corresponding to end-shortening strain of $\varepsilon_0 = 2.69 \times 10^{-3}$ after which the delamination propagates stably as pure-mode-II to the clamped ends at an end-shortening strain of $\varepsilon_0 = 2.97 \times 10^{-3}$. The Timoshenko beam partition theory predicts mode-II-dominated unstable propagation occurring at an end-shortening strain of $\varepsilon_0 = 2.92 \times 10^{-3}$, during which the delamination extends to a total length of 46.68 mm. Then the delamination propagates as pure-mode-II to the clamped ends at an end-shortening strain of $\varepsilon_0 = 2.97 \times 10^{-3}$. The 2D elasticity partition theory predicts a fairly mixed-mode unstable propagation occurring at an end-shortening strain of $\varepsilon_0 = 2.56 \times 10^{-3}$, during which the delamination extends to a total crack length of 37.45 mm. Then the delamination propagates as mode-II-dominated to the clamped ends at an end-shortening strain of $\varepsilon_0 = 2.96 \times 10^{-3}$. In a sense, the 2D elasticity partition theory is an ‘average’ of the two beam partition theories. The propagation behaviour is also shown graphically in Fig. 6 as delamination length $a$ vs. the end-shortening strain $\varepsilon_0$. It is seen that the predictions from the three partition theories are considerably different from each other. In contrast with the prediction for Case 1, for Case 2 the Timoshenko beam partition theory gives very different predictions to those from the Euler beam partition theory.

Similar to the study for Case 1, experimental test data in Ref. [9] are used to assess the accuracy of each partition theory. Fig. 7 shows the histories of the compression force per unit width $F$ against the upper surface mid-span axial strain $\varepsilon_{sc}$ as measured in testing and as predicted by the three partition theories. In general, it is seen that the predictions from the Euler beam partition theory agree quite well with the test results, that the predictions from the Timoshenko beam partition theory are poor, and that the predictions from the 2D-elasticity partition theory are somewhere in the middle.

As was seen for Case 1, the critical local-buckling compression force predicted analytically may not agree very well with the experimentally observed value. In order to examine the partition theories more thoroughly, it is necessary to correct for any discrepancy between the analytical and experimental critical local-buckling compression forces. Fig. 7, however, shows that an imperfection-type initial buckling is observed in experiments (whereas a bifurcation-type initial buckling is predicted by the analytical theories). To account for this, the intersection point of the linear regions of the pre-buckling and post-buckling responses in the experimental data in Fig. 7 (data markers 1 to 6, and 15 to 17 respectively) is used to approximate the experimental values of the upper-surface mid-span axial strain $\varepsilon_{sc}$ and the compression force $F$ at the point of
bifurcation-type local buckling, which are found to be $\varepsilon_{sc} = -1.834 \times 10^{-3}$ and $F = 672$ N. As before for Case 1, these values give two approximate critical local-buckling end-shortening strains $\varepsilon_{ce}$. When averaged, $\varepsilon_{ce} = 1.867 \times 10^{-3}$ is obtained with $\alpha_\varepsilon = 0.893$ and $\alpha_\varepsilon/\alpha = 0.949$. The critical local-buckling strain $\varepsilon_{ce}$ at any delamination length is then calculated from Eq. (52) as $\varepsilon_{ce} = 0.949^2 \varepsilon_\alpha$.

Fig. 8 shows the comparisons between the three partition theories and the test results [9]. In general, it is seen that the predictions from the Euler beam partition theory agree well with the test results, that the predictions from the Timoshenko beam partition theory are poor, and that the predictions from the 2D-elasticity partition theory are, again, somewhere in the middle.

4 Conclusions

Based on the Euler beam, Timoshenko beam and 2D-elasticity mixed-mode fracture partition theories [3,11-17], analytical theories have been developed for predicting the propagation behaviour of post-local buckling-driven delamination in bilayer composite beams. The conclusions are as follows: (1) accurate calculation of the total ERR $G$ is essential in order to obtain accurate predictions. This work has presented a more accurate analytical formula for total ERR $G$ than that in Refs. [2,3] by developing a more accurate expression for the post-buckling mode shape and also by including the axial strain energy contribution from the intact part of beam. Very good agreement is observed between the present analytical results and the numerical results [5]. (2) The accuracy of critical local-buckling strain is also a key factor in making accurate predictions. Empirical values, obtained either numerically or experimentally for particular cases, give more accurate predictions. (3) The method used to partition the total ERR $G$ into $G_\iota$ and $G_\iota'$ is another key factor for making accurate predictions. This work presents three partition theories, namely, the Euler beam, Timoshenko beam and 2D elasticity partition theories. Independent experimental tests by Kutlu and Chang [9] show that, in general, the analytical theory based on the Euler beam partition theory predicts the propagation behaviour very well and much better than the theories based on the Timoshenko beam and 2D elasticity partition theories, when using the critical local-buckling strain derived with the aid of experimental results. (4) Buckling-driven delamination is a major form of failure in engineering structures made of composite materials. One important example is the thermal buckling-driven cracking of thermal barrier coatings used in aero-engines. The present Euler beam analytical theory provides a valuable tool for the engineering design of such material structures. The
present work is being extended to buckling-driven delamination in generally laminated composite beams and will be reported in the near future.

References


Fig. 1: A post-locally buckled bilayer composite beam due to delamination under compression.
Fig. 2: Free-body diagram of a symmetrical half of the buckled upper layer.
Fig. 3: Delamination length vs. end-shortening strain for Case 1.
Fig. 4: Compression force per unit width $F$ vs. upper-surface mid-span strain $\varepsilon_{sc}$ for Case 1 using the analytical buckling strain $\varepsilon_c$. 
Fig. 5: Compression force per unit width $F$ vs. upper-surface mid-span strain $\varepsilon_{sc}$ for Case 1 using the experimental buckling strain $\varepsilon_{sc}$. 
Fig. 6: Delamination length vs. end-shortening strain for Case 2.
Fig. 7: Compression force per unit width $F$ vs. upper-surface mid-span strain $\varepsilon_{sc}$ for Case 2 using the analytical buckling strain $\varepsilon_c$. 
Fig. 8: Compression force per unit width $F$ vs. upper-surface mid-span strain $\varepsilon_{sc}$ for Case 2 using the experimental buckling strain $\varepsilon_{ce}$. 
Table 1: Configurations of two composite beams containing a central through-width delamination.

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Table 2: Total ERR $G$ results for Case 1.

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Table 3: Total ERR $G$ results for Case 2.

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Table 4: Delamination propagation behaviour of Case 1.

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Table 5: Delamination propagation behaviour of Case 2.

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The table shows the delamination propagation behaviour for Case 2, with data for $\varepsilon_0$, Euler, Timoshenko, and 2D Elasticity models.