Collision of plane gravitational and electromagnetic waves in a Minkowski background: solution of the characteristic initial value problem

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Abstract

We consider the collisions of plane gravitational and electromagnetic waves with distinct wavefronts and of arbitrary polarizations in a Minkowski background. We first present a new, completely geometric formulation of the characteristic initial value problem for solutions in the wave interaction region for which initial data are those associated with the approaching waves. We present also a general approach to the solution of this problem which enables us in principle to construct solutions in terms of the specified initial data. This is achieved by re-formulating the nonlinear dynamical equations for waves in terms of an associated linear problem on the spectral plane. A system of linear integral "evolution" equations which solve this spectral problem for specified initial data is constructed. It is then demonstrated explicitly how various colliding plane wave space-times can be constructed from given characteristic initial data.

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1 Introduction

The collisions of plane gravitational or gravitational and electromagnetic waves in general relativity have been studied extensively over recent decades by many authors. These studies have revealed many interesting properties concerning the nonlinear interaction between such waves. These relate to both the physical and geometrical features of space-times with colliding plane waves. The great majority of these properties have been deduced from detailed studies of particular solutions of Einstein's equations, which were derived by some very special particular methods and tricks. However, a general mathematical solution of this problem, in which solutions in the wave interaction region are constructed from "input data" (the parameters characterizing the waves before their collision) has not been presented until now (although reported briefly in [1]).

In this paper, we present a general scheme for a solution of this problem for gravitational or gravitational and electromagnetic waves of arbitrary polarizations propagating with distinct wavefronts and colliding in a Minkowski background. This method is based on the well known integrability properties of the hyperbolic reductions of Einstein's vacuum and electrovacuum field equations with a two-dimensional Abelian space-time isometry group. Mathematical tools for the solution of the characteristic initial value problems for these equations have been developed in [2]. However, this approach has had to be adapted (with some necessary generalizations) for colliding plane waves in a Minkowski background.

In the remaining part of this introduction, we recall some elements of the theory of integrable hyperbolic reductions of the Einstein and Einstein–Maxwell equations and give some references to preceding results. In the subsequent sections, we present a detail description of our new construction, the corresponding solution generation procedure, and we demonstrate how the procedure can be implemented. This demonstration includes the derivation of a previously unknown family of solutions.

Reviewing what is known about colliding plane waves, one could initially recall an appreciable number of known solutions of Einstein's equations which can be interpreted as the outcomes of a collision of plane waves. A large number of such solutions and their physical and geometrical interpretations can be found in [3]. This list may be extended by further publications appearing since 1991. In particular, infinite hierarchies of exact vacuum and electrovacuum solutions with an arbitrary number of free parameters have recently been found [4], and many of these are of the type that is appropriate for colliding plane waves. However, all of these colliding plane wave solutions have been found using an "inverse" approach in which a formal solution in the interaction region is directly obtained and the initial parameters of the corresponding approaching waves were calculated afterwards.

These solutions describe pairs of plane pure gravitational or mixed gravitational and electromagnetic waves with distinct wavefronts which approach each other from opposite spatial directions on the Minkowski background. The problem of constructing the corresponding solution in the wave interaction region is a well formulated characteristic initial value problem. (See [3] and the sections below for a detailed formulation of this problem including the general properties of approaching waves, typical matching conditions and the structure of the governing symmetry reduced field equations.)

In the important particular case in which the colliding waves are purely gravitational and possess constant and aligned polarizations, Einstein's equations in the interaction region can be reduced to a linear Euler–Poisson–Darboux equation. In this case, the solution can be constructed from the characteristic initial data using the generalized version of Abel's transform [5], [6]. However, if the polarizations of the approaching gravitational waves are not constant and aligned, or in the presence of electromagnetic waves, the governing Einstein or Einstein–Maxwell field equations are essentially nonlinear and this makes the problem much more complicated.

Fortunately, these equations (which can be represented conveniently in the form of hyperbolic Ernst equations) have been found to be completely integrable. In fact, the integrability of the hyperbolic vacuum and electrovacuum Ernst equations facilitates a number of solution generating techniques such as the generation of vacuum [7] and electrovacuum [8] solitons, Bäcklund or the symmetry transformations [9, 10, 11]. However, no techniques have been developed which provide algorithms for explicitly generating solutions from initial data. Nonetheless, this integrability gives rise also to some powerful methods for a general analysis of the governing field equations. In particular, Hauser and Ernst analyzed the characteristic initial value problem for the vacuum hyperbolic Ernst equation [12]. They generalized their group-theoretical approach (which had been developed earlier for stationary axisymmetric fields) and constructed a homogeneous Hilbert problem with corresponding matrix linear integral equations. Many global aspects of the characteristic initial value for vacuum fields, including the existence and uniqueness of solutions and a detailed proof of the Geroch conjecture, have been elaborated in this way [13]. However, this approach does not immediately lead to any effective methods for the solution of the corresponding characteristic initial value problem.

Other schemes for the solution of characteristic initial value problems for the vacuum and electrovavuum Ernst equations [14] or for the σ-model form of the vacuum equations [15] have also been considered. However, in these cases, characteristic initial value problems were analyzed only for initial data which are regular functions of the geometrically defined coordinates (4) below. As will be discussed below, this restriction does not cover the collisions of plane waves with distinct wavefronts in a Minkowski background. In such cases, physically relevant initial data, being regular on the wavefronts in the local frame, must possess some singularities as functions of the coordinates adopted.

The solution of the characteristic initial value problem for colliding plane waves that will be presented here originates from another approach to the analysis of the structure of integrable reductions of the Einstein and Einstein–Maxwell field equations that was developed in [16]–[18]. In this approach, every solution can be characterized by a set of functions of an auxiliary (spectral) parameter. These functions are interpreted as the monodromy data on the spectral plane of the fundamental solution of an associated linear system with a spectral parameter. These monodromy data are nonevolving (i.e. coordinate independent) and, generally, can be chosen arbitrarily or specified in accordance with the properties of the solution being sought. In particular, these data can be determined (at least in principle) from the initial or boundary data. In this scheme, the solution of the initial or boundary value problem is determined by the solution of some linear singular integral equations whose scalar kernel is constructed using these (specified) monodromy data. In this original form of the "monodromy transform" approach, a conjecture of local analyticity of solutions near the point of normalization was used very essentially. However, for waves with distinct wavefronts, the solutions of the linear equations are normalized at the point at which the waves collide and, for this situation, these are nonanalytic at this point. This violation of analyticity at the point of normalization gives rise to some difficulties with the construction of the master singular integral equation.

Our resolution of this difficulty is achieved by the introduction of some "dressing" or "scattering" matrices (see [2]) into the structure of the fundamental solution of the associated linear system. The problem can then be overcome because, even in the nonanalytical case, these matrices possess much more convenient analytical properties. For locally analytic solutions, this construction is completely equivalent to the previous approach. However, the new construction introduces some interesting features. In particular, the monodromy data which characterize the scattering matrices are not conserved quantities. They evolve, and are therefore referred to as "dynamical monodromy data". Moreover, the evolution of these data is determined completely by the behaviour of fields on the characteristics which cross at the point of normalization. Again, the usual nonlinear partial differential form of the field equations are replaced by a pair of linear integral equations which determine the solution uniquely. With this new construction, the scalar kernels of these new quasi-Fredholm equations are built from this new kind of monodromy data. These linear integral equations then essentially possess a different, simpler structure. Moreover, they are much better adapted to the context of a characteristic initial value problem.

2 Space-time geometry for colliding plane waves

Symmetry conjecture. In the situation under consideration, each of the approaching plane waves possesses two commuting space-like Killing vector fields with noncompact orbits, and we expect this symmetry to be global and retained during the collision and subsequent interaction of the waves. Thus, throughout the space-time, the metric and all nonmetric field components can be considered locally to be functions of two null coordinates, say u and v, while the two other space-like coordinates, say y and z (which we assume to be defined globally) are completely ignorable. An additional simplification of the metric and electromagnetic field components arises if we extend the conjecture of y- and z- independence from the metric and Maxwell tensor components to include their related potentials.

Conjectured structure of Einstein–Maxwell fields. For both colliding plane gravitational waves in vacuum and a coupled system of colliding plane gravitational and electromagnetic waves, the components of the metric and the 1-form of the electromagnetic vector potential can be considered globally in the forms

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{ab} dx^a dx^b, \qquad \underline{\mathbf{A}} = A_b dx^b \tag{1}$$

where $\mu, \nu, \ldots = 0, 1$ with $x^0, x^1 = u, v$, and $a, b, \ldots = 2, 3$ with $x^2, x^3 = y, z$. The orbit space metric $g_{\mu\nu}$, the metric on the orbits g_{ab} and the components A_b of the vector electromagnetic potential are functions of x^μ only.

General and geometrically defined coordinates. As is well known, the local coordinates x^μ can be specified to reduce $g_{\mu\nu}$ to a conformally flat form:

$$g_{\mu\nu} dx^\mu dx^\nu = f(u, v) du dv. \tag{2}$$

It is clear that these null coordinates are defined up to an arbitrary transformation of the form

$$\widetilde{u} = \widetilde{u}(u) \quad \text{and} \quad \widetilde{v} = \widetilde{v}(v). \tag{3}$$

For the analysis of the reduced Einstein–Maxwell equations and their integrability properties in the wave interaction region, it is convenient to use also some geometrically defined null coordinates ξ and η which can be introduced as follows:

$$\begin{cases} \xi = \beta + \alpha \\ \eta = \beta - \alpha \end{cases} \quad \text{where} \quad \alpha = \sqrt{\det \|g_{ab}\|} \quad \text{and} \quad \begin{cases} \partial_u \beta = \partial_u \alpha \\ \partial_v \beta = -\partial_v \alpha. \end{cases} \tag{4}$$

The function $\alpha(u,v)$ is the area measure on the orbits of the isometry group and the function $\beta(u,v)$ is then determined (up to an additive real constant). (It may be recalled that the Einstein–Maxwell equations imply that the function $\alpha(u,v)$ is a “harmonic” function in a sense that it satisfies the equation $\partial_u\partial_v\alpha = 0$ which also provides the integrability condition for the above β-equations to be satisfied.)

Parametrization of the metric components. It is convenient to parametrize the metric on the orbits by three scalar functions $\alpha(u,v) > 0$, $H(u,v) > 0$ and $\Omega(u,v)$ as

$$g_{ab} = -\begin{pmatrix} H & H\Omega \\ H\Omega & H\Omega^2 + \dfrac{\alpha^2}{H} \end{pmatrix}. \tag{5}$$

Newman–Penrose tetrad and scalars. With the metric (1) it is convenient to adopt the Newman–Penrose null tetrad whose vector components with respect to coordinates (u,v,y,z) are:

$$\begin{array}{lll} \mathbf{l}_j = \sqrt{\dfrac{f}{2}}\{1,0,0,0\} & \mathbf{n}_j = \sqrt{\dfrac{f}{2}}\{0,1,0,0\} & \mathbf{m}_j = \sqrt{\dfrac{H}{2}}\{0,0,1,\Omega + \dfrac{i\alpha}{H}\} \\ \mathbf{l}^j = \dfrac{1}{\sqrt{2f}}\{0,1,0,0\} & \mathbf{n}^j = \dfrac{1}{\sqrt{2f}}\{1,0,0,0\} & \mathbf{m}^j = \dfrac{i}{\alpha}\sqrt{\dfrac{H}{2}}\{0,0,\Omega + \dfrac{i\alpha}{H}, -1\} \end{array}$$

where $j = 0,1,2,3$. The expressions for the standard projections of the self-dual parts of the Weyl tensor (Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4) and of the Maxwell tensor (ϕ_0, ϕ_1, ϕ_2) then take the forms

$$\begin{array}{ll} \Psi_0 = -\dfrac{i}{\alpha}(\partial_v + i\alpha^{-1}H\partial_v\Omega)\left[\dfrac{H}{f}\partial_v(\Omega + \dfrac{i\alpha}{H})\right] & \phi_0 = -\dfrac{2}{\sqrt{fH}}\partial_v\Phi \\ \Psi_1 = 0 & \phi_1 = 0 \\ \Psi_2 = \dfrac{H^2}{2f\alpha^2}\partial_v(\Omega + \dfrac{i\alpha}{H})\partial_u(\Omega - \tfrac{i\alpha}{H}) - \dfrac{\partial_u\alpha\partial_v\alpha}{2f\alpha^2} & \phi_2 = \dfrac{2}{\sqrt{fH}}\partial_u\Phi \\ \Psi_3 = 0 & \\ \Psi_4 = \dfrac{i}{\alpha}(\partial_u - i\alpha^{-1}H\partial_u\Omega)\left[\dfrac{H}{f}\partial_u(\Omega - \dfrac{i\alpha}{H})\right] & \end{array} \tag{6}$$

where Φ is the Ernst scalar electromagnetic potential to be defined below. (This potential coincides with the y-component ($\Phi \equiv \Phi_y$) of the complex self-dual electromagnetic vector potential $\Phi_i = \{0,0,\Phi_y,\Phi_z\}$.)

The symmetry reduced Einstein–Maxwell equations. It is well known that, for these space-times, the nontrivial part of the Einstein–Maxwell equations decouples into two parts. One of these contains the constraint equations which can be considered as determining the conformal factor $f(u,v)$ (up to a multiplicative constant) in terms of the other functions:

$$\begin{cases} \dfrac{f_u}{f} = \dfrac{\alpha_{uu}}{\alpha_u} - \dfrac{H_u}{H} + \dfrac{\alpha}{2\alpha_u}\left[\dfrac{\left|\mathcal{E}_u + 2\overline{\Phi}\Phi_u\right|^2}{H^2} + \dfrac{4}{H}|\Phi_u|^2\right] \\ \dfrac{f_v}{f} = \dfrac{\alpha_{vv}}{\alpha_v} - \dfrac{H_v}{H} + \dfrac{\alpha}{2\alpha_v}\left[\dfrac{\left|\mathcal{E}_v + 2\overline{\Phi}\Phi_v\right|^2}{H^2} + \dfrac{4}{H}|\Phi_v|^2\right] \end{cases} \tag{7}$$

where all suffices mean the derivatives, and the right hand sides are expressed in terms of the complex Ernst potentials $\mathcal{E}(u,v)$ and $\Phi(u,v)$ which are determined by the relations:

$$\begin{cases} \operatorname{Re}\mathcal{E} = & -H - \Phi\overline{\Phi} \\ \partial_u(\operatorname{Im}\mathcal{E}) = & \alpha^{-1}H^2\partial_u\Omega + i(\overline{\Phi}\partial_u\Phi - \Phi\partial_u\overline{\Phi}) \\ \partial_v(\operatorname{Im}\mathcal{E}) = & -\alpha^{-1}H^2\partial_v\Omega + i(\overline{\Phi}\partial_v\Phi - \Phi\partial_v\overline{\Phi}) \end{cases} \qquad \begin{cases} \operatorname{Re}\Phi = & A_y \\ \partial_u(\operatorname{Im}\Phi) = & -\alpha^{-1}H(\partial_u A_z - \Omega\partial_u A_y) \\ \partial_v(\operatorname{Im}\Phi) = & \alpha^{-1}H(\partial_v A_z - \Omega\partial_v A_y) \end{cases} \tag{8}$$

where A_y, A_z are the nonzero components of a real electromagnetic vector potential.

The remaining part of the reduced electrovacuum Einstein–Maxwell equations are the dynamical equations for the metric functions $\alpha(u,v)$, $H(u,v)$ and $\Omega(u,v)$. It is very convenient to present these in the form of the hyperbolic Ernst equations for the Ernst potentials:

$$\begin{cases} (\operatorname{Re}\mathcal{E} + \Phi\overline{\Phi})\left(2\mathcal{E}_{uv} + \dfrac{\alpha_u}{\alpha}\mathcal{E}_v + \dfrac{\alpha_v}{\alpha}\mathcal{E}_u\right) - \left(\mathcal{E}_u + 2\overline{\Phi}\Phi_u\right)\mathcal{E}_v - \left(\mathcal{E}_v + 2\overline{\Phi}\Phi_v\right)\mathcal{E}_u = 0 \\ (\operatorname{Re}\mathcal{E} + \Phi\overline{\Phi})\left(2\Phi_{uv} + \dfrac{\alpha_u}{\alpha}\Phi_v + \dfrac{\alpha_v}{\alpha}\Phi_u\right) - \left(\mathcal{E}_u + 2\overline{\Phi}\Phi_u\right)\Phi_v - \left(\mathcal{E}_v + 2\overline{\Phi}\Phi_v\right)\Phi_u = 0 \\ \alpha_{uv} = 0 \end{cases} \tag{9}$$

These dynamical equations play the primary role in studies of colliding plane waves because they govern the nonlinear processes of the interaction between the waves. They also provide the integrability conditions for the constraint equations (7) and the compatibility of the relations (8) which relate all the metric functions and other field variables with the solutions of the Ernst equations (9). The constraint equations (7) determine the remaining metric component – the conformal factor $f(u,v)$.

Matching conditions at the wavefronts. We say that a plane wave with a distinct wavefront, say $u = 0$ or $v = 0$, propagates through a given background if any wave-like solution of (9) defined for $u \geq 0$ or $v \geq 0$ respectively matches appropriately on the corresponding wavefront to the given background solution which is defined in the region $u \leq 0$ or $v \leq 0$ respectively.

For colliding plane waves, it is well known that when considering the junction conditions across the null hypersurfaces which correspond to the wavefronts, the familiar Lichnerowicz conditions, which require that there should exist a coordinate system in which the components of the metric and electromagnetic potential are at least of class C^1 on the wavefront, should be relaxed in favour of the O'Brien–Synge conditions. These admit also some other types of waves, such as impulsive gravitational waves or gravitational and electromagnetic shock waves. For the class of metrics defined above, these conditions imply that across the wavefront

(i) the function α is of the class C^1

(ii) the Ernst potentials $\mathcal{E}$ and Φ are continuous

(iii) the conformal factor f is continuous

It is important to note here (see below for more details) that the condition (iii), through the relations (7), may impose some further restrictions (additional to the conditions (i) and (ii)) on the behaviour of function α and the Ernst potentials near the wavefront.

3 Plane waves in a Minkowski background

The Minkowski background. Within the class of metrics (1), the Minkowski space-time can be represented in the form

$$ds^2 = dudv - dy^2 - dz^2 \tag{10}$$

or, in the notations introduced above, by the values

$$\alpha = 1, \qquad \mathcal{E} = -1, \qquad \Phi = 0, \qquad f = 1 \tag{11}$$

(and therefore, $H = 1$ and $\Omega = 0$). Everywhere below, we shall consider this space-time as the background for the waves in the sense that was formulated in the previous section.

Travelling waves. It is well known that the Ernst equations (9) admit a wide class of plane wave solutions which can also be called "travelling waves". For these solutions all potentials depend on one null coordinate u or v only and, therefore, correspond to plane waves travelling without any evolution in the positive or negative x-direction respectively. To distinguish these similar solutions, we shall call the waves depending on u, and travelling therefore in the positive x-direction, the "left waves", and the waves depending on v and travelling in the negative x-direction the "right waves".

The left waves. For these waves all metric components and potentials depend only on u. It is very convenient to use the coordinate freedom $u \to \widehat{u}(u)$ to specify the conformal factor $f(u) = 1$. Geometrically this means that the coordinate u is chosen to be an affine parameter on the null geodesics $v =$ const, $y =$ const, $z =$ const which cross the wave. This class of waves can be described completely by the Ernst potentials

$$\{\mathcal{E}(u),\ \Phi(u)\} \tag{12}$$

which are functions of the affine parameter u and which should satisfy only the signature condition

$$H(u) \equiv -\mathrm{Re}\,\mathcal{E}(u) - \Phi(u)\overline{\Phi}(u) > 0. \tag{13}$$

The corresponding function $\alpha(u)$ is determined for $u > 0$ as the solution of a linear differential equation which follows from the first of the constraint equations (7) for the case $f(u) = 1$:

$$\alpha_{uu} - \frac{H_u}{H}\alpha_u + \frac{1}{2}\left[\frac{\left|\mathcal{E}_u + 2\overline{\Phi}\Phi_u\right|^2}{H^2} + \frac{4}{H}|\Phi_u|^2\right]\alpha = 0. \tag{14}$$

with the initial data at $u = 0$ which follows from the matching conditions.

These waves for $u \geq 0$ are combined with a Minkowski background for $u < 0$ as illustrated in Fig.(1). These two regions are joined on the null hypersurface (wavefront) $u = 0$ where all metric functions and the Ernst potentials (12) defined for $u \geq 0$ (the wave interior region) should be matched with their Minkowski values (11).

The matching conditions (i)–(iii) of the previous section imply the following conditions on the wavefront $u = 0$

$$\mathcal{E}(0) = -1, \quad \Phi(0) = 0, \quad \alpha(0) = 1, \quad \alpha'(0) = 0 \tag{15}$$

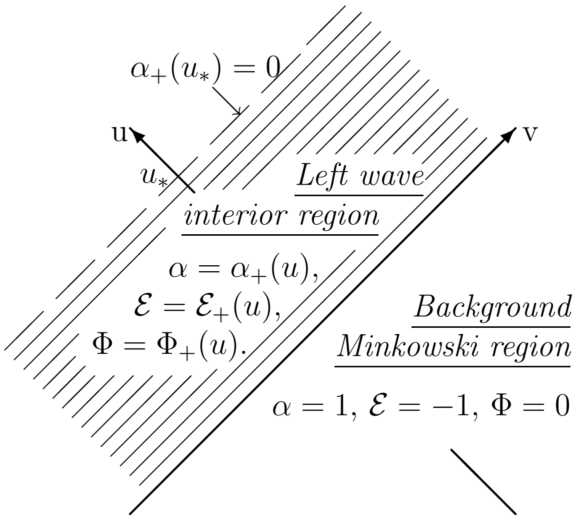

Figure 1: The left wave propagating on the Minkowski background. The coordinate u is chosen to be an affine parameter on the geodesics $v =$ const. In this case, $f = 1$ everywhere.

(together with the conditions $H(0) = 1$ and $\Omega(0) = 0$). The last two conditions in (15) provide initial conditions for the equation (14) for $\alpha(u)$ which is then determined uniquely. It follows from the structure of equation (14) for $H(u) > 0$ that, for any nonconstant data $\mathcal{E}(u)$, $\Phi(u)$, the corresponding function $\alpha(u)$ decreases monotonically from its initial value $\alpha(0) = 1$ with $\alpha'(0) = 0$, when u runs from 0 to some finite critical value u_* where $\alpha(u_*)$ vanishes. Since α is a measure of the area of two-dimensional sections of the null geodesic tubes, its decrease is a manifestation of the well known general focusing property of gravitational fields. The surface $u = u_*$ corresponds to the presence of a caustic, and represents a boundary of the space-time being considered.

The right waves. Solutions in which the metric components and the Ernst potentials are functions of the null coordinate v only are exactly equivalent to those described above, except that they propagate in the opposite, negative x-direction. In an exactly equivalent way to that described above, we impose the coordinate condition $f(v) = 1$ so that v is an affine parameter and any solution is then described by the corresponding Ernst potentials $\{\mathcal{E}(v),\ \Phi(v)\}$ and the function $\alpha(v) > 0$ satisfies an equation equivalent to (14). Again, we introduce a wave of this type for $v \geq 0$ by matching it with a Minkowski region for $v < 0$ exactly as above.

Other coordinates. Sometimes the wave solution simplifies considerably if we relax our choice of the null coordinate u (or v) as the affine parameter on the null geodesics crossing the travelling wave interior region.

In practice, it is often convenient to avoid having to solve the equation (14) for α. Instead, the freedom $u \to \tilde{u}(u)$ can be used to specify the form of $\alpha(u)$. However, this introduces a non-constant conformal function $f(u)$ which, for any wave functions $\mathcal{E}(u)$, $\Phi(u)$, can be calculated in quadratures from the constraint equations (7). Of course, the choice of the function α is restricted by the condition that it and its first derivative must be continuous on the wavefront and monotonically decreasing through the wave region. In addition, the choice of the Ernst potentials $\mathcal{E}(u)$ and $\Phi(u)$ should be also restricted by the condition for the regularity of the space-time geometry and the electromagnetic field components near the wavefront, and this imposes further restrictions on the character of α at the wavefront. This alternative approach will not generally be adopted in the remainder of this paper.

4 Characteristic initial value problem for colliding plane waves

Now we consider the situation in which two shock waves exist in the same space-time, are initially separated by some Minkowski region, and approach each other from opposite spatial directions. The left wave comes from the negative x-direction and has the wavefront $u = 0$, while the right wave comes from the positive x-direction and has the wavefront $v = 0$.

Before their collision, their interior regions ($u > 0$, $v < 0$ for the left wave and $v > 0$, $u < 0$ for the right wave) are described completely by the Ernst potentials expressed as functions of the affine parameters on the null geodesics crossing these regions. These are represented by the functions $\mathcal{E}_+(u)$, $\Phi_+(u)$ for $u > 0$ and $\mathcal{E}_-(v)$, $\Phi_-(v)$ for $v > 0$ for the left and right waves respectively. (For convenience, we denote functions associated with the left and right waves using the subscripts $+$ and $-$ respectively.) As discussed in the previous section, the function α which appears in the dynamical equations as an additional dynamical variable, is not an independent function. Rather, in the approaching waves, the functions $\alpha_+(u)$ and $\alpha_-(v)$ are determined uniquely as solutions of the corresponding constraint equations with initial data determined by the matching conditions on their wavefronts.

A collision of these waves at the point $u = 0$, $v = 0$ gives rise to the existence in space-time of a wave interaction region ($u > 0, v > 0$). As in the approaching wave regions, this region also possesses a boundary which is determined by the condition $\alpha(u, v) = 0$. However, unlike the boundaries $u = u_*$ and $v = v_*$ of the approaching wave regions, this boundary is generally a curvature singularity (see [3] for more details). The complete space-time is illustrated in Fig.(3).

The problem of the construction of the solution in the wave interaction region for a given pair of approaching travelling plane waves is clearly a characteristic initial value problem. The matching conditions applied on the boundaries ($u = 0, v > 0$) and ($v = 0, u > 0$) of the wave interaction region imply the continuity of the function α together with its first derivatives and continuity of the Ernst potentials on these characteristics. These conditions provide all the necessary characteristic initial values for these functions.

First of all, it is easy to see that the characteristic initial value problem for the function $\alpha(u, v)$ separates from that for the Ernst potentials and its solution is trivial:

$$\alpha(u, v) = \alpha_+(u) + \alpha_-(v) - 1, \tag{16}$$

because the matching conditions for the approaching waves with the Minkowski background have been satisfied, and the functions $\alpha_+(u)$ and $\alpha_-(v)$ possess the properties

$$\alpha_+(0) = \alpha_-(0) = 1, \qquad \alpha'_+(0) = \alpha'_-(0) = 0. \tag{17}$$

In particular, at the point of collision we have the "normalization" $\alpha(0, 0) = 1$.

We also have the characteristic initial data for the Ernst potentials in the interaction region:

$$\begin{aligned} \mathcal{E}(u, 0) &= \mathcal{E}_+(u), \qquad & \mathcal{E}(0, v) &= \mathcal{E}_-(v), \\ \Phi(u, 0) &= \Phi_+(u), \qquad & \Phi(0, v) &= \Phi_-(v). \end{aligned} \tag{18}$$

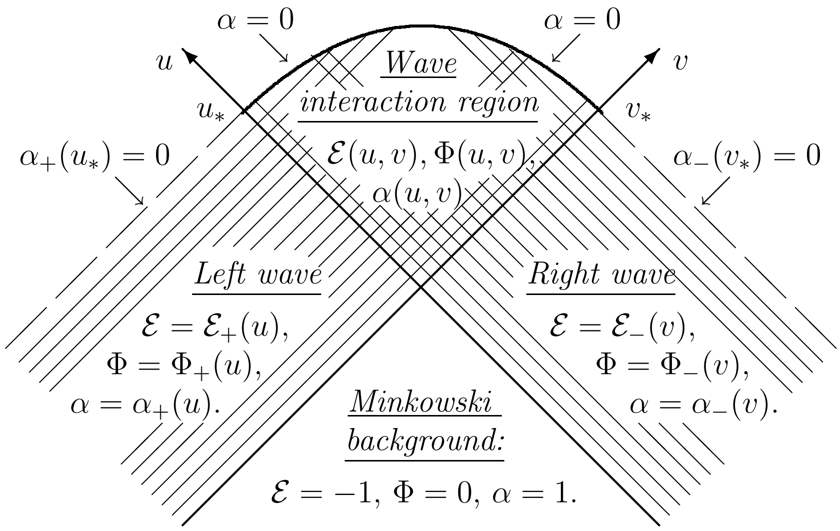

Figure 2: The collision of travelling plane waves propagating into a Minkowski background. The coordinates u and v in the wave regions before their collision are chosen to be affine parameters along the geodesics $v = \text{const}$ and $u = \text{const}$ respectively. The conformal factors in these regions are $f_+(u) = 1$ and $f_-(v) = 1$ respectively. However, in the wave interaction region, $f(u,v) \neq 1$ and $\alpha(u,v) = \alpha_+(u) + \alpha_-(v) - 1$.

These are identical to the functions which determine the structure of the approaching waves. The compatibility of these data at the point of collision follows from a correct matching of the original travelling waves with the Minkowski background. In this case, these data satisfy the conditions

$$\mathcal{E}_+(0) = \mathcal{E}_-(0) = -1, \qquad \Phi_+(0) = \Phi_-(0) = 0.$$

Thus, these also provide normalization conditions for the Ernst potentials at the point of collision: $\mathcal{E}(0,0) = -1$ and $\Phi(0,0) = 0$.

Geometrically defined null coordinates (u,v) **and** (ξ,η)**.** In the above formulation of the characteristic initial value problem, it seems appropriate to emphasize that two different pairs of geometrically defined null coordinates can be used. The coordinates (u,v) have been adapted to the orbit space of the isometry group where the dynamics of waves effectively takes place. These coordinates have been introduced in the approaching wave interior regions, prior to their collision, as the affine parameters on the plane wave rays and on the geodesics crossing the corresponding wavefronts. In the wave interaction region, these coordinates can then be defined uniquely using their continuity on the initial characteristics. However, in this region, these coordinates lose their property as affine parameters.

It is also convenient to introduce another system of null coordinates (ξ,η). These are also defined geometrically but, unlike the (u,v) coordinates which are adapted to the orbit space of the isometry group, the definition of these coordinates is adapted to the internal geometry of the orbits themselves and, therefore, to the internal geometry of the wave surfaces. More importantly for us, the coordinates (ξ,η) are better adapted to the analysis of the field equations.

The relation between these two coordinate systems can be found using the definition (4) and the expression (16) derived above. We can take the function β, which is conjugate to α, as

$$\beta = \alpha_+(u) - \alpha_-(v).$$

A constant of integration in this expression has been chosen to give $\beta = 0$ at the point of collision. With this expression, ξ and η have the forms:

$$\xi(u) = 2\alpha_+(u) - 1, \qquad \eta(v) = 1 - 2\alpha_-(v). \tag{19}$$

Since $\alpha_+(u)$ and $\alpha_-(v)$ are monotonically decreasing functions, ξ and η can be adopted as coordinates throughout the interaction region, with $\xi = 1$ and $\eta = -1$ on the wavefronts $u = 0$ and $v = 0$ respectively.

Unfortunately, when formulated in the coordinates (ξ, η), the characteristic initial value problem under consideration possesses some unpleasant properties. The problem arises from the fact that these coordinates, when extended to the initial regions, take the forms $\xi = 1$ for $u \leq 0$ and $\eta = -1$ for $v \leq 0$, so that the expressions for the inverse coordinate transformations $u(\xi)$ and $v(\eta)$ have nonanalytic behaviour near the wavefronts. This means that, even for physically acceptable solutions which possess no singularities on the wavefronts and which, therefore, are described by regular functions of the coordinates u and v, the metric components and potentials possess some singular behaviour near the wavefronts when expressed in terms of the coordinates ξ and η. In particular, although they are continuous on the wavefronts, they possess unbounded ξ- or η-derivatives there.

5 Monodromy transform and the integral evolution equations

In our present construction, we use the general approach developed in [16], [17] and [18] for the analysis of the structure and construction of solutions for integrable hyperbolic reductions of the vacuum Einstein and electrovacuum Einstein–Maxwell field equations. In this approach, which is called the monodromy transform (because of its analogy with the well known inverse scattering transform), every solution of the reduced field equations is characterized by two (for vacuum) or by four (for electrovacuum) functions which depend on the spectral parameter only. These are interpreted as the monodromy data of the corresponding fundamental solution of the associated linear system. The problem of the solution of the nonlinear field equations has been substituted in this approach by a solution of some linear singular integral equations whose scalar kernel is built from these monodromy data.

A further development of this approach, which provides some new mathematical tools for the solution of characteristic initial value problems, has been presented in [2]. In that paper, the general solution of the associated spectral problem was represented by two scattering matrices. These dress the initial values of the fundamental solution of the associated linear system given on two characteristics which pass through some arbitrarily chosen point where the solutions are normalized. The monodromy structure of these scattering matrices is characterised by "dynamical" monodromy data whose evolution is determined completely by the initial data for the fields on these characteristics. In this context, the problem of the solution of the nonlinear field equations reduces to a solution of some linear integral "evolution" equations of quasi-Fredholm type. The scalar kernels of these equations are built from the dynamical monodromy data and the initial values of the fundamental solution of the associated linear system on the above-mentioned initial characteristics. In terms of solutions of these linear integral evolution equations, the corresponding solution of the nonlinear field equations can be calculated in quadratures. Thus, many properties which are revealed by these constructions are in principle well adapted for the solution of the characteristic initial value problem for various integrable reductions of Einstein's equations.

As formulated in the cited papers, however, these constructions cannot be applied to a solution of the colliding plane wave problem. This is because the local analyticity of the solution is violated on the wavefronts. Moreover, as mentioned at the end of the previous section, although the components of physically acceptable solutions are regular functions of the (u, v) coordinates, their ξ- and η-derivatives have singular behaviour on the wavefronts. Fortunately, however, it has been found possible to derive a generalization of the previous construction which overcomes this difficulty. This was reported briefly in [1] where the corresponding generalized linear integral evolution equations were presented. It was shown how the character of the singularities of the kernels of the quasi-Fredholm integral equations can be explicitly related to that of the characteristic initial data. This construction, whose detailed description and some applications we present below, remains valid and, eventually, opens the way for the consideration of a wide range of types of colliding waves with nonanalytical behaviour on their wavefronts.

5.1 Associated linear system for the Ernst equations

Consider a linear system for a complex 3×3 matrix function $\mathbf{\Psi}(\xi, \eta, w)$ which depends on two real null coordinates ξ, η (as mentioned above) and a free complex (spectral) parameter w. To represent the electrovacuum Ernst equations, this system should be supplemented by additional conditions of the following two kinds. The first consists of algebraic restrictions on the matrix coefficients of the system. Together with the linear system itself, this can be presented in the form

$$\begin{cases} \partial_\xi \mathbf{\Psi} = \dfrac{\mathbf{U}(\xi,\eta)}{2i(w-\xi)}\mathbf{\Psi}, \qquad \operatorname{rank}\mathbf{U} = 1, \qquad \operatorname{tr}\mathbf{U} = i, \\ \partial_\eta \mathbf{\Psi} = \dfrac{\mathbf{V}(\xi,\eta)}{2i(w-\eta)}\mathbf{\Psi}, \qquad \operatorname{rank}\mathbf{V} = 1, \qquad \operatorname{tr}\mathbf{V} = i \end{cases} \tag{20}$$

The other conditions imply the existence of a Hermitian matrix integral of (20) which possesses the following structure

$$\begin{cases} \mathbf{\Psi}^\dagger\, \mathbf{W}\, \mathbf{\Psi} = \mathbf{W}_0(w), \\ \mathbf{W}_0^\dagger(w) = \mathbf{W}_0(w), \end{cases} \qquad \frac{\partial \mathbf{W}}{\partial w} = 4i\mathbf{\Omega}, \qquad \mathbf{\Omega} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{21}$$

where the Hermitian conjugation † is defined as $\mathbf{W}_0^\dagger(w) \equiv \overline{\mathbf{W}_0^T(\overline{w})}$, and $\mathbf{W}_0(w)$ is an arbitrary (nondegenerate) Hermitian 3×3 matrix function of w.

Equivalence of the matrix problem to the field equations. The conditions (20), (21) are equivalent to the symmetry reduced Einstein–Maxwell equations [17, 18]. For pure vacuum gravitational fields, the same set of conditions arise but for 2×2 matrices for which the third rows and columns of all matrices in (20) and (21) are omitted. In the fully general case, these conditions imply that the matrices $\mathbf{U}$ and $\mathbf{V}$ have the specific forms

$$\mathbf{U}=\mathcal{F}_+\cdot\widehat{\mathbf{U}}\cdot\mathcal{F}_+^{-1},\qquad \widehat{\mathbf{U}}=\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes(i,\,-\partial_\xi\mathcal{E},\,\partial_\xi\Phi),$$
$$\mathbf{V}=\mathcal{F}_-\cdot\widehat{\mathbf{V}}\cdot\mathcal{F}_-^{-1},\qquad \widehat{\mathbf{V}}=\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes(i,\,-\partial_\eta\mathcal{E},\,\partial_\eta\Phi),\qquad \mathcal{F}_\pm=\begin{pmatrix}1&0&0\\p_\pm&1&0\\q_\pm&0&1\end{pmatrix},\tag{22}$$

in which $p_\pm=\Omega\pm\dfrac{i\alpha}{H}$ and $q_\pm=2\overline{\widetilde{\Phi}}-2\overline{\Phi}\left(\Omega\pm\dfrac{i\alpha}{H}\right)$. Also, the matrix $\mathbf{W}$ must be a linear function of the spectral parameter w with the structure

$$\mathbf{W}=4i(w-\beta)\mathbf{\Omega}+\mathbf{G},\qquad \mathbf{G}=\begin{pmatrix}4(H\Omega^2+\dfrac{\alpha^2}{H})+4\widetilde{\Phi}\overline{\widetilde{\Phi}} & -4H\Omega-4\widetilde{\Phi}\overline{\Phi} & -2\widetilde{\Phi}\\ -4H\Omega-4\overline{\widetilde{\Phi}}\Phi & 4H+4\Phi\overline{\Phi} & 2\Phi\\ -2\overline{\widetilde{\Phi}} & 2\overline{\Phi} & 1\end{pmatrix}\tag{23}$$

If in (22)–(23), α, β, H, Ω, $\mathcal{E}$, Φ and $\widetilde{\Phi}$, are identified respectively with the previously defined functions $\alpha\equiv\frac{1}{2}(\xi-\eta)$, $\beta\equiv\frac{1}{2}(\xi+\eta)$, the metric functions H and Ω in (5), the Ernst potentials $\mathcal{E}$ and $\Phi\equiv\Phi_y$, and another component $\widetilde{\Phi}\equiv\Phi_z$ of the complex electromagnetic potential, then (20)–(21) imply that all the field equations and relations given above are satisfied for these functions. The inverse statement is also true, so that all of the functions in (22)–(23) can be calculated for any solution of the reduced Einstein–Maxwell equations. Thus, the set of conditions (20)–(21) for the four complex matrix functions

$$\mathbf{\Psi}(\xi,\eta,w),\quad \mathbf{U}(\xi,\eta),\quad \mathbf{V}(\xi,\eta),\quad \mathbf{W}(\xi,\eta,w)$$

constitute a problem whose solution is equivalent to the solution of the symmetry reduced Einstein–Maxwell equations. However, there is a significant advantage in substituting an analysis of the matrix problem (20)–(21) in place of studying the nonlinear partial differential equations which govern the collision of gravitational or gravitational and electromagnetic waves. It was just this step that was made in previous formulations of the monodromy transform approach, and we develop this further here.

5.2 Normalization conditions

Before we analyze the matrix problem (20)–(21), it is important to remove the gauge freedom which exists in these equations. Without loss of generality, we can impose normalization conditions for the values at some "initial" or "reference" space-time point of the matrix functions $\mathbf{\Psi}$ and $\mathbf{W}$, as well as of the components of the metric and electromagnetic potentials and the Ernst potentials. For the problem considered here, it is convenient to choose this point of normalization as the point at which the waves collide. i.e. with the point $u=0$, $v=0$, or $\xi=1$, $\eta=-1$. First, we recall that we have already normalized the metric functions and the Ernst potentials at $u=0$, $v=0$ as follows:

$$\begin{array}{llll}\alpha(0,0)=1, & H(0,0)=1, & \mathcal{E}(0,0)=-1, & \widetilde{\Phi}(0,0)=0\\ \beta(0,0)=0, & \Omega(0,0)=0, & \Phi(0,0)=0. & \end{array}$$

For the matrix function $\mathbf{\Psi}$, we can adopt the normalization

$$\mathbf{\Psi}(1,-1,w)=\mathbf{I}.\tag{24}$$

by using the transformation $\mathbf{\Psi}(\xi,\eta,w)\to\mathbf{\Psi}(\xi,\eta,w)\mathbf{\Psi}^{-1}(1,-1,w)$.

From (21) and (23), it is now easy to show that the above normalization conditions imply that the matrix integral $\mathbf{W}_0(w)$ in (21) satisfies

$$\mathbf{W}_0(w)=4iw\mathbf{\Omega}+\text{diag}\,(4,4,1)\ .$$

5.3 The analytical structure of $\mathbf{\Psi}$ on the spectral plane

Everywhere below, $\mathbf{\Psi}$ denotes the fundamental solution of the linear system (20), normalized at the point of collision as in (24). The general analytical structure of this solution was described in detail in [16], [17], [18]. However, in those formulations, $\mathbf{\Psi}$ was normalized at some arbitrarily chosen space-time point at which all components of the solution of the field equations are analytic functions of the coordinates ξ and η. However, in the present case we specifically choose the point of normalization to be the point $(\xi,\eta)=(1,-1)$ at which the waves collide. It is obvious that this point is not a point at which the field components are locally analytic.

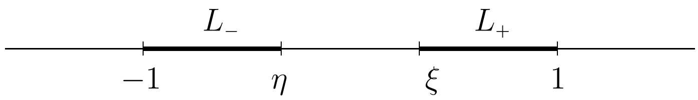

Figure 3: Cuts in the spectral plane w

As in the regular case, with the normalization (24), the structure of the associated linear system implies that the fundamental solution $\mathbf{\Psi}(\xi,\eta,w)$ (and also its inverse) possesses four branch points on the spectral plane w. The relative positions of these points and the cuts $L_\pm$ which join them are indicated in Fig. 3. These singular points include the two "variable" points $w=\xi$ and $w=\eta$ which correspond to the poles of the coefficients of the linear system (20). As in the regular case, these singularities for $\mathbf{\Psi}$ are branch points of order $-1/2$. In addition, there are two fixed branch points of $\mathbf{\Psi}$ at $w=1$ and $w=-1$ which arise from the normalization (24) imposed at the point of the collision. However, the order of the branching of $\mathbf{\Psi}$ at these two points may differ from 1/2 as occurs in the regular case. Moreover, the character of these branch points is not universal in this case. In fact, it becomes dependent on the solutions and differs from one solution to another.

Fortunately, we can proceed with our analysis without needing to specify the order of branching of $\mathbf{\Psi}$ at the points $w=1$ and $w=-1$ on the spectral plane. Following mainly the same simple arguments given in [16, 17] and the ideas suggested for the regular case in [2], we present below a generalization of this construction that is available for the analysis of fields which have nonanalytic behaviour on the characteristics.

Because the rank of the matrix coefficients of the linear equations (20) is everywhere not more than 1, near each of the singular points of $\mathbf{\Psi}$ only one of three linearly independent solutions of the associated linear system can be branching, while the two others should be holomorphic. This immediately implies that the normalized fundamental solution $\mathbf{\Psi}$ in general has the local structure on the cuts $L_\pm$ which can be expressed in the forms:

$$\begin{aligned}L_+:\quad &\mathbf{\Psi}(\xi,\eta,w)=\widetilde{\boldsymbol{\psi}}_+(\xi,\eta,w)\otimes\mathbf{k}_+(w)+\mathbf{M}_+(\xi,\eta,w)\\ L_-:\quad &\mathbf{\Psi}(\xi,\eta,w)=\widetilde{\boldsymbol{\psi}}_-(\xi,\eta,w)\otimes\mathbf{k}_-(w)+\mathbf{M}_-(\xi,\eta,w)\end{aligned}\tag{25}$$

where, as in the regular case, the coordinate independent components of the row vectors $\mathbf{k}_+(w)$ and $\mathbf{k}_-(w)$ are regular near the cuts L_+ and L_- respectively or, at least, are not branching. However, they can have isolated singularities at the endpoints $w=1$

and $w = -1$ respectively. The same is true for the matrices $\mathbf{M}_+$ and $\mathbf{M}_-$ on L_+ and L_- respectively. In the regular case, the vectors $\widetilde{\boldsymbol{\psi}}_\pm(\xi,\eta,w)$ always have the structures $\widetilde{\boldsymbol{\psi}}_+(\xi,\eta,w) = \sqrt{\frac{w-1}{w-\xi}}\boldsymbol{\psi}_+(\xi,\eta,w)$ and $\widetilde{\boldsymbol{\psi}}_-(\xi,\eta,w) = \sqrt{\frac{w+1}{w-\eta}}\boldsymbol{\psi}_-(\xi,\eta,w)$ where the vectors $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ are holomorphic on the cuts L_+ and L_- respectively. In the general case, the components of the column vectors $\widetilde{\boldsymbol{\psi}}_\pm(\xi,\eta,w)$ also have branch points at the endpoints of the corresponding cuts $L_\pm$. However, the characters of their singularities at the points $w = 1$ and $w = -1$ respectively can be different for different solutions. Their specific characters are determined by the physical properties of the wavefronts as incorporated in the characteristic initial data for the approaching waves.

5.4 Characteristic initial value problem for $\boldsymbol{\Psi}$ and analytical structure of its initial data

Following a similar construction as for the regular case given in [2], we now reformulate the characteristic initial value problem for the hyperbolic Ernst equations described in section 4 as an equivalent characteristic initial value problem for the matrix function $\boldsymbol{\Psi}$. Later it will be shown that the solution of this problem can be reduced to a solution of some simple linear integral equations.

First, it is easy to observe that the given initial data (18) on the characteristics enable us to calculate the matrix coefficients in (20) on these characteristics. i.e. we can determine the values of $\mathbf{U}(\xi,\eta=-1)$ and $\mathbf{V}(\xi=1,\eta)$. This enables us to introduce two matrix functions $\boldsymbol{\Psi}_+(\xi,w)$ and $\boldsymbol{\Psi}_-(\eta,w)$ which are the normalized fundamental solutions of the associated linear system on the characteristics $\xi = 1$ and $\eta = -1$:

$$\begin{cases} \partial_\xi \boldsymbol{\Psi}_+(\xi,w) = \dfrac{\mathbf{U}(\xi,-1)}{2i(w-\xi)} \cdot \boldsymbol{\Psi}_+(\xi,w) \\ \boldsymbol{\Psi}_+(1,w) = \mathbf{I} \end{cases} \qquad \begin{cases} \partial_\eta \boldsymbol{\Psi}_-(\eta,w) = \dfrac{\mathbf{V}(1,\eta)}{2i(w-\eta)} \cdot \boldsymbol{\Psi}_-(\eta,w) \\ \boldsymbol{\Psi}_-(-1,w) = \mathbf{I} \end{cases} \tag{26}$$

The coefficients of each of these systems are completely determined by the initial data for the fields on the corresponding characteristic. Thus these matrices form the characteristic initial data for the required solution for $\boldsymbol{\Psi}(\xi,\eta,w)$:

$$\boldsymbol{\Psi}_+(\xi,w) \equiv \boldsymbol{\Psi}(\xi,-1,w), \quad \boldsymbol{\Psi}_-(\eta,w) \equiv \boldsymbol{\Psi}(1,\eta,w)$$

Applying now the same arguments that were used in [16, 17], we conclude that the boundary values $\boldsymbol{\Psi}_+(\xi,w)$ and $\boldsymbol{\Psi}_-(\eta,w)$ should possess analytical structures on the spectral plane which are very similar to that of $\boldsymbol{\Psi}(\xi,\eta,w)$. Namely, $\boldsymbol{\Psi}_\pm(w=\infty) = \mathbf{I}$ and $\boldsymbol{\Psi}_+$ and its inverse are holomorphic outside L_+, while $\boldsymbol{\Psi}_-$ and its inverse are holomorphic outside L_-. Moreover, their local structures on the cuts are given by the expressions

$$\begin{aligned} L_+ : \quad & \boldsymbol{\Psi}_+(\xi,w) = \widetilde{\boldsymbol{\psi}}_{0+}(\xi,w) \otimes \mathbf{k}_+(w) + \mathbf{M}_{0+}(\xi,w), \\ L_- : \quad & \boldsymbol{\Psi}_-(\eta,w) = \widetilde{\boldsymbol{\psi}}_{0-}(\eta,w) \otimes \mathbf{k}_-(w) + \mathbf{M}_{0-}(\eta,w) \end{aligned} \tag{27}$$

where $\mathbf{k}_\pm(w)$ are the same as for $\boldsymbol{\Psi}$ in (25), and $\mathbf{M}_{0+}(\xi,w)$ and $\mathbf{M}_{0-}(\eta,w)$ are non-branching on L_+ and L_- respectively. However, $\boldsymbol{\Psi}_\pm$ may have isolated singularities at the endpoints $w = 1$ and $w = -1$ of these cuts respectively. As for the general case described above, the vectors $\widetilde{\boldsymbol{\psi}}_{0+}$ and $\widetilde{\boldsymbol{\psi}}_{0-}$ have branch points at the ends of the cuts L_+ or L_- respectively. Also, the character of their branching at these points is completely determined by the initial data.

5.5 The "scattering" matrices and their structures

We now introduce, in analogy with the regular case, the "evolution" or "scattering" matrices $\chi_\pm(\xi,\eta,w)$, which represent $\boldsymbol{\Psi}(\xi,\eta,w)$ in two alternative forms

$$\boldsymbol{\Psi}(\xi,\eta,w)=\chi_+(\xi,\eta,w)\cdot\boldsymbol{\Psi}_+(\xi,w),\qquad \boldsymbol{\Psi}(\xi,\eta,w)=\chi_-(\xi,\eta,w)\cdot\boldsymbol{\Psi}_-(\eta,w). \tag{28}$$

To understand the analytical structures of these matrices on the spectral plane, we express them using the above relations in the form

$$\chi_+(\xi,\eta,w)\equiv\boldsymbol{\Psi}(\xi,\eta,w)\cdot\boldsymbol{\Psi}_+^{-1}(\xi,w),\qquad \chi_-(\xi,\eta,w)\equiv\boldsymbol{\Psi}(\xi,\eta,w)\cdot\boldsymbol{\Psi}_-^{-1}(\eta,w). \tag{29}$$

It may be noted that these matrices are solutions of linear equations with well defined initial conditions. We obtain these equations for $\chi_\pm$ after multiplication of the second of two linear systems in (20) by $\boldsymbol{\Psi}_+^{-1}(\xi,w)$ and the first one by $\boldsymbol{\Psi}_-^{-1}(\eta,w)$ from the right. We then find that each of the matrices χ_+ and χ_- are determined by the linear systems with obvious initial conditions

$$\begin{cases}\partial_\xi\chi_-=\dfrac{\mathbf{U}(\xi,\eta)}{2i(w-\xi)}\cdot\chi_-\\ \chi_-(1,\eta,w)=\mathbf{I}\end{cases}\qquad \begin{cases}\partial_\eta\chi_+=\dfrac{\mathbf{V}(\xi,\eta)}{2i(w-\eta)}\cdot\chi_+\\ \chi_+(\xi,-1,w)=\mathbf{I}\end{cases} \tag{30}$$

For any given local solution of the Ernst equations each of these two systems admits a unique solution. In both of these cases, the structures of the linear systems and the initial conditions show immediately that the corresponding fundamental solutions χ_+ or χ_- should be holomorphic on the spectral plane w everywhere outside the cut L_- or L_+ respectively. Each of these fundamental solutions possesses only two branch points, which are located at the endpoints of the corresponding cut, and a finite jump on this cut. In addition, it is easy to show that these matrix functions should also possess the property $\chi_\pm(\xi,\eta,w=\infty)\equiv\mathbf{I}$.

All of the above features enable us to represent each of these matrix functions as the Cauchy integrals of their jumps over the corresponding cut:

$$\chi_+(\xi,\eta,w)=\mathbf{I}+\frac{1}{i\pi}\int\limits_{L_-}\frac{[\chi_+]_{\zeta_-}}{\zeta_--w}d\zeta_-\qquad \chi_-(\xi,\eta,w)=\mathbf{I}+\frac{1}{i\pi}\int\limits_{L_+}\frac{[\chi_-]_{\zeta_+}}{\zeta_+-w}d\zeta_+ \tag{31}$$

where $[\ldots]_\zeta$ is the jump (a half of the difference between the left and right limits) of a function at the point $w=\zeta$ on a cut. It may be noted that we usually choose directions on the paths L_+ and L_- shown in Fig. 3 from $w=1$ to $w=\xi$ and from $w=-1$ to $w=\eta$ respectively. However, the above definition of the jumps makes the integrals (31) insensitive to the choice of these directions. It is also necessary to note here that the convergence of the Cauchy integrals in (31) at $w=-1$ and $w=1$ respectively has been assumed. This conjecture has been proved for the regular case, and is confirmed for all known examples which possess different degrees of non-smooth behaviour on the characteristics.

Let us now explicitly calculate the structure of the jumps of $\chi_\pm$ on the cuts using the definitions (29), the local structures (25) as well as the analyticity of $\boldsymbol{\Psi}_+$ and its inverse on L_- and $\boldsymbol{\Psi}_-$ and its inverse on L_+. Simple calculations lead to the expressions

$$[\chi_+]_{\tau_-}=[\widetilde{\boldsymbol{\psi}}_-]_{\tau_-}\otimes\mathbf{m}_-(\xi,\tau_-),\qquad [\chi_-]_{\tau_+}=[\widetilde{\boldsymbol{\psi}}_+]_{\tau_+}\otimes\mathbf{m}_+(\eta,\tau_-), \tag{32}$$

where $\tau_+ \in L_+$ and $\tau_- \in L_-$. Substituting these into the integrands of (31), we obtain

$$\begin{aligned} \boldsymbol{\chi}_+(\xi,\eta,w) &= \mathbf{I} + \frac{1}{\pi i}\int\limits_{L_-} \frac{[\widetilde{\boldsymbol{\psi}}_-]_{\zeta_-} \otimes \mathbf{m}_-(\xi,\zeta_-)}{\zeta_- - w}\, d\zeta_- \\ \boldsymbol{\chi}_-(\xi,\eta,w) &= \mathbf{I} + \frac{1}{\pi i}\int\limits_{L_+} \frac{[\widetilde{\boldsymbol{\psi}}_+]_{\zeta_+} \otimes \mathbf{m}_+(\eta,\zeta_+)}{\zeta_+ - w}\, d\zeta_+. \end{aligned} \tag{33}$$

These expressions involve a new, evolving kind of vector-functions

$$\mathbf{m}_+(\eta,w) = \mathbf{k}_+(w)\cdot\boldsymbol{\Psi}_-^{-1}(\eta,w), \qquad \mathbf{m}_-(\xi,w) = \mathbf{k}_-(w)\cdot\boldsymbol{\Psi}_+^{-1}(\xi,w). \tag{34}$$

These "dynamical" monodromy data replace the coordinate independent vectors $\mathbf{k}_\pm(w)$ which arose earlier in similar integral representations for $\boldsymbol{\Psi}$ normalized at a regular point. (The interpretation of these vector functions was explained in [2] for the regular case.)

5.6 Integral "evolution" equations

The alternative representations of $\boldsymbol{\Psi}$ that are given in (28) implies an obvious consistency condition

$$\boldsymbol{\chi}_+(\xi,\eta,w)\boldsymbol{\Psi}_+(\xi,w) = \boldsymbol{\chi}_-(\xi,\eta,w)\boldsymbol{\Psi}_-(\eta,w). \tag{35}$$

It is useful to recall that the matrix functions $\boldsymbol{\Psi}_+$ and $\boldsymbol{\chi}_-$ are holomorphic on the spectral plane everywhere outside L_+, while $\boldsymbol{\Psi}_-$ and $\boldsymbol{\chi}_+$ are holomorphic outside L_-. In addition, the values of these four matrices at $w = \infty$ are all equal to the unit matrix $\mathbf{I}$. Therefore, the right and left hand sides of (35) are analytical functions outside the cuts L_+ and L_-. Hence, for the condition (35) to be satisfied, it is necessary that (here and everywhere below, $\tau_+ \in L_+$ and $\tau_- \in L_-$)

$$\boldsymbol{\chi}_+(\tau_+)[\boldsymbol{\Psi}_+]_{\tau_+} = [\boldsymbol{\chi}_-]_{\tau_+}\boldsymbol{\Psi}_-(\tau_+), \qquad [\boldsymbol{\chi}_+]_{\tau_-}\boldsymbol{\Psi}_+(\tau_-) = \boldsymbol{\chi}_-(\tau_-)[\boldsymbol{\Psi}_-]_{\tau_-}. \tag{36}$$

Here (and in some expressions below) we omit for simplicity the parametric dependence of the functions on the coordinates ξ , η, or both ξ and η.

We now substitute into (36) the integral representations (33), and the expressions (32). The expressions for the jumps of $\boldsymbol{\Psi}_\pm$ on the cuts $L_\pm$ can then be derived easily from the local representations (27):

$$[\boldsymbol{\Psi}_+]_{\tau_+} = [\widetilde{\boldsymbol{\psi}}_{0+}]_{\tau_+} \otimes \mathbf{k}_+(\tau_+), \quad \tau_+ \in L_+ \qquad [\boldsymbol{\Psi}_-]_{\tau_-} = [\widetilde{\boldsymbol{\psi}}_{0-}]_{\tau_-} \otimes \mathbf{k}_-(\tau_-). \tag{37}$$

It is convenient to denote the jumps of the column-vector functions $\widetilde{\boldsymbol{\psi}}_\pm$ on the cuts $L_\pm$ as

$$\boldsymbol{\phi}_+(\xi,\eta,\tau_+) \equiv [\widetilde{\boldsymbol{\psi}}_+]_{\tau_+}, \qquad \boldsymbol{\phi}_-(\xi,\eta,\tau_-) \equiv [\widetilde{\boldsymbol{\psi}}_-]_{\tau_-},$$

and the initial values of these jumps as respectively

$$\boldsymbol{\phi}_{0+}(\xi,\tau_+) \equiv [\widetilde{\boldsymbol{\psi}}_{0+}]_{\tau_+}, \qquad \boldsymbol{\phi}_{0-}(\eta,\tau_-) \equiv [\widetilde{\boldsymbol{\psi}}_{0-}]_{\tau_-}.$$

Substituting the expressions (33), (32) and (37) into (36), we arrive at the following coupled pair of linear integral equations for the functions $\boldsymbol{\phi}_+(\xi,\eta,\tau_+)$ and $\boldsymbol{\phi}_-(\xi,\eta,\tau_-)$, which are defined on the cuts L_+ and L_- respectively:

$$\left\{ \begin{aligned} \boldsymbol{\phi}_+(\tau_+) - \int\limits_{L_-} S_+(\tau_+,\zeta_-)\boldsymbol{\phi}_-(\zeta_-)\, d\zeta_- &= \boldsymbol{\phi}_{0+}(\tau_+) \\ \boldsymbol{\phi}_-(\tau_-) - \int\limits_{L_+} S_-(\tau_-,\zeta_+)\boldsymbol{\phi}_+(\zeta_+)\, d\zeta_+ &= \boldsymbol{\phi}_{0-}(\tau_-) \end{aligned} \right. \tag{38}$$

where $\tau_+, \zeta_+ \in L_+$ and $\tau_-, \zeta_- \in L_-$. Also the scalar kernels of these equations take the forms

$$S_+(\xi,\tau_+,\zeta_-) = \frac{\Big(\mathbf{m}_-(\xi,\zeta_-)\cdot \boldsymbol{\phi}_{\,0+}(\xi,\tau_+)\Big)}{i\pi(\zeta_- - \tau_+)}, \qquad S_-(\eta,\tau_-,\zeta_+) = \frac{\Big(\mathbf{m}_+(\eta,\zeta_+)\cdot \boldsymbol{\phi}_{\,0-}(\eta,\tau_-)\Big)}{i\pi(\zeta_+ - \tau_-)}. \tag{39}$$

The equations (38) can easily be decoupled into two separate equations for the vector functions $\boldsymbol{\phi}_{\,+}(\xi,\eta,\tau_+)$ and $\boldsymbol{\phi}_{\,-}(\xi,\eta,\tau_-)$. Substituting $\boldsymbol{\phi}_{\,-}(\xi,\eta,\tau_-)$ from the second equation in (38) into the first, and inversely, we obtain the decoupled integral equations

$$\begin{aligned} \boldsymbol{\phi}_{\,+}(\tau_+) - \int\limits_{L_+} \mathcal{F}_+(\tau_+,\zeta_+)\boldsymbol{\phi}_{\,+}(\zeta_+)\, d\zeta_+ &= \mathbf{f}_+(\tau_+) \\ \boldsymbol{\phi}_{\,-}(\tau_-) - \int\limits_{L_-} \mathcal{F}_-(\tau_-,\zeta_-)\boldsymbol{\phi}_{\,-}(\zeta_-)\, d\zeta_- &= \mathbf{f}_-(\tau_-) \end{aligned} \tag{40}$$

in which the kernels and right hand sides are given by

$$\begin{aligned} \mathcal{F}_+(\tau_+,\zeta_+) &= \int\limits_{L_-} \mathcal{S}_+(\tau_+,\chi_-)\mathcal{S}_-(\chi_-,\zeta_+)\, d\chi_-, \\ \mathcal{F}_-(\tau_-,\zeta_-) &= \int\limits_{L_+} \mathcal{S}_-(\tau_-,\chi_+)\mathcal{S}_+(\chi_+,\zeta_-)\, d\chi_+, \\ \mathbf{f}_+(\tau_+) &= \boldsymbol{\phi}_{\,0+}(\tau_+) + \int\limits_{L_-} \mathcal{S}_+(\tau_+,\chi_-)\boldsymbol{\phi}_{\,0-}(\chi_-)\, d\chi_-, \\ \mathbf{f}_-(\tau_-) &= \boldsymbol{\phi}_{\,0-}(\tau_-) + \int\limits_{L_+} \mathcal{S}_-(\tau_-,\chi_+)\boldsymbol{\phi}_{\,0+}(\chi_+)\, d\chi_+. \end{aligned} \tag{41}$$

However, although they are decoupled, these equations possess more complicated structures than those in (38).

All the coefficients of the integral equations (38), and the equivalent decoupled pair (40), are determined by the initial data functions $\mathbf{\Psi}_+(\xi,w)$ and $\mathbf{\Psi}_-(\eta,w)$. These are determined, in turn, by the characteristic initial data for the field variables (e.g., the Ernst potentials). The relations of the coefficients with the characteristic initial data will be clarified in more detail in subsequent sections where we consider particular examples of the calculation procedure.

6 Principal algorithm for solution of the problem

6.1 Specification of the initial data

As explained in sections 3 and 4, we may consider the initial data to consist of the Ernst potentials, given as functions of the affine parameters u and v along the initial null characteristics

$$\{\mathcal{E}_+(u), \Phi_+(u)\}, \qquad \{\mathcal{E}_-(v), \Phi_-(v)\}. \tag{42}$$

Alternatively, the initial data may consist of the metric functions and the complex electromagnetic potential (again as functions of the affine parameters u and v)

$$\{H_+(u), \Omega_+(u), \Phi_+(u)\}, \qquad \{H_-(v), \Omega_-(v), \Phi_-(v)\} \tag{43}$$

In either case, we require the continuity conditions

$$\mathcal{E}_\pm(0) = -1, \quad \Phi_\pm(0) = 0, \quad H_\pm(0) = 1, \quad \Omega_\pm(0) = 0. \tag{44}$$

Because of the choice of u and v as affine parameters on the initial characteristics, for the conformal factor f, we have $f_+(u) = 1$ and $f_-(v) = 1$. Also, the function $\alpha(u,v)$ is given explicitly by (16), where the functions $\alpha_+(u)$ and $\alpha_-(v)$ are determined from the linear equations with the well defined initial conditions:

$$\begin{cases} \alpha_+'' - \alpha_+' \dfrac{H_+'}{H_+} + \dfrac{\alpha_+}{2H_+}\left[\dfrac{\left|\mathcal{E}_+' + 2\overline{\Phi}_+\Phi_+'\right|^2}{H_+} + 4|\Phi_+'|^2\right] = 0, \\ \alpha_+(0) = 1, \quad \alpha_+'(0) = 0. \end{cases} \tag{45}$$

where $H_+(u) \equiv -\mathrm{Re}\,\mathcal{E}_+(u) - \Phi_+(u)\overline{\Phi}_+(u)$ and

$$\begin{cases} \alpha_-'' - \alpha_-' \dfrac{H_-'}{H_-} + \dfrac{\alpha_-}{2H_-}\left[\dfrac{\left|\mathcal{E}_-' + 2\overline{\Phi}_-\Phi_-'\right|^2}{H_-^2} + 4|\Phi_-'|^2\right] = 0, \\ \alpha_-(0) = 1, \quad \alpha_-'(0) = 0. \end{cases} \tag{46}$$

where $H_-(v) \equiv -\mathrm{Re}\,\mathcal{E}_-(v) - \Phi_-(v)\overline{\Phi}_-(v)$. Having obtained the functions $\alpha_+(u)$ and $\alpha_-(v)$, the functions $\xi(u)$ and $\eta(v)$ must now be determined using the definitions (4).

A nice feature of the above specification is that the functions (42) or (43) can be taken as initial data functions and chosen arbitrarily, provided only that the conditions (44) are satisfied. In this case, we have to solve the linear equations (45) and (46) to find the corresponding functions $\alpha_+(u)$ and $\alpha_-(v)$.

6.2 Characteristic initial values for $\mathbf{\Psi}$

Everything is now ready for the calculation of the characteristic initial values for $\mathbf{\Psi}(u,v,w)$. From the initial data, we must calculate the matrix coefficients $\mathbf{U}(\xi(u),-1) = \mathbf{U}(u, v=0)$ and $\mathbf{V}(1,\eta(v)) = \mathbf{V}(u=0,v)$ of the linear systems (26), using the definitions (22) and expressing them in terms of the u, v coordinates. We should then solve the equations (26) and determine the normalized fundamental solutions $\mathbf{\Psi}_+(u,w)$ and $\mathbf{\Psi}_-(v,w)$.

It is clear, of course, that this step is crucial for a practical realization of the method. In practice, however, for a great majority of the initial data, it is found to be impossible to solve these equations explicitly in a closed form. Nonetheless, it may be argued that there should exist some infinite hierarchies of the initial data for which the solutions of these linear systems can be found in a desired form. In this case, it may be hoped that such cases correspond to sets of initial data which are reasonably dense in the whole space of the initial data and at least contain cases that are of particular physical significance.

6.3 Construction of the integral evolution equations and their solution

Once the matrix functions $\mathbf{\Psi}_+(u,w)$ and $\mathbf{\Psi}_-(v,w)$ have been found explicitly, we are able to calculate the fragments of their local structures (27), i.e. the monodromy data vectors $\mathbf{k}_\pm(w)$, the dynamical monodromy data vectors $\mathbf{m}_+(v,w)$ and $\mathbf{m}_-(u,w)$ as well as the characteristic initial values $\widetilde{\boldsymbol{\psi}}_{0+}(u,w)$, $\widetilde{\boldsymbol{\psi}}_{0-}(v,w)$ of the functions $\widetilde{\boldsymbol{\psi}}_+(u,w)$, $\widetilde{\boldsymbol{\psi}}_-(v,w)$

and the initial values of their jumps $\boldsymbol{\phi}_{0+}(u, w = \tau_+)$, $\boldsymbol{\phi}_{0-}(v, w = \tau_-)$ on the cuts L_+ and L_- respectively. This is all we need to construct the kernels (39) and the right hand sides of the main integral evolution equations (38) or the kernels and right hand sides of the decoupled form (40) of these equations.

It is then necessary to solve the linear integral equations (38) or (40) with the corresponding kernels and right hand sides. This is the second crucial step in our algorithm because, even in the case in which we are able to find $\mathbf{\Psi}_+(u, w)$ and $\mathbf{\Psi}_+(u, w)$ explicitly, there are no guarantees that the solution of the linear integral equations (38) or (40) can be found in an explicit form. However, if the particular forms of the coefficients of these integral equations (derived for the specified initial data) enables us to find their solution explicitly, then the Ernst potentials and all the components of the metric and the electromagnetic vector potential in the wave interaction region can be expressed in quadratures (or even explicitly) in terms of these solutions.

6.4 Calculation of the field components

As in the regular case [2], all components of the solution of the characteristic initial value problem can be expressed in terms of the matrices $\mathbf{R}_\pm(u, v)$ defined by the asymptotic expansions of the scattering matrices

$$\boldsymbol{\chi}_\pm(u, v, w) = \mathbf{I} + \frac{1}{w}\mathbf{R}_\pm(u, v) + O(\frac{1}{w^2}). \tag{47}$$

The integral representations (33) suggest the following expressions for the matrices $\mathbf{R}_\pm(u, v)$ in terms of solutions of the integral evolution equations (38) or (40) and the dynamical monodromy data vectors $\mathbf{m}_+(v, \tau_+)$ and $\mathbf{m}_-(u, \tau_-)$

$$\begin{aligned} \mathbf{R}_+(u, v) &= -\frac{1}{\pi i}\int\limits_{L_-} \boldsymbol{\phi}_-(u, v, \zeta_-) \otimes \mathbf{m}_-(u, \zeta_-)\, d\zeta_- \\ \mathbf{R}_-(u, v) &= -\frac{1}{\pi i}\int\limits_{L_+} \boldsymbol{\phi}_+(u, v, \zeta_+) \otimes \mathbf{m}_+(v, \zeta_+)\, d\zeta_+ \end{aligned} \tag{48}$$

Using these expansions in (20) and (21) leads to the expressions for the matrices $\mathbf{U}$ and $\mathbf{V}$:

$$\mathbf{U}(u, v) = \mathbf{U}(u, 0) + 2i\partial_\xi \mathbf{R}_+, \qquad \mathbf{V}(u, v) = \mathbf{V}(0, v) + 2i\partial_\eta \mathbf{R}_-, \tag{49}$$

where we have to transform the ξ- and η-derivatives into u- and v-derivatives respectively. In this way we also obtain the alternative forms of the matrix $\mathbf{W}$

$$\mathbf{W}(u, v, w) = \mathbf{W}(u, 0, w) - 4i(\mathbf{\Omega}\mathbf{R}_+ + \mathbf{R}_+^\dagger\mathbf{\Omega}) = \mathbf{W}(0, v, w) - 4i(\mathbf{\Omega}\mathbf{R}_- + \mathbf{R}_-^\dagger\mathbf{\Omega}) \tag{50}$$

where the constant matrix $\mathbf{\Omega}$ was defined in (21). In accordance with (23), the values of the matrices $\mathbf{W}(u, 0, w)$ and $\mathbf{W}(0, v, w)$ in (50) are determined completely by the characteristic initial data for the fields.

The expressions (49) provide us with the following alternative forms of the expressions for the Ernst potentials:

$$\begin{aligned} \mathcal{E}(u, v) &= \mathcal{E}(u, 0) - 2i(\mathbf{e}_1 \cdot \mathbf{R}_+(u, v) \cdot \mathbf{e}_2) = \mathcal{E}(0, v) - 2i(\mathbf{e}_1 \cdot \mathbf{R}_-(u, v) \cdot \mathbf{e}_2) \\ \Phi(u, v) &= \Phi(u, 0) + 2i(\mathbf{e}_1 \cdot \mathbf{R}_+(u, v) \cdot \mathbf{e}_3) = \Phi(0, v) + 2i(\mathbf{e}_1 \cdot \mathbf{R}_-(u, v) \cdot \mathbf{e}_3) \end{aligned} \tag{51}$$

where $\mathbf{e}_1 = \{1,0,0\}$, $\mathbf{e}_2 = \{0,1,0\}$ and $\mathbf{e}_3 = \{0,0,1\}$. It may be helpful here to mention also some additional interesting relations for the matrices $\mathbf{R}_\pm(u,v)$, such as

$$\begin{aligned} \mathcal{E}(u,0) &= -1 - 2i(\mathbf{e}_1 \cdot \mathbf{R}_-(u,0)\cdot \mathbf{e}_2), \qquad & \Phi(u,0) &= 2i(\mathbf{e}_1 \cdot \mathbf{R}_-(u,0)\cdot \mathbf{e}_3),\\ \mathcal{E}(0,v) &= -1 - 2i(\mathbf{e}_1 \cdot \mathbf{R}_+(0,v)\cdot \mathbf{e}_2), \qquad & \Phi(0,v) &= 2i(\mathbf{e}_1 \cdot \mathbf{R}_+(0,v)\cdot \mathbf{e}_3). \end{aligned} \tag{52}$$

For later reference, we give here also the expressions (51) in a more explicit form

$$\begin{aligned} \mathcal{E}(u,v) &= \mathcal{E}_+(u) + \frac{2}{\pi}\int\limits_{L_-} (\mathbf{e}_1 \cdot \boldsymbol{\phi}_-(u,v,\zeta_-))(\mathbf{m}_-(u,\zeta_-)\cdot \mathbf{e}_2) d\zeta_-\\ &= \mathcal{E}_-(v) + \frac{2}{\pi}\int\limits_{L_+} (\mathbf{e}_1 \cdot \boldsymbol{\phi}_+(u,v,\zeta_+))(\mathbf{m}_+(v,\zeta_+)\cdot \mathbf{e}_2) d\zeta_+\\ \Phi(u,v) &= \Phi_+(u) - \frac{2}{\pi}\int\limits_{L_-} (\mathbf{e}_1 \cdot \boldsymbol{\phi}_-(u,v,\zeta_-))(\mathbf{m}_-(u,\zeta_-)\cdot \mathbf{e}_3) d\zeta_-\\ &= \Phi_-(v) - \frac{2}{\pi}\int\limits_{L_+} (\mathbf{e}_1 \cdot \boldsymbol{\phi}_+(u,v,\zeta_+))(\mathbf{m}_+(v,\zeta_+)\cdot \mathbf{e}_3) d\zeta_+ \end{aligned} \tag{53}$$

where we have returned to our basic notations, with $\mathcal{E}_+(u)$, $\Phi_+(u)$ and $\mathcal{E}_-(v)$, $\Phi_-(v)$ denoting the characteristic initial data for the Ernst potentials on the characteristics. A more detailed description of the various steps of this algorithm is given in the following section, where we present various particular examples.

7 Examples of solutions

7.1 Collision of two gravitational impulses

As a first example, let us consider the collision of two impulsive gravitational waves. This case is well known. It is described by the solution of Khan and Penrose [19] if the polarizations of the approaching waves are aligned. Its generalization to the case when the waves possess nonaligned polarizations was found later by Nutku and Halil [20]. We consider these cases here to illustrate that our algorithm works in this situation without introducing any essential difference between the linear (aligned) or nonlinear (nonaligned) cases.

Gravitational impulses before their collision. The gravitational waves under consideration have impulsive components located prior to their collision on two characteristics, and the space-time behind these is again part of a Minkowski space-time. Although the metric functions are continuous on the wavefronts, some of their derivatives in directions which cross the wavefronts are discontinuous and the curvature possesses a distributional character. In this case, the equation (45) for the left wave as well as the equation (46) for the right wave should be solved together with the conditions that all projections (6) of the Weyl tensor should vanish behind the wavefront for each of the approaching waves. The solutions subject to the conditions (44) take the forms

$$\begin{aligned} &\alpha_+(u) = 1 - k_+^2 u^2 & &\alpha_-(v) = 1 - k_-^2 v^2,\\ &\mathcal{E}_+(u) = -1 - 2k_+ e^{-i\delta_+} u - k_+^2 u^2, & &\mathcal{E}_-(v) = -1 - 2k_- e^{-i\delta_-} v - k_-^2 v^2,\\ &\Phi_+(u) = 0, \qquad f_+(u) = 1, & &\Phi_-(v) = 0, \qquad f_-(v) = 1, \end{aligned} \tag{54}$$

where $k_\pm$ and $\delta_\pm$ are arbitrary real constants. The corresponding expressions for $H_\pm(u)$ and $\Omega_\pm(u)$ are

$$
\begin{aligned}
H_+(u) &= 1 + 2uk_+\cos\delta_+ + k_+^2u^2, \qquad & \Omega_+(u) &= \frac{2uk_+\sin\delta_+}{H_+(u)} \\
H_-(v) &= 1 + 2vk_-\cos\delta_- + k_-^2v^2, \qquad & \Omega_-(v) &= -\frac{2vk_-\sin\delta_-}{H_-(v)}
\end{aligned}
\tag{55}
$$

It may be noted that the constants $k_\pm$ play the role of amplitudes for these waves. They determine the scales (distances) in the affine parameters u and v at which the waves can focus initially parallel null geodesic rays which cross them. The parameters $\delta_\pm$ determine the polarization of the waves. It is clear that either δ_+ or δ_- can be transformed to zero by a rotation of the ignorable coordinates y and z. However, it is the difference $\delta_+ - \delta_-(\mathrm{mod}2\pi)$ which has physical significance, determining the relative polarizations of the approaching waves. However, we keep below both the parameters $\delta_\pm$ to maintain a symmetry of the subsequent expressions.

Using (16) and (54), we obtain the functions α and β in the interaction region as

$$
\alpha(u,v) = 1 - k_+^2u^2 - k_-^2v^2, \qquad \beta(u,v) = k_-^2v^2 - k_+^2u^2.
$$

From this we obtain the relation between the null coordinates ξ and η and u and v respectively:

$$
\xi = 1 - 2k_+^2u^2, \qquad \eta = -1 + 2k_-^2v^2. \tag{56}
$$

Calculation of the "in-states" $\mathbf{\Psi}_+(u,w)$ and $\mathbf{\Psi}_-(v,w)$. With the characteristic initial data (54) together with (55), we can calculate the matrix coefficients $\mathbf{U}(\xi,\eta=-1)$ and $\mathbf{V}(\xi=1,\eta)$ of the linear systems (26) on the characteristics $\eta=-1$ (i.e. $v=0$) and $\xi=1$ (i.e. $u=0$) respectively:

$$
\begin{aligned}
\mathbf{U}(u,0) &= -\frac{1}{2ik_+u}\begin{pmatrix} e^{-i\delta_+} + k_+u \\ i(e^{-i\delta_+} - k_+u) \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1, & i, & 0 \end{pmatrix} \\
\mathbf{V}(0,v) &= -\frac{1}{2ik_-v}\begin{pmatrix} e^{-i\delta_-} + k_-v \\ -i(e^{-i\delta_-} - k_-v) \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1, & -i, & 0 \end{pmatrix}
\end{aligned}
\tag{57}
$$

Substituting differentiation with respect to u and v rather than with respect to ξ and η in (26) and using the coefficients (57), these equations can be solved explicitly. The normalized fundamental solutions take the forms:

$$
\begin{aligned}
\mathbf{\Psi}_+ &= \frac{\lambda_+^{-1}}{2(w-1)}\begin{pmatrix} w - 1 - 2k_+e^{-i\delta_+}u \\ -i(w - 1 + 2k_+e^{-i\delta_+}u) \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1, & i, & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathbf{\Psi}_- &= \frac{\lambda_-^{-1}}{2(w+1)}\begin{pmatrix} w + 1 + 2k_-e^{-i\delta_-}v \\ i(w + 1 - 2k_-e^{-i\delta_-}v) \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1, & -i, & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{i}{2} & 0 \\ -\frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
\tag{58}
$$

while the inverse matrices are

$$\mathbf{\Psi}_+^{-1} = \frac{1}{2}\lambda_+ \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & i & 0 \end{pmatrix} + \begin{pmatrix} \dfrac{w-1+2k_+e^{-i\delta_+}u}{2(w-1)} & -\dfrac{i(w-1-2k_+e^{-i\delta_+}u)}{2(w-1)} & 0 \\ \dfrac{i(w-1+2k_+e^{-i\delta_+}u)}{2(w-1)} & \dfrac{w-1-2k_+e^{-i\delta_+}u}{2(w-1)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{\Psi}_-^{-1} = \frac{1}{2}\lambda_- \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & -i & 0 \end{pmatrix} + \begin{pmatrix} \dfrac{w+1-2k_-e^{-i\delta_-}v}{2(w+1)} & \dfrac{i(w+1+2k_-e^{-i\delta_-}v)}{2(w+1)} & 0 \\ -\dfrac{i(w+1-2k_-e^{-i\delta_-}v)}{2(w+1)} & \dfrac{(w+1+2k_-e^{-i\delta_-}v)}{2(w+1)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{59}$$

where the functions $\lambda_+(u,w)$ and $\lambda_-(v,w)$ are determined by the expressions (see also (56))

$$\lambda_+(u,w) = \sqrt{\frac{w-\xi(u)}{w-1}}, \quad \lambda_-(v,w) = \sqrt{\frac{w-\eta(v)}{w+1}} \tag{60}$$

which satisfy the additional conditions $\lambda_+(u, w=\infty) = 1$ and $\lambda_-(v, w=\infty) = 1$.

Construction of the integral evolution equations. For our construction, we now need to determine certain fragments of the structures of these matrix functions on the cuts $L_\pm$ defined in (27). These are the conserved monodromy data vectors

$$\mathbf{k}_+(w) = \begin{pmatrix} 1, & i, \, 0 \end{pmatrix}, \qquad \mathbf{k}_-(w) = \begin{pmatrix} 1, & -i, \, 0 \end{pmatrix}$$

and the initial values for the vector functions whose jumps on the cuts L_+ and L_- respectively are the unknowns in the integral evolution equations:

$$\widetilde{\boldsymbol{\psi}}_{0+}(u,w) = \frac{\lambda_+^{-1}}{2(w-1)} \begin{pmatrix} w-1-2k_+e^{-i\delta_+}u \\ -i(w-1+2k_+e^{-i\delta_+}u) \\ 0 \end{pmatrix}$$

$$\widetilde{\boldsymbol{\psi}}_{0-}(v,w) = \frac{\lambda_-^{-1}}{2(w+1)} \begin{pmatrix} w+1+2k_-e^{-i\delta_-}v \\ i(w+1-2k_-e^{-i\delta_-}v) \\ 0 \end{pmatrix}$$

We now calculate the dynamical monodromy data vectors using the definitions (34), giving

$$\mathbf{m}_+(v,\zeta_+) = \begin{pmatrix} 1 - \dfrac{2k_-e^{-i\delta_-}v}{\zeta_+ + 1}, & i\big(1 + \dfrac{2k_-e^{-i\delta_-}v}{\zeta_+ + 1}\big), & 0 \end{pmatrix}$$

$$\mathbf{m}_-(u,\zeta_-) = \begin{pmatrix} 1 + \dfrac{2k_+e^{-i\delta_+}u}{\zeta_- - 1}, & -i\big(1 - \dfrac{2k_+e^{-i\delta_+}u}{\zeta_- - 1}\big), & 0 \end{pmatrix}$$

and the kernels $S_\pm$ defined in (39), for which we obtain the expressions

$$S_+(u,v,\tau_+,\zeta_-) = -\frac{1}{i\pi}\frac{2k_+e^{-i\delta_+}u[\lambda_+^{-1}]_{\tau_+}}{(\zeta_- - 1)(\tau_+ - 1)}, \qquad S_-(u,v,\tau_-,\zeta_+) = \frac{1}{i\pi}\frac{2k_-e^{-i\delta_-}v[\lambda_-^{-1}]_{\tau_-}}{(\zeta_+ + 1)(\tau_- + 1)}.$$

For this case, the kernels $\mathcal{F}_\pm$ and the right hand sides $\mathbf{f}_\pm$ of the decoupled integral evolution equations (40) defined in (41) take the forms

$$\mathcal{F}_+(u,v,\tau_+,\zeta_+) = -\frac{1}{i\pi}\frac{2k_+k_-e^{i(-\delta_+-\delta_-)}uv}{\widetilde{v}(\tau_+-1)(\zeta_++1)}\,[\lambda_+^{-1}]_{\tau+}$$
$$\mathcal{F}_-(u,v,\tau_-,\zeta_-) = \frac{1}{i\pi}\frac{2k_+k_-e^{i(-\delta_+-\delta_-)}uv}{\widetilde{u}(\tau_-+1)(\zeta_--1)}\,[\lambda_-^{-1}]_{\tau-}$$
$$\mathbf{f}_+(\tau_+) = \tfrac{1}{2}[\lambda_+^{-1}]_{\tau+}\left[\begin{pmatrix}1\\-i\end{pmatrix} - \frac{2k_+e^{-i\delta_+}u}{\widetilde{v}(\tau_+-1)}\begin{pmatrix}1+k_-e^{-i\delta_-}v\\ i(1-k_-e^{-i\delta_-}v)\end{pmatrix}\right]$$
$$\mathbf{f}_-(\tau_-) = \tfrac{1}{2}[\lambda_-^{-1}]_{\tau-}\left[\begin{pmatrix}1\\i\end{pmatrix} - \frac{2k_-e^{-i\delta_-}v}{\widetilde{u}(\tau_-+1)}\begin{pmatrix}-1-k_+e^{-i\delta_+}u\\ i(1-k_+e^{-i\delta_+}u)\end{pmatrix}\right]$$

where $\widetilde{u} = \sqrt{1-k_+^2u^2}$ and $\widetilde{v} = \sqrt{1-k_-^2v^2}$.

Solution of the integral evolution equations. The equations (38) and (40) show unambiguously that the unknown functions $\boldsymbol{\phi}_\pm$ should possess the following structure:

$$\boldsymbol{\phi}_+(\tau_+) = [\lambda_+^{-1}]_{\tau+}\left\{\frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix} + \frac{1}{\tau_+-1}\begin{pmatrix}A_+\\B_+\end{pmatrix}\right\},\quad \boldsymbol{\phi}_-(\tau_-) = [\lambda_-^{-1}]_{\tau-}\left\{\frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix} + \frac{1}{\tau_-+1}\begin{pmatrix}A_-\\B_-\end{pmatrix}\right\},$$

where $A_\pm$ and $B_\pm$ are unknown functions of u and v. Substituting these expressions into (38) and explicitly calculating the integrals using analytical continuations and the elementary theory of residues leads to algebraic equations whose solution is

$$A_+ = -\frac{k_+e^{-i\delta_+}u(\widetilde{u}+k_-e^{-i\delta_-}v)}{\widetilde{u}\widetilde{v}-k_+k_-e^{i(-\delta_+-\delta_-)}uv} \qquad A_- = \frac{k_-e^{-i\delta_-}v(\widetilde{v}+k_+e^{-i\delta_+}u)}{\widetilde{u}\widetilde{v}-k_+k_-e^{i(-\delta_+-\delta_-)}uv}$$
$$B_+ = -\frac{ik_+e^{-i\delta_+}u(\widetilde{u}-k_-e^{-i\delta_-}v)}{\widetilde{u}\widetilde{v}-k_+k_-e^{i(-\delta_+-\delta_-)}uv} \qquad B_- = -\frac{ik_-e^{-i\delta_-}v(\widetilde{v}-k_+e^{-i\delta_+}u)}{\widetilde{u}\widetilde{v}-k_+k_-e^{i(-\delta_+-\delta_-)}uv}$$

Calculation of the Ernst potential. It remains now to substitute this solution into the general expression (53) for the Ernst potential $\mathcal{E}$. Calculating the corresponding integral leads to the expression

$$\mathcal{E}(u,v) = 1-k_+^2u^2-k_-^2v^2-\frac{2(\widetilde{u}+k_-ve^{-i\delta_-})(\widetilde{v}+k_+ue^{-i\delta_+})}{\widetilde{u}\widetilde{v}-k_+k_-e^{i(-\delta_+-\delta_-)}uv}$$

It is not difficult to check directly that this solution satisfies all of the characteristic initial conditions of the problem under consideration and that it coincides with the corresponding expression for the Nutku–Halil solution. The case $\delta_+=\delta_-=0$ can be seen to correspond to the Ernst potential of the Khan–Penrose solution.

7.2 Collision of gravitational impulse with electromagnetic step-like wave of constant amplitude and polarization

As a second example, let us now include a step electromagnetic wave with one of the impulsive gravitational waves as described in the previous example. Specifically, let us consider a step electromagnetic wave to come from the right such that, behind the wave-front, it has constant amplitude and polarization. This is expressed by the condition

$$\phi_0 = \text{const}, \qquad \Psi_0 = 0, \tag{61}$$

where the nonzero value of ϕ_0 determines the constant amplitude and polarization of the electromagnetic wave, and the vanishing of Ψ_0 indicates that there is no associated gravitational wave component behind the wavefront.

It is first necessary to determine the initial metric functions which give rise to the specific components (61). These can be determined using the expressions (6), together with the field equation (46) and the definition of the Ernst potential (8). It is found that a general solution of these equations, which satisfies the junction conditions (44) is given by

$$\begin{aligned}
&\alpha_-(v) = 1 - (1+k_-^2)\sin^2(\ell_- v), \qquad\qquad f_-(v) = 1,\\
&\mathcal{E}_-(v) = -1 - 2k_-\sin(\ell_- v) - 2k_-^2\big(1-\cos(\ell_- v)\big),\\
&\Phi_-(v) = e^{i\gamma_-}\big[\sin(\ell_- v) + k_-\big(1-\cos(\ell_- v)\big)\big],\\
&H_-(v) = \big[\cos(\ell_- v) + k_-\sin(\ell_- v)\big]^2, \qquad \Omega_-(v) = 0,\\
&\widetilde{\Phi}_-(v) = -ie^{i\gamma_-}\left[\sin(\ell_- v) - k_-(1-\cos(\ell_- v))\right].
\end{aligned} \tag{62}$$

In the above expressions, ℓ_- and γ_- are real constants which determine the values of the constant amplitude and phase of the electromagnetic field such that the above scalar is $\phi_0 = -2\ell_- e^{i\gamma_-}$. In addition to these parameters, a nonzero value of the real constant parameter k_- represents an impulsive gravitational wave localized on the wavefront. If we want to consider a purely electromagnetic wave, we simply put $k_- = 0$. However, in this subsection we do not use this simplification, and permit the parameter k_- to be nonzero.

It is now necessary to obtain the geometrically defined coordinates in the interaction region. Using (16) with (54) and (62) as the characteristic initial data, we obtain

$$\alpha(u,v) = 1-(1+k_-^2)\sin^2(\ell_- v) - k_+^2u^2, \qquad \beta(u,v) = (1+k_-^2)\sin^2(\ell_- v) - k_+^2u^2.$$

This leads to the following expressions for ξ and η:

$$\xi = 1-2k_+^2u^2, \qquad \eta = -1+2(1+k_-^2)\sin^2(\ell_- v). \tag{63}$$

Calculation of the "in-states" $\boldsymbol{\Psi}_+(u,w)$ and $\boldsymbol{\Psi}_-(v,w)$. Using the characteristic initial data (54), (55) and (62), we can again calculate directly, first the matrix coefficients $\mathbf{U}(\xi,\eta=-1)$ and $\mathbf{V}(\xi=1,\eta)$, and then the corresponding fundamental solutions $\boldsymbol{\Psi}_+(u,w)$ and $\boldsymbol{\Psi}_-(v,w)$ of the linear systems (26) on the characteristics $\eta=-1$ (i.e. $v=0$) and $\xi=1$ (i.e. $u=0$). However, since the left wave in this case is the same as that of the first example, the expressions for $\boldsymbol{\Psi}_+(u,w)$ and its inverse are given by (58) and (59). The result of similar calculations for $\boldsymbol{\Psi}_-(v,w)$ and its inverse for the right wave is

$$\begin{aligned}
&\boldsymbol{\Psi}_-(v,w) = \widetilde{\boldsymbol{\psi}}_{0-}(v,w)\otimes\mathbf{k}_-(w) + \mathbf{M}_{0-}(v,w), \qquad \mathbf{k}_-(w) = \left(1,\,-i,\,-\frac{i}{2k_-}\right),\\
&\widetilde{\boldsymbol{\psi}}_{0-}(v,w) = \lambda_-^{-1}\left\{\frac{k_-\sin(\ell_- v)}{w+1}\begin{pmatrix}1\\-i\\0\end{pmatrix} - \frac{k_-^2\cos(\ell_- v)}{w-w_{1-}}\begin{pmatrix}1\\i\\-4ik_-\end{pmatrix} + \begin{pmatrix}0\\0\\2ik_-\end{pmatrix}\right\},\\
&\mathbf{M}_{0-}(v,w) = \frac{k_-^2}{w-w_{1-}}\begin{pmatrix}1\\i\\-4ik_-\end{pmatrix}\otimes\left(1,\,-i,\,-\frac{i}{2k_-}\right) + \begin{pmatrix}1&0&0\\0&1&0\\-2ik_-&-2k_-&0\end{pmatrix}
\end{aligned}$$

and for the inverse matrix we obtain

$$\boldsymbol{\Psi}_-^{-1}(v,w)=\mathbf{l}_-(w)\otimes\widetilde{\boldsymbol{\varphi}}_{0-}(v,w)+\mathbf{N}_{0-}(v,w),\qquad \mathbf{l}_-(w)=(w+1)\left(1,\,i,\,-\frac{2i}{k_-}(w-1)\right)$$

$$\widetilde{\boldsymbol{\varphi}}_{0-}(v,w)=-\frac{\lambda_-k_-^2}{(w+1)(w-w_{1-})}\left(1,\,-i,\,-\frac{i}{2k_-}\right)$$

$$\mathbf{N}_{0-}(v,w)=-\frac{k_-\sin(\ell_-v)}{w+1}\begin{pmatrix}1\\-i\\0\end{pmatrix}\otimes\left(1,\,-i,\,-\frac{i}{2k_-}\right)$$
$$+\frac{k_-^2\cos(\ell_-v)}{w-w_{1-}}\begin{pmatrix}1\\i\\-4ik_-\end{pmatrix}\otimes\left(1,\,-i,\,-\frac{i}{2k_-}\right)+\begin{pmatrix}1&0&0\\0&1&0\\-2ik_-&-2k_-&0\end{pmatrix}$$

where $w_{1-}=1+2k_-^2$, and the function $\lambda_-(v,w)$ was defined in (60).

Construction of the integral evolution equations. We now need the conserved monodromy data vectors

$$\mathbf{k}_+(w)=\left(1,\;\; i,\,0\right),\qquad \mathbf{k}_-(w)=\left(1,\,-i,\,-\frac{i}{2k_-}\right)$$

and the jumps $\boldsymbol{\phi}_{0+}$ and $\boldsymbol{\phi}_{0-}$ of the column-vector functions $\widetilde{\boldsymbol{\psi}}_{0+}$ at $\tau_+\in L_+$ and $\widetilde{\boldsymbol{\psi}}_{0-}$ at $\tau_-\in L_-$ respectively:

$$\boldsymbol{\phi}_{0+}(u,\tau_+)=[\lambda_+^{-1}]_{\tau_+}\left\{\frac{1}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix}-\frac{k_+e^{-i\delta_+}u}{\tau_+-1}\begin{pmatrix}1\\i\\0\end{pmatrix}\right\}$$

$$\boldsymbol{\phi}_{0-}(v,\tau_-)=[\lambda_-^{-1}]_{\tau_-}\left\{\frac{k_-\sin(\ell_-v)}{\tau_-+1}\begin{pmatrix}1\\-i\\0\end{pmatrix}-\frac{k_-^2\cos(\ell_-v)}{\tau_--w_{1-}}\begin{pmatrix}1\\i\\-4ik_-\end{pmatrix}+\begin{pmatrix}0\\0\\2ik_-\end{pmatrix}\right\}$$

which are the characteristic initial values for the unknowns in our integral evolution equations. Then, we obtain the dynamical monodromy data vectors using (34) and the above expressions for the matrix functions $\boldsymbol{\Psi}_+(u,w)$ and $\boldsymbol{\Psi}_-(v,w)$ and their inverse:

$$\mathbf{m}_+(v,\zeta_+)=\left(1,\,i,\,0\right)-\frac{2k_-\sin(\ell_-v)}{\zeta_++1}\left(1,\,-i,\,-\frac{i}{2k_-}\right)$$
$$\mathbf{m}_-(u,\zeta_-)=\left(1,\,-i,\,-\frac{i}{2k_-}\right)+\frac{2k_+e^{-i\delta_+}u}{\zeta_--1}\left(1,\,i,\,0\right)$$

Then, for the kernels $S_\pm$ defined in (39), we obtain the expressions

$$S_+(u,\tau_+,\zeta_-)=-\frac{2k_+e^{-i\delta_+}u[\lambda_+^{-1}]_{\tau_+}}{i\pi(\zeta_--1)(\tau_+-1)},\qquad S_-(v,\tau_-,\zeta_+)=\frac{2k_-\sin(\ell_-v)[\lambda_-^{-1}]_{\tau_-}}{i\pi(\zeta_++1)(\tau_-+1)}.$$

We now have to solve the integral equations (38) for the functions $\boldsymbol{\phi}_+$ and $\boldsymbol{\phi}_-$ using the above expressions for the kernels and the characteristic initial values $\boldsymbol{\phi}_{0+}$ and $\boldsymbol{\phi}_{0-}$. The procedure involved can be reduced to the solution of an algebraic system if we observe that the structure of the kernels calculated above and the right hand sides of these integral equations imply the following dependence of the unknown functions on the spectral parameter:

$$\mathbf{\Phi}_+ = [\lambda_+^{-1}]_{\tau_+} \left\{ \mathbf{A}_+ + \frac{\mathbf{B}_+}{\tau_+ - 1}, \right\}, \qquad \mathbf{\Phi}_- = [\lambda_-^{-1}]_{\tau_-} \left\{ \mathbf{A}_- + \frac{\mathbf{B}_-}{\tau_- + 1} + \frac{\mathbf{C}_-}{\tau_- - w_{1-}} \right\},$$

where the vector coefficients $\mathbf{A}_\pm$, $\mathbf{B}_\pm$ and $\mathbf{C}_-$ are independent of $\tau_\pm$, although they can be functions of the coordinates u and v. Substituting these expressions back into the linear integral equations, we obtain that the coefficients $\mathbf{A}_\pm$ and $\mathbf{C}_-$ should be expressed as

$$\mathbf{A}_+ = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \qquad \mathbf{A}_- = \begin{pmatrix} 0 \\ 0 \\ 2ik_- \end{pmatrix}, \qquad \mathbf{C}_- = -k_-^2 \cos(\ell_- v) \begin{pmatrix} 1 \\ i \\ -4ik_- \end{pmatrix}$$

whereas the other coefficients should satisfy algebraic systems whose coefficients are explicitly calculated integrals and whose solutions are

$$\mathbf{B}_+ = -\frac{k_+ e^{-i\delta_+} u}{Z(u,v)} \begin{pmatrix} \widetilde{u}\cos(\ell_- v) + k_- \sin(\ell_- v) \\ i\widetilde{u}\cos(\ell_- v) - ik_- \sin(\ell_- v) \\ 4ik_-\widetilde{u}\big(1 - \cos(\ell_- v)\big) \end{pmatrix}$$

$$\mathbf{B}_- = \frac{k_- \sin(\ell_- v)}{Z(u,v)} \begin{pmatrix} k_+ e^{-i\delta_+} u \cos(\ell_- v) + \widetilde{v} \\ ik_+ e^{-i\delta_+} u \cos(\ell_- v) - i\widetilde{v} \\ 4ik_+ k_- e^{-i\delta_+} u\big(1 - \cos(\ell_- v)\big) \end{pmatrix}$$

where we have used the notations $\widetilde{u} = \sqrt{1 - k_+^2 u^2}$, $\widetilde{v} = \sqrt{1 - (1 + k_+^2)\sin^2(\ell_- v)}$, and $Z(u,v) = \widetilde{u}\widetilde{v} - k_+ k_- e^{-i\delta_+} u \sin(\ell_- v)$. This solution of the integral evolution equations, in accordance with (53), leads to the following expressions for the Ernst potentials:

$$\begin{aligned}
\mathcal{E}(u,v) = &-1 - k_+^2 u^2 - 2k_-^2\big(1 - \cos(\ell_- v)\big) - \frac{2k_- \sin(\ell_- v)}{Z(u,v)}(\widetilde{v} + e^{-i\delta_+} k_+ u) \\
&- \frac{2e^{-i\delta_+} k_+ u \cos(\ell_- v)}{Z(u,v)}(\widetilde{u} + k_- \sin(\ell_- v)) \\
\Phi(u,v) = &\; k_-(1 - \cos(\ell_- v)) + \frac{(\widetilde{v} + e^{-i\delta_+} k_+ u \cos(\ell_- v))\sin(\ell_- v)}{Z(u,v)}
\end{aligned}$$

This is a new family of solutions which, for particular values of its parameters, reduces to various known solutions. Specifically, for pure vacuum limits, if we substitute $k_- \to k_-/\ell_-$ and then take the limit $\ell_- \to 0$, we obtain the Khan–Penrose solution [19] if $\delta_+ = 0$ or the Nutku–Halil solution [20] if $\delta_+ \neq 0$ as in the previous example. The limit $k_- = 0$, $\delta_+ = 0$ leads to a known solution [3] for the collision of a step electromagnetic wave with an impulsive gravitational wave. The case $k_- \neq 0$, $\delta_+ = 0$ corresponds to a generalization of the solution of Hogan, Barrabès and Bressange [21].

7.3 Collision of electromagnetic step-like waves of constant amplitudes and polarizations

As a final example, let us consider the collision of two step electromagnetic waves of constant amplitudes and polarizations which are not accompanied by gravitational impulses.

It can be seen from above that, for this case, we can take

$$\begin{aligned} &\alpha_+(u)=\cos^2(\ell_+u) && \alpha_-(v)=\cos^2(\ell_-v),\\ &\mathcal{E}_+(u)=-1 && \mathcal{E}_-(v)=-1\\ &\Phi_+(u)=e^{i\gamma_+}\sin(\ell_+u) && \Phi_-(v)=e^{i\gamma_-}\sin(\ell_-v) \end{aligned} \tag{64}$$

with $f_+(u)=1$ and $f_-(v)=1$. The metric functions and additional component of the complex electromagnetic potential then take the forms

$$\begin{aligned} &H_+(u)=\cos^2(\ell_+u), && H_-(v)=\cos^2(\ell_-v),\\ &\Omega_+(u)=0, && \Omega_-(v)=0),\\ &\widetilde{\Phi}_+(u)=ie^{i\gamma_+}\sin(\ell_+u) && \widetilde{\Phi}_-(v)=-ie^{i\gamma_-}\sin(\ell_-v) \end{aligned} \tag{65}$$

where the real parameters $\ell_\pm$ and $\gamma_\pm$ determine the initial amplitude and phase respectively of the two approaching electromagnetic waves.

In this case, the geometrically defined coordinates in the interaction region are

$$\begin{aligned} \alpha(u,v)&=1-\sin^2(\ell_+u)-\sin^2(\ell_-v),\\ \beta(u,v)&=\sin^2(\ell_-v)-\sin^2(\ell_+u), \end{aligned}$$

and the corresponding expressions for ξ and η are

$$\xi=1-2\sin^2(\ell_+u),\qquad \eta=-1+2\sin^2(\ell_-v). \tag{66}$$

Calculation of the "in-states" $\mathbf{\Psi}_+(u,w)$ and $\mathbf{\Psi}_-(v,w)$ As in the previous examples, with the characteristic initial data (64), we can calculate, first the matrix coefficients $\mathbf{U}(\xi,\eta=-1)$ and $\mathbf{V}(\xi=1,\eta)$, and then the corresponding fundamental solutions $\mathbf{\Psi}_+(u,w)$ and $\mathbf{\Psi}_-(v,w)$ of the linear systems (26) on the characteristics $\eta=-1$ (i.e. $v=0$) and $\xi=1$ (i.e. $u=0$). The result of these calculations for $\mathbf{\Psi}_+(u,w)$ and its inverse is

$$\mathbf{\Psi}_+(u,w)=\widetilde{\boldsymbol{\psi}}_{0+}(u,w)\otimes\mathbf{k}_+(w)+\mathbf{M}_{0+}(u,w),\qquad \mathbf{k}_+(w)=(0,\,0,\,1)\,,$$

$$\widetilde{\boldsymbol{\psi}}_{0+}(u,w)=\lambda_+^{-1}(u,w)\begin{pmatrix}-\dfrac{ie^{i\gamma_+}\sin(\ell_+u)}{2(w-1)}\\ \dfrac{e^{i\gamma_+}\sin(\ell_+u)}{2(w-1)}\\ 1\end{pmatrix},\qquad \mathbf{M}_{0+}(u,w)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

where $\lambda_+(u,w)$ is defined in (60). The inverse matrices possess the expressions

$$\mathbf{\Psi}_+^{-1}(u,w)=\mathbf{l}_+(w)\otimes\widetilde{\boldsymbol{\varphi}}_{0+}(u,w)+\mathbf{N}_{0+}(u,w),\qquad \mathbf{l}_+(w)=-4(w^2-1)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\widetilde{\boldsymbol{\varphi}}_{0+}(u,w)=-\frac{\lambda_+(u,w)}{4(w^2-1)}\,(0,\,0,\,1)\,,\qquad \mathbf{N}_{0+}(u,w)=\begin{pmatrix}1&0&\dfrac{ie^{i\gamma_+}\sin(\ell_+u)}{2(w-1)}\\ 0&1&-\dfrac{e^{i\gamma_+}\sin(\ell_+u)}{2(w-1)}\\ 0&0&0\end{pmatrix}$$

The equivalent expressions for $\mathbf{\Psi}_-(v,w)$ and its inverse have the same structure except that

$$\widetilde{\boldsymbol{\psi}}_{0-}(v,w)=\lambda_-^{-1}(v,w)\begin{pmatrix}-\dfrac{ie^{i\gamma_-}\sin(\ell_- v)}{2(w+1)}\\ -\dfrac{e^{i\gamma_-}\sin(\ell_- v)}{2(w+1)}\\ 1\end{pmatrix},\qquad \mathbf{N}_{0-}(v,w)=\begin{pmatrix}1&0&\dfrac{ie^{i\gamma_-}\sin(\ell_- v)}{2(w+1)}\\ 0&1&\dfrac{e^{i\gamma_-}\sin(\ell_- v)}{2(w+1)}\\ 0&0&0\end{pmatrix}.$$

Construction of the integral evolution equations. We now need certain fragments of the structures of the matrix functions $\mathbf{\Psi}_+(u,w)$ and $\mathbf{\Psi}_-(v,w)$ on the cuts $L_\pm$. These are the conserved monodromy data vectors

$$\mathbf{k}_+(w)=\begin{pmatrix}0, & 0, & 1\end{pmatrix},\qquad \mathbf{k}_-(w)=(0,\,0,\,1)$$

and the jumps $\boldsymbol{\phi}_{0+}$ and $\boldsymbol{\phi}_{0-}$ of the column-vector functions $\widetilde{\boldsymbol{\psi}}_{0+}$ at $\tau_+\in L_+$ and $\widetilde{\boldsymbol{\psi}}_{0-}$ at $\tau_-\in L_-$ respectively:

$$\boldsymbol{\phi}_{0+}(u,\tau_+)=[\lambda_+^{-1}]_{\tau_+}\begin{pmatrix}-\dfrac{ie^{i\gamma_+}\sin(\ell_+ u)}{2(\tau_+-1)}\\ \dfrac{e^{i\gamma_+}\sin(\ell_+ u)}{2(\tau_+-1)}\\ 1\end{pmatrix},\qquad \boldsymbol{\phi}_{0-}(v,\tau_-)=[\lambda_-^{-1}]_{\tau_-}\begin{pmatrix}-\dfrac{ie^{i\gamma_-}\sin(\ell_- v)}{2(\tau_++1)}\\ -\dfrac{e^{i\gamma_-}\sin(\ell_- v)}{2(\tau_++1)}\\ 1\end{pmatrix},$$

which are the characteristic initial values for the unknowns in our integral evolution equations. After that, we calculate the dynamical monodromy data vectors

$$\mathbf{m}_+(v,\zeta_+)=\lambda_-(\zeta_+)\begin{pmatrix}0,\,0,\,1\end{pmatrix},\qquad \mathbf{m}_-(u,\zeta_-)=\lambda_+(\zeta_-)\begin{pmatrix}0,\,0,\,1\end{pmatrix}.$$

Then, for the kernels $S_\pm$ defined in (39), we obtain the expressions

$$S_+(u,\tau_+,\zeta_-)=\frac{\lambda_+(u,\zeta_-)}{i\pi(\zeta_--\tau_+)},\qquad S_-(v,\tau_-,\zeta_+)=\frac{\lambda_-(v,\zeta_+)}{i\pi(\zeta_+-\tau_-)}$$

With these expressions, the structure of the integral evolution equations (38) implies the following dependence of the unknown functions on the spectral parameter:

$$\boldsymbol{\phi}_+(\tau_+)=[\lambda_+^{-1}]_{\tau_+}\lambda_-^{-1}(\tau_+)\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}+\frac{\mathbf{A}_+}{\tau_+-1}+\frac{\mathbf{B}_+}{\tau_++1}\right\}$$
$$\boldsymbol{\phi}_-(\tau_-)=[\lambda_-^{-1}]_{\tau_-}\lambda_+^{-1}(\tau_-)\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}+\frac{\mathbf{A}_-}{\tau_--1}+\frac{\mathbf{B}_-}{\tau_-+1}\right\}$$

where the vector coefficients $\mathbf{A}_\pm$ and $\mathbf{B}_\pm$ are independent of $\tau_\pm$ but can be functions of u and v. Substituting these expressions back into the linear integral equations, we obtain

$$\mathbf{A}_+=\mathbf{A}_-=-\frac{i}{2}e^{i\gamma_+}\sin(\ell_+ u)\cos(\ell_- v)\begin{pmatrix}1\\ i\\ 0\end{pmatrix},\quad \mathbf{B}_+=\mathbf{B}_-=-\frac{i}{2}e^{i\gamma_-}\cos(\ell_+ u)\sin(\ell_- v)\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}$$

This solution of the integral evolution equations, in accordance with (53) leads to the following expressions for the Ernst potentials:

$$\mathcal{E} = -1, \qquad \Phi = e^{i\gamma_+} \sin(\ell_+ u) \cos(\ell_- v) + e^{i\gamma_-} \cos(\ell_+ u) \sin(\ell_- v).$$

In the particular case in which the electromagnetic waves possess aligned polarizations (i.e. for $\gamma_+ = \gamma_-$), this solution reduces to the Bell–Szekeres solution while, for $\gamma_+ \neq \gamma_-$, it coincides with another solution of Griffiths (see [3]) with the metric functions

$$\alpha(u,v) = 1 - \sin^2(\ell_+ u) - \sin^2(\ell_- v),$$

$$H(u,v) = \cos^2(\ell_+ u) \cos^2(\ell_- v) + \cos^2(\ell_+ u) \cos^2(\ell_- v) - \tfrac{1}{2} \cos(\gamma_+ - \gamma_-) \sin(2\ell_+ u) \sin(2\ell_- v),$$

$$\Omega(u,v) = \frac{\sin(\gamma_+ - \gamma_-) \sin(2\ell_+ u) \sin(2\ell_- v)}{2H(u,v)}.$$

8 Concluding remarks

We have presented in this paper a general scheme for the construction of solutions of the characteristic initial value problem for the Einstein or Einstein–Maxwell field equations for vacuum or electrovacuum space-times respectively with two-dimensional spatial symmetries. Specifically, we have demonstrated the application of this technique in the construction of solutions of the characteristic initial value problem for the collision and subsequent interaction of given plane gravitational or gravitational and electromagnetic waves with distinct wavefronts, which are propagating initially towards each other in a Minkowski background.

The main feature of this approach is that it enables a solution to be constructed starting directly from given characteristic initial data. In the case of plane wave collisions, these data can be determined via the matching conditions from certain functions and parameters that are specified on the two intersecting null characteristics which correspond to the wavefronts of the approaching waves. Then, to construct the corresponding solution in the wave interaction region, it is necessary to work through three essential steps.

- First, we must solve two systems of linear ordinary differential equations with a spectral parameter which determine the characteristic initial values $\mathbf{\Psi}_+(u,w)$ and $\mathbf{\Psi}_-(v,w)$ of the matrix function $\mathbf{\Psi}(u,v,w)$.

- Next, the solutions $\mathbf{\Psi}_+(u,w)$ and $\mathbf{\Psi}_-(v,w)$ should be used in the construction of the kernels and right hand sides of the integral evolution equations derived above, and then we have to solve these integral equations.

- As the last step of our construction, we have to calculate the quadratures which express all components of the solution of the characteristic initial value problem in terms of the solution of the integral evolution equations.

It is clear, that these three steps solve our (effectively two-dimensional) characteristic initial value problem, at least in principle. However, it is also clear that a practical realization of all these steps, for more or less nontrivial initial data, may meet a number of technical difficulties. First of all, a problem which frequently arises is that, for given initial data, the ordinary linear differential equations for $\mathbf{\Psi}_+(u,w)$ and (or) $\mathbf{\Psi}_-(v,w)$ may not possess explicit solutions. The algorithm obviously fails in this case. However, even if we are successful in obtaining an explicit solution of these equations, we may meet a similar difficulty in the solution of the corresponding integral evolution equations.

Despite the above-mentioned difficulties, however, the algorithm presented here can work in practice, as is demonstrated in the examples given above. Of course, as may be noted with a grain of salt, most of our examples correspond to physically important but well known solutions. Thus, it has not been demonstrated in the paper that the suggested algorithm can work effectively as a new explicit solution generating (or more correctly, – solution constructing) technique, although the general solution given in section 7.2 is new. It may be noted however, that our examples were selected using the simplest physically significant structures of the approaching waves. Some further physically reasonable but more complicated examples lead to differential and integral equations which seem not to admit explicit solutions in terms of elementary functions. In these cases further work may be useful. On the other hand, it is not difficult now to find many formally new examples for specially constructed, but more artificial, explicitly "integrable" initial data. However, looking for such examples is not of interest for us because it returns to an approach which is opposite to our strategy of construction of solutions *starting from the initial data*.

It may thus occur that our approach can be useful for a direct construction of some new physically interesting (exact or possibly approximate) examples. However, it must be pointed out that the main purpose of our construction is a systematic description of the steps which are necessary to actually construct the solution from the initial data. In addition, this approach to the solution of the characteristic initial value problems for vacuum and electrovacuum fields in space-times with two commuting isometries suggests some general frameworks for a consideration of various questions concerning the dynamics of these fields. Among these, we can mention for example, the relation between the structure of the initial waves and the asymptotic behaviour of the solution in the wave interaction region near the singularity, the conditions for the approaching waves leading to the creation of an (unstable) horizon instead of a singularity etc. It may also be mentioned that the basic constructions of the approach suggested above, such as the spectral problems which give rise to the characterisation of every solution in terms of its monodromy data and the linear integral evolution equations, can be used for a similar consideration (and possibly for a solution) of the analogous problems for the collision and interaction of non-plane waves or various waves propagating through non-Minkowskian backgrounds. All these questions seem to be very interesting topics for further investigations.

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References

[1] G. A. Alekseev and J. B. Griffiths, Phys. Rev. Lett. **87**, 221101 (2001).

[2] G. A. Alekseev, Teor.Mat.Fiz., **129**, 184 (2001); Theor. Math. Phys. **129**, 1466,(2001).

[3] J. B. Griffiths, *Colliding plane waves in general relativity*, (Oxford University Press, 1991) (Available at `www-staff.lboro.ac.uk/~majbg`)

[4] G. A. Alekseev and J. B. Griffiths, Phys. Rev. Lett. **84**, 5247 (2000).

[5] I. Hauser and F. J. Ernst, J. Math. Phys., **30**, 872, 2322 (1989).

[6] J. B. Griffiths and M. Santano-Roco, Class. Quantum Grav., **19**, 4273 (2002).

[7] V. A. Belinskii and V. E. Zakharov, Sov. Phys. JETP **48**, 985, (1978).

[8] G. A. Alekseev, Abstracts of GR9, pp. 2-3 (1980); Pis'ma JETP 32, 301 (1980).

[9] B. K. Harrison, Phys. Rev. Lett. **41**, 1197 (1978).

[10] G. Neugebauer, J. Phys. A: Math. Gen., **12**, L67, (1979).

[11] B. K. Harrison, J. Math. Phys. **24**, 2178 (1983).

[12] I. Hauser and F. J. Ernst, J. Math. Phys., **31**, 871, (1990); **32**, 198 (1991).

[13] I. Hauser and F. J. Ernst, Gen. Rel. Grav., **33**, 195 (2001).

[14] G. A. Alekseev, Proceedings of the workshop NEEDS'92, p5 (World Scientific, Singapore, 1993).

[15] A.S. Fokas, L.-Y. Sung and D. Tsoubelis, Mathematical Physics, Analysis and Geometry **1**, 313 (1999).

[16] G. A. Alekseev, Sov. Phys. Dokl. **30**, 565 (1985).

[17] G. A. Alekseev, Proc. Steklov Inst. Maths. **3**, 215 (1988).

[18] G. A. Alekseev, Proceedings of the workshop "Nonlinearity, Integrability and all that: Twenty years after NEEDS'79", p12 (World Scientific, Singapore, 2000); gr-qc/9911045.

[19] K. A. Khan and R. Penrose, Nature, **229**, 185, (1971).

[20] Y. Nutku and M. Halil, Phys. Rev. Lett., **39**, 1379, (1977).

[21] P. A. Hogan, C. Barrabès and G. F. Bressange, Lett. Math. Phys. **43**, 263, (1998).