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Vibration of a rectangular plate with a central power-law profiled groove by the Rayleigh–Ritz method

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ABSTRACT

The prediction of natural frequencies of rectangular plates where a profiled indentation is present is made using a Rayleigh–Ritz variational energy method. Panels with holes are often found for access cables and access gaps, and it is shown that the application of damping to the profile leads to a more efficient method of reducing vibration than covering a whole rectangular plate, which is advantageous where weight saving is necessary.

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1. Introduction

A significant amount of research has been undertaken into the study of structural vibrations and methods to reduce the resonant amplitudes, including analyses which determine how to modify the structure to avoid coupling resonant frequencies with excitation sources, incorporating damping materials or the inclusion of special geometries to attenuate energy at specified frequencies [1]. More emphasis is also now placed on the optimisation of the design, in terms of the placement of the natural frequencies [2] and reducing the amount of damping material, in order to decrease the carbon footprint and installation time.

Many structural plate elements are formed from rectangular plates with various shapes machined into them, either for mass reduction or so that access holes are present for cables or control boxes. There is then the problem of determining the natural frequencies and minimising the coupling of different resonances. In order to reduce the vibration amplitude, damping material with a high stiffness and loss factor can be attached to the plate surface. A significant amount of the literature on thick and thin plates is available for regular shapes, see for example the work of Leissa [3,4] or Liew et al. [5].

When part of a structure containing flexural waves tapers to a sharp point, care must be taken in a finite element representation [6] to avoid errors. For example, as the local flexural wavenumber at a position increases, the nodal density must also increase (this rapid increase in mesh density is computationally undesirable) and without the variation in mesh density, either the whole structure must be meshed to an extremely high fidelity (computationally significantly undesirable) or the finite element representation at the local position will be incorrect. In this paper, a relatively simple method is used to obtain the natural frequencies of a plate which has a section where the thickness varies with position using a continuous integration. While few papers consider the variation of thickness with position [7], even fewer consider the inclusion of variable thickness sections into constant thickness plates, see O'Boy et al. for examples [8,9], despite the interesting engineering applications for such plate structures. The aim of this paper is to illustrate how simple designs for plates with indentations may be tested rapidly with good accuracy using existing methods.

1.1. Vibration advantages for plates with profiled indentations.

It has previously been shown that a profiled indentation can be attached to a plate edge which can yield a greater loss factor for flexural vibrations, especially at higher mode numbers (a more efficient damping method) [8]. Although higher attenuation was obtained, the rectangular plate was left with a sharp edge which is relatively weak. By incorporating the tapering thickness into the centre of a structure, where access holes are required for cables/ conduits, this disadvantage can be overcome.
When a flexural wave travels into a plate where the thickness is reduced, the amplitude of vibration increases and the phase speed of the wave decreases. There exist a limited number of mathematical profiles where analytical solutions to the simplified bending plate equation of motion can be found, one such group is a power-law profile, where the thickness varies according to \( h(x) = cx^\gamma \), where \( \gamma \) and \( c \) are positive constants. In beams for example, an analytical solution involving Bessel's functions is available for the case of a linear tapered profile \( \gamma = 1 \) but higher powers require a numerical approximation, while Mironov [11] has shown that the group velocity of a wave moving towards the end of the beam \( x = 0 \) asymptotically decreases to zero, when the power-law is \( \gamma \geq 2 \). It therefore never reaches the end and cannot reflect back from the free edge of the beam. When a truncation to the profile is included, the reflection coefficient again increases, but this can be negated through the application of polymeric damping layers [12,13]. As the incident flexural wave travels into the profile, the decreased thickness of material results in larger amplitude displacement away from the neutral layer and the damping material is extended and compressed at an amplified rate.

The application of thin damping layers to beam and plate structures is a well known damping method [14,15], however, the integration of a specified change in thickness as a means of reducing reflections from free edges is a novel concept. This tapering of the thickness has been reported as a viable means of reducing vibration, where the thickness of a section of a rectangular or cylindrical plate is reduced according to the power-law profile, where \( \gamma \geq 2 \). Experimental measurements and numerical predictions were carried out on a rectangular plate with and without a wedge of quadratic power-law profile added to one side, as shown in Fig. 1. It was shown that when a layer of damping tape was applied to the plain plate, only a reduction of 1–5 dB of cross-point mobility amplitude was found. When the damping tape was applied to the wedge profile, a reduction of 10–15 dB was found [8].

In this paper, a simple Rayleigh–Ritz variational model is documented which determines the vibration characteristics of an asymmetric or symmetric profile into a rectangular plate in any arbitrary position, such as Fig. 2. The first natural frequency is predicted for the case of a double groove machined into the centre of the plate, utilising both linear and quadratic power-laws, of varying sizes, to an accuracy which would be acceptable for use by most design requirements.

The numerical Rayleigh–Ritz variational model is detailed in Section 2, both for a constant thickness plate and the modifications required for the addition of a profiled section. Then, numerical results are shown in Section 2.3 with comparisons to the other results in the literature, as a means of validation of the methods used.

In Section 3, the numerical method is then used to predict the first natural frequency for a range of plate dimensions, where the length of the tapering profiles are varied. An example of the industrial use of such plates with access holes, in terms of the damping of vibration, is provided in Section 4.

Fig. 1. Rectangular plate incorporating a quadratic wedge of power-law profile machined onto the end. Damping tape is applied to the very tip of the wedge.

Fig. 2. Dimensions of the structural plate idealised in the numerical model, with a central quadratic power-law profile with truncation position leaving a gap in the centre of the plate.

1.2. Numerical solutions to plate vibration involving a change in thickness

The use of rectangular plates is widespread, many of which suffer from excessive vibration causing joint fatigue, tactile discomfort and acoustic problems [16] [17, pp. 2-42]. The behaviour of flexural and extensional plate vibrations has been explored by Mansfield [18] using infinite series solutions to determine the natural frequencies where the thickness varies exponentially and an annular plate where the thickness varies as a power of the radius, including a variation in temperature. Additional numerical solutions to variable thickness plates can also be found in Trapezon [19], Lardner [20] or Wang [21].

The most common numerical solutions to the vibration of tapered plates use this method of an infinite power series in the displacement function. Jain and Soni [22] have a thorough analysis of the free vibrations of rectangular plates where the thickness varies according to a parabolic curve. Tomar et al. [23] have also used a similar method to analyse the vibrations of an infinite plate with a linear variation in thickness, with results for the first two modes of vibration determined.

The Rayleigh–Ritz variational method has been applied to rectangular plates by a range of authors and the methods used in this paper build upon this work, adding to the literature, see for example Young [24] and Warburton [25] for rectangular plates or Gupta and Bhardwaj [26] for orthotropic elliptical plates (see also Chakraverty et al. [7] for a generalised method for inhomogeneous orthotropic circular plates). In order to remove a rectangular section of plate, the Ritz method by Laura et al. [27,28] will be employed and is applicable with the simple shape function used in this paper for a variety of rectangular aspect ratios, in different positions on the plate [29]. As the method is based on energy distribution in the plate, it readily lends itself to the analysis of complex plate shapes. It is not the intention to demonstrate the most efficient convergence, rather that complicated shapes can be obtained without recourse to more complicated techniques such as finite element.

2. Numerical model of a plate with a profiled section in the centre

A numerical model is required to determine the natural frequencies of three distinct plates; a plain rectangular plate, a plate with a square shouldered hole and a plate with a double quadratic power-law profile across the width of the plate, the subject of this paper illustrated in Fig. 2.
The overall plate has material properties of Young’s modulus $E$, Pa, density $\rho$, kg/m$^3$, Poisson’s ratio $\nu$, and dimensions in (x, y) of (a, b) m respectively. The flexural displacement from the neutral layer is denoted by $w(x,y)$ m assuming harmonic vibration of frequency $\omega$ rad/s. The bending stiffness of the plate is $D_1 = Eh_1^4/12(1 - \nu^2)$ Nm where $h_1$ m is the constant thickness.

The Rayleigh–Ritz variational method is an energy method, so that a central rectangular section can be removed from this plate, to be replaced with another plate with properties denoted $E_2$, $\rho_2$, $v_2$ and thickness which varies with position $h_2(x,y)$. The properties of plate 1 will always be the same as plate 2 and the solutions gathered are the undamped natural frequencies.

2.1. Rayleigh–Ritz variational method for uniform plate vibration

The Rayleigh–Ritz variational method is documented by Young [24], although a simpler set of admissible functions are utilised, as also discussed by Warburton [30], which provides an approximation of the natural frequencies and locations of nodal points for a given geometry by equating the maximum potential energy with the maximum kinetic energy at resonant frequencies. The maximum potential energy of the (constant thickness) plate flexural vibration is given by

$$U_{\text{Max}} = \frac{D_1}{2} \int \frac{\partial^2 W}{\partial x^2}^2 + \frac{\partial^2 W}{\partial y^2}^2 + 2(1 - v_1) \left( \frac{\partial^2 W}{\partial x \partial y}^2 \right) \, dx \, dy$$

and the maximum kinetic by,

$$T_{\text{Max}} = \frac{1}{2} \rho_1 h_1 \omega^2 \int w^2 \, dx \, dy.$$

The displacement is assumed to be an infinite series of admissible shape functions in the x and y directions. This numerical method constrains the total number of shape functions in the two directions to a maximum of $M, N$ terms respectively, with amplitudes $b_{mn}$.

$$w(x,y,t) = \sum_{m=1}^{M} \sum_{n=1}^{N} X_m(x) Y_n(y) b_{mn} e^{i\omega t}.$$  

When the series solution is substituted into Eqs. (1) and (2), an expression is obtained which is a function of the amplitudes $b_{mn}$. If the difference in the potential and kinetic energy is denoted $J$, then the Rayleigh–Ritz method requires the minimisation of $J$ with respect to these amplitudes, obtained by taking the partial derivatives of $J$ with respect to $b_{mn}$ and setting equal to zero.

$$\frac{\partial J}{\partial b_0} = 0 = \frac{\partial U_{\text{Max}}}{\partial b_0} - \frac{\partial T_{\text{Max}}}{\partial b_0},$$

where the indices i, j represent any of m, n. This latter equation produces an $M \times N$ system of linear homogeneous equations in the unknown amplitudes $b_{mn}$. For a suitable set of orthogonal shape functions, the matrices determining the natural frequencies of vibration may be written as,

$$\sum_{m=1}^{M} \sum_{n=1}^{N} [C_{mn}] - \lambda \delta_{mn} b_{mn} = 0,$$  

where $\lambda = \omega^2$ and for the case of a solid rectangular plate,

$$\delta_{mn} = \{ \rho_1 h_1 ab/4 \} \text{ for } mn = ij, \text{ 0 for } mn \neq ij.$$  

Two simple shape functions are implemented in this paper, $X(x) = \sin(mx/\pi)$, $Y(y) = \sin(n\pi y/b)$, which completely satisfy the simply supported boundary conditions on the outer edges of the plates, but only approximate the edges on the inside of the plate. This is admissible when using the Rayleigh–Ritz method as the approximate boundary condition becomes more accurate as the number of terms in the series increases to infinity (subject to numerical considerations), see Larrondo et al. [31] who also employs a double Fourier series of shape functions to investigate a rectangular plate with a linear variation in thickness. The individual terms in the matrix are given by,

$$C_{mn} = \left\{ \begin{array}{ll} D_1 \int \int \left( \frac{\partial^2 W}{\partial x^2}^2 + \frac{\partial^2 W}{\partial y^2}^2 + 2(1 - v_1) \frac{\partial^2 W}{\partial x \partial y}^2 \right) \, dx \, dy \hspace{1cm} \text{for } mn = ij, \\ D_1 \left[ v_1(E_{mn} F_{mn} + E_{nm} F_{nm}) + 2(1 - v_1)H_{mn} K_{mn} \right] \hspace{1cm} \text{for } mn \neq ij. \end{array} \right.$$  

Eq. (5) takes the form of a secular determinant [24], with a separation of odd and even modes of vibration where these decouple [27,30] due to lines of symmetry. The numerical solution utilised a size for $M$ and $N$ of 30 terms to obtain both the resonant frequency for each mode number and also the mode shapes. The natural frequency for the plate is non-dimensionalised and represented by $\Omega = \alpha^2 \omega / \rho_1 h_1 / D_1$.

2.2. Inclusion of a power-law profiled section in the plate

The numerical model must be able to calculate not just the natural frequencies of a rectangular plate, but the modifications which arise when a linear or quadratic power-law profile is machined in the plate centre, see Fig. 2. For the double profile along the coordinate $x$, it is assumed that it extends from $x_1$ to $x_2$, so that the thickness is given by $h(x) = c(|x - x_2|^\gamma)$ m where the constant $c = (x_2 - x_1)/2$ m locates the profile in the centre of the plate, $\gamma = 1$ for a linear wedge and $\gamma = 2$ for a quadratic profile.

The expressions for the maximum potential and kinetic energy are also valid where the plate thickness varies with the position, provided the varying bending stiffness in Eq. (1) is taken inside the integral and the integration is carried out over the whole plate area. Similarly, the change in thickness in the kinetic energy, Eq. (2) is also required to be calculated over the whole region of the integral, as detailed by Nallim et al. [32] through the application of the Rayleigh–Ritz method in polygonal plates with a linear variation in thickness.

The inclusion of a profiled section in the centre of the plate as shown in Fig. 2 requires the energy terms associated with a hole of dimensions $(x = x_1$ to $x_2$, $y = y_1$ to $y_2)$ to be removed from
Eq. (5), and the energy terms associated with the profiled plate substituted in, these having bending stiffness $D_2$. The modified integrals $E_2, E_3$ are introduced and $H_m$ where the limits now exist between $x_1$ and $x_2$. Similarly in the $y$ axis, the modified integrals are now $F_{y1}, F_{y2}$ and $K_{y}$ with limits from $y_1$ to $y_2$. Then following the procedure by Laura [33], the method to remove a central hole of the plate and replace it with a different constant thickness plate modifies the matrix element, Eq. (7), as follows,

$$C_{mn}^{(m,n)} = \left[ \int_{0}^{a} \int_{0}^{b} \left( \frac{\partial^2 X_m}{\partial x^2} \right)^2 Y_n + X_m^2 \left( \frac{\partial^2 Y_n}{\partial y^2} \right)^2 \, dx \, dy \right]^{1/2}$$

$$- (D_1 - D_2) \left( \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left( \frac{\partial^2 X_m}{\partial x^2} \right)^2 Y_n^2 + X_m^2 \left( \frac{\partial^2 Y_n}{\partial y^2} \right)^2 \, dx \, dy \right)^{1/2}$$

$$+ 2v_1D_1[\varepsilon_{mn}F_{m1} - 2D_1v_1D_1 - 2D_1v_2] \left[ \varepsilon_{mm}F_{m1} \right]$$

$$-2[D_1(1 - v_1) - D_2(1 - v_2)]H_{mn}K_{mn}$$

$$+ 2D_1(1 - v_1)H_{mn}K_{mn} \text{ for } mn = \bar{y},$$

$$C_{m0}^{(m,0)} = \left[ \int_{0}^{a} \int_{0}^{b} \left( \frac{\partial^2 X_m}{\partial x^2} \right)^2 Y_0 + X_m^2 \left( \frac{\partial^2 Y_0}{\partial y^2} \right)^2 \, dx \, dy \right]^{1/2}$$

$$+ D_1v_1[\varepsilon_{mm}F_{m} + \varepsilon_{mn}F_{m}] - [\varepsilon_{mm}F_{m} - 2D_1v_1D_1 - 2D_1v_2] \left[ \varepsilon_{mm}F_{m} + \varepsilon_{mn}F_{m} \right]$$

$$-2(1 - v_1)D_1 - 2(1 - v_2)D_2]H_{mn}K_{mn} + 2(1 - v_1)D_1H_{mn}K_{mn},$$

$$\delta_{mn} = \begin{cases} \rho h_1ab/4 - \left( \rho_1h_1 - \rho_2h_2 \right) \int_{x_1}^{x_2} \int_{y_1}^{y_2} X_m^2 Y_n^2 \, dx \, dy & \text{for } mn = \bar{y}, \\
(-\rho_1h_1 - \rho_2h_2) \int_{x_1}^{x_2} \int_{y_1}^{y_2} X_m^2 Y_n^2 \, dx \, dy & \text{for } mn \not= \bar{y}. \end{cases}$$

An analytical form of this integration (with the bending stiffness term inside the integral) is used at all times for this inner section based on the summation of each individual power of $x$ with the shape function.

Finally, it is impossible to machine a power-law profile with such precision that it can extend to zero thickness, rather, in the manufacturing process, the profile is truncated at a position $x = x_{bf}$ on one side and at a position $x = x_{bf}$ on the other, where the length of the power-law profile can be manipulated by varying this truncation position in terms of the limits of the integration in Eqs. (10) and (11). The truncation is necessary for convergence, as discussed in Krylov [12], while Zhou describes the application of a boundary condition to a beam with a sharp point. It is pointed out that any solution found in this case would be inconsistent as the beam with a sharp point cannot sustain a bending moment or shear force and that the deflection is required to be finite at the end of the beam [34]. In the case of finite elements, the mesh density would still have to be increased significantly closer to the truncation point, even with the truncation. The analytical integration is efficient in this case.

### 2.3. Series convergence

Results are shown for a steel material, a Poisson’s ratio of $\nu = 0.3$, with the convergence of natural frequency for a square plate $(a = b)$ and central hole incorporating a linear and quadratic power-law profile in Fig. 3. The size of the central hole is given as a ratio $a_1/a, b_1/b$ where $a_1 = x_2 - x_1$, $b_1 = y_2 - y_1$. The truncation position of the power-law profile is set at $(x_{bf} - x_1)/a = 0.1a_1/a$.

The first mode is obtained with sufficient accuracy after approximately $M = N = 21$ terms. The following results in this paper are obtained with 30 terms, sufficient to resolve the change in resonant frequency to within one percent. As an example, a square plate with a central hole $a_1/a = 0.2$ and truncation point $(x_{bf} - x_1)/a = 0.1a_1/a$ with a linear wedge has first natural frequency $\Omega_1(M = N = 21) = 19.013$ and $\Omega_1(M = N = 30) = 19.008$, a reduction of 0.03 percent. For the case of a quadratic power-law, the convergence is $\Omega_1(M = N = 28) = 18.954$ and $\Omega_1(M = N = 30) = 18.949$, a reduction of 0.02 percent. Although this simple method always overestimates the resonant frequency through an upper bound, it still can provide an estimate to within acceptable accuracies for most engineering applications. For the ninth frequency, the reduction between 28 and 30 terms is 0.32 percent.

### 2.4. Validation

The variational method has been successfully used to predict the natural frequencies of a plate with a central square or rectangular hole, see for example the work of Laura et al. [27,28]. The method used in this paper does not differ in the mathematical approach, only the likely numerical implementation. The natural frequencies of a rectangular plate with rectangular central hole are provided in Table 1, where it may be seen that using $N = 30$ terms, the natural frequencies in [28] can be obtained (the variation is due to a different number of terms in the variational method).

It remains to provide a validation for the use of the variational Rayleigh–Ritz method for a thickness varying profile. The comparison is made with the finite element results published by Larrodo et al. [31] for a single linear rectangular wedge. In this case, the linear profile extends from $x = x_1 = 0$ to $x = x_2 = a$ (the second profile using dimensions $x_3$ and $x_4$ are not utilised), where the thickness variation $h(x) = \epsilon(x - x)$ is dictated by the constant $\zeta$, the taper constant $\epsilon = \zeta h_1/a$ and $\zeta = (1 + \zeta)a/c$. The natural frequencies predicted by this paper for the range of taper geometries

![Fig. 3. Convergence of $\Omega_1$ with increasing size of the secular determinant $M$, for the case of (solid line) a linear wedge and (dashed line) a quadratic wedge in a square plate with variously sized inner profile, $0 < a_1/a = 0.1$, $a_1/a = 0.2$.](image)

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Table 1

First non-dimensionalised natural frequency $\Omega_1$ for a rectangular plate with rectangular hole. Values in brackets are given by Laura [28].
and rectangular plate sizes are compared to the finite element calculations in Table 2 with excellent accuracy and agreement.

The energy method has been validated in stages against published results from the literature, showing that the method can be used to determine the plate shape for variable thickness sections, providing that the integrals in Eq. (10) can be completed successfully. In order to improve accuracy for the natural frequencies of the higher order modes, alternative shape functions in Eq. (3) and non-dimensionalised coordinates could be incorporated. For example, polynomial co-ordinate functions can yield excellent accuracy for the fundamental frequency in addition to allowing a simple integration method for the case of a non-linear thickness or inhomogeneous medium (note that in this paper, the shape function also allows a highly computationally efficient analytical integration method to be employed). Often the specific choice of shape functions is determined by the boundary conditions or numerical implementation [34].

### 3. Results

The natural frequencies are presented for various configurations of the plate with a profiled inclusion, including both square and rectangular plate sizes and various ratios \( a_1 / a \) to vary the outer size of the profile. The truncation positions are also varied which has the effect of increasing or decreasing the overall size of the inner hole.

A table of results showing the variation of non-dimensionalised natural frequency for a square plate with a variable size inner hole containing a linear profile section is shown in Table 3 where the truncation position of the linear wedge is also altered. Varying the truncation position also varies the size of the inner aperture. The same comparison is produced for a quadratic power-law profile in Table 4. It can be seen that the natural frequency of the plate with a linear wedge is generally higher than with a quadratic power-law profile for small inner profile sizes, as with the quadratic profile the material becomes thinner approximately half way down the wedge length, with the associated reduction in flexural stiffness. As the size of the inner profile is increased, the natural frequency varies, initially decreasing and then increasing. As the truncation positions are increased (increasing the aperture hole in the centre of the plate) the natural frequency decreases as the overall stiffness is decreased.

### 4. Impact of a profiled inclusion on the damped natural frequencies

Damping added to the plate, which is consistent with typical industrial solutions to noise and vibration issues is now shown. It is assumed that the structure represents a panel with access holes for cables or conduits, see Fig. 6 for an example of an automotive panel with holes for weight saving.

The cases will be compared for a plate dimension \((a, b) = (1.0, 0.5)\) m, with central inner profile size \((a_1, b_1) = (0.3, 0.3)\) m with reference to Fig. 2. The double profile is quadratic in the \(x\) direction with truncation points \((x_1, x_2) = (0.485, 0.515)\) m. The plate materials are considered to be rolled mild steel with Young’s modulus, density and Poisson’s ratio given by \(E = 190 \text{ GPa}, \rho = 7850 \text{ kg/m}^3\) and \(\nu = 0.3\) respectively. The inherent damping in the metal is approximated as \(\eta = 0.01\).
The properties of the viscoelastic damping layer reflect a material such as butyl rubber, with thickness $h_d = 1$ mm, Young’s modulus $E_d = 15$ MPa and loss factor $\eta_d = 0.8$, where the loss factor is incorporated into the numerical prediction through making the Young’s modulus complex, $E_d = E_d(1 + i\eta_d)$. By fully attaching the damping layer to a steel sheet, the overall composite loss factor $\eta_{COMP}$ for flexural waves at any local position can be obtained (composite defined as steel and damping layer). Oberst [14] provides an approximation dependent on the extensional stiffness of the two layers as,

$$
\eta_{COMP} = \frac{\eta_d x [12 \beta^2 + \Gamma^2 (1 + x)^2]}{[1 + x][12 \beta^2 + (1 + x)(1 + \alpha \Gamma^2)]}
$$

(12)

where the subscript $p$ denotes plate material and $d$ the damping material. The terms $\Gamma = h_d/h_p$, $\alpha = E_d/E_p$ and $\beta = (h_d + h_p)/2h_p$.

Three test cases are shown as industrial examples. The first is a rectangular plate with a thin rectangular slot at the truncation positions. The only damping is the inherent metal damping. This is termed the “plain plate”. The second case is the plain plate with a layer of damping material of thickness 1 mm applied to the whole surface, termed the “plain damped”. Finally the last case shown is the rectangular plate with a double quadratic profile extending to the truncation positions, which has a layer of damping material on top. This is termed the “profiled plate”.

All three cases are illustrated as cross-sections on the left-hand side of Fig. 7 (note axes not to scale). On the right-hand side are the

Fig. 4. Mode shapes for increasing natural frequencies of a square plate with a square hole filled with a double quadratic power-law profile section. The size of the profile is $a_i/a = 0.2$ where the truncation points are given by $(a_i - x)/a = 0.1a_i/a$. Note dimensions not to scale.

Fig. 5. Variation of natural frequencies with a variation in the truncation position for a square plate. The outer size of the inclusion is $a_i/a = 0.2$, with the power-law of the profile being –x linear and –o quadratic, for (a) $\Omega_1$, (b) $\Omega_2$, and (c) $\Omega_3$. 
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Table 6

First natural frequency $\Omega_1$ for a rectangular plate with profiled inner double linear profile where $b < a$. Variation of the size of the outer profile and truncation length.

Table 7

First natural frequency $\Omega_1$ for a rectangular plate with profiled inner double quadratic power-law profile where $b < a$. Variation of the size of the outer profile and truncation length.

Table 8

First natural frequency $\Omega_1$ for a rectangular plate with profiled inner double quadratic power-law profile where $b > a$. Variation of the size of the outer profile and truncation length.
composite loss factors for every position on the cross section. For the plain plate, only inherent damping $\eta = 0.01$ is present. For the plain plate with damping layer, the composite damping rises to $\eta_{\text{comp}} = 0.0101$ (an increase of only 1%) as the plate properties dominate Eq. (12). This shows the relative futility of covering the whole plate with damping material, as is the usual solution to plate vibration problems.

The composite damping when applied to the profile shows a significant rise close to the truncation point. Any flexural waves which have a sufficiently high wavelength will be able to propagate into the profile and be attenuated. Indeed, such is the lack of impact covering the constant thickness sections, it would make as much sense to just apply damping to the tapering thickness areas.

This composite loss factor, which varies according to position on the structure, leads to an overall modal loss factor for each natural frequency. The damped natural frequency $\omega_r$ for the $r_{th}$ mode is related to the undamped natural frequency $\omega_r$ and the modal loss factor $\eta_r$ through $\omega_r^2 = \omega_r^2(1 + i\eta_r)$ (with associated non-dimensional parameter $\Omega_r$).

The damped natural frequency and the modal loss factors are shown in Table 9 for the three case studies. It can be seen that when the mode shape indicates an incident angle for the flexural wave into the profiled section which is large, the modal loss factor is relatively low (yet still higher than for the constant thickness plate). However, for higher modes, the profiled plate shows a more significant and more effective increase in the modal damping. This is consistent with shorter wavelength flexural waves propagating into the tapering section where the damping layer is then forced to extend and compress significantly.

The mass of the damping material for the damped plain plate and profiled plates are approximately equal, however, the mass of damping material could be further reduced by not covering the constant thickness sections of the profiled plate, as these parts do not generate a significant reduction in amplitude. Along with an appreciable increase in the modal damping, the reduction in damping mass will lead to financial and carbon cost reductions. It is a simple step to generate the frequency response functions for the forced case through a modal summation, to obtain the point

Fig. 6. Illustration of a typical automotive panel (bonnet shown) with holes for weight saving and attachment of the heavy trim liner (bottom of the photo). Automotive bulkheads typically have access holes for cables and heating ducts. Typical solutions for noise and vibration include attachment of viscoelastic damping layers.

Fig. 7. Three industrial cases compared, a plain plate with only inherent damping (top), the same with a layer of damping tape applied (middle) and a plain plate with a damped profiled indentation (bottom). Cross sections through the material are shown on the left and the composite damping at each position on the right.
mobility of each plate, where the larger modal loss factor would lead to a reduction in vibration amplitude.

5. Conclusions

It has been shown that the natural frequencies of vibration of a complex plate structure can be approximately determined using a variational Rayleigh–Ritz method. The plate is a square or rectangular structure of constant thickness incorporating a central double profiled indentation of power-law profile (linear or quadratic). The size of the truncation position of the power-law profile can be moved to alter the size of the inner aperture.

The natural frequencies have been predicted for a range of plate dimensions and truncation sizes using a variational Rayleigh–Ritz method incorporating 30 terms, sufficient to resolve the change in natural frequency with subsequent additional terms to within one percent. The numerical model has been validated in sections against numerical predictions published in the literature with excellent agreement against finite element calculations. It has been shown that a plate with a double linear power-law profile has a higher natural frequency than a similar plate with a quadratic power-law indentation for small profile sizes.

The method is sufficiently adaptable to allow the movement of the aperture to any location on the plate and many other thickness variations can be employed provided the numerical integrals can be completed.

A demonstration of the use of tapered indentations as highly efficient damping mechanisms for flexural waves has been shown.

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