Cohomology for multicontrolled stratified spaces

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Cohomology for Multicontrolled Stratified Spaces

by

Vladimir Lukiyanov

A Doctoral Thesis

Submitted in partial fulfilment
of the requirements for the award of

Doctor of Philosophy

of

Loughborough University

22 March 2016

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Abstract

In this thesis an extension of the classical intersection cohomology of Goresky and MacPherson, which we call multiperverse cohomology, is defined for a certain class of depth 1 controlled stratified spaces, which we call multicontrolled stratified spaces. These spaces are spaces with singularities – this being their controlled structure – with additional multicontrol data. Multiperverse cohomology is constructed using a cochain complex of $\tau$-multiperverse forms, defined for each case $\tau$ of a parameter called a multiperversity. For the spaces that we consider these multiperversities, forming a lattice $\mathcal{M}$, extend the general perversities of intersection cohomology.

Multicontrolled stratified spaces generalise the structure of (the compactifications of) $\mathbb{Q}$-rank 1 locally symmetric spaces. In this setting multiperverse cohomology generalises some of the aspects of the weighted cohomology of Harder, Goresky and MacPherson.

We define two special cases of multicontrolled stratified spaces: the product-type case, and the flat-type case. In these cases we can calculate the multiperverse cohomology directly for cones and cylinders, this yielding the local calculation at a singular stratum of a multicontrolled space. Further, we obtain extensions of the usual Mayer-Vietoris sequences, as well as a partial Künneth Theorem.

Using the concept a dual multiperversity we are able to obtain a version of Poincaré duality for multiperverse cohomology for both the flat-type and the product-type case. For this Poincaré duality there exist self-dual multiperversities in certain cases, such as for non-Witt spaces, where there are no self-dual perversities.

For certain cusps, called double-product cusps, which are naturally compactified to multicontrolled spaces, the multiperverse cohomology of the compactification of the double-product cusp for a certain multiperversity is equal to the $L^2$-cohomology, analytically defined, for certain doubly-warped metrics.
I would like to thank my supervisor E. Hunsicker for her help and advice. I would also like to thank my parents, O. Umnova and A. Lukyanov, and my partner, C. Guitton, for their encouragement.
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Chapter 1

Introduction

The interaction between $L^2$-cohomology, harmonic forms and intersection (co)homology is classical and can be found in many works, spanning from algebraic applications, such as locally symmetric spaces and arithmetic groups, to applications to spaces arising in mathematical physics. The correspondence between $L^2$-cohomology and harmonic forms, arising from Hodge theory, is an interesting source of analytic problems; furthermore, though harmonic forms are not amenable to sheaf theoretic methods, $L^2$-forms are, and $L^2$-cohomology can often be compared to intersection cohomology by passing to local calculations, which in certain cases determine intersection (co)homology.

1 Outline

The aim of this thesis is to describe a construction of a cochain complex, the complex of multiperverse forms, whose cohomology is a generalisation of intersection cohomology for a certain class of spaces which we call multicontrolled spaces. The motivating example for this construction is the computation of the (weighted) $L^2$-cohomology of a double-product cusp, that is of $Y := \mathbb{R}_{>0} \times F \times L'$, where $F$ and $L'$ are compact Riemannian manifolds, and where the metric on $Y$ is $dr^2 + e^{-2a_1 r} ds_F^2 + e^{-2a_2 r} ds_{L'}^2$, with $a_2 < a_1$. Using the natural compactification of $Y$ to a cone $c(F \times L')$, a direct computation using Zucker’s Künneth formula ([32], Corollary 2.36) shows that the (weighted) $L^2$-cohomology is not the intersection cohomology for any perversity on the compactification $c(F \times L')$. However this $L^2$ cohomology is isomorphic to the multiperverse cohomology for a certain multiperversity.
The outline of this thesis is as follows. In Chapter 2 we will cover the background on controlled stratified spaces, in Chapter 3 we will cover the background on liftable forms, in Chapter 4 we will introduce multicontrolled spaces in depth 1; following this are two chapters where we define and study multiperverse cohomology, in Chapter 5 we define multiperverse cohomology, and in Chapter 6 we study certain properties arising from the definition.

2 Background

2.1 Intersection (co)homology

Intersection (co)homology is an invariant associated to a complex defined on a stratified space, and in particular this can be defined for various kinds of pseudomanifolds. Intersection homology was developed in the 1970s by Goresky and MacPherson ([17]), their aim being to define a theory for pseudomanifolds which was an invariant that was stratification independent, and which furthermore preserved a kind of Poincaré duality for the spaces that they considered. For the pseudomanifolds that they considered, unlike for smooth manifolds, the ordinary cohomology and homology theories did not satisfy a Poincaré duality. For the purposes of this thesis we will not need to consider questions relating to stratification independence of pseudomanifolds, so we work directly with stratified spaces and a fixed stratification. The background and definition of stratified spaces will be covered in Chapter 2. A simple and locally typical example is the cone: let $S$ be a compact smooth manifold, then the quotient of $S \times \mathbb{R}_{\geq 0}$, under the equivalence relation which identifies the 0 coordinate to a $pt$, is the cone $cS$. The singular stratum of $cS$ is $pt$. For general stratified spaces the singularities are more complicated but they are required to possess a structure so that locally they are a product of a cone and $\mathbb{R}^n$.

To define intersection (co)homology one needs an extra parameter; this extra parameter is called a perversity. The background for perversities will be covered in Subsection 1.1 in Chapter 5 and also in Subsection 2.3 of this introduction. For the cone $cS$ with the fixed stratum $pt$, a perversity can be thought of as an integer $\overline{p} \in \mathbb{Z}$, though in this subsection, for technical reasons\(^1\), we will assume that $\overline{p} \in [0, \dim(S) - 1]$. An abundance of constructions which have been used by various authors to define intersection homology or cohomology. All such constructions begin with some complex of chains or cochains. Here are some examples; we state how these are defined for $cS$ only:

\(^1\)See comments in [1] (4. Intersection Homology) and [13] for details. Note also that the zero perversity $\overline{0}$ is given by 0 and that the top perversity $\overline{1}$ is given by $\dim(S) - 1$. 
1. **Using finite simplicial chains.** For the cone $cS$, assuming that $cS$ has a PL-structure (originally in [17]; see also [23], pp. 51-53), let $T$ be a triangulation of $cS$. If $\gamma$ is a simplicial $k$-chain $\gamma = \sum_{\sigma} \gamma_{\sigma} \sigma$ for $T$ on $cS$, where $\sigma$ ranges over $k$-simplices in $T$, then we say that $\gamma$ is $\overline{p}$-allowable if its support satisfies:
\[
\dim\text{(supp}(\gamma) \cap \text{pt}) \leq k - (\dim(S) + 1) + \overline{p}.
\]
(1.1)

We let $I^pC^*_k(X)$ be the set of those $k$-simplices which are $\overline{p}$-allowable such that their boundary is also $\overline{p}$-allowable. After taking the filtered colimit over all compatible triangulations $T$, we obtain a chain complex $I^pC_\bullet(cS)$ of $\overline{p}$-perverse simplices. From $I^pC_\bullet(cS)$ we obtain the simplicial intersection homology groups $I^pH_k(cS)$. For $cS$ it is immediate that these groups vanish in degree $k \geq \dim(S) - \overline{p}$, a little more computation yields:
\[
I^pH_k(cS) \cong \begin{cases} 
H_k(S) & \text{if } k < \dim(S) - \overline{p}, \\
0 & \text{otherwise}.
\end{cases}
\]
(1.2)

For details see [23] (Sections 4.4 and 4.7) and [8] (pp. 14-16). This construction is heuristically the simplest but requires a PL-structure. Note that we also need to work with equivalence classes of chains at all times.

2. **Using singular chains.** These chains are also known as compact chains. One can use singular simplices without the requirement for a PL-structure, with a similar condition to the above Equation 1.1, to define the perverse singular chains. See [22] for the original definition, see also [23] (Section 4.3). For $cS$ this chain complex satisfies the same homology calculation as in Equation 1.2 above ([22], Proposition 5).²

3. **Using Borel-Moore chains.** These chains are also known as BM-chains or closed chains. Borel-Moore chains are more delicate to define than singular (compact) chains but form a sheaf (unlike the compact chains) as they are locally finite rather than globally finite, and thus can be restricted to open subsets; the original construction of Borel-Moore chains was motivated by sheaf theory ([6]). In the example of the cone $cS$ this yields a chain complex

²In [22] King considers slightly more general perversities, this leads to the discrepant middle line in the calculation for the cone ($IH^p_{\overline{p}}(CX)$ in the reference), which can be ignored if $\overline{p} \in [0, \dim(S) - 1]$. 
CHAPTER 1. INTRODUCTION

$I\mathcal{P}S^B_M(cS)$ whose homology satisfies:

\[ I\mathcal{P}H^B_M(cS) \cong \begin{cases} 
H_{k-1}(S) & \text{if } k > \dim(S) - \bar{p}, \\
0 & \text{otherwise},
\end{cases} \]

\[ = \begin{cases} 
H_{k-1}(S) & \text{if } k \geq \dim(S) + 1 - \bar{p}, \\
0 & \text{otherwise}.
\end{cases} \]  

(1.3)

For details see [8] (pp. 14-16). Analogues of the BM-chains can also be defined using a PL-structure and were the starting point for the sheaf-theoretic treatment of intersection homology, as in [18] (2.1). All the above constructions, including this one, can be achieved using stratified local coefficient systems ([18], 2.2); in the above we have tacitly assumed that we take values in the coefficient system $\mathbb{Z}$ or $\mathbb{Q}$.

4. Using smooth differential forms. One starts out with smooth forms on the regular part, so for $cS$ we start with forms on $\text{reg}(cS) := cS \setminus \text{pt} = \mathbb{R}_{>0} \times S$, and then require certain conditions. In the example of $cS$ a smooth form $\omega \in \Omega^k(\text{reg}(cS))$ is said to be $\bar{p}$-admissible if for any set $X_1, \ldots, X_{\bar{p}+1}$ of smooth vector fields in $T(\text{reg}(cS))$ we have $\omega[X_1, \ldots, X_{\bar{p}+1}] = 0$, where the square brackets denote contraction. Letting $\mathcal{P}\Omega^k(cS)$ be the set of $k$-forms which are $\bar{p}$-admissible such that exterior derivative is also $\bar{p}$-admissible, one obtains a cochain complex whose cohomology satisfies the following calculation:

\[ \mathcal{P}H^k(cS) \cong \begin{cases} 
H^k(S) & \text{if } k \leq \bar{p}, \\
0 & \text{otherwise},
\end{cases} \]  

(1.4)

\[ = \begin{cases} 
H^k(S) & \text{if } k < \bar{p} + 1, \\
0 & \text{otherwise},
\end{cases} \]  

(1.5)

The above Equation 1.4 is the description of intersection cohomology which we will use throughout this thesis; this is also, visually speaking, the simplest formula amongst the ones that we have stated in this list. If we compare the above Equation 1.5 with Equation 1.2, noting also that $i < \dim(S) - \bar{p}$ is equivalent to $i \leq \dim(S) - \bar{p} - 1$, then letting $\bar{q} = \dim(S) - 1 - \bar{p}$ we obtain

\[ \mathcal{P}H^k(cS) \cong \mathcal{P}H^B_{\dim(S)+1-k}(cS). \]

This is not a coincidence and is obtained in more generality in [26] (Theorem 7.1), where $\bar{q}$ is known as the dual perversity. This construction was known to Goresky and MacPherson, the reference that we use here is [26].
5. **Using regular differential forms.** In [10] the approach is to start out with smooth forms on the regular part and on each singular stratum. Conditions similar to the ones above are applied, but there are also certain glueing conditions. This approach requires fixing certain trivializations called $\Sigma$-charts, but the resulting cohomology is independent of this choice.

6. **Using liftable forms.** In [11] the approach is to start out with smooth forms on the regular part, as in the case two above, but then the requirement is added so that these forms lift to a smooth forms on a certain desingularization of the space, known as the deshirring (or unfolding). Conditions similar to the ones above are applied.

In [18] Goresky and MacPherson develop the sheaf theoretic approach to intersection cohomology using the Deligne sheaf. The Deligne sheaf ([18], 3.1) is constructed using homological algebra, utilising the truncation functor, and producing a complex of constructible sheaves $P$ ([18], 3.1, Lemma). This complex of sheaves is characterised by four axioms ([18], 3.3, Definition.), so that if a constructible complex of sheaves satisfies these four axioms, then it is quasi-isomorphic to $P$ (in other words uniquely determined in the derived category); further, if the constructible complex is fine, then its cohomology groups are naturally isomorphic to the intersection homology groups (Borel-Moore or the equivalent PL-definition; [18], 3.6, Theorem). The sheaf formulation of intersection cohomology provides a framework for studying intersection cohomology using sheaf theory, in particular for applying Verdier duality to obtain Poincaré duality ([18], 5.3).

### 2.2 $L^2$-cohomology

Let $Y$ be a complete Riemannian manifold with metric $g_Y$. At each point $y \in Y$ the metric $g_Y$ defines a scalar product on the $k$-th exterior product $\Lambda^k T^*_y(Y)$, from where a global scalar product is defined:

$$ (\alpha, \beta) = \int_Y (\alpha_y, \beta_y)_y \quad \text{for} \quad \alpha, \beta \in \Omega^*(Y). $$

Using this global scalar product, which may be infinite, the space of smooth $L^2$-forms on $Y$ is \{ $\omega \in \Omega^k(Y)$ and $(\omega, \omega) < \infty$ \}. It is then possible to define the $L^2$-cohomology groups directly ([12], pp.314-316; [32], (1.2)):

$$ H^k_{(2)}(Y, g_Y) := \{ \omega \in \Omega^k(Y) : d\omega = 0 \} \bigg/ \{ \eta \in \Omega^k(Y) : \eta = d\eta', \| \eta' \|_{L^2} < \infty, \| \eta' \|_{L^2} < \infty \}. $$

The $L^2$-cohomology groups are not always finite dimensional, depending on the metric, even for $\mathbb{R}$ ([12], Example 3.1).
The space of smooth $L^2$ forms is not closed with respect to the norm $||\omega||_{L^2} := \sqrt{\langle \omega, \omega \rangle}$, as defined from the $L^2$ scalar product; the closure with respect to this norm is denoted by $L^2\Omega^k(Y, g_Y)$, and this is the domain of the closure of the exterior derivative $d$ also, written conventionally as $d$. When $(Y, g_Y)$ is geodesically complete the closure of $d$ is unique. From this we can define the reduced $L^2$-cohomology groups ([32], (1.4)):

$$H^k_{(2)}(Y, g_Y) := \frac{\{ \omega \in L^2\Omega^k(Y, g_Y) : d\omega = 0 \}}{\text{closure of } dL^2\Omega^{k-1}(Y, g_Y) \text{ in } L^2\Omega^k(Y, g_Y)}.$$  

The reduced $L^2$-cohomology groups and $L^2$-cohomology groups are isomorphic if $dL^2\Omega^{k-1}(Y, g_Y)$ is closed, and in this case they are also isomorphic to $L^2$-Harmonic forms (though some care must be taken in defining the adjoint to $d$ in this case).

If $Y$ is compact, the $L^2$ growth conditions are vacuous, so the spaces of interest for $L^2$-cohomology are non-compact. If $Y$ is a complete non-compact manifold with a compactification $\overline{Y}$, then one of the techniques used in comparing the $L^2$-cohomology of $Y$ to the intersection cohomology of $\overline{Y}$ is to use the sheaf-based axioms of intersection cohomology on $\overline{Y}$ ([18], 3.3, Definition.). These essentially distil to a local calculation at the singularities; to do this calculation one must consider the complex of sheaves of $L^2$-forms on $\overline{Y}$, which is done by a stalk-like construction (by pushing forward the sheaf of $L^2$-forms on $Y$). This sheaf may fail to be a fine sheaf on $\overline{Y}$, depending on the particular compactification chosen, but when it is fine the comparison can be made. Let us now consider two examples of $L^2$-cohomology ([30], 2.1-2.2):

1. **Metric cones.** Let $S$ be a compact manifold of odd dimension. Further let $cS$ be the cone on $S$ and let $\text{reg}(cS) := cS \setminus \text{pt}$ be the interior, which we call the regular stratum on $cS$ (this is just $\mathbb{R}_{>0} \times S$.) We reparametrise this interior and consider the non-compact manifold $Y := (0, 1) \times S$. If the conical metric on $Y$ is $g_Y := dr^2 + r^2 g_S$, where $g_S$ is a Riemannian metric on $S$, then the $L^2$-cohomology satisfies ([30], Theorem 2.3; [23], Proposition 6.2.2):

$$H^k_{(2)}(Y, g_Y) \cong \begin{cases} 0 & \text{if } k \geq \frac{\dim(S)-1}{2}, \\ H^k(S) & \text{otherwise.} \end{cases}$$

Comparing this to Equation 1.3 we see that that $H^k_{(2)}(Y, g_Y)$ can be naturally compared with intersection cohomology on $cS$; the perversity is $\overline{p} = \frac{\dim(S)+1}{2}$, where we recall that $\dim(S)$ is assumed to be odd, so that this is an integer (this also holds without the assumption on $\dim(S)$ but is more delicate). Notice that the “infinity” here is at $r = 0$ (in the other weighted $L^2$ calculation we put it at $r = \infty$ by reparametrising, as appropriate).
2. **Locally symmetric spaces.** Let $D$ be a Hermitian symmetric space and $\Gamma$ an arithmetic subgroup. The Bergman metric on $D$ induces a locally symmetric metric on $Y := \Gamma \backslash D$. If the Baily-Borel Satake compactification of $Y$ is $Y^*$, then the $L^2$-cohomology satisfies ([30], 4.2.3):

$$H^k_{(2)}(Y, g_Y) \cong \overline{\mathcal{H}}_k^{BM} (Y^*) .$$

This is the statement of the Zucker conjecture. In [32] (6.20) Zucker conjectured that for a Hermitian locally symmetric space the $L^2$ cohomology is isomorphic to the intersection cohomology of its Baily-Borel Satake compactification; this conjecture was proven independently by Saper-Stern in 1990 ([29]) and by Looijenga in 1988 ([24]).

For the example of the metric cone we can examine the calculation in more detail. Considering the metric $g_Y = dr^2 + r^2 g_S$, the determinant $|g_Y|$ is given by

$$|g_Y| = |g_S| \cdot \frac{r^2 \cdots \cdot r^2}{\text{dim}(S)\text{-times}} = |g_S| \cdot r^{2 \cdot \text{dim}(S)} ,$$

so that, if $\eta \in \Omega^k(S)$, and if $p_S : Y \to S$ is the natural projection, then we obtain:

$$\|p_S^*(\eta)\|_{L^2}^2 = \int_Y \|p_S^*(\eta)\|^2 d\text{vol}_{g_Y} = \int_{r=0}^{r=1} \int_Y r^{-2k} \|\eta\|^2 \left( \sqrt{r^{2 \cdot \text{dim}(S)}} \cdot d\text{vol}_{g_S} \right) dr = \int_{r=0}^{r=1} \int_Y r^{(\text{dim}(S) - 2k)} \|\eta\|^2 d\text{vol}_{g_S} dr .$$

The $r^{-2k}$ is obtained by considering that the inner product extending $g_Y$ to the exterior algebra is defined as

$$(x_1 \wedge \ldots \wedge x_k, x'_1 \wedge \ldots \wedge x'_k) := \left| \left( x_i, x'_j \right)_{i,j} \right| ,$$

so that, by similar considerations as with the volume form for $g_Y$, we obtain that the $r^2$ is repeated $k$ times:

$$\|\eta\|^2 = r^{2k} \cdot \|p_S^*\eta\|^2 .$$

From this we can obtain the following observation: $\|p_S^*(\eta)\|_{L^2}^2 < \infty$ if and only if $\eta = 0$ or

$$\int_{r=0}^{r=1} r^{(\text{dim}(S) - 2k)} dr < \infty .$$
Thus $||p^*_S(\eta)||^2_{L^2} < \infty$ if and only if $k \leq \frac{\dim(S)+1}{2}$. This provides heuristic evidence for the calculation; the proof is achieved by using the standard Poincaré homotopy operators, which one must verify map $L^2$-forms to $L^2$-forms in an appropriate way.

### 2.3 Perversities

For a fixed space $X$ the standard homology and cohomology groups have one parameter (in addition to the choice of coefficients, which we will consider fixed). This parameter is degree, so that we obtain groups $H_k(X)$ and $H^k(X)$. If the space $X$ is a Riemannian manifold then the metric $g_X$ on $X$ can alter the (weighted) $L^2$-cohomology groups $H^k_{(2)}(X, g_X)$, one would also obtain different groups by taking relative groups, and also by changing the subset relative to which the (co)homology is computed. The notion of a perversity for intersection (co)homology takes this a step further and introduces formally a parameter called *perversity*, so that for each perversity there is a (co)chain complex, as we have seen briefly in Subsection 2.1 for $cS$.

The original perversities of Goresky and MacPherson, as in [17] and [18], were designed with two purposes in mind: stratification independence of the resulting intersection homology groups for pseudomanifolds (which are stratifiable, but do not have a fixed stratification), and the existence of a dual perversity for Poincaré duality. For certain applications where the stratification is fixed, such as studying the relationship between $L^2$-cohomology and intersection (co)homology, the stratification independence aspect is less important, so more general perversities have been used. These permit a certain amount of “extra perversity”, but do not alter the ideas of Goresky and MacPherson in any consequential way, so that most constructions, except the aforementioned stratification independence, are carried forward; a thorough summary can be found in [13]. Formal properties of perversities become more important when general perversities are considered, or when more abstract properties of standard perversities are considered; some formal properties that one can consider involve studying the lattice of perversities, as well as studying the homological algebra associates to perversities ([20]). Other definitions of perversities have also been considered, for example in [14], where certain multiperversities are considered (these do not appear to relate directly the multiperversities that we define).
2.4 Locally symmetric spaces

It was noticed that for naturally constructed metrics on Hermitian locally symmetric spaces the $L^2$-cohomology groups were isomorphic to the intersection homology groups for certain compactifications; an overview and definition can be found in [7], where this is shown by algebraic means. From a geometric point of view we include two examples from [28] (5.1, 5.2):

1. **$\text{SL}_2(\mathbb{Z})\backslash \text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$**. The underlying space $H := \text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$ is the upper half-plane, so is identified with $\mathbb{R}_{>0} \times \mathbb{R}$. The action of $\text{SL}_2(\mathbb{Z})$ wraps up the $\mathbb{R}$ direction into an $S^1$, so topologically we obtain that $\text{SL}_2(\mathbb{Z})\backslash H$ is the cusp $\mathbb{R}_{>0} \times S^1$. The naturally constructed Riemannian metric on this cusp is of the form $dr^2 + e^{-2r} g_{S^1}$. The Borel-Serre compactification is obtained by adding in the boundary, resulting in the manifold with boundary $(0, \infty] \times S^1$. As $r \to \infty$ the metric on the $S^1$ component vanishes, so that the reductive Borel-Serre compactification is $cS^1$, appropriately reparametrised.

2. **Hilbert modular surface**. This example yields the cusp $\mathbb{R}^+ \times Y$ where $Y$ is a flat fibre bundle over $S^1$ with fibre $T^2$. The natural Riemannian metric on this cusp is of the form $dr^2 + g_{S^1} + e^{-r} g_{T^2}$. The Borel-Serre compactification is obtained by adding in the boundary, resulting in the manifold with boundary $(0, \infty] \times Y$. As $r \to \infty$ the metric on the fibre $T^2$ vanishes, so that the reductive Borel-Serre compactification is obtained by adding the singular stratum $S^1$ at infinity, whose link bundle is then $Y$.

Both of these examples have $\mathbb{Q}$-rank 1, where this rank (defined algebraically) refers to the number of directions at infinity; the compactifications of the above spaces have depth 1, the compactifications of higher rank locally symmetric spaces have higher depth. In the above examples the metric is not doubly-warped; for a doubly-warped metric we include an example from [32] (Appendix):

3. **Doubly-warped cusp**. The space which occurs is $\mathbb{R}^+ \times Y$, obtained as a locally symmetric space, where $Y$ is a bundle over $T^2$ with fibre $S^1$, with the natural metric induced from the metric $dr^2 + e^{-2r}(g_{T^2}) + e^{-4r} g_{S^1}$ on the universal cover. Further, in [32], the author considers more general doubly-warped metrics $dr^2 + e^{-2k_1 r}(g_{T^2}) + e^{-2k_2 r} g_{S^1}$, where $k_2 > 2k_1 > 0$.

The above metrics, without the algebraic conditions imposed by the locally symmetric space structure, provide the motivation for studying the $L^2$-cohomology of cusps equipped with such metrics and the intersection cohomology of their natural compactifications; the natural compactification is one on which the sheaf of $L^2$-forms is fine, see for example [19].
2.5 Weighted cohomology

For locally symmetric spaces, through their work to introduce invariants for arithmetic groups, Goresky, Harder, and MacPherson, defined weighted cohomology ([15]). This weighted cohomology is constructed on the reductive Borel-Serre compactification $\hat{X}$ of a locally symmetric space $X$ using special forms satisfying conditions depending on a weight profile on the Borel-Serre compactification $X$, which are then pushed forward as a sheaf onto $\hat{X}$. These special forms are partly geometric and partly algebraic in nature ([15], 13.2) but the weight profiles are algebraic in nature, and resemble perversities. Further in [15] it is shown that the weighted cohomology (with either of the two middle weight profiles) computes the intersection cohomology with middle perversity on the Baily-Borel compactification of a locally symmetric space, which is itself related to $L^2$-cohomology of the locally symmetric space.

For a $\mathbb{Q}$-rank 1 locally symmetric space, whose Borel-Serre and reductive Borel-Serre compactifications are depth 1 stratified spaces, a weight profile ([15], 1.A.) is an integer in $\mathbb{Z} + \frac{1}{2}$; this parameter alters the way a complex of sheaves is truncated, which is done not by degree, but by weight ([15], 1.C).

3 Main properties of multiperverse cohomology

The motivating problem for the construction of multiperverse cohomology in this thesis, as mentioned earlier, was to describe certain naturally arising (weighted) $L^2$-cohomology groups of a double-product cusp by using a topologically defined cochain complex on the natural compactification of this cusp. We compute the $L^2$-cohomology of double-product cusps in Section 5 of Chapter 6 and compare this to the topologically defined multiperverse cohomology in Subsection 5.2 of Chapter 6. Let us briefly consider the main aspects of the construction and the differences between multiperverse cohomology and intersection (co)homology:

1. **Truncation not by degree only.** Recall from Equation 1.5 that for the stratified cone $cS$ the intersection cohomology satisfies:

$$pH^k(cS) \cong \begin{cases} 
H^k(S) & \text{if } k < \overline{\rho} + 1, \\
0 & \text{otherwise}.
\end{cases}$$

In this sense the truncation, here determined by an integer $\overline{\rho}$, occurs only by degree of forms in $S$. In the construction of multiperverse cohomology for the cone $cS$, where $S = L' \times F$ for two compact manifolds $L'$ and $F$, we introduce a decreasing function $\tau : \mathbb{Z} \to \mathbb{Z}$, satisfying $-1 \leq \tau(k) \leq \dim(F)$, which is $-1$ for $k > \dim(S)$, and which is $\dim(F)$ for $k < 0$; we call such a
function a *multiperversity* (Definition 5.2 in Chapter 5). The multiperverse cohomology with multiperversity $\tau$ satisfies (Theorem 5.43 in Chapter 5):

$$MH^k_\tau(cS) \cong \bigoplus_{\begin{subarray}{l} i+j=k \\ i \leq \tau(k) \end{subarray}} (H^i(F) \otimes H^j(L')).$$

In this calculation the truncation is no longer necessarily achieved by degree in $S$. This is then, in some sense, a topological analogue (of sorts) of the weighted cohomology that is defined for locally symmetric spaces. As the vanishing of certain groups underlies the isomorphism between $L^2$-cohomology and intersection cohomology on the Baily-Borel Satake compactification for Hermitian locally symmetric spaces (see [16] for details relating to this isomorphism), then it may be possible to study aspects of this using this multiperverse cohomology.

2. **Existence of many self-dual multiperversities.** For certain multicontrolled spaces we will demonstrate, in Sections 2 and 3 of Chapter 6, that there is a Poincaré duality for multiperverse cohomology, utilising a dual multiperversity. We will also show that there exist many self-dual multiperversities, in certain cases there exists a self-dual multiperversity where there do not exist any self-dual perversities (Subsection 4.2 in Chapter 6).

3. **A more complex lattice.** We will show that multiperversities form a lattice (Proposition 5.10 in Chapter 6) and we will also consider how this lattice is different to the lattice of perversities.

The spaces for which we can study the properties of multiperverse cohomology are the flat-type multicontrolled spaces; these have quite rigid structure. For these spaces, apart from the aforementioned Poincaré duality for multiperverse cohomology (Sections 2 and 3 of Chapter 6), we also demonstrate extensions of the usual Mayer-Vietoris sequences (Lemmas 6.1 and 6.4 in Chapter 6) and prove a partial Künneth formula (Lemma 6.8 in Chapter 6).
4 Higher depth

In Chapter 3 we will recall the complex of liftable forms in depth 1; this follows the presentation in [11]. The complex of liftable can be defined in arbitrary (finite) depth by using a desingulrisation of the controlled stratified space known as the deshirring or unfolding. The multiperverse forms we construct are based on this complex. It thus seems plausible that the definition of multiperverse cohomology can be achieved in higher depth. However, it seems likely that the lattices of multiperversities and the definition of multiperverse cohomology will be more complex for higher depth, especially from a purely combinatorial perspective. It is also expected that the multicontrol data will be more complex. In the depth 1 case we use certain properties which follow from the assumption that the depth is 1 to simplify the definitions that we use (see Remark 5.29 to Definition 5.28 in Chapter 5).
Chapter 2

Controlled stratified spaces

In this chapter we will recall the background on controlled stratified spaces. In Section 1 we will recall the definition of a controlled stratified space and of associated constructions, while in Section 2 we will recall the deshirring of a controlled stratified space.

1 Stratified spaces and control data

In Subsection 1.1 we will recall the general definition of a controlled stratified space and the depth 1 case that we consider, while in Subsection 1.2 we will recall the constructions of the controlled cone and cylinder, recalling also some results on the local structure of a controlled stratified space.

1.1 General definition

In this document we will be concerned with depth 1 controlled stratified spaces. To recall the definition of a depth 1 controlled stratified space, in this subsection we will first recall the general definition of a controlled stratified space, then recalling the definition of the special case that we consider. This material is standard and we generally follow the notation of [11].

Definition 2.1 ([11], A.I.1; [26], Definition 1.1). A topological space $X$ is a controlled stratified space if it satisfies the following eight numbered conditions:

1. $X$ is locally compact, Hausdorff and admits a countable basis for its topology. Recall that, in the presence of the Hausdorff separability condition, local compactness is equivalent to the statement that every point in $X$ has a closed compact neighbourhood.
2. There exists a locally finite partition of $X$ into locally closed subsets, called strata, each of which has the structure of a smooth manifold. Recall that a subset $A \subseteq X$ is said to be locally closed if it is closed in an open subspace of $X$. The set of strata is written as $S(X)$.

If we let $X_i$ denote the union of those elements of $S(X)$ whose dimensions, as smooth manifolds, are less than or equal to $i$, then there exists a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots.$$  \hfill (2.1)

We place the following condition on the filtration in Equation 2.1:

3. There exists an integer $n \in \mathbb{Z}_{\geq 1}$ such that the following conditions (a) to (c) are satisfied:

(a) $X_n = X$.

(b) $X \setminus X_{n-1}$ is a dense open subset of $X$.

Concerning the above condition 3, we will use the following notation, writing

$$\text{reg}(X) := (X \setminus X_{n-1})$$  \hfill (2.2)

to denote the dense open subset of $X$, as specified above, calling this the regular subset of $X$; we also define a subset of singular strata as

$$SS(X) := \left\{ A \in S(X) : \dim(A) \leq n-1 \right\} \subseteq S(X).$$  \hfill (2.3)

The next conditions specify how the strata, that is elements of the set $S(X)$, fit together in $X$:

4. There exists a set $D(X)$ of control data, indexed by the set of strata $S(X)$, of the form

$$D(X) = \left\{ (T_A, \pi_A : T_A \to A, \rho_A : T_A \to \mathbb{R}_{\geq 0}) \right\}_{A \in S(X)}$$

such that, for each stratum $A \in S(X)$, the following conditions (a) to (c) are satisfied:

(a) $T_A \subseteq X$ is an open subset of $X$ containing $A$. This subset is called the tubular neighbourhood of $A$.

(b) $\pi_A$ is a continuous retraction of $T_A$ onto $A$. This map is called the retraction to the stratum for $A$. 

(c) $\rho_A$ is a continuous function which satisfies $A = \rho_A^{-1}[0]$. This function is called the singularity defining function for $A$. The function $\rho_A$ is similar in purpose to the boundary defining function and gives, in essence, a measure of “distance” from the stratum.

5. If two strata $A, B \in \mathcal{S}(X)$ satisfy $A \cap \text{cl}_X(B) \neq \emptyset$, then $A \subset \text{cl}_X(B)$. This condition is called the frontier condition.

6. For all pairs of strata $A, B \in \mathcal{S}(X)$ the following properties (a) and (b) are verified:
   (a) $A \cap \text{cl}_X(B) \neq \emptyset$ if and only if $T_A \cap B \neq \emptyset$.
   (b) $T_A \cap T_B \neq \emptyset$ if and only if one of the following conditions i. to iii. is satisfied:
      i. $A \subset \text{cl}_X(B)$.
      ii. $B \subset \text{cl}_X(A)$.
      iii. $A = B$.

7. If a pair of strata $A, B \in \mathcal{S}(X)$ satisfies $T_A \cap B \neq \emptyset$, so that $T_A \cap T_B \neq \emptyset$, then the continuous map
   $$ (\pi_A, \rho_A) : T_A \cap B \longrightarrow A \times \mathbb{R}_{>0} $$

   is a smooth submersion. Note that both $T_A \cap B$ and $A \times \mathbb{R}_{>0}$ are smooth manifolds, and that the above map is induced from the universal property of a product, mapping an element $b \in T_A \cap B$ to $(\pi_A(b), \rho_A(b)) \in A \times \mathbb{R}_{>0}$.

8. For all strata $A, B, C \in \mathcal{S}(X)$ and elements
   $$ x \in T_A \cap T_B \cap C \quad \text{and} \quad \pi_B(x) \in T_A \cap B, $$

   the following relations (a) and (b) are verified:
   (a) $\pi_A \pi_B(x) = \pi_A(x)$.
   (b) $\rho_A \pi_B = \rho_A(x)$.

The data of a controlled stratified space is a tuple $(X, \mathcal{S}(X), D(X))$.

**Definition 2.2** ([11], A.I.2; [26], Definition 1.7). Let $X$ and $Y$ be a pair of controlled stratified spaces. An isomorphism of controlled stratified spaces is a homeomorphism $f : X \rightarrow Y$ which satisfies the following four numbered conditions:
1. The morphism, $f$, maps each stratum of $X$ onto a stratum of $Y$, that is, for each stratum $A \in S(X)$, the image under $f$ of $A$ satisfies $f[A] \in S(Y)$.

2. The restriction $f|_A$, for each stratum $A \in S(X)$, is a smooth map between the smooth manifolds $A$ and $f[A]$.

3. For each stratum $A \in S(X)$ there exists an open subset $W \subseteq T_A$ such that, for all $x \in W$, the following two conditions (a) and (b) are satisfied:

   (a) $\pi_{f[A]}f(x) = f\pi_A(x)$.

   (b) $\rho_A(x) = \rho_{f[A]}f(x)$.

**Definition 2.3.** Let $X$ be a controlled stratified space. For a pair of strata $A, B \in S(X)$ we write $A \preceq B$ if both $A \subseteq \text{cl}_X(B)$ and $A \neq B$ are verified; in this case it is said that $A$ is incident to $B$.

Let $A \in S(X)$ be a stratum in a controlled stratified space $X$. The depth of $A$ in $X$ is the largest integer $j \in \mathbb{Z}_{\geq 0}$ for which there exists a chain

$$A_0 < A_1 < A_2 < \ldots < A_j = A,$$

with all $A_i \in S(X)$. We write $\text{depth}_X(A)$ to denote the the depth of $A$ in $X$, dropping the subscript $X$ to write $\text{depth}(A)$ when no confusion can arise. We remark that if $A \preceq B$ is satisfied, then $\text{depth}_X(A) < \text{depth}_X(B)$. Using the notion of depth for a stratum, one can then define the global depth of a controlled stratified space $X$ to be

$$\text{depth}(X) := \sup \left\{ \text{depth}_X(A) : A \in S(X) \right\} \in \mathbb{Z}_{\geq 0}.$$  

(2.5)

### 1.2 Controlled cylinder and cone

In this subsection we will recall the construction of the controlled cylinder and the controlled cone.

**Definition 2.4** (Controlled cylinder; [26], Example 1.5). Let $X$ be a controlled stratified space. The product $X \times \mathbb{R}$, called the cylinder on $X$, can be given the structure of a controlled stratified space. The strata of $X \times \mathbb{R}$ are

$$S(X \times \mathbb{R}) := \left\{ (A \times \mathbb{R}) : A \in S(X) \right\},$$

and the set of control data $D(X \times \mathbb{R})$ is produced from $D(X)$ as follows. For a stratum $A \times \mathbb{R} \in S(X \times \mathbb{R})$ the tubular neighbourhood is $T_{A \times \mathbb{R}} := T_A \times \mathbb{R}$, with
the retraction
\[ \pi_{A \times I} : T_A \times \mathbb{R} \rightarrow A \times \mathbb{R} \]
\[ (x, t) \mapsto (\pi_A(x), t). \]

The singularity defining function is \( \rho_{A \times \mathbb{R}}(x, t) = \rho(x) \). The validity of the conditions of Definition 2.1 for \( X \times \mathbb{R} \) is verified directly from their validity in \( X \).

The cylinder on a controlled stratified space \( X \) has the same depth as \( X \). In fact, for each stratum \( A \in \mathcal{S}(X) \), the following expression is verified:
\[ \text{depth}_X(A) = \text{depth}_{X \times \mathbb{R}}(A \times \mathbb{R}). \tag{2.6} \]

**Definition 2.5** (Controlled cone; [26], Example 1.6). Let \( X \) be a controlled stratified space. We define the following topological space:
\[ cX := \frac{X \times \mathbb{R}_{\geq 0}}{\sim_{\text{cone}}} \text{, with } (x, t) \sim_{\text{cone}} (x', t') \text{ if and only if } t = t' = 0. \]

The above-defined space \( cX \) is called the *cone on \( X \)*, constructed by identifying all the points with \( t \) coordinate equal to 0. Writing \([x, t]\) for \([x, t]_{\sim_{\text{cone}}} \), we set \( \text{pt} := [x, 0] \in cX \) to be the unique *cone point*. The cone \( cX \) can be stratified by letting the strata be
\[ \mathcal{S}(cX) := \{ \text{pt} \} \cup \left\{ A \times \mathbb{R}_{>0} : A \in \mathcal{S}(X) \right\}. \]

The control data \( \pi_{A \times \mathbb{R}_{>0}} \) and \( \rho_{A \times \mathbb{R}_{>0}} \), for a stratum \( A \neq \text{pt} \), are defined as in the above Definition 2.4. For the singular stratum \( \text{pt} \) we define \( \pi_{\text{pt}} : \mathcal{S} \rightarrow \text{pt} \) to be the map which maps \([s, t]\) to \([s, 0]\), while the singularity defining function \( \rho_{\text{pt}} \) is defined to be the function which maps \([s, t]\) to \( t \).

We observe that for cone \( cX \), as defined in the above Definition 2.5, and for each stratum \( A \in \mathcal{S}(X) \), the expression \( \text{pt} < (A \times \mathbb{R}_{>0}) \) holds. In fact the relation \(< \) on \( \text{pt} \) is defined by these expressions, from which we deduce, for each stratum \( A \in \mathcal{S}(X) \), the validity of the expression
\[ \text{depth}_{c(X)}(A \times \mathbb{R}_{>0}) = \text{depth}_X(A) + 1, \]
this further yielding the global expression
\[ \text{depth}(cX) = \text{depth}(X) + 1. \]
Thus the cone has strictly greater depth and is consequently always a space with at least one singular stratum, in other words $\mathcal{S}(cX) \neq \emptyset$.

**Lemma 2.6** ([11], A.I.6; [31], p.16). Let $X$ be a controlled stratified space. For each stratum $A \in \mathcal{S}(X)$ there exists a set

$$\left\{ U_i \subseteq A, \phi_i : \pi_A^{-1}[U_i] \to (U_i \times cL_A) \right\}_{i \in I}$$

satisfying the following conditions:

1. $L_A$ is a controlled stratified space.
2. $\{U_i\}_{i \in I}$ is an open covering of $A$.
3. For each $i \in I$, the map $\phi_i$ is an isomorphism of controlled stratified spaces.
4. $\pi_A : T_A \to A$ is a fibre bundle with fibre $cL_A$.
5. The transitions functions for the above covering and $\pi_A$ take their values in the group of stratified automorphisms of $T_A$.

The concept of an isomorphism of controlled stratified spaces was defined in Definition 2.2; a stratified automorphism is simply an automorphism which is a stratified isomorphism.

### 1.3 Depth 1 case

As mentioned before, in this document we will be principally concerned with controlled stratified spaces of depth 1; to simplify the notation involved we will include certain technical assumptions in the definition that we use. Thus, we reformulate the contents of the above Definition 2.1 in the depth 1 case, including therein the additional assumptions that we make.

**Definition 2.7.** Let $X$ be a topological space. We will say that $X$ is a depth 1 controlled stratified space if satisfies the conditions of Definition 2.1, with depth($X$) = 1, and further satisfies the following numbered conditions:

1. The only stratum of depth 1 is the set $\text{reg}(X) = X_n \setminus X_{n-1}$. Thus, for all $A \in \mathcal{S}(X)$, we have $A < \text{reg}(X)$, this defining fully the relation $<$ on $\mathcal{S}(X)$. Note that $\mathcal{S}(X)$ is non-empty and so $\text{reg}(X)$ cannot have dimension 0 or be empty.
2. There is a fixed local structure, as defined Lemma 2.6. There is also a fixed set of identifications between $T_A$ and the cylinder on $\pi_A$, see Lemma 2.11 below.
3. There is a fixed $\beta \in (0, 1)$. For each stratum $A \in S(X)$ we fix $S_A = \rho^{-1}_A[\beta]$ to be a fixed link bundle of $A$, with the restriction $\pi_A|_{S_A} : S_A \to A$ written just $\pi_A$ when no confusion can arise.

Remarks 2.8. We make the following remarks concerning the implications of the above Definition 2.7:

1. From the assumption that $\text{reg}(X)$ is the sole stratum of depth 1, we conclude that $SS(X) = S(X) \setminus \text{reg}(X)$.

2. For a pair of singular strata $A, B \in SS(X)$ it is impossible that $A < B$ or $B < A$, from where we have $T_A \cap T_B = \emptyset$, unless $A = B$.

3. The control data for $\text{reg}(X)$ is defined as follows. The tubular neighbourhood $T_{\text{reg}(X)} = \text{reg}(X)$, the retraction $\pi_{\text{reg}(X)}$ is the identity and $\rho_{\text{reg}(X)}$ is the zero map.

4. The numbered condition 7 in Definition 2.1 reduces to the requirement that, for all $A \in SS(X)$, the map
   \[(\pi_A, \rho_A) : T_A \cap \text{reg}(X) \to A \times \mathbb{R}_{>0}\]
   be a smooth submersion.

5. The numbered condition 8 in Definition 2.1 is not applicable when $X$ satisfies depth$(X) = 1$.

Hereinafter the words depth 1 controlled stratified space will always refer to a space satisfying the conditions of the above Definition 2.7.

Example 2.9. Let $S$ be a smooth manifold. The cone $cS$ is a depth 1 controlled stratified space in accord with Definition 2.7. The sole singular stratum is $\text{pt}$, while the regular stratum $\text{reg}(cS) = cS \setminus \text{pt}$ is diffeomorphic to $S \times \mathbb{R}_{>0}$ or, by reparametrisation of $\mathbb{R}_{>0}$ to $[0, 1]$, to $S \times (0, 1)$.

Definition 2.10 (Mapping cylinder). Let $S$ be a smooth manifold which is the total space of a fibration:

\[
\begin{array}{ccc}
F & \longrightarrow & S \\
\downarrow & & \downarrow \pi \\
A & \end{array}
\]

Assume that both $A$ and $F$ are smooth manifolds, and that $\pi$ is a smooth submersion. Consider the following quotient space:

\[
cyl \pi := \frac{S \times \mathbb{R}_{\geq 0}}{s \sim_t s' \text{ if and only if } t = t' = 0 \text{ and } \pi(s) = \pi(s')}.
\]
This space has a singular stratum $A$ at the coordinate $t = 0$. The strata of $\text{cyl} \pi$ are 
$$S(\text{cyl} \pi) := \left\{ A, \ (\text{cyl} \pi \backslash A) \right\},$$
with $\text{reg} (\text{cyl} \pi) = (\text{cyl} \pi \backslash A)$, and we define $\pi_{\text{reg}(\text{cyl} \pi)}$ to be the identity, while we define $\pi_A : \text{cyl} \pi \rightarrow A$ to be the map which sends $[s, t]$ to $[\pi(s), 0]$, where we write $[s, t]$ for the equivalence class $[s, t]_{\sim_A}$. The map $\rho_{\text{reg}(\text{cyl} \pi)}$ is defined to be identically 0, while $\rho_A$ is defined to be the map $[s, t] \mapsto t$.

**Lemma 2.11** ([11], A.I.6.3). *Let $X$ be a depth 1 controlled stratified space. For each stratum $A \in S(X)$ there exists an isomorphism of stratified spaces

$$C_A : \text{cyl}(\pi_A) \longrightarrow T_A,$$

where the cylinder is on $\pi_A|_{S_A} : S_A \rightarrow A$.*

We note the definition of $C_A$, which can be extracted from the proof of A.I.6.3 in [11], and the properties that this entails; for this we fix an element $[s, t] \in \text{cyl} (\pi_A)$, with $s \in \pi_A^{-1}[U_i]$. If $\phi_{U_i}(s) = (u, [l, \beta]) \in U_i \times cL_A$, where $\phi_{U_i}$ is a stratified isomorphism as in Lemma 2.6, then we define 
$$C_A([s, t]) := \phi_{U_i}^{-1}(u, [l, t]).$$

Note also that every stratified isomorphism $f : X \rightarrow Y$ satisfies the following property in $T_{f[A]}$:

$$fC_A([s, t]_{\sim_A}) = C_{f[A]}([f(s), t]_{\sim_{f[A]}}).$$

## 2 Deshirring

Deshirring, also known as unfolding, is a general process to resolve the singularities of a controlled stratified space. Elementary deshirring is a process which removes the singular strata of lowest depth, resulting in a deshirred space; the full deshirring of a controlled stratified space can be achieved by iteratively applying the elementary deshirring, reducing the depth of the controlled stratified space at each iteration, so that after a finite number of iterations one obtains a smooth manifold. The presentation in this section follows A.II (“Déplissage d’un Espace Stratifié”) of [11], adapted to the depth 1 case of Definition 2.7.

**Definition 2.12** (Deshirring; [11], A.II.1). *Let $X$ be a depth 1 controlled stratified space and recall that $\mathcal{S}(X) \subseteq S(X)$ denotes the set of singular strata, that is the*
strata of depth 0. Consider the space constructed as
\[ D := \left( (X \setminus \bigsqcup_{A \in \mathcal{S}(X)} A) \times \{-1, 1\} \right) \sqcup \left( \bigsqcup_{A \in \mathcal{S}(X)} (S_A \times (-1, 1)) \right), \]

defining an equivalence relation \( \sim_D \) on this space by
\[
(C_A(s, |t|), j) \sim_D (s, t)
\]

where \( |t| = jt \), for all \( t, A \in \mathcal{S}(X) \) and \( s \in S_A \). The space \( D \) modulo \( \sim_D \), equipped with the quotient topology, is then called the deshirring of \( X \) and is written \( \mathcal{D}(X) \).

**Proposition 2.13** ([11], A.II.3). \( \mathcal{D}(X) \) is a smooth manifold.

**Proposition 2.14** ([11], A.II.1). There exists a continuous surjective map
\[
\theta_X : \mathcal{D}(X) \to X.
\]

Furthermore, if \( f : X \to X' \) is an isomorphism of stratified spaces of depth 1, then there exists a diffeomorphism \( \mathcal{D} f : \mathcal{D}(X) \to \mathcal{D}(X') \), fitting into the following commutative diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow{\theta_X} & & \downarrow{\theta_{X'}} \\
\mathcal{D}(X) & \xrightarrow{\mathcal{D} f} & \mathcal{D}(X')
\end{array}
\]

From the proof of the above Proposition 2.13 in [11], it is possible to remark that, for each \( A \in \mathcal{S}(X) \), we have \( (C_A(s, 0), \mathcal{U}) \not\sim_D (s, 0) \) and \( (C_A(s, 0), \mathcal{L}) \not\sim_D (s, 0) \), for \( C_A(s, 0) \notin D \). From the proof of Proposition 2.14 in [11], it is possible to obtain the explicit construction of \( \theta_X \). For this let \( A \in \mathcal{S}(X) \) and recall that \( C_A : S_A \times [0, 1]_{\sim} \to T_A \) is the mapping cylinder for \( \pi_A|_{S_A} : S_A \to A \). The restriction of \( C_A \) to the interior of the subset \([0, 1)_{\prec}\) written
\[
C_A|_{S_A \times [0, 1)_{\prec}} : S_A \times (0, 1) \to \text{reg}(T_A),
\]

is defined without passing to equivalence classes in \( S_A \times [0, 1)_{\prec} \); we also note that this restriction is a diffeomorphism. We define \( \theta_X \) as follows:

\[
\theta_X([y]_{\sim_D}) := \begin{cases} 
C_A(s, |t|) & \text{if } y \sim_D (s, t) \in S_A \times (-1, 1), A \in \mathcal{S}_0(X) \\
x & \text{if } y \sim_D (x, 1) \\
x & \text{if } y \sim_D (x, -1).
\end{cases} \quad (2.7)
\]
Let us show that this is both correctly defined and well-defined. Firstly, note that the three possibilities in the above Equation 2.7 are (i) mutually exclusive and (ii) cover all the requisite possibilities; this follows directly from the definition of \( \sim_D \). Secondly, note that the only well-definition issue that can arise is that, if \((s,t) \sim_D (s',t')\), then we must verify that \( C_A(s,t) = C_A(s',t') \) in \( X \), but this follows directly from the definition of \( C_A \).

**Example 2.15.** Let \( X \) be a depth 1 controlled stratified space with a sole singular stratum \( A \in S\mathcal{S}(X) \) and with \( \text{reg}(X) = \text{reg}(T_A) \). At first consider the simplifying assumption \( \text{reg}(T_A) = S_A \times \mathbb{R}_{>0} \), so that \( C_A \) is the identity on \( \text{reg}(T_A) \). Then following the above Definition 2.12, we obtain:

\[
D = \left( (X \setminus A) \times \{1,1\}\right) \sqcup \left( S_A \times \{1,1\}\right) = \left( \text{reg}X \times \{1,1\}\right) \sqcup \left( S_A \times \{1,1\}\right) = \left( S_A \times \mathbb{R}_{>0} \times \{1,1\}\right) \sqcup \left( S_A \times \{1,1\}\right).
\]

From this we conclude that \( \mathcal{D}(X) = (D/ \sim_D) \) is the smooth manifold \( S_A \times \mathbb{R} \). Dropping the simplifying assumption \( \text{reg}(T_A) = S_A \times \mathbb{R}_{>0} \), we obtain that \( \mathcal{D}(X) \) is glued from two copies of \( C_A^{-1}[S_A, \mathbb{R}_{>1}] \), one \( \mathcal{D}(X)_\psi \) and other \( \mathcal{D}(X)_\varphi \), and a single copy of \( S_A \times (-1,1) \).

Let \( A \in S(X) \) be a stratum and let \( U \subseteq A \) be an open subset. We introduce the following notation:

\[
\pi_A^{-1}[U] \cap S_A := \left( \pi_A^{-1}[U] \cap \rho_A^{-1}[\beta] \right) \subseteq T_A,
\]

where \( \beta \in (0,1) \) is a fixed constant and we have previously defined \( S_A := \rho_A^{-1}[\beta] \).

**Definition 2.16** (Distinguished subset). Let \( A \in S(X) \) be a stratum and let \( U \subseteq A \) be open subset, fixing an \( \epsilon \in (0,1) \). A subset of the form

\[
W_{U,\epsilon} := \left( \pi_A^{-1}[U] \cap \rho_A^{-1}[[0,\epsilon]] \right) \subseteq T_A
\]

is a **distinguished subset of** \( X \), which we write as \( W \) when \( U \) and \( \epsilon \) are not emphasised.
Chapter 3

Liftable forms

In this chapter we will recall the definition and properties of liftable forms, the retraction to the boundary for liftable forms and liftable versions of the Poincaré homotopy operators. In Section 1 we will recall the definition of the complex of liftable forms and recall some of its properties; all of this material is background. In Section 2 we will recall the classical Poincaré homotopy operator for de Rham cohomology, applying this to obtain Poincaré homotopy operators for liftable forms on the cone and the cylinder; most of this material is again background.

1 Complex of liftable forms

In this section we will recall the definition and properties of the complex of liftable forms. This is a complex associated to a controlled stratified space, and is a subcomplex of the complex of forms on the regular part of the controlled stratified space. The material in Subsection 1.1 follows [11] (B.IV), but we restrict ourselves to depth 1 controlled stratified spaces only; the material in Subsection 1.2 follows some of the methods of [10].

1.1 Definition of liftable forms

In this subsection we will define the complex of liftable forms. For an open subset \( V \subseteq X \), we will henceforth let the regular part of \( V \) be the intersection \( V \cap \text{reg}(X) \), written \( \text{reg}(V) \). The subset \( \text{reg}(V) \subseteq V \), if non-empty, is a smooth manifold of the same dimension as \( \text{reg}(X) \).

**Definition 3.1** ([11], IV.1). Let \( V \subseteq X \) be an open subset such that \( \text{reg}(V) \) is non-empty. A smooth form \( \omega \in \Omega^k(\text{reg}(V)) \) on the regular part of \( V \) is said to be liftable, if there exists a smooth form \( \tilde{\omega} \in \Omega^k(\theta_X^{-1}[V]) \), called the lift of \( \omega \), satisfying

\[
(\theta_X|_{\theta_X^{-1}[\text{reg}(V)]})^*(\omega) = (\tilde{\omega})|_{\theta_X^{-1}[\text{reg}(V)]}.
\]  

(3.1)
The set of liftable forms on $V$ is written $\Omega^k(V)$. To simplify notation, we will henceforth write $\theta^*_X$ to denote $(\theta_X|_{\theta^{-1}_X[\text{reg}(V)]})^*$, where appropriate.

**Remark 3.2.** It is salient to note that a liftable form is defined only on the regular part $\text{reg}(V)$ of $V$; a liftable form does not extend to the singular part of $V$. Notwithstanding, if a form $\omega$ is liftable, then its lift $\tilde{\omega}$ is uniquely determined by denseness. Note also, if $V$ is disjoint from all the singular strata of $X$, then the condition of being liftable is vacuous, with $\Omega^*_i(V) = \Omega^*(V)$.

From the observation that the restriction of $\theta^*_X$ to $\Omega^*(\text{reg}(V))$ is a morphism of complexes, further commuting with $\wedge$, we obtain the following result.

**Proposition 3.3 ([11], IV.1).** Let $\omega, \omega' \in \Omega^k_i(V)$, and $\gamma \in \Omega^k_i(V)$, be three liftable forms on an open subset $V \subseteq X$ of a controlled stratified space $X$. The following three statements are verified:

1. $d\omega \in \Omega^{k+1}_i(V)$ with $d(\tilde{\omega}) = \tilde{d}\omega$.

2. $\omega \wedge \gamma \in \Omega^{k+k'}_i(V)$ with $\tilde{\omega} \wedge \tilde{\gamma} = \tilde{\omega} \wedge \tilde{\gamma}$.

3. $\omega + \omega' \in \Omega^k_i(V)$ with $\tilde{\omega} + \tilde{\omega}' = \tilde{\omega} + \tilde{\omega}'$.

**Proof.** The proof of this result uses the commutativity of $\theta^*_X$ with the operations $d, \wedge$ and $+$ on the complex $\Omega^*(\text{reg}(V))$. For part 1, note that $\omega \in \Omega^k(\text{reg}(V))$, so that Equation 3.1 in Definition 3.1 yields

$$\theta^*_X(d\omega) = d\theta^*_X(\omega) = (d\tilde{\omega})|_{\theta^{-1}_X[\text{reg}(V)]}. \tag{3.2}$$

We deduce part 1 from the above Equation 3.2 by the uniqueness of lifts. A similar argument, using alternately the equality

$$\theta^*_X(\omega \wedge \gamma) = \theta^*_X(\omega) \wedge \theta^*_X(\gamma) = (\tilde{\omega} \wedge \tilde{\gamma})|_{\theta^{-1}_X[\text{reg}(V)]},$$

proves part 2. Analogously, from the equality

$$\theta^*_X(\omega + \omega') = \theta^*_X(\omega) + \theta^*_X(\omega') = (\tilde{\omega} + \tilde{\omega}')|_{\theta^{-1}_X[\text{reg}(V)]},$$

we obtain part 3.

As a direct consequence of the above Proposition 3.3, the object $\Omega^*_i(V)$ is a differential graded algebra in the operations induced from the differential graded algebra $\Omega^*(\text{reg}(V))$ of smooth forms on $V$. 

1.2 Pullback of liftable forms on distinguished open subsets

Throughout this subsection let $W := W_{U, \epsilon} \subseteq X$ be a distinguished open subset of a fixed singular stratum $A \in SS(X)$ of a depth 1 controlled stratified space $X$; we remark that

$$\theta_X^{-1}[W_{U, \epsilon}] = (\pi_A^{-1}[U] \cap S_A) \times (-\epsilon, \epsilon). \quad (3.3)$$

Further, as a consequence of the above Equation 3.3, we can obtain the following result; in this result $p_{SA}$ is the projection

$$p_{SA} : (\pi_A^{-1}[U] \cap S_A) \times (-\epsilon, \epsilon) \rightarrow (\pi_A^{-1}[U] \cap S_A).$$

**Proposition 3.4.** The lift $\tilde{\omega} \in \Omega^k(\theta_X^{-1}[W_{U, \epsilon}])$ of a liftable form $\omega \in \Omega^k_1(W_{U, \epsilon})$ is a finite linear combination of forms

$$f(s, t) \wedge p_{SA}^*(\eta) \wedge dt \quad \text{and} \quad f(s, t) \wedge p_{SA}^*(\eta),$$

where the following three numbered conditions are satisfied:

1. $f : (\pi_A^{-1}[U] \cap S_A) \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ is a smooth function and satisfies the following three conditions (a) to (c):

   (a) In the first case, where $f(s, t) \wedge p_{SA}^*(\eta) \wedge dt$ is considered, the identity $f(s, t) = -f(s, -t)$.

   (b) In the second case, where $f(s, t) \wedge p_{SA}^*(\eta)$ is considered, the identity $f(s, t) = f(s, -t)$.

   (c) In all cases, the restriction $f(s, 0)|_{\pi_A^{-1}[U] \cap S_A}$ is a smooth function on $\pi_A^{-1}[U] \cap S_A$.

2. $\eta \in \Omega^k(S_A \cap \pi_A^{-1}[U])$ is a smooth form and satisfies the following two conditions (a) and (b):

   (a) In the first case, where $f(s, t) \wedge p_{SA}^*(\eta) \wedge dt$ is considered, the identity

   $$\eta \in \Omega^{k-1}(S_A \cap \pi_A^{-1}[U]).$$

   (b) In the second case, where $f(s, t) \wedge p_{SA}^*(\eta)$ is considered, the identity

   $$\eta \in \Omega^k(S_A \cap \pi_A^{-1}[U]).$$

3. $dt \in \Omega^1((-\epsilon, \epsilon))$ is the 1-form corresponding to a standard coordinate $t$ on $(-\epsilon, \epsilon)$. 
Proof. It is standard that a form on \((\pi^{-1}_A[U] \cap S_A) \times (-\epsilon, \epsilon)\) is a combination of forms of the two types above, what is to be shown is that their constituent parts satisfy the numbered conditions. These conditions follow directly, in particular 1(a) follows by applying \(\theta_X^*\) of \(dt\) on \((\pi^{-1}_A[U] \cap S_A) \times (-\epsilon, 0)\) and \((\pi^{-1}_A[U] \cap S_A) \times (0, \epsilon)\); this itself uses the identity \(\theta_X(s, |t|) = \theta_X(s, t)\).

For the fixed distinguished open subset \(W := W_{U,\epsilon}\) of a singular stratum \(A \in SS(X)\) of a depth 1 controlled stratified space \(X\), the smooth map

\[
r : (S_A \cap \pi^{-1}_A[U]) \rightarrow (S_A \cap \pi^{-1}_A[U]) \times (-\epsilon, \epsilon),
\]

defined by the correspondence \(s \mapsto (s, 0)\), permits us to introduce the pullback to the boundary at \(U\) for liftable forms on \(W\). This pullback to the boundary is defined by the composition of the correspondence

\[
\omega \mapsto \tilde{\omega} \mapsto r^*(\tilde{\omega}),
\]

leading to an arrow

\[
R_{U,\epsilon} : \Omega^k(W_{U,\epsilon}) \rightarrow \Omega^k(S_A \cap \pi^{-1}_A[U]).
\]

The map above is called the pullback to the boundary or the retraction to the boundary\(^1\). One can then make the following observations concerning the above-defined map \(R_{U,\epsilon}\).

**Lemma 3.5.** The map \(R_{U,\epsilon}\) is well-defined, linear and commutes with \(d\). Hence the map \(R_{U,\epsilon}\) extends to a morphism of complexes

\[
R_{U,\epsilon} : \Omega^\bullet(W_{U,\epsilon}) \rightarrow \Omega^\bullet(S_A \cap \pi^{-1}_A[U]),
\]

from the complex of liftable forms on the distinguished open subset \(W_{U,\epsilon}\) to the complex of smooth forms on the corresponding link bundle \(S_A \cap \pi^{-1}_A[U]\).

**Proof.** Note that the map \(R_{U,\epsilon}\), as defined by the correspondence of Equation 3.4, is well defined by the uniqueness of lifts. Using the above Proposition 3.4, we then compute \(R_{U,\epsilon}\) on the two types of forms therein:

\(^1\)Note that this nomenclature refers to the lift of the form. In [10] a similar operation (pp.299-300) is called reflection in the singular stratum, in this sense each liftable form leaves a “trace”, its reflection, in the fixed link bundle \(S_A\).
1. In the first case, where $\tilde{\omega} = f \wedge p^*_{S\Lambda}(\eta) \wedge dt$ is considered, we have

$$r^*(f \wedge p^*_{S\Lambda}(\eta) \wedge dt) = r^*(f \wedge p^*_{S\Lambda}(\eta)) \wedge r^*(dt)$$

$$= r^*(f \wedge p^*_{S\Lambda}(\eta)) \wedge 0$$

$$= 0.$$

2. In the second case, where $\tilde{\omega} = f \wedge p^*_{S\Lambda}(\eta)$ is considered, we have

$$r^*(f \wedge p^*_{S\Lambda}(\eta)) = r^*(f) \wedge r^*(p^*_{S\Lambda}(\eta))$$

$$= fr \wedge (p_{S\Lambda})^*(\eta)$$

$$= fr \wedge \eta$$

$$= f(s,0) \wedge \eta.$$

It follows directly that $R_{U,\epsilon}$ is linear and commutes with $d$. \hfill \Box

Remark 3.6. From the proof of the above Lemma 3.5, we note the definition of $R_{U,\epsilon}$ on the lifts of the two types of forms in Proposition 3.4:

$$f(s,t) \wedge p^*_{S\Lambda}(\eta) \longrightarrow f(s,0) \wedge \eta.$$  

$$f(s,t) \wedge p^*_{S\Lambda}(\eta) \wedge dt \longrightarrow 0.$$  

This description will be used later.

2 Poincaré lemmas

In this section we will recall the usual Poincaré homotopy operator for de Rham cohomology, then consider the extension of this homotopy operator to liftable forms on the controlled stratified cone and on the controlled stratified cylinder. The material in Subsections 2.2 and 2.3 follows some of the methods of [10]; the material in Subsection 2.1 is standard. Similar results to the ones in Subsections 2.2 and 2.3, for general depth, and concerning intersection cohomology, using also liftable forms, can be found in [11] (C.IV.2).

2.1 Poincaré operator for smooth forms

In this subsection we will recall the usual Poincaré homotopy operator for forms on a smooth manifold. The standard reference for the contents of this subsection is [9] (Section I.4), but some aspects of the presentation below follow an alternate presentation, as in [10], for example.
Let $M$ be a smooth manifold and let $M \times \mathbb{R}$ be the cylinder on $M$. Further, let $Q$ be the pullback to 0, that is the map which is the pullback along $s \mapsto (s, 0)$:

$$Q : \Omega^k(M \times \mathbb{R}) \longrightarrow \Omega^k(M).$$

Consider also the pullback along natural projection $p_M : M \times \mathbb{R} \to M$:

$$p_M^* : \Omega^k(M) \longrightarrow \Omega^k(M \times \mathbb{R}).$$

With this notation in place, we can recall the utility of the usual Poincaré homotopy operator for de Rham cohomology ([9], Proposition 4.1), which is defined by making use of the retraction

$$A : (M \times \mathbb{R}) \times [0, 1] \longrightarrow M \times \mathbb{R}$$

$$(m, t, u) \longmapsto (m, tu),$$

where we note that $A(m, t, 0) = (m, 0)$ and $A(m, t, 1) = (m, t)$. We fix the names of the usual coordinates on $\mathbb{R}$ and $[0, 1]$ to be $t$ and $u$ respectively.

**Definition 3.7** (Poincaré homotopy of forms). For each $k \in \mathbb{Z}_{\geq 1}$ and $\omega \in \Omega^k(M \times \mathbb{R})$ we define the Poincaré homotopy of $\omega$ to be

$$K^k(\omega) := (-1)^{k-1} \cdot \int_{u=0}^{v=1} (A^* \omega)[\partial u] \, du.$$  

(3.6)

In the above Equation 3.6 the notation $(A^* \omega)[\partial u]$ denotes the contraction of $A^* \omega$ with the vector field $\partial u \in T((0, 1))$, the latter corresponding to the 1-form $du \in \Omega^1((0, 1))$.

The above-defined Poincaré homotopy of Equation 3.6 extends to a linear map

$$K^k : \Omega^k(M \times \mathbb{R}) \longrightarrow \Omega^{k-1}(M \times \mathbb{R});$$

this follows by the combined linearity of $A^*$ and the integral. The map $K^k$, where we omit the subscript to write $K$ when no confusion can arise, is called the Poincaré homotopy operator.

**Remark 3.8.** Consider the explicit local description of $K^k$, where we let $(m, t) \in M \times \mathbb{R}$ and $\omega \in \Omega^k(M \times \mathbb{R})$. Locally at $(m, t)$ the above $K^k(\omega)$ corresponds to a map

$$K^k(\omega)_{(m,t)} : \prod_{k-1} T_{(m,t)}(M \times \mathbb{R}) \longrightarrow \mathbb{R},$$
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so that, for \( Z_1, \ldots, Z_{k-1} \in T_{(m,t)}(M \times \mathbb{R}) \), the object \( K^k(\omega)_{(m,t)} \) is given by

\[
(Z_1, \ldots, Z_{k-1}) \mapsto (-1)^{k-1} \int_{u=0}^{u=1} (A^* \omega)_{(m,t,u)}(Z_1, \ldots, Z_{k-1}, \partial u) \, du .
\]

Remark 3.9. Consider a concise way of using \( K_k \). Firstly, recall that forms in \( \Omega^k(M \times \mathbb{R}) \) are a finite linear combination of

\[
f(m, w) \wedge \eta^* \] \( \eta \) \( \in \Omega^* (M) \) \( \) and \( \) \( \eta \)

where \( f : M \times \mathbb{R} \to \mathbb{R} \) is smooth, \( \eta \in \Omega^* (M) \) and \( dw \in \Omega^1 (\mathbb{R}) \) is the 1-form corresponding to a coordinate \( w \) on \( \mathbb{R} \). The pullback along \( A \) of \( f(m, w) \wedge \eta^* \) is

\[
A^* (f(m, w) \wedge \eta^*) = f(m, tu) \wedge \eta^* ,
\]

so that \( A^* \omega [\partial u] = 0 \), and thus

\[
K^k (f(m, w) \wedge \eta^*) = 0 .
\]

Similarly, the pullback along \( A \) of \( f(m, w) \wedge \eta^* \wedge dw \) is

\[
A^* (f(m, w) \wedge \eta^* \wedge dw) = f(m, tu) \wedge \eta^* \wedge (t \wedge du + u \wedge dt) ,
\]

so that \( A^* \omega [\partial u] = t \cdot f(m, tu) \wedge \eta^* \), and thus

\[
K^k (f(m, w) \wedge \eta^* \wedge dw) = \eta^* \wedge \int_{w=0}^{w=t} f(m, w) \, dw .
\]

Lemma 3.10. For each \( k \in \mathbb{Z}_{\geq 0} \) and \( \omega \in \Omega^k (M \times \mathbb{R}) \) the following identity is verified

\[
K^{k+1} (d\omega) - dK^k (\omega) = (-1)^k \cdot (\omega - \eta^* Q \omega) . \tag{3.7}
\]

Proof. As it is sufficient to verify this on the basis of \( \Omega^k (M \times \mathbb{R}) \), we consider the two cases. Further, as it suffices to verify this in local coordinates, we fix coordinates \( m_1, \ldots, m_n \) on a patch of \( M \), letting \( w \) be the coordinate on \( \mathbb{R} \).

1. In the case that \( \omega = f(m, w) \wedge \eta^* \) we compute:

   (a) The exterior derivative of \( \omega \) is

   \[
d\omega = df \wedge \eta^* + f \wedge \eta^*(d\eta) .
\]
In the above, \( df \) is the 1-form \( d_M(f) + f'(m, w) \wedge dw \), with \( f'(m, w) \) denoting \( \frac{\partial f}{\partial w}(m, w) \).

(b) The pullback of \( d\omega \) by \( A \) is

\[
A^*(d\omega) = d(fA) \wedge p^*_M(\eta) + f(m, tu) \wedge p^*_M(d\eta)
\]
\[
= d_M(f) \wedge p^*_M(\eta)
\]
\[
+ f'(m, tu) \wedge (u \wedge dt + t \wedge du) \wedge p^*_M(\eta)
\]
\[
+ f(m, tu) \wedge p^*_M(d\eta).
\]

When we contract with \( \hat{c}u \) we obtain

\[
A^*(d\omega)[\hat{c}u] = t \cdot f'(m, tu) \wedge p^*_M(\eta).
\]

Hereinafter, to simplify the expressions used in this proof, by a slight abuse of notation, we write \( p^*_M(\eta) \) for \( p^*_{p_M}(\eta) \), and \( d_M(f) \) for \( A^*d_M(f) \).

(c) Noting that \( K(\omega) = 0 \), so that \( d(K\omega) = d(0) = 0 \), it suffices to show that

\[
K(d\omega) = (-1)^{k-1} \cdot (\omega - f(m, 0) \wedge p^*_M(\eta)),
\]

to verify the validity of Equation 3.7 in this case. We obtain this by direct computation as

\[
K(d\omega) = (-1)^{k-1} \cdot \sum_{u=0}^{u=t} \left( t \cdot f'(m, tu) \wedge p^*_M(\eta) \right) du
\]
\[
= (-1)^{k-1} \cdot p^*_M(\eta) \wedge \int_{w=0}^{w=t} f'(m, w) \, dw
\]
\[
= (-1)^{k-1} \cdot p^*_M(\eta) \wedge (f(m, t) - f(m, 0))
\]
\[
= (-1)^{k-1} \cdot (\omega - f(m, 0) \wedge p^*_M(\eta)).
\]

In the above, by slight abuse of notation, we have used the change of coordinates \( w = tu \).

2. In the case that \( \omega = f(m, w) \wedge p^*_M(\eta) \wedge dw \) we compute:

(a) The exterior derivative of \( \omega \) is

\[
d\omega = d(p^*_M(\eta) \wedge (f \wedge dw))
\]
\[
= f \wedge p^*_M(d\eta) \wedge dw + (-1)^{(k-1)} \cdot p^*_M(\eta) \wedge df \wedge dw
\]
\[
= f \wedge p^*_M(d\eta) \wedge dw + (-1)^{(k-1)} \cdot p^*_M(\eta) \wedge d_M(f) \wedge dw.
\]

The contents of the final line, labelled as Equation 3.8 above, is justified
by noting that $df$ is the 1-form

$$d_M(f) + f'(m, w) \wedge dw.$$  

(b) The pullback of $d\omega$ by $A$ is

$$A^*(d\omega) = f(m, tu) \wedge p^*_M(d\eta) \wedge d(tu) + (-1)^{(k-1)} \cdot p^*_M(\eta) \wedge d_M(f) \wedge d(tu)$$

$$= f(m, tu) \wedge p^*_M(d\eta) \wedge (t \wedge du + u \wedge dt)$$

$$\quad + (-1)^{(k-1)} \cdot p^*_M(\eta) \wedge d_M(f) \wedge (t \wedge du + u \wedge dt).$$

When we contract with $\hat{v}u$ we obtain

$$A^*(d\omega)[\hat{v}u] = (-1)^k \cdot t \cdot f(m, tu) \wedge p^*_M(d\eta) - t \cdot p^*_M(\eta) \wedge d_M(f).$$

(c) Firstly, we compute the homotopy operator applied to $d\omega$:

$$K(d\omega) = (-1)^k \cdot \int_{u=0}^{w=t} (-1)^k \cdot t \cdot f(m, tu) \wedge p^*_M(d\eta) \, du$$

$$\quad - (-1)^k \cdot \int_{u=0}^{w=t} t \cdot p^*_M(\eta) \wedge d_M(f) \, du$$

$$= \int_{w=0}^{w=t} f(m, w) \wedge p^*_M(d\eta) \, dw$$

$$\quad + (-1)^{k+1} \cdot \int_{w=0}^{w=t} p^*_M(\eta) \wedge d_M(f) \, dw$$

$$= p^*_M(d\eta) \wedge \int_{w=0}^{w=t} f(m, w) \, dw$$

$$\quad + (-1)^{k+1} \cdot p^*_M(\eta) \wedge \int_{w=0}^{w=t} d_M(f) \, dw. \quad (3.9)$$

In the above, by slight abuse of notation, we have used the change of coordinates $w = tu$. 

(d) Secondly we compute $d$ applied to $K(\omega)$ as

\[
dK(\omega) = d\left( \int_{w=0}^{w=t} f(s, w) \wedge p^*_M(\eta) \, dw \right)
\]
\[
= d\left( p^*_M(\eta) \wedge \int_{w=0}^{w=t} f(s, w) \, dw \right)
\]
\[
= p^*_M(d\eta) \wedge \int_{w=0}^{w=t} f(m, w) \, dw
\]
\[+ (-1)^{(k-1)} \cdot p^*_M(\eta) \wedge d \int_{w=0}^{w=t} f(s, w) \wedge dw
\]
\[
= p^*_M(d\eta) \wedge \int_{w=0}^{w=t} f(m, w) \, dw
\]
\[+ (-1)^{(k-1)} \cdot f(m, t) \wedge p^*_M(\eta) \wedge dt
\]
\[+ (-1)^{(k-1)} \cdot p^*_M(\eta) \wedge \int_{w=0}^{w=t} d_M(f) \, dw. \tag{3.10}
\]

The contents of the final line, labelled as Equation 3.10 above, is justified by applying the Leibniz integral rule, noting that the limits of integration are independent of the coordinates $m$.

(e) Computing the expression for $K(d\omega) - dK(\omega)$, using the above Equations 3.9 and 3.10, we obtain $(-1)^{(k-1)} \cdot f(m, t) \wedge p^*_M(\eta) \wedge dt$, which is as required to show the validity of Equation 3.7.

As mentioned earlier, the two cases form a basis of $\Omega^k(M \times \mathbb{R})$. As all the operations involved in the expressions are linear, proving this locally for the two cases proves the result and verifies the validity of Equation 3.7 globally.

The above Lemma 3.10, and in particular the contents of Equation 3.7, proves the Poincaré lemma for de Rham cohomology, where the isomorphism below sends a cohomology class $[\omega] \in H^k(M \times \mathbb{R})$ to $[Q\omega] \in H^k(M)$.

**Corollary 3.11** (Poincaré lemma). For each $k \in \mathbb{Z}_{\geq 0}$ there exists an isomorphism between $H^k(M \times \mathbb{R})$ and $H^k(M)$.

### 2.2 Poincaré operator for liftable forms on the cone

Using the Poincaré homotopy operator defined in the preceding Subsection 2.1 we can obtain Poincaré lemmas for liftable forms on a controlled stratified cone and on a controlled stratified cylinder; in this subsection we will deal with the case of a controlled stratified cone and in the following subsection with the case of a controlled stratified cylinder.

Throughout this subsection we let $S$ be a smooth manifold, with $cS$ the controlled stratified cone on $S$ with the structure of a depth 1 controlled stratified
space, as in Definition 2.5. Further, recall that \( pt \in S(cS) \) is the cone point and the sole singular stratum of \( cS \), defined as the equivalence class \( pt := \{(s, 0)\} \subseteq cS \). Note that, following directly from Definition 2.12, the deshirring of the cone \( cS \) is

\[
\mathcal{D}(cS) = S \times \mathbb{R}.
\]

To work with the deshirring, throughout this subsection we fix \( p_S : S \times \mathbb{R} \to S \) to be the projection onto the first factor in \( \mathcal{D}(cS) \). Recall that \( R_{pt} \) refers to the pullback to \( S \) defined in Lemma 3.5, here the whole of \( cS \) is the distinguished open subset so we omit the \( \epsilon \).

**Proposition 3.12.** Let \( \omega \in \Omega^k_l(cS) \) be a liftable form. The smooth form

\[
p^*_S R_{pt}(\omega) \in \Omega^k_l(\mathcal{D}(cS))
\]

defines the lift of a form, denoted \( R\omega \in \Omega^k_l(cS) \), this yielding an endomorphism

\[
R : \Omega^*_l(cS) \to \Omega^*_l(cS).
\]

**Proof.** That \( R_{pt} \) is well defined and linear follows by the second and third part of Proposition 3.3. From Lemma 3.5 it follows that the map \( R_{pt} \) is a map of complexes, from where

\[
\frac{p^*_S R_{pt}(d\omega)}{R} = d(p^*_S R_{pt}(\omega)). \tag{3.12}
\]

From the above Equation 3.12, combined with the first part of Proposition 3.3, we deduce that \( R \) commutes with \( d \). \( \square \)

**Proposition 3.13.** The subcomplex \( (\text{im } R) \subseteq \Omega^*_l(cS) \) is isomorphic to \( \Omega^*(S) \).

**Proof.** We begin by remarking that, for a form \( \omega \in (\text{im } R) \), the correspondence

\[
R\omega \mapsto R_{pt}(\omega) \tag{3.13}
\]

defines a surjective map onto \( \Omega^*(S) \). To prove that this correspondence is also injective, assume that \( R\omega = R\omega' \). Then, by the uniqueness of lifts, we have

\[
\frac{p^*_S R_{pt}(\omega)}{R} = \frac{p^*_S R_{pt}(\omega')}{R},
\]

and since \( p^*_S \) is injective, we deduce that the above-defined correspondence of Equation 3.13 is injective; hence the above-defined correspondence of Equation 3.13 defines the appropriate isomorphism between \( (\text{im } R) \) and \( \Omega^*(S) \). \( \square \)
In consequence of the definition of the deshirring, we note that particular care is to be taken in defining the appropriate Poincaré homotopy operator between $\mathbb{R}$ and the identity; this is a direct consequence of parts 1(a) and 1(b) of Proposition 3.4. In essence, the definition of the Poincaré homotopy operator for liftable forms on $cS$ is direct, but care must be taken to verify that it maps liftable forms to liftable forms.

**Definition 3.14.** Let $k \in \mathbb{Z}_{\geq 1}$ and let $\omega \in \Omega^k_l(cS)$ be a liftable form with lift $\tilde{\omega} \in \Omega^k(\mathcal{D}(cS))$. We define the Poincaré homotopy of $\omega$ to be

$$K^k_l(\omega) := K^k(\tilde{\omega}),$$

(3.14)

this yielding a map

$$K^k_l : \Omega^k_l(cS) \rightarrow \Omega^{k-1}(\mathcal{D}(cS)).$$

In the above Equation 3.14, the symbol $K^k$ refers to the standard Poincaré homotopy operator, as in Definition 3.7.

To verify that the morphism $K^k_l$ maps liftable forms to liftable forms, we begin with some preliminaries. Firstly, recall that $\theta_{cS}$ is the deshirring map, which restricts to a smooth map

$$\theta_{cS} : \theta^{-1}_{cS}[\text{reg}(cS)] \rightarrow \text{reg}(cS),$$

where $\theta^{-1}_{cS}[\text{reg}(cS)]$ is the union of what we will call the upper part, diffeomorphic to $S \times \mathbb{R}_{>0}$, and the lower part, diffeomorphic to $S \times \mathbb{R}_{<0}$. Secondly, recall that there is a diffeomorphism between $\text{reg}(cS)$ and $S \times \mathbb{R}_{>0}$, inducing an isomorphism of their form bundles. A liftable $k$-form $\omega \in \Omega^k_l(cS)$ is a $k$-form

$$\omega \in \Omega^k(\text{reg}(cS)) \cong \Omega^k(S \times \mathbb{R}_{>0}),$$

with lift $\tilde{\omega} \in \Omega^k(S \times \mathbb{R})$. In what follows the case of $\omega = f(s,t) \wedge p^*_S(\eta)$ will not be complicated, what we must consider is the other case; to this end let

$$\omega = f(s,t) \wedge p^*_S(\eta) \wedge dt.$$

(3.15)

Consider the lift of this $\omega$, when restricted to the upper and lower parts:

$$\tilde{\omega}|_{S \times \mathbb{R}_{>0}} = \theta^*_S(\omega)|_{S \times \mathbb{R}_{>0}} = f(s,t) \wedge p^*_S(\eta) \wedge dt.$$  

(3.16)

$$\tilde{\omega}|_{S \times \mathbb{R}_{<0}} = \theta^*_S(\omega)|_{S \times \mathbb{R}_{<0}} = -f(s,-t) \wedge p^*_S(\eta) \wedge dt.$$  

(3.17)

The change of sign is imperative, as noted in Proposition 3.4, and arises from the
expression \( \theta^*_c(\mathcal{S}) dt = d(\theta_c \mathcal{S}) \) which introduces a change of sign on the lower half, in turn arising from the expression \( \theta_c\mathcal{X}(s, t) = \theta_c\mathcal{X}(s, -t) \), which holds for all \( s \in \mathcal{S} \). Finally, we note that furthermore we may write

\[
\tilde{\omega} = \tilde{f}(s, t) \wedge p^*_S(\eta) \wedge dt
\]

to denote the lift of \( \omega \), where \( \omega \) is as in Equation 3.15, with \( \tilde{f} \) agreeing with \( f \theta_c \mathcal{S} \) on the interior (up to the change of sign, as detailed above). Note that we may write \( f(s, 0) := \tilde{f}(s, 0) = 0 \).

**Lemma 3.15.** Let \( k \in \mathbb{Z}_{\geq 1} \) and let \( \omega \in \Omega^k_f(\mathcal{C}\mathcal{S}) \) be a liftable k-form. The smooth form \( K^k_f(\omega) \in \Omega^{k-1}(S \times \mathbb{R}) \) is the lift of a form on \( \mathcal{C}\mathcal{S} \), leading to a well defined morphism

\[
K^k_f : \Omega^k_f(\mathcal{C}\mathcal{S}) \longrightarrow \Omega^{k-1}_f(\mathcal{C}\mathcal{S}). \tag{3.18}
\]

The morphism in the above Equation 3.18 is called the Poincaré operator for liftable forms on \( \mathcal{C}\mathcal{S} \).

**Proof.** Following Proposition 3.4 and Remark 3.6, there are two cases to consider; note that we use the contents of Remark 3.9 for \( K^k_f \). If \( \omega = f(s, t) \wedge p^*_S(\eta) \), then \( K^k_f(\omega) = 0 \), from where the result follows for this case. To prove the result it will suffice to prove the result for the case when

\[
\omega = f(s, t) \wedge p^*_S(\eta) \wedge dt.
\]

For this case we will keep to the notation of the preceding paragraph, which followed Equation 3.15. First we define a form

\[
\gamma := \left( p^*_S(\eta) \wedge \int_{w=0}^{u=t} f(s, w) \, dw \right) \in \Omega^{k-1}(\text{reg}(\mathcal{C}\mathcal{S})),
\]

and we aim to demonstrate that

\[
\theta^*_c(\gamma)|_{\theta^{-1}_c(\text{reg}(\mathcal{C}\mathcal{S}))} = K^k_f(\tilde{\omega}). \tag{3.19}
\]

The verification of the above Equation 3.19 would imply that \( \tilde{\gamma} = K^k_f(\tilde{\omega}) \), which in turn would imply that \( K^k_f(\omega) \) is the lift of a form, which is then unique. To verify the validity of Equation 3.19, we will verify its validity separately on \( S \times \mathbb{R}_{>0} \) and
CHAPTER 3. LIFTABLE FORMS

\( S \times \mathbb{R}_{<0} \). We begin by computing the pullbacks of \( \gamma \):

\[
\theta^{*}_{\text{CS}}(\gamma)|_{S \times \mathbb{R}_{>0}} = p_{S}^{*}(\eta) \wedge \int_{w=0}^{w=t} f(s, w) \, dw, \tag{3.20}
\]

\[
\theta^{*}_{\text{CS}}(\gamma)|_{S \times \mathbb{R}_{<0}} = p_{S}^{*}(\eta) \wedge \int_{w=0}^{w=-t} f(s, w) \, dw. \tag{3.21}
\]

In the above Equation 3.20 we have \( t > 0 \), and in the above Equation 3.21 we have \( t < 0 \); the orientations of the regions of integration are the standard ones.

Now we compute the restrictions of \( K^{k} \) to \( S \times \mathbb{R}_{>0} \) and \( S \times \mathbb{R}_{<0} \), noting that \( \check{f}(s, t) = -\check{f}(s, -t) \) as a consequence of Equations 3.16 and 3.17. The final computations are the following:

\[
K^{k}(\check{\omega})|_{S \times \mathbb{R}_{>0}} = (p_{S}^{*}\eta)|_{S \times \mathbb{R}_{>0}} \wedge \int_{w=0}^{w=t} \check{f}(s, w) \, dw \quad (t > 0)
\]

\[
= p_{S}^{*}(\eta) \wedge \int_{w=0}^{w=t} f(s, w) \, dw. \tag{3.22}
\]

\[
K^{k}(\check{\omega})|_{S \times \mathbb{R}_{<0}} = (p_{S}^{*}\eta)|_{S \times \mathbb{R}_{<0}} \wedge \int_{w=0}^{w=-t} \check{f}(s, w) \, dw \quad (t < 0)
\]

\[
= (p_{S}^{*}\eta)|_{S \times \mathbb{R}_{<0}} \wedge -\int_{w=0}^{w=-t} f(s, w) \, dw
\]

\[
= (p_{S}^{*}\eta)|_{S \times \mathbb{R}_{<0}} \wedge \int_{w=0}^{w=-t} f(s, w) \, dw
\]

\[
= p_{S}^{*}(\eta) \wedge \int_{w=0}^{w=-t} f(s, w) \, dw. \tag{3.23}
\]

Comparing Equation 3.20 and Equation 3.22, and further comparing Equation 3.21 and Equation 3.23, we obtain the validity of Equation 3.19. As noted earlier, this proves the result.

At this point we also introduce the following result, which will be used when the above Poincaré homotopy operator for liftable forms will be extended to a Poincaré homotopy operator for multiperverse forms.

**Proposition 3.16.** For all \( k \in \mathbb{Z}_{>1} \) and \( \omega \in \Omega^{k}_{t}(cS) \) we have \( R(K^{k}_{t}(\omega)) = 0 \).

**Proof.** Following Proposition 3.4 and Remark 3.6, there are two cases to consider; we also note that we use Remark 3.9 for \( K^{k} \). If \( \tilde{\omega} = f(s, w) \wedge p_{S}^{*}(\eta) \), in which case \( A^{*}(\tilde{\omega}) = f(s, tu) \wedge p_{S}^{*}(\eta) \), then we have \( K^{k}_{t}(\omega) = 0 \), so that \( R(K^{k}_{t}(\omega)) = 0 \) holds in consequence of the linearity of \( R \). Thus it suffices to prove the result in the case when

\[
\tilde{\omega} = f(s, w) \wedge p_{S}^{*}(\eta) \wedge dw.
\]
For this case we compute:

\[ A^*(\tilde{\omega}) = f(s, tu) \wedge p_S^*(\eta) \wedge (t \wedge du + u \wedge dt), \]
\[ A^*(\tilde{\omega})[\tilde{\zeta}u] = t \cdot f(s, tu) \wedge p_S^*(\eta). \]

In the above we take the convention that the change of coordinates from \( p_S^*(\eta) \) to \((p_S A)^*\eta\) is understood, where appropriate. Further, applying \( K_k^t \), we obtain:

\[ K_k^t(\omega) = \int_{u=0}^{u=1} f(s, tu) \wedge t \cdot p_S^*(\eta) \wedge du = \int_{w=0}^{w=t} f(s, w) \wedge p_S^*(\eta) \wedge dw \]
\[ = p_S^*(\eta) \wedge \left( \int_{w=0}^{w=t} f(s, w) dw \right). \]

In the above we have used that \( dw = t \wedge du \). We then deduce the validity of \( R(K_k^t(\omega)) = 0 \) from the equality

\[ \int_{w=0}^{w=0} f(s, w) dw = 0. \]

\[ \square \]

**Lemma 3.17.** For all \( k \in \mathbb{Z}_{\geq 1} \) and \( \omega \in \Omega_k^c(cS) \) the following identity holds

\[ K_k^{k+1}(d\omega) + dK_k^t(\omega) = (-1)^{k-1} \cdot (\omega - R^\omega). \] (3.24)

Thus \( K^\cdot \) defines a chain homotopy between the identity and \( R \) endomorphisms of \( \Omega^c(cS) \).

**Proof.** It suffices to verify this on the lifts of liftable forms, thus it suffices to verify that

\[ K_k^{k+1}(d\tilde{\omega}) + dK_k^t(\tilde{\omega}) = (-1)^{k-1} \cdot \left( \tilde{\omega} - R^\tilde{\omega} \right). \]

By applying Proposition 3.3 and the above Lemma 3.15 this reduces to verifying that

\[ K^{k+1}(d\tilde{\omega}) + dK^k(\tilde{\omega}) = (-1)^{k-1} \cdot \left( \tilde{\omega} - p_S^*(\tilde{Q}(\tilde{\omega})) \right), \]

which follows directly from the validity of an analogous identity in the standard case, as in Lemma 3.10. \( \square \)

**Corollary 3.18.** For all \( k \in \mathbb{Z}_{\geq 0} \) the cohomology computed using the complex of liftable forms satisfies

\[ H^k(\Omega^c(cS)) = H^k(\Omega^c(S)), \]
where the cohomology on the left hand side is computed using liftable forms on the singular space $cS$ and the cohomology on the right hand side is computed using smooth forms on the (fixed) link bundle $S$ of $cS$.

Proof. Let $[\omega] \in H^k(\Omega^*_i(cS))$ be a cohomology class. From Equation 3.24 in Lemma 3.17 we have

$$R\omega + (-1)^k \cdot dK^k(\omega) = \omega,$$

so that $[\omega] = [R\omega]$ in $H^k(\Omega^*_i(cS))$. Note that $R$ and $R_{pt}$ are morphisms of complexes and that the map

$$[\omega] = [R\omega] \mapsto [R_{pt}\omega]$$

is linear, well-defined and injective. To show that the correspondence is surjective, let $[\eta] \in H^k(\Omega^*(S))$, from where we obtain

$$[p^*_S(\eta)] = [R\omega] \mapsto [R_{pt}\omega] = [\eta].$$

From this we conclude that $H^k(\Omega^*_i(cS))$ is isomorphic to $H^k(\Omega^*(S))$. \qed

### 2.3 Poincaré operator for liftable forms on the cylinder

Let $X \times \mathbb{R}$ be the controlled cylinder over a depth 1 controlled stratified space $X$. Note that $X \times \mathbb{R}$ is a depth 1 controlled stratified space, as in Definition 2.4. We will use a similar approach to the preceding Subsection 2.2 to prove the Poincaré lemma for the cylinder, as before we first define the appropriate Poincaré homotopy operator, and then prove that it maps liftable forms to liftable forms.

**Definition 3.19.** Let $\omega \in \Omega^k_i(X \times \mathbb{R})$ be a liftable $k$-form with $k \in \mathbb{Z}_{\geq 1}$ and lift $\tilde{\omega} \in \Omega^k(\mathcal{D}(X) \times \mathbb{R})$. We define the Poincaré homotopy of $\omega$ to be

$$K^k_i(\omega) := K^k(\tilde{\omega}),$$

yielding a map

$$K^k_i : \Omega^k_i(X \times \mathbb{R}) \rightarrow \Omega^{k-1}_i(\mathcal{D}(X) \times \mathbb{R}).$$

The next step is to verify that $K^k_i$ maps liftable forms on $X \times \mathbb{R}$ to liftable forms on $X \times \mathbb{R}$.

**Lemma 3.20.** Let $\omega \in \Omega^k_i(X \times \mathbb{R})$ be a liftable $k$-form with $k \in \mathbb{Z}_{\geq 1}$, then $K^k_i(\omega)$ is the lift of a form, leading to a well defined map

$$K^k_i : \Omega^k_i(X \times \mathbb{R}) \rightarrow \Omega^{k-1}_i(X \times \mathbb{R}).$$
Proof. If \( \tilde{\omega} = f(x, t) \wedge p_{\overline{\varphi}(X)}^*(\eta) \), where \( \eta \in \Omega^k_1(\mathcal{D}(X)) \), then \( K^k_1(\omega) = 0 \), so it will suffice to prove this for the case when

\[
\tilde{\omega} = f(x, t) \wedge p_{\overline{\varphi}(X)}^*(\eta) \wedge dt ,
\]

where \( \eta \in \Omega^{k-1}(\mathcal{D}(X)) \) and \( f(x, t) \) is a liftable function in \( X \), that is a liftable function for each fixed \( t \). Now we apply the Poincaré homotopy operator to \( \omega \), obtaining

\[
K^k(f(x, t) \wedge p_{\overline{\varphi}(X)}^*(\eta) \wedge dt) = p_{\overline{\varphi}(X)}^*(\eta) \wedge \int_{w=0}^{w=t} f(x, w) \, dw .
\]

To prove that this is a lift of a form on \( X \times \mathbb{R} \), it will suffice to show that the function

\[
(x, t) \mapsto \int_{w=0}^{w=t} f(x, w) \, dw
\]

is liftable, where \((x, t) \in \text{reg}(X) \times \mathbb{R} \). We claim that its lift on \( \mathcal{D}(X) \times \mathbb{R} \) is the function

\[
(y, t) \mapsto \int_{w=0}^{w=t} f(y, w) \, dw ,
\]

where \((y, t) \in \mathcal{D}(X) \times \mathbb{R} \). This is verified by noting that, for a fixed \( y \in \theta^{-1}_X[\text{reg}(X)] \), we have \( f(\theta_X(y), t) = f(y, t) \), so that

\[
\int_{w=0}^{w=t} f(\theta_X(y), w) \, dw = \int_{w=0}^{w=t} f(y, w) \, dw .
\]

For the result below we let \( p_{\text{reg}(X)} : \text{reg}(X) \times \mathbb{R} \rightarrow \text{reg}(X) \) be the natural projection and let \( Q \) be the pullback along the map defined by \( x \mapsto (x, 0) \), as for the usual Poincaré lemma of Subsection 2.1.

**Proposition 3.21.** The endomorphism \( p_{\text{reg}(X)}^*Q \) maps liftable forms to liftable forms, yielding a map of complexes

\[
p_{\text{reg}(X)}^*Q : \Omega^*_1(X \times \mathbb{R}) \rightarrow \Omega^*_1(X \times \mathbb{R}) .
\]

**Proof.** If the form \( \omega \) is defined by

\[
\omega = f(x, t) \wedge p_{\text{reg}(X)}^*(\eta) \wedge dt ,
\]

then \( p_{\text{reg}(X)}^*(Q(\omega) = 0 \); here 0 is a liftable form, as it is constant. On the other hand, if \( \omega \) is defined by

\[
\omega = f(x, t) \wedge p_{\text{reg}(X)}^*(\eta) ,
\]
then we compute $Q(\omega) = f(x, 0) \wedge \eta$, so that

$$p_{\text{reg}(X)}^* Q(\omega) = f(x, 0) \wedge p_{\text{reg}(X)}^*(\eta),$$

which is liftable, for $f(x, 0)$ is a liftable function in $x$.

**Lemma 3.22.** For all $k \in \mathbb{Z}_{\geq 1}$ and $\omega \in \Omega^k_\text{l}(X \times \mathbb{R})$ the following identity holds

$$K^{k+1}_i d\omega + dK^k_i(\omega) = (-1)^{k-1}(\omega - p_{\text{reg}(X)}^* Q\omega).$$

(3.25)

Thus $K_i^\bullet$ defines a chain homotopy between the identity and $p_{\text{reg}(X)}^* Q$ endomorphisms on $\Omega_i^\bullet(cS)$.

**Proof.** This follows by applying Lemma 3.10 on lifts, analogously to the proof of Lemma 3.17.

**Corollary 3.23.** For the cylinder $X \times \mathbb{R}$ and for all for all $k \in \mathbb{Z}_{\geq 0}$ the cohomology computed using the complex of liftable forms satisfies

$$H^k(\Omega^\bullet(X \times \mathbb{R})) = H^k(\Omega^\bullet(cS)),$$

where the cohomology on the left hand side is computed using liftable forms on the singular space $X \times \mathbb{R}$ and the cohomology on the right hand side is computed using liftable forms on the singular space $X$.

**Proof.** This follows from the above Lemma 3.22, in the same was as Corollary 3.23 follows from Lemma 3.17.
Chapter 4

Multicontrolled spaces

In this chapter we will define multicontrol data for depth 1 controlled stratified space and consider the multicontrolled cylinder and cone.

1 Depth 1 multicontrol data

1.1 Definition of multicontrol data

In subsection we will define multicontrol data for a depth 1 controlled stratified space, thus obtaining the definition of a multicontrolled controlled stratified space.

Definition 4.1 (Multicontrol data). Let $X$ be a depth 1 controlled stratified space. A set of multicontrol data for $X$ is a set of objects and arrows

\[ \mathcal{M}(X) = \left\{ \left( T'_A, \pi'_A : T'_A \to A, \rho'_A : T'_A \to \mathbb{R}_{\geq 0}, \mu_A : T_A \to T'_A \right) \right\}_{A \in \mathcal{SS}(X)} \tag{4.1} \]

satisfying the following four numbered conditions:

1. For each singular stratum $A \in \mathcal{SS}(X)$ the object $T'_A$ is a depth 1 controlled stratified space and the arrows $\pi_A, \rho_A$, and $\mu_A$ are continuous maps.

2. For each singular stratum $A \in \mathcal{SS}(X)$ the stratified space $T'_A$ satisfies $\mathcal{SS}(T'_A) = \{A\}$, with the tubular neighbourhood of $A$ equal to the whole of $T'_A$. The control data in $T'_A$ for the singular stratum $A \in \mathcal{SS}(T'_A)$ is given by

\[ \pi'_A : T'_A \to A \quad \text{and} \quad \rho'_A : T'_A \to \mathbb{R}_{\geq 0}. \]

3. For each singular stratum $A \in \mathcal{SS}(X)$ the map $\mu_A : T_A \to T'_A$ restricts to the identity map on $A$ and to a smooth map on $\text{reg}(T_A)$, and furthermore satisfies the following properties:

4. For each singular stratum $A \in \mathcal{SS}(X)$ the property of $\mu_A$ is that $\mu_A$ is smooth on $\text{reg}(T_A)$ and that $\mu_A$ is a diffeomorphism on $\text{reg}(T_A)$.
(a) The restriction of $\mu_A$ onto $\text{reg}(T_A)$ is a fibre bundle with smooth manifold fibre $F_A$, as illustrated on the following diagram.

\[
\begin{array}{ccc}
F_A & \longrightarrow & \text{reg}(T_A) \\
\mu_A|_{\text{reg}(T_A)} & \downarrow & \\
\text{reg}(T_A') & \end{array}
\]

(b) $\pi_A$ is the composition of $\pi'_A$ and $\mu_A$, so

\[\pi_A = \pi'_A \mu_A.\]

(c) $\mu_A$ commutes with $\rho_A$ and $\rho_A'$, so that for all $x \in \text{reg}(T_A)$ we have

\[\rho_A(x) = \rho_A'((\mu_A(x))).\]

4. For each singular stratum $A \in SS(X)$ the (locally trivial) fibre bundle $\pi_A : T_A \longrightarrow A$ has the following structure

\[
\begin{array}{ccc}
cL_A & \longrightarrow & T_A \\
\mu_A & \downarrow & \\
cL'_A & \overset{\pi_A}{\longrightarrow} & T_A \\
\pi'_A & \downarrow & \\
& & A \\
\end{array}
\]

So that $\pi_A = \pi'_A \mu_A$ as above, where both $L_A'$ and $L_A$ are smooth manifolds.

In the above $cL_A$ is the fibre of $\pi_A$ and $cL'_A$ is the fibre of $\pi'_A$.

A depth 1 controlled stratified space endowed with a set of multicontrol data is called a depth 1 multicontrolled stratified space, or (for brevity) a multicontrolled space.

1.2 Definition of multicontrolled isomorphisms

In this subsection we will define multicontrolled isomorphisms. Let $f : X \rightarrow Y$ be a controlled isomorphism between two controlled stratified spaces. Henceforth, let $f_A$ denote the restriction of $f$ to $T_A$, noting that the following diagram commutes.

\[
\begin{array}{ccc}
T_A & \overset{\pi_A}{\longrightarrow} & A \\
\downarrow f_A & & \downarrow f|_A \\
T_{f[A]} & \overset{\pi_{f[A]}}{\longrightarrow} & f[A] \\
\end{array}
\]
From this we note that $f_A : T_A \to T_{f[A]}$ is itself an isomorphism of stratified spaces.

**Definition 4.2** (Multicontrolled isomorphism). Let $X$ and $Y$ be two depth 1 multicontrolled spaces and $f : X \to Y$ a controlled isomorphism between them. We will say that $f$ is *multicontrolled isomorphism*, if it satisfies the following additional conditions:

1. There exists a set of controlled morphisms
   \[
   \left\{ \ f'_A : T'_A \to T'_{f[A]} \right\}_{A \in \SS(X)},
   \]
   indexed by the set $\SS(X)$ of singular strata, such that $(f'_A)|_A = f|_A$.

2. For each singular stratum $A \in \SS(X)$, consider the following diagram.

   \[
   \begin{array}{ccc}
   T_A & \xrightarrow{f_A} & T_{f[A]} \\
   \downarrow{\mu_A} & \downarrow{\mu_{f[A]}} & \downarrow{\mu_{f[A]}} \\
   T'_A & \xrightarrow{f'_A} & T'_{f[A]} \\
   \downarrow{\pi'_A} & \downarrow{\pi'_{f[A]}} & \downarrow{\pi'_{f[A]}} \\
   A & \xrightarrow{f[A]} & f[A]
   \end{array}
   \]

   We require that the top and bottom square commute (note also that the outer square commutes, for $f_A$ is a controlled isomorphism).

**Proposition 4.3.** Let $f : X \to Y$ and $g : Y \to Z$ be two multicontrolled isomorphisms. Their composition $gf : X \to Z$, with $gf'_A : T'_A \to T'_{gf[A]}$ defined by

   \[(gf)'_A := g'_{f[A]}f'_A,
   \]

is itself a multicontrolled isomorphism. Furthermore, the composition of multicontrolled isomorphisms is associative.

**Proof.** The given data populate the following commutative diagram.

\[
\begin{array}{ccc}
T_A & \xrightarrow{f_A} & T_{f[A]} & \xrightarrow{g_{f[A]}} & T_{gf[A]} \\
\downarrow{\mu_A} & \downarrow{\mu_{f[A]}} & \downarrow{\mu_{f[A]}} & \downarrow{\mu_{gf[A]}} \\
T'_A & \xrightarrow{f'_A} & T'_{f[A]} & \xrightarrow{g'_{f[A]}} & T'_{gf[A]} \\
\downarrow{\pi'_A} & \downarrow{\pi'_{f[A]}} & \downarrow{\pi'_{f[A]}} & \downarrow{\pi'_{gf[A]}} \\
A & \xrightarrow{f[A]} & f[A] & \xrightarrow{g_{f[A]}} & gf[A]
\end{array}
\]
If the shaded arrows are removed from the above diagram, then the diagram so defined still commutes, thus proving that $gf$ is a multicontrolled isomorphism. The final part follows directly, for the composition of multicontrolled isomorphisms is actualised by pasting together commutative diagrams.

## 2 Local structure

### 2.1 Multicontrolled cylinder and cone

In the controlled setting the cylinder of a controlled stratified space has the same depth as the original space, whereas the cone on a controlled space increments the depth of the original space by 1. We begin with the multicontrolled cylinder, where this operation is defined upon a given a depth 1 multicontrolled stratified space and yields a depth 1 multicontrolled stratified space.

**Definition 4.4 (Multicontrolled cylinder).** Let $X$ be a depth 1 multicontrolled stratified space. Let the product $X \times \mathbb{R}$ be given the structure of a controlled stratified space, as in Definition 2.4. For this controlled stratified structure on $X \times \mathbb{R}$ we can define a set of multicontrol data $M(X \times \mathbb{R})$ for $X \times \mathbb{R}$; for this we use the multicontrol data $M(X)$ for $X$:

1. For each singular stratum $A \times \mathbb{R} \in \mathbb{S}\!(X \times \mathbb{R})$, we set $T_A' \times \mathbb{R} = T_A' \times \mathbb{R}$.

2. The retraction from $T_A' \times \mathbb{R}$ to the singular stratum $A \times \mathbb{R} \in \mathbb{S}\!(X \times \mathbb{R})$, having the form

   $$\pi'_{A \times \mathbb{R}} : T_A' \times \mathbb{R} \longrightarrow A \times \mathbb{R},$$

   is defined by the correspondence $(x, t) \mapsto (\pi_A'(x), t)$, that is to say as the product morphism $\pi_A' \times 1_{\mathbb{R}}$.

3. The continuous map between the two tubular neighbourhoods $T_A$ and $T_A'$, having the form

   $$\mu'_{A \times \mathbb{R}} : T_A \times \mathbb{R} \longrightarrow T_A' \times \mathbb{R},$$

   is defined by the correspondence $(x, t) \mapsto (\mu_A(x), t)$, that is to say as the product morphism $\mu_A' \times 1_{\mathbb{R}}$.

For each singular stratum $A \in \mathbb{S}\!(X)$, the above-defined data populates the fol-
The conditions of Definition 4.1 are then verified directly, noting that $\mu_{A \times R}|_{\text{reg}(T_A \times R)}$ is still a fibre bundle with fibre $F_A$.

The construction of the multicontrolled cone does not extend directly from the controlled setting. To see this, we remark that a sensible definition of a depth 0 multicontrolled space is as a manifold\(^1\). If we then take the cone over the smooth manifold, the resulting controlled space can only be furnished with the trivial multicontrol data, that is, there is no candidate for $T_A'$ except that of $T_A$ itself. To rectify this we introduce the concept of fibering data.

\textbf{Definition 4.5} (Fibering data). The data of a smooth manifold $S$ and a smooth fibration $\lambda : S \to S'$ with smooth manifold fibre $F$ is a set of depth 0 fibering data. For two fibering data, the first $\lambda_1 : S_1 \to S'_1$, and the second $\lambda_2 : S_2 \to S'_2$, a morphism of depth 0 fibering data is an arrow $f : S_1 \to S_2$ which is the data of the following commutative diagram.

$$
\begin{array}{ccc}
S_1 & \xrightarrow{f_t} & S_2 \\
\downarrow{\lambda_1} & & \downarrow{\lambda_2} \\
S'_1 & \xrightarrow{f_b} & S'_2
\end{array}
$$

Thus, in the above, $f_t$ and $f_b$ are smooth maps; so is a morphism of fibering data is simply a morphism of smooth fibre bundles.

\textbf{Definition 4.6} (Multicontrolled cone). Let $\lambda : S \to S'$ be a set of depth 0 fibering data and let the cone $cS$ be given the structure of a controlled stratified space as in Definition 2.5. For this controlled stratified structure on $cS$ we can define a set of multicontrol data $M(cS)$ by using the fibering data $\lambda$:

1. For the sole singular stratum $pt \in SS(cS)$ we set $T'_pt = cS'$.

2. The retraction from $cS'$ to the singular stratum $pt \in SS(cS')$ is defined as in Definition 2.5, giving $cS'$ the controlled stratified structure of a cone.

\(^1\)This justified by considering that a depth 0 controlled stratified space is simply a manifold.
3. The continuous map between the two tubular neighbourhoods, \(cS\) and \(cS'\), having the form

\[
\mu_{pt} : cS \to cS',
\]

is defined by the correspondence \([x, t] \mapsto [\lambda(x), t]\), that is to say as the product morphism \(\lambda \times 1\) applied to the equivalence classes in \(cS\) (this is well defined because \(pt = [\lambda(x), 0]\)).

This defines a multicontrolled space \(cS\), which we say is associated to the fibering data \(\lambda : S \to S'\).

### 2.2 Multicontrolled mapping cylinder and local structure

**Definition 4.7** (Multicontrolled mapping cylinder). Let \(\mu : S \to S'\) be a morphism between \(\pi : S \to A\) and \(\pi' : S' \to A\), thus forming the following commutative diagram.

\[
\begin{array}{ccc}
S & \xrightarrow{\mu} & S' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
A & \xrightarrow{1_A} & A
\end{array}
\]

In the above \(1_A \pi = \pi' \mu\), or more indicatively \(\pi = \pi' \mu\). Let us also be given the datum of the following diagram.

\[
\begin{array}{ccc}
F & \xrightarrow{\mu} & S \\
\downarrow & & \downarrow{\mu} \\
S' & & S'
\end{array}
\]

Assume that the \(\mu\) appearing in Equation 4.2 is a smooth fibre bundle with fibre \(F\), where \(F\) is a smooth manifold. From this data we form the controlled stratified spaces \(\text{cyl}(\pi)\) and \(\text{cyl}(\pi')\), as in Definition 2.10, between which we then define a continuous map

\[
\mu_A : \text{cyl}(\pi) \to \text{cyl}(\pi')
\]

\([s, t]_{\pi} \mapsto [\mu(s), t]_{\pi'}\),

using which we can define a multicontrolled structure on \(\text{cyl}(\pi)\). We recall from Definition 2.10 that \(SS(\text{cyl}(\pi)) = \{A\}\), with \(T_A = \text{cyl}(\pi)\), and similarly for \(\text{cyl}(\pi')\). Hence we set \(T'_A = \text{cyl}(\pi')\) and \(\mu_A\) as above to obtain the multicontrolled mapping cylinder on \(\mu\).
Remark 4.8. One can alternatively construct the first datum in Equation 4.2 of the above Definition 4.7 as the following double fibration diagram.

\[ \begin{array}{ccc}
L & \longrightarrow & S \\
\mu \downarrow & & \downarrow \pi \\
L' & \longrightarrow & S' \\
\pi' \downarrow & & \downarrow A \\
& & \end{array} \]

(4.3)

In the above \( L \) is the fibre of \( \pi \) and \( L' \) is the fibre of \( \pi' \).

Fix a depth 1 multicontrolled stratified space \( X \). From Lemma 2.11 there are isomorphism of stratified spaces

\[ C_A : \text{cyl}(\pi_A|_{S_A}) \longrightarrow T_A \quad \text{and} \quad C'_A : \text{cyl}(\pi'_A|_{S'_A}) \longrightarrow T_A, \]

(4.4)

fitting into the following diagram.

\[ \begin{array}{ccc}
cL & \longrightarrow & \text{cyl}(\pi_A|_{S_A}) \\
\mu_A \times 1 \downarrow & & \downarrow \pi_A \times 1 \\
cL' & \longrightarrow & \text{cyl}(\pi'_A|_{S'_A}) \\
\pi'_A \times 1 \downarrow & & \downarrow A \\
& & \end{array} \]

In the above diagram the large-dashed arrow \( \mu_A \times 1 \) is defined by making use of the morphism \( \mu_A : T_A \rightarrow T'_A \) and is defined by the correspondence

\[ [s, t]_{\pi_A} \longmapsto [\mu_A(s), t]_{\pi'_A}, \]

making \( \text{cyl}(\pi_A|_{S_A}) \) into a multicontrolled stratified space, which, in the notation of Definition 4.7, is just \( \text{cyl}(\mu_A|_{S_A}) \). For the following result we also note that for each \( A \in SS(X) \) the tubular neighbourhood \( T_A \) has the induced structure of a multicontrolled space.

**Lemma 4.9.** There exists an isomorphism of multicontrolled spaces

\[ B_A : \text{cyl}(\mu_A|_{S_A}) \longrightarrow T_A, \]

between the above-defined \( \text{cyl}(\mu_A|_{S_A}) \) and the multicontrolled stratified space \( T_A \).
Proof. Consider the following diagram.

In this diagram the square is commutative and hence satisfies

\[ \mu_A C_A([s, t]_{\pi_A}) = C'_A([\mu_A(s), t]_{\pi'_A}). \]

We then define \( B_t = C_A \) and \( B_b = C'_A \), and it follows that \( B_A \) is an isomorphism, for both \( C_A \) and \( C'_A \) are.
Chapter 5

Multiperverse cohomology

In this chapter we will define the lattice of multiperversities and the complex of multiperverse forms, from where we will be able to compute multiperverse cohomology. In Section 1 we will define the lattice of multiperversities and consider the properties of this lattice. Following this, in Section 2, we will define multiperverse cohomology in a reduced product-type case, calculating the multiperverse cohomology of the cone and of the cylinder. In the final Section 3 we study multiperverse cohomology in the flat-type case, which is a generalisation of the product-type case.

1 Lattice of multiperversities

In this section we will define the concept of a multiperversity, study the properties of the resulting lattice of multiperversities, and define the dual multiperversity. For the multicontrolled spaces defined in Chapter 4, the concept of a multiperversity extends the concept of a perversity, taking the same rôle in the definition of multiperverse cohomology as the one taken by perversities in the definition of intersection (co)homology.

1.1 General perversities

In this subsection we will recall the definition and some properties of general perversities. The concept of a perversity, as used in the original definition of intersection homology in [17], can be generalised in a number of ways; one such generalisation is the following.

Definition 5.1 (General perversities; [13], p.24). Let $X$ be a controlled stratified space. A general perversity is a function

$$\overline{p} : S\mathcal{S}(X) \rightarrow \mathbb{Z},$$
satisfying, for each singular stratum $A \in \mathcal{SS}(X)$, the identity
\[-1 \leq \mathcal{p}(A) \leq \text{codim}_X(A) - 1.\] (5.1)

The identity in the above Equation 5.1 is known as the *efficiency condition*. The set of general perversities for $X$ is written as $\mathcal{P}(X)$, or just $\mathcal{P}$ when $X$ is understood.

The above-defined general perversities are extensible to multiperversities, in the sense that the definition of a multiperversity is based on the idea of the above-defined general perversities. Furthermore, we will show in Proposition 5.10 that a general perversity can be naturally considered as a multiperversity. Henceforth by the word perversity we will mean, unless otherwise specified, a general perversity as in Definition 5.1.

We continue recalling the properties of the set $\mathcal{P}$. The set $\mathcal{P}$ can be made into a partially ordered set by taking the partial order relation to be pointwise comparison; $\mathcal{P}$ is then a lattice where the greatest lower bound, written $\land$, and the least upper bound, written $\lor$, are computed pointwise. As a consequence of the efficiency condition in Equation 5.1, the lattice $\mathcal{P}$ has both a highest and a lowest element; the highest element is defined as $\overline{h}(A) = \text{codim}(A) - 1$ and the lowest element is defined as $\overline{l}(A) = -1$, for $A \in \mathcal{SS}(X)$. It is important not to confuse $\overline{h}$ with the top perversity $t = \text{codim}(p) - 2$ and $\overline{l}$ with the zero perversity $0 = 0$; these are the highest and lowest perversities in the setting of standard perversities, but the general perversities permit two extra perversities\(^1\).

We recall the result of the cone calculation for the general perversities of Definition 5.1. Restricting ourselves to the depth 1 case, and to the case of intersection cohomology, the calculation for the cone over a compact manifold $S$ has the form:

\[
IH^k_p(cS) = \begin{cases} 
H^k(S) & \text{if } k \leq \mathcal{p}(pt), \\
0 & \text{if } k > \mathcal{p}(pt).
\end{cases}
\] (5.2)

This result can be found in [21] (local Poincaré lemma formula on page 2 and also in Section 5.2) and will also be obtained in Corollary 5.68 in the flat-type case. In the above Equation 5.2, if the perversity $\mathcal{p}$ is the highest perversity

\[
\overline{h}(pt) = (\text{codim } pt) - 1 = \dim(S),
\]

\(^1\text{Thus, if all four are defined, then } \overline{l} < \mathcal{0} < \overline{t} < \overline{h}.\)
then we obtain absolute cohomology of $cS$; on the contrary, if the perversity $\bar{p}$ is the smallest perversity $\bar{p}(pt) = -1$, then we obtain relative cohomology of $cS$, which is zero. Because $IH^k_{\bar{p}}(cS)$ is defined using smooth forms, these calculations can be compared to the calculation using Borel-Morrey chains (also known as closed chains), for example in [8] (top of p.17) or in Equation 1.4 in Subsection 2.1 of Chapter 1; for details see Equation 1.5 in Subsection 2.1 of Chapter 1.

### 1.2 Definition of depth 1 multiperversities

In this subsection we will define the concept of a multiperversion for a depth 1 multicontrolled stratified space; for the definition of a depth 1 multicontrolled stratified space see Definition 4.1 in Chapter 4. To define the concept of a multiperversion we will first need to recall some notation. Let $P$ be a partially ordered set. The partially ordered set $P^{\text{op}}$ will denote the opposite partially ordered set to $P$, where $x \leq y$ holds in $P$ if and only if $y \leq x$ holds in $P^{\text{op}}$; while $(P, Q)$, for two partially ordered sets $P$ and $Q$, will denote the set of order preserving maps from $P$ to $Q$. By combining the two preceding sentences it is immediate that $(P^{\text{op}}, Q)$ denotes the set of order reversing maps from $P$ to $Q$. There is a natural pointwise order on $(P, Q)$, and hence on $(P^{\text{op}}, Q)$, inherited from the partial orders for $P$ and $Q$, and described in the following way. An order preserving map $e : P \to Q$ is less than or equal to an order preserving map $f : P \to Q$, this being written $e \leq f$, if and only if, for all elements $p \in P$, $e(p) \leq f(p)$ holds in the order relation of $Q$.

In order to simplify the notation involved, we will henceforth use the following overloading of the codim notation. If $A \in \text{SS}(X)$ is a singular stratum in a depth 1 multicontrolled stratified space $X$, then we will write

$$\text{codim}_X(S'_A) := \dim(\text{reg}(X)) - \dim(S'_A),$$

where $\dim(M)$ denotes the dimension of the manifold $M$. By codimension we will always mean codimension with respect to the ambient controlled stratified space.

**Definition 5.2 (Multiperversities).** Let $X$ be a depth 1 multicontrolled stratified space. A *multiperversion* for $X$ is a function

$$\tau : \text{SS}(X) \longrightarrow (\mathbb{Z}^{\text{op}}, \mathbb{Z})$$

satisfying two efficiency conditions. The *first efficiency condition* is the requirement that the identity

$$-1 \leq \tau(A)(k) \leq \text{codim}_X(S'_A) - 1$$

(5.4)
hold for all singular strata \( A \in \mathcal{S}(X) \) and integers \( k \in \mathbb{Z} \); equivalently the first efficiency condition requires that the image of \( \tau(A) \) be contained in the interval \([-1, \text{codim}(S'_A) - 1]\). The second efficiency condition is the requirement that the following two identities be satisfied for all singular strata \( A \in \mathcal{S}(X) \):

\[
\begin{align*}
  k < 0 & \implies \tau(A)(k) = \text{codim}_X(S'_A) - 1. & (5.5) \\
  k > \text{codim}_X(A) - 1 & \implies \tau(A)(k) = -1. & (5.6)
\end{align*}
\]

The set of multiperversities for \( X \) is written as \( M(X) \), or just \( M \) when \( X \) is understood.

**Remark 5.3.** The two efficiency conditions in the above Definition 5.2, in their essence, serve to discard some of the values at which the multiperversity will have no highestological meaning in the definition of multiperverse cohomology. However, for a given space, the two efficiency conditions do not discard all the values at which the multiperversity will have no highestological meaning in the definition of multiperverse cohomology; for further details see Example 5.30 in Subsection 2.1 of this chapter.

To illustrate the above Definition 5.2 we consider an example. Note that the definition of a multiperversity depends only on the values of \( \text{codim}_X(A) \) and \( \text{codim}_X(S'_A) \) for singular strata \( A \in \mathcal{S}(X) \). These values completely determine the efficiency conditions in Equations 5.4, 5.5 and 5.6.

**Example 5.4.** Consider a depth 1 multicontrolled stratified \( X \) with a single singular stratum \( A \), such that \( \text{codim}_X(A) = 5 \) and \( \text{codim}_X(S'_A) = 3 \). We can define a multiperversity \( \tau \in M(X) \) by specifying the following data:

\[
\tau(A)(k) = \begin{cases} 
  2 & \text{if } k < 0 \\
  2 & \text{if } k = 0 \\
  1 & \text{if } k = 1 \\
  1 & \text{if } k = 2 \\
  0 & \text{if } k = 3 \\
  -1 & \text{if } k = 4 \\
  -1 & \text{if } k > 4
\end{cases}
\]

Writing \( \tau(k) \) for \( \tau(A)(k) \), we can further represent the above-defined multiperversity \( \tau \) by using the following diagram.
To read the above Figure 5.1 we count vertically the hatched or solid grey cubes at a particular value of \( k \); the highest value which is hatched or solid grey is the value of \( \tau \) at this \( k \). The hatched part denotes the part of the multiperversity which is not subject to either the first or the second efficiency condition of Equations 5.4, 5.5 or 5.6; the solid grey part denotes the part of the multiperversity which is subject to the aforementioned efficiency conditions. To define a multiperversity it is sufficient to define the part which is not subject to the efficiency conditions.

Multiperveristies on \( X \), that is elements of the set \( \mathcal{M}(X) \), admit a pointwise-defined order relation.

**Definition 5.5.** Let \( \tau, \tau' \in \mathcal{M}(X) \) be two multiperversities. We will say that \( \tau \) is less than or equal to \( \tau' \), written \( \tau \leq \tau' \), if for all singular strata \( A \in \mathcal{SS}(X) \) the relation \( \tau(A) \leq \tau'(A) \) holds in \( (\mathbb{Z}^{op}, \mathbb{Z}) \).

**Remark 5.6.** As the set \( (\mathbb{Z}^{op}, \mathbb{Z}) \) of order reversing functions from \( \mathbb{Z} \) to \( \mathbb{Z} \) itself inherits a natural structure of a partially ordered set under pointwise-defined comparison, the expression \( \tau(A) \leq \tau'(A) \) in the above Definition 5.5 is meaningful.

We will now define an embedding of the set \( \mathcal{P}(X) \) into \( \mathcal{M}(X) \); this will show that every general perversity, as in Definition 5.1, can be considered as a multiperversity.

**Definition 5.7.** Let \( X \) be a depth 1 multicontrolled stratified space. The embedding of \( \mathcal{P}(X) \) into \( \mathcal{M}(X) \) is an arrow

\[
\mathbf{m} : \mathcal{P}(X) \longrightarrow \mathcal{M}(X),
\]

which, for a perversity \( \bar{p} \in \mathcal{P}(X) \), is defined to be the arrow

\[
\mathbf{m}(\bar{p}) : \mathcal{SS}(X) \longrightarrow (\mathbb{Z}^{op}, \mathbb{Z}),
\]
itself defined by the following correspondence, for all singular strata \( A \in \mathcal{S}(X) \) and all integers \( k \in \mathbb{Z} \):

\[
\mathcal{m}(\overline{p})(A)(k) = \begin{cases} 
\text{codim}(S'_A) - 1 & \text{if } k \leq \overline{p}(A), \\
-1 & \text{if } k > \overline{p}(A).
\end{cases}
\] (5.7)

Let us briefly consider why the two efficiency conditions in Definition 5.2 are verified for \( \mathcal{m}(\overline{p}) \). The validity of the first efficiency condition of Equation 5.4 follows directly from the definition of \( \mathcal{m}(\overline{p}) \) in the above Equation 5.7. The validity of the second efficiency condition of Equations 5.5 and 5.6 follows from the efficiency condition for \( \overline{p} \) in Equation 5.1 of Definition 5.1.

The embedding \( \mathcal{m} \) is utile in identifying the least and highest multiperversities, as shown below. Recall that the least and highest perversities are \( \underline{l} \) and \( \overline{h} \) respectively, which correspond to absolute and relative cohomology in intersection cohomology (see also Subsection 1.1). One can interpret the succeeding result as follows: the set of perversities parametrises a gradation between the absolute and relative cohomology and, for the spaces which we consider, the same is true of multiperversities.

**Proposition 5.8.** The element \( \mathcal{m}(\overline{h}) \) is the highest element in \( \mathcal{M}(X) \), while the element \( \mathcal{m}(\underline{l}) \) is the least element in \( \mathcal{M}(X) \).

**Proof.** From Equation 5.7, for each singular stratum \( A \in \mathcal{S}(X) \), we have:

\[
\mathcal{m}(\overline{h})(A)(k) = \begin{cases} 
\text{codim } S'_A - 1 & \text{if } k \leq \text{codim } A - 1, \\
-1 & \text{if } k > \text{codim } A - 1.
\end{cases}
\] (5.8)

The above implies that pointwise \( \mathcal{m}(\overline{h}) \) attains the maximum possible values within the constraints of the two efficiency conditions in Definition 5.2. Similarly, for each singular stratum \( A \in \mathcal{S}(X) \), we have the following:

\[
\mathcal{m}(\underline{l})(A)(k) = \begin{cases} 
\text{codim } S'_A - 1 & \text{if } k < 0, \\
-1 & \text{if } k \geq 0.
\end{cases}
\] (5.9)

The above implies that pointwise \( \mathcal{m}(\underline{l}) \) attains the minimum possible values within the constraints of the two efficiency conditions in Definition 5.2. This proves the result. \( \Box \)
Example 5.9. Consider again an example where a depth 1 multicontrolled stratified space $X$ has a sole single singular stratum $A$, such that $\text{codim}(A) = 5$ and $\text{codim}(S_A') = 3$; following on from Example 5.4. The greatest multiperversity $m(\overline{h}) \in \mathcal{M}(X)$, as defined in the above Equation 5.8, is illustrated on the following diagram.

![Diagram representing the multiperversity $m(\overline{h})$.]

Similarly, the least multiperversity $m(\overline{l}) \in \mathcal{M}(X)$, as defined in the above Equation 5.9, is illustrated on the following diagram.

![Diagram representing the multiperversity $m(\overline{l})$.]

In general, to simplify the notation involved, and in light of Proposition 5.8, we will write $h$ for $m(\overline{h})$ and $l$ for $m(\overline{l})$, wherever this does not lead to confusion between these respective elements considered within $\mathcal{P}(X)$ and $\mathcal{M}(X)$.

Proposition 5.10. The set of multiperversities $\mathcal{M}(X)$ is a complete lattice and the map

$$m : \mathcal{P}(X) \rightarrow \mathcal{M}(X)$$

is an injective lattice morphism.

Proof. As $\mathbb{Z}$ is a totally ordered set, which is in particular a lattice, then $(\mathbb{Z}^{op}, \mathbb{Z})$ is a lattice, with the lattice operations computed pointwise. From this we conclude that $\mathcal{M}(X)$ is a lattice with the lattice operations computed pointwise. To verify
that $\mathcal{M}(X)$ is a complete lattice, we proceed as follows. The second efficiency condition of Equations 5.5 and 5.6 implies that a multiperversity is allowed to vary only a finite number of elements, while the first efficiency condition of Equation 5.4 implies that a multiperversity may take only a finite number of values; from this we obtain that there are a finite number of distinct multiperversities. Combining this with the result of Proposition 5.8, which states that $\mathcal{M}(X)$ has a least and greatest element, we conclude that $\mathcal{M}(X)$ is complete as a lattice. For the second part, we first note that $m$ is injective, as this follows directly from the definition of $m$ in Equation 5.7. To verify that $m$ is a lattice morphism, we must verify ([4], Section 2.4) that the following hold for all perversities $p, q \in \mathcal{P}(X)$:

\begin{align*}
m(p \wedge q) &= m(p) \wedge m(q), \\
m(p \vee q) &= m(p) \vee m(q).
\end{align*}

In the above Equations 5.10 and 5.11, the greatest lower bound and least upper bound $\wedge$ and $\vee$ are computed in $\mathcal{P}(X)$ and $\mathcal{M}(X)$ as appropriate. From the pointwise definition of $\wedge$ in $\mathcal{M}(X)$ the following series of equalities is obtained for each singular stratum $A \in SS(X)$:

\[
(m(p) \wedge m(q))(A)(k) = m(p)(A)(k) \wedge m(q)(A)(k)
\]

\[
= \begin{cases} 
\text{codim} S'_A - 1 & \text{if } k \leq p(A) \text{ and } k \leq q(A), \\
-1 & \text{otherwise}.
\end{cases}
\]

This establishes the identity in Equation 5.10. From the definition of $\vee$ in $\mathcal{M}(X)$ the following series of equalities is obtained for each singular stratum $A \in SS(X)$:

\[
(m(p) \vee m(q))(A)(k) = m(p)(A)(k) \vee m(q)(A)(k)
\]

\[
= \begin{cases} 
\text{codim} S'_A - 1 & \text{if } k \leq p(A) \text{ or } k \leq q(A), \\
-1 & \text{otherwise}.
\end{cases}
\]

This establishes the identity in Equation 5.11. Altogether this proves the result.

Finally, let us consider the differences between the lattice of perversities and the lattice of multiperversities.

**Example 5.11.** Consider again an example where a depth 1 multicontrolled stratified space $X$ has a sole single singular stratum $A$, such that $\text{codim}(A) = 5$ and $\text{codim}(S'_A) = 3$; following on from Examples 5.4 and 5.9. Consider the upper and lower middle perversities $\overline{m}$ and $\underline{m}$, considered as multiperversities $\tau^* := m(\overline{m})$ and $\tau^* := m(\underline{m})$, as illustrated on the following two diagrams.
In the lattice $\mathcal{P}(X)$ there are no elements between $\overline{m}$ and $m$, so $\overline{m}$ covers\(^2\) $m$ in $\mathcal{P}(X)$. However in the lattice $\mathcal{M}(X)$ there are two elements between $\overline{m}$ and $m$, forming a chain
\[
m(\overline{m}) > \tau > \tau' > m(m),\]
as illustrated on the following two diagrams.

\(^2\)An element $y \in P$ is said to cover an element $x \in P$ in a partially ordered set $P$ if there is no intermediate element “in-between $x$ and $y$”, that is if there does not exist a $z \in P$ with $y > z > x$; this is equivalent to the condition that $[x, y]$ be equal to $\{x, y\}$. 

---

Figure 5.4: Diagrams representing the multiperversities $\tau^{**}$ and $\tau^*$. 

...
In this sense multiperversities can be seen as interpolating between two perversities, one of which covers the other; however not all multiperversities interpolate between two such perversities.

1.3 Operations on multiperversities

In this subsection we will consider the operations of subtraction and addition for multiperversities. These are similar to, but do not extend, the operations of subtraction and addition for perversities. For the lattice of perversities $P$ the operations of subtraction and addition are defined in [20] and we briefly recall their definitions\(^3\). The lattice $P$ can be equipped with an operation of addition, written $p \oplus q$, for two perversities $p$ and $q$, $p \oplus q$ being the greatest perversity below their pointwise sum; the pointwise sum is a function defined by $p(A) + q(A) + 1$ for each $A \in SS(X)$. Analogously the operation of subtraction, written $-$, can be defined on the lattice $P$ by replacing, directly in the preceding sentence, $p + q$ by $p - q$, greatest by least, below by above and using the pointwise subtraction $p(A) - q(A) - 1$.

\(^3\)In [20] (paragraph between Lemma 1-1 and Proposition 1-2) the operations of subtraction and addition are defined for standard perversities, but these can be defined for $P(X)$ as in Definition 5.1. Note also that the numbering in [20] is different, in particular we have two extra perversities, so the formulae for addition and subtraction there do not have the $\pm 1$ term of the formulae we use for $P(X)$. 
**Definition 5.12.** Let \( \tau, \tau' \in \mathcal{M}(X) \) be two multiperversities. For each singular stratum \( A \in \mathcal{SS}(X) \) and each integer \( k \in \mathbb{Z} \) we define the pointwise subtraction of \( \tau' \) from \( \tau \) as follows. If \( -1 \leq k \leq \text{codim}(A) - 1 \), then we define

\[
(\tau - \tau')(A)(k) := \tau(A)(k) - \tau'(A)(\text{codim}(A) - k - 1) - 1.
\]

(5.12)

To make this into a function with domain \( \mathbb{Z} \), we also define:

\[
k < -1 \implies (\tau - \tau')(A)(k) := \text{codim}(S'_A) - 1,
\]

(5.13)

\[
k > \text{codim}(A) - 1 \implies (\tau - \tau')(A)(k) := -1.
\]

(5.14)

The function defined by the above Equations 5.12, 5.13 and 5.14 is then an arrow

\[
(\tau - \tau') : \mathcal{SS}(X) \to \text{Set}(\mathbb{Z}, \mathbb{Z}),
\]

but \( (\tau - \tau') \) is not necessarily a multiperversity.

**Example 5.13.** Consider again an example where \( X \) has a sole singular stratum \( A \) and satisfies \( \text{codim}(A) = 5 \), \( \text{codim}(S') = 3 \); following on from Examples 5.4 and 5.9. Consider two multiperversities \( \tau \) and \( \tau' \) as defined on the following diagrams.

![Diagram](image)

Figure 5.6: Diagram representing the multiperversities \( \tau \) and \( \tau' \).

We can obtain the pointwise subtraction \( \tau - \tau' \), as represented on the following diagram.
The function $\tau - \tau'$ does not define a multiperversity, to see this we compute:

\[
(\tau - \tau')(4) = \tau(4) - \tau'(5 - 4 - 1) - 1,
\]
\[
= \tau(4) - \tau'(0) - 1,
\]
\[
= 1 - 2 - 1 = -2.
\]

This is not within the bounds of the efficiency conditions of Definition 5.2; the same is true of $(\tau - \tau')(3) = -2$.

**Proposition 5.14.** For all singular strata $A \in SS(X)$ and for all multiperversities $\tau, \tau' \in \mathcal{M}(X)$ the function

\[
(\tau - \tau')(A) : [-1, \text{codim}(A) - 1] \rightarrow \mathbb{Z}
\]

is a decreasing function; here $[-1, \text{codim}(A) - 1]$ denotes the set of integers $k \in \mathbb{Z}$ satisfying $-1 \leq k \leq \text{codim}(A) - 1$.

**Proof.** We must show that $(\tau - \tau')(A)(k)$ is a decreasing function in $k \in \mathbb{Z}$ for $-1 \leq k \leq \text{codim}(A) - 1$; by the definition of pointwise subtraction this is equivalent to showing

\[
\tau(A)(k + 1) - \tau'(A)(\text{codim}(A) - (k + 1) - 1) - 1 \leq \tau(A)(k) - \tau'(A)(\text{codim}(A) - k - 1) - 1.
\]

To demonstrate the validity of the above Equation 5.15, we note that

\[
\tau(A)(k + 1) - 1 \leq \tau(A)(k) - 1,
\]

and that furthermore

\[
-\tau'(A)(\text{codim}(A) - k - 2) \leq -\tau'(A)(\text{codim}(A) - k - 1),
\]
following on from the inequality
\[
\tau'(A) (\text{codim}(A) - k - 2) \geq \tau'(A) (\text{codim}(A) - k - 1).
\]

The final line above is obtained because \(\text{codim}(A) - k - 2 \leq \text{codim}(A) - k - 1\).

We also recall that, for integers \(a, b, c, d \in \mathbb{Z}\), if \(a \leq b\) and \(c \leq d\), then \(a + c \leq b + d\); the result now follows from this observation by adding Equation 5.16 to Equation 5.17, obtaining thus the validity of Equation 5.15.

**Proposition 5.15.** Let \(\tau, \tau' \in \mathcal{M}(X)\) be two multiperversities. There exists a least multiperversity \(\tau \ominus \tau'\) that completes \(\tau - \tau'\) to a multiperversity, in the sense that for all singular strata \(A \in \mathcal{S}(X)\) and integers \(k \in \mathbb{Z}\) we have
\[
(\tau - \tau')(A)(k) \leq (\tau \ominus \tau')(A)(k),
\]
and there is no smaller multiperversity which satisfies these conditions.

**Proof.** This follows from the fact that \(\mathcal{M}\) is a complete lattice, as shown in Proposition 5.10.

We can now define the dual multiperversity. We remark that subtraction of perversities and subtraction of multiperversities do not correspond, except for defining the dual perversity.

**Definition 5.16.** Let \(\tau \in \mathcal{M}\) be a multiperversity. The dual multiperversity to \(\tau\) is defined to be
\[
\mathcal{D}(\tau) := \overline{\tau} \ominus \tau \in \mathcal{M}(X).
\]

**Proposition 5.17.** The function \((\overline{\tau} - \tau)(A) : \mathbb{Z} \to \mathbb{Z}\) satisfies the inequality
\[
-1 \leq (\overline{\tau} - \tau)(A)(k) \leq \text{codim}(S'_A) - 1
\]
for all multiperversities \(\tau \in \mathcal{M}(X)\) and for all singular strata \(A \in \mathcal{S}(X)\).

**Proof.** From the efficiency condition for \(\tau\) we deduce that for all \(j \in \mathbb{Z}\) the following inequality holds
\[
-1 \leq \tau(A)(j) \leq \text{codim}(S'_A) - 1,
\]
and rearranging this inequality we have
\[
1 \geq -\tau(A)(j) \geq 1 - \text{codim}(S'_A).
\]

\(^4\)For each singular stratum \(A \in \mathcal{S}(X)\) the comparison is realised in the lattice of decreasing functions.
From the preceding line, by adding $\text{codim}(S_A')$ to both sides and writing from right to left, we obtain

$$1 \leq \text{codim}(S_A') - \tau(A)(j) \leq \text{codim}(S_A') + 1,$$

from where we finally conclude that

$$-1 \leq \text{codim}(S_A') - \tau(A)(j) - 2 \leq \text{codim}(S_A') - 1,$$

which we can rewrite as

$$-1 \leq (\text{codim}(S_A') - 1) + \tau(A)(j) - 1 \leq \text{codim}(S_A') - 1,$$

(5.18)

The result now follows by comparing the above Equation 5.18 with Equation 5.12, setting $j = \text{codim}(A) - k - 1$, where $j \leq \text{codim}(A) - 1$, so that $k \geq 0$. \qed

Directly from the above Proposition 5.17 we obtain that for the dual multiperversity the formalities of Proposition 5.15 are unnecessary, so that the dual multiperversity $\mathcal{D}(\tau)$ can be computed pointwise.

**Corollary 5.18.** The dual multiperversity $\mathcal{D}(\tau)$ can be computed using pointwise subtraction, that is

$$\mathcal{D}(\tau) = h \ominus \tau = h - \tau.$$

**Proof.** This follows directly from Proposition 5.17; heuristically this can also be deduced by considering Example 5.13. \qed

**Corollary 5.19.** Let $\bar{p} \in \mathcal{P}(X)$ be a perversity, then

$$m(h) \ominus m(\bar{p}) = \left. m(h) - m(\bar{p}) \right|_{\mathcal{M}(X)} = m(\bar{h} - \bar{p}).$$

(5.19)

**Proof.** The first equality follows directly from Proposition 5.17; the second equality is obtained directly, for this we consider the definition of $m$ in Definition 5.7 and
compute, for a fixed singular stratum $A \in \mathcal{SS}(X)$:

$$\left(\mathfrak{m}(\overline{h}) - \mathfrak{m}(\overline{p})\right)(A)(k) = \mathfrak{m}(\overline{h})(A)(k) - \mathfrak{m}(\overline{p})(A)(\operatorname{codim}(A) - k - 1) - 1$$

$$= \begin{cases} 
-1 & \text{if } \operatorname{codim}(A) - k - 1 \leq \overline{p}(A), \\
\operatorname{codim}(S'_A) - 1 & \text{if } \operatorname{codim}(A) - k - 1 > \overline{p}(A),
\end{cases}$$

$$= \begin{cases} 
-1 & \text{if } \overline{h}(A) - k \leq \overline{p}(A), \\
\operatorname{codim}(S'_A) - 1 & \text{if } \overline{h}(A) - k > \overline{p}(A),
\end{cases}$$

$$= \begin{cases} 
\operatorname{codim}(S'_A) - 1 & \text{if } k \leq (\overline{h} - \overline{p})(A), \\
-1 & \text{if } k > (\overline{h} - \overline{p})(A).
\end{cases}$$

The final line is obtained by noting that $\overline{h}(A) = \operatorname{codim}(A) - 1$ (see also Subsection 1.1), so that

$$(\overline{h} - \overline{p})(A) = \overline{h}(A) - \overline{p}(A) - 1.$$ Comparing this final line with the definition of $\mathfrak{m}$ in Definition 5.7 proves the result. □

Using the same approach, we can extend the definition of addition to multiperversities.

**Definition 5.20.** Let $\tau, \tau' \in \mathcal{M}(X)$ be two multiperversities. For each singular stratum $A \in \mathcal{SS}(X)$ and each integer $k \in \mathbb{Z}$ we define the *pointwise addition of $\tau$ to $\tau'$* as follows. If $-1 \leq k \leq \operatorname{codim}(A) - 1$, then we define

$$(\tau + \tau')(A)(k) := \tau(A)(k) + \tau'(A)(\operatorname{codim}(A) - k - 1) + 1. \quad (5.20)$$

To make this into a function with domain $\mathbb{Z}$, we also define:

$$k < -1 \implies (\tau + \tau')(A)(k) := \operatorname{codim}(S'_A) - 1, \quad (5.21)$$

$$k > \operatorname{codim}(A) - 1 \implies (\tau + \tau')(A)(k) := -1. \quad (5.22)$$

The function defined by the above Equations 5.20, 5.21 and 5.22 is then an arrow

$$(\tau + \tau') : \mathcal{SS}(X) \longrightarrow \mathcal{Set}(\mathbb{Z}, \mathbb{Z}),$$

but $(\tau + \tau')$ is not necessarily a multiperversity.

**Example 5.21.** Consider again an example where $X$ has a sole singular stratum $A$ and satisfies $\operatorname{codim}(A) = 5$ and $\operatorname{codim}(S') = 3$; following on from Examples 5.4 and 5.9. Consider two multiperversities defined by the following diagrams.
We can obtain the pointwise sum $\tau + \tau'$, as represented on the following diagram.

Notice that $\tau + \tau'$ does not define a multiperversity, for it exceeds the bounds of the efficiency conditions for a multiperversity and is furthermore not a decreasing function.

**Proposition 5.22.** Let $\tau, \tau' \in \mathcal{M}(X)$ be two multiperversities. There exists a multiperversity $\tau \oplus \tau'$ such that for all singular strata $A \in \mathcal{S}(X)$ and integers $k \in \mathbb{Z}$ we have

$$(\tau + \tau')(A)(k) \geq (\tau \oplus \tau')(A)(k),$$
and there is no greater multiperversity that satisfies these conditions.

Proof. This follows as in Proposition 5.10. □

For the multiperversities $\tau$ and $\tau'$ in Example 5.21 the multiperversity $\tau + \tau'$ is completed to the multiperversity $\tau \oplus \tau'$, as represented on the following diagram.

![Diagram representing the multiperversity $\tau \oplus \tau'$.

It is salient to note that this does not extend the addition of perversities. Further, the following result can be obtained.

**Proposition 5.23.** Let $X$ be a depth 1 multicontrolled stratified space and let $\bar{p}, \bar{q} \in \mathcal{P}(X)$ be two perversities. If $\bar{p} + \bar{q} = \bar{h}$, then

$$m(\bar{p}) \oplus m(\bar{q}) = m(\bar{h}).$$

Proof. Firstly, for $-1 \leq k \leq \text{codim}(A) - 1$ we obtain the following:

$$(m(\bar{p}) + m(\bar{q}))(A)(k) := m(\bar{p})(A)(k) + m(\bar{q})(A)(\text{codim}(A) - k - 1) + 1.$$ 

From the equality

$$p(A) + q(A) + 1 = \text{codim}_X(A) - 1,$$

we obtain that if $k \leq p(A)$, then $\text{codim}_X(A) - k - 1 \geq q(A) + 1$, so that $\text{codim}_X(A) - k - 1 > q(A)$. Furthermore, we also obtain that if $k \leq q(A)$, then $\text{codim}_X(A) - k - 1 > p(A)$. By considering the definition of $m$ in Definition 5.7, the result now follows. □
2 Product-type case

In this section we will define the main case of depth 1 multicontrolled stratified spaces that we consider, called the product-type case, in which multiperverse cohomology can be calculated; we will present the calculations for the cone and the cylinder, when these spaces satisfy the product-type assumptions. In the succeeding section we will consider a generalisation of the product-type case, called the flat-type case, where the assumptions there will permit most of the proofs in the product-type case to be extended to the flat-type case without modification.

2.1 Definition of multiperverse forms on a product-type depth 1 multicontrolled space

In this subsection we will define the product-type conditions for a depth 1 multicontrolled stratified space, consider the necessary prerequisites to define multiperverse forms, which we then define, and finally state some immediate properties of this definition.

Definition 5.24. Let $X$ be a depth 1 multicontrolled stratified space. We will say that $X$ is of product-type if, for each singular stratum $A \in \mathcal{SS}(X)$, the following numbered conditions are verified:

1. The link bundle $S_A$ of $\pi_A$ is a product:

$$S_A = F_A \times L'_{A} \times A.$$  \hspace{1cm} (5.23)

2. The link bundle $S'_A$ of $\pi'_A$ is a product:

$$S'_A = L'_{A} \times A.$$  \hspace{1cm} (5.24)

3. The restriction of $\pi_A$ to $S_A$ is the projection onto the third factor in Equation 5.23:

$$\pi_A|_{S_A} : (F_A \times L'_{A} \times A) \rightarrow A.$$  

4. The restriction of $\pi'_A$ to $S'_A$ is the projection onto the second factor in Equation 5.24:

$$\pi'|_{S'_A} : (L'_{A} \times A) \rightarrow A.$$  

5. The restriction of $\mu_A$ to $S_A$ is the projection onto the factor $L'_{A} \times A$ in Equation 5.24:

$$\mu_A|_{S_A} : (F_A \times (L'_{A} \times A)) \rightarrow (L'_{A} \times A).$$
6. There exists a finite good cover of $A$ and the manifolds $S_A$, $S'_A$ and $F_A$ are oriented, each with a fixed compatible orientation.

Throughout this subsection we will let $X$ be a product-type depth 1 multicontrolled stratified space, as in the above Definition 5.24. For a fixed singular stratum $A \in \mathcal{S}(X)$, from the product structure on the link bundle $S_A$, as in Equation 5.23, there exists a canonical decomposition on tangent spaces, which induces, for $x \in F_A$, $y \in L'_A$ and $z \in A$, the canonical decomposition on cotangent spaces:

$$T^*_T(x,y,z)(S_A) \cong (T^*_x F_A) \times (T^*_y L'_A) \times (T^*_z A).$$

(5.25)

By considering the $k$-th exterior product, for $k \in \mathbb{Z}_{\geq 0}$, the above Equation 5.25 further induces the following decomposition on the $k$-form bundle:

$$\Lambda^k (T^*_T(x,y,z)(S_A)) \cong \bigoplus_{i+j+h=k} (\Lambda^i (T^*_x F_A) \times \Lambda^j (T^*_y L'_A) \times \Lambda^h (T^*_z A)).$$

(5.26)

Moreover, the decomposition in the above Equation 5.26 itself entails that the complex of smooth differential forms on $S_A$ satisfies the following decomposition:

$$\Omega^k(S_A) = \bigoplus_{i+j+h=k} \Gamma(S_A, (\Lambda^i F_A \wedge \Lambda^j L'_A \wedge \Lambda^h A)).$$

In the above, we denote the $(i, j, h)$-th component of $\Omega^k(S_A)$ as

$$\Omega^{i,j,h}(S_A) = \Gamma(S_A, (\Lambda^i F_A \wedge \Lambda^j L'_A \wedge \Lambda^h A)),$$

(5.27)

from where $\Omega^k(S_A) = \bigoplus_{i+j+h=k} \Omega^{i,j,h}(S_A)$. For the $(i, j, h)$-th component, as in the above Equation 5.27, we will say that a form contained in $\Omega^{i,j,h}(S_A)$ has total degree $k = i + j + h$, has total fibre degree $i + j = k - h$, has lower fibre degree $j$, and has upper fibre degree $i$. We remark that in the preceding paragraph, one can replace $S_A$ with

$$S_A \cap \pi_A^{-1}[U] = F_A \times L'_A \times U,$$

for an open subset $U \subseteq A$ of the singular stratum $A$, obtaining thus a decomposition $\Omega^{i,j,h}(S_A \cap \pi_A^{-1}[U])$.

The use of the grading of the above paragraph is standard and appears in [1] (Section 6), for example. Our next aim is to obtain a decomposition for the exterior derivative

$$d_{S_A} : \Omega^k(S_A) \rightarrow \Omega^{k+1}(S_A)$$

according to the tri-grading above.

**Lemma 5.25.** Let $X$ be a depth 1 product-type multicontrolled stratified space.
There exists a decomposition of the exterior derivative $d_{S_A}$ acting on $\Omega^{i,j,h}(S_A)$ into components

$$d_{S_A} = d_{F_A} + (-1)^i \cdot d_{L'_A} + (-1)^{i+j} \cdot d_A. \quad (5.28)$$

In the right hand side of the above Equation 5.28 each component acts as follows:

$$d_{F_A} : \Omega^{i,j,h}(S_A) \rightarrow \Omega^{i+1,j,h}(S_A). \quad (5.29)$$

$$d_{L'_A} : \Omega^{i,j,h}(S_A) \rightarrow \Omega^{i,j+1,h}(S_A). \quad (5.30)$$

$$d_A : \Omega^{i,j,h}(S_A) \rightarrow \Omega^{i,j,h+1}(S_A). \quad (5.31)$$

Consequently the exterior derivative $d_{S_A}$ acts as

$$d_{S_A} : \Omega^{i,j,h}(S_A) \rightarrow \Omega^{i+1,j,h}(S_A) \oplus \Omega^{i,j+1,h}(S_A) \oplus \Omega^{i,j,h+1}(S_A).$$

Proof. Another proof of this result can be found in Lemma 5.59, where we consider the more general setting of a flat-type depth 1 multicontrolled stratified space. For this proof, let there be defined three coordinate patches $U \subseteq F_A$, $V \subseteq L'_A$ and $W \subseteq A$; respectively with coordinates $u = (u_1, \ldots, u_f) \in \mathbb{R}^f$, $v = (v_1, \ldots, v_{l'}) \in \mathbb{R}^{l'}$ and $w = (w_1, \ldots, w_a) \in \mathbb{R}^a$; in this $f = \text{dim}(F_A)$, $l' = \text{dim}(L'_A)$ and $a = \text{dim}(A)$. The $k$-form $\omega$, given by the expression

$$\omega := f(u,v,w) \wedge du_I \wedge dv_J \wedge dw_H, \quad (5.32)$$

represent a basis $i+j+h=k$-form in $\Omega^{i,j,h}(S_A)$, where $|I| = i$, $|J| = j$ and $|H| = h$. Each component in the decomposition of $d_{S_A}$, as in Equation 5.28, acting on the form $\omega$ in Equation 5.32, can be written as follows:

$$d_{F_A}(f \wedge du_I \wedge dv_J \wedge dw_H) := \sum_{i=1}^{f} \frac{\partial f}{\partial u_i} \wedge du_i \wedge du_I \wedge dv_J \wedge dw_H. \quad (5.33)$$

$$d_{L'_A}(f \wedge du_I \wedge dv_J \wedge dw_H) := \sum_{j=1}^{l'} \frac{\partial f}{\partial v_j} \wedge dv_j \wedge du_I \wedge dv_J \wedge dw_H. \quad (5.34)$$

$$d_A(f \wedge du_I \wedge dv_J \wedge dw_H) := \sum_{h=1}^{a} \frac{\partial f}{\partial w_h} \wedge dw_h \wedge du_I \wedge dv_J \wedge dw_H. \quad (5.35)$$

From this we deduce the appropriate forms of the components, as in Equations 5.29, 5.30 and 5.31; the conclusion of Equation 5.28 follows directly from the local coordinate description.
Remark 5.26. The two sign changes, the \((-1)^i\), and the \((-1)^{i+j}\), which occur in the above Equation 5.28 are in consequence of the convention that the order of the operations is \(F_A, L'_A\) and then \(A\), from left to right. Note that the content of the above Lemma 5.25 remains unchanged if we were to replace \(S_A\) by \(S_A \cap \pi_A^{-1}[U]\), for some open subset \(U \subseteq A\) of the singular stratum \(A\).

Remark 5.27. Let \(\omega \in \Omega^k(S_A)\) be a \(k\)-form and let \(Z_i \in T(S_A)\) be a collection of smooth vector fields on \(S_A\). The exterior derivative \(d_{S_A}\) satisfies

\[
(d_{S_A}\omega)(Z_1, \ldots, Z_{k+1}) = \sum_{c=1}^{k+1} (-1)^{c-1} Z_c \omega(Z_1, \ldots, \hat{Z}_c, \ldots, Z_{k+1}) + \sum_{1 \leq c < d \leq k+1} (-1)^{c+d} \omega([Z_c, Z_d], Z_1, \ldots, \hat{Z}_c, \ldots, \hat{Z}_d, \ldots, Z_{k+1}).
\]  

(5.36)

In the above Equation 5.36 the notation \(\hat{Z}_c\) and \(\hat{Z}_d\) denotes that these vector fields are omitted, while \([Z_c, Z_d]\) denotes the standard commutator of \(Z_c\) and \(Z_d\). In this setting the components \(d_{F_A}, d_{L'_A}\) and \(d_A\), as in Equation 5.28, have the following presentations:

\[
d_{F_A}(\omega)(Z_1, \ldots, Z_i, Z_{i+1}; Z'_1, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_{h+1}) = \sum_{c=1}^{i+1} (-1)^{c-1} Z'_c \omega(Z_1, \ldots, \hat{Z}'_c, \ldots, Z_{i+1}; Z'_1, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_{h+1}) + \sum_{1 \leq c < d \leq i+1} (-1)^{c+d} \omega([Z'_c, Z'_d], Z_1, \ldots, \hat{Z}'_c, \ldots, \hat{Z}'_d, \ldots, Z_{i+1}; Z'_1, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_{h+1})
\]  

(5.37)

\[
d_{L'_A}(\omega)(Z_1, \ldots, Z_i; Z'_1, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_{h+1}) = \sum_{c=1}^{j+1} (-1)^{c-1} Z'_c \omega(Z_1, \ldots, \hat{Z}'_c, \ldots, Z'_1, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_{h+1}) + \sum_{1 \leq c < d \leq j+1} (-1)^{c+d} \omega([Z'_c, Z'_d], Z_1, \ldots, \hat{Z}'_c, \ldots, \hat{Z}'_d, \ldots, Z_{j+1}; Z''_1, \ldots, Z''_{h+1})
\]  

(5.38)

\[
d_A(\omega)(Z_1, \ldots, Z_i; Z'_1, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_{h+1}) = \sum_{c=1}^{h+1} (-1)^{c-1} Z''_c \omega(Z_1, \ldots, \hat{Z}''_c, \ldots, Z'_1, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_{h+1}) + \sum_{1 \leq c < d \leq h+1} (-1)^{c+d} \omega([Z''_c, Z''_d], Z_1, \ldots, \hat{Z}''_c, \ldots, \hat{Z}''_d, \ldots, Z_{h+1})
\]  

(5.39)
This approach will be used in the flat-type case, for this see the proof of Lemma 5.59 in Section 3 of the current chapter.

With the appropriate notation in place, we can define the set of \(\tau\)-multiperverse forms for a depth 1 product-type multicontrolled stratified space \(X\) and a multiperversity \(\tau \in M(X)\).

**Definition 5.28.** Let \(\omega \in \Omega^k(X)\) be a liftable form on a product-type depth 1 multicontrolled stratified space \(X\) and let \(\tau \in M(X)\) be a multiperversity for \(X\). We will say that the liftable form \(\omega\) is \(\tau\)-admissible if, for each singular stratum \(A \in SS(X)\), we have

\[
R_A(\omega|_{\text{reg}(T_A)}) \in \bigoplus_{i \leq \tau(A)(k-h)} \Omega^{i,j,h}(S_A).
\]  

(5.40)

In the above Equation 5.40 the integer \(i\) is the upper fibre degree, \(j\) is the lower fibre degree and \(h\) is the base degree; \(i + j\) is the total fibre degree and \(k = i + j + h\) is the total degree. Furthermore, we will say that the liftable form \(\omega\) is \(\tau\)-multiperverse if both \(\omega\) and \(d\omega\) are \(\tau\)-admissible. We write \(M\Omega^\tau_\bullet(X)\) for the set of \(\tau\)-multiperverse forms on \(X\).

It is immediate that the set of \(\tau\)-multiperverse forms is a cochain complex \(M\Omega^\tau_\bullet(X)\) under the exterior derivative inherited from \(\Omega^\tau_\bullet(X)\), which is just the exterior derivative on \(\text{reg}(X)\). The cohomology of the cochain complex \(M\Omega^\tau_\bullet(X)\) is called the \(\tau\)-multiperverse cohomology of \(X\), where we write

\[
MH^\tau_k(X) := H^k(M\Omega^\tau_\bullet(X)).
\]

**Remark 5.29.** Where intersection forms are defined from liftable or smooth forms, as in [11](B.III.2) or [26](Definition 3.2), the definitions are local to each point on the singular stratum. In the setting of our Definition 5.28, this type of definition would be as follows: for each singular stratum \(A \in SS(X)\), and for each \(a \in A\), there exists some distinguished open subset \(W := W_{U,s}\) of \(A\), with \(a \in W_{U,s}\), such that

\[
R_{U,s}(\omega|_{\text{reg}(W)}) \in \bigoplus_{i \leq \tau(A)(k-h)} \Omega^{i,j,h}(S_A \cap \pi_A^{-1}[U]).
\]  

(5.41)

However, because we have assumed that \(A\) has a finite cover, and because

\[
\widetilde{\omega|_{\text{reg}(W)}} = \tilde{\omega}|_{\theta_X^{-1}[\text{reg}(W)]},
\]

this definition would be equivalent to Definition 5.28 above.
If a multicontrolled space $X$ has a sole singular stratum $A$, in other words if $SS(X) = \{A\}$, then we will adopt the following notation. To simplify the presentation of formulae for this reduced case we will not write the subscripts corresponding to $A$, so that the following notation is maintained: $S := S_A$, where this space is the link bundle of $A$; $S' := S'_A$; $F := F_A$, the fibre of $\mu := \mu_A$; $L = L_A$, the fibre of $\pi = \pi_A$; and $L' = L'$, the fibre of $\pi' = \pi'_A$. Furthermore, when working with the lattice of multiperversities $M(X)$ for this reduced case, we will adopt the following notation. For a multiperversity $\tau \in M(X)$ we will write $\tau(k)$ for $\tau(A)(k)$, thus considering a multiperversity $\tau$ as fixed by the element $\tau(A) \in (\mathbb{Z}^{op}, \mathbb{Z})$.

**Example 5.30.** Consider again an example where the depth 1 multicontrolled stratified space $X$ has a sole singular stratum $A$ and satisfies $\text{codim}(A) = 5$, $\text{codim}(S') = 3$; following on from Examples 5.4 and 5.9. We make a further assumption that $\dim(\text{reg}(X)) = \dim(S) + 1$, from which we derive:

\[
\begin{align*}
\text{codim}(A) &= \dim(S) + 1 - \dim(A) \\
&= \dim(F) + \dim(L') + 1, \\
\text{codim}(S') &= \dim(S) + 1 - \dim(S') \\
&= \dim(F) + 1.
\end{align*}
\]

This implies that $\dim(F) = 2$ and $\dim(L') = 2$. Let $\tau$ be the multiperversity specified as follows:

\[
\tau(A)(k) = \begin{cases} 
2 & \text{if } k < 0 \\
2 & \text{if } k = 0 \\
1 & \text{if } k = 1 \\
1 & \text{if } k = 2 \\
0 & \text{if } k = 3 \\
-1 & \text{if } k = 4 \\
-1 & \text{if } k > 4 
\end{cases}
\]

This multiperversity can also represented on the following diagram.

![Diagram representing the multiperversity $\tau$.](image-url)
The integer \((i,j)\)-indices, where \(0 \leq i, j \leq 2\), which satisfy the identity

\[ i \leq \tau(k - h) = \tau(i + j), \]

as involved in Definition 5.28, can be represented on the following diagram; in this diagram the numbers within the shaded squares represent \(k = i + j\).

![Diagram representing the multiperversity \(\tau\) on the fibre degrees of \(F\) and \(L'\).](image1)

Consider the following observations. The above Figure 5.11 contains more information than the resulting condition on the \((i, j)\)-indices in Figure 5.12. This can be explained by noting that certain aspects of a multiperversity have no topological meaning within the scope of Definition 5.28; we can illustrate this on the following diagram.

![Diagram representing the topologically relevant part of the multiperversity \(\tau\).](image2)
In the above Diagram 5.13 the lightly shaded part has topological meaning in Definition 5.28; the hatched part is the intersection of the multiperversity $\tau$ of Figure 5.11 and the lightly shaded part, that is the part of the multiperversity $\tau$ which has topological meaning within the scope of Definition 5.28 above$^5$.

**Example 5.31.** Consider again an example where $X$ has a sole singular stratum $A$ and satisfies $\text{codim}(A) = 5$, $\text{codim}(S') = 3$; following on directly from the above Example 5.30 and Examples 5.4, 5.9. As in Example 5.30, we make a further assumption that $\dim(\text{reg}(X)) = \dim(S) + 1$, so that $\dim(F) = 2$ and $\dim(L') = 2$. Consider the highest multiperversity $\mathbf{m}(\mathbf{h})$, as represented on the following diagram.

![Diagram](image)

Figure 5.14: Diagram representing the multiperversity $\mathbf{m}(\mathbf{h})$.

The integer $(i, j)$-indices, where $0 \leq i, j \leq 2$, which satisfy the identity

$$i \leq \mathbf{m}(\mathbf{h})(k - h) = \mathbf{m}(\mathbf{h})(i + j),$$

as involved in Definition 5.28, can be represented on the following diagram; in this diagram the numbers within the shaded squares represent $k = i + j$.

$^5$Note that in the above Figure 5.13 we have not included the medium grey guide shading, as in the other diagrams of this type that appear in this document; this is done to indicate that there can be a topological difference between $\tau(0) = -1$ and $\tau(0) = 0$ within the scope of Definition 5.28.
Figure 5.15: Diagram representing the multiperversity $m(\bar{h})$ on fibre degrees of $F$ and $L'$.

Now consider the bottom multiperversity $m(\bar{l})$, as represented on the following diagram.

Figure 5.16: Diagram representing the multiperversity $m(\bar{l})$.

The integer $(i,j)$-indices, where $0 \leq i, j \leq 2$, which satisfy the identity

$$ i \leq m(\bar{l})(k - h) = m(\bar{l})(i + j), $$

as involved in Definition 5.28, can be represented on the following diagram.
Finally, consider the multiperversity \( m(\bar{p}) \), where \( \bar{p}(A) = 2 \), as represented on the following diagram.

The integer \((i, j)\)-indices, where \( 0 \leq i, j \leq 2 \), which satisfy the identity

\[
i \leq m(\bar{p})(k - h) = m(\bar{p})(i + j),
\]

as involved in Definition 5.28, can be represented on the following diagram.
As $\mathcal{M}(X)$ is a lattice, and in particular a partially ordered set, we can note the following relationship between the order relation on multiperversities and the inclusion of the complexes of multiperverse forms.

**Proposition 5.32.** Let $\tau, \tau' \in \mathcal{M}(X)$ be two multiperversities. If $\tau \leq \tau'$ holds, then there is an inclusion of complexes $M\Omega^*_\tau(X) \hookrightarrow M\Omega^*_{\tau'}(X)$.

**Proof.** This follows directly from the implication

$$i \leq \tau(A)(k-h) \implies i \leq \tau'(A)(k-h),$$

valid for all singular strata $A \in SS(X)$, combined with the content of above Definition 5.28. \qed

To aid the presentation of the computations which follow, we define an intermediary complex; this complex will be used in the computation of the multiperverse cohomology of a cone.

**Definition 5.33.** Let $\tau \in \mathcal{M}(X)$ be a multiperversity and let $S_A$ be the link bundle of a singular stratum $A \in SS(X)$ of a depth 1 product-type multicontrolled stratified space $X$. Define $M\Omega^k_{\tau}(S_A)$ to be the set of forms $\omega \in \Omega^k(S_A)$ which satisfy:

$$\omega \in \left( \bigoplus_{i \leq \tau(A)(k-h)} \Omega^{i,j,h}(S_A) \right) \text{ and } d_{S_A}(\omega) \in \left( \bigoplus_{i' \leq \tau(A)((k+1)-h')} \Omega^{i',j',h'}(S_A) \right).$$

This extends to a cochain complex $M\Omega^*_\tau(S_A)$, with the coboundary operator the restriction of the exterior derivative $d_{S_A}$.

**Remark 5.34.** Concerning the above Definition 5.33. For clarity, if $\omega \in \Omega^{i,j,h}(S_A)$, $i \leq \tau(A)(k-h)$ and $i+j+h = k$, then $d_{S_A}(\omega) \in \Omega^{i',j',h'}(S_A)$, for some indices $i', j'$, and $h'$, which satisfy $i' \leq \tau((k+1)-h)$ and $i'+j'+h' = k+1$. Note that without the condition on $d_{S_A}(\omega)$ this would not define a complex. An analogous remark applies to Definition 5.28.

**Proposition 5.35.** Let $\tau \in \mathcal{M}(X)$ be a multiperversity for a depth 1 product-type multicontrolled stratified space $X$ and $\tau' \in \mathcal{M}(Y)$ a multiperversity for a depth 1 product-type multicontrolled stratified space $Y$. Consider a morphism of complexes

$$f : \Omega^*_\tau(X) \rightarrow \Omega^*_\tau(Y).$$

Then $f$ descends to $\tau$-multiperverse forms, yielding

$$f : M\Omega^*_\tau(X) \rightarrow M\Omega^*_\tau(Y),$$
if and only if for all $\tau$-multiperverse forms $\omega \in M\Omega^k_\tau(X)$ the form $f(\omega) \in \Omega^1_\tau(Y)$ is $\tau'$-admissible.

Proof. It suffices to show that $d(f(\omega))$ is $\tau'$-admissible, assuming that $\omega$ is $\tau$-multiperverse and $f(\omega)$ is $\tau'$-admissible. Consider for each singular stratum $B \in \mathcal{SS}(Y)$ the expression

$$R_B(d(f(\omega))|_{\text{reg}(T_B)}) = R_B(f(d(\omega))|_{\text{reg}(T_B)}).$$

The result is obtained directly from the observation that $d(\omega)$ is $\tau$-multiperverse, for $d(\omega)$ is $\tau$-admissible and $dd\omega = 0$ is $\tau$-admissible also; from this it follows that $d(f(\omega)) = f(d(\omega))$ is $\tau'$-admissible. \qed

### 2.2 Calculation of $\tau$-multiperverse cohomology for a product-type cone

In this subsection we will compute the $\tau$-multiperverse cohomology of a product-type cone, wherein Definition 5.28 can be applied. The computation will proceed via the intermediary complex of Definition 5.33.

Let $cS$ be the depth 1 multicontrolled fibered cone over the depth 0 fibering data $\mu : S \rightarrow S' = L'$, where $S = L' \times F$ is a product with $S' = L'$ a compact smooth manifold and $F$ a compact smooth manifold also; we refer to Definition 4.6 for the definition of the multicontrolled fibered cone and to Definition 4.5 for the definition of the fibering data. Let us assume that the space $cS$ is a product-type depth 1 multicontrolled stratified space, so that $cS$ is a multicontrolled space satisfying the conditions of Definition 5.24. As the sole singular stratum of $cS$ is $\mathbf{pt} \in \mathcal{SS}(cS)$, and $\mathbf{pt}$ is 0-dimensional, for the purpose of Definition 5.28 it will suffice to consider, in lieu of the threefold decomposition $\Omega^{i,j,k}(S)$, simply the twofold decomposition $\Omega^{i,j}(S) := \Omega^{i,j,0}(S)$. In this setting the content of Equation 5.28 takes the form

$$d_S = d_F + (-1)^i \cdot d_{L'},$$

for $d_{\mathbf{pt}}$ acts as the zero map.

We begin the computation of $MH^*_\tau(cS)$ by recalling some preliminaries. Firstly, recall that the deshirring $\mathcal{D}(cS)$ of the cone $cS$ is the cylinder $S \times \mathbb{R}$. Let

$$p_S : S \times \mathbb{R} \rightarrow S$$

be the natural projection. To simplify notation, for $\omega \in M\Omega^k_\tau(S)$, and for $p_S^*(\omega) \in M\Omega^k_\tau(cS)$, we will interchangeably write:

1. The form $\omega$ is $\tau$-multiperverse.
2. The lift $p^*_S(\omega)$ is $\tau$-multiperverse.

3. The restriction of the lift $p^*_S(\omega)$ to $\theta^{-1}_c[S \cap \text{reg}(cS)]$ defines a $\tau$-multiperverse form.

This reduces the notational overhead in working with the intermediary complex $M\Omega^k_{\tau}(S)$, which we will use throughout this subsection. By translating the definition of $\tau$-admissibility from the trigrading $\Omega^{i,j,h}(S)$ to the bigrading $\Omega^{i,j}(S)$, we obtain the following description of the $\tau$-multiperverse forms on $cS$.

**Proposition 5.36.** Let $\tau \in M(cS)$ be a multiperversity for $cS$ and let $\omega^{i,j} \in \Omega^{i,j}(S)$ be a $k$-form, where $i+j = k$. Then $p^*_S(\omega^{i,j}) \in \Omega^k_l(\mathcal{P}(cX))$ is the lift of a $\tau$-admissible form if and only if $i \leq \tau(k)$. Furthermore, the following two numbered statements are verified:

1. Let $p^*_S(\omega^{i,j})$ be the lift of a $\tau$-admissible form and let

$$d_S(\omega^{i,j}) = \eta^{i+1,j} + \eta^{i,j+1},$$

with $\eta^{i+1,j} \in \Omega^{i+1,j}(S)$, and with $\eta^{i,j+1} \in \Omega^{i,j+1}(S)$; then $p^*_S(\omega^{i,j})$ is the lift of a $\tau$-multiperverse form if and only if $d_S(\omega^{i,j})$ is $\tau$-admissible if and only if $i+1 \leq \tau(k+1)$. If $\eta^{i+1,j} = 0$, then the condition is weakened to $i \leq \tau(k+1)$.

2. Let $p^*(\omega^{i,j})$ be the lift of a $\tau$-admissible form and let

$$d_S(\omega^{i,j}) = \eta^{i+1,j} + \eta^{i,j+1},$$

with $\eta^{i+1,j} \in \Omega^{i+1,j}(S)$, and with $\eta^{i,j+1} \in \Omega^{i,j+1}(S)$. If $j = \tau(k)$ and if $p^*(\omega^{i,j})$ is the lift of a $\tau$-multiperverse form, then $d_F(\omega^{i,j}) = 0$, that is $\eta^{i+1,j} = 0$. If further $\tau(k+1) < \tau(k)$, then $d_L(\omega^{i,j}) = 0$, that is $\eta^{i,j+1} = 0$. If $\eta^{i,j+1} = 0$.

**Proof.** The statement that $p^*_S(\omega^{i,j})$ is the lift of a $\tau$-admissible form if and only if $i \leq \tau(k)$ follows directly from Definition 5.28 and the definition of $\Omega^{i,j}(S)$ above, as does the numbered part 1, where it is helpful to consider the following diagram.

$$
\begin{array}{c}
\eta^{i,j+1} \\
\downarrow_{d_L} \\
\omega^{i,j} \\
\downarrow_{d_F} \\
\eta^{i+1,j}
\end{array}
$$

In the above diagram $\omega^{i,j} \in \Omega^{i,j}(S)$, $\eta^{i+1,j} \in \Omega^{i+1,j}(S)$ and $\eta^{i,j+1} \in \Omega^{i,j+1}(S)$. Further, we illustrate the situation which arises in part 1 on the following two Figures 5.20 and 5.21; in these two diagrams the shaded grey denotes the $(i,j)$-indices which are required to be $\tau$-admissible.
For the numbered part 2 we proceed as follows. If we assume that $\eta^{i+1,j} \neq 0$, then, from $\tau(k + 1) \leq \tau(k)$, we would obtain that $i + 1 \leq \tau(k + 1) \leq \tau(k) = i$; this statement implies that $i + 1 \leq i$ and this is a contradiction. The above Figure 5.21 also illustrates this situation. If additionally $\tau(k + 1) < \tau(k)$ holds, then similarly, assuming that $\eta^{i,j+1} \neq 0$, we would obtain the statement that $i \leq \tau(k + 1) < \tau(k) = i$; this statement implies that $i < i$ and this is again a contradiction. This last part is illustrated on the following Figure 5.22.

Fix a Riemannian metric $g_F$ on $F$, with the orientation of $F$ fixed as in Definition 5.24. The metric $g_F$ defines the linear operator $*_F$ on the complex of smooth differential forms $\Omega^*(F)$; this operation is known as the Hodge star

$$*_F : \Omega^k(F) \longrightarrow \Omega^{f-k}(F).$$  \hspace{1cm} (5.42)
Further, for all \( k \in \mathbb{Z}_{\geq 0} \), the Hodge star \( *_F \) satisfies the following three numbered properties ([27], pp.19-22):

1. For all \( \omega, \omega' \in \Omega^k(F) \) the identity:
   \[
   \omega \wedge *_F(\omega') = *_F(\omega) \wedge \omega'.
   \]

2. For all \( \omega \in \Omega^k(F) \) the identity:
   \[
   *_F *_F(\omega) = (-1)^{k(d(F) - k)} \wedge \omega = (-1)^{k(f-k)} \wedge \omega.
   \]

3. For all \( \omega \in \Omega^k(F) \), if the identity \( \omega \wedge *_F(\omega) = 0 \) holds, then \( \omega = 0 \).

These three numbered properties imply a local description for \( *_F \); for this consider an oriented coordinate patch \( U \subseteq F \) with coordinates \( \{u_i\}_{i=1}^f \), as used in Equation 5.32. For an integer \( k \in \mathbb{Z} \) satisfying \( k \in [1, f] \), we can consider an ordered set of indices

\[
1 \leq i_1 < i_2 < \ldots < i_{k-1} < i_k \leq f,
\]

which we call \( I := (i_1, \ldots, i_k) \), this corresponding to a permutation \( \sigma \in \text{Sym}(f) \) of \( (1, \ldots, f) \); the full permutation is written \( (i_1, \ldots, i_f) \). It suffices to consider the expression

\[
(du_{i_1} \wedge \ldots \wedge du_{i_k}) \wedge *_F (du_{i_1} \wedge \ldots \wedge du_{i_k}) = ||du_{i_1} \wedge \ldots \wedge du_{i_k}||^2 \cdot \sqrt{\det ((g_F)_{ij})} \cdot du_{i_1} \wedge \ldots \wedge du_f,
\]

in order to derive that

\[
*_{F} (du_{i_1} \wedge \ldots \wedge du_{i_k})
= ||du_{i_1} \wedge \ldots \wedge du_{i_k}||^2 \cdot \text{sgn}(\sigma) \cdot \sqrt{\det ((g_F)_{ij})} \cdot du_{i_{k+1}} \wedge \ldots \wedge du_{i_f} \quad (5.43)
= G_I(u) \cdot du_{i_{k+1}} \wedge \ldots \wedge du_{i_f}. \quad (5.44)
\]

In the above Equation 5.43 the notation \( \text{sgn}(\sigma) \) denotes the sign of the permutation, this evaluates to 1 if the permutation is even and \(-1\) if the permutation is odd. In the above Equation 5.44 we have written

\[
G_I(u) := ||du_{i_1} \wedge \ldots \wedge du_{i_k}||^2 \cdot \text{sgn}(\sigma) \cdot \sqrt{\det ((g_F)_{ij})}.
\]

The next step is to extend \( *_F \) to \( \Omega^\bullet(S) \). For this we fix oriented local coordinate
patches \( U \subseteq F \) and \( V \subseteq L' \), and consider a base form \( \omega \) in \( \Omega^{i,j}(S) \subseteq \Omega^k(S) \):

\[
\omega := f(u,v) \wedge du_I \wedge dv_J,
\]

with \(|I| = i, |J| = j\), and with \( i + j = k \). The extension of \(*_F\) to \( \Omega^k(S) \) is defined on this \( \omega \) and extended linearly:

\[
*_F \left( f(u,v) \wedge du_I \wedge dv_J \right) := f(u,v) \wedge *(du_I) \wedge dv_J.
\]  (5.46)

Using the content of Equation 5.43, for a permutation \( \sigma \in \text{Sym}(f) \) of \((1,\ldots,f)\), with the notation as in Equation 5.43, we can rewrite the contents of Equation 5.46:

\[
*_F \left( f(u,v) \wedge du_{i_1} \wedge \ldots \wedge du_{i_k} \wedge dv_J \right)
= (f(u,v) \cdot G_{1}(u)) \wedge (du_{i_k+1} \wedge \ldots \wedge du_{i_f}) \wedge dv_J.
\]  (5.47)

It is important to note that in the above Equation 5.47 the function \( G_{1}(u) \) is dependent only on the \( u \) coordinates. Globally this extension of \(*_F\) defines an arrow

\[
*_F : \Omega^{i,j}(S) \longrightarrow \Omega^{f-i,j}(S).
\]

From this, by using the extension of \(*_F\) to \( \Omega^{i,j}(S) \), we can extend \( \delta_F \) to \( \Omega^{i,j}(S) \):

\[
\delta_F = (-1)^{(f+i+f+1)} \cdot *_F d_F *_F = (-1)^{(f(i+1)+1)} \cdot *_F d_F *_F.
\]  (5.48)

We obtain an arrow

\[
\delta_F : \Omega^{i,j}(S) \longrightarrow \Omega^{i-1,j}(S).
\]

Next we will consider the commutativity of \(*_F\) with \( d_{L'} \), deriving that this holds up to a sign term. From this we derive the commutativity of \( \delta_F \) and \( d_{L'} \), which also holds up to a sign term.

**Lemma 5.37.** For a depth 1 multicontrolled product-type cone \( cS \), with link bundle \( S = F \times L' \), the following three numbered identities are verified:

1. \( d_F d_{L'} = (-1) \cdot d_{L'} d_F \).

2. \( d_{L'} *_F = (-1)^f \cdot *_F d_{L'} \).

3. \( d_{L'} \delta = (-1)^{f+1} \cdot \delta d_{L'} \).

**Proof.** It suffices to verify this in the oriented local coordinate patches given by the open subsets \( U \subseteq F \) and \( V \subseteq L' \), with coordinates \( \{u_i\}_{i=1}^f \) and \( \{v_j\}_{j=1}^f \) respectively.
For identity 1 we reproduce here the content of Equations 5.33 and 5.34, for \( d_F \) and \( d_L \) acting on \( f(u, v) \wedge du_I \wedge dv_J \):

\[
d_F(f(u, v) \wedge du_I \wedge dv_J) = \sum_{i=1}^{f} \left( \frac{\partial f}{\partial u_i} \wedge du_i \right) \wedge du_I \wedge dv_J, \tag{5.49}
\]

\[
d_L(f(u, v) \wedge du_I \wedge dv_J) = \sum_{j=1}^{l'} \left( \frac{\partial f}{\partial v_j} \wedge dv_j \right) \wedge du_I \wedge dv_J. \tag{5.50}
\]

Using the above Equations 5.49 and 5.50 we compute the two expressions for \( d_Fd_L \) and \( d_Ld_F \) applied to \( f(u, v) \wedge du_I \wedge dv_J \):

\[
d_Fd_L(f(u, v) \wedge du_I \wedge dv_J) = d_F \left( \sum_{j=1}^{l'} \left( \frac{\partial f}{\partial v_j} \wedge dv_j \right) \wedge du_I \wedge dv_J \right)
= \sum_{i=1}^{f} \sum_{j=1}^{l'} \left( \frac{\partial f}{\partial u_i} \wedge du_i \wedge dv_j \right) \wedge du_I \wedge dv_J
= (-1)^{f} \sum_{i=1}^{f} \sum_{j=1}^{l'} \left( \frac{\partial f}{\partial v_j} \wedge dv_j \wedge du_i \right) \wedge du_I \wedge dv_J, \tag{5.51}
\]

\[
d_Ld_F(f(u, v) \wedge du_I \wedge dv_J) = d_L \left( \sum_{i=1}^{f} \left( \frac{\partial f}{\partial u_i} \wedge du_i \right) \wedge du_I \wedge dv_J \right)
= \sum_{j=1}^{l'} \sum_{i=1}^{f} \left( \frac{\partial f}{\partial u_i} \wedge du_i \wedge dv_j \right) \wedge du_I \wedge dv_J
= \sum_{i=1}^{f} \sum_{j=1}^{l'} \left( \frac{\partial f}{\partial v_j} \wedge dv_j \wedge du_i \right) \wedge du_I \wedge dv_J. \tag{5.52}
\]

By comparing the expression in the above Equation 5.51 with the expression in the above Equation 5.52, we obtain identity 1.

For identity 2 we compute the expressions for \( d_L*F \) and \( *Fd_L \) applied to

\[
f(u, v) \wedge du_{i_1} \wedge \ldots \wedge du_{i_k} \wedge dv_J,
\]

where \( i_1, \ldots, i_k \) is an ordered set of indices corresponding to a permutation \( \sigma \):

\[
d_L*F \left( f(u, v) \wedge du_{i_1} \wedge \ldots \wedge du_{i_k} \wedge dv_J \right)
= d_L \left( f(u, v) \cdot G_1(u) \right) \wedge (du_{i_{k+1}} \wedge \ldots \wedge du_{i_l}) \wedge dv_J
= \sum_{j=1}^{l'} \left( \frac{\partial f}{\partial v_j} \cdot G_1(u) \wedge dv_j \right) \wedge (du_{i_{k+1}} \wedge \ldots \wedge du_{i_l}) \wedge dv_J, \tag{5.53}
\]
\[ *_F d_{L'} ( f(u, v) \wedge (du_{i_1} \wedge \ldots \wedge du_{i_k}) \wedge dv_j ) \]

\[ = *_F \left( \sum_{j=1}^{p'} \left( \frac{\partial f}{\partial v_j} \wedge dv_j \right) \wedge (du_{i_1} \wedge \ldots \wedge du_{i_k}) \wedge dv_j \right) \]

\[ = *_F \left( (-1)^k \cdot \sum_{j=1}^{p'} \frac{\partial f}{\partial v_j} \cdot G_I(u) \wedge (du_{i_{k+1}} \wedge \ldots \wedge du_{i_f}) \wedge dv_j \right) \]

\[ = (-1)^k \cdot (-1)^{f-k} \cdot \sum_{j=1}^{p'} \left( \frac{\partial f}{\partial v_j} \cdot G_I(u) \wedge dv_j \right) \wedge (du_{i_{k+1}} \wedge \ldots \wedge du_{i_f}) \wedge dv_j \]

\[ = (-1)^{f} \cdot \sum_{j=1}^{p'} \left( \frac{\partial f}{\partial v_j} \cdot G_I(u) \wedge dv_j \right) \wedge (du_{i_{k+1}} \wedge \ldots \wedge du_{i_f}) \wedge dv_j. \quad (5.54) \]

In the above Equations 5.53 and 5.54 the function \( G_I \) is dependent only on the \( u \) coordinates, as in Equation 5.47, thus the partial derivatives of \( G_I \) with respect to the \( v \) coordinates vanish. By comparing the expression in the above Equation 5.53 with the expression in the above Equation 5.54, we obtain identity 2.

For identity 3 we combine the two preceding identities:

\[ d_L \delta_F = (-1)^{(f(i+1)+1)} \cdot d_{L'} (*_F d_{L'} *_F) \]

\[ = (-1)^f \cdot (-1)^{(f(i+1)+1)} \cdot *_F d_{L'} *_F \]

\[ = (-1) \cdot (-1)^{(f(i+1)+1)+f} \cdot *_F d_{L'} *_F \]

\[ = (-1)^{(f(i+1)+1)+f} \cdot *_F d_{L'} d_L' \]

\[ = (-1)^{f+1} \cdot \delta_F d_{L'}. \]

In the above Equation 5.55 we have used identity 2; in the above Equation 5.56 we have used identity 1; in the above Equation 5.57 we have used the fact that \((-1)^{2f+2} = 1\). Altogether this proves the final identity 3, establishing the result.

\[ * \]

\textbf{Lemma 5.38} ([1], Lemma 6.4). \textit{The Hodge decomposition for} \( \Omega^*(F) \) \textit{induces the following decomposition:}

\[ \Omega^{i,j}(S) = d_F(\Omega^{i-1,j}(S)) \oplus \delta_F(\Omega^{i+1,j}(S)) \oplus (\mathcal{H}^i(F) \otimes \Omega^j(L')). \quad (5.58) \]

Furthermore the decomposition in the above Equation 5.58 is preserved by the operation \( d_{L'} \).
**Proof.** We reproduce here a brief outline of the proof, which can be found in the aforementioned reference. Firstly, note that we may work in local coordinates of the form $\mathbb{R}^l \times F$, in which the metric is a product metric. The Hodge decomposition with respect to the $F$ direction comes from the application of the projection $\Pi_F$ onto $F$-harmonic forms and the general inverse $P_F$ of $\Delta_F \otimes 1_{\mathbb{R}^l}$, so that $\omega = \Delta_F P_F \omega + \Pi_F \omega$. The three operators $\Pi_F \otimes 1_{\mathbb{R}^l}$, $d_F \delta_F \otimes 1_{\mathbb{R}^l}$ and $\delta_F d_F \otimes 1_{\mathbb{R}^l}$, when applied to $\omega$ give smooth results, so that we obtain $\omega = d_F \alpha + \delta_F \beta + \gamma$.

In Definition 3.14 of Subsection 2.2 we have defined the Poincaré homotopy operator for liftable forms on $cS$; this Poincaré homotopy operator is

$$K^k_i : \Omega^k_i(cS) \longrightarrow \Omega^{k-1}_i(cS),$$

taking liftable $k$-forms on $cS$ to liftable $(k-1)$-forms on $cS$, as shown in Lemma 3.15. $K^k_i$ also satisfies the definition of a chain homotopy operator, as shown in Lemma 3.17. To compute the multiperverse cohomology of $cS$ we will need to go through an argument analogous to that of Subsection 2.2, with the aim of proving a result analogous to Corollary 3.23; to do this we will need to extend the Poincaré homotopy operator from liftable forms on $cS$ to multiperverse forms on $cS$. Firstly we note the requisite extension of $R$, the endomorphism of $\Omega^*_\tau(cS)$ defined in Proposition 3.12, which we redefine here for convenience.

**Proposition 5.39.** Let $\tau \in \mathcal{M}(cS)$ be a multiperversity and let $\omega \in M\Omega^k_\tau(cS)$ be a $\tau$-multiperverse form. The smooth form $p^*_S R_{\text{pt}}(\omega) \in \Omega^k(\mathscr{D}(cS))$ defines the lift of a $\tau$-multiperverse form, denoted $\overline{R}\omega \in M\Omega^k_\tau(cS)$, this yielding an endomorphism

$$\overline{R} : M\Omega^*_\tau(cS) \longrightarrow M\Omega^*_\tau(cS).$$

**Proof.** Let $\omega$ be a $\tau$-multiperverse $k$-form. From Proposition 5.36 we obtain that

$$R_{\text{pt}}(\omega) \in \bigoplus_{i+j=k, i \leq \tau(k)} \Omega^{i,j}(S),$$

but, by the definition of $R_{\text{pt}}$, we have $R_{\text{pt}}(p^*_S R_{\text{pt}}) = R_{\text{pt}}$, from where we obtain

$$R_{\text{pt}}(p^*_S R_{\text{pt}}(\omega)) \in \bigoplus_{i+j=k, i \leq \tau(k)} \Omega^{i,j}(S). \quad (5.59)$$

From the above Equation 5.59, and using again Proposition 5.36, we conclude that $p^*_S R_{\text{pt}}(\omega)$ is $\tau$-admissible. The result now follows from Proposition 5.35, where we note that $\overline{R}$ is a morphism of complexes, for it is a composition of morphisms of complexes.
Let \( R[M\Omega^*_\tau(cS)] \) denote the image of the endomorphism \( R \) on \( M\Omega^*_\tau(cS) \). Using an argument analogous to Proposition 3.13 we obtain the following result.

**Proposition 5.40.** The subcomplex \( R[M\Omega^*_\tau(cS)] \subseteq M\Omega^*_\tau(cS) \) is isomorphic to \( M\Omega^*_\tau(S) \).

**Proof.** Let \( \omega \in M\Omega^k_\tau(cS) \) be a \( \tau \)-multiperverse form and consider the definition of \( R_{\text{pt}} \). The map defined by the correspondence \( R\omega \mapsto R_{\text{pt}}\omega \), which extends to a map of complexes, is surjective onto \( M\Omega^k_\tau(S) \). The map defined by this correspondence is injective by the uniqueness of lifts, as in Proposition 3.13, and is thus the required isomorphism. \( \square \)

**Lemma 5.41.** Let \( \tau \in M(cS) \) be a multiperversity for a product-type cone \( cS \) and consider the homotopy operator \( K^k_I \) on liftable forms on \( cS \). The following two numbered statements are verified:

1. The operator \( K^k_I \) descends to an operator

\[
K^k_I : M\Omega^k_\tau(cS) \rightarrow M\Omega^{k-1}_\tau(cS),
\]

that is to say, \( K^k_I \) takes \( \tau \)-multiperverse \( k \)-forms on \( cS \) to \( \tau \)-multiperverse \( (k-1) \)-forms on \( cS \).

2. For all \( \omega \in M\Omega^k_\tau(cS) \) the following identity is verified:

\[
K^{k+1}_I(d\omega) - dK^k_I(\omega) = (-1)^{k-1}(\omega - R\omega). \tag{5.60}
\]

Hence \( K^*_I \) defines a chain homotopy between the identity and \( R \) endomorphisms of the complex \( M\Omega^*_\tau(cS) \) of \( \tau \)-multiperverse forms on \( cS \).

**Proof.** Let \( \omega \in M\Omega^k_\tau(cS) \) be a \( \tau \)-multiperverse \( k \)-form. To prove that \( K^k_I \) descends to \( \tau \)-multiperverse forms we must verify that both \( K^k_I(\omega) \) and \( dK^k_I(\omega) \) are \( \tau \)-admissible. From Proposition 3.16 we deduce that \( R_{\text{pt}}(K^k_I(\omega)) = 0 \), so that \( K^k_I(\omega) \) is \( \tau \)-admissible. Applying the result of Proposition 3.16 to the result of Lemma 3.17 one obtains the following:

\[
R_{\text{pt}}(dK^k_I(\omega)) = R_{\text{pt}}(K^{k+1}_I(d\omega)) + (-1)^k \cdot R_{\text{pt}}(\omega - p^*SR_{\text{pt}}(\omega)) = 0 + (-1)^k \cdot R_{\text{pt}}(\omega) - (-1)^k \cdot R_{\text{pt}}(\omega) = 0.
\]
From this we conclude that $dK^k_\tau(\omega)$ is $\tau$-admissible and hence that $K^k_\tau(\omega)$ is $\tau$-multiperverse; this proves the first numbered part. The identity in Equation 5.60 now follows from Lemma 3.17; this proves the second numbered part and establishes the result.

\begin{corollary}
Let $\tau \in \mathcal{M}(cS)$ be a multiperversity for a product-type cone $cS$. For all integers $k \in \mathbb{Z}_{\geq 0}$ there exists an isomorphism

$$MH^k_\tau(cS) \cong H^k(M\Omega^\ast_\tau(S)),$$

so that, for the cone $cS$, the two complexes defined in Definitions 5.28 and 5.33 are quasi-isomorphic.

\end{corollary}

\begin{proof}
This follows from the above Lemma 5.41 in the same way as Corollary 3.23 follows from Lemma 3.17. For convenience we reproduce the details. Let $[\omega] \in MH^k_\tau(cS)$ be a cohomology class. From Equation 5.60 in Lemma 5.41 we obtain

$$R\omega + (-1)^k \cdot dK^k_\tau(\omega) = \omega,$$

so that $[\omega] = [R\omega]$ in $MH^k_\tau(cS)$. Note that $R$ and $R_{pt}$ are morphisms of complexes and that the map

$$[\omega] = [R\omega] \mapsto [R_{pt}\omega]$$

is linear, well-defined and injective. To show that it is surjective, let $[\eta] \in H^k(M\Omega^\ast_\tau(S))$ be a cohomology class, so that:

$$[p^*_S(\eta)] = [Rp^*_S(\eta)] \mapsto [R_{pt}p^*_S(\eta)] = [\eta].$$

From this we conclude that $MH^k_\tau(cS)$ is isomorphic to $MH^k_\tau(cS)$. \hfill \Box

The above Corollary 5.42 reduces the computation of $MH^\bullet_\tau(cS)$ to the computation of $H^\bullet(M\Omega^\ast_\tau(S))$. The direct computation of the cohomology of $M\Omega^\ast_\tau(S)$ will occupy the remainder of this subsection.

\begin{theorem}
Let $cS$ be the depth 1 multicontrolled fibered cone over the depth 0 fibering data $\mu : S \to S' = L'$, where $S = L \times F$ is a product, with $L'$ and $F$ compact smooth manifolds; assume further that $cS$ is of product-type. Let $\tau \in \mathcal{M}(cS)$ be a multiperversity for $cS$. Then, for all $k \in \mathbb{Z}_{\geq 0}$, there exists an isomorphism

$$H^k(M\Omega^\ast_\tau(S)) \cong \bigoplus_{i+j=k \atop i \leq \tau(k)} (H^i(F) \otimes H^j(L')),$$

(5.61)

\end{theorem}
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and hence there exists an isomorphism

\[ MH^k_{\tau}(cS) \cong \bigoplus_{i+j=k} \left( H^i(F) \otimes H^j(L') \right). \]  

(5.62)

Proof. Note that the isomorphism in Equation 5.62 will follow from the isomorphism in Equation 5.61; this is a consequence of Corollary 5.42. Thus we need to establish the isomorphism of Equation 5.61 to prove the result.

Fix the natural projections \( p_F : F \times L' \to F \) and \( p_{L'} : F \times L' \to L' \). Without considering \( \tau \)-multiperversity conditions, from the Künneth Theorem ([9], p.47), there is an isomorphism

\[ \Psi : \bigoplus_{i+j=k} \left( H^i(F) \otimes H^j(L') \right) \to H^k(\Omega^* (S)), \]

defined by the correspondence

\[ (\alpha \otimes \beta) \to [p^*_F(\alpha) \wedge p^*_{L'}(\beta)]. \]

Furthermore, it is a statement of the Künneth Theorem that every element of \( H^k(\Omega^* (S)) \) is a sum of elements of the form \([p^*_F(\alpha) \wedge p^*_{L'}(\beta)]\). For clarity, we also fix the inclusion

\[ \Upsilon : H^k(M\Omega^* _{\tau}(S)) \to H^k(\Omega^* (S)), \]

from where we obtain a composition

\[ H^k(M\Omega^* _{\tau}(S)) \xrightarrow{\Upsilon} H^k(\Omega^* (S)) \xrightarrow{\Psi^{-1}} \bigoplus_{i+j=k} \left( H^i(F) \otimes H^j(L') \right), \]

and we aim to establish that \( \Psi^{-1} \Upsilon \) is an isomorphism onto \( \bigoplus_{i+j=k} \left( H^i(F) \otimes H^j(L') \right) \), that is that \( \Psi^{-1} \Upsilon \) is injective and surjective onto \( \bigoplus_{i+j=k} \left( H^i(F) \otimes H^j(L') \right) \). Firstly, we will verify the surjectivity of \( \Psi^{-1} \Upsilon \) onto \( \bigoplus_{i+j=k} \left( H^i(F) \otimes H^j(L') \right) \), assuming that the injectivity of \( \Psi^{-1} \Upsilon \) is verified; thus we reduce the proof of the result to the proof of the injectivity of \( \Psi^{-1} \Upsilon \). To prove the surjectivity of \( \Psi^{-1} \Upsilon \) we consider
the following commutative diagram.

\[
\begin{array}{ccc}
\bigoplus_{i+j=k} (H^i(F) \otimes H^j(L')) & \xrightarrow{\Psi} & H^k(M\Omega^\bullet_r(S)) \\
\downarrow & & \downarrow \gamma \\
\bigoplus_{i+j=k} (H^i(F) \otimes H^j(L')) & \xrightarrow{\Psi} & H^k(\Omega^\bullet(S)) \\
\downarrow \psi^{-1} & & \\
\bigoplus_{i+j=k} (H^i(F) \otimes H^j(L')) & & \\
\end{array}
\]

The arrow $\Lambda^|$ in the top row is defined identically to $\Psi$, that is as the map $(\alpha \otimes \beta) \mapsto [p^*_F(\alpha) \wedge p^*_{L'}(\beta)]$, which maps into $H^k(M\Omega^\bullet_r(S))$ by its definition; this can be seen by noting that $\Lambda^|$ maps onto the set

\[
\left\{ [p^*_F(\alpha) \wedge p^*_{L'}(\beta)] : i + j = k, \ [\alpha] \in H^i(F), \ [\beta] \in H^j(L') \right\}.
\]

We can deduce that the top right vertical arrow $\Upsilon$ is injective, this follows from the assumption that $\Psi^{-1} \Upsilon$ is injective. Following a similar argument we also deduce that the top row $\Lambda^|$ is injective; this completes the proof of the surjectivity of $\Psi^{-1} \Upsilon$ onto $\bigoplus_{i+j=k} (H^i(F) \otimes H^j(L'))$, for the square in the above diagram commutes and $\Psi^{-1} \Psi$ is the identity.

Secondly we verify the injectivity of $\Psi^{-1} \Upsilon$ into $\bigoplus_{i \in \tau(k)} (H^i(F) \otimes H^j(L'))$, proceeding by reducing the proof further as follows. Consider two $\tau$-multiperverse cohomology classes $[\gamma], [\gamma'] \in H^k(M\Omega^\bullet_r(S))$ which satisfy

\[
\Psi^{-1} \Upsilon ([\gamma] - [\gamma']) = 0;
\]

to prove injectivity we must show that $[\gamma - \gamma'] = [0]$ in $H^k(M\Omega^\bullet_r(S))$. Because $\Psi^{-1}$ is an isomorphism, and by basic linear algebra, it will then suffice to verify that if $\Upsilon[\gamma] = [0]$ in $H^k(\Omega^\bullet(S))$, then $[\gamma] = [0]$ in $H^k(M\Omega^\bullet_r(S))$. This reduces to showing that, if $\omega \in \Omega^k_r(S)$ is a closed $\tau$-multiperverse form, that is a form satisfying $d_S(\omega) = 0$, and if we have $\omega = d_S(\eta)$ for some $\eta \in \Omega^{k-1}(S)$, then there exist a $\tau$-multiperverse form $\eta' \in \Omega^{k-1}_r(S)$ such that $\omega = d_S(\eta')$. The rest of the proof is devoted to the verification of this property and to the construction $\eta'$, where we note that it will suffice to find a $\tau$-admissible $\eta' \in \Omega^{k-1}(S)$ satisfying this property.
To construct $\eta'$, we fix decompositions of $\omega$ and $\eta$ into their respective components:

$$
\omega = \sum_{i+j=k} \omega^{i,j} \quad \text{with} \quad \omega^{i,j} \in \Omega^{i,j}(S),
$$

$$
\eta = \sum_{i+j=k-1} \eta^{i,j} \quad \text{with} \quad \eta^{i,j} \in \Omega^{i,j}(S).
$$

If $\tau(k) \geq k$ is satisfied then, from the identity $\tau(k-1) \geq \tau(k)$, it would follow automatically that $\eta \in \Omega^{k-1}_c(S)$, that is that $\eta$ is $\tau$-admissible and that $p^*_S(\eta)$ is the lift of a $\tau$-multiperverse form. For the proof we assume that $\tau(k) < k$, that is: $\omega^{i,k-i} = 0$ for all $i > \tau(k)$. To simplify the presentation we define, for an integer $0 \leq N \leq k - 1$, and an integer $0 \leq M \leq k$, a diagonal index of $\omega$ and $\eta$ with $k \in \mathbb{Z}_{\geq 0}$ fixed:

$$
\omega_M := \omega^{M,k-M},
$$

$$
\eta_N := \eta^{N,(k-1)-N}.
$$

In this notation the preceding assumption $\tau(k) < k$ translates as: $\omega_M = 0$ for $M > \tau(k)$. Next we obtain the following list of observations; each observation follows from the fact that $p^*_S(\omega)$ is the lift of a $\tau$-multiperverse form:

1. For integers $0 \leq M \leq \tau(k)$ we consider the following diagram.

   \[\text{Figure 5.23: Diagram of } d_F \text{ and } d_{L'} \text{ for } \omega_M.\]

From this Figure 5.23 we conclude that

$$
\omega_M = d_F (\eta_{M-1}) + (-1)^M \cdot d_{L'} (\eta_M), \tag{5.63}
$$

---

Footnote: This relates to aspect of the multiperversity that do not have topological meaning in Definition 5.28, see also Figure 5.13 in Example 5.30.
where the change of sign follows from Lemma 5.25.

2. For \( \tau(k) \leq M < k \), we have \( \omega_{(M)} = 0 \), so that we can consider the following diagram.

\[
d_F \eta_{(M-1)} - d_{L'} = 0.
\]

(5.64)

3. For \( M = k \), since we assume that \( \tau(k) < k \), so that \( \omega_{(k)} = 0 \), we obtain the following diagram.

\[
d_F \eta_{(k-1)} - d_{L'} = 0.
\]

(5.65)
For each integer $0 \leq N \leq k-1$, using Lemma 5.38, we fix the Hodge decomposition

$$\eta(N) := d_F(\alpha_{(N-1)}) + \delta_F(\beta_{(N+1)}) + \gamma(N), \tag{5.66}$$

where the following elements are fixed:

1. $\alpha_{(N-1)} \in \Omega^{(N-1),(k-N-1)}(S)$.
2. $\beta_{(N+1)} \in \Omega^{(N+1),(k-N)}(S)$.
3. $\gamma(N) \in \mathcal{H}^N(F) \otimes \Omega^{(k-N-1)}(L')$.

The next step of the proof is to present an iteration. We consider first the case when $N = k - 1$. The contents of Equation 5.66 yields

$$\eta_{(k-1)} = d_F(\alpha_{(k-2)}) + \delta_F(\beta_{(k)}) + \gamma_{(k-1)}, \tag{5.67}$$

which we combine with $d_F(\eta_{(k-1)}) = 0$ from Equation 5.65, to derive

$$d_F(\eta_{(k-1)}) = d_F\left(d_F(\alpha_{(k-2)}) + \delta_F(\beta_{(k)}) + \gamma_{(k-1)}\right)$$

$$= 0 + d_F\delta_F(\beta_{(k)}) + 0 = 0.$$

From this we conclude that $d_F\delta_F(\beta_{(k)}) = 0$. Further, by applying the Laplacian $\Delta_F = \delta_F d_F + d_F \delta_F$ to the element $\delta_F(\beta_{(k)})$, we obtain

$$\Delta_F(\delta_F(\beta_{(k)})) = \delta_F d_F \delta_F(\beta_{(k)}) + d_F \delta_F \delta_F(\beta_{(k)})$$

$$= \delta_F(0) + d_F(0) = 0.$$

From the above we have $\Delta_F(\delta_F(\beta_{(k)})) = 0$, whence $\delta_F(\beta_{(k)}) = 0$ by the uniqueness of the Hodge decomposition. Consequently we deduce that Equation 5.67 takes the form

$$\eta_{(k-1)} = d_F(\alpha_{(k-2)}) + \gamma_{(k-1)}. \tag{5.68}$$

We take $d_{L'}$ of both sides in the above Equation 5.68, using also part 1 of Lemma 5.37, to obtain

$$d_{L'}(\eta_{(k-1)}) = d_{L'}d_F(\alpha_{(k-2)}) + d_{L'}(\gamma_{(k-1)})$$

$$= -d_Fd_{L'}(\alpha_{(k-2)}) + d_{L'}(\gamma_{(k-1)}), \tag{5.69}$$

from where we deduce

$$d_{L'}(\gamma_{(k-1)}) = d_{L'}(\eta_{(k-1)}) + d_Fd_{L'}(\alpha_{(k-2)}). \tag{5.70}$$
Further, from Equation 5.64 for $M = k - 1$, we also obtain

$$0 = d_F(\eta_{k-2}) + (-1)^{k-1} \cdot d_{L'}(\eta_{k-1})$$

$$= (-1)^{k-2} \cdot d_F(\eta_{k-2}) + (-1)^{k-2} \cdot (-1)^{k-1} \cdot d_{L'}(\eta_{k-1})$$

$$= (-1)^{k-2} \cdot d_F(\eta_{k-2}) - d_{L'}(\eta_{k-1}) ;$$

from where we conclude that

$$d_{L'}(\eta_{k-1}) = (-1)^{k-2} \cdot d_F(\eta_{k-2}) . \quad (5.71)$$

Combining the above Equation 5.71 with Equation 5.70 we obtain

$$d_{L'}(\gamma_{k-1}) = d_{L'}(\eta_{k-1}) + d_F d_{L'}(\alpha_{k-2})$$

$$= (-1)^{k-2} \cdot d_F(\eta_{k-2}) + d_F d_{L'}(\alpha_{k-2})$$

$$= d_F \left((-1)^{k-2} \cdot \eta_{k-2} + d_{L'}(\alpha_{k-2})\right).$$

From this we conclude that $d_{L'}(\gamma_{k-1}) \in \text{im}(d_F)$. We now aim to show that $d_S(\gamma_{k-1}) = 0$, as this will be used in the final part of the proof. For this it will suffice to show that both $d_F(\gamma_{k-1}) = 0$ and $d_{L'}(\gamma_{k-1}) = 0$: the first of these expressions follows from Equations 5.65 and 5.68, so that we are reduced to showing that $d_{L'}(\gamma_{k-1}) = 0$. To show this we apply the Laplacian $\Delta_F$ to the element $d_{L'}(\gamma_{k-1})$:

$$\Delta_F \left(d_{L'}(\gamma_{k-1})\right) = \delta_F d_F d_{L'}(\gamma_{k-1}) + d_F \delta_F d_{L'}(\gamma_{k-1})$$

$$= \delta_F d_F d_{L'} \left((-1)^{k-2} \cdot \eta_{k-2} - d_{L'}(\alpha_{k-2})\right)$$

$$+ d_F \delta_F d_{L'}(\gamma_{k-1})$$

$$= 0 + d_F \delta_F d_{L'}(\gamma_{k-1}) \quad (5.72)$$

$$= (-1)^{f+1} \cdot d_F d_{L'} \delta_F(\gamma_{k-1}) \quad (5.73)$$

$$= (-1) \cdot (-1)^{f+1} \cdot d_{L'} d_F \delta_F(\gamma_{k-1}) \quad (5.74)$$

$$= (-1)^{f+1} \cdot d_{L'} \delta_F d_F(\gamma_{k-1}) \quad (5.75)$$

$$= (-1)^{f+1} \cdot (-1)^{f+1} \cdot \delta_F d_{L'} d_F(\gamma_{k-1}) \quad (5.76)$$

$$= -\delta_F d_F d_{L'}(\gamma_{k-1}) \quad (5.77)$$

$$= -\delta_F d_F d_{L'} \left((-1)^{k-2} \cdot \eta_{k-2} - d_{L'}(\alpha_{k-2})\right) \quad (5.78)$$

$$= 0.$$

In the above Equation 5.75 we have used the identity $d_F \delta_F(\gamma_{k-1}) = -\delta_F d_F(\gamma_{k-1})$, which follows from $\Delta_F(\gamma_{k-1}) = 0$; in the above Equations 5.73 and 5.76 we have
used part 3 of Lemma 5.37; further, in Equations 5.74 and 5.77 we have used part 1 of Lemma 5.37; finally, in Equations 5.72 and 5.78 we have used that $d_F d_F = 0$. We combine $\Delta_F \left( d_{L'} \left( \gamma_{(k-1)} \right) \right) = 0$ with Equations 5.69 and 5.71 to obtain the following:

$$
d_{L'}(\eta_{(k-1)}) = \frac{d_F(\Omega^{k-2,2}(S))}{d_F(\Omega^{k-2,2}(S))} (-1)^{k-2} \cdot d_F(\eta(k-2))
$$

$$
d_{L'}(\eta_{(k-1)}) = -\frac{d_F d_{L'}(\alpha_{(k-2)})}{d_F(\Omega^{k-2,2}(S))} + \frac{d_{L'}(\gamma_{(k-1)})}{\Omega^{k-1}(F) \otimes \Omega^0(L')}
$$

By the uniqueness of Hodge decomposition we conclude that $d_{L'} \left( \gamma_{(k-1)} \right) = 0$. Thus $d_S \left( \gamma_{(k-1)} \right) = 0$. Using this conclusion and Equation 5.68, we further obtain

$$
\eta_{(k-1)} - d_S(\alpha_{(k-2)}) - \gamma_{(k-1)} = \eta_{(k-1)} - d_F(\alpha_{(k-2)}) - d_{L'}(\alpha_{(k-2)}) - \gamma_{(k-1)}
$$

$$
= \delta_F(\beta_{(k)}) - d_{L'}(\alpha_{(k-2)})
$$

$$
= -d_{L'}(\alpha_{(k-2)}).
$$

The above is the final computational part of the proof; the final stage of the proof of the first iteration is to define

$$
\overline{\eta}_{k-1} := \eta - d_S(\alpha_{(k-2)}) - \gamma_{(k-1)}
$$

$$
= \eta(0) + \eta(1) + \ldots + \eta_{(k-2)} + \eta_{(k-1)} - d_S(\alpha_{(k-2)}) - \gamma_{(k-1)}
$$

$$
= \eta(0) + \eta(1) + \ldots + \eta_{(k-2)} - d_{L'}(\alpha_{(k-2)}),
$$

where we also note that

$$
(\overline{\eta}_{k-1})_{(k-1)} := \left( \eta_{(k-2)} - d_{L'}(\alpha_{(k-2)}) \right) \in \Omega^{(k-2,1)}(S).
$$

Correspondingly $(\overline{\eta}_{k-1})_{(k-1)} = 0$. However the form $\overline{\eta}_{k-1}$, so constructed, still satisfies

$$
d_S(\overline{\eta}_{k-1}) = d_S(\eta - d_S(\alpha_{(k-1)}) - \gamma_{(k-1)})
$$

$$
= d_S(\eta) - d_S(\gamma_{(k-1)})
$$

$$
= \omega.
$$

This completes the first iteration, where we have verified that the cohomology class $[\eta]$ may be represented by $[\overline{\eta}_{k-1}]$, which satisfies $(\overline{\eta}_{k-1})_{(k-1)} = 0$, as above. We can iterate the above procedure with $\overline{\eta}_{k-1}$ in place of $\eta$ and $k - 2$ in place of $k - 1$; the only modification that is necessary is to substitute Equation 5.65, by noting that, provided we have not reached the subscript index $(k-1) - \tau(k)$ at
We continue this procedure all the way to, but not including, the subscript index 
\((k - 1) - \tau(k)\); at the final stage we obtain

\[
\overline{\eta} = \overline{\eta}_{\tau(k)} = 
\eta_{(0)} + \ldots + \eta_{(\tau(k))} + \kappa,
\]

with the \(\kappa \in \Omega^{(k),(k-1)-\tau(k)}(S)\). Furthermore we still have \(d\overline{\eta} = d\eta = \omega\) and 
\(\overline{\eta} \in \Omega_{\tau}^{k-1}(S)\), so that \(p_S^\tau(\overline{\eta})\) is \(\tau\)-admissible. This verifies injectivity, implying

surjectivity and this proves the result. \(\Box\)

**Remark 5.44.** The computation of the cohomology \(MH^*_\tau(cS)\) in the above Theorem 5.43 is independent of the choice of metric \(g\) on \(F\). The metric \(g\) was fixed in order to define \(*_F\) in Equation 5.42 on page 79.

**Example 5.45.** Let \(S = S^1 \times S^1\), where \(F = L' = S^1\), and consider the product-type depth 1 multicontrolled cone \(cS\). We obtain directly that \(\dim(cS) = 3\), \(\dim(S) = 2\), \(\codim_{cS}(pt) = 3\), and \(\codim_{cS}(L') = 2\), so that a multiperversity \(\tau \in \mathcal{M}(cS)\) may be defined as:

\[
\tau(k) = \begin{cases} 
1 & \text{if } k < 0 \\
1 & \text{if } k = 0 \\
0 & \text{if } k = 1 \\
-1 & \text{if } k = 2 \\
-1 & \text{if } k > 2
\end{cases}
\]

Further, we can illustrate the multiperversity \(\tau\) on the following two diagrams.

Figure 5.26: Diagram representing the multiperversity \(\tau\).
Using the result of the above Theorem 5.43 we obtain:

\[ MH^0_\tau(cS) = \mathbb{R}, \quad MH^1_\tau(cS) = \mathbb{R} \quad \text{and} \quad MH^2_\tau(cS) = 0. \]

**Remark 5.46.** Consider the setting of the above Example 5.45. Replace the multiperversity \( \tau \) by multiperversity \( \tau' \):

\[
\tau'(k) = \begin{cases} 
1 & \text{if } k < 0 \\
0 & \text{if } k = 0 \\
0 & \text{if } k = 1 \\
-1 & \text{if } k = 2 \\
-1 & \text{if } k > 2 
\end{cases}
\]

The multiperversity \( \tau' \) can be illustrated on the following two diagrams.

Figure 5.27: Diagram representing the multiperversity \( \tau \) on the fibre degrees of \( F \) and \( L' \).

Figure 5.28: Diagram representing the multiperversity \( \tau' \).
Thus we obtain $M\Omega^\bullet_m(cS) = M\Omega^\bullet_{\tau'}(cS)$. This again illustrates the comment in Example 5.30.

If we consider $cS$ equipped with trivial multicontrol data, then the multiperverse cohomology computed for $cS$ with a multiperversity $m(\mathfrak{p})$, where $\mathfrak{p}$ is a general perversity for $cS$, should agree with intersection cohomology. This is indeed the case, if we compare Equation 5.80 below with the computation in [1](Equation 13, p.24).

**Corollary 5.47.** Let $cS$ be a product-type depth 1 multicontrolled cone. Let $m(\mathfrak{p}) \in \mathcal{M}(cS)$ be the multiperversity associated to a perversity $\mathfrak{p}$ for $cS$; this is specified by an integer $\mathfrak{p}(\mathfrak{p}) \in [-1, \dim(S)]$. Then, for all $k \in \mathbb{Z}_{\geq 0}$, there exists an isomorphism

$$MH^k_{m(\mathfrak{p})}(cS) \cong \begin{cases} H^k(S) & \text{if } k \leq \mathfrak{p}(\mathfrak{p}), \\ 0 & \text{if } k > \mathfrak{p}(\mathfrak{p}). \end{cases}$$ (5.80)

**Proof.** There exists an isomorphism $MH^k_{m(\mathfrak{p})}(cS) \cong H^k(M\Omega^\bullet_{m(\mathfrak{p})}(S))$, and further

$$H^k(M\Omega^\bullet_{m(\mathfrak{p})}(S)) \cong \bigoplus_{i + j = k} (H^i(F) \otimes H^j(L')).$$

However, from the definition of $m$ in Definition 5.7, we have that $m(\mathfrak{p})(k) = -1$ for $k > \mathfrak{p}(\mathfrak{p})$, and that $m(\mathfrak{p})(k) = \text{codim}(S) - 1$ for $k \leq \mathfrak{p}(\mathfrak{p})$. Thus for $k > \mathfrak{p}(\mathfrak{p})$ we obtain 0, while for $k \leq \mathfrak{p}(\mathfrak{p})$ we obtain $\bigoplus_{i+j=k} (H^i(F) \otimes H^j(L')) = H^k(S)$. This proves the result. \[\square\]

**Remark 5.48.** Note that in the above Corollary 5.47 the condition that the multicontrol data for $cS$ be trivial is not required. If this assumption were made, then the computation would simplify further, for the concept of a multiperversity would be redundant for this space.
2.3 Poincaré lemma for $\tau$-multiperverse forms in the product-type case

In this subsection we will compute the $\tau$-multiperverse cohomology of a cylinder $X \times \mathbb{R}$, where $X$ is a product-type depth 1 multicontrolled stratified space. We begin with an observation concerning the lattices of multiperversities for $X$ and for $X \times \mathbb{R}$.

**Proposition 5.49.** There is an isomorphism between the lattice $\mathcal{M}(X)$ and the lattice $\mathcal{M}(X \times \mathbb{R})$.

**Proof.** Firstly, note that there is a bijection from $\mathcal{S}(X)$ to $\mathcal{S}(X \times \mathbb{R})$; this bijection is defined by the correspondence $A \mapsto (A \times \mathbb{R})$ for $A \in \mathcal{S}(X)$. Secondly, by considering Equations 5.4, 5.5, and 5.6, in Definition 5.2, we remark that the result will follow if we can show that the two efficiency conditions are identical in $X$ and $X \times \mathbb{R}$. It will suffice to show this for a fixed singular stratum $A \in \mathcal{S}(X)$. The two efficiency conditions for $X$ and $X \times \mathbb{R}$ involve only the numerical values of $\text{codim}_X(S_A')$ and $\text{codim}_{X \times \mathbb{R}}(S_A' \times \mathbb{R})$. We compute the expression

$$\text{codim}_X(S_A') = \dim(X) - \dim(S_A),$$

followed by the expression

$$\text{codim}_{X \times \mathbb{R}}(S_A' \times \mathbb{R}) = \dim(X \times \mathbb{R}) - \dim(S_A' \times \mathbb{R})$$

$$= \dim(X) + \dim(\mathbb{R}) - \dim(S_A') - \dim(\mathbb{R})$$

$$= \dim(X) + 1 - \dim(S_A') - 1$$

$$= \dim(X) - \dim(S_A).$$

The expressions for $\text{codim}_X(S_A')$ and $\text{codim}_{X \times \mathbb{R}}(S_A' \times \mathbb{R})$ are thus identical, from where the result follows. $\square$

For clarity, we briefly remark that for $X$ of product-type, the cylinder $X \times \mathbb{R}$ is of product-type. From Equations 5.23 and 5.24, we have

$$S_{A \times \mathbb{R}} = F_A \times L_A' \times (A \times \mathbb{R}),$$

$$S_{A \times \mathbb{R}}' = L_A' \times (A \times \mathbb{R}).$$

The maps $\pi_{A \times \mathbb{R}}$ and $\pi_{A \times \mathbb{R}}'$ are defined as $\pi_A \times 1_{\mathbb{R}}$ and $\pi_A' \times 1_{\mathbb{R}}$ respectively. Thus all of the conditions in Definition 5.24 are satisfied. For $X \times \mathbb{R}$, the contents of
Equation 5.26 has the following form, where $x \in F_A$, $y \in L'_A$ and $(z_1, z_2) \in A \times \mathbb{R}$:

$$\Lambda^k \left( T^*_{(x,y,z_1,z_2)} \right) \cong \bigoplus_{i+j+h=k} \Lambda^i \left( T^*_{x} F_A \right) \times \Lambda^j \left( T^*_{y} L'_A \right) \times \Lambda^h \left( T^*_{(z_1,z_2)} (A \times \mathbb{R}) \right).$$

To prove the Poincaré lemma for $\tau$-multiperverse forms on $X \hat{\times} \mathbb{R}$ we will use the results of Subsection 2.3, verifying as before that the Poincaré homotopy operator for liftable forms $K^k_{X} : \Omega^k(X \times \mathbb{R}) \rightarrow \Omega^{k-1}(X \times \mathbb{R})$ descends to a homotopy operator on the complex of $\tau$-multiperverse forms $\Omega^\tau_\ast(X \times \mathbb{R})$. Firstly, we note the requisite extension of the endomorphism $p_\text{reg}(X)Q$ of $\Omega^\tau_\ast(X \times \mathbb{R})$, where $p_\text{reg}(X) : \text{reg}(X) \times \mathbb{R} \rightarrow \text{reg}(X)$ is the natural projection and

$$Q : \Omega^k(X \times \mathbb{R}) \rightarrow \Omega^k(X)$$

is defined as the pullback along the map $x \mapsto (x, 0)$.

**Lemma 5.50.** Let $\tau \in \mathcal{M}(X)$ be a multiperversity for $X$ and $X \times \mathbb{R}$, and let $\omega \in M\Omega^k_\ast(X \times \mathbb{R})$ be a $\tau$-multiperverse form. The liftable form $p_\text{reg}(X)Q(\omega) \in \Omega^k(\text{reg}(X) \times \mathbb{R})$ is $\tau$-multiperverse, this yielding an endomorphism $p_\text{reg}(X)Q : M\Omega^\tau_\ast(X \times \mathbb{R}) \rightarrow M\Omega^\tau_\ast(X \times \mathbb{R})$.

**Proof.** By Proposition 5.35 it suffices to verify that $p_\text{reg}(X)Q$ maps $\tau$-multiperverse forms to $\tau$-admissible forms. If $\omega = f(x,t) \wedge p_\text{reg}(X)(\eta) \wedge dt$, then $Q(\omega) = 0$, so that it will suffice to verify this result in the case when

$$\omega = f(x,t) \wedge p_\text{reg}(X)(\eta).$$

In this case $Q(\omega) = f(x,0) \wedge \eta$, so that

$$p_\text{reg}(X)Q(\omega) = f(x,0) \wedge p_\text{reg}(X)(\eta).$$

For each singular stratum $A \in \mathcal{S}(X)$, from the assumption $\omega \in M\Omega^k_\ast(X \times \mathbb{R})$, we
obtain
\[ R_{A \times \mathbb{R}}(\omega) = F(s, t) \cdot R_A(\eta) \in \bigoplus_{i \leq \tau(k-h)}^\oplus \Omega^{i,j,h}(S_A), \]
for some function \( F \), but we also obtain by direct computation
\[ R_{A \times \mathbb{R}}(p^*_\text{reg}(X)Q(\omega)) = R_{A \times \mathbb{R}}(f(x, 0) \land \eta) = F^+(s, 0) \land R_A(\eta), \]
for some function \( F^+ \). Since \( F \) and \( F^+ \) are 0-forms, this proves the result, as it verifies that \( p^*_\text{reg}(X)Q(\omega) \) is \( \tau \)-admissible.

\[ \text{Remark 5.51.} \]
Note the distinction between \( R \) defined on \( cS \), as in Proposition 5.39, and \( p^*_\text{reg}(X)Q \), as defined in the above Lemma 5.50. Diagrammatically the operation \( R \) is the pullback to the singular boundary of \( cS \) in the deshirring, or a kind of reflection of the liftable form in the singular stratum; the action of \( p^*_\text{reg}(X)Q \) is, in this sense, perpendicular to the singular stratum of \( X \).

\[ \text{Lemma 5.52.} \]
Let \( \tau \in M(X) \) be a multiperversity for \( X \) and \( X \times \mathbb{R} \). The homotopy operator \( K^k_i \) for liftable forms descends to an operator
\[ K^k_i : M\Omega^k_\tau(X \times \mathbb{R}) \longrightarrow M\Omega^{k+1}_\tau(X \times \mathbb{R}), \]
so that, for all \( \omega \in M\Omega^k_\tau(X \times \mathbb{R}) \), the following identity holds
\[ K^{k+1}_i(d\omega) - dK^k_i(\omega) = (-1)^{k-1} \cdot (\omega - p^*_\text{reg}(X)Q\omega). \]

Thus \( K^*_i \) defines a chain homotopy between the identity and \( p^*_\text{reg}(X)Q \) endomorphisms on the complex \( M\Omega^*_\tau(X \times \mathbb{R}) \).

\[ \text{Proof.} \]
If \( \omega = f(x, t) \land p^*_\text{reg}(X)(\eta) \), then \( K^k_i(\omega) = 0 \), so that it will suffice to prove this result in the case when
\[ \omega = f(x, t) \land p^*_\text{reg}(X)(\eta) \land dt. \]
By direct computation we obtain
\[ K^k_i(\omega) = K^k_i(f(x, t) \land p^*_\text{reg}(X)(\eta) \land dt) = p^*_\text{reg}(X)(\eta) \land \int_{w=0}^{w=t} f(x, w) \, dw. \]
It will suffice to verify that both \( K^k_i(\omega) \) and \( dK^k_i(\omega) \) are \( \tau \)-admissible for a fixed singular stratum \( A \times \mathbb{R} \in \mathcal{S}(X \times \mathbb{R}) \). To verify that \( K^k_i(\omega) \) is \( \tau \)-admissible we
compute

\[ R_{A \times \mathbb{R}}(K^k_t(\omega)) = R_{A \times \mathbb{R}} \left( p^*_{\text{reg}(X)}(\eta) \land \int_{w=0}^{w=t} f(x, w) \, dw \right) \]

\[ = R_A(\eta) \land \int_{w=0}^{w=t} f(s, w) \, dw , \]

where \( s \in S_A \), and we denote the lift of \( f \) by \( f \), with \( R_{A \times \mathbb{R}}(K^k_t(\omega)) \in \Omega^k(S_A \times \mathbb{R}) \). As the expression \( \int_{w=0}^{w=t} f(s, w) \, dw \) is a 0-form, it will suffice to verify that

\[ R_A(\eta) \in \bigoplus_{i \leq \tau(k-1-h)} \bigoplus_{i+j+h' = k} \Omega^{i,j,h}(S_A) , \]

but this follows directly from the following computation:

\[ R_{A \times \mathbb{R}}(p^*_{\text{reg}(X)}(\eta) \land dt) = (R_A(\eta) \land dt) \in \bigoplus_{i \leq \tau(k-h')} \bigoplus_{i+j+h' = k} \Omega^{i,j,h}(S_A) . \quad (5.84) \]

In the above Equation 5.84, as a consequence of the \( dt \) term, we have \( h' = h + 1 \). To verify that \( dK^k_t(\omega) \) is \( \tau \)-admissible, we compute:

\[ R_{A \times \mathbb{R}}(dK^k_t(\omega)) = R_{A \times \mathbb{R}}(K^{k+1}_t(d\omega)) + R_{A \times \mathbb{R}} \left( (-1)^k \cdot \left( \omega - p^*_\text{reg}(X) Q \omega \right) \right) \]

\[ = R_{A \times \mathbb{R}}(K^{k+1}_t(d\omega)) + (-1)^k \cdot R_{A \times \mathbb{R}}(\omega) . \]

We obtain that \( R_{A \times \mathbb{R}}(\omega) \) is contained \( \bigoplus_{i \leq \tau(k-h')} \bigoplus_{i+j+h' = k} \Omega^{i,j,h}(S_A) \) by assumption, and we obtain that \( R_{A \times \mathbb{R}}(K^{k+1}_t(d\omega)) \) is contained in \( \bigoplus_{i \leq \tau(k-h')} \bigoplus_{i+j+h' = k} \Omega^{i,j,h}(S_A) \) by using the same argument as for \( K^k_t(\omega) \); in this we recall that \( d\omega \) is \( \tau \)-multiperverse because \( dd\omega = 0 \). This proves the result. \( \square \)

Corollary 5.53. Let \( \tau \in \mathcal{M}(X) \) be a multiperversity for \( X \) and \( X \times \mathbb{R} \). For all \( k \in \mathbb{Z}_{\geq 0} \) there exists an isomorphism

\[ MH^k_\tau(X \times \mathbb{R}) \cong MH^k_\tau(X) . \]

Furthermore, by induction, there exists an isomorphism

\[ MH^k_\tau(X \times \mathbb{R}^n) \cong MH^k_\tau(X) . \]

Proof. This follows from the above Lemma 5.52 in the same way as Corollary 3.18 follows from Lemma 3.17. For convenience we reproduce the details. Let \( [\omega] \in MH^k_\tau(X \times \mathbb{R}) \) be a cohomology class. From Equation 5.83 in Lemma 5.52
we have
\[ p_{\text{reg}}^* (X) Q \omega + (-1)^k \cdot dK^k_i (\omega) = \omega, \]
so that \([\omega] = [p_{\text{reg}}^* (X) Q \omega]\) in \(MH^k_+ (X \times \mathbb{R})\). Note that \(p_{\text{reg}}^* (X)\) and \(Q\) are morphisms of complexes and that the map
\[ [\omega] = [p_{\text{reg}}^* (X) Q \omega] \mapsto [Q \omega] \]
is linear, well-defined and injective. To show that it is surjective let \([\eta] \in MH^k_+ (X)\), then we obtain
\[ [p_{\text{reg}}^* (X) (\eta)] = [p_{\text{reg}}^* (X) Q p_{\text{reg}}^* (X) (\eta)] \mapsto [Q p_{\text{reg}}^* (X) (\eta)] = [\eta]. \]
From this we conclude that \(MH^k_+ (X \times \mathbb{R})\) is isomorphic to \(MH^k_+ (X)\).

From the above Poincaré lemma and the previous computation for the cone \(cS\), we obtain the usual local calculation for intersection cohomology (compare to, for example, with the calculation in [26], Theorem 6.1).

**Corollary 5.54.** Let \(cS\) be a product-type depth 1 multicontrolled cone. Let \(m(p) \in M(cS)\) be the multiperversity associated to a perversity \(p\) for \(cS\) and \(cS \times \mathbb{R}^n\); this is specified by an integer \(\overline{p}(\text{pt}) \in [-1, \dim(S)]\). Then, for all \(k \in \mathbb{Z}_{\geq 0}\), there exists an isomorphism
\[
MH^k_{m(p)} (cS \times \mathbb{R}^n) \cong MH^k_{m(p)} (cS) \cong \begin{cases} H^k (S) & \text{if } k \leq \overline{p}(\text{pt}) \\ 0 & \text{if } k > \overline{p}(\text{pt}) \end{cases}
\]

**Proof.** This follows from the above Lemma 5.52 and Corollary 5.47.

## 3 Flat-type case

In this section we will define the more general case of depth 1 multicontrolled stratified spaces that we consider, called the *flat-type* case, in which multiperverse cohomology can be calculated; as for the product-type case we will present the calculations for the cone and the cylinder when these spaces satisfy the flat-type assumption. In the preceding section we introduced the product-type case and presented explicit calculations for the cone and the cylinder, once appropriate properties have been verified for the flat-type case, these calculations can be carried from the product-type case to the flat-type case without excessive modification. Additionally, multiperverse cohomology can be defined in a more general setting still; we present this in Subsection 3.1.
3.1 Definition of multiperverse forms on a general depth 1 multicontrolled space

In this subsection we will define the complex of multiperverse forms on an arbitrary depth 1 multicontrolled stratified space. For product-type spaces this definition will be concordant with the definition in Subsection 2.1.

Let $X$ be an arbitrary depth 1 multicontrolled stratified space and consider the following diagram for the link bundle $S_A$, recalling that $L_A$ is the fibre of $\pi_A$ and $L'_A$ is the fibre of $\pi'_A$; we also recall that the fibre of $\mu_A$ is $F_A$.

\[
\begin{align*}
L_A & \longrightarrow S_A \\
\mu_A & \\
L'_A & \longrightarrow S'_A \\
\pi'_A & \\
& \downarrow \pi_A \downarrow \ A
\end{align*}
\]

For each singular stratum $A \in SS(X)$ choose horizontal subbundles $Q_{1,A}, Q_{2,A} \subseteq T(S_A)$ such that the identities

\[
T(S_A) = \ker(d\pi_A) \oplus Q_{2,A} \quad \text{and} \quad \ker(d\pi_A) = \ker(d\mu_A) \oplus Q_{1,A}
\]

are verified, from where

\[
T(S_A) = \ker(d\mu_A) \oplus Q_{1,A} \oplus Q_{2,A}.
\] (5.85)

Hereinafter we write $Q := \{Q_{1,A}, Q_{2,A}\}_{A \in SS(X)}$ to denote the set of horizontal subbundles consisting of the various $Q_{1,A}$ and $Q_{2,A}$ involved, further writing $(X, Q)$ to denote a depth 1 multicontrolled stratified space with a fixed choice of horizontal subbundles $Q$. Throughout the rest of this subsection we fix an $(X, Q)$.

For a fixed singular stratum $A \in SS(X)$, and for each $s \in S_A$, from the choice of $Q_{1,A}$ and $Q_{2,A}$ fitting into the decomposition of Equation 5.85, we obtain an induced decomposition on cotangent spaces:

\[
T^*_s(S_A) \cong \frac{\ker(d\mu_A)_s^*}{\ker(d\mu_A)_s^*} \times \left( (Q_{1,A}^*)_s \right) \times \left( (Q_{2,A}^*)_s \right).
\]

From there, by considering the $k$-th exterior product for $k \in \mathbb{Z}_{\geq 0}$, we obtain the following decomposition of the $k$-form bundle:

\[
\Lambda^k(T^*_s(S_A)) \cong \bigoplus_{i+j+b=k} \Lambda^i \left( \ker(d\mu_A)_s^* \right) \times \Lambda^j \left( (Q_{1,A}^*)_s \right) \times \Lambda^b \left( (Q_{2,A}^*)_s \right).
\] (5.86)
The decomposition in the above Equation 5.86 further entails that the complex of smooth differential forms on $S_A$ itself satisfies the decomposition

$$\Omega^k(S_A, Q) = \bigoplus_{i+j+h=k} \Gamma\left(S_A, \left(\Lambda^i(\ker(d\mu_A)^*_{s}) \times \Lambda^j((Q^*_{1,A})_s) \times \Lambda^h((Q^*_{2,A})_s)\right)\right),$$

where we denote the $(i, j, h)$-th component of $\Omega^k(S_A)$ as

$$\Omega^{i,j,h}(S_A, Q) := \Gamma\left(S, \left(\Lambda^i(\ker(d\mu_A)^*_{s}) \times \Lambda^j((Q^*_{1,A})_s) \times \Lambda^h((Q^*_{2,A})_s)\right)\right), \tag{5.87}$$

from where $\Omega^k(S_A) = \bigoplus_{i+j+h=k} \Omega^{i,j,h}(S_A, Q)$. This splitting is a generalisation of the splitting in Subsection 2.1, the above Equation 5.87 being analogous to Equation 5.27 in the aforementioned subsection. For the $(i, j, h)$-th component in the above Equation 5.87, we will say that a form in $\Omega^{i,j,h}(S, Q)$ has total degree $k = i+j+h$, has total fibre degree $i+j = k-h$, has lower fibre degree $j$ and has upper fibre degree $i$. Finally, we remark that in the above paragraph one can replace $S_A$ with $S_A \cap \pi^{-1}_A[U]$, for an open subset $U \subseteq A$ of the singular stratum $A$, obtaining thus a decomposition $\Omega^{i,j,h}(S_A \cap \pi^{-1}_A[U], Q)$.

**Definition 5.55.** Let $\omega \in \Omega^i_\tau(X)$ be a liftable form on a depth 1 multicontrolled stratified space $X$ and let $\tau \in \mathcal{M}(X)$ be a multiperversity for $X$. We will say that the liftable form $\omega$ is $\tau$-admissible if, for each singular stratum $A \in \mathbb{S}\mathbb{S}(X)$, the following is satisfied:

$$R_A(\omega|_{\text{reg}(T_A)}) \in \bigoplus_{i \leq \tau(A)(k-h)} \Omega^{i,j,h}(S_A, Q). \tag{5.88}$$

In the above Equation 5.88 the integer $i$ is the upper fibre degree, $j$ is the lower fibre degree and $h$ is the base degree; $i+j$ is the total fibre degree and $k = i+j+k$ is the total degree. Furthermore, we will say that the liftable form $\omega$ is $\tau$-multiperverse if both $\omega$ and $d\omega$ are $\tau$-admissible. We write $\mathcal{M}\Omega^k_\tau(X, Q)$ for the set of $\tau$-multiperverse forms on $X$. It is immediate that the set of $\tau$-multiperverse forms is a cochain complex $\mathcal{M}\Omega^*_\tau(X, Q)$ under the exterior derivative inherited from $\Omega^*_\tau(X, Q)$, which is just the exterior derivative on $\text{reg}(X)$. The cohomology of the cochain complex $\mathcal{M}\Omega^*_\tau(X, Q)$ is called the $\tau$-multiperverse cohomology of $X$, where we write

$$\mathcal{M}H^k_\tau(X, Q) := H^k(\mathcal{M}\Omega^*_\tau(X, Q)).$$

**Definition 5.56.** Let $\tau \in \mathcal{M}(X)$ be a multiperversity and let $S_A$ be the link bundle of a singular stratum $A \in \mathbb{S}\mathbb{S}(X)$ of a depth 1 multicontrolled stratified space $X$ with fixed horizontal subbundles $Q$. Define $\mathcal{M}\Omega^*_\tau(S_A, Q)$ to be the set of
forms $\omega \in \Omega^k(S_A)$ which satisfy

$$\omega \in \left( \bigoplus_{i \leq \tau(A)(k-h)} \Omega^{i,j,h}(S_A, Q) \right) \quad \text{and} \quad d_S \omega \in \left( \bigoplus_{i' \leq \tau(A)((k+1)-h')} \Omega^{i',j',h'}(S_A, Q) \right).$$

This extends to a cochain complex $M\Omega^*_\tau(S_A, Q)$ with the boundary operator the restriction of the exterior derivative $d_S$.

To avoid extraneous notation, as the $Q$ will often be fixed, hereinafter we will permit ourselves to write, for example, $MH^*_\tau(X)$ or $M\Omega^*_\tau(S_A)$, where the choice of $Q$ is clear from the context. See also Remark 5.67, which states that flat-type cone calculation is independent of certain choices, including $Q$.

Remark 5.57. Let $X$ be a product-type depth 1 multicontrolled stratified space and choose $Q$ by using the product structure for $S_A$. The contents of Definitions 5.55 and 5.56, as above, agrees with the analogous Definitions 5.28 and 5.33 for the product-type case.

Finally, we note that Proposition 5.32 remains valid for the general case of $M\Omega^*_\tau(X, Q)$; the proof is unchanged.

### 3.2 Flat-type depth 1 multicontrolled spaces

In this subsection we will define the flat-type case. This case is more general than the product-type case, but the assumptions will permit us to carry over the proofs of the results from the product-type case without excessive modification.

Definition 5.58. Let $X$ be a depth 1 multicontrolled stratified space and consider the following diagram for the link bundle $S_A$, recalling that $L_A$ is the fibre of $\pi_A$ and $L'_A$ is the fibre of $\pi'_A$; we also recall that the fibre of $\mu_A$ is $F_A$.

$$
\begin{array}{ccc}
L_A & \longrightarrow & S_A \\
\mu_A \downarrow & & \downarrow \pi_A \\
L'_A & \longrightarrow & S'_A \\
\pi'_A \downarrow & & \downarrow A
\end{array}
$$

With reference to the above diagram, we will say that $X$ is of flat-type if, for each singular stratum $A \in \mathcal{SS}(X)$, the following conditions are verified:

1. There is a metric $g_A$ on $F_A$ such that for all $z \in A$ the bundle $\mu_A|_{\pi_A^{-1}(z)}$ is flat with respect to the group $\text{Isom}(F_A, g_A)$; we assume that $S_A$, $S'_A$ and $F_A$ are oriented with fixed compatible orientations.
2. The map $\pi_A$ is flat with respect to the group $\text{BDiff}(L_A)$ of bundle diffeomorphisms of $L_A \to L'_A$.

3. There exists a finite good cover of $A$.

From this we define $Q_{1,A}$ and $Q_{2,A}$ by patching together the horizontal subspaces arising from local trivialisations of $\mu_A$ and $\pi_A$. Then $Q$ is flat and we can assume that $\pi_A$ is flat with respect to the subgroup whose derivatives are diagonal with respect to the splitting $\ker(d\pi_A) = \ker(d\mu_A) \oplus Q_{1,A}$.

Throughout the rest of this subsection we fix a flat-type depth 1 multicontrolled stratified space $(X, Q)$. Following the same outline as in the product-type case, our next aim is to obtain a decomposition for the exterior derivative $d_{S_A} : \Omega^k(S_A) \to \Omega^{k+1}(S_A)$ according to the splitting $\Omega^{i,j,h}(S_A)$. To obtain this decomposition we will use the following description of the exterior derivative. Let $\omega \in \Omega^k(S_A)$ be a $k$-form and let $Z_i \in T(S_A)$ be smooth vector fields on $S_A$. The exterior derivative $d_{S_A}$ satisfies

\[
(d_{S_A}\omega)(Z_1, \ldots, Z_{k+1}) = \
\sum_{c=1}^{k+1} (-1)^{c-1} Z_c \omega(Z_1, \ldots, \hat{Z}_c, \ldots, Z_{k+1}) + \sum_{1 \leq c < d \leq k+1} (-1)^{c+d} \omega([Z_c, Z_d], Z_1, \ldots, \hat{Z}_c, \ldots, \hat{Z}_d, \ldots, Z_{k+1}).
\]

**Lemma 5.59.** Let $(X, Q)$ be a flat-type depth 1 multicontrolled stratified space with horizontal subbundles $Q$. There exists a decomposition of the exterior derivative $d_{S_A}$, acting on $\Omega^{i,j,h}(S_A)$, into components

\[
d_{S_A} = d_{F_A,Q} + (-1)^i \cdot d_{L'_A,Q} + (-1)^{i+j} \cdot d_{A,Q}.
\]

In the above, each component acts as follows:

\[
d_{F_A,Q} : \Omega^{i,j,h}(S_A) \to \Omega^{i+1,j,h}(S_A),
\]

\[
d_{L'_A,Q} : \Omega^{i,j,h}(S_A) \to \Omega^{i,j+1,h}(S_A),
\]

\[
d_{A,Q} : \Omega^{i,j,h}(S_A) \to \Omega^{i,j,h+1}(S_A).
\]

Consequently the exterior derivative $d_{S_A}$ acts as

\[
d_{S_A} : \Omega^{i,j,h}(S_A) \to \Omega^{i+1,j,h}(S_A) \oplus \Omega^{i,j+1,h}(S_A) \oplus \Omega^{i,j,h+1}(S_A).
\]
From this, by rewriting the content of Equation 5.89, we obtain expressions for $Q$ if the various $Z$ integers of flatness: $Z_{r}$, $Z_{e}$, $Z_{i}$ are elements of $Q_{1,A}$ and the various $Z''$ are elements of $Q_{2,A}$. Throughout we let $t$ be elements of $\ker(d\mu_{A})$, the various $Z'_{i}$ be elements of $Q_{1,A}$ and the various $Z''_{e}$ be elements of $Q_{2,A}$. We obtain the following identities from flatness: $[Z_{c}, Z_{d}] \in \ker(d\mu_{A})$, $[Z'_{c}, Z'_{d}] \in Q_{1,A}$ and $[Z''_{d}, Z''_{e}] \in Q_{2,A}$: $[i_{1}Z_{c}, i_{2}Z'_{d}] = 0$, $[i_{1}Z_{c}, i_{3}Z''_{e}] = 0$ and $[i_{2}Z'_{c}, i_{3}Z''_{e}] = 0$. Following a slight abuse of notation, we may rewrite the final three identities: $[Z_{c}, Z'_{d}] = 0$, $[Z_{c}, Z''_{e}] = 0$ and $[Z'_{c}, Z''_{e}] = 0$.

The next step in the proof is to decompose $d_{S_{A}}$ acting on $\Omega^{i,j,k}(S_{A})$ using the content of Equation 5.89; due to the trigrading this produces a large number of expressions, most of which will vanish as a consequence of flatness. The computation is split up into three steps which will yield the three terms in Equations 5.91, 5.92 and 5.93. Throughout we let $Z_{i}$ be elements of $\ker(d\mu_{A})$, the various $Z'_{i}$ be elements of $Q_{1,A}$ and the various $Z''_{e}$ be elements of $Q_{2,A}$. For a fixed sum of integers $e_{1} + e_{2} + e_{3} = i + j + h + 1$, we write

$$(d_{S_{A}})(\omega)(Z_{1}, \ldots, Z_{e_{1}}, Z'_{1}, \ldots, Z'_{e_{2}}, Z''_{1}, \ldots, Z''_{e_{3}}) = C_{F_{A},Q} + C'_{L_{A},Q} + C_{A,Q}.$$  

From this, by rewriting the content of Equation 5.89, we obtain expressions for $C_{F_{A},Q}$, $C'_{L_{A},Q}$ and $C_{A,Q}$:

$$C_{F_{A},Q} = \sum_{c=1}^{s} (-1)^{c-1} Z_{c} \omega(Z_{1}, \ldots, Z_{e_{1}}, Z'_{1}, \ldots, Z'_{e_{2}}, Z''_{1}, \ldots, Z''_{e_{3}}) + \sum_{1 \leq c < d \leq e_{1}} (-1)^{c+d} \omega([Z_{c}, Z_{d}], Z_{1}, \ldots, Z'_{c}, \ldots, Z'_{d}, \ldots, Z_{e_{1}}, Z'_{1}, \ldots, Z'_{e_{2}}, Z''_{1}, \ldots, Z''_{e_{3}}) + \sum_{1 \leq c \leq e_{1}} \sum_{1 \leq d \leq e_{2}} (-1)^{c+d} \omega([Z_{c}, Z'_{d}], Z_{1}, \ldots, Z'_{c}, \ldots, Z'_{d}, \ldots, Z_{e_{1}}, Z'_{1}, \ldots, Z'_{e_{2}}, Z''_{1}, \ldots, Z''_{e_{3}}) + \sum_{1 \leq c \leq e_{1}} \sum_{1 \leq d \leq e_{2}} (-1)^{c+d} \omega([Z_{c}, Z''_{d}], Z_{1}, \ldots, Z'_{c}, \ldots, Z'_{d}, \ldots, Z_{e_{1}}, Z'_{1}, \ldots, Z'_{e_{2}}, Z''_{1}, \ldots, Z''_{e_{3}})$$

There are $3 + P(3,2)$ expressions involved, so $3 + 6 = 9$ expressions.
\[ C_{A,Q} = \sum_{c=1}^{\epsilon_2} (-1)^{c+e_1-1} Z_c' \omega(Z_1, \ldots, Z_{e_1}, Z'_1, \ldots, \hat{Z}_c', \ldots, Z'_{e_2}, Z''_1, \ldots, Z''_{e_2}) + \]
\[ \sum_{1 \leq c < d \leq \epsilon_2} (-1)^{c+e_2} \omega([Z'_c, Z''_d], Z_1, \ldots, Z_{e_1}, Z'_1, \ldots, \hat{Z}_c', \ldots, Z'_{e_2}, Z''_1, \ldots, Z''_{e_2}) + \]
\[ \sum_{1 \leq c < d \leq \epsilon_3} (-1)^{c+e_1+1} \omega([Z'_c, Z''_d], Z_1, \ldots, Z_{e_1}, Z'_1, \ldots, \hat{Z}_c', \ldots, Z'_{e_2}, Z''_1, \ldots, Z''_{e_2}) \]
\[
(5.97)
\]

In the above Equations 5.95, 5.96 and 5.97, we have not gathered the expressions into the decomposition; we do this in tandem with applying the identities which arise from flatness assumption. From this we obtain three expressions:

\[ C_{F_A,Q} = \sum_{c=1}^{\epsilon_1} (-1)^{c-1} Z_c \omega(Z_1, \ldots, \hat{Z}_c, \ldots, Z_{e_1}, Z'_1, \ldots, Z''_1, \ldots, Z_{e_3}) + \]
\[ \sum_{1 \leq c < d \leq \epsilon_1} (-1)^{c+e_1} \omega([Z_c, Z_d], Z_1, \ldots, \hat{Z}_c, \ldots, Z_{e_1}, Z'_1, \ldots, Z_{e_2}, Z''_1, \ldots, Z''_{e_1}) \]
\[
(5.98)
\]

\[ C'_{A,Q} = \sum_{c=1}^{\epsilon_3} (-1)^{c+e_1-1} Z_c' \omega(Z_1, \ldots, Z_{e_1}, Z'_1, \ldots, \hat{Z}_c', \ldots, Z'_{e_2}, Z''_1, \ldots, Z''_{e_3}) + \]
\[ (-1)^{e_1} \sum_{1 \leq c < d \leq \epsilon_2} (-1)^{c+d} \omega(Z_1, \ldots, Z_{e_1}, [Z'_c, Z''_d], Z'_1, \ldots, \hat{Z}_c, \ldots, Z''_1, \ldots, Z''_{e_2}) \]
\[
(5.99)
\]

\[ C_{A,Q} = \sum_{c=1}^{\epsilon_3} (-1)^{c+e_2+e_1-1} Z_c'' \omega(Z_1, \ldots, Z_{e_1}, Z''_1, \ldots, Z''_{e_2}, Z'_1, \ldots, Z''_{e_3}) + \]
\[ (-1)^{e_1+e_2} \sum_{1 \leq c < d \leq \epsilon_2} (-1)^{c+d} \omega(Z_1, \ldots, Z_{e_1}, Z'_1, \ldots, Z_{e_2}, [Z''_c, Z''_d], Z''_1, \ldots, \hat{Z}_c, \ldots, Z''_1, \ldots, Z''_{e_2}) \]
\[
(5.100)
\]

Having reduced Equations 5.95, 5.96 and 5.97 to Equations 5.98, 5.99 and 5.100, we can now remark that, if \( \omega \in \Omega^{i,j,h}(S_A) \), then the form in Equation 5.98 is in
\( \Omega^{i+1,j,h}(S_A) \), the form in Equation 5.99 is in \( \Omega^{i,j+1,h}(S_A) \), and the form in Equation 5.99 is in \( \Omega^{i,j,h+1}(S_A) \). Hence we may define \( d_{F_A, \mathbf{Q}} \), \( d_{L_A', \mathbf{Q}} \) and \( d_{A, \mathbf{Q}} \):

\[
d_{F_A, \mathbf{Q}}(\omega)(Z_1, \ldots, Z_i; Z_i, Z_{i+1}; Z'_1, \ldots, Z'_j; Z''_1, \ldots, Z''_h) = \sum_{c=1}^{i+1} (-1)^{c-1} Z_c \omega(Z_1, \ldots, \hat{Z}_c, \ldots, Z_{i+1}; Z'_1, \ldots, Z'_j; Z''_1, \ldots, Z''_h) + \sum_{1 \leq c < d \leq i+1} (-1)^{c+d} \omega([Z_c, Z_d], Z_1, \ldots, \hat{Z}_c, \ldots, Z_{i+1}; Z'_1, \ldots, Z'_j; Z''_1, \ldots, Z''_h) \tag{5.101}
\]

\[
d_{L_A', \mathbf{Q}}(\omega)(Z_1, \ldots, Z_i; Z'_1, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_h) = \sum_{j+1}^{i+1} (-1)^{c-1} Z'_c \omega(Z_1, \ldots, Z'_i; Z'_1, \ldots, \hat{Z}'_c, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_h) + \sum_{1 \leq c < d \leq j+1} (-1)^{c+d} \omega(Z_1, \ldots, Z_k; [Z'_c, Z'_d], Z'_1, \ldots, \hat{Z}'_c, \ldots, Z'_{j+1}; Z''_1, \ldots, Z''_h) \tag{5.102}
\]

\[
d_{A, \mathbf{Q}}(\omega)(Z_1, \ldots, Z_i; Z'_1, \ldots, Z'_j; Z''_1, \ldots, Z''_{h+1}) = \sum_{h+1}^{i+1} (-1)^{c-1} Z''_c \omega(Z_1, \ldots, Z'_i; Z'_1, \ldots, Z'_j; Z''_1, \ldots, \hat{Z}_c, \ldots, Z''_{h+1}) + \sum_{1 \leq c < d \leq h+1} (-1)^{c+d} \omega(Z_1, \ldots, Z_i; Z'_1, \ldots, Z''_d; [Z''_c, Z''_d], Z''_1, \ldots, \hat{Z}_c, \ldots, Z''_{d}, \ldots, Z''_{h+1}) \tag{5.103}
\]

By the previous comment and by the content of Equations 5.98, 5.99 and 5.100, combined with Equation 5.94, the definitions in Equations 5.101, 5.102 and 5.103 give us the appropriate arrows of Equations 5.91, 5.92 and 5.93, which satisfy the contents of Equation 5.90.

It is salient to note that \( d_{F_A, \mathbf{Q}} \), \( d_{L_A', \mathbf{Q}} \) and \( d_{A, \mathbf{Q}} \) are defined using \( \mathbf{Q} \); the possibility to define these three operations depends on flatness and on the particular choice of \( \mathbf{Q} \). Let us consider the local presentation of \( d_{F_A, \mathbf{Q}} \), \( d_{L_A', \mathbf{Q}} \) and \( d_{A, \mathbf{Q}} \); for this we fix a stratum \( A \in \mathcal{S}(X) \) and let there be defined a coordinate patch \( Y \subseteq S_A \) around \( y \in Y \). From the flatness assumption, as in Definition 5.58, we can find coordinate vector fields arising locally from trivialisations of the bundles to span \( \ker(d\mu_A), Q_{1,A} \) and \( Q_{2,A} \); let these be \( \{ \hat{\partial} u_i \}_{i=1}^f \), \( \{ \hat{\partial} v_j \}_{j=1}^t \) and \( \{ \hat{\partial} w_h \}_{h=1}^a \). A basis form in \( \Omega^{i,j,h}(S_A) \) is

\[
\omega := f(u, v, w) \wedge du \wedge dv \wedge dw_H, \tag{5.104}
\]

this representing a basis \( i+j+h = k \)-form, where \( |I| = i \), \( |J| = j \) and \( |H| = h \). Consider the action of each component in the decomposition of \( d_{S_A} \), as in Equation
5.90, acting on the form $\omega$ in Equation 5.104:

\[
d_{F,A,Q}(f \wedge du_I \wedge dv_J \wedge dw_H) := \sum_{i=1}^{l} \frac{\partial f}{\partial u_i} \wedge du_i \wedge du_J \wedge dv_J \wedge dw_H. \quad (5.105)
\]

\[
d_{L,A,Q}(f \wedge du_I \wedge dv_J \wedge dw_H) := \sum_{j=1}^{p} \frac{\partial f}{\partial v_j} \wedge dv_j \wedge du_J \wedge dv_J \wedge dw_H. \quad (5.106)
\]

\[
d_{A,Q}(f \wedge du_I \wedge dv_J \wedge dw_H) := \sum_{h=1}^{q} \frac{\partial f}{\partial w_h} \wedge dw_h \wedge du_I \wedge dv_J \wedge dw_H. \quad (5.107)
\]

This is identical to the local presentation in Subsection 2.1, one may respectively compare Equation 5.105 above with Equation 5.33, Equation 5.106 above with Equation 5.34 and Equation 5.107 above with Equation 5.35. Furthermore, the local coordinate changes preserve the $(i,j,h)$-splitting, as $\pi_A$ is flat with respect to the subgroup of bundle diffeomorphisms.

### 3.3 Calculation of $\tau$-multiperverse cohomology for a flat-type cone

In this subsection we will consider the extension of the results in Subsection 2.2 to the flat-type case; the aim is to obtain the calculation of $MH^\tau_*(cS,Q)$ for a flat-type cone $cS$. For this we will use the intermediary complex of Definition 5.56.

Let $cS$ be the depth 1 multicontrolled fibered cone over the depth 0 fibering data $\mu : S \rightarrow S' = L'$, with $S' = L'$ and $F$ compact smooth manifolds. Further, assume that $cS$ satisfies Definition 5.58. For clarity we reiterate the contents of Definition 5.58 for $cS$:

1. There is a Riemannian metric $g$ on $F$ such that the map $\mu$ is flat with respect to the group $\text{Isom}(F,g)$.

2. The map $\pi : S \rightarrow \text{pt}$ is flat with respect to the subgroup of $\text{BDiff}(S')$ of bundle diffeomorphisms.

As the sole singular stratum of $cS$ is $\text{pt} \in SS(cS)$, which is 0-dimensional with $Q_2 = 0$, then for the purpose of Definition 5.55 it will suffice to consider, in lieu of the three fold decomposition $\Omega^{i,j,h}(S)$, simply the two fold decomposition $\Omega^{i,j}(S) := \Omega^{i,j,0}(S)$. In this setting the content of Equation 5.90 takes the form

\[
d_S = d_{F,Q} + (-1)^i \cdot d_{L',Q},
\]

for the map $d_{\text{pt},Q}$ acts as the zero map.
To compute $MH^\tau_\bullet(cS, Q)$ we note that a number of intermediary results and definitions remain unchanged from the product-type case to the flat-type case. The definition of $R$ and the definition of $K^k_1$ on $\tau$-multiperverse forms remain unchanged; we will not repeat these definitions here. Further, the results that remain unchanged are Proposition 5.36, Proposition 5.39 and Lemma 5.41; for convenience we will restate these results here.

**Proposition 5.60.** Let $\tau \in M(cS)$ be a multiperversity for a flat-type cone $(cS, Q)$ and let $\omega \in M\Omega^\tau_\bullet(cS)$ be a $k$-form. Then $p^*_\tau(\omega) \in \Omega^k(\mathcal{D}(cX))$ is the lift of a $\tau$-admissible form if and only if $i \leq \tau(k)$.

*Proof.* The proof is identical to that of Proposition 5.36.

**Proposition 5.61.** Let $\tau \in M(cS)$ be a multiperversity for a flat-type cone $(cS, Q)$ and let $\omega \in M\Omega^\tau_\bullet(cS, Q)$ be a $\tau$-multiperverse form. The smooth form $p^*_\tau R\omega \in \Omega^k(\mathcal{D}(cS))$ defines the lift of a $\tau$-multiperverse form, denoted $R\omega \in M\Omega^\tau_\bullet(cS, Q)$, this yielding an endomorphism

$$R : M\Omega^\tau_\bullet(cS, Q) \rightarrow M\Omega^\tau_\bullet(cS, Q).$$

*Proof.* The proof is identical to that of Proposition 5.39.

**Lemma 5.62.** Let $\tau \in M(cS)$ be a multiperversity for a flat-type cone $(cS, Q)$ and consider the homotopy operator $K^k_1$ on liftable forms on $cS$. The following two numbered statements are verified:

1. The operator $K^k_1$ descends to an operator

$$K^k_1 : M\Omega^\tau_k(cS, Q) \rightarrow M\Omega^\tau_{k-1}(cS, Q),$$

that is to say $K^k_1$ takes $\tau$-multiperverse $k$-forms on $cS$ to $\tau$-multiperverse $(k-1)$-forms on $cS$.

2. For all $\omega \in M\Omega^\tau_k(cS, Q)$ the following identity is verified:

$$K^k_1(d\omega) - dK^k_1(\omega) = (-1)^{k-1}(\omega - R\omega).$$

(5.108)

Hence $K^\tau_\bullet$ defines a chain homotopy between the identity and $R$ endomorphisms of the complex $M\Omega^\tau_\bullet(cS, Q)$ of $\tau$-multiperverse forms on $cS$.

*Proof.* The proof is identical to that of Lemma 5.41.

From the above Lemma 5.62, we obtain directly the validity of the following result.
Corollary 5.63. Let $\tau \in \mathcal{M}(cS)$ be a multiperversity for a flat-type cone $(cS, Q)$. For all integers $k \in \mathbb{Z}_{\geq 0}$ there exists an isomorphism

$$MH^k_\mathcal{T}(cS, Q) \cong H^k(M\Omega^*_\mathcal{T}(S, Q)),$$

so that, for the cone $cS$, the two complexes defined in Definitions 5.55 and 5.56 are quasi-isomorphic.

Proof. The proof is identical to that of Corollary 5.42. \hfill \Box

To complete the computation we will need to verify that the proof of Theorem 5.43 can be adapted to the flat-type case. To do this we must define $*_F$, $\delta_F$, and further verify a result analogous to Lemma 5.37. The operator $*_F$ is defined locally for $z \in Z$ and $y \in Y \subseteq \pi^{-1}[z]$, where $Y$ is an oriented coordinate patch with $\{\hat{e}u_i\}_{i=1}^f$ locally spanning $\ker(d\mu)$, and with $\{\hat{e}v_j\}_{j=1}^g$ locally spanning $Q_1$. Given a permutation $\sigma \in \text{Sym}(f)$ of $(1, \ldots, f)$ and an ordered set of indices $\{i_k\}_{k=1}^h$, as in the product-type case, we define:

$$*_F(du_{i_1} \wedge \ldots \wedge du_{i_h})$$

$$= ||du_{i_1} \wedge \ldots \wedge du_{i_h}||_g^2 \cdot \text{sgn}(\sigma) \cdot \sqrt{\det((g_{ij}))} \cdot du_{i_{k+1}} \wedge \ldots \wedge du_{i_f} \quad (5.109)$$

$$= G_1(u) \cdot du_{i_{k+1}} \wedge \ldots \wedge du_{i_f}. \quad (5.110)$$

The above Equation 5.109 takes the same form as Equation 5.43 in the product-type case, while Equation 5.110 takes the same form as Equation 5.44 in the product-type case. Furthermore, in the above Equation 5.110 we have written

$$G_1 := ||du_{i_1} \wedge \ldots \wedge du_{i_h}||_g^2 \cdot \text{sgn}(\sigma) \cdot \sqrt{\det((g_{ij}))},$$

noting that this function is dependent only on the $u_i$ coordinates; this follows because $\mu|_{\pi^{-1}[z]}$ is flat with respect to the group of isometries $\text{Isom}(F, g)$. The definition of $\delta_F$ is as in Equation 5.48.

Lemma 5.64. For a depth 1 multicontrolled flat-type cone $(cS, Q)$ the following three numbered identities are verified:

1. $(d_{F,Q})(d_{L',Q}) = (-1) \cdot (d_{L',Q})(d_{F,Q})$.
2. $(d_{L',Q})*_F = (-1)^f \cdot *_F(d_{L',Q})$.
3. $(d_{L',Q})\delta_F = (-1)^{f+1} \cdot \delta_F(d_{L',Q})$.

Proof. The proof is identical to that of Lemma 5.37. \hfill \Box
Proposition 5.65 ([2], Theorem 5.1). For a depth 1 multicontrolled flat-type cone $(cS, Q)$ there is an isomorphism

$$H^k(\Omega^*(S)) \cong \bigoplus_{i+j=k} H^j(L', H^i(F)),$$

where every element $[\alpha] \in H^j(L', H^i(F))$ can be represented by a $\gamma \in \Omega^{i,j}(S)$ satisfying: (i) $d_{F,Q}(\gamma) = 0$, (ii) $d_{L',Q}(\gamma) = 0$, (iii) $\delta_{F}(\gamma) = 0$, (iv) if $\delta_{S}(\gamma) = 0$.

Theorem 5.66. Let $cS$ be the depth 1 multicontrolled fibered cone over the depth 0 fibering data $\mu : S \to S' = L'$, with $S' = L'$ a compact smooth manifold and $F$ a compact smooth manifold also. Fix a choice of horizontal subbundles $Q$, noting that $Q_2 = 0$, and assume that $cS$ is of flat-type. Further let $\tau \in \mathcal{M}(cS)$ be a multiperversity for $cS$. Then, for all $k \in \mathbb{Z}_{\geq 0}$, there exists an an isomorphism

$$H^k(M\Omega^*_\tau(S), Q) \cong \bigoplus_{i+j=k \atop i \leq \tau(k)} H^j(L', H^i(F)),$$

and hence there exists an isomorphism

$$MH^k_\tau(cS, Q) \cong \bigoplus_{i+j=k \atop i \leq \tau(k)} H^j(L', H^i(F)).$$

Proof. Note that the isomorphism in Equation 5.112 follows from the isomorphism in Equation 5.111; this is a consequence of Corollary 5.63. Thus we need to establish the isomorphism of Equation 5.111 to prove the result.

From the above Proposition 5.65 there is an isomorphism

$$\Psi : \bigoplus_{i+j=k} H^j(L', H^i(F)) \longrightarrow H^k(\Omega^*(S)),$$

and, for clarity, we also fix the inclusion

$$\Upsilon : H^k(M\Omega^*_\tau(S), Q) \longrightarrow H^k(\Omega^*(S)).$$

From this obtaining a composition

$$H^k(M\Omega^*_\tau(S), Q) \xrightarrow{\Upsilon} H^k(\Omega^*(S)) \xrightarrow{\Psi^{-1}} \bigoplus_{i+j=k \atop i \leq \tau(k)} H^j(L', H^i(F)).$$

We aim to establish that $\Phi^{-1}\Upsilon$ is an isomorphism onto $\bigoplus_{i+j=k \atop i \leq \tau(k)} H^j(L', H^i(F))$, that is that $\Psi^{-1}\Upsilon$ is injective and surjective onto $\bigoplus_{i+j=k \atop i \leq \tau(k)} H^j(L', H^i(F))$. Firstly, we verify the surjectivity of $\Psi^{-1}\Upsilon$ onto $\bigoplus_{i+j=k \atop i \leq \tau(k)} H^j(L', H^i(F))$, assuming that the
in injectivity of $\Psi^{-1} \Upsilon$ is verified; thus we reduce the proof of the result to the proof of injectivity. To prove surjectivity we consider the following commutative diagram.

$$
\begin{array}{ccc}
\bigoplus_{i+j=k} H^j(L', \mathbf{H}^i(F)) & \xrightarrow{\Psi|} & H^k(M \Omega^\bullet_r(S, \mathbf{Q})) \\
\downarrow & & \downarrow \\
\bigoplus_{i+j=k} H^j(L', \mathbf{H}^i(F)) & \xrightarrow{\Psi} & H^k(\Omega^\bullet(S)) \\
\downarrow & & \downarrow \Psi^{-1} \\
\bigoplus_{i+j=k} H^j(L', \mathbf{H}^i(F)) & & \\
\end{array}
$$

The top row $\Psi|$ is defined identically to $\Psi$, which maps into $H^k(M \Omega^\bullet_r(S, \mathbf{Q}))$ by its definition. We can deduce that the top right vertical arrow $\Upsilon$ is injective, this is as we have assumed that $\Psi^{-1} \Upsilon$ is injective. Following a similar argument, we also deduce that the top row $\Psi|$ is injective; this completes the proof of the surjectivity of $\Psi^{-1} \Upsilon$ onto $\bigoplus_{i+j=k} H^j(L', \mathbf{H}^i(F))$, for the square in the above commutative diagram commutes and $\Psi^{-1} \Psi$ is the identity.

Now we verify the injectivity of $\Psi^{-1} \Upsilon$ into $\bigoplus_{i+j=k} H^j(L', \mathbf{H}^i(F))$. As in the proof of Theorem 5.43 this reduces to the following: if $\omega \in \Omega^k_r(S)$ is a closed $\tau$-multiperverse form, that is a form satisfying $d_S(\omega) = 0$, and if we have $\omega = d_S(\eta)$ for some $\eta \in \Omega^{k-1}(S)$, then there exist a $\tau$-multiperverse form $\eta' \in \Omega^{k-1}(S, \mathbf{Q})$ such that $\omega = d_S(\eta')$. The proof of this statement is identical to the proof of the corresponding statement in Theorem 5.43, using instead of $d_L$, the operation $d_{L, \mathbf{Q}}$, and instead of $d_F$ the operation $d_{F, \mathbf{Q}}$, and further applying the above Lemma 5.64 instead of Lemma 5.37.

Remark 5.67. The computation of the cohomology $MH^\bullet_r(cS, \mathbf{Q})$ in the above Theorem 5.43 is independent of the choice of metric $g$ on $F$ and the choice of horizontal subbundles $\mathbf{Q}$.

If we consider $cS$ with trivial multicontrol data, then the multiperverse cohomology computed for $cS$ with a multiperversity $m(p)$, where $p$ is a general perversity for $cS$, should agree with intersection cohomology.

**Corollary 5.68.** Let $cS$ be a flat-type depth 1 multicontrolled cone with $S' = S$ and $\mu = 1_S$, so that the multicontrol data are redundant. Let $m(p) \in \mathcal{M}(cS)$ be the multiperversity associated to a perversity $p$ for $cS$; this is specified by an integer $p(pt) \in [-1, \dim(S)]$. Then, for all $k \in \mathbb{Z}_{\geq 0}$, there exists an isomorphism

$$
MH^k_{m(p)}(cS) \cong \begin{cases} 
H^k(S) & \text{if } k \leq p(pt), \\
0 & \text{if } k > p(pt). 
\end{cases} 
$$

(5.113)
Proof. The proof is analogous to that of Corollary 5.47 in the product-type case.\hfill\qed

\section*{3.4 Poincaré lemma for $\tau$-multiperverse forms in the flat-type case}

In this subsection we will compute the $\tau$-multiperverse cohomology of a cylinder $X \times \mathbb{R}$ where $X$ is a flat-type depth 1 multicontrolled stratified space.

Firstly, recall that by Corollary 5.49 the lattices $M(X)$ and $M(X \times \mathbb{R})$ are isomorphic. We will be able to use the proofs in Subsection 2.3 without modification, but we must verify that the product $X \times \mathbb{R}$ of a flat-type $X$ with $\mathbb{R}$ is still of flat-type, noting the changes that occur to the horizontal subbundles. As the link bundles in $X \times \mathbb{R}$ are $S_{A \times \mathbb{R}} = S_A \times \mathbb{R}$ and $S'_{A \times \mathbb{R}} = S'_A \times \mathbb{R}$, then from the equality

$$T(S_A) = \ker(d\mu_A) \oplus Q_{1,A} \oplus Q_{2,A},$$

we may define

$$T(S_{A \times \mathbb{R}}) := \ker(d\mu_A) \oplus Q_{1,A} \oplus (Q_{2,A} \oplus T(\mathbb{R})).$$

Considering the conditions in Definition 5.58 we see that condition 1 is still valid for $X \times \mathbb{R}$, as is condition 2, because $\mathbb{R}$ has the trivial bundle structure over $\mathbb{R}$ in $X \times \mathbb{R}$. Finally, condition 3 is still valid because if $A$ has a finite good cover, then so does $A \times \mathbb{R}$.

\begin{corollary}
Let $X$ be a flat-type depth 1 multicontrolled stratified space and let $\tau \in M(X)$ be a depth 1 multiperversity for $X$ and $X \times \mathbb{R}$. For all $k \in \mathbb{Z}_{\geq 0}$ there is an isomorphism

$$MH^k_\tau(X \times \mathbb{R}) \cong MH^k_\tau(X).$$

Furthermore, by induction, there exists an isomorphism

$$MH^k_\tau(X \times \mathbb{R}^n) \cong MH^k_\tau(X).$$

\end{corollary}

Proof. This follows exactly as in Corollary 5.53, noting that the contents of Lemma 5.52 and 5.50 remain valid.\hfill\qed
Chapter 6
Properties of multiperverse cohomology

In this chapter we will study properties of multiperverse cohomology, where multiperverse cohomology has been defined in Chapter 5. For this we will use results from the theory of de Rham cohomology, combined with the cone calculation of Theorem 5.66 and the cylinder calculation of Corollary 5.69. In Section 1 we will obtain Mayer-Vietoris sequences and a K"unneth formula for multiperverse cohomology; these results will be obtained by modifying the classical proofs for de Rham cohomology. In Section 2 we will obtain a Poincaré duality for multiperverse cohomology for the cone and the cylinder, defining the complexes of relative multiperverse and lift-compact multiperverse forms, while in Section 3 we extend these results to a global duality. In the final two Sections 4 and 5 we consider self-dual multiperversities and the $L^2$-cohomology of certain cusps.

1 Mayer-Vietoris sequence and a K"unneth formula

In this section we will obtain two Mayer-Vietoris sequences and a K"unneth formula for multiperverse cohomology.

1.1 Mayer-Vietoris sequence for multiperverse cohomology

In this subsection we will obtain two Mayer-Vietoris sequences. In the first sequence of Lemma 6.1 the subsets may simultaneously intersect the same singular stratum, while in the second sequence of Lemma 6.4 the subsets are required not to intersect at a singular stratum.
Lemma 6.1. Let $X$ be a flat-type depth 1 multicontrolled stratified space and let $A \in \SS(X)$ be a singular stratum. Let $U_1, U_2 \subseteq A$ be open subsets and consider $Y_1 = \pi_A^{-1}[U_1]$ and $Y_2 = \pi_A^{-1}[U_2]$, with $Z = Y_1 \cup Y_2 = \pi_A^{-1}[U_1 \cup U_2]$. There exists a short exact sequence

$$0 \to M\Omega^*_Y(Z) \xrightarrow{\alpha} M\Omega^*_Y(Y_1) \oplus M\Omega^*_Y(Y_2) \xrightarrow{\beta} M\Omega^*_Y(Y_1 \cap Y_2) \to 0,$$

(6.1)

where $\alpha$ is defined by $\omega \mapsto (\omega|_{Y_1}, \omega|_{Y_2})$, and where $\beta$ is defined by $(\eta, \eta') \mapsto (\eta|_{Y_1 \cap Y_2} - \eta'|_{Y_1 \cap Y_2})$.

Proof. It is salient to note the multicontrol structures for $Y_1, Y_2, Z$, and $Y_1 \cap Y_2$; these are induced from the multicontrol structure for $X$. The singular stratum in $Y_1$ is $U_1$, with link bundle $\pi_A^{-1}[U_1] \cap S_A$, where we also recall the tri-grading

$$\Omega^k(\pi_A^{-1}[U_1] \cap S_A) = \bigoplus_{i+j+k=h} \Omega^{i,j,k}(\pi_A^{-1}[U_1] \cap S_A).$$

The multicontrol structures for $Y_2, Z$, and $Y_1 \cap Y_2$ are defined analogously. We remark that the restriction of a $\tau$-multiperverse form from $Y_1$ to $Y_1 \cap Y_2$, for example, is again $\tau$-multiperverse.

The morphisms $\alpha$ and $\beta$ map $\tau$-admissible forms to $\tau$-admissible forms; this follows because the $\tau$-multiperversity conditions are local to the singular strata, as noted in Remark 5.29. Furthermore, since $\alpha$ and $\beta$ are morphisms of complexes, as in the smooth case, we can deduce that they map $\tau$-multiperverse forms to $\tau$-multiperverse forms by Proposition 5.35. Next we verify that Equation 6.1 defines a short exact sequence, where we begin by noting that $\alpha$ is injective and that the statement $\im(\alpha) \subseteq \ker(\beta)$ follows as in the smooth case. To verify the statement $\ker(\beta) \subseteq \im(\alpha)$, let $\beta(\eta, \eta') = 0$, so that $\eta|_{Y_1 \cap Y_2} = \eta'|_{Y_1 \cap Y_2}$, from where it follows that we may patch $\eta$ and $\eta'$ into a $\tau$-admissible form on $Z$. Furthermore, we have

$$d\beta(\eta, \eta') = d(\eta|_{Y_1 \cap Y_2} - \eta'|_{Y_1 \cap Y_2}) = (d\eta)|_{Y_1 \cap Y_2} - (d\eta')|_{Y_1 \cap Y_2},$$

so that we may patch $d\eta$ and $d\eta'$ to a $\tau$-admissible form on $Z$, proving that the patching of $\eta$ and $\eta'$ is $\tau$-multiperverse. Finally, it remains to verify the surjectivity of $\beta$, that is to show that $\im(\beta) = M\Omega^*_Y(Y_1 \cap Y_2)$; for this we fix two functions

$$\chi : \reg(Z) \to \mathbb{R} \text{ and } \chi' : \reg(Z) \to \mathbb{R}. $$

For the two functions above, we can assume that $\chi + \chi' = 1$, $\supp(\chi) \subseteq \reg(Y_1)$ and $\supp(\chi') \subseteq \reg(Y_2)$; we can further assume that $\chi$ and $\chi'$ lift to smooth functions on the deshirring, that is that they are controlled, and furthermore
that they lift from a partition of unity for $U_1$ and $U_2$. For a $\tau$-multiperverse form $\omega \in M\Omega_\tau^*(Y_1 \cap Y_2)$, we verify that the forms $\chi \cdot \omega$ and $\chi' \cdot \omega$ are both $\tau$-multiperverse forms, in $M\Omega_\tau^*(Y_2)$ and $M\Omega_\tau^*(Y_1)$ respectively. It suffices to verify this for $\chi \cdot \omega$ as the verification for $\chi' \cdot \omega$ is analogous. Firstly we note that $\chi \cdot \omega$ is $\tau$-admissible, this follows because the $\tau$-multiperversity conditions are local to $A$, noting also that $d_\pi \chi = \tau$-admissible.

\begin{equation}
  (6.2)
\end{equation}

To verify that $d(\chi) \wedge \omega$ is $\tau$-admissible, consider that $\chi$ depends only on the $U_1$-variables, that is

\[ R_A(d(\chi)) \in \Omega^{0,0,1}(\pi_A^{-1}[U_1 \cup U_2] \cap S_A), \]

so that if

\[ R_A(\omega) \in \bigoplus_{i \leq \tau(k-h)} \Omega^{i,j,h}(\pi_A^{-1}[U_1 \cup U_2] \cap S_A), \]

then we have

\[ R_A(d(\chi) \wedge \omega) \in \bigoplus_{i \leq \tau(k-h)} \Omega^{i,j,h+1}(\pi_A^{-1}[U_1 \cup U_2] \cap S_A). \]

Having verified that $\chi \cdot \omega$ and $\chi \cdot \omega'$ are $\tau$-multiperverse forms, we note that

\[ \beta(\chi' \cdot \omega, -\chi \cdot \omega) = \chi' \cdot \omega + \chi \cdot \omega \]

\[ = (\chi + \chi') \cdot \omega \]

\[ = \omega. \]

This verifies the surjectivity of $\beta$ and proves the result.

\[ \square \]

Remark 6.2. The validity of the short exact sequence in Equation 6.1 induces a long exact sequences on cohomology

\[ \ldots \rightarrow MH^k_\tau(X) \xrightarrow{\alpha} MH^k_\tau(Y_1) \oplus MH^k_\tau(Y_2) \xrightarrow{\beta} MH^k_\tau(Y_1 \cap Y_2) \xrightarrow{\delta} MH^{k+1}_\tau(X) \rightarrow \ldots. \]

\[ (6.3) \]

It is useful to recall the definition of $\delta$, the connecting morphism obtained via the snake lemma. Let $[\eta] \in MH^k_\tau(Y_1 \cap Y_2)$ be a cohomology class. By the surjectivity of $\beta$ we have $\beta(\chi \cdot \eta, \chi' \cdot \eta) = \eta$, but also

\[ \beta(d(\chi \cdot \eta, \chi' \cdot \eta)) = d(\eta) = 0, \]
implying that \( d(\chi \cdot \eta, \chi' \cdot \eta) \in \ker(\beta) \). From this we conclude that there exists some \( \omega \in M\Omega^\tau_r(X) \) with \( \alpha(\omega) = d(\chi \cdot \eta, \chi' \cdot \eta) \); this is essentially the \( \tau \)-multiperverse form patched together from \( d(\chi \cdot \eta) \) and \( d(\chi' \cdot \eta) \). Then \( \delta([\eta]) \) is defined to be this \([\omega] \).

**Example 6.3.** Consider the flat-type depth 1 multicontrolled stratified spaces \( Y_1 = cS \times U_1 \) and \( Y_2 = cS \times U_2 \), where \( U_1 \) and \( U_2 \) are smooth manifolds and \( cS \) is a flat-type depth 1 multicontrolled cone. If further

\[
X = (cS \times U_1) \cup (cS \times U_2)
\]

and \( cS \times (U_1 \cap U_2) \) are flat-type depth 1 multicontrolled stratified spaces, then using the above Lemma 6.1 we obtain a short exact sequence

\[
0 \to M\Omega^\tau_r(X) \xrightarrow{\alpha} M\Omega^\tau_r(Y_1) \oplus M\Omega^\tau_r(Y_2) \xrightarrow{\beta} M\Omega^\tau_r(Y_1 \cap Y_2) \to 0.
\]

**Lemma 6.4.** Let \( X \) be a flat-type depth 1 multicontrolled stratified space. Let \( Y_1 \) and \( Y_2 \) be open subsets with \( \cl_X(Y_1 \cap Y_2) \subseteq \reg(X) \) disjoint from all the singular strata of \( X \) and \( Z = Y_1 \cup Y_2 \). There exists a short exact sequence

\[
0 \to M\Omega^\tau_r(Z) \xrightarrow{\alpha} M\Omega^\tau_r(Y_1) \oplus M\Omega^\tau_r(Y_2) \xrightarrow{\beta} \Omega^\tau_r(Y_1 \cap Y_2) \to 0,
\]

where \( \alpha \) is defined by \( \omega \mapsto (\omega|_{Y_1}, \omega|_{Y_2}) \), and where \( \beta \) is defined by \( (\eta, \eta') \mapsto (\eta|_{Y_1 \cap Y_2} - \eta'|_{Y_1 \cap Y_2}) \). Note that \( M\Omega^\tau_r(Y_1 \cap Y_2) = \Omega^\tau_r(Y_1 \cap Y_2) \), for we have assumed that \( Y_1 \cap Y_2 \) does not approach the singular strata, so that the \( \tau \)-multiperversity conditions on \( Y_1 \cap Y_2 \) are null.

**Proof.** The proof is similar to that of Lemma 6.1. The morphism \( \alpha \) maps \( \tau \)-multiperverse forms to \( \tau \)-multiperverse forms, this follows because the \( \tau \)-multiperversity conditions are local to each singular stratum and the intersection \( Y_1 \cap Y_2 \) does not intersect any singular stratum. The map \( \beta \) maps \( \tau \)-multiperverse forms to smooth forms, this follows because the intersection \( Y_1 \cap Y_2 \) does not meet any singular stratum. Next we verify that Equation 6.4 defines a short exact sequence, where we begin by noting that \( \alpha \) is injective and that \( \text{im}(\alpha) \subseteq \ker(\beta) \). To verify the statement \( \ker(\beta) \subseteq \text{im}(\alpha) \), let \( \beta(\eta, \eta') = 0 \), so that \( \eta|_{Y_1 \cap Y_2} = \eta'|_{Y_1 \cap Y_2} \), so that we may patch \( \eta \) and \( \eta' \) into a \( \tau \)-admissible form on \( Z \). Furthermore, we have

\[
d\beta(\eta, \eta') = d(\eta|_{Y_1 \cap Y_2} - \eta'|_{Y_1 \cap Y_2}) = (d\eta)|_{Y_1 \cap Y_2} - (d\eta')|_{Y_1 \cap Y_2},
\]

so that we may patch \( d\eta \) and \( d\eta' \) into a \( \tau \)-admissible form on \( Z \), proving that the patching of \( \eta \) and \( \eta' \) is \( \tau \)-multiperverse. Finally, it remains to verify the surjectivity.
of $\beta$, that is to show that $\text{im}(\beta) = \Omega^*(Y_1 \cap Y_2)$; for this we fix two functions

$$\chi: \text{reg}(Z) \rightarrow \mathbb{R} \text{ and } \chi': \text{reg}(Z) \rightarrow \mathbb{R}.$$ 

For the two functions above we can assume that $\chi + \chi' \equiv 1$, $\text{supp}(\chi) \subseteq \text{reg}(Y_1)$ and $\text{supp}(\chi') \subseteq \text{reg}(Y_2)$; we can also assume that these functions are liftable. For a form $\omega \in \Omega^*(Y_1 \cap Y_2)$ we verify that the forms $\chi \cdot \omega$ and $\chi' \cdot \omega$ are both $\tau$-multiperverse forms, in $M\Omega^\tau_p(Y_2)$ and $M\Omega^\tau_p(Y_1)$ respectively. It suffices to verify this for $\chi \cdot \omega$ as the verification for $\chi' \cdot \omega$ is analogous. To deduce that $\chi \cdot \omega$ is $\tau$-multiperverse, consider that for a singular stratum $A$ we have $\chi|_{\theta_X^{-1}[A]} \equiv 1$ (in fact this is valid in an open neighbourhood of $\theta_X^{-1}[A]$ in deshirring), whence $\chi$ does not modify $\omega$ near any singular stratum, where $\chi \cdot \omega$ vanishes, and $\chi \cdot \omega$ is hence $\tau$-multiperverse. This verifies the surjectivity of $\beta$ and proves the result.

Remark 6.5. As in Remark 6.2, we note that the validity of the short exact sequence in Equation 6.4 induces a long exact sequence on cohomology

$$\ldots \rightarrow MH^k_p(X) \xrightarrow{\alpha} MH^k_p(Y_1) \oplus MH^k_p(Y_2) \xrightarrow{\beta} H^k(Y_1 \cap Y_2) \xrightarrow{\delta} MH^{k+1}_p(X) \rightarrow \ldots.$$ 

As a consequence of the two Mayer-Vietoris sequences of Lemmas 6.1 and 6.4 we can deduce, under certain assumptions, that the cohomology of a flat-type depth 1 multicontrolled stratified space is finite dimensional.

Corollary 6.6. Let $X$ be a flat-type depth 1 multicontrolled stratified space. If the cohomology groups of all link bundles $S_A$ and of $\text{reg}(X)$ are finite dimensional, then the $\tau$-multiperverse cohomology groups $MH^\tau_p(X)$ are finite dimensional.

Proof. For a fixed singular stratum $A \in \mathcal{SS}(X)$ fix a finite good cover $\{U_i\}_{i=1}^N$ of $A$ which trivialises $\pi_A$; we note that a good cover trivialises a bundle. This yields a finite cover of $T_A$ of the form $\{\pi_A^{-1}[U_i]\}_{i=1}^N$, where each $\pi_A^{-1}[U_i]$ is the multicontrolled space $U_i \times cS_A$. Firstly we assume that $N = 1$, where $A = U_1$ is $\mathbb{R}^n$, so that $X = U_1 \times cS_A$, so that $MH^\tau_p(X)$ is finite dimensional because the cohomology of $S_A$ is finite dimensional. From there we proceed by induction on the number of elements in the finite cover $\{U_i\}_{i=1}^N$, using the Mayer-Vietoris sequence of Lemma 6.1. If the result is valid for a finite good cover of $N$ open subsets, then consider a manifold $A$ having a good cover by $N + 1$ open subsets $\{U_i\}_{i=1}^{N+1}$. Now the intersection

$$(U_1 \cup U_2 \cup \ldots \cup U_N) \cap U_{N+1}$$
has a finite good cover with $N$ open sets, namely $U_1 \cup U_{N+1}$ to $U_N \cup U_{N+1}$. From this we deduce that $MH^*_\tau(\pi^{-1}_A[U_1 \cup \ldots \cup U_{N+1}])$ is finite dimensional using the Mayer-Vietoris sequence of Lemma 6.1 for the two subsets $U_1 \cup U_2 \cup \ldots \cup U_N$ and $U_{N+1}$. For this we note that $MH^*_\tau(\pi^{-1}_A[U_{N+1}])$ is finite dimensional since $U_{N+1}$ is $\mathbb{R}^n$, and $MH^*_\tau(\pi^{-1}_A(U_1 \cup U_2 \cup \ldots \cup U_N) \cap U_{N+1})$ is finite dimensional by assumption, for it has a finite good cover by $N$ open subsets. This establishes that $MH^*_\tau(T_A)$ is finite dimensional for each singular stratum $A \in \mathcal{SS}(X)$. Finally we apply the Mayer-Vietoris sequence of Lemma 6.4 to the regular stratum $\text{reg}(X)$ and the union $\bigcup_{A \in \mathcal{SS}(X)} T_A$ of all the $T_A$ (note here that the $T_A$ are disjoint). Since $\text{reg}(X) \cap \bigcup_{A \in \mathcal{SS}(X)} T_A$ is diffeomorphic to a disjoint union of the $S_A$, then it has finite dimensional cohomology, as does $\text{reg}(X)$ by assumption, and thus the $\tau$-multiperverse cohomology of $X$ is finite dimensional. \hfill \Box

1.2 Partial K"{u}nneth formula for multiperverse cohomology

In this subsection we will obtain a partial\footnote{The word partial here serves to distinguish this from the full K"{u}nneth formula for intersection homology, as in [18] (6.3); the result there is obtained using sheaf theory.} K"{u}nneth formula for multiperverse cohomology. A similar result for intersection cohomology, where the intersection forms are also based on smooth forms, can be found in [26] (Theorem 8.4); the result herein differs from the result therein by the presence of details related to the definition of multiperverse forms and the use of the pullback to the boundary $R$. The proof is based on the classical presentation of the K"{u}nneth formula in [9](pp. 47 - 50).

Firstly we recall the multicontrol data for $X \times M$. In Definition 4.4 we have defined the multicontrol data for the cylinder $X \times \mathbb{R}$ based on the multicontrol data for $X$. If $M$ is a smooth manifold and $X$ is a depth 1 multicontrolled stratified space, and we consider $X \times M$, with the strata given by

$$S(X \times M) := \{ A \times M : A \in S(X) \},$$

then, by replacing the identity map $1_{\mathbb{R}}$ by $1_M$ in the arguments of Definition 4.4, we obtain the multicontrol data for $X \times M$. Immediately, if $X$ is of flat-type, then so is $X \times M$, following similar arguments to Subsection 3.4, and furthermore, for $X \times M$, by following an analogous argument to Proposition 5.49, we obtain the following result.

**Proposition 6.7.** Let $M$ be a smooth manifold. There exists an isomorphism between the lattice $\mathcal{M}(X)$ and the lattice $\mathcal{M}(X \times M)$.
Proof. In the proof of Proposition 5.49, replace $\dim(R)$ by $\dim(M)$ in Equation 5.81, obtaining the same cancellation in Equation 5.82, finally yielding that

$$\text{codim}_{X \times M}(S'_A \times M) = \dim(X) - \dim(S_A) = \text{codim}_X(S'_A).$$

The conclusion now follows as in the proof of Proposition 5.49. \qed

Lemma 6.8. Let $M$ be a compact smooth manifold and let $X$ be a flat-type depth 1 multicontrolled cone. There exists an isomorphism

$$M^{k}\tau_pX^\hat{\tau}M^q \cong \bigoplus_{i+j=k} (M^{i}\tau_pX^\hat{\tau}M^j \otimes H^j(M)).$$

Proof. The verification of this result follows the outline of the proof of the usual Künneth formula. However, because there are two inductions, and extra details related to the multiperversity conditions, we split the proof up into four parts.

Part 1. The first part of the proof is to define the appropriate map $\Psi$, which we will then demonstrate to be an isomorphism. In this part we can assume that $M$ is an arbitrary smooth manifold. The map has the form

$$\Psi : M^{k}\tau_pX^\hat{\tau}M^q \cong M^{k+k'}\tau_pX^\hat{\tau}M^q \quad (6.5)$$

and is defined by the correspondence

$$([\omega], [\eta]) \mapsto [p_{\text{reg}(X)}^*(\omega) \wedge p_M^*(\eta)],$$

where $p_M$ and $p_{\text{reg}(X)}$ denote the natural projections from $\text{reg}(X \times M) = (\text{reg}(X) \times M)$ onto $M$ and $\text{reg}(X)$. These two maps are the following:

$$p_M : \text{reg}(X) \times M \rightarrow M,$$
$$p_{\text{reg}(X)} : \text{reg}(X) \times M \rightarrow \text{reg}(X).$$

Next we verify that the morphism in Equation 6.5 is well-defined; this is similar to the smooth case, but where appropriate we must verify the preservation of the $\tau$-multiperversity conditions.

Part 2. Fix two cohomology classes $[\omega] \in M^{k}\tau_pX^\hat{\tau}M^q$ and $[\eta] \in H^{k'}(M)$. We begin the verification of the well-definition of $\Psi$ by showing that $p_{\text{reg}(X)}^*(\omega) \wedge p_M^*(\eta)$ is $\tau$-admissible. For this we fix a singular stratum $A \in SS(X)$, where we obtain directly that

$$R_{A \times M}(p_{\text{reg}(X)}^*(\omega) \wedge p_M^*(\eta)) = R_A(\omega) \wedge p_M^*(\eta). \quad (6.6)$$

In the above Equation 6.6 and hereinafter we let $R$ denote either of $R_{A \times M}$ or $R_A$. 

as appropriate, these are as defined in Subsection 1.2 in Chapter 2. From the
\( \tau \)-multiperversity of \( \omega \) we obtain that
\[
R(\omega) \in \bigoplus_{i \leq \tau(k-h)} \bigoplus_{i+j+h=k} \Omega^{i,j,h}(S_A),
\]
from where, for the above Equation 6.6, we deduce that
\[
R(\omega) \wedge p^*_M(\eta) \in \bigoplus_{i \leq \tau(k-h)} \bigoplus_{i+j+h+k'=k+k'} \Omega^{i,j,h+k'}(S \times M),
\]
showing that \( p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(\eta) \) is \( \tau \)-admissible on \( X \times M \). To show that \( p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(\eta) \) is \( \tau \)-multiperverse, we must also show that the form
\[
d(p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(\eta))
\]
is \( \tau \)-admissible on \( X \times M \). To show this we apply the Leibniz rule to obtain
\[
d(p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(\eta)) =
\]
\[
p^*_{\text{reg}(X)}(d\omega) \wedge p^*_M(\eta) + (-1)^k \cdot p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(d\eta),
\]
and then note the following:

\[
R\left( p^*_{\text{reg}(X)}(d\omega) \wedge p^*_M(\eta) \right) = R(d\omega) \wedge p^*_M(\eta) \in \bigoplus_{i \leq \tau(k+1)} \bigoplus_{i+j+h+k'=k+1+k'} \Omega^{i,j,h+k'}(S \times M),
\]
\[
R\left( p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(d\eta) \right) = R(\omega) \wedge p^*_M(d\eta) \in \bigoplus_{i \leq \tau(k+1)} \bigoplus_{i+j+h+k'+1=k+k'+1} \Omega^{i,j,h+k'+1}(S \times M).
\]
From this we deduce that the \( \Psi \) in Equation 6.5 maps \( \tau \)-multiperverse forms to \( \tau \)-multiperverse forms.

Part 3. The next step is to verify that \( \Psi \) is well-defined on cohomology classes. First we verify that \( \Psi \) maps closed forms to closed forms. If we assume that
\( d(\eta) = 0 \), and that \( d(\omega) = 0 \), and then consider the expression
\[
d(p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(\eta)) =
\]
\[
p^*_{\text{reg}(X)}(d\omega) \wedge p^*_M(\eta) + (-1)^k \cdot p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(d\eta) = 0,
\]
we obtain the required conclusion. Next we verify the validity of the equality
\[
p^*_{\text{reg}(X)}(\omega + d\gamma') \wedge p^*_M(\eta + d\gamma) - p^*_{\text{reg}(X)}(\omega) \wedge p^*_M(\eta) = d\gamma'',
\]
(6.7)
where \( \eta \) and \( \omega \) are closed forms, \( \gamma' \) is \( \tau \)-multiperverse by assumption, and \( \gamma'' \) is to be constructed and to be shown to be \( \tau \)-multiperverse, that is to be shown to be contained in \( M\Omega_{\tau}^{k+k'-1}(X \times M) \). The above Equation 6.7 reduces to
\[
(p^*_\text{reg}(X)(\omega) \wedge p^*_M(d\gamma')) + (p^*_\text{reg}(X)(d\gamma') \wedge p^*_M(\eta)) + (p^*_\text{reg}(X)(d\gamma') \wedge p^*_M(d\gamma)) = d\gamma'',
\]
and to obtain \( \gamma'' \) we remark that the following equations are verified:
\[
\begin{align*}
p^*_\text{reg}(X)(\omega) \wedge p^*_M(d\gamma) &= (-1)^k \cdot d(p^*_\text{reg}(X)(\omega) \wedge p^*_M(\gamma))., \quad (6.8) \\
p^*_\text{reg}(X)(d\gamma') \wedge p^*_M(\eta) &= d(p^*_\text{reg}(X)(\gamma') \wedge p^*_M(\eta))., \quad (6.9) \\
p^*_\text{reg}(X)(d\gamma') \wedge p^*_M(d\gamma) &= d(p^*_\text{reg}(X)(\gamma') \wedge p^*_M(d\gamma)).. \quad (6.10)
\end{align*}
\]
Using the right-hand sides in the above yields
\[
\gamma'' = (-1)^k \cdot (p^*_\text{reg}(X)(\omega) \wedge p^*_M(\gamma)) + p^*_\text{reg}(X)(\gamma') \wedge p^*_M(\eta) + p^*_\text{reg}(X)(\gamma') \wedge p^*_M(d\gamma).
\]
Let us verify that the right-hand side in the above Equation 6.11 defines a \( \tau \)-multiperverse form; this would prove the validity of Equation 6.7 as required. In Equation 6.8, which holds because \( d\omega = 0 \), the form \( p^*_\text{reg}(X)(\omega) \wedge p^*_M(\gamma) \) is \( \tau \)-multiperverse because \( \omega \) is \( \tau \)-multiperverse. In Equation 6.9, which holds because \( d\eta = 0 \), the form \( p^*_\text{reg}(X)(\gamma') \wedge p^*_M(\eta) \) is \( \tau \)-multiperverse because \( \gamma' \) is \( \tau \)-multiperverse. Finally, in Equation 6.10 the form \( p^*_\text{reg}(X)(\gamma') \wedge p^*_M(d\gamma) \) is \( \tau \)-multiperverse because \( \gamma' \) is \( \tau \)-multiperverse. From this we deduce that the term on the right-hand side in Equation 6.11 is \( \tau \)-multiperverse; this proves that \( \Psi \) is a well-defined bilinear map between the cohomology groups, as in Equation 6.5. Using the fact that \( \Psi \) is bilinear, we note that \( \Psi \) further induces a natural map
\[
\Psi : MH^k_\tau(X) \otimes H^{k'}(M) \rightarrow MH^{k+k'}_\tau(X \times M).
\]

**Part 4.** The next and final stage in the proof is to show that \( \Psi \) defines an isomorphism; this is achieved as follows:

1. By an induction on the number of open subsets in the finite good cover of \( M \) for \( X = cS \). In this part it is useful to weaken the condition on \( M \) from compact to having a finite good cover.

2. By induction on the number of open subsets in the finite good cover of \( A \) for \( X = T_A \), using Lemma 6.1.

3. By induction on the number of cusps in \( X \), using Lemma 6.4.
Part 4.1. Let us now proceed with the first induction. Here we let $X$ be $cS$, with $SS(X) = \{pt\}$, and we assume that $M$ has a finite\footnote{Any smooth manifold has a good cover ([9], Theorem 5.1). If $M$ is compact this good cover can be chosen to be finite.} good cover $\{U_i\}_{i=1}^N$. From the good cover $\{U_i\}_{i=1}^N$ of $M$ we have an induced cover $\{cS \times U_i\}_{i=1}^N$ of $cS \times M$.

To begin with let $U_1$ and $U_2$ be two arbitrary subsets of $M$, with $Y_1 = cS \times M$ and $Y_2 = cS \times U_2$. Then we consider the standard long exact sequence for $U_1$ and $U_2$:

$$\ldots \to H^j(U_1 \cup U_2) \xrightarrow{\alpha} H^j(U_1) \oplus H^j(U_2) \xrightarrow{\beta} H^j(U_1 \cap U_2) \xrightarrow{\delta} H^{j+1}(U_1 \cup U_2) \to \ldots .$$

We can tensor this long exact sequence with $MH^{k-j}(cS)$ to obtain a long exact sequence\footnote{Note that all the objects here are $\mathbb{R}$-vector spaces and that every $\mathbb{R}$-vector space is flat as an $\mathbb{R}$-module, so that the functor $MH^{k-j}(cS) \otimes -$ is an exact functor, thus preserving long exact sequences.}

$$\ldots \to MH^{k-j}(cS) \otimes H^j(U_1 \cup U_2) \to \left( MH^{k-j}(cS) \otimes H^j(U_1) \right) \oplus \left( MH^{k-j}(cS) \otimes H^j(U_2) \right) \to MH^{k-j}(cS) \otimes H^j(U_1 \cap U_2) \xrightarrow{\delta} MH^{k-j}(cS) \otimes H^{j+1}(U_1 \cup U_2) \to \ldots ,$$

which we sum over $j = 0, \ldots , k$ to obtain

$$\ldots \to \bigoplus_{j=0}^k MH^{k-j}(cS) \otimes H^{j-1}(U_1 \cap U_2) \to \bigoplus_{j=0}^k MH^{k-j}(cS) \otimes H^j(U_1 \cup U_2) \to \bigoplus_{j=0}^k \left( MH^{k-j}(cS) \otimes H^j(U_1) \right) \oplus \left( MH^{k-j}(cS) \otimes H^j(U_2) \right) \to \bigoplus_{j=0}^k MH^{k-j}(cS) \otimes H^{j+1}(U_1 \cup U_2) \to \ldots .$$
In order to proceed with the next step, we introduce the following notation:

\[
D^k_1 := \bigoplus_{j=0}^{k} MH^k_{T}(cS) \otimes H^j_{\ast}(U_1 \cup U_2).
\]

\[
D^k_2 := \bigoplus_{j=0}^{k} \left( MH^k_{T}(cS) \otimes H^j_{\ast}(U_1) \right) \oplus \left( MH^k_{T}(cS) \otimes H^j_{\ast}(U_2) \right).
\]

\[
D^k_3 := \bigoplus_{j=0}^{k} MH^k_{T}(cS) \otimes H^j_{\ast}(U_1 \cap U_2).
\]

\[
C^k_1 := MH^k_{T}(cS \times (U_1 \cup U_2)).
\]

\[
C^k_2 := MH^k_{T}(cS \times U_1) \oplus MH^k_{T}(cS \times U_2).
\]

\[
C^k_3 := MH^k_{T}(cS \times (U_1 \cap U_2)).
\]

The \( D^k_i \) define objects linked by the long exact sequence described above, while the \( C^k_i \) are linked by the long exact sequence of Equation 6.3 which followed Lemma 6.1. Recalling the definition of \( \Psi \) as in Equation 6.12 for \( X = cS \), we then consider the following diagram:

\[
\begin{array}{ccccccc}
\cdots & D^k_1 & \xrightarrow{\alpha^\ast} & D^k_2 & \xrightarrow{\beta^\ast} & D^k_3 & \xrightarrow{\delta} & D^{k+1}_1 & \xrightarrow{\psi} & \cdots \\
& | & \psi & | & \psi & | & \psi & & \\
\cdots & C^k_1 & \xrightarrow{\alpha^\ast} & C^k_2 & \xrightarrow{\beta^\ast} & C^k_3 & \xrightarrow{\delta} & C^{k+1}_1 & \xrightarrow{\psi} & \cdots
\end{array}
\]  

(6.13)

By a slight abuse of notation we let \( \alpha, \beta, \) and \( \Psi \), denote the various maps, as in Equation 6.3, and their extensions to \( D^k_i \), as in Equation 6.13 above. Let us verify the commutativity of the squares in the diagram in Equation 6.13 above.

We begin with the leftmost square \((1)\), where we are required to verify that \( \Psi \alpha = \alpha \Psi \); for this we fix an element \([\omega] \otimes [\eta] \in D^k_1\), with \([\omega] \in MH^k_{T}(cS)\) and \([\eta] \in H^j(U_1 \cup U_2)\). Then we compute \( \Psi \alpha([\omega] \otimes [\eta]) \) and \( \alpha \Psi([\omega] \otimes [\eta]) \):

\[
\Psi(\alpha([\omega] \otimes [\eta])) = \Psi\left( ([\omega] \otimes [\eta_{|U_1}]) \oplus ([\omega] \otimes [\eta_{|U_2}]) \right)
\]

\[
= \left[ p_{reg(cS)}^\ast(\omega) \wedge p_{U_1}^\ast(\eta_{|U_1}) \right] \oplus \left[ p_{reg(cS)}^\ast(\omega) \wedge p_{U_2}^\ast(\eta_{|U_2}) \right],
\]

\[
\alpha \Psi([\omega] \otimes [\eta]) = \alpha\left( [\omega_{reg(cS)}](\omega) \wedge p_{U_1 \cup U_2}^\ast(\eta) \right)
\]

\[
= \left[ (p_{reg(cS)}^\ast(\omega) \wedge p_{U_1 \cup U_2}^\ast(\eta))_{|cS \times U_1} \right] \oplus \left[ (p_{reg(cS)}^\ast(\omega) \wedge p_{U_1 \cup U_2}^\ast(\eta))_{|cS \times U_2} \right]
\]

\[
= \left[ p_{reg(cS)}^\ast(\omega) \wedge p_{U_1}^\ast(\eta_{|U_1}) \right] \oplus \left[ p_{reg(cS)}^\ast(\omega) \wedge p_{U_2}^\ast(\eta_{|U_2}) \right].
\]

This verifies the commutativity of the leftmost square.

For the middle square \((2)\), we verify that \( \Psi \beta = \beta \Psi \); for this we fix an element

\[
([\omega] \otimes [\eta]) \oplus ([\omega'] \otimes [\eta']) \in D^k_2,
\]  

(6.14)
with $[\omega] \in MH_\delta^{k-1}(cS)$, $[\omega'] \in MH_\delta^{k-j'}(cS)$ $[\eta] \in H^j(U_1)$, and $[\eta'] \in H^{j'}(U_2)$. Then we compute $\Psi \beta$ and $\beta \Psi$ applied to the element in Equation 6.14:

$$\Psi \beta\left([(\omega] \otimes [\eta]) \oplus ([\omega'] \otimes [\eta'])\right) = \Psi((\omega] \otimes [\eta]|_{U_1 \cap U_2}) - \Psi([\omega'] \otimes [\eta']|_{U_1 \cap U_2})$$

$$= [p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta)|_{U_1 \cap U_2}) - [p^*_\text{reg}(cS)(\omega') \wedge p^*_\text{U_1 \cap U_2}(\eta')|_{U_1 \cap U_2})],$$

$$\beta \Psi\left([(\omega] \otimes [\eta]) \oplus ([\omega'] \otimes [\eta'])\right) = \beta\left([p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1}(\eta)] \oplus [p^*_\text{reg}(cS)(\omega') \wedge p^*_\text{U_2}(\eta')\right]$$

$$= [(p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1}(\eta))]|_{cS \times (U_1 \cap U_2)} - [(p^*_\text{reg}(cS)(\omega') \wedge p^*_\text{U_2}(\eta'))]|_{cS \times (U_1 \cap U_2)}$$

$$= [p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta)|_{U_1 \cap U_2}) - [p^*_\text{reg}(cS)(\omega') \wedge p^*_\text{U_1 \cap U_2}(\eta')|_{U_1 \cap U_2})].$$

This verifies the commutativity of the middle square.

For the rightmost square (3), we recall the definition of $\delta$ that appears in Remark 6.2 to Lemma 6.1. We verify that $\Psi \delta = \delta \Psi$; for this we fix an element $[\omega] \otimes [\eta] \in D^1_\delta$, with $[\omega] \in MH_\delta^{k-1}(cS)$ and $[\eta] \in H^j(U_1 \cap U_2)$. Then we compute $\Psi \delta([\omega] \otimes [\eta])$ and $\delta \Psi([\omega] \otimes [\eta])$:

$$\Psi \delta([\omega] \otimes [\eta]) = \begin{cases} -\Psi([\omega] \otimes [d(\chi' \cdot \eta)]) & \text{on } U_1 \\ \Psi([\omega] \otimes [d(\chi \cdot \eta)]) & \text{on } U_2 \end{cases}$$

$$= \begin{cases} [p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(d(\chi' \cdot \eta))] & \text{on } cS \times U_1 \\ [p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(d(\chi \cdot \eta))] & \text{on } cS \times U_2 \end{cases}$$

$$= \begin{cases} [d\chi' \wedge p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta)] & \text{on } cS \times U_1 \\ [d\chi \wedge p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta)] & \text{on } cS \times U_2 \end{cases}$$

$$\delta \Psi([\omega] \otimes [\eta]) = \delta\left([p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta)]\right)$$

$$= \begin{cases} [d(\chi' \cdot p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta))] & \text{on } cS \times U_1 \\ [d(\chi \cdot p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta))] & \text{on } cS \times U_2 \end{cases}$$

$$= \begin{cases} [d\chi' \wedge p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta)] & \text{on } cS \times U_1 \\ [d\chi \wedge p^*_\text{reg}(cS)(\omega) \wedge p^*_\text{U_1 \cap U_2}(\eta)] & \text{on } cS \times U_2 \end{cases}$$

This follows because in the above $\chi$ and $\chi'$ are liftable smooth functions chosen\footnote{Recall that under this choice $\chi + \chi' = 1$, $\text{supp}(\chi) \subseteq \text{reg}(cS \times U_1)$, $\text{supp}(\chi') \subseteq \text{reg}(cS \times U_2)$, and $\chi$ and $\chi'$ lift from a partition of unity for $U_1$ and $U_2$.} as in Lemma 6.1, we have omitted indicating their restrictions to $U_1$ and $U_2$ for readability. This proves the commutativity of the diagram in Equation 6.13.

For the final part of the first induction we apply the five-lemma (25, p.205, Lemma 4), where from the commutativity of the diagram in Equation 6.13, if $\Psi$ is an isomorphism for $cS \times U_1$, $cS \times U_2$, and $cS \times (U_1 \cap U_2)$, then $\Psi$ is an isomorphism for $cS \times (U_1 \cup U_2)$; this follows directly from the statement of the
five-lemma. If \( M = \mathbb{R}^m \), that is if \( M \) has a good cover by 1 open subset, then \( \Psi \) is an isomorphism by the Poincaré lemma for \( cS \times \mathbb{R}^m \) in Corollary 5.69. If \( \Psi \) is an isomorphism for a good cover of \( N \) open subsets, then consider a manifold \( M \) having a good cover by \( N + 1 \) open subsets \( \{ U_i \}_{i=1}^{N+1} \). Now the intersection

\[
(U_1 \cup U_2 \cup \ldots \cup U_N) \cap U_{N+1}
\]

has a good cover with \( N \) open sets, namely \( U_1 \cap U_{N+1} \) to \( U_N \cap U_{N+1} \). But then \( \Psi \) is an isomorphism for \( cS \times (U_1 \cup \ldots \cup U_N) \), for \( cS \times U_{N+1} = cS \times \mathbb{R}^m \) and for \( cS \times (U_1 \cup U_2 \cup \ldots \cup U_N) \cap U_{N+1} \), so that it is an isomorphism for \( cS \times M = cS \times (U_1 \cup \ldots \cup U_{N+1}) \) as well. This completes the proof of the first induction.

Part 4.2. Let us now proceed with the second induction. For this, let \( X \) again be an arbitrary flat-type depth 1 multicontrolled stratified space. For each singular stratum \( A \subseteq \mathcal{SS}(X) \) we fix the finite good cover \( t_{V,A,i} \) of \( A \), as in Definition 5.58, so that \( T_A \hat{\times} M \) can be expressed as

\[
\bigcup_{i=1,\ldots,N_A} (cS \times (V_{A,i} \times M)) = \bigcup_{i=1,\ldots,N_A} U_{A,i},
\]

where \( U_{A,i} := cS \times (V_{A,i} \times M) \). The second induction is an induction on \( N_A \), where our aim is to prove that \( \Psi \) is an isomorphism for \( T_A \times M \). For \( N = 1 \) this follows directly from Part 4.1. For the induction step we proceed similarly to Part 4.1, where we consider \( X \) with a finite good cover of \( N + 1 \) open subsets, so that we consider

\[
\bigcup_{i=1,\ldots,N_A} U_i \cap U_{N_A+1}.
\]

The above Equation 6.15 has a decomposition into \( N \) open subsets of the form \( U_i \cap U_{N_A+1} \) to \( U_{N_A} \cap U_{N_A+1} \). Then we apply the Mayer-Vietoris sequence of Lemma 6.1, proceeding as before to obtain that since \( \Psi \) is an isomorphism for \( \bigcup_{i=1,\ldots,N_A} U_i \cap U_{N+1} \), and \( \bigcup_{i=1,\ldots,N_A} U_i \cap U_{N+1} \), then we obtain a commutative diagram to which the five lemma can be applied, deducing that \( \Psi \) is an isomorphism for \( \bigcup_{i=1,\ldots,N_A+1} U_i = T_A \times M \).

Part 4.3. Now let \( Z \) be such that \( X \times M \) is the union of all the \( T_A \times M \), for \( A \in \mathcal{SS}(X) \), such that \( Z \) is a smooth manifold obtained by enlarging \( \text{reg}(X) \times M \) slightly into the \( T_A \times M \), so that \( Z \cap (T_A \times M) \) is diffeomorphic to \( S_A \times M \). Now we apply the Mayer-Vietoris sequence from Lemma 6.4 to \( Z \cup \bigcup_{i=1,\ldots,N} T_A \times M \), where there are a finite number \( N \) of cusps \( T_A \times M \), to obtain the final result. \( \square \)
2 Local Poincaré duality

In this section we will define the complex of relative multiperverse forms and the complex of lift-compact multiperverse forms for a depth 1 multicontrolled stratified space; we will also obtain a version of Poincaré duality for certain flat-type multicontrolled spaces.

2.1 Relative multiperverse forms

In this subsection we will define the complex of relative multiperverse forms, where these forms will be relative to a subset whose closure is disjoint from the singular strata. We will also introduce a long exact sequence to calculate relative multiperverse cohomology.

**Definition 6.9** (Relative multiperverse forms). Let $X$ be a depth 1 multicontrolled stratified space and let $E \subset X$ be an open subset with $\text{cl}_X(E) \subset \text{reg}(X)$. We define the complex of $E$-relative $\tau$- multiperverse forms on $X$ by

$$M\Omega^k_{\tau}(X; E) := M\Omega^k_{\tau}(X) \oplus \Omega^{k-1}(E),$$

with the coboundary operator $d : M\Omega^k_{\tau}(X; E) \to M\Omega^{k+1}_{\tau}(X; E)$ defined as

$$d(\omega, \omega') := \left( d_{\text{reg}(X)}(\omega), i^*_E(\omega) - d_E(\omega') \right),$$

where $i^*_E : \Omega^k(X) \to \Omega^k(E)$ is the pullback by the inclusion of $E$ into $X$. For each integer $k \in \mathbb{Z}_{>0}$ the cohomology of the above complex $M\Omega^*_\tau(X; E)$ is written as

$$MH^k_{\tau}(X; E) := H^k(M\Omega^*_\tau(X; E)).$$

If for each singular stratum $A \in \text{SS}(X)$ we have $E \cap T_A \neq \emptyset$, and if $E \cap T_A$ is $E_A := (0.5, 1) \times S_A$, where $\text{reg}(T_A)$ is reparametrised as $(0, 1) \times S_A$, then we will say that $E$ is adapted to the control data of $X$, or just that $E$ is adapted to $X$. If the identity $X = E \cup \bigcup_{A \in \text{SS}(X)} T_A$ is verified, then we will say that $E$ partitions $X$.

**Remark 6.10.** The statement $\text{cl}_X(E) \subset \text{reg}(X)$ should be interpreted as specifying that the open subset $E$ does not approach the singular strata, whereby the support of a form in the relative complex is not specified to be zero at any of the singular strata of $X$.

For the complex of $E$-relative $\tau$-multiperverse forms $M\Omega^k_{\tau}(X; E)$ one obtains
a short exact sequence of complexes

$$0 \to \Omega^{k-1}(E) \overset{\alpha}{\to} M\Omega^k_{\tau}(X; E) \overset{\beta}{\to} M\Omega^k_{\tau}(X) \to 0,$$

(6.18)

where $\alpha(\omega') := (0, \omega')$ and $\beta(\omega, \omega') := \omega$. One verifies directly that $\ker(\beta) = \text{im}(\alpha)$, for $\beta(0, \omega') = 0$. Note that

$$\alpha(d\omega) = (0, d\omega) = d(0, -\omega),$$

but that, nevertheless, the short exact sequence in Equation 6.18 induces a long exact sequence on cohomology:

$$\ldots \to H^{k-1}(E) \overset{\alpha}{\to} MH^k_{\tau}(X; E) \overset{\beta}{\to} MH^k_{\tau}(X) \overset{\delta}{\to} H^k(E) \to \ldots .$$

(6.19)

For certain $E$ and $X$ we will be able to calculate the $E$-relative $\tau$-multiperverse cohomology $MH^*_{\tau}(X; E)$ using the $\tau$-multiperverse cohomology $MH^*_{\tau}(X)$ and the above long exact sequence. With this purpose in mind, we verify the definition of the snake map $\delta$ ([25], p.206, Lemma 5), following [9] (Claim 6.48, p.78).

**Proposition 6.11.** In the above Equation 6.19 we have $\delta = i^*_E$.

**Proof.** Note that $\delta$ is a well-defined map. Consider the following commutative diagram formed from the short exact sequence of complexes in Equation 6.18.

$$
\begin{array}{ccc}
0 & \longrightarrow & \Omega^{k-1}(E) \\
\downarrow{d} & & \downarrow{d} \\
0 & \longrightarrow & \Omega^k(E)
\end{array}
\quad
\begin{array}{ccc}
\alpha & \longrightarrow & M\Omega^k_{\tau}(X; E) \\
\beta & \longrightarrow & M\Omega^k_{\tau}(X) \\
\delta & \longrightarrow & H^k(E)
\end{array}
\quad
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow{d} & & \downarrow{d} \\
0 & \longrightarrow & 0
\end{array}
\]

Fix an $\eta \in M\Omega^k_{\tau}(X)$ which is closed, that is an $\eta$ which satisfies $d(\eta) = 0$. By the surjectivity of $\beta$ there is some $(\eta, \eta') \in M\Omega^k_{\tau}(X; E)$ with $\beta(\eta, \eta') = \eta$; now $d(\eta, \eta') = (0, i^*_E(\eta) - d(\eta'))$, because $d(\eta) = 0$, so that

$$\beta(d(\eta, \eta')) = d(\beta(\eta, \eta')) = 0,$$

and thus $(0, i^*_E(\eta) - d(\eta')) \in \ker(\beta)$. From this we deduce that there exists some $\omega \in \Omega^k(E)$ with

$$\alpha(\omega) = (0, \omega) = (0, i^*_E(\eta) - d(\eta')) ,$$

so that $\delta[\eta] = [\omega]$. Further we compute

$$\delta[\eta] = [i^*_E(\omega) - d(\eta')] = [i^*_E(\omega)],$$
which proves the result.

Remark 6.12. Above we have used the notation $\delta$ to denote the snake map ([25], p.206, Lemma 5), as is standard. Previously the notation $\delta_F$ has been used to denote the adjoint of $d_F$. The two maps are unrelated.

### 2.2 Relative multiperverse cohomology of $cS$ and $cS \times M$

In this subsection we will compute the relative multiperverse cohomology of $cS$ and $M \times cS$, where these spaces are of flat-type and $E$ is naturally chosen. To achieve this we will use the long exact sequence of Equation 6.19, as introduced in Subsection 2.1.

**Proposition 6.13.** Let $cS$ be a flat-type depth 1 multicontrolled cone. For the open subset $E = S \times (1, \infty) \subseteq cS$ the cohomology of the relative complex $M\Omega^*_r(cS; E)$ satisfies

$$MH^k_r(cS; E) \approx \frac{H^{k-1}(E)}{i^*_E MH^{k-1}_r(cS)}.$$  \hspace{1cm} (6.20)

Furthermore, the above may be expressed as

$$MH^k_r(cS; E) \approx \bigoplus_{i+j=k-1} \left(H^j(L', H^i(F))\right).$$ \hspace{1cm} (6.21)

**Proof.** First we consider the long exact sequence of Equation 6.19:

$$\ldots \rightarrow MH^k_r(cS) \xrightarrow{i^*_E} H^{k-1}(E) \xrightarrow{\alpha} MH^k_r(cS; E) \xrightarrow{\beta} MH^k_r(cS) \rightarrow \ldots$$

To prove the result it will suffice to show that $\ker(i^*_E) = 0$; in this case $\text{im}(i^*_E) = MH^k_r(cS)$, from where we would obtain

$$\text{im}(\alpha) = \frac{H^{k-1}(E)}{\ker(\alpha)} = \frac{H^{k-1}(E)}{\text{im}(i^*_E)} = \frac{H^{k-1}(E)}{i^*_E MH^{k-1}_r(cS)}.$$  \hspace{1cm} (6.20)

Moreover, by considering the long exact sequence

$$\ldots \rightarrow H^{k-1}(E) \xrightarrow{\alpha} MH^k_r(cS; E) \xrightarrow{\beta} MH^k_r(cS) \xrightarrow{i^*_E} H^k(E) \rightarrow \ldots,$$

we would obtain that

$$\text{im}(\beta) = \ker(i^*_E) = 0,$$

so that $\text{im}(\alpha) = \ker(\beta) = MH^k_r(cS; E)$, from where

$$MH^k_r(cS; E) = \text{im}(\alpha) = \frac{H^{k-1}(E)}{i^*_E MH^{k-1}_r(cS)}.$$
CHAPTER 6. PROPERTIES OF MULTIVERSE COHOMOLOGY

To show that $\ker(i_E^* \sim E_q)^0$ we first consider the following diagram.

\[
\begin{array}{ccc}
MH_k^{k-1}(cS) & \xrightarrow{i_E^*} & H^{k-1}(E) \\
\downarrow \cong & & \downarrow \cong \\
H^{k-1}(M\Omega_r^*(S)) & \hookrightarrow & H^{k-1}(S) \\
\end{array}
\]

The commutativity of the above diagram is verified as follows. Let \( r \omega s P MH_k^{k-1}(cS); c S \).

From Corollary 5.63 we have \( r \omega s P MH_k^{k-1}(cS); c S \).

From there \( r R p \omega q S H_k^{k-1}(M\Omega_r^*(S)) \).

And moreover \( r i_E^* \omega s P H_k^{k-1}(M\Omega_r^*(S)) \).

From this we deduce that, under the assumption \( r i_E^* \omega s P H_k^{k-1}(M\Omega_r^*(S)) \).

We have \( r \eta s = 0 \) in \( H^{k-1}(S) \).

To show that \( r \omega s = 0 \) in \( MH_k^{k-1}(cS) \).

It will suffice to show that \( r \eta s = 0 \) in \( H^{k-1}(S) \).

This last part follows by the same argument as in Theorem 5.66, which itself followed the same argument as in Theorem 5.43.

Remark 6.14. In the above Proposition 6.13, for \( k = 0 \), we have

\[ MH_0^0(cS; E) = 0. \] (6.22)

Let us remark that the proof still applies in this case. For this we consider the following long exact sequence

\[ \ldots \to H^{-1}(E) \xrightarrow{\alpha} MH_0^0(cS; E) \xrightarrow{\beta} MH_0^0(cS) \xrightarrow{i_E^*} H^0(E) \to \ldots , \]

where \( \text{im}(\beta) = 0 \) and \( \text{im}(\alpha) = 0 \) by definition, from where we obtain that \( MH_0^0(cS; E) = 0 \).

Corollary 6.15. Let \( cS \) be a product-type depth 1 multicontrolled cone. For the open subset \( E = S \times (1, \infty) \subseteq cS \) the \( E \)-relative \( r \)-multiperverse cohomology satisfies

\[ MH_r^k(cS; E) = \bigoplus_{i+j=k-1 \atop i \geq r(k-1)+1} (H^i(L') \otimes H^j(F)). \]

Proof. This follows directly from Proposition 6.13.

Corollary 6.16. Let \( cS \) be a flat-type depth 1 multicontrolled cone and let \( m(p) \in M(cS) \) be the multiperversity associated to a perversity \( p \) for \( cS \); this is specified
by an integer $\overline{p}(pt) \in [-1, \dim(S)]$. For the open subset $E = S \times (1, \infty) \subseteq cS$ the $E$-relative $\tau$-multiperverse cohomology satisfies

\[
MH^k_{m(\overline{p})}(cS; E) \cong \begin{cases} 
0 & \text{if } k \leq \overline{p}(pt) + 1, \\
H^{k-1}(S) & \text{if } k > \overline{p}(pt) + 1,
\end{cases}
\]

\[
\cong \begin{cases} 
0 & \text{if } k < \overline{p}(pt) + 2, \\
H^{k-1}(S) & \text{if } k \geq \overline{p}(pt) + 2. 
\end{cases}
\] (6.23)

Proof. For this result we consider Equation 6.21 in Proposition 6.13. We claim that $MH^k_{m(\overline{p})}(cS; E)$ vanishes if $k - 1 \leq \overline{p}(pt)$, for in this case $m(\overline{p}) = \text{codim}(S') - 1$, so the identity $i \geq m(\overline{p}) + 1$ reduces to $i \geq \text{codim}(S') = \dim(F) + 1$, from where the claim follows. Similarly we obtain that if $k - 1 > \overline{p}(pt)$, then the value of $MH^k_{m(\overline{p})}(cS; E)$ is $H^{k-1}(S)$.

Proposition 6.17. For the open subset

\[
E^\circ = (E \times M) = (S \times (1, \infty) \times M) \subseteq (cS \times M),
\]

the cohomology of the $E^\circ$-relative complex $M\Omega^*_\tau(M \times cS; E^\circ)$ satisfies

\[
MH^k_\tau(M \times cS; E^\circ) \cong \bigoplus_{i+j=k} H^i(M) \otimes MH^j_\tau(cS; E),
\] (6.24)

where we recall that $E = S \times (1, \infty) \subseteq cS$, as in the above Proposition 6.13.

Proof. By Lemma 6.8 there is an isomorphism

\[
MH^k_\tau(cS \times M) \cong \bigoplus_{i+j=k} MH^i_\tau(cS) \otimes H^j(M),
\]

so that it will suffice to show that $\ker(i^*_E) = 0$ to prove the result; in this case $\text{im}(i^*_E) \cong MH^k_\tau(M \times cS)$, from where

\[
\text{im}(\alpha) = \frac{H^{k-1}(E^\circ)}{\ker(\alpha)} = \frac{H^{k-1}(E^\circ)}{\text{im}(i^*_E)} \cong \frac{H^{k-1}(E^\circ)}{i^*_E MH^{k-1}_\tau(cS \times M)}.
\]

Moreover, as $\text{im}(\beta) = \ker(i^*_E)$, we would obtain that $\text{im}(\alpha) = \ker(\beta) \cong MH^k_\tau(cS \times M; E^\circ)$, so that

\[
MH^k_\tau(cS \times M; E^\circ) = \text{im}(\alpha) = \frac{H^{k-1}(E^\circ)}{i^*_E MH^{k-1}_\tau(cS \times M)}.
\]
To show that \( \ker(i_{E^0}^* \circ p) = 0 \) we will consider a commutative diagram, where we let

\[
C^k := \bigoplus_{i+j=k} H^i(M \Omega^*_\tau(S)) \otimes H^j(M),
\]

\[
D^k := \bigoplus_{i+j=k} H^i(S) \otimes H^j(M),
\]

further letting the \( k \)-form \( p^*_S(\eta) \wedge p^*_M(\eta') \) be a base form, with \( \eta \) an \( i \)-form and \( \eta' \) a \( j \)-form.

Assuming that \( i_{E^0}^*[p^*_S(\eta) \wedge p^*_M(\eta')] = 0 \) in \( H^{k-1}(E^0) \), we obtain that \( [\eta] \otimes [\eta'] = 0 \) in \( D^{k-1} \). To show that \( [p^*_S(\eta) \wedge p^*_M(\eta')] = 0 \), it will then suffice to show that under these assumptions \( [\eta] = 0 \) in \( H^i(M \Omega^*_\tau(S)) \); this follows by the same argument as in Theorem 5.66.

### 2.3 Lift-compact multiperverse forms

In this subsection we will introduce the complex of multiperverse forms with compactly supported lifts; we will call these forms lift-compact multiperverse forms.

**Definition 6.18 (Multiperverse forms with compactly supported lifts).** Let \( X \) be a depth \( 1 \) flat-type multicontrrolled stratified space. Denote by \( M \Omega^k_{\text{lc,} \tau}(X) \) the subset of \( \Omega^k(\mathcal{D}(X)) \) consisting of those elements \( \omega \in M \Omega^k_{\tau}(X) \) whose lift \( \tilde{\omega} \in \Omega^k(\mathcal{D}(X)) \) is a compactly supported form on \( \mathcal{D}(X) \). Note that this subset is naturally a subcomplex. Elements of \( M \Omega^k_{\text{lc,} \tau}(X) \) are called *lift-compact \( \tau \)-multiperverse forms*, where we denote the cohomology of the above complex \( M \Omega^k_{\text{lc,} \tau}(X) \) by

\[
MH^k_{\text{lc,} \tau}(X) := H^k(M \Omega^*_\tau(X)).
\]
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Remark 6.19. If the support \(\text{supp}(\tilde{\omega})\) of the lift of a lift-compact \(\tau\)-multiperverse form \(\omega\) is disjoint from the preimages in \(\mathcal{D}(X)\) of all singular strata of \(X\), then the \(\tau\)-multiperversity conditions on \(\omega\) are null. From this we deduce that any liftable form \(\omega \in \Omega^k_r(X)\), the support of whose lift \(\tilde{\omega}\) is compact and disjoint from the preimages in \(\mathcal{D}(X)\) of the singular strata of \(X\), is automatically a lift-compact \(\tau\)-multiperverse form.

To compute the lift-compact cohomology of the multicontrolled cone \(cS\), we will need to construct an extended space \(\overline{cS}\), which is defined as follows, where we reparametrise \(cS\) to \(S \hat{\times}^r_0, 1\) with the 0-coordinate collapsed:

\[
\overline{cS} := cS \cup (S \times [1, 2]) .
\]

Using the Mayer-Vietoris sequence of Lemma 6.4 for \(Y_1 = cS\) and \(Y_2 = S \times (0.5, 2)\), so that \(Y_1 \cup Y_2 = \overline{cS}\), we obtain, as in Equation 6.3, the long exact sequence:

\[
\cdots \rightarrow MH^k_r(\overline{cS}) \xrightarrow{\alpha} MH^k_r(cS) \oplus H^k(S \times (0.5, 2)) \xrightarrow{\beta} H^k(S \times (0.5, 1)) \xrightarrow{\delta} MH^{k+1}_r(\overline{cS}) \rightarrow \cdots .
\]

Using a standard argument we obtain that \(\alpha\) is injective and surjective onto \(MH^k_r(cS)\), so that \(MH^k_r(\overline{cS}) \cong MH^k_r(cS)\).

Let \(\text{reg}(cS)\) be \(S \times (0, 1)\) and \(\text{reg}(\overline{cS})\) be \(S \times (0, 2)\). Fix an \(\overline{E}\) which is adapted to \(\overline{cS}\) and partitions \(\overline{cS}\), we can assume that \(\overline{E} = S \times (1.5, 2)\) and we will write \(\overline{E}\) to denote both the subset considered in \(\text{reg}(\overline{cS})\) and its preimage in \(\mathcal{D}(\overline{cS})\), where the latter is

\[
\theta_{\overline{cS}}^{-1}[\overline{E}] = (S \times (1.5, 2)) \cup (S_A \times (-1.5, -2)).
\]

Furthermore, fix three further open subsets which satisfy

\[
\overline{E} \subsetneq E' \subsetneq \text{reg}(\overline{cS}),
\]

so that \(E' = E_{0.5}\), where for \(t \in (0, 2)\) we fix the identification

\[
\theta_{\overline{cS}}^{-1}[E_t] = (S \times (t, 2)) \cup (S_A \times (t, -2)).
\]

Using the five lemma we obtain that \(MH^k_r(\overline{cS}; \overline{E})\) is isomorphic to \(MH^k_r(cS; E_{0.5} \cap cS)\), which we write as \(MH^k_r(cS; E_{0.5})\) for brevity; for this we can consider the
following commutative diagram, in which the squares \(1\), \(2\) and \(3\) commute.

\[
\begin{array}{cccccc}
\cdots & \rightarrow & H^{k-1}(E) & \rightarrow & MH^k_\tau(cS; E) & \rightarrow & MH^k_\tau(cS) & \rightarrow & H^k(E) & \rightarrow & \cdots \\
& \simeq & 1 & & 2 & \simeq & 3 & \simeq & \simeq & \simeq & \cdots \\
\cdots & \rightarrow & H^{k-1}(E_{0.5}) & \rightarrow & MH^k_\tau(cS; E_{0.5}) & \rightarrow & MH^k_\tau(cS) & \rightarrow & H^k(E_{0.5}) & \rightarrow & \cdots 
\end{array}
\]

**Proposition 6.20.** For each \(t \in (0, 1)\) there exists an invertible smooth map

\[
q_t : \mathcal{D}(cS) \rightarrow \mathcal{D}(cS)
\]

with inverse

\[
s_t : \mathcal{D}(cS) \rightarrow \mathcal{D}(cS)
\]

such that for some small \(\epsilon > 0\):

1. \(q|_{\theta^{-1}_{cS}[E_{-\epsilon}]}\) is the identity.

2. \(q|_{\mathcal{D}(cS) \setminus \theta^{-1}_{cS}[E_t]}\) is the identity.

3. \(q\) maps \(\theta^{-1}_{cS}[E_t]\) into \(\theta^{-1}_{cS}[E]\)

4. \(q\) lifts from a function defined on \(\text{reg}(cS)\).

5. \(q\) is smoothly homotopic to the identity on \(\mathcal{D}(cS)\).

**Proof.** It suffices to define the map \(q : S \times (-2, 2) \rightarrow S \times (-2, 2)\) satisfying the properties above. Furthermore, it suffices to define an endomorphism of \((0, 2)\) satisfying the appropriate properties, this is done by specifying the function directly and smoothing out where appropriate; this also defines the appropriate smooth homotopy to the identity.

For the maps \(q_t\) and \(s_t\) as in the above Proposition 6.20, we note that the following diagrams commute.

\[
\begin{array}{ccc}
\mathcal{D}(cS) & \xrightarrow{q_t} & \mathcal{D}(cS) \\
\downarrow i_{E_t} & & \downarrow i_{E_t} \\
\theta^{-1}_{cS}[E_t] & \xrightarrow{q_t|_{E_t}} & \theta^{-1}_{cS}[E]
\end{array}
\quad
\begin{array}{ccc}
\mathcal{D}(cS) & \xrightarrow{s_t} & \mathcal{D}(cS) \\
\downarrow i_{E_t} & & \downarrow i_{E_t} \\
\theta^{-1}_{cS}[E_t] & \xrightarrow{s_t|_{E_t}} & \theta^{-1}_{cS}[E_t]
\end{array}
\]

From this we can deduce that the pullback \(q^*_t\) induces a morphism of complexes

\[
q^*_t : M\Omega^*(cS; E) \rightarrow M\Omega^*(cS; E_t),
\]
to verify this we compute directly:

$$dq^*_t(\omega, \omega') = d(q^*_t(\omega), q^*_t(\omega')) = (dq^*_t(\omega), i^*_{E_1} q^*_t(\omega) - dq^*_t(\omega'))$$

$$= (q^*_t(d\omega), q^*_t i^*_{E_1}(\omega) - q^*_t(d\omega'))$$

$$= q^*_t(d\omega, i^*_{E_1}(\omega) - d\omega') = q^*_t d(\omega, \omega').$$

Using the long exact sequence for relative $\tau$-multiperverse cohomology, as in Equation 6.19, we also obtain that $q^*_t$ induces an isomorphism between $MH^{k}_{\tau}(\overline{cS}; \overline{E})$ and $MH^{k}_{\tau}(cS; E_1)$. Analogous remarks hold for $s^*_t$.

**Lemma 6.21.** Let $cS$ be a flat-type cone and let $\tau \in \mathcal{M}(cS)$ a multiperversity for $cS$. Assume that $S$ is a compact manifold and let $E$ be a subset which partitions and is adapted to $\overline{cS}$. Then there exists an isomorphism of cohomology groups

$$MH^{k}_{lc, \tau}(cS) \cong MH^{k}_{\tau}(cS; E_{0.5}).$$

**Proof.** Let us write $q$ for $q_{1.0}$ and $s$ for $s_{1.0}$. Then we define the map $\Theta : MH^{k}_{lc, \tau}(cS) \rightarrow MH^{k}_{\tau}(\overline{cS}; \overline{E})$ by first mapping a lift-compact cohomology class $[w]$ to the element $[(\omega, 0)]$ of $MH^{k}_{\tau}(\overline{cS}; E_{1.0})$ and then applying $s^*$ to obtain $[(s^* \omega, 0)] \in MH^{k}_{\tau}(\overline{cS}; \overline{E})$. This map is well defined and we will verify that it is injective and surjective.

Firstly let us verify that $\Theta$ is injective. If $[(\omega, 0)] \in MH^{k}_{\tau}(\overline{cS}; E_{1.0})$ is a class such that $(\omega, 0) = d(\eta, \eta')$, then

$$d(\eta) = \omega \quad \text{and} \quad i^*_{E_{1.0}}(\eta) = d(\eta').$$

Fix a function $\chi : \operatorname{reg}(\overline{cS}) \rightarrow \mathbb{R}$ which is equal to 1 on a neighbourhood of $\overline{E}$, say $E_{1.4}$, and is equal to 0 on $\operatorname{reg}(cS) \setminus E_{1.1}$. Then we consider the following calculation:

$$i^*_{E_1}(\eta - d(\chi \cdot \eta')) = 0.$$

From this calculation we deduce that the form $\eta - d(\chi \cdot \eta')$ is supported away from $\overline{E}$, so is represented by a class $[(\eta - d(\chi \cdot \eta'), 0)] \in MH^{k}_{\tau}(\overline{cS}; \overline{E})$. However, if we then map this class to $[q^*(\eta - d(\chi \cdot \eta')), 0] \in MH^{k}_{\tau}(\overline{cS}; E_{1.0})$, then we obtain an element of $MH^{k}_{\tau}(cS)$, since the support of $q^*(\eta - d(\chi \eta'))$ is disjoint from $E_{1.0}$.

$$d(q^*(\eta - d(\chi \cdot \eta'))) = d(q^*(\eta)) - ddq^*(\chi \cdot \eta')$$

$$= q^*(d\eta) - 0 = q^*(\omega).$$

But $[(q^* \omega, 0)] = [(\omega, 0)]$, for example by directly computing $q^* p^*_S \gamma$ on a representative $[\omega] = [p^*_S \gamma]$, or by noting that $q^*$ is homotopic to the identity, descending to $\tau$-multiperverse forms. Furthermore $q^*(\eta - d(\chi \eta'))$ is still $\tau$-multiperverse, since
by construction it is identically equal to \( \eta \) in a neighbourhood of \( \text{pt} \). This shows that \([\omega] = 0\) and completes the proof of injectivity.

Secondly let us verify the surjectivity of \( \Theta \); for this we will need the result of Proposition 6.13 in Subsection 2.2. Every element \( MH^k(cS; E) \) can be represented by \([0, p^*_S \eta]\) where

\[
\eta \in \bigoplus_{i+j=k-1 \atop i \geq \tau(k-1)+1} (H^j(L', H^i(F))) .
\]

Fix \( \chi : \text{reg}(X) \to \mathbb{R} \) as above, so that \( d(\chi) \) is a 1-form with compact support in \( E_{1,0} \). Then we claim that \( \Theta([d(\chi) \wedge p^*_S \eta]) = [(0, p^*_S \eta)] \), which we verify by noting that

\[
d(\chi \cdot p^*_S \eta, 0) = (d(\chi \cdot p^*_S \eta), i^*_E(\chi \cdot p^*_S \eta))
= (d\chi \wedge p^*_S \eta, p^*_S \eta)
= (d\chi \wedge p^*_S \eta, 0) + (0, p^*_S \eta).
\]

This completes the proof of surjectivity, from where we deduce that \( \Theta \) is an isomorphism.

**Remark 6.22.** In Proposition 6.13 every class \([\omega] \in MH^k(cS; E)\) is represented by \([0, p^*_S \eta]\) where

\[
\eta \in \bigoplus_{i+j=k-1 \atop i \geq \tau(k-1)+1} (H^j(L', H^i(F))) .
\]

To represent this class by a lift-compact form, we follow the proof of Lemma 6.21, obtaining, for some compactly supported \( d\chi \) with support in \((0.25, 0.5)\):

\[
[(0, p^*_S \eta)] = [(d\chi \wedge p^*_S \eta, 0)] \text{ in } MH^k(cS; E).
\]

Thus the lift-compact representative \( d\chi \wedge p^*_S \eta \) does not intersect the singular stratum, that is \( R(d\chi \wedge p^*_S \eta) = 0 \). The reason that only forms with the condition in Equation 6.25 make up the lift-compact \( \tau \)-multiperverse classes follows because forms satisfying the opposite condition are not exact.

### 2.4 Poincaré duality for cS

In the first part of this subsection we will briefly demonstrate that there is an isomorphism between \( MH^m(cS) \) and \( MH^{\dim(S)+1-m}(cS; E) \) when cS is of product-type. In the remainder of the subsection we will demonstrate this result for cS in the flat-type case including the explicit construction of the pairing.

Let us begin by briefly presenting the Poincaré duality isomorphism for a product-type cone cS. Fix a multiperversity \( \tau \in \text{M}(cS) \), recalling the definition
of the dual multiperversity $\mathcal{D}(\tau)$ from 6.26 on page 138:

$$\mathcal{D}(\tau)(k) = \text{codim}(S') - 1 - \tau(\text{codim}(\text{pt}) - k - 1) - 1$$

$$= \dim(S) + 1 - \dim(S') - 1 - \tau(\dim(S) + 1 - k - 1) - 1$$

$$= (\dim(S) - \dim(L')) - \tau(\dim(S) - k) - 1. \quad (6.26)$$

Using Corollary 6.15 from the preceding Subsection 2.2 we obtain

$$MH^k_{\mathcal{D}(\tau)}(cS; E) \cong \bigoplus_{i+j=k-1} (H^i(F) \otimes H^j(L'))$$

$$\cong \bigoplus_{i \geqslant \dim(S) - \dim(L')} (H^i(F) \otimes H^j(L')). \quad (6.27)$$

Next we fix an integer $m \in \mathbb{Z}$ satisfying $m \in [0, \dim(S) + 1]$, and then substitute $k = \dim(S) + 1 - m$ into the above Equation 6.27, obtaining

$$MH^{\dim(S) + 1 - m}_{\mathcal{D}(\tau)}(cS; E) = \bigoplus_{i+j=\dim(S) - m} (H^i(F) \otimes H^j(L')).$$

Now we apply the Poincaré duality for $F$ and $L'$. First we assume that

$$i + j = \dim(S) - m,$$

$$i \geqslant (\dim(S) - \dim(L')) - \tau(m),$$

so that, from these assumptions, there exists an isomorphism

$$\left( H^i(F) \otimes H^j(L') \right) \cong \left( H^{\dim(F)-i}(F) \otimes H^{\dim(L')-j}(L') \right),$$

where, we claim, $\dim(L') - j + \dim(F) - i = m$ and $\dim(F) - i \leqslant \tau(m)$. The claim is established as follows. The equality $\dim(L') - j + \dim(F) - i = m$ follows by direct computation:

$$-j - i = m - \dim(S),$$

$$\dim(L') - j + \dim(F) - i = m - \dim(S) + \underbrace{\dim(F) + \dim(L')}_{= \dim(S)},$$

$$\dim(L') - j + \dim(F) - i = m.$$
The inequality \( \dim(F) - i \leq \tau(m) \) also follows by direct computation:

\[
- i \leq -\dim(S) + \dim(L') + \tau(m), \\
\dim(F) - i \leq -\dim(S) + \frac{\dim(F) + \dim(L')}{\dim(S)} + \tau(m), \\
\dim(F) - i \leq \tau(m).
\]

From this we deduce that there exists an isomorphism

\[
MH_r^{m}(cS) \cong MH_{\mathcal{D}(\tau)}^{\dim(S)+1-m}(cS; E) \cong MH_{\mathcal{D}(\tau)}^{\dim(S)+1-m}(cS). (6.28)
\]

Let us now present the explicit pairing for the flat-type cone \( cS \), where we follow the above outline, but verify the extra details that arise from the flat-type assumption and construct the pairing explicitly.

**Proposition 6.23.** There exists an isomorphism

\[
(H^i(L', H^i(F))) \cong \left( H^{\dim(L')-j}(L', H^{\dim(F)-i}(F)) \right)^*. 
\]

**Proof.** Firstly consider the bilinear pairing

\[
H^j(L', H^i(F)) \otimes H^{j'}(L', H^{i'}(F)) \rightarrow H^{j+j'}(L', H^{i+i'}(F)),
\]

defined by the correspondence

\[
[\alpha^{i,j}] \otimes [\beta^{i',j'}] \mapsto [\alpha^{i,j} \wedge \beta^{i',j'}].
\]

This map is well-defined on cohomology classes, for if \( \beta^{i',j'} = d(\eta) \) and \( d(\alpha^{i,j}) = 0 \), then we compute:

\[
[\alpha^{i,j} \wedge \beta^{i',j'}] = [\alpha^{i,j} \wedge d(\eta)] = [(-1)^{i+j} \cdot d(\alpha) \wedge \beta + \alpha \wedge d(\beta)] \\
= (-1)^{i+j} \cdot [d(\alpha^{i,j} \wedge \beta^{i',j'})].
\]

Note further that for the fixed orientation of \( S \), so for a fixed fundamental class \([S]\), the cohomology group \( H^{\dim(L')}(L', H^{\dim(F)}(F'))) \) is the cohomology group \( H^{\dim(S)}(S) \), which is isomorphic to \( \mathbb{R} \) since \( S \) is compact. Let \( j' = \dim(L') - j \) and \( i' = \dim(F) - i \), then we want to show that the pairing

\[
H^j(L', H^i(F)) \otimes H^{\dim(L')-j}(L', H^{\dim(F)-i}(F)) \rightarrow \mathbb{R}
\]

is nondegenerate. To construct the pairing we will use a submersion metric \( g_S \)
on $S$; because $\mu : S \rightarrow S'$ is flat with respect to the group $\text{Isom}(F, g)$, we can assume this metric to be geometrically flat ([1], p.3), so that the metric is locally a product. Note that by [3] (9.15) this submersion metric is uniquely determined by a Riemannian metric on $S' = L'$. From $g_S$ we obtain the Hodge star $*_S$ satisfying:

$$\text{if } \omega \in \Omega^{i,j}(S) \text{ then } *_S(\omega) \in \Omega^{\dim(F) - i, \dim(L') - j}(S).$$

Note that $*_S$ is an isomorphism from $\Omega^k(S)$ to $\Omega^{\dim(S) - k}(S)$, we further note that it is an isomorphism from $\Omega^{i,j}(S)$ to $\Omega^{\dim(F) - i, \dim(L') - j}(S)$. Further, any class in $[\omega] \in H^j(L', \mathbf{H}^i(F))$ may be represented by a unique harmonic $(i,j)$-form with respect to $g_S$. This follows by considering the Hodge decomposition $\omega = d_S(\alpha) + \delta_S(\beta) + \gamma$, where $\gamma$ is harmonic, that is $\Delta_S(\gamma) = 0$. From the closedness of $\omega$ we obtain:

$$d_S(\omega) = \underbrace{d_S d_S(\alpha)}_{=0} + d_S \delta_S(\beta) + d_S(\gamma) = 0.$$

From this we deduce that $d_S \delta_S(\beta) = 0$, so that

$$\int_S \delta_S(\beta) \wedge *_S(\delta_S(\beta)) = \int_S d_S \delta_S(\beta) \wedge *_S(\beta) = 0,$$

so that further $\delta_S(\beta) = 0$, from where the Hodge decomposition for $\omega$ is $\omega = d_S(\alpha) + \gamma$. This shows that $[\omega] = [\gamma]$. Let us now verify that $*_S$ is an isomorphism on cohomology classes in $H^j(L', \mathbf{H}^i(F))$. For a class $[\gamma] \in H^j(L', \mathbf{H}^i(F))$ represented by a harmonic form $\gamma$, we obtain:

$$\delta_S(\gamma) = 0 \text{ implies } d_S(*_S \gamma) = 0,$$

$$d_S(\gamma) = 0 \text{ implies } \delta_S(*_S \gamma) = 0.$$

From this we deduce that $*_S(\omega)$ represents a harmonic form and thus a class in $H^{\dim(L') - j}(L', \mathbf{H}^{\dim(F) - i}(F))$. Thus we obtain that

$$[\omega] \otimes [*_S \omega] \mapsto [\omega \wedge *_S \omega] \mapsto \int_S \omega \wedge *_S \omega = ||\omega||^2_{L^2(S, g_S)} = 0$$

if and only if $\omega = 0$. Thus the pairing is nondegenerate and the map given by integration

$$H^j(L', \mathbf{H}^i(F)) \rightarrow (H^{\dim(L') - j}(L', \mathbf{H}^{\dim(F) - i}(F)))^*,$$

that is by $[\omega] \mapsto \int_S \omega \wedge -$ , is an isomorphism. 

**Lemma 6.24.** Let $cS$ be a depth 1 flat-type multicntrolled cone. There exists an isomorphism

$$MH_{l,cS(r)}^{\dim(S) + 1 - m}(cS; E) \cong MH_{r}^{m}(cS)^*.$$
Furthermore, this isomorphism is given by the map
\[ \Psi : MH_{\dim(p) + 1 - m}(cS) \rightarrow MH_{\dim(p)}(cS)^*, \]
which is defined by integration on lifts \([\tilde{\omega}] \mapsto \int_{\mathcal{D}(\tau)} \tilde{\omega} \wedge \cdot \).

Proof. As in the first part of this subsection, we can obtain the existence of an isomorphism combinatorially by using the isomorphism from Proposition 6.23:
\[ (H^j(L', H^i(F))) \cong \left(H^{\dim(L') - j}(L', H^{\dim(F) - i}(F))\right)^*. \]

Let us now verify that the pairing
\[ \int_{\mathcal{D}(\tau)} : MH_{\dim(S) + 1 - m}(cS) \otimes MH_{\dim(S)}(cS) \rightarrow \mathbb{R} \]
given by \([\omega] \otimes [\omega'] \mapsto \int_{\mathcal{D}(\tau)} \omega \wedge \omega' \) is nondegenerate. Note that this pairing is well defined on cohomology classes. By Remark 6.22 every cohomology class in \( MH_{\dim(S) + 1 - m}(cS; E) \) is of the form \([\omega] = [d\chi \wedge p^*_S(\eta)]\), where
\[ \eta' \in \bigoplus_{i+j=\dim(S)-m \atop i \geq \dim(S) - m + 1} \left(H^i(L', H^j(F))\right). \]

Similarly, every cohomology class in \( MH_{\dim(S)}(cS) \) is of the form \([\omega'] = [p^*_S(\eta')]\), where
\[ \eta \in \bigoplus_{i+j=m \atop i \leq \tau(m)} \left(H^i(L', H^j(F))\right). \]

Thus, if we assume that the pairing \( \int_{\mathcal{D}(\tau)} \) degenerates, that is we assume
\[ \int_{\mathcal{D}(\tau)} p^*_S(\eta) \wedge d\chi \wedge p^*_S(\eta') = 0, \]
then, since the integral of the one form \( d\chi \) is 1, we obtain that
\[ \int_S \eta \wedge \eta' = 0, \]
so that, by applying Proposition 6.23, the pairing \( \int_{\mathcal{D}(\tau)} \) is nondegenerate. \( \square \)
2.5 Lift-compact cohomology and Poincaré duality for $cS \times \mathbb{R}$

To compute the lift-compact multiperverse cohomology of $cS \times \mathbb{R}$ we will apply a modification of the standard Poincaré homotopy operator for compactly supported forms ([9], 37-40). For this we fix a flat-type depth 1 multicontrolled stratified space $X$ and then we consider integration along the fibre $p_\ast : \Omega^k_{lc}(X \times \mathbb{R}) \rightarrow \Omega^{k-1}_{lc}(X)$,

which is defined on the basis forms of $\mathcal{D}(X) \times \mathbb{R}$ as follows:

\[ f(x, t) \wedge p^{\ast}_{\mathcal{D}(X)}(\eta) \rightarrow 0, \]
\[ f(x, t) \wedge p^{\ast}_{\mathcal{D}(X)}(\eta) \wedge dt \rightarrow \eta \wedge \int_{w=-\infty}^{w=\infty} f(x, w) \, dw. \]

Furthermore, we also consider the map $P : \Omega^{k-1}_{lc}(X) \rightarrow \Omega^k_{lc}(X \times \mathbb{R})$,

which is defined on lifts by $\eta \mapsto p^{\ast}_{\mathcal{D}(X)}(\eta) \wedge \psi$, where $\psi = e(t) \wedge dt \in \Omega^1_c(\mathbb{R})$ is a compactly supported 1-form with total integral equal to 1. Then we define the homotopy operator

\[ K_{lc}^k : \Omega^k_{lc}(X \times \mathbb{R}) \rightarrow \Omega^{k-1}_{lc}(\mathcal{D}(X) \times \mathbb{R}) \]

by the following correspondence on lifts:

\[ f(x, t) \wedge p^{\ast}_{\mathcal{D}(X)}(\eta) \rightarrow 0, \]
\[ f(x, t) \wedge p^{\ast}_{\mathcal{D}(X)}(\eta) \wedge dt \rightarrow p^{\ast}_{\mathcal{D}(X)}(\eta) \wedge \left( \int_{w=-\infty}^{w=t} f(x, w) \, dw - \int_{w=-\infty}^{w=t} e(w) \, dw \cdot \int_{w=-\infty}^{w=\infty} f(x, w) \, dw \right). \]

For the above it is necessary to remark that $p_\ast P \eta$ is $\eta$, this is obtained by the choice of $\psi = e \wedge dt$ with total integral 1.

**Lemma 6.25.** Let $\omega \in \Omega^k_{lc}(X \times \mathbb{R})$ be a lift-compact $k$-form with $k \in \mathbb{Z}_{\geq 1}$. Then the form $K_{lc}^k(\omega) \in \Omega^{k-1}_{lc}(\mathcal{D}(X) \times \mathbb{R})$ is the lift of a form on $\text{reg}(X)$, leading to a well-defined map

\[ K_{lc}^k : \Omega^k_{lc}(X \times \mathbb{R}) \rightarrow \Omega^{k-1}_{lc}(X \times \mathbb{R}). \]

**Proof.** It suffices to prove the result when $\tilde{\omega} = f(x, t) \wedge p^{\ast}_{\mathcal{D}(X)}(\eta) \wedge dt$, where $\eta \in \Omega^{k-1}(\mathcal{D}(X))$ and $f(x, t)$ is the lift of a liftable function in $X$, that is a lift-
able function for each fixed \( t \); note also that \( f(x, t) \) is compactly supported (the restriction to \( \theta^{-1}_c[\text{reg}(X)] \) may not be compactly supported, however). Now we apply the Poincaré homotopy operator \( K^k_{lc} \) to \( \omega \), obtaining

\[
K^k_{lc}(f(x, t) \wedge p^*_{\theta(X)}(\eta) \wedge dt) = p^*_{\theta(X)}(\eta) \wedge \left( \int_{w=-\infty}^{w=t} f(x, w) \, dw - \int_{w=-\infty}^{w=t} e(w) \, dw \cdot \int_{w=-\infty}^{w=\infty} f(x, w) \, dw \right).
\]

To prove that this is a lift of a form on \( X \times \mathbb{R} \) it will suffice to show that the function

\[
(x, t) \mapsto \left( \int_{w=-\infty}^{w=t} f(x, w) \, dw - \int_{w=-\infty}^{w=t} e(w) \, dw \cdot \int_{w=-\infty}^{w=\infty} f(x, w) \, dw \right)
\]

is liftable, where \( (x, t) \in \text{reg}(X) \times \mathbb{R} \). The expression \( \int_{w=-\infty}^{w=t} f(x, w) \, dw \) is liftable as in Lemma 3.20, the expression \( \int_{w=-\infty}^{w=t} e(w) \, dw \cdot \int_{w=-\infty}^{w=\infty} f(x, w) \, dw \) is liftable by a similar argument, while the expression \( \int_{w=-\infty}^{w=t} e(w) \, dw \) is liftable because it is independent of \( x \).

\[\square\]

**Lemma 6.26.** For all \( k \in \mathbb{Z}_{\geq 1} \) and \( \omega \in \Omega^k_{lc}(X \times \mathbb{R}) \) the following identity holds

\[
K^{k+1}_{lc}(d\omega) + dK^k_{lc}(\omega) = (-1)^{k-1}(\omega - Pp_*\omega).
\]

(6.29)

Thus \( K^*_{lc} \) defines a chain homotopy between the identity and \( Pp_* \) endomorphisms of \( \Omega^*_{lc}(X \times \mathbb{R}) \).

**Proof.** This follows by applying the result of [9] (Proposition 4.6) on lifts. \( \square \)

**Lemma 6.27.** Let \( \tau \in \mathcal{M}(X) \) be a multiperversity for \( X \). The endomorphism \( Pp_* \) maps \( \tau \)-multiperverse forms to \( \tau \)-multiperverse forms.

**Proof.** If the lift of \( \omega \) is \( \tilde{\omega} = f(x, t) \wedge p^*_{\theta(X)}(\eta) \), then \( p_*(\omega) = 0 \). On the other hand, if the lift of \( \omega \) is \( \tilde{\omega} = f(x, t) \wedge p^*_{\theta(X)}(\eta) \wedge dt \), then

\[
Pp_*(\omega) = p^*_{\theta(X)}(\eta) \wedge \left( \int_{w=-\infty}^{w=\infty} f(x, w) \, dw \right) \wedge e(t) \wedge dt.
\]

Then, for each singular stratum \( A \in \mathcal{S}(X) \), we compute the pullback to the boundary:

\[
R_{A \times \mathbb{R}}(Pp_*\omega) = R_{A \times \mathbb{R}} \left( p^*_{\theta(X)}(\eta) \wedge \int_{w=-\infty}^{w=\infty} f(x, w) \, dw \wedge (e(t) \wedge dt) \right)
= R_A \left( \int_{w=-\infty}^{w=\infty} f(x, w) \, dw \wedge \eta \right) \wedge (e(t) \wedge dt). \tag{6.30}
\]
From the assumption that $\omega$ is $\tau$-multiperverse, the following pullback to the boundary is obtained:

$$R_{A \times \mathbb{R}}(\tilde{\omega}) = R_{A \times \mathbb{R}}(f(x, t) \wedge p_\mathcal{G}(X)_e(\eta) \wedge dt)$$

$$= F(s, t) \cdot R_A(\eta) \wedge dt \in \bigoplus_{i \leq \tau(k-h-1)} \wedge_{i+j+h+1 = k} \Omega^{i,j,h+1}(S_A \times \mathbb{R}),$$

for some function $F$, so that $R_A(\eta) \in \bigoplus_{i \leq \tau(k-h)} \Omega^{i,j,h}(S)$. We apply this to Equation 6.30, obtaining

$$R_{A \times \mathbb{R}}(Pp_*\omega) = F(s, t) \cdot R_A(\eta) \wedge dt,$$

so that $Pp_*$ is then $\tau$-admissible. This proves that $Pp_*$ maps lift compact $\tau$-multiperverse to lift-compact $\tau$-admissible forms, which proves the result.

**Lemma 6.28.** The homotopy operator $K_{lc}^k$ descends to lift-compact $\tau$-multiperverse forms:

$$K_{lc}^k : M\Omega^k_{\mathcal{G},\tau}(X \times \mathbb{R}) \to M\Omega^{k-1}_{\mathcal{G},\tau}(X \times \mathbb{R}).$$

Thus, for a lift-compact $\tau$-multiperverse form $\omega \in M\Omega^k_{\mathcal{G},\tau}(X \times \mathbb{R})$:

$$K_{lc}^{k+1}(d\omega) + dK_{lc}^k(\omega) = (-1)^{k-1}(\omega - Pp_*\omega).$$

**Proof.** If $\omega \in M\Omega^k_{\mathcal{G},\tau}(X \times \mathbb{R})$ and $\tilde{\omega} = f(x, t) \wedge p_\mathcal{G}(X)_e(\eta)$, then $K_{lc}^k(\omega) = 0$; thus it will suffice to prove this result in the case when

$$\tilde{\omega} = f(x, t) \wedge p_\mathcal{G}(X)_e(\eta) \wedge dt.$$

In this case we obtain by direct computation:

$$K_{lc}^k(\omega) = K_{lc}^k(f(x, t) \wedge p_\mathcal{G}(X)_e(\eta) \wedge dt)$$

$$= p_\mathcal{G}(X)_e(\eta) \wedge F(x, t),$$

where $F(x, t)$ is the 0-form part of the expression. The rest of the verification proceeds as in Lemma 5.52.

**Corollary 6.29.** Let $\tau \in \mathcal{M}(X)$ be a multiperversity for $X$ and $X \times \mathbb{R}$. For all integers $k \in \mathbb{Z}_{\geq 0}$ there exists an isomorphism

$$MH_{lc,\tau}^k(X \times \mathbb{R}) \cong MH_{lc,\tau}^{k-1}(X).$$

**Proof.** This follows from the above Lemma 6.26 in the same way as Corollary
3.23 follows from Lemma 3.17. For convenience we reproduce the details. Let $[\omega] \in MH_{lc,\tau}^k(X \times \mathbb{R})$ be a cohomology class. From Equation 6.29 in Lemma 6.26 we have

$$Pp_*\omega + (-1)^k \cdot dK_{lc}^k(\omega) = \omega,$$

so that $[\omega] = [Pp_*\omega]$ in $MH_{lc,\tau}^k(X \times \mathbb{R})$. Note that $P$ and $p_*$ are morphisms of complexes ([9], p.38) and that the map

$$[\omega] = [Pp_*\omega] \mapsto [p_*\omega]$$

is linear and well-defined; this correspondence is also injective because $\psi = e \wedge dt \neq 0$. To show that it is surjective let $[\eta] \in MH_{lc,\tau}^{k-1}(X)$. Then we obtain

$$[P\eta] = [Pp_*P\eta] \mapsto [p_*P\eta] = [\eta],$$

from where we conclude that $MH_{lc,\tau}^k(X \times \mathbb{R})$ is isomorphic to $MH_{lc,\tau}^{k-1}(X)$.

**Corollary 6.30.** Let $\tau \in \mathcal{M}(X)$ be a multiperversity for $X$ and $X \times \mathbb{R}^n$. Then for all $k \in \mathbb{Z}_{\geq 0}$ there exists an isomorphism

$$MH_{lc,\tau}^k(X \times \mathbb{R}^n) \cong MH_{lc,\tau}^{k-n}(X).$$

In particular, if $X = cS$, then there exists an isomorphism

$$MH_{lc,\tau}^k(cS \times \mathbb{R}^n) \cong MH_{lc,\tau}^{k-n}(cS) \cong MH_{\tau}^{k-n}(cS; E) \cong \bigoplus_{i+j=k-n-1 \atop i \geq \tau(k-n-1)+1} (H^j(L', H^i(F))).$$

**Proof.** The first part follows from the above Corollary 6.29 by induction, while the second part, for $X = cS$, follows from the first part, Lemma 6.21, and Proposition 6.13.

**Corollary 6.31.** Let $\tau \in \mathcal{M}(cS \times \mathbb{R}^n)$ be a multiperversity. There exists an isomorphism

$$MH_{lc,\mathcal{D}(\tau)}^{\dim(S)+n+1-m}(cS \times \mathbb{R}^n) \cong MH_{\tau}^m(cS)^*.$$

Furthermore, the isomorphism is given by the map

$$\Psi : MH_{lc,\mathcal{D}(\tau)}^{\dim(S)+n+1-m}(cS \times \mathbb{R}^n) \longrightarrow MH_{\tau}^m(cS)^*,$$

which is defined by integration on lifts $[\tilde{\omega}] \mapsto \int_{\mathcal{D}(cS \times \mathbb{R}^n)} \tilde{\omega} \wedge -$. 
**Proof.** This follows from the above Corollary 6.30 by the same proof as in Lemma 6.24, using the non-degeneracy of the pairing for \( S \) in Proposition 6.23.

\[ \square \]

## 3 Global Poincaré duality

In this section we will use Lemma 6.21 and the Mayer-Vietoris argument to deduce a global Poincaré duality for certain types of flat-type depth 1 multicontrolled spaces.

### 3.1 Isolated singularities

In this subsection we will consider a depth 1 flat-type multicontrolled stratified space \( X \) with isolated singularities. Let us assume that \( \dim(\text{reg}(X)) = n \) and further that \( X \) has \( N \in \mathbb{Z} \) isolated singularities \( \text{pt}_i \in S\mathcal{S}(X), \) so that \( X \) is a union

\[ X = E \cup \bigcup_{1 \leq i \leq N} T_{\text{pt}_i}. \]

First consider the case when \( N = 1 \) and \( X = E \cup T_{\text{pt}_1}. \) To prove the Poincaré duality for \( X \) we will need to consider two short exact sequences:

\[
0 \to M\Omega^*_c(T_{\text{pt}} \cup E) \xrightarrow{\alpha^\prime} M\Omega^*_r(T_{\text{pt}}) \oplus \Omega^*(E) \xrightarrow{\beta^\prime} \Omega^*(E_{\text{pt}}) \to 0, \tag{6.31}
\]

\[
0 \to \Omega^*_c(E_{\text{pt}}) \xrightarrow{\alpha^\prime} M\Omega^*_c(T_{\text{pt}}) \oplus \Omega^*_c(E) \xrightarrow{\beta^\prime} M\Omega^*_c(T_{\text{pt}} \cup E) \to 0. \tag{6.32}
\]

In the above \( E_{\text{pt}} \) is the intersection of \( T_{\text{pt}} \) and \( E, \) this can be chosen to be \( (0.5, 1) \times S_A, \) where we reparametrize \( \text{reg}(T_A) \) to \( (0, 1) \times S_A. \) The above Equation 6.31 is the short exact sequence from Lemma 6.4. The above Equation 6.32 is the analogous short exact sequence for lift-compact forms, where the map \( \alpha^\prime \) is defined by the correspondence

\[
\omega \longmapsto ( -j_{\text{reg}(T_{\text{pt}})}(\omega), j_E(\omega) ), \tag{6.33}
\]

where \( j_- \) denotes extension by zero, while the map \( \beta^\prime \) is defined by the correspondence

\[
(\eta, \eta^\prime) \longmapsto j_{\text{reg}(T_{\text{pt}}) \cup E}(\eta) - j_{\text{reg}(T_{\text{pt}}) \cup E}(\eta^\prime). \tag{6.34}
\]

**Proposition 6.32.** The maps \( \alpha^\prime \) and \( \beta^\prime \) are appropriately defined and Equation 6.32 defines a short exact sequence.

**Proof.** The map \( \alpha^\prime \) is appropriately defined because \( j_{\text{reg}(T_{\text{pt}})}(\omega) \) is automatically liftable and \( \tau \)-multiperverse, its pullback to the boundary being 0. For \( \beta^\prime, \) as in Equation 6.34, we note that \( j_{\text{reg}(T_{\text{pt}}) \cup E}(\eta) \) is automatically \( \tau \)-multiperverse and lift-compact as \( \eta \) is \( \tau \)-multiperverse and lift-compact; further \( j_{\text{reg}(T_{\text{pt}}) \cup E}(\eta^\prime) \) is also

\[ \square \]
automatically $\tau$-multiperverse and lift-compact as $\eta'$ is trivially $\tau$-multiperverse. To verify exactness, we note that $\alpha'$ is injective and that $\text{im}(\alpha') \subseteq \ker(\beta')$, for $\ker(\beta') \subseteq \text{im}(\alpha')$ we note that, if $\beta'(\eta, \eta') = 0$, then necessarily $\text{supp}(\eta) \subseteq E$ and $\text{supp}(\eta') \subseteq \text{reg}(T_{pt})$, so that they are in fact the same compactly supported form on $E_{pt}$. It remains to verify the surjectivity of $\beta'$. For this we fix $\chi$ and $\chi'$ as in the proof of Lemma 6.1. For $\omega \in M\Omega_{lc,\tau}^*(X)$, we note that

$$(\chi' \cdot \omega, -\chi \cdot \omega) \in M\Omega_{lc,\tau}^*(T_{pt}) \oplus \Omega_c^*(E),$$

and that $\beta'(\chi' \omega, -\chi \omega) = \omega$. In the above we have used that $\chi' \cdot \omega$ is liftable with the support of its lift is compact, this follows because it is a closed subset in a compact subset, hence compact; a similar remark applies to $-\chi \cdot \omega$. This proves the result.

From the short exact sequences in Equations 6.31 and 6.32 we obtain long exact sequences on cohomology:

$$\ldots \to MH^k_c(X) \xrightarrow{\alpha} MH^k_c(T_{pt}) \oplus H^k_c(E) \xrightarrow{\beta} H^k_c(E_{pt}) \xrightarrow{\delta} MH^k_{c+1}(X) \to \ldots,$$

$$\ldots \to H^k_c(E_{pt}) \xrightarrow{\alpha'} MH^k_{lc,\tau}(T_{pt}) \oplus H^k_c(E) \xrightarrow{\beta'} MH^k_{lc,\tau}(X) \xrightarrow{\delta'} H^k_{c+1}(E_{pt}) \to \ldots.$$

Using these two sequences, Poincaré duality for manifolds and the local Poincaré duality for $T_{pt} = cS$ we obtain the following diagram:

\[
\begin{array}{ccccccccc}
H_c^{n-k}(E_{pt}) & \xrightarrow{\beta} & H^k(E_{pt})^* & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
MH_{lc,\tau}^{n-k}(T_{pt}) \oplus H^k_c(E) & \xrightarrow{\beta'} & MH^k_{c}(T_{pt})^* \oplus H^k_c(E)^* & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
MH_{lc,\tau}^{n-k}(X) & \xrightarrow{\beta'} & MH^k_{lc,\tau}(X)^* & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H_c^{n-k-1}(E_{pt}) & \xrightarrow{\beta'} & H^k_c(E_{pt})^* & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\end{array}
\]

In the above the $\int$ maps denote the pairings, mapping a form $\omega$ to the $\mathbb{R}$-valued function $\int \omega \wedge -$. Let us briefly indicate the commutativity of the three squares.
1, 2 and 3. For the top square 1 we compute:

\[
\left( \beta^* \int_{E_{pt}} \right) (\omega)(\eta, \eta') = \int_{E_{pt}} \omega \wedge i_{E_{pt}}^*(\eta) + \int_{E_{pt}} \omega \wedge i_{E_{pt}}^*(\eta') \quad (6.36)
\]

\[
\left( \int_{\text{reg}(T_{pt})} + \int_{E} \right) (\alpha' \omega)(\eta, \eta') = \int_{\text{reg}(T_{pt})} \omega \wedge \eta + \int_{E} \omega \wedge \eta' \quad (6.37)
\]

In the above \( \omega \in H_c^{n-k}(E_{pt}), \eta \in MH^k(T_{pt}) \) and \( \eta' \in H^k(E) \). The expression on the right-hand side in the above Equation 6.36 is equal to the Equation on the right-hand side in the above Equation 6.37 because the (compact) support of the forms \( \omega \wedge i_{E_{pt}}^*(\eta') \) and \( \omega \wedge \eta \) is equal and within this support they are equal, so the integrals are equal, and the same for \( \eta' \).

For the middle square 2 we compute:

\[
\alpha^* \left( \int_{\text{reg}(T_{pt})} + \int_{E} \right) (\omega, \omega')(\eta) = \int_{\text{reg}(T_{pt})} \omega \wedge i_{\text{reg}(T_{pt})}^*(\eta) - \int_{E} \omega' \wedge i_{E}^*(\eta). \quad (6.38)
\]

\[
\left( \int_{\text{reg}(X)} \right) (\omega, \omega')(\eta) = \int_{\text{reg}(X)} \omega \wedge \eta - \int_{\text{reg}(X)} \omega' \wedge \eta \quad (6.39)
\]

In the above \( \omega \in MH_c^{n-k}(T_{pt}), \omega' \in H_c^{n-k}(E) \) and \( \eta \in MH^k(X) \). The expression on the right-hand side in the above Equation 6.38 is equal to the expression on the right-hand side in the above Equation 6.39 because the (compact) support of the forms \( \omega \wedge i_{\text{reg}(T_{pt})}^*(\eta) \) and \( \omega \wedge \eta \) is equal and within this support they are equal, so the integrals are equal, and the same for \( \omega' \).

For the bottom square 3 we compute:

\[
\delta^* \left( \int_{\text{reg}(X)} \right) (\omega)(\eta) = \int_{\text{reg}(X)} \omega \wedge \delta(\eta) \quad (6.40)
\]

\[
= \int_{E_{pt}} \omega \wedge d(\chi \eta) \quad (6.41)
\]

\[
= \int_{E_{pt}} \omega \wedge (d(\chi) \wedge \eta) \quad (6.42)
\]

\[
\left( \int_{E_{pt}} \right) (\delta' \omega)(\eta) = \int_{E_{pt}} \delta' \omega \wedge \eta \quad (6.43)
\]

\[
= \int_{E_{pt}} d(\chi) \wedge \omega \wedge \eta \quad (6.44)
\]
In the above $\omega \in MH^{n-k}_{lc,\mathcal{D}(\tau)}(X)$ and $\eta \in H^{k+1}(E_{pt})$. The above Equation 6.42 is equal to Equation 6.44 up to sign. In Equations 6.43 and 6.40 we have used the definition of $\delta$ and $\delta'$, via the snake lemma (see also Remark 6.2); in Equation 6.41 we have used the fact that $\eta$ is a closed form, as we have done in the conclusion of Equation 6.44. Thus the square commutes up to sign, but this is sufficient. Using the five lemma we obtain the following:

**Lemma 6.33.** Let $X$ be a depth 1 flat-type multicontrolled stratified space with isolated singularities. For a multiperversity $\tau \in M(X)$ there exists a Poincaré duality isomorphism

$$MH^{\dim(X)-m}_{lc,\mathcal{D}(\tau)}(X) \cong MH^m_{\tau}(X)^*.$$  

Furthermore, this isomorphism is induced by integration on $\text{reg}(X)$.

**Proof.** Consider first the case with a single isolated singularity and recall the commutative diagram in Equation 6.35 on 147 above. Using the five lemma we obtain that if $\mathcal{f}$ is an isomorphism for $H^c_\tau(E_{pt}), MH^{\bullet}_{lc,\mathcal{D}(\tau)}(T_{pt})$ and $H^c_\tau(E)$, then $\mathcal{f}$ is an isomorphism for $MH^{\bullet}_{lc,\mathcal{D}(\tau)}(X)$. Note that a multiperversity for $X$ is a multiperversity for $T_{pt}$. For the general case we proceed by iteration on the number of cusps and using Lemma 6.1 instead of Lemma 6.4, replacing $H^c_\tau(E)$ with the $\tau$-multiperverse lift-compact cohomology and glueing on a single cusp at a time; the verification of the commutativity of the analogous diagram follows as above for Equation 6.35. Note that the multiperversity $\tau$ for $X$ can be naturally considered as multiperversity for any cusp of $X$. \hspace{1cm} \square

### 3.2 General case

In this subsection we will consider the general case of global Poincaré duality for multiperverse cohomology. We will use an argument analogous to that of Equation 6.35, but using also Corollary 6.31 and a similar inductive proof as for the Künneth formula in Lemma 6.8.

**Theorem 6.34.** Let $X$ be a depth 1 flat-type multicontrolled stratified space. For a multiperversity $\tau \in M(X)$ there exists a Poincaré duality isomorphism

$$MH^{\dim(X)-m}_{lc,\mathcal{D}(\tau)}(X) \cong MH^m_{\tau}(X)^*.$$  

Furthermore, this isomorphism is induced by integration on $\text{reg}(X)$.

**Proof.** The verification of this results follows the outline of the proof of Lemma 6.33, in particular we use the argument to obtain the commutative diagram in
Equation 6.35, which we do not repeat here. However, because there are two inductions, we split the proof up into two parts.

**Part 1.** The first part of the proof is to obtain Poincaré duality for a general cusp $T_A$ of $X$, where $A \in S^s(X)$ is a fixed singular stratum of $X$. This is done by induction on the number $N = N_A \in \mathbb{Z}_{\geq 1}$ of elements in a finite good cover $\{U_i\}_{i=1}^{N_A}$ of $A$. If $N = 1$ then the result follows by Corollary 6.31. If we assume that the result holds for a good cover by $N$ open subsets, then using the five lemma applied to a diagram as in Equation 6.35. Using the Mayer-Vietoris argument, as in Part 4.2 of Lemma 6.8, we obtain the result for a good cover by $N + 1$ open subsets. This establishes the duality for $T_A$, so we have verified that

$$MH_{lc, \delta}(T_A) \cong MH^m_{\tau}(T_A)^*.$$ 

**Part 2.** Fix a subset $E$, which is a slightly enlarged $\text{reg}(X)$, so that $E \cap T_A = E_A$ is $S_A \times (0, 1)$, where $\text{reg}(T_A)$ is $S_A \times (0, 1)$. Then $E$ is a smooth manifold, so that Poincaré duality holds. Then we proceed by iteration on the number of cusps, as outlined in the proof of Lemma 6.33. This establishes the result. 

\[\square\]

## 4 Self-dual multiperversities

In this section we consider the concept of a self-dual multiperversity, which extends some aspect the concept of the self-dual middle perversity on a Witt space ([23], Section 5.2; in particular Corollary 5.2.4).

### 4.1 Further properties of multiperversities

In this brief subsection we introduce some further properties of the lattice of multiperversities $M(X)$, these will be relevant when we consider the self-dual multiperversities.

The first property we consider is the extension of Lemma 1.1 in [20] to multiperversities. The content of the result below states that any multiperversity $\tau \in M(X)$, which is not the top multiperversity $m(\overline{t})$, may be increased by a “single step”, though this increase may be done in a number of distinct ways, this number being governed by the number of elements which cover $\tau$ in $M(X)$.

**Proposition 6.35.** Let $X$ be a multicontrolled stratified space. There exists an order preserving function

$$\text{rk}: M(X) \rightarrow \mathbb{Z}_{\geq 0}$$

which satisfies the following condition: if $\tau < \tau'$ in $M(X)$ then there exists $\tau'' \in M(X)$ with $\text{rk}(\tau') = \text{rk}(\tau'') + 1$ and $\tau \leq \tau'' < \tau'$. Hence $M(X)$ is a ranked lattice.
Proof. The rank function is defined as a double sum

$$\text{rk}(\tau) := \sum_{A \in \mathcal{SS}(X)} \sum_{i \in \mathbb{Z}} \tau(A)(i),$$

where we note that this iterates the rank function on \( (\mathbb{Z}^\text{op}, \mathbb{Z}) \). If \( \tau < \tau' \), then there exists some singular stratum \( A \in \mathcal{SS}(X) \) such that \( \tau(A) < \tau'(A) \) is satisfied; fix this singular stratum \( A \). For this fixed singular stratum \( A \) there exists a smallest integer \( i \in \mathbb{Z} \) such that \( \tau(A)(i) \leq \tau'(A)(i) \). We now fix the smallest integer \( j \in \mathbb{Z} \) which satisfies (i) \( i < j \) and (ii) for all \( j' \in \mathbb{Z} \) which satisfy \( j \leq j' \) we have \( \tau(A)(j') < \tau'(A)(j) \); if there does not exists a \( j \) satisfying the two preceding conditions, then then we take \( j = \text{codim}(A) \). The element \( \tau''(A) \in (\mathbb{Z}^\text{op}, \mathbb{Z}) \) is then defined by the following expression

$$\tau''(A)(k) := \begin{cases} \tau'(A)(k) & \text{if } k \neq j \\ \tau(A)(k) + 1 & \text{if } k = j \end{cases}$$

while, for \( B \in \mathcal{SS}(X) \) with \( B \neq A \), we define \( \tau''(B) = \tau'(B) \). From the construction of \( \tau'' \), we see that it satisfies the required properties.

The above result implies that all maximal chains in \( \mathcal{M}(X) \) are finite and of equal length.

### 4.2 Self-dual multiperversities

In this subsection we consider the concept of a self-dual multiperversity, that is of a multiperversity \( \tau \in \mathcal{M}(X) \) such that \( \mathcal{D}(\tau) = \tau \).

**Example 6.36.** Consider a depth 1 controlled stratified space \( X \) in which is a pseudomanifold such that all strata have even codimension, then ([23], Example 5.2.2) \( X \) is a Witt space. To consider this situation explicitly, consider the cone \( cS \) over \( S = T^3 \), then \( \text{codim}_{cS}(\text{pt}) = 4 \) is even, so \( cS \) is an example of a Witt space. In fact, it is a Witt space simply because the upper and lower middle perversities take the same value, \( \overline{m} = m = \frac{4}{2} - 1 = 1 \). Let us further assume that \( S' = S^1 \), so that \( \text{codim}_{cS}(S') = 3 \) and \( cS \) is a depth 1 multicontrolled cone. Then if we consider the multiperversity

$$\tau = m(\overline{m}) = m(m),$$

we can illustrate \( \tau \) on the following diagram.
Figure 6.1: Diagram representing the multiperversities $m(m)$ and $m(m)$.

As the dashed lines illustrate, this multiperversity $\tau$ cuts the multiperversity $\bar{1}$ in half, from this remark it is clear that $\mathcal{D}(\tau) = \tau$.

**Example 6.37.** Consider a depth 1 controlled cone $cS$ over $S$, where $\dim(S) = 2r$ but $H^r(S) \neq 0$. The cone $cS$ is a pseudomanifold, but not a Witt space because $H^r(S) \neq 0$, so the two middle perversities do not induce the same intersection cohomology; there is no self-dual perversity. Take, for example $S = T^4$, which has non-zero cohomology in degree 2 (and all degrees from 0 and 4). Consider a product-type multicontrolled structure for $cS$ given by letting $S_1 = L_1 = S_1$, so that $F = T^3$, with $\text{codim}_{cS}(pt) = 5$ and $\text{codim}_{cS}(S') = 5 - 1 = 4$. Define the multiperversity $\tau$ as follows:

$$\tau(k) = \begin{cases} 
3 & \text{if } k < 0 \\
3 & \text{if } k = 0 \\
3 & \text{if } k = 1 \\
1 & \text{if } k = 2 \\
-1 & \text{if } k \geq 3 
\end{cases}$$

We can illustrate $\tau \in M(cS)$ on the following diagram.
To illustrate that $\tau$ is self-dual, we recall the formula for $\mathfrak{D}(\tau)$, as in Equation 6.26 on page 138:

$$\mathfrak{D}(\tau)(k) = (\dim(S) - \dim(L')) - \tau(\dim(S) - k) - 1$$

$$= (4 - 1) - \tau(4 - k) - 1$$

$$= 2 - \tau(4 - k).$$

Using this formula we compute the values of the multiperversity $\mathfrak{D}(\tau)$:

$$\mathfrak{D}(\tau)(-1) = 2 - \tau(4 - (-1)) = 2 - (-1) = 3,$$

$$\mathfrak{D}(\tau)(0) = 2 - \tau(4 - 0) = 2 - (-1) = 3,$$

$$\mathfrak{D}(\tau)(1) = 2 - \tau(4 - 1) = 2 - (-1) = 3,$$

$$\mathfrak{D}(\tau)(2) = 2 - \tau(4 - 2) = 2 - 1 = 1,$$

$$\mathfrak{D}(\tau)(3) = 2 - \tau(4 - 3) = 2 - 3 = -1,$$

$$\mathfrak{D}(\tau)(4) = 2 - \tau(4 - 4) = 2 - 3 = -1,$$

$$\mathfrak{D}(\tau)(5) = 2 - \tau(4 - 5) = 2 - 3 = -1.$$
As the Betti numbers\(^5\) in the cohomology of \(S^1\) and \(T^3\) are all non-zero, we obtain that this multiperversity \(\tau\) gives \(\tau\)-multiperverse cohomology that does not correspond to any intersection cohomology group.

5 \(L^2\)-cohomology of fibre cusps

In this section we will compute the weighted \(L^2\)-cohomology of certain cusps whose natural compactifications are depth 1 product-type multicontrolled stratified spaces.

5.1 \(L^2\)-cohomology of double-product cusps

Fix two compact manifolds \(F, L'\) with \(\dim(F) = f\), \(\dim(L') = l'\). Consider a double-product cusp

\[
Y := \mathbb{R}_{>0} \times F \times L',
\]

equipped with the doubly-warped Riemannian metric

\[
ds_Y := dr^2 + e^{-2a_1 r} ds_F^2 + e^{-2a_2 r} ds_{L'}^2,
\]

where \(a_1, a_2 \in \mathbb{R}_{>0}\) are two constants with \(a_1 \geq a_2\) ([3], 9.11 Warped Products); if \(a_1 = a_2\) we obtain intersection cohomology, so we assume that \(a_1 > a_2\). Using Zucker’s Künneth formula ([32], Corollary 2.36), we obtain the following computation.

**Proposition 6.38.** The weighted \(L^2\)-cohomology of the double-product cusp \(Y\)

\(^5\)The Betti number for the cohomology of \(T^n = (S^1)^n\) is given by coefficients of its Poincaré polynomial \((1 + x)^n\).
with the Riemannian metric $ds_Y$ satisfies

$$WH^{k}_{(2)}(Y; a_0) = \bigoplus_{i+j=k \atop c(i,j)<0} H^i(F) \otimes H^j(L'),$$

where $c(j,k)$ is defined as

$$c(i,j) = a_1 \left( i - \frac{f}{2} \right) + a_2 \left( j - \frac{l'}{2} \right) + a_0.$$

Here $a_0$ is a small constant so that we have $c(i,j) \neq 0$ for all $i, j \in \mathbb{Z}_{\geq 0}$.

**Proof.** Using Zucker’s Künneth formula ([32], Corollary 2.36) the computation is obtained as follows:

$$WH^{k}_{(2)}(Y; a_0) = \bigoplus_{k=0}^{l'} WH^{k-j}_{(2)}(\mathbb{R}_{>0} \times F; a_2 j - \frac{l'}{2}) \otimes H^j(L')$$

$$= \bigoplus_{i=0}^{j} \bigoplus_{j=0}^{l'} WH^{k-i-j}_{(2)}(\mathbb{R}_{>0}; c(i,j)) \otimes H^i(F) \otimes H^j(L').$$

The result then follows by using the following result ([19], in proof of Proposition 2), which computes the weighted $L^2$-cohomology of $\mathbb{R}_{>0}$ with non-zero weight:

$$WH^0_{(2)}(\mathbb{R}_{>0}; c(i,j)) = \begin{cases} \mathbb{R} & \text{if } c(i,j) < 0 \\ 0 & \text{if } c(i,j) \geq 0 \end{cases}$$

Furthermore we have $WH^1_{(2)}(\mathbb{R}_{>0}; c(i,j)) = 0$ because $c(i,j) \neq 0$ ([19], in proof of Proposition 2). From this the result follows. \qed

In the above Proposition 6.38 the weight function

$$c(i,j) = a_1 \left( i - \frac{f}{2} \right) + a_2 \left( j - \frac{l'}{2} \right) + a_0$$

can be presented as a function $\tau(k)$ of the argument $k = i + j$, with the condition $c(i,j) < 0$ replaced by $i < \tau(k)$. To construct $\tau$, firstly we rearrange the inequality

$$a_1 \left( i - \frac{f}{2} \right) + a_2 \left( j - \frac{l'}{2} \right) + a_0 < 0,$$

by using the substitution $j = k - i$ to obtain

$$a_1 i + a_2 (k - i) < \frac{a_1 f}{2} + \frac{a_2 l'}{2} - a_0,$$
from where we obtain the inequality

\[ i < \frac{a_1 f + a_2 f'}{2} - a_0 - a_2 k \frac{a_1}{a_2} . \]

We define \( \tau(k) \) to be the floor of the right-hand side in the above equation. Thus we arrive at the following.

**Corollary 6.39.** The weighted \( L^2 \)-cohomology of the double-product cusp \( Y \) with the Riemannain metric \( ds_Y \) satisfies

\[
WH_{(2)}^k(Y; a_0) = \bigoplus_{i+j=k \atop i \in \tau(k)} H^i(F) \otimes H^j(L').
\]

### 5.2 Compactifications of double-product cusps and the relation to multiperverse cohomology

The cusp \( Y := \mathbb{R}_{>0} \times F \times L' \) of the preceding subsection 5.1 has two natural compactifications; these are in some way analogues of the Borel-Serre ([5],III.5) and the reductive Borel-Serre ([5],III.6) compactifications for locally symmetric spaces. The first compactification is obtained by adjoining to \( Y \) the space \( F \times L' \) at the zero-coordinate to obtain

\[ \overline{Y} := (\mathbb{R}_{>0} \cup \infty) \times F \times L', \]

which is analogous to the Borel-Serre compactification; we will call \( \overline{Y} \) the topological compactification of \( Y \). Note that \( \overline{Y} \) can be stratified by a single singular stratum \( \partial \overline{Y} := \{ \infty \} \times F \times L' \), with \( \text{reg}(\overline{Y}) = Y \). The second compactification is obtained by collapsing \( \partial \overline{Y} \) to a point \( \text{pt} \), thus obtaining

\[ \hat{Y} := c(F \times L'). \]

This is analogous to the reductive Borel-Serre compactification; we will call \( \hat{Y} \) the metric compactification of \( Y \) and \( \hat{Y} \) can be stratified as the cone, with the sole singular stratum \( \text{pt} \). Furthermore, given the structure of \( Y \), we obtain that \( \hat{Y} \) is a depth 1 product-type multicontrolled stratified space.

**Corollary 6.40.** Let \( Y \) and \( \tau \) be as in Corollary 6.39. There exists an isomorphism

\[
WH_{(2)}^k(Y; a_0) \simeq MH^k_\tau(\hat{Y}).
\]

**Proof.** This follows because the calculation in Theorem 5.43 and Corollary 6.39 yield identical computations. \( \square \)
References


