Effects of magnetic field on electron transport in semiconductor superlattices

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Metadata Record: [https://dspace.lboro.ac.uk/2134/21922](https://dspace.lboro.ac.uk/2134/21922)

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Effects of Magnetic Field on Electron Transport in
Semiconductor Superlattices

by

Liang Zhang

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of
Doctor of Philosophy of Loughborough University

October 2015
Abstract

Quantum superlattice with a narrow energy band is an artificial semiconductor structure demonstrating both nonlinear and active high-frequency electromagnetic properties. These types of superlattices are used as key elements in various miniature electronic devices including frequency multipliers and quantum cascade lasers. Interaction between terahertz radiation and magnetic field in semiconductor superlattices has been the subject of growing research interest, both theoretical and experimental. In this thesis, we study the nonlinear dynamics of electrons in minibands of the semiconductor superlattices subjected to a terahertz electric field and a magnetic field.

Electron transport in a semiconductor superlattice with an electric field and a tilted magnetic field has been studied using semiclassical equations. In particular, we consider how dynamics of electron in superlattices evolve with changing the strength and the tilt of a magnetic field. In order to investigate the influence of a tilted magnetic field on electron transport, we calculate the drift velocity for different values of the magnetic field. Studies have shown that the resonance of Bloch oscillations and cyclotron oscillations produces additional peaks in drift velocity. We also found out that appearance of these resonances can promote amplification of a small ac signal applied to the superlattice.

In the presence of the electromagnetic field, the superlattice is expected to demonstrate the Hall effect, which however should have a number of very specific features due to an excitation of Bloch oscillations and a significant electric anisotropy. Here, we theoretically study the Hall effect in a semiconductor superlattice both for the steady electron transport and for the transient response. We studied the coherent Hall effect in an extraordinary configuration where the electric field is applied in the transverse...
direction of the superlattice growth direction. By mapping the momentum dynamics to the pendulum equivalent, we distinguished the two regimes of the oscillations from the viewpoint of the effective potentials. We discuss the experimental manifestation of the Hall effect in a realistic superlattice. We also made the numerical simulations of the polarized THz field and the time-resolved internal electro-optic sampling (TEOS) signals where we found the unusual shaped waveforms of the THz signals.
Acknowledgements

I would like to express my gratitude to all those who helped me during the writing of this thesis.

My deepest gratitude goes first and foremost to Dr Alexander Balanov and Dr Cyril Alekseev, my supervisors, for their constant encouragement and guidance. They have walked me through all the stages of the writing of this thesis and offered me valuable suggestions in the academic studies.

Second, I feel grateful to all the teachers in the physics department of Loughborough University who once offered me valuable courses and advice during my study.

Last my thanks would go to my beloved family for their loving considerations and great confidence in me all through these years. I also owe my sincere gratitude to my friends and my fellow classmates who gave me their help and time in listening to me and helping me work out my problems during the difficult course of the thesis.
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<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$V$</td>
<td>Potential energy.</td>
</tr>
<tr>
<td>$\mathbf{r}$</td>
<td>Position vector</td>
</tr>
<tr>
<td>$\mathbf{R}_n$</td>
<td>Lattice vector.</td>
</tr>
<tr>
<td>$\mathbf{a}$</td>
<td>Unit lattice vector.</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Electron wavefunction.</td>
</tr>
<tr>
<td>$k$</td>
<td>Wavevector.</td>
</tr>
<tr>
<td>$\psi(\mathbf{r})$</td>
<td>Bloch wave.</td>
</tr>
<tr>
<td>$m^*$</td>
<td>Effective mass.</td>
</tr>
<tr>
<td>$\hbar$</td>
<td>Planck's constant divided by $2\pi$.</td>
</tr>
<tr>
<td>$d$</td>
<td>Superlattice period.</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Miniband width.</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Electron scattering time.</td>
</tr>
<tr>
<td>$F$</td>
<td>Electric field magnitude.</td>
</tr>
<tr>
<td>$e$</td>
<td>Charge on an electron.</td>
</tr>
<tr>
<td>$\omega_B$</td>
<td>Bloch frequency in ordinary configuration ($\omega_B = eFd/\hbar$).</td>
</tr>
<tr>
<td>$\omega_y$</td>
<td>Bloch frequency in extraordinary configuration.</td>
</tr>
<tr>
<td>$N_0, N(t)$</td>
<td>Electron density.</td>
</tr>
<tr>
<td>$v_d$</td>
<td>Electron drift velocity.</td>
</tr>
<tr>
<td>$B$</td>
<td>Magnetic field magnitude.</td>
</tr>
<tr>
<td>$E(\mathbf{p})$</td>
<td>The energy versus crystal momentum dispersion relation for the lowest miniband.</td>
</tr>
<tr>
<td>$p_x, p_y, p_z$</td>
<td>Momentum components along the $x$, $y$ and $z$ axes respectively.</td>
</tr>
<tr>
<td>$v_x, v_y$ and $v_z$</td>
<td>Velocity components along the $x$, $y$ and $z$ axes respectively.</td>
</tr>
<tr>
<td>$\omega_c, \omega_{cx}$, $\omega_{cy}$</td>
<td>Cyclotron frequency.</td>
</tr>
<tr>
<td>$x$</td>
<td>Axis aligned along the superlattice growth direction.</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Magnetic field tilt angle relative to the x axis.</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$r$</td>
<td>Ratio defined by $r = \frac{\omega_B}{\omega_c}$.</td>
</tr>
<tr>
<td>$A$</td>
<td>Absorption of the probe ac field.</td>
</tr>
<tr>
<td>$W_x$</td>
<td>Defined by $W_x = -\cos\left(\frac{p_x d}{\pi}\right)$.</td>
</tr>
<tr>
<td>$U(k_x)$</td>
<td>Potential energy.</td>
</tr>
<tr>
<td>$j_x, j_y$</td>
<td>Current density components along the $x$ and $y$ axes respectively.</td>
</tr>
<tr>
<td>$E_{THz,x}$</td>
<td>The electric field strength of the radiated electromagnetic field.</td>
</tr>
<tr>
<td>$P_x$</td>
<td>Component of the polarization.</td>
</tr>
<tr>
<td>$\tau_d$</td>
<td>The dephasing time.</td>
</tr>
</tbody>
</table>
1. Background

1.1. Terahertz frequency range

Terahertz (THz) range occupies a quite wide interval on the frequency scale of electromagnetic waves from 100 GHz to 10 THz. The position of terahertz range in the electromagnetic spectrum is between the far infrared and microwave regions. This means that while the long-wave band of terahertz range overlaps with millimeter wave, which belongs to electronics, the short-wave band of terahertz range reaches the infrared region, which belongs to photonics, as shown in figure 1.

![Terahertz Spectrum Diagram](image)

FIG. 1. The position of terahertz range in electromagnetic spectrum.
Terahertz electromagnetic waves, are also known as T-Rays, have important applications in physics [1], life sciences [2], astronomy [3, 4], material science [5] and information technology [6]. At the moment, the most of wave bands in the electromagnetic spectrum have been well studied and applied, except the terahertz range. This can be explained by the difficulties in developing efficient and practical methods to generate, amplify and detect THz radiation. Therefore, the terahertz region in the electromagnetic spectrum is also named as terahertz gap [7]. This gap is the last frequency range needed to be comprehensively studied in the research of the electromagnetic spectrum region [7].

THz electromagnetic waves attract a lot of research attentions due to the following important applications [8, 9]:

1. THz spectroscopy is an efficient good tool to study the physical phenomena in condensed materials, for example nonlinear dynamics of electron associated with charge transport [10].

2. THz imaging technology is one of the most important applications of THz radiation [11, 12]. THz radiation is non-ionising, and it can pass through many materials. Terahertz body scanners can be used as a security facility [13]. Moreover, THz radiation has little damage to biological tissue which makes the T-rays attractive in medical imaging [12].

3. THz waves are promising for high speed of wireless transmission and high security communication [14, 15]. Although it cannot be widely used yet because
of lacking of efficient THz source, it is an important potential application.

The Bloch oscillations make the superlattice be a promising candidate of THz generator. Interaction between terahertz radiation and semiconductor nanostructures has been a subject of growing interest \cite{16-18}. Superlattice is a good component to generate THz radiation \cite{19}. In this thesis, we focus especially on amplifying properties of superlattices, Hall effect and the related THz dynamics of electrons.

1.2. Superlattices

1.2.1. History of superlattices research

In 1970, the concept of semiconductor superlattice was first proposed by Esaki and Tsu, as a structure obtained by the alternating growth of two kinds of semiconductor materials \cite{20}. The aims of Esaki and Tsu’s experiments in semiconductor superlattices were intended to use the negative differential conductivity to achieve high-frequency electromagnetic radiation. Because of the limitation of the experimental setup, defects in materials, non-uniform interfaces and other factors, they cannot make a high quality structure of the superlattice. Nowadays with the development of molecular beam epitaxial technology \cite{21}, various high quality periodic structures can be successfully grown in the laboratory. The very fine structured semiconductor superlattices have been made and investigated \cite{22, 23}. These semiconductor superlattices are made from the alternatively stacking of different semiconductors having different band gaps. For example the heterostructure constructed from alternating n and p type doped semiconductors. Here ‘fine’ means the length scale of the variation of the chemical potential.
is smaller than the electron’s mean free path. Therefore the superlattices exhibit the quantum properties. The charge carriers are confined in the quantum wells of a staggered well/barrier structures. When a biased voltage is applied align the axial of the superlattice, subbands scatterings are excited and the device can be used as electromagnetic wave generators [19].

Since adjusting the period of superlattice, the energy band structure of the superlattice can also be accurately controlled. It means that experimental researches of the electron transport phenomena in superlattices will enter a new stage. Studies on the phenomenon of electrons transport under static bias and periodic field in superlattices revealed many interesting effects, including negative differential conductivity, Bloch oscillation, multiphoton-assisted tunneling, dynamic localization and absolute negative conductance[25–32].

In this thesis we focus on the electron transport in semiconductor superlattices. Because of the existence of the mini-Brillouin zone and minibands in semiconductor superlattices [33], the electron can be accelerated by an electric field and enter a negative effective mass region, which leads to the appearance of negative differential conductivity. This negative differential conductivity in semiconductor superlattices can cause a number of very interesting physical phenomena, including Bloch oscillations, which cannot be observed in natural solid [34–36].

1.2.2. Materials

The materials usually used in semiconductor superlattices are well-known semiconductors and their alloys. They include Ge, Si and Ge-Si alloys. For
superlattices growth one uses molecular beam epitaxy \[20\]. Since the period of superlattice can be adjusted according to researchers’ requirements, one can control miniband width and energy gap of the device. In other words, the wave function of an electron can be controlled. This is a new feature never seen in other materials. In this sense, semiconductor superlattice is a new artificial material.

In this thesis, we theoretically study the electron transport in the GaAs/(AlGa)As superlattice systems. For these materials, the differences on the lattice constant and the thermal expansion coefficient are small.

1.2.3. The types of semiconductor superlattice

Esaki and Tsu proposed two kinds of superlattices, one is the component superlattice; the other is doping superlattices \[20\]. The component superlattice has the multilayer film structure which is composed of different materials with close lattice period. One of the most popular typical compositional superlattice is a multi-layer structure formed by alternating growth of GaAs and AlAs materials \[20\].

Figure 2 shows the superlattice structure and band structure. We can see that GaAs and AlAs will form a potential well, as shown in figure 2(b), this is because of the different band gap in these two materials. Furthermore, the motion of the charged carrier in superlattice quantum wells will be effected by potential barriers on both side of the potential well, so that it will be hindered.

In some semiconductor materials, for example silicon, the superlattice structure can be formed by alternating doping types (n and p). Figure 3 shows the
1.2.4. Main applications

Semiconductor superlattices have many applications in various fields [37–42], including electron devices and optical devices.

Heterostructure laser [37]: Perhaps the most important application of semiconductor superlattice materials are used to produce multiple quantum well lasers [38]. Quantum well lasers can be divided into the single quantum well laser and
the multi quantum well laser. The multi quantum well laser is also called superlattice laser.

Quantum cascade laser [39]: Quantum cascade lasers are semiconductor lasers based on the theory of electron transition in quantum well of superlattice which are first demonstrated by Jerome Faist, Federico Capasso, Deborah Sivco, Carlo Sirtori, Albert Hutchinson, and Alfred Cho at Bell Laboratories in 1994 [40]. Unlike the traditional semiconductor laser, only electrons execute the stimulated radiation in quantum cascade laser and the wavelength is determined by the band
gap of semiconductor superlattices.

Bistable device [41]: One of main applications of superlattices in the optical device is used to make the optical bistable device. This kind of the nonlinear optical element is the key element in the optical logic computer [41].

Avalanche photodiode [42]: Because of the different height of voltage barriers in superlattices, superlattices can be used to make avalanche photodiodes in optical communication system.

1.3. Band theory

Band theory is one of the main foundations of solid state electronics. It plays an invaluable role in the development of microelectronics technology.

1.3.1. Energy band formation

As we know, when two atoms stay close, the each of the energy levels of both atoms will divide into two narrowly separated levels under the influence of the Coulomb interaction. Thus, in solid, the interactions between all atoms cause the energy spectra of the individual atoms to split into discrete levels. Those levels are very closed to each other and can be approximated as continuous. Therefore, these energy levels form bands [43].
1.3.2. **Bloch theorem**

Instead of being trapped by a single atom, the electrons move in the periodic potential of crystal. The periodic potential has the following form:

\[ V(r + R_n) = V(r), \]  

where \( V \) is the periodic potential, \( r \) is a position vector and \( R_n \) is a lattice vector. \( R_n \) can be expressed by:

\[ R_n = n \mathbf{a}, \]

where \( \mathbf{a} \) is unit lattice vector and \( n \) is an integer. In this periodic potential, the eigenfunction \( \psi(r) \) of single electron’s *Schrödinger* equation can be expressed as:

\[ \psi(r) = e^{ik \cdot r} u(r), \]

where \( k \) is the wavevector and \( u(r) \) is a periodic function which can be given by:

\[ u(r + R_n) = u(r). \]

\( \psi(r) \) is Bloch wave. \( \psi \) in equation (3) also can be viewed as the free electron wavefunction \( e^{ik \cdot r} \) modulated by \( u(r) \).

1.3.3. **Conductors, insulators and semiconductors**

Band theory provides a way to understand the difference between conductors, insulators and semiconductors.

All solids consist of atoms and the atoms are composed of nuclei and electrons.
The electrons move in the Coulomb field of the nucleus. The further the electron away from the nucleus, the higher the energy level it has. These electrons which have higher energy levels are easier to get away from the nucleus and become free electrons. The state of an electron in the atom determines the conductive properties of solid matter, while the energy band is a conceptual representation of electronic states in semiconductor physics.

In solid state, the valence band and conduction band determine the electrical conductivity of the solid. In conductors, there is a portion of overlap between valence band and conduction band. Thus, electrons in the vicinity of Fermi surface can contribute the conductivity. In insulators and semiconductors the electrons in the valence band are separated by a gap from the conduction band. This gap is known as band gap where no electron states can exist. Fermi level is in the band gap. In insulators there is a large band gap, so that the electrons in the valence band is hard to travel to the conduction band. This is the reason why insulators is non-conductive.

The band gap width of semiconductor is small. Thus electrons in valence band can be excited to conduction band by thermal or other excitations. This is the mechanism of conductance of semiconductors. Moreover, the presence of a doping material also can increase the conductivity of semiconductor.

1.3.4. Minibands in superlattice

Because of the different energy band gaps of the two materials, the conduction band edge of an ideal superlattice is periodically modulated. This periodic
potential leads to the formation of narrow energy bands, known as minibands. The conduction bands in superlattices are shown in figure 4. Along the superlattice growth direction (x axis), potential barriers and wells are alternating (see figure 4(b)). Thus the motion of an electron in x axis in a superlattice is affected by a spatio-periodic field.

**FIG. 4.** (a) The alternating growth GaAs and AlAs layers. (b) The conduction bands in semiconductor superlattices

Figure 4(a) shows the two kinds of component materials, for example, GaAs and AlAs, alternating growth along x-direction. In superlattices, electrons are
affected by a spatially periodic potential field in the $x$-direction, thus the wave-function of an electron is not localized. The electron performs Bloch waves in every quantum well in the $x$-direction. The wave function of a particle in the quantum well can be written as \[46\] \[47\]

\[\psi_w = A \exp(ik_w x) - B \exp(-ik_w x). \tag{5} \]

In the barrier,

\[\psi_b = C \exp(ik_b x) + D \exp(-ik_b x), \tag{6} \]

where

\[k_w = \sqrt{\frac{2m_w^*}{\hbar^2}} \varepsilon, \tag{7} \]

\[k_b = \sqrt{\frac{2m_b^*}{\hbar^2}(\varepsilon - V)}. \tag{8} \]

In equations (7)-(8), $m_w^*$ and $m_b^*$ are effective mass of an electron in the potential well and barrier, $\varepsilon$ is energy and $V$ is barrier length. In an ideal superlattice, the probability of electrons in each of the quantum wells is equivalent, therefore the wave function of an electron is periodic in space

\[\psi(x) = \psi(x + d), \tag{9} \]

where $d = w + b$ is superlattice period, as show in figure 4. The wave function $\psi(x + d)$ can be written as

\[\psi(x + d) = \exp[ik(x + d)] = \exp(ikx) \exp(ikd) = \psi(x) \exp(ikd). \tag{10} \]

According to the BenDaniel-Duke first boundary condition \[48\] when $x = d,$
\[ \psi_w(d) = \psi_w(0) \exp(ikd) = \psi_b(d). \] (11)

We can substitute equation (11) to the wave function defined by equations (5) and (6)

\[ (A + B) \exp(ikd) = C \exp(ik_b d) - D \exp(-ik_b d). \] (12)

According to the Ben Daniel Duke second boundary condition [48] for \( x = d, \)

\[ \frac{ik_w}{m_w} (A - B) \exp(ikd) = \frac{ik_b}{m_b} \bigl[C \exp(ik_b d) - D \exp(-ik_b d)\bigr]. \] (13)

We can also apply Ben Daniel-Duke boundary condition at \( x = w \) to obtain

\[ A \exp(ik_w w) + B \exp(-ik_w w) = C \exp(ik_b w) + D \exp(-ik_b w), \] (14)

\[ \frac{ik_w}{m_w^*} [A \exp(ik_w w) - B \exp(-ik_w w)] = \frac{ik_b}{m_b^*} \bigl[C \exp(ik_b w) + D \exp(-ik_b w)\bigr]. \] (15)

After rearrangement the set of equations (12), (13), (14) and (15), we obtain [46]:

\[ \cos(k_w w) \cos(k_b w) - \sin(k_w w) \sin(k_b w) \left( \frac{m_b^* k_w^2 + m_w^* k_b^2}{2 m_w^* m_b^* k_w k_b} \right) = \cos(kd). \] (16)

If \( \varepsilon < V, \) \( k_b \) is imaginary number, it can be expressed by \( k_b = i \kappa, \) namely

\[ \kappa = \sqrt{\frac{2m_b^*}{\hbar^2}} (V - E), \] (17)

we can substitute \( k_b \) to equation (16) and use the equations \( \cos(i \kappa b) \equiv \cosh(\kappa b) \) and \( \sin(i \kappa b) \equiv i \sinh(\kappa b), \)
\[ \cos(k_w w) \cosh(\kappa b) - \sin(k_w w) \sin(\kappa b) \left( \frac{m_b^* k_w^2 + m_w^* \kappa^2}{2m_w^* m_b^* k_w \kappa} \right) = \cos(k d). \] (18)

Based on solving equation (17) and (18), we can find the dispersion relation in superlattices. The minibands structure of superlattices can be expressed by the cosine function. Within the tight-binding approximation, the dispersion relation of the miniband energy \( E \) and Bloch vector \( k \) can be given by the following equation:

\[ E_x(k) \approx E(0) - \frac{\Delta}{2} \cos(k d), \] (19)

where \( \Delta \) is miniband width.

1.4. Electron transport

1.4.1. Electron scattering

One main source of electron-ion scattering is the imperfect lattices. Thus, electron-ion scattering can occur in a real semiconductor structure. Besides, the thermal vibrations of the ions in semiconductors can also promote the electron-ion scattering. The lattice is not perfectly periodic due to the lattice ions vibrate in different directions. This nonperiodic lattice cause the electron-ion interactions. But the thermal vibrations is weak as the temperature approaches absolute zero. In this thesis, we do not consider the scattering caused by the thermal vibrations.

From the Fermi liquid theory, the independent electron approximation is used in this thesis and the charge carriers are still treated as electrons. Thus, we think the electron-electron scattering can be neglected in this thesis. In our calculations we use a realistic value \( \tau = 10^{-12} \text{s} \) [20] for scattering time in GaAs/(AlGa)As.
1.4.2. Bloch oscillations

Bloch oscillations describe the oscillations of a quantum particle confined in a periodic potential when a constant force is acting on it. Generally, the period of natural crystals is so small that Bloch oscillations can not occur because the Bloch frequency is much less than the electron scattering rate. Due to the longer superlattice period, the superlattice material has both a mini Brillouin zone and a miniband. Therefore, the electron can be accelerated from the bottom of the miniband to the top of the miniband. If the electron does not experience scattering before it arrives the edge of the miniband, it will have Bragg reflection and perform periodic motion. These oscillations are known as Bloch oscillations. In absence of scattering the motion of an electron in this superlattice can be described by the following equation for crystal momentum $k$:

$$\hbar \frac{dk}{dt} = -eF, \quad (20)$$

where $e$ is the charge on an electron, we solve equation (20) and get

$$k(t) = \frac{eFt}{\hbar}. \quad (21)$$

According to generally kinetic equation $\frac{1}{\hbar} \frac{dE}{dk} = v(t)$, we can get the velocity equation of electron moving along $x$-direction

$$v_x = \frac{\Delta d}{2\hbar} \sin \frac{eFd}{\hbar} t. \quad (22)$$

Then the electron position in real space can be given by

$$x(t) = -\frac{\Delta}{2eF} \cos \frac{eFd}{\hbar} t. \quad (23)$$
Equation (23) reveals that the electron oscillates in real space at the absence of scattering. The Bloch frequency is \( \omega_B = \frac{eFd}{\hbar} \). The parameter \( \Delta \) is the amplitude of the Bloch oscillations, which is inversely proportional to the strength of the electric field. The maximum speed can be only determined by the band structure, however, the frequency of the Bloch oscillations increases with the enhancing of the electric field strength.

If there is no electron scattering in superlattices, the electron performs the Bloch oscillations without drifting. Indeed, due to the scattering, the oscillation process cannot proceed all the time. The electron scattering tends to restore the electron to thermal equilibrium. Thus, we will introduce the electron drift.

1.4.3. Drift velocity

In the previous section we discussed the Bloch oscillations in superlattices which electrons perform at the absence of scattering. However, the scattering event can cause the electrons drift. To characterise this drift we introduce the drift velocity in this section.

We define the initial number of electrons within a superlattice miniband as \( N_0 \), the number of unscattered electrons at time \( t \) as \( N(t) \). After a period of time \( dt \), some electrons are scattering and the number of the rest electrons at this time is

\[
N(t + dt) = N(t) - N(t) \frac{dt}{\tau},
\]

It means during the \( dt \) of time, there will be a number of \( N(t) \frac{dt}{\tau} \) of electrons
scatter. Therefore the rate of change of unscattered electrons is

\[ \frac{dN}{dt} = \frac{N(t + dt) - N(t)}{dt} = -\frac{N(t)}{\tau}. \] (26)

This equation of rate can be given by a different form

\[ \frac{dN(t)}{N(t)} = -\frac{dt}{\tau}. \] (27)

The number of unscattered electrons can be found by solving equation (27):

\[ N(t) = N_0 e^{-\frac{t}{\tau}}. \] (28)

Now we need to calculate the probability of an electron scattering within this time interval

\[ P(t)dt = \frac{N(t)dt}{\tau \cdot N_0} = \frac{1}{\tau} e^{-\frac{t}{\tau}} dt. \] (29)

Thereby the electron drift velocity can be expressed as

\[ v_d = \int_0^\infty v_x(t)P(t)dt = \frac{1}{\tau} \int_0^\infty v_x(t)e^{-\frac{t}{\tau}} dt. \] (30)

In [20], the authors used path integral to calculate the drift velocity in the miniband of the superlattice under an applied electric field [20]. Using the semiclassical equations of motion,

\[ v_x = \frac{1}{\hbar} \frac{dE}{dk_x}, \] (31)

\[ \hbar \frac{dk_x}{dt} = eF, \] (32)

and the energy at the first miniband we derived before
\[
E(k) = \frac{\Delta}{2} (1 - \cos(k(t)))d, \quad (33)
\]

assuming that the initial position of the electron is at the bottom of the miniband, the electron drift velocity in the \(x\)-direction can be given by

\[
v_d = \frac{\Delta d}{2\hbar} \int_0^\infty \frac{1}{\tau} \sin(\omega_B t) e^{-\frac{t}{\tau}} dt, \quad (34)
\]

which yields

\[
v_d = \frac{\Delta d}{2\hbar} \frac{\omega_B \tau}{(\omega_B \tau)^2 + 1}. \quad (35)
\]

From equation (35) it is easy to see that the drift velocity has a peak when \(\omega_B \tau = 1\). Figure 5 displays the dependence of \(v_d\) on \(\omega_B \tau\) for GaAs/(AlGa)As superlattice with \(\Delta = 26.2\,meV\) and \(d = 10.3\,nm\). In the small electric field, the electrons accelerate slowly and can not pass through the whole miniband. They scatter when they are still near the miniband bottom, hence producing no Bloch oscillations. Therefore, current density is proportional to the electric field strength. Essentially, it follows Ohm’s Law. When electric field is large enough, in other words \(\omega_B \tau > 1\), electrons can reach the top of the miniband before scattering. Due to repeated Bragg reflection, electrons can perform Bloch oscillations. It leads to the drift velocity decreasing with the increasing electric field. This phenomenon in superlattice is well known as negative differential velocity.

The negative differential velocity can cause the electrons in superlattice to perform high frequency oscillations, which makes superlattice promising for generation in THz frequency range \([49]\). An accurate analysis of superlattices can be made by using quantum mechanics. By using the method similar to the one of
studying the cascade superlattices, the characteristic properties include the band structures, local density of states and voltage-current curve can be revealed. In these calculations, the frequency of the emitted electromagnetic wave is corresponding to the width of the gaps among the subband states, i.e. the Wannier-Stark states. These results can be very well reproduced through the semiclassical approach. In such treatments the frequency of the electromagnetic wave, which is originated from the scattering between the subband states established in superlattices, has the same value as the frequency of the Bloch frequency.
1.4.4. Numerical method of integration

For solution of nonlinear transport equation which cannot be solved analyti-
cally we need to involve numerical integration. In this thesis we use fourth orders
Runge-Kutta method \[50\] for numerical calculation.

We define a general differential equation

\[
\frac{dy}{dt} = f(t, y).
\]  

(36)
y is an unknown function of time \(t\) which we would like to approximate. The
rate of change of \(y\) is a function of \(t\) and \(y\). Then, the fourth order Runge-Kutta
method for this function is given by the following equations, where \(h\) is the step
of variation of \(t\).

\[
y_{n+1} = y_n + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4) \tag{37}
\]

\[t_{n+1} = t_n + h \tag{38}\]

\[k_1 = f(t_n, y_n) \tag{39}\]

\[k_2 = f\left(t_n + \frac{1}{2} h, y_n + \frac{1}{2} hk_1\right) \tag{40}\]

\[k_3 = f\left(t_n + \frac{1}{2} h, y_n + \frac{1}{2} hk_2\right) \tag{41}\]

\[k_4 = f(t_n + h, y_n + hk_3). \tag{42}\]

Where \(k_1 \cdots k_4\) are the increments based on the slope, and \(n\) is the number of
time steps.
1.5. Superlattices in electric and tilted magnetic fields

1.5.1. Chaos phenomenon in superlattices

The modern theory of deterministic chaos is, in many respects, based on seminal works of H. Poincaré, Lyapunov and Birkhoff [51]. The chaos phenomena were found in semiconductor superlattice both in theory and experiment [52–62].

Chaos charge transport in a semiconductor superlattice was perhaps for the first time discovered by Bulashenko and Bonilla [54]. They added a weak ac signal in superlattices when the electron did periodic oscillation. The electron would do a non-periodic motion. Zhang has proven the evidence of the spatiotemporal chaos phenomenon through experiments [56].

In this thesis, we investigate chaotic electron transport in the lowest miniband of a semiconductor superlattice with a tilted magnetic field. The configuration of the field is schematically presented in figure 6. In this configuration, the electric field $F$ is parallel to the $x$-direction and perpendicular to the semiconductor superlattice layers. The magnetic field $B$ lies in the $x − z$ plane at an angle $\theta$ to the $x$-direction. In this thesis, in order to distinguish with the model which we use in Chapter 4 and 5, we call this model ordinary fields configuration.

1.5.2. Absorption in superlattices

The semiclassical theory predicts that electrons performing Bloch oscillations can provide THz Bloch gain [63]. Bloch oscillations can potentially lead to the
amplification of a weak ac field at frequencies smaller than the Bloch frequency, whereas absorption occurs for frequencies that are larger than the Bloch frequency [64]. In recent studies and experiments, the Bloch gain was also found in intersubband transitions of quantum cascade lasers [65, 66]. Thus the Bloch gain becomes an interesting and attractive research subject.

Previously Alekseev and Hyart studied the influence of the magnetic field on the small-signal absorption and gain [67]. They calculated the dynamical conductivity of electrons and observed the cyclotron gain by solving the Boltzmann
equation numerically.

In this thesis, we consider the applied field as a dc field $F_{dc}$ and an ac field $F_{ac}\cos\omega t$, as shown in figure 6.

$$F = F_{dc} + F_{ac}\cos\omega t,$$

where $\omega$ is frequency of the ac field. Our aim is to investigate the influence of a tilted magnetic filed on the gain of the ac field. The energy transferred from the ac field to the medium is:

$$E = \int j(t)F_{ac}\cos(\omega t)dt.$$  \hspace{1cm} (44)

If $E < 0$, the energy is transferred from the medium to the ac field, which causes the amplification. On the contrary, $E > 0$, the energy is transferred from the ac field to the medium, causing absorption. We define the $A$ which is related to the energy transferred from the alternating field to the medium as

$$A = < j(t) \cos \omega t >_t,$$

where $j(t)$ is the current density induced in the superlattice by the total field $F$, and $< \cdots >_t$ represents time averaging. The absorption (gain) $\alpha$ is related to $A$:

$$\alpha = \frac{2}{n_r \epsilon_0 c} \frac{A}{F_{ac}},$$

where $\epsilon_0$ and $c$ are the permittivity and the speed of light in vacuum, and $n_r$ is the refractive index of the superlattice material. So, in this thesis, we calculate the $A$ to represent the absorption and gain phenomena in the superlattice. When $A > 0$, the energy is transferred from the alternating field to the medium (absorption). When $A < 0$, the energy is transferred from the medium to the alternating field (gain).
1.6. Superlattices in electric and perpendicular magnetic fields

1.6.1. Superlattices model

In most studies of superlattices [69, 70], the applied external electric field is directed along the superlattice axis. We will call such orientation of field "the ordinary configuration". In this thesis we study an extraordinary configuration which was mentioned in Epshtein’s paper [71]. We apply the electric field $F$ perpendicular to the superlattice growth direction, see figure 7. The superlattice growth direction is denoted as the $x$ axis and the direction the electric field $F$ aligned with is defined as the $y$ direction. The magnetic field $B$ is applied in the $z$ direction which is orthogonal to the $x$ and $y$ directions. The Hall field is considered in the $x$-direction, see figure 8.

1.6.2. Hall effect

The classical Hall effect is one of the most famous phenomena related to interaction between the solid state and electromagnetic field. In 1879, Edwin Hall discovered that when current passes through the conductor perpendicular to external magnetic field, electric potential difference will be produced in two sides of conductor which are perpendicular to magnetic field and current direction [72, 73].

In the Hall effect system, the electrons were affected by a superposition of electric field force and Lorentz force

$$F_T = eF_H + evB,$$  \hspace{1cm} (47)
FIG. 7. The schematic diagram for the different configurations which we used in this thesis. The left one is ordinary configuration and the right one is extraordinary configuration. The arrows show the orientation of applied electric field. The red line is superlattice axis.

where $F_T$ is the total force, $F_H$ is the Hall electric field, $B$ is the magnetic field and $v$ is the velocity of electrons. The Hall field can be given by

$$F_H = -vB.$$  \hspace{1cm} (48)

Due to the influence of the Lorentz force, the trajectory of electrons in the solid material under an applied magnetic field will have circular excursion. This causes charge accumulation in both sides of the material and produce an electric field, which is perpendicular to the current direction. Thus there is a stable potential difference on the both sides of solid when electric field force produced by
FIG. 8. The orientations of the electric and magnetic fields with respect to the super-lattice structure. F is electric field and B is magnetic field.

the accumulative electrons in both sides of solid equals Lorentz force acting on electrons. This electric potential difference is called the Hall voltage.

At low temperatures and strong magnetic field, the quantum Hall effect, which is a quantum mechanical version of the Hall effect, can be observed in very specific structures such as two-dimensional electron gases. The discovery of both integer and fractional quantum Hall effects leads to a number of important applications [69, 74], including the method for high-accuracy determination of the fine structure constant [69].

The Hall effect plays an important role both in science and technology [75, 76], for example the Hall sensors, which are produced based on the Hall effect,
has widespread applications in industrial automation, detection techniques and information processing [77]. The Hall sensor is a transducer that varies its output voltage in response to a magnetic field. So far, the main applications of the Hall sensor are proximity switching, autometer, odometer and automotive ignition system [78].

Superlattice materials play an important role in modern technology, especially in semiconductor lasers [79]. With further researches related to superlattice materials, more semiconductor electronic devices with excellent features can be manufactured. In this thesis, Hall effect in the semiconductor superlattice is discussed in Chapter 4 and Chapter 5.

1.7. Structure of thesis

The structure of the thesis is as follows. In Chapter 2, we consider the superlattice model with an electric field applied perpendicular to the plane of the layers and antiparallel to the superlattice axis (x axis) and a tilted magnetic field which lies in the x – z plane. We introduce the semiclassical equations of motion for a miniband electron in this model and study the electron transport. We will discuss the influence of the magnetic field on the motion of an electron. In Chapter 3, we will introduce the absorption of probe signal in superlattices and compare it with the electron drift velocity. Then, we discuss the influence of the magnetic field on the absorption. In Chapter 4, we structure an extraordinary configuration which has the applied electric field antiparallel to the y axis and magnetic field perpendicular to x – y plane in a superlattice. We study the Hall effect in this model and give an approximate analytical solution of drift velocity.
In Chapter 5, we study the coherence Hall effect and electron temporal behaviour in our extraordinary model. In Chapter 6, we summarize our results and draw conclusions.
2. Chaotic transport in a superlattice

Recent theoretical and experimental works have revealed that the chaotic electron transport in the lowest miniband of a semiconductor superlattice with a tilted magnetic field has unique properties [80–84]. For example, an unusual type of Hamiltonian chaos, which does not obey the Kolmogorov-Arnold-Moser (KAM) theorem, has been observed in such devices. This non-KAM chaos is of great interest due to diverse applications in the theory of plasma physics, tokamak fusion, turbulent fluid dynamics, ion traps, and quasicrystals [85–87]. In particular, Fromhold et.al. has discovered that miniband transport in a tilted magnetic field can have a chaotic characteristic [81]. The chaos switches on abruptly when the field parameters satisfy certain resonance conditions.

In this chapter, we analyze the electron trajectories for different field parameters. The calculations in this chapter are primarily performed within the semiclassical approach. Within this approach the dynamics of charge carries is treated classically, however the Hamiltonian was derived using quantum mechanics. This approximation has been previously used in a number of works including [80–82].

2.1. Dynamics model

In this thesis we consider superlattices constructed from the alternating GaAs and (GaAl)As layers. The layers in the transverse direction is much larger in size than the mean free path and the magnetic length of the charge carriers, and the length of the lattices in $x$ direction is long enough to neglect the influence of the edge states. Further, we assume only the single particle approximation, which is valid in case of low charge density. Within the tight-binding model, the energy
spectrum of the electrons in the conduction bands can be expressed as

$$E(p) = \Delta \left[ 1 - \cos \left( \frac{p_x d}{\hbar} \right) \right] + \left( p_y^2 + p_z^2 \right)/2 m^*,$$

(49)

where \(E(p)\) is the energy versus crystal momentum dispersion relation for the lowest miniband, \(p_x\) is the momentum in \(x\) direction, \(p_y\) is the momentum in \(y\) direction, \(p_z\) is the momentum in \(z\) direction and \(m^*\) is the effective electron mass for motion in the \(y - z\) plane for GaAs. Here \(m^* = 0.067 m_e\) and \(m_e\) is the mass of a free electron. The equation (49) allows a semiclassical approach in the description of the electron dynamics.

The geometric configuration of fields is shown in figure 6. Here the applied electric field is \(\mathbf{F} = (F_x, F_y, F_z) = (-F, 0, 0)\) and the tilted magnetic field is \(\mathbf{B} = (B_x, B_y, B_z) = (B \cos \theta, 0, B \sin \theta)\). In this ordinary configuration the dynamics of the electron is determined by the Lorentz force and an electric field \[45\],

$$\frac{dp}{dt} = -e \left\{ \mathbf{F} + [\nabla p E(p) \times \mathbf{B}] \right\},$$

(50)

In three-dimensional space, the following expression is valid

$$e \nabla p E(p) \times \mathbf{B} = e \begin{vmatrix} i & j & k \\ \frac{\partial E(p)}{\partial p_x} & \frac{\partial E(p)}{\partial p_y} & \frac{\partial E(p)}{\partial p_z} \\ B_x & B_y & B_z \end{vmatrix},$$

(51)

In its component form, equation (50) can be presented as

$$\dot{p}_x = e F - \omega_c^x p_y \tan \theta,$$

(52)

$$\dot{p}_y = \frac{d \Delta m^* \omega_c^x}{2 \hbar} \sin \left( \frac{p_x d}{\hbar} \right) \tan \theta - \omega_c^x p_z,$$

(53)
\[
\dot{p}_z = \omega_c^2 p_y, \tag{54}
\]

where \( \omega_c^x = \frac{eB}{m^*} \cos \theta \) is the cyclotron frequency defined by the component of magnetic field directed along \( x \) axis. From equations (52)-(54), we can deduce the following second order differential equation

\[
\ddot{p}_z + \omega_c^x p_z = C \sin(Kp_z - \omega_B t + \phi), \tag{55}
\]

where \( C = -m^* \omega_c^x d\Delta \tan \theta / 2\hbar \), \( K = d \tan \theta / \hbar \) and \( \phi \) is phase. The phase \( \phi = d[p_x(t = 0) + p_z(t = 0) \tan \theta] / \hbar \), which depends on the initial conditions, will be zero if the initial velocity of electrons is zero. As a consequence, our dynamic equations (52)-(55) describe the electron motion with different \( \theta \), which is the angle between magnetic field and superlattice axis (see figure 6). The equation (55) describes the motion of electrons completely because the momentums along other axes can be expressed in terms of \( p_z(t) \) and \( \dot{p}_z(t) \) as

\[
p_x = p_x(t = 0) + eFt - [p_z - p_z(t = 0)] \tan \theta, \tag{56}
\]

\[
p_y = \frac{\dot{p}_z}{\omega_c}, \tag{57}
\]

We can also use equations (52)-(54) to determine the electron velocity components.

\[
\dot{x} = \frac{d\Delta}{2\hbar} \sin(Kp_z - \omega_B t + \phi), \tag{58}
\]

\[
\dot{y} = \frac{\dot{p}_z}{\omega_c^x m^*}, \tag{59}
\]

\[
\dot{z} = \frac{p_z}{m^*}. \tag{60}
\]
Thus, as stated above, all momentum components and velocity components can be determined by \( p_z(t) \) and \( \dot{p}_z(t) \). Therefore, we can find the whole real-space trajectories of electrons in a superlattice by solving equations (58)-(60) numerically. Furthermore for the fixed electric and magnetic fields, the shape of the trajectories are also affected by the initial velocity of electrons. Either the Fermi-Dirac statistics or the Boltzmann statistics is used in the system is determined by the comparison between the Fermi temperature and the thermal (environmental) temperature. When the environmental temperature is close or lower than the Fermi temperature, the system should be treated in the quantum limit and the Fermi-Dirac statistics are used. In the opposite case, Boltzmann statistics can give a acceptable approximation. In this thesis we assume that temperature is close to zero. The Fermi energy of electrons in the superlattice is close to the bottom of the conductance band. The corresponding Fermi temperature has the magnitude of the order of milli-Kelvin. In the case of low temperature of several Kelvin, the temperature is still much higher than the Fermi temperature. Therefore, Boltzmann statistics can provide a rather accurate description of the system and the system can be treated in the classical limit. Further, because the temperature is low enough, the variations of the rest momentum is small and close to zero. Consequently, we assume that the electrons start moving from rest and with \( \phi = 0 \).

2.2. Electron trajectories

According to our analysis and derivation in section 2.1, we can simulate the trajectories of an electron numerically. At different parameters \( F, B \) and \( \theta \), there
are two types of trajectories exist in this system.

The first type of trajectories is shown in figure 9. In this case, the ratio \( r = \omega_B/\omega_c^* \) between Bloch frequency and cyclotron frequency is an integer. This is also the resonance condition. This resonance confines the electrons to delocalized the quasiregular motion in real space. In the figure, the electrons start from rest and show helical trajectories with increased radius. The unbounded motion cause the electron transmit through the superlattice rapidly.

![Graph](image)

FIG. 9. Delocalized electron orbits in real space calculated for the semiconductor superlattice with \( B = 2T, \theta = 30^\circ, F = 2.9 \times 10^5V/m \) and \( r = 1 \). The ratio \( r \) satisfies the resonance condition.

The other type of trajectory presented in figure 10 is realized when the ratio \( \omega_B/\omega_c^* \) is not an integer and therefore dose not satisfy resonance condition. The electron does an irregular motion in the superlattice and can not escape from the miniband, thereby the trajectory is localized in a bounded space region.

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FIG. 10. Localized electron orbit in real space calculated for the semiconductor super-lattice with $B = 4.75T$, $\theta = 45^\circ$, $F = 7.5 \times 10^5 V/m$ and $r \approx 1.33$. The ratio $r$ does not satisfy the resonance condition.

2.3. Poincarè section

We study the chaotic transport of electron by analyzing the Poincarè section in this section. Due to the equation (55), we know that the electron motion depends on the momentum $p_z$. We obtain the Poincarè section of the trajectories starting from different initial conditions by plotting the momentum components $p_z$ and $p_y = \frac{\dot{p}_z}{\omega_c^x}$, which is calculated by numerical simulation at discrete time moments $T_n = 2n\pi/\omega_c^x$, $n = 0, 1, 2, 3,\ldots$. 

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FIG. 11. Poincarè section by plotting the momentum components $p_z$ and $p_y$ in the plane of the superlattice layers when $B = 2T$, $\theta = 30^\circ$ and $F = 2.9 \times 10^5 V m^{-1}$ and $r = 1$.

Figure 11 shows the Poincarè section for $B = 2T$, $\theta = 30^\circ$ and $F = 2.9 \times 10^5 V m^{-1}$ and $r = 1$, which is satisfied the resonance condition. There are both closed curves and radial filaments in this Poincarè section. The closed curves indicate the regular quasiperiodic motion of electrons in real space. Due to the influence of the magnetic field, when the electron travels along its trajectory in real space, it diffuses along the radial filaments and goes far away from the centre of circles. Thereby the value of momentum becomes bigger and the trajectory of electron goes to infinity after a period of time.
FIG. 12. Poincarè section by plotting the momentum components $p_z$ and $p_y$ in the plane of the superlattice layers when $B = 4.75T$, $\theta = 45^\circ$ and $F = 7.5 \times 10^5 \text{Vm}^{-1}$ and $r = 1.33$.

Figure 12 shows the Poincarè section for $B = 4.75T$, $\theta = 45^\circ$ and $F = 7.5 \times 10^5 \text{Vm}^{-1}$ and $r = 1.33$ which is not satisfied the resonance condition. In this Poincarè section, the low energy trajectories are strongly affected by the magnetic field, so that there is "chaotic sea" in the center. But the magnetic field has weak perturbation on high energy trajectories. Thus, the curves around the chaotic sea are stable circular rings. In this Poincarè section we can see that the circular rings enclose the chaotic sea, it illustrates the electron trajectory in real space is localized, as shown in figure 10.
2.4. Drift velocity at different magnetic field

In Section 1.4.2, we discuss the drift velocity and Esaki-Tus’s result, when magnetic field is absent. In this section we will calculate the drift velocity at magnetic field and discuss the influence of magnetic field.

In this section we study the drift velocity for different magnetic field strengths and a fixed angle $\theta$. We calculated the drift velocity for electrons starting from rest by using the following equation:

$$v_{dx} = \frac{1}{\tau} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) v_x(t) dt.$$  \hspace{1cm} (61)

Figure 13 shows the calculated dependencies $v_{dx}(F)$ from equation (61) at different magnetic fields. At small electric field, the electric force is too weak to accelerate the electrons to travel fast. Thus, in this condition the electrons have short mean free path and can not go far away from the bottom of minibands before scattering. Thus in this regime there are no Bragg reflections, therefore it demonstrates positive differential conductivity. All the curves of drift velocity are linear growth and follow Ohm’s law in this regime. As the electric field increases, the electrons have longer mean free path in real space. When the electrons scatter at the edge of miniband, the drift velocity has the maximum value. It is corresponding to the first peak in every curve and we call it local maximum. When electric field is stronger the electrons can pass through the entire miniband before scattering, which cause the Bragg reflections. Thus, in this regime there are Bloch oscillations and it presents negative differential conductivity. As shown in figure 13 the drift velocity decreases with increasing electric field in large electric
FIG. 13. The drift velocity versus the electron field $F \frac{dx}{d\tau}$ calculated in superlattice for different magnetic fields and fixed angle $\theta = 30^\circ$. For different colour, $B = 0T$ (red), $B = 1T$ (green), $B = 2T$ (blue) and $B = 3T$ (purple). Arrows mark resonant peaks for $r=1$ and 2.

The red curve in figure 13 acts the Esaki-Tsu relation which has a maximum at the field $\omega_B \tau = 1$ when magnetic field is absent. The green curve shows the drift velocity at $B = 1T$. The cyclotron oscillations produced by the magnetic field influence the motion of the electron. Because the magnetic field bends the electron trajectory from $x$-direction, the green curve lies below the red curve in small electric field. We consider that Bloch-cyclotron resonance at $r = 1$ is close
to the local maximum, thus, we just observe one peak in green curve. The blue and purple curves present the drift velocity at $B = 2T$ and $B = 3T$ which have a significant distinction from the red one. Due to the larger magnetic field, the local maximum (the first peak when $B = 2T$ and $B = 3T$) lies below the red curve in these curves. We can observe pronounced additional maximum produced by Bloch-cyclotron resonance at $r = 1, 2$, whose position is indicated by arrow in figure 13. For the purple curve the magnetic field is large enough to make the electron undergo rapid diffusive motion and travel farther along $x$ axis before scattering. Thus the purple curve lies above the red curve in large electric field.

2.5. Conclusion

We observe that miniband electron in a tilted magnetic field can demonstrate non-KAM chaos. Following earlier works [81], we find electrons able to demonstrate two types of phase portraits. In this chapter, we study the trajectories of electrons in three dimensional space. Compare to the results of Fromhold’s research, we find the electrons perform two kinds of trajectories in space. When the cyclotron and Bloch frequencies are resonant, the electrons completely delocalizes. In the Poincarè section, this delocalizes motion performs radial filaments. In other cases, the trajectory is localized in a limited space. In the Poincarè section we can just find concentric circles in this case. The resonances between cyclotron and Bloch frequencies produce additional peaks in the curves of drift velocity.
3. The absorption of electromagnetic radiation in a superlattice

In section 1.5.2 we discuss the gain and absorption of the ac field in the superlattice and introduce some previous studies. In this chapter, we study the influence of a tilted magnetic field on the absorption for different probe signals in the superlattice.

3.1. Calculations of absorption/gain

In this chapter, we study the electron transport under the action of a dc field and a time-dependent electric field, \( F = F_{dc} + F_{ac} \cos \omega t \). The dependence of the dc current density on the dc bias in the absence of magnetic field \( B \) is given by the familiar Esaki-Tus formula, see equation (35), which we introduced in the section 1.4.

\[
j_{dc}(eF_{dc}) = \frac{\Delta dN}{2h} \frac{eF_{dc}d\tau / h}{1 + \left( eF_{dc}d\tau / h \right)^2}.
\]

(62)

For low frequency case, namely \( \omega \tau \ll 1 \), we can use the total field \( F \) instead of \( F_{dc} \) to this equation and get

\[
j = \frac{\Delta dN}{2h} \frac{e\tau (F_{dc} + F_{ac} \cos \omega t)d/h}{1 + (e\tau (F_{dc} + F_{ac} \cos \omega t)d/h)^2}.
\]

(63)

We can substitute equation (63) to equation (45), thus the value of \( A \) can be calculated. Then we add the magnetic field \( B \) applied at an angel \( \theta \) to the superlattice axis. We can replace the applied electric field in equations (52)-(54) with the total field \( (F_{dc} + F_{ac} \cos \omega t) \) and get the expression of momentum.

\[
\dot{p}_x = e(F_{dc} + F_{ac} \cos \omega t) - \omega_c p_y \tan \theta,
\]

(64)
\[ \dot{p}_y = \frac{d\Delta m^{*} \omega_c^z}{2\hbar} \sin\left(\frac{p_y d}{\hbar}\right) \tan \theta - \omega_c^z p_z, \tag{65} \]

\[ \dot{p}_z = \omega_c^z p_y. \tag{66} \]

Based on the relation between drift velocity and current density

\[ j = N e v_d, \tag{67} \]

the value of \( A \) in this system can be calculated numerically using equations (45) and (64)-(67). We are interested in the influence of the magnetic field on the absorption in semiconductor superlattices. Figure 14 shows the comparison between the drift velocity and \( A \) with a small probe signal when \( B = 16T, \theta = 30^\circ, \omega\tau = 0.01 \) and \( F_{ac} = \frac{1}{2} F_{cr} \), where \( F_{cr} = \frac{h}{ed\tau} \). The red vertical lines denote the extreme points of drift velocity in the upper figure and the zero point of \( A \) in the lower figure. We can find in negative differential conductivity regime in upper figure, the increase of electric field causes decrease of the current. Thus the energy is transferred from the medium to the alternating field. One can see that negative \( A \) manifesting amplification appears for negative differential velocity.

### 3.2. Contour map for drift velocity and absorption

In figure 14 in order to better understand the effect of \( F_{ac} \) on \( v_d \) and \( A \), we draw the contour map, which describe the value of the drift velocity and the absorption in different dc and ac field. The different colors represent the different intervals of velocity or absorption. In the drift velocity contour map, the red color presents the larger drift velocity. In the absorption contour map, the red color
FIG. 14. (a) Plot of $v_d$ vs. dc electric field $F$ calculated in the semiconductor super-lattice with $B = 16\, T$, $\theta = 30^\circ$, $\omega\tau = 0.1$ and $F_{ac} = \frac{1}{2}F_{cr}$. (b) Plot of absorption vs. dc electric field $F$ calculated for the SL with $B = 16\, T$, $\theta = 30^\circ$, $\omega\tau = 0.1$ and $F_{ac} = \frac{1}{2}F_{cr}$.

stands for positive and blue color stands for negative values of the absorption.

Figure 15 presents the contour map of velocity and absorption for the case when there is no magnetic field. We can easily find that the drift velocity has
only one red color region as the dc field increases. Because the drift velocity has one maximum value in the system without magnetic field, which is discussed by Esaki and Tsu. As the ac field increases, the maximum drift velocity is realized for larger dc field applied. This value decreasing with increase of ac field. This evidences that the oscillations of electron in superlattices and the oscillations of ac signal interact with each other. The larger ac field localizes electrons, and therefore decreases their drift velocity, which leads to increase of the absorption (see low panel of figure 16).
FIG. 15. Contour map for the drift velocity and absorption when the magnetic field is absent. $x$ coordinate is dc field and $y$ coordinate is ac field. The color stands for the value of drift velocity and absorption.
FIG. 16. Contour map for the drift velocity and absorption when $B = 16T$ and $\theta = 30^\circ$. $x$ coordinate is dc field and $y$ coordinate is ac field. The color stands for the value of drift velocity and absorption. The white lines correspond to contour line.
Figure 16 shows the contour map of drift velocity and absorption with $B = 16T$ and $\theta = 30^\circ$. In these contour maps, the color division is inhomogenous and not regular as figure 15. The tilted magnetic field has significant influence on drift velocity and absorption. There are several visible ringlike areas in the drift velocity map because of the resonance between Bloch oscillations and cyclotron oscillations. It presents the drift velocity has several growth intervals. As we discussed, the large ac field also weakens the oscillations of the electron in superlattices in this system with a tilted magnetic field. We can even not observe other maximum of drift velocity at large ac field in this contour map. Compare the contour map of absorption in this figure with figure 15, we can find the positive and negative regimes of absorption are affected by the magnetic field. It enhances the controllability of absorption. In fixed electric field and probe signal, we can modulate the absorption by changing the magnetic field.

3.3. Conclusion

In this section, we study the influence of an external tilted magnetic field on the absorption and amplification of terahertz radiation in semiconductor superlattices. We plot the curve of absorption at a tilted magnetic field and compare with the electron drift velocity. We find that the positive and negative regimes of absorption are related to the slope of drift velocity. Thus, the absorption also can be changes by the applied magnetic field. Then we plot the absorption map to analyse the influence of every parameter on absorption. Because the strong Bloch oscillations, at the large dc field, the absorption is negative. The magnetic field can extend the amplification area and lead to some negative islands in positive
area. Our study shows can control the amplification regime to a certain degree.

It would be useful to the absorption with different angle and value of magnetic field. By changing the parameters of magnetic field, we can modulate the absorption more convenient.
4. Hall effect in a semiconductor superlattice

In this chapter, we study the Hall effect in an extraordinary configuration of the field applied to the superlattice, which is illustrated in figure 8. A similar configuration has been investigated by E. M. Epshtein [71]. He solved the Boltzmann equation by iteration with respect to the magnetic field and estimated the Hall field as a function of the current density. In particular, he predicted that the Hall field is proportional to the current density when the latter is small. However, for the large value of current density, the Hall field becomes ambiguous. This points to possible different regimes of electron dynamics, which were not considered in Epshtein’s work. In this chapter we classify regime of the electron transport and describe them by simple pendulum model.

We found out that these regimes are the result of competition of two oscillatory processes involved in dynamics, namely Bloch and cyclotron oscillations. The competition of the two processes lead to two regimes of the electrons’ oscillations: 1) cyclotron-like regime and 2) magneto-Bloch regime. Roughly speaking, the former regime is covered mainly the cases, when the electron’s cyclotron frequency is faster than the Bloch frequency, while the latter regime corresponds to the opposite cases. Additional specific feature of the superlattice in the extraordinary configuration is that Bloch oscillations are excited not by the electric field, but by $F \times B$ drift force.
4.1. Dynamics equations

In the extraordinary configuration shown in the figure we consider an electric field \( \mathbf{F} = (F_x, F_y, F_z) = (0, -F, 0) \) and magnetic field \( \mathbf{B} = (B_x, B_y, B_z) = (0, 0, B) \). In this case the dynamics of the electron is determined by the Lorentz force and an electric field force, see equation (50). The corresponding equation of motion can be expressed in terms of the momentum:

\[
\dot{p}_x = -\frac{eB}{m^*}p_y, \quad (68)
\]

\[
\dot{p}_y = -eF + \frac{\Delta deB}{2\hbar} \sin\left(\frac{p_x d}{\hbar}\right), \quad (69)
\]

where \( p_x \) and \( p_y \) are the momentum components along \( x \) and \( y \) direction in figure 8. Based on the general dispersion relation,

\[
v = \frac{1}{\hbar} \frac{\partial \varepsilon(k)}{\partial k}, \quad (70)
\]

where \( k = \mathbf{p}/\hbar \), we can find the electron velocity components as

\[
v_x = \frac{\Delta d}{2\hbar} \sin\left(\frac{p_x d}{\hbar}\right), \quad (71)
\]

\[
v_y = -\frac{\dot{p}_x}{eB}. \quad (72)
\]

The equations of motion can be presented in dimensionless form using the following substitutions

\[
v_x \rightarrow \tilde{v}_x = \frac{v_x}{v_0}, \quad v_0 = \frac{\Delta d}{2\hbar}, \quad (73)
\]

\[
v_y \rightarrow \tilde{v}_y = \frac{v_y}{v_{sp}}, \quad v_{sp} = \frac{\hbar}{m^* d}, \quad (74)
\]

\[k_x \rightarrow \tilde{k}_x = k_x d, \quad (75)\]
\[ k_y \rightarrow \tilde{k}_y = k_y d, \quad (76) \]
\[ \tilde{t} = \frac{t}{\tau}. \quad (77) \]

Note that for simplicity in Chapter 4 we use normalized time \( \tilde{t} \). Then equations (71) and (131) can be rewritten as

\[ \dot{\tilde{v}}_x = \tilde{W}_x (\omega_{cy} \tau) \tilde{v}_y, \quad (78) \]
\[ \dot{\tilde{v}}_y = -\omega_y \tau + (\omega_{cx} \tau) \tilde{v}_x, \quad (79) \]
\[ \dot{\tilde{W}}_x = -\tilde{v}_x \tilde{v}_y (\omega_{cy} \tau). \quad (80) \]

Here \( \omega_{cy} = eB/m^* \) is the frequency of cyclotron oscillations along \( y \)-direction, \( \omega_{cx} = eB/m_x \) is the frequency of cyclotron oscillations along \( x \)-direction, \( m_x^{-1} = \Delta d^2/2\hbar^2 \), \( \omega_y = eFd/\hbar \) is the Bloch frequency and \( \tilde{W}_x = -\cos(\frac{eFd}{\hbar}) \) is potential energy of an electron.

In \( k \)-space, these equations (78)-(80) have form

\[ \dot{\tilde{k}}_x = \omega_{cy} \tau \tilde{k}_y, \quad (81) \]
\[ \dot{\tilde{k}}_y = \omega_y \tau - \omega_{cx} \tau \sin \tilde{k}_x, \quad (82) \]

where \( \tilde{k}_x \) and \( \tilde{k}_y \) are the dimensionless wave vector components. We want to analysis the influence of electric field and magnetic field on electron motion. For this aim, we multiply both sides of the equation (78) by \( \tilde{v}_x \) and both sides of equation (80) by \( \tilde{W}_x \):

\[ \dot{\tilde{v}}_x \tilde{v}_x = \tilde{v}_x \tilde{W}_x (\omega_{cy} \tau) \tilde{v}_y, \quad (83) \]
\[ \dot{\tilde{W}}_x \tilde{W}_x = -\tilde{v}_x \tilde{W}_x \tilde{v}_y (\omega_{cy} \tau). \quad (84) \]
After adding equation (83) to equation (84) we obtain

\[ \dot{\tilde{v}}_x \tilde{v}_x + \dot{\tilde{W}}_x \tilde{W}_x = 0. \tag{85} \]

Since the temperature is assumed to be close to zero, the initial conditions of the dynamical system are \( \tilde{W}_x(0) = -1 \) and \( \tilde{v}_x(0) = 0 \). Without loss of generality \( \tilde{v}_x \) and \( \tilde{W}_x \) can be expressed as

\[ \tilde{v}_x = \sin \theta(t), \tag{86} \]
\[ \tilde{W}_x(t) = -\cos \theta(t). \tag{87} \]

The phase \( \theta = \tilde{k}_x \) reflects momentum component in \( x \)-direction. Substituting equations (86)-(87) to equation (78) yields

\[ \dot{\theta} = -\tilde{v}_y(\omega_{cy}\tau), \tag{88} \]
\[ \ddot{\theta} = -\omega_{cy}\tau \omega_{cx} \tau \sin \theta + \omega_y \tau \omega_{cy} \tau. \tag{89} \]

In \( k \)-space, the equation (89) can be written as

\[ \ddot{k}_x = -\omega_{cy} \tau \omega_{cx} \tau \sin(k_x) + \omega_y \tau \omega_{cy} \tau. \tag{90} \]

From mechanical point of view equation (90) describe a pendulum driven by a drag force, where the normalized wavevector \( \tilde{k}_x \) plays the role of the deflection angle. Figure 17 illustrates this analogy. This mechanism is similar to one discussed in the previous research [96], however there is clear difference related to the effective potential of the momentum \( U(\tilde{k}_x) \). In our model, the effective potential \( U(\tilde{k}_x) \) can be expressed as

\[ U(\tilde{k}_x) = -\omega_{cy} \omega_{cx} \tau^2 (\cos(\tilde{k}_x) - 1) - \omega_B \omega_{cy} \tau^2 \tilde{k}_x. \tag{91} \]
where we choose the boundary condition $U(\tilde{k}_x)|_{\tilde{k}_x=0} = 0$. There are two cases of the charge carrier motion.

In the first case, the pendulum equivalent does not have enough energy to overcome the 1st potential barrier of $U(\tilde{k}_x)$, see the figure 17 (bottom). Therefore the wave vector $\tilde{k}_x$ oscillates in the valley of one potential well and the charge carrier is localized within the range of a single Brillouin zone. This behavior is related to the harmonic oscillation of the pendulum and happens when the system has a small ratio of $F/B$. This regime can be called the cyclotron-like regime because that the motion of the charge carrier is similar to it in semiconductor with a strong magnetic field [96].

In another case, the charge carrier gains enough energy to cross the barrier, see figure 17 (top). The wave vector $\tilde{k}_x$ will increase monotonically and cross the Brillouin zone boundaries consequently. This behavior is related to the rotary motion of the pendulum and happens when the system has a larger ratio of $F/B$. This regime can be called the magneto-Bloch regime because in this case the motion of the charge carrier is similar to the Bloch oscillations.

The transition occurs, when the pendulum equivalent has right the amount energy to reach the top of the barrier, see figure 17 (middle left). In the pendulum analogy, the velocity of the pendulum is zero when it reaches the upper balance point, figure 17 (middle right). Mathematically the transition condition is thus defined as $U(\tilde{k}_x) = 0$, where $\tilde{k}_x$ has the value of the first maximum of the effective potential barrier. The numerical calculation gives the approximated value of the transition condition, as

$$\frac{\omega_y}{\omega_{cx}} \approx 0.78, \quad \text{or} \quad F/B \approx 0.78 \frac{\Delta d}{2\hbar} \quad (92)$$
Hence, at the transition point between the two regimes, the ratio of the electric field and the magnetic field strength is fixed, and depends on the hopping energy and the distance between the superlattice layers. In most details the transition could be understood with help of figure 17.

The left panels of figure 17 illustrate the dependencies $E_{\tilde{k}_x}$ for different $\omega_{c_x}$. The red lines represent the effective potential $U(\tilde{k}_x)$ and the black lines show the energy spectrum of the electron in superlattices. The colored balls, grey, orange and blue ones represent the state of the system at three different times, the arrows indicate the evolution directions. On the top panel, the barrier of the effective potential is lower than the initial position of the potential, therefore, the ball representing $\tilde{k}_x$ can move continuously to the right and cross Brillouin zone boundaries. The system is thus in the magneto-Bloch regime and is related to the rotating pendulum in the analogy. The middle and the bottom panel are the cases of transition and the cyclotron-like regime, respectively. Note that in the transition case, the barrier has the same height as the initial energy and the peak position does not correspond to the Brillouin zone boundary. In the pendulum analogy to the right, the balance position of the pendulum is not at the top of the circle where the corresponding $\tilde{k}_x$ is equal to $\pi$.

In the next section we calculate the Hall field in pendulum model with a constant magnetic field and for various electric field strength.

4.2. Hall field in a steady state

According to the earlier study of Epshtein [71], we know the Hall field is a function of stationary current density, measured after some relaxation time,
FIG. 17. The connection between the semiclassical approximation (left panels) and the pendulum analogy (right panels). From top to bottom, $\omega_{cx}\tau = 0.7, 0.89, 1.2, \omega_y = 0.7$. 
which is much longer than the scattering time. Such state of the system can be called ”the steady state”. In this section we want to calculate the Hall field in the steady state of the pendulum model, which we analyzed in section 4.1. Based on the general relation between current density and drift velocity, the normalized Hall field $\xi$ and current density along y-direction $\beta$, can be expressed by equations

$$\xi = \frac{j_x}{\sigma_0} = \frac{nev_{dx}}{\sigma_0} = v_{dx} \left( \frac{m^*}{m_x} \right),$$  \hspace{1cm} (93)$$

$$\beta = \frac{j_y}{j_0} = \frac{nev_{dy}}{j_0} = \omega_{cy} \tau \tilde{v}_{dy},$$  \hspace{1cm} (94)$$

where $j_x$ and $j_y$ is current density along $x$ and $y$ direction, $\sigma_0 = \frac{e^2 n \tau}{m^*}$ and $j_0 = \frac{\hbar}{\beta dr}$.

According to equations (93)-(94), the Hall field is proportional to $\tilde{v}_{dx}$ and current density along $y$-direction is proportional to $\tilde{v}_{dy}$. These drift velocity can be estimated in the way described in Chapter 2, see equation (61). After normalization of parameters $v_x$, $v_y$ and $t$, the equation (61) can be expressed in the dimensionless form:

$$v_{dx(y)} = \int_0^\infty v_{x(y)}(\tilde{t}) e^{-\tilde{t}} d\tilde{t}.$$  \hspace{1cm} (95)$$

Figure 18 (a)-(d) show the dependence of the Hall field $\xi$ on the current density $\beta$ calculated numerically for different values of the magnetic field represented by the value of $\omega_{cy}$. In all cases the curves have both descending and ascending parts. For small currents, the Hall field is almost linearly decrease with $\beta$, as it should be in usual conductors [88]. When the Hall field reaches the minimum value, there is a transition point, after which, the Hall field starts to ascend as $\beta$ increases. Comparing four curves in (a)-(d), which describe the Hall field
FIG. 18. Dependence of $\beta$ vs. $\xi$, the curves are calculated for $m^*/m_x = 1$ and for (a) $\omega_{cy}\tau = 1$, (b) $\omega_{cy}\tau = 3$, (c) $\omega_{cy}\tau = 5$, (d) $\omega_{cy}\tau = 10$.

at different magnetic field, we notice that for different $\omega_{cy}\tau$ the transition points occur in different positions. We will discuss the transition point of electron motion and give an analytical solution of magnetic field in transition point in Chapter 5.

We show that in these two distinctive parts of $\xi(\beta)$ curve, the electrons demonstrate different types of motion. In order to observe these different regimes of motion corresponding to the pendulum model, we simulate the drift velocity along $x$ and $y$ directions, which are proportional to the Hall electric field $\xi$ and current
density $\beta$, for different value of applied electric field along $y$-direction.

FIG. 19. Drift velocity along $x$ and $y$ directions as a function of $\omega_y \tau$. for (a), (b) $\omega_{cy} \tau = 3$, (c), (d) $\omega_{cy} \tau = 5$ and (e), (f) $\omega_{cy} \tau = 10$.

The dependence of the drift velocity $\tilde{v}_{dx(y)}$ on the Bloch frequency $\omega_y$ calculated
for different value of the magnetic field are plotted in the figure [19]. With the small Bloch frequency $\omega_y$ (i.e. the small electric field), the Hall drift velocity $\tilde{v}_{dx}$ increases linearly with the frequency $\omega_y$, so does the Hall current $j_x$. The conduction velocity $|\tilde{v}_{dy}|$ is almost zero in this region. With the large Bloch frequency $\omega_y$, the Hall drift velocity $\tilde{v}_{dx}$ decays exponentially with $\omega_y$, while the conduction velocity $|\tilde{v}_{dy}|$ increases linearly with $\omega_y$. There is a clear phase change in the transition of the two cases. For the various values of the $\omega_{cy}$, the phase transition point shift (compare the figures in the same column). When the electric field is small, the charge carrier does not gain enough energy to overcome the effective potential barrier, figure [17] (Bottom). Thus the Bloch oscillations are not initiated and the drift velocity is mostly determined by the $F \times B$ drift. The trajectory of the electrons in this regime is schematically illustrated in figure [20](a). Because of domination of $F \times B$ drift, the change of $\tilde{v}_{dx}$ is proportional to change of electric field $F$, which explains the linear behavior of $\tilde{v}_{dx}$ in the low field region in the figure [19](a, c, e). Since there is no drift in $y$-direction, a flat plateau of $|\tilde{v}_{dy}|$ is observed in the low field region (figure [19](b, d, f)). With the increase of the electric field $F$, a critical position is reached, where the largest Hall drift velocity $\tilde{v}_{dx}$ is detected. Beyond this point, the Bloch oscillations dominate. The Bloch oscillations periodically reverse the direction of the movement $\tilde{v}_x$, thus, suppress the $F \times B$ drift. A schematic trajectory of the electron in this regime is illustrated in the figure [20](b). It shows that the Bloch oscillations suppressed $\tilde{v}_x$ enhance the drift in the $y$ direction, $\tilde{v}_y$. This explains an exponential decrease $\tilde{v}_{dx}$ and linear increased $|\tilde{v}_{dy}|$ demonstrated in figure [19]. The transition point depends on the ratio between the electric field $F$ and the magnetic field $B$. This ratio is given by the equation (92), which accurately predicts the transition points.
FIG. 20. (a) The schematic show of the $F \times B$ drift. (b) The inhibited $x$ drift and the enhanced $y$ drift due to the Bragg reflection. The dotted line represent the Brillouin zone boundaries.

From the numerical simulations, we observe that the Hall field and the drift velocity of electron have two regimes. These two regimes are similar to ones discussed in the pendulum model. We will study the transition point and the electron trajectories in both those regimes in Chapter 5.
4.3. Analytical solutions

In this section we will calculate the Hall field in the steady state analytically and compare it with the numerical solutions, which we discussed in Chapter 4.2. Firstly, we need to get the basic analytical formula connecting the Hall field and the current density. For this aim we apply a path integral, which is expressed as $< f(t) >= \int_0^\infty \frac{1}{\tau} f(t) e^{-\frac{t}{\tau}} dt$, to equation (79),

$$< \dot{\tilde{v}}_y >= < -\omega_y \tau > + < (\omega_{cx} \tau) \tilde{v}_x > .$$

(96)

After integrating equation (96), we obtain $< \dot{\tilde{v}}_y >= \frac{1}{\tau} < \nu_y >$. Thus, after simplification, equation (96) can be rewritten as

$$\tilde{v}_{dx} = \frac{\tilde{v}_{dy} + \omega_y \tau}{\omega_{cx} \tau} .$$

(97)

In equation (97), all parameters and variables are dimensionless. To restore the measurable quantity, one can use the following expressions:

$$v_0 = \frac{\Delta d}{2\hbar}, v_{sp} = \frac{\hbar}{m^* d}, m_{-1} = \frac{\Delta d^2}{2\hbar^2} ,$$

(98)

$$\tilde{v}_x = \frac{\nu_x}{v_0}, \tilde{v}_y = \frac{\nu_y}{v_{sp}} .$$

(99)

Here we use an approximation $m_x \approx m^*$. Thus, we obtain

$$v_{dx} = \frac{v_{dy} + \omega_y e_{dy} \nu_y}{\omega_{cx} \tau},$$

(100)

$$v_{dx} = \left( \frac{m^* \Delta d^2}{2\hbar^2} \nu_{dy} + \frac{\Delta d}{2\hbar} \nu_{yy} \tau \right) \frac{1}{\omega_{cx} \tau},$$

(101)

$$v_{dx} = \left( \frac{m^*}{m_x} \nu_{dy} + \frac{\Delta d \nu \nu_y}{2\hbar} \frac{1}{\omega_{cx} \tau} ,$$

(102)
\[ v_{dx} = \left( v_{dy} + \frac{\Delta d \cdot e F_y d}{2\hbar} \frac{1}{\hbar} \right) \frac{1}{\omega_{cx} \tau}, \]  
(103)

\[ J_x = \frac{1}{w_{cx} \tau} \left( J_y + \frac{\Delta d^2}{2\hbar^2} ne^2 F_y \tau \right), \]  
(104)

\[ F_x = \frac{1}{w_{cx} \tau} \left( \frac{J_y}{\sigma_0} + F_y \right). \]  
(105)

Equation (105) expresses the Hall field \( F_x \) as a function of the current density along \( y \)-direction, where \( J_x \) and \( J_y \) are current density along \( x \) and \( y \) direction, respectively. Then we will calculate the analytical solution of drift velocity with the help of pendulum equation and compare it with equation (97).

4.3.1. Cyclotron-like regime

In the cyclotron-like regime, the wavepackets oscillate at the bottom of the Brillouin zone, therefore the value of \( \tilde{k}_x \) is small, \( \tilde{k}_x \ll 1 \). Using the approximation \( \sin(\tilde{k}) = \sin(\theta) \approx \theta \), we can write

\[ \ddot{\theta} = -\omega_{cy} \tau \omega_{cx} \tau \theta + \omega_y \tau \omega_{cy} \tau. \]  
(106)

For the chosen initial conditions,

\[ \theta(0) = 0, \]  
(107)

\[ \dot{\theta}(0) = 0, \]  
(108)

the equation (106) has a solution

\[ \theta = -\frac{a}{b} \cos(b\tilde{t}) + \frac{a}{b}, \]  
(109)

where \( a = \omega_y \tau, b = \omega_{cy} \tau \). This expression for \( \theta \) can be substituted to equation (95).
\[ \tilde{v}_{dx} = \int_0^\infty e^{-t} \sin\left(-\frac{a}{b} \cos(b\tilde{t}) + \frac{a}{b}\right) d\tilde{t}. \]  \hspace{1cm} (110)

Since we can assume that in cyclotron-like regime \( a \ll 1 \), equation (110) can be rewritten taking into account the Jacobi-Anger expansion \[89–91\]

\[ \cos(z \cos \theta) = J_0(z) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(z) \cos(2n\theta), \]  \hspace{1cm} (111)

\[ \sin(z \cos \theta) = -2 \sum_{n=1}^{\infty} (-1)^n J_{2n-1}(z) \cos((2n-1)\theta)]. \]  \hspace{1cm} (112)

In this case the equation (110) becomes the sum of the series of Bessel functions

\[ \tilde{v}_{dx} = \int_0^\infty e^{-t} \sin\left[-\frac{a}{b} \cos(b\tilde{t}) + \frac{a}{b}\right] d\tilde{t} \]

\[ = \int_0^\infty e^{-t} \left[ -\sin\left(\frac{a}{b} \cos(b\tilde{t})\right) \cos \frac{a}{b} + \cos\left(\frac{a}{b} \cos(b\tilde{t})\right) \sin \frac{a}{b} \right] d\tilde{t} \]

\[ = \int_0^\infty e^{-t} \left[ 2 \sum_{n=1}^{\infty} (-1)^n J_{2n-1}\left(\frac{a}{b}\right) \cos((2n-1)b\tilde{t}) \cos \frac{a}{b} + J_0\left(\frac{a}{b}\right) \sin \frac{a}{b} \right. \]

\[ + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}\left(\frac{a}{b}\right) \cos(2nb\tilde{t}) \sin \left(\frac{a}{b}\right) ] \]

\[ = 2 \sum_{n=1}^{\infty} (-1)^n J_{2n-1}\left(\frac{a}{b}\right) \cos\left(\frac{a}{b}\right) \frac{1}{1 + 4n^2b^2 - 4nb^2 + b^2} ] \]

\[ + J_0\left(\frac{a}{b}\right) \sin\left(\frac{a}{b}\right) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}\left(\frac{a}{b}\right) \sin\left(\frac{a}{b}\right) \frac{1}{1 + b^2}, \]  \hspace{1cm} (113)
\[ J_n(z) \] is the n-th Bessel function of the first kind. In the cyclotron-like regime, \( \omega_y/\omega_{cy} \ll 1 \), we keep only the terms containing up to the second order of \( \omega_y/\omega_{cy} \). The equation (113) is reduced to

\[
\tilde{v}_{dx} = -2J_1 \left( \frac{a}{b} \right) \cos \left( \frac{a}{b} \right) \frac{1}{1 + b^2} + J_0 \left( \frac{a}{b} \right) \sin \left( \frac{a}{b} \right) - 2J_2 \left( \frac{a}{b} \right) \sin \left( \frac{a}{b} \right) \frac{1}{1 + b^2}
\]

\[
= -\frac{a}{b} \frac{1}{1 + b^2} + \frac{a}{b}
\]

\[
= \frac{ab}{1 + b^2}
\]

\[
= \frac{\omega_y \tau \omega_{cy} \tau}{1 + (\omega_{cy} \tau)^2}.
\]

The drift velocity in \( y \) direction is derived similarly and the result is

\[
\tilde{v}_{dy} = \frac{-\omega_y \tau}{1 + (\omega_{cy} \tau)^2}.
\]

4.3.2. Magneto-Bloch regime

In the magneto-Bloch regime, \( \omega_y/\omega_{cy} \gg 1 \). The first term of pendulum equation is much smaller compared with the second term and can be neglected. Hence, the pendulum equation is simplified to

\[
\ddot{\theta} = -\omega_{cy} \tau \omega_{cx} \tau \sin \theta + \omega_y \tau \omega_{cy} \tau \approx \omega_y \tau \omega_{cy} \tau.
\]

Equation (116) can be solved exactly to obtain

\[
\theta = \frac{\omega_y \tau \omega_{cy} \tau t^2}{2}.
\]
We substitute equation (117) to equation (89) for obtaining more accurate solution.

\[ \ddot{\theta} = -\omega_c y \omega_c \tau \sin(\frac{ab}{2} t^2) + \omega_y \tau \omega_c y \tau. \]  
(118)

After solving the equations (88) and (118) we obtain

\[ \tilde{v}_y = b \sqrt{\frac{\pi}{ab}} S(\frac{ab\tilde{t}}{\sqrt{ab\pi}}) - a\tilde{t}. \]  
(119)

Where \( S(\frac{ab\tilde{t}}{\sqrt{ab\pi}}) \) is Fresnel integral. \( S(x) \) calculator \( S(x) = \int_0^x \sin\left(\frac{1}{2} \pi t^2\right) dt, \) for which \( S(\infty) = \frac{1}{2} \) [92]. Assuming that \( \frac{ab\tilde{t}}{\sqrt{ab\pi}} \) tends to infinity, thus

\[ S(\frac{ab\tilde{t}}{\sqrt{ab\pi}}) = \frac{1}{2}, \]  
(120)

and

\[ \tilde{v}_y = \frac{b}{2} \sqrt{\frac{\pi}{ab}} - a\tilde{t}. \]  
(121)

Using equations (121) and (95) we can calculate the drift velocity of electron along \( y \)-direction

\[ \tilde{v}_{dy} = \int_0^\infty e^{-\frac{1}{2}}(\frac{b\pi}{a} - a\tilde{t})d\tilde{t}, \]  
(122)

which after integration gives

\[ \tilde{v}_{dy} = \frac{1}{2} \sqrt{\frac{b\pi}{a}} - a. \]  
(123)

According to equations (86) and (117), we obtain

\[ \tilde{v}_x = \sin(\frac{ab}{2} t^2), \]  
(124)

thus the drift velocity along \( x \)-direction can be expressed
\[ v_{dx} = \int_0^{\infty} e^{-\tilde{t}} (\sin(ab/2)) \, dt. \] (125)

Equation (125) can be solved analytically

\[
v_{dx} = \frac{1}{4} \sqrt{\frac{2\pi}{c}} \sin\left(\frac{1}{4c}\right) + \frac{1}{4} \sqrt{\frac{2\pi}{c}} \cos\left(\frac{1}{4c}\right)
- \frac{1}{2} \sqrt{\frac{2\pi}{c}} \sin\left(\frac{1}{4c}\right) S\left(\frac{1}{2} \sqrt{\frac{2}{\pi c}}\right)
- \frac{1}{2} \sqrt{\frac{2\pi}{c}} \cos\left(\frac{1}{4c}\right) C\left(\frac{1}{2} \sqrt{\frac{2}{\pi c}}\right)
\approx \frac{1}{2} \sqrt{\frac{\pi}{ab}},
\] (126)

where \( C(x) = \int_0^x \cos\left(\frac{1}{2} \pi t^2\right) \, dt. \)

Figure 21 illustrates the comparison of the numerical simulation with the analytical results. The analytical calculations in both the small and large field regimes are denoted by the dotted red lines, which have good agreement with the numerically derived blue curve.

4.4. Analytical calculation with the electron energy

In section 4.3 we get the analytical solutions of drift velocity by solving dynamics equations. The different regimes of electron dynamics can be also described in terms of energy. We know that in equation (80), \( W_x \) is the potential energy of an electron. Now we apply path integral to the both sides of the equation (80)

\[ v_{dx} = \omega_{cy} \tau < W_x v_y >, \] (127)

which returns the expression of drift velocity \( v_{dx} \) in terms of energy \( W_x \).
FIG. 21. The comparison between the numerical simulation and the analytical calculation. The solid curve is the numerical results for the cyclotron frequency $\omega_{cy} = 5$ while the dotted lines are the analytical proximity far from the transition point.

4.4.1. Cyclotron-like regime

In cyclotron-like regime the approximate solution of $\theta$ can be expressed by equation (109). Since the electric field is small and electron is closed to the bottom of miniband, the velocity of electron along $x$ axis is very low. Therefore we can assume

$$W_x = -\cos \theta = -\cos\left[-\frac{a}{b} \cos(b\tilde{t}) + \frac{a}{b}\right] = -1.$$  \hspace{1cm} (128)

From equations (88), (127) and (128), it follows that
\[ v_{dx} = \omega_c r < W_x v_y > = b \int_0^\infty e^{-i(-1)(-\frac{a}{b} \sin(b\tilde{t}))}d\tilde{t} = \frac{ab}{1 + b^2}, \quad (129) \]

which is the result identical to equation (114).

4.4.2. Magneto-Bloch regime

In cyclotron-like regime the approximate solution of \( \theta \) can be expressed by equation (117). Therefore the energy is

\[
W_x = -\cos \theta = -\cos\left(\frac{ab}{2} \tilde{t}^2\right). \quad (130)
\]

We know in the magneto-Bloch regime, \( a \) is a large parameter, thus we use an approximation of \( \ddot{v_y} \) based on equation (121)

\[
\ddot{v}_y \approx -\tilde{a}. \quad (131)
\]

From equations (127), (130) and (131)

\[
v_{dx} = b \int_0^\infty e^{-i(-a\tilde{t})\left(-\cos\left(\frac{ab}{2} \tilde{t}^2\right)\right)}d\tilde{t}, \quad (132)
\]

Taking the integral in equation (132), we obtain

\[
v_{dx} = \frac{1}{4} \sqrt{\frac{2\pi}{c}} \sin\left(\frac{1}{4c}\right) + \frac{1}{4} \sqrt{\frac{2\pi}{c}} \cos\left(\frac{1}{4c}\right)
- \frac{1}{2} \sqrt{\frac{2\pi}{c}} \sin\left(\frac{1}{4c}\right)S\left(\frac{1}{2} \sqrt{\frac{2}{\pi c}}\right) - \frac{1}{2} \sqrt{\frac{2\pi}{c}} \cos\left(\frac{1}{4c}\right)C\left(\frac{1}{2} \sqrt{\frac{2}{\pi c}}\right)
\approx \frac{1}{2} \sqrt{\frac{\pi}{ab}}. \quad (133)
\]

Therefore, we get the same results of drift velocity as given in equation (126)
4.5. Conclusion

In this chapter, we investigate the Hall current in a steady state in the extraordinary configurations of the semiconductor superlattices and analyse the two regimes of the electrons’ oscillations. Compare to the Epshtein’s result [71], we give a detailed analysis of physical process for the Bloch-magneto regime and cyclotron-like regime which is not mentioned in Epshtein’s paper. In these two regimes, electrons exhibit two different type of motion. We explain this by \( F \times B \) and scheme it in figure 20. Furthermore, in this chapter, both the qualitative interpretations and the numerical simulations for these two regimes are given with the help of pendulum model.
5. Coherent Hall effect and electron temporal behaviour

We study the coherent Hall effect, which is the transient response to an impulsive charge carrier excitation. The coherent Hall effect is the ac current phenomenon in the route to the onset of the dc classical Hall effect. From viewpoint of quantum mechanics, this phenomenon occurs due to the coherence of the wavepackets of electrons. The Hall current is static with a dc biased voltage applied to the ends of the superlattices in the classical limit, however, it is not static current immediately after an excitation of the charge carriers. The validity of the classical limit requires the decoherence of the charge carriers’ wavepackets, which implies that the average scattering time should be taken into account within the classical limit. Kosevich et. al. considered this phenomenon theoretically and confirmed it through experiment in superlattices [93–95]. Hummel et. al. analyzed the electron motion numerically and computed the real-space and k-space trajectories of the electrons [96].

In this chapter, we investigate the coherent Hall effect in the extraordinary configuration of the superlattice. Based on the previous results of Kosevich and Hummel [93, 96], the trajectory of the wavepackets can be determined in both $k$ and real space by the numerical calculations. The temporal behavior of wavepackets after the excitation of the charge carrier is also visualized. We define a vector indicating the state of the system

$$\mathbf{u} = (u_1, u_2, u_3, u_4)^T = (\tilde{k}_x, \tilde{k}_y, x, y)^T,$$

and consider the temperature close to absolute zero, thereby the electron can be regard as motionless before applying electric field. For the initial condition $\mathbf{u}(0) = 0$, which corresponds to the excitation of a wavepacket at the bottom of
the miniband, the differential equation for \( u \), based on the calculation discussed in Chapter 4, can be written as

\[
\begin{align*}
\frac{d\mathbf{u}}{dt} = & \begin{pmatrix} \dot{k}_x \\ \dot{k}_y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \omega_{cy}u_2 \\ \omega_y - \omega_{cx}\sin(u_1) \\ \frac{\Delta d}{2\hbar}\sin(u_1) \\ \frac{\hbar}{md^2}u_2 \end{pmatrix} 
\end{align*}
\] (135)

All calculations in this chapter are performed for \( e = 1.6 \times 10^{-19} C \), the miniband width \( \Delta = 26.2 meV \), a superlattice period \( d = 11.4 nm \) and \( m^* = m_x = 0.067 m_e \). These parameters correspond to a realistic superlattice discussed in [45].

5.1. Trajectories in k-space

In order to study the temporal behaviour and the nonlinear motion of wavepacket at different value of magnetic field and fixed electric field in k-space, we analyze the time dependence of dimensionless wavenumber \( \tilde{k}_x \) after solving equation (135) numerically.
FIG. 22. The trajectories of the wavepackets in $k$ space. In the magneto-Bloch regime, $\tilde{k}_x$ increase monotonously. The increasing rates are promoted with the higher magnetic field but reach a turning point for the transition regime $B \approx 0.78T$ (green curve).

Figure 22 shows the time dependence of the wave vector $\tilde{k}_x$ in the presence of a transverse electric field $1 \text{kV/cm}$ and a magnetic field of strength ranging from $0.1T$ to $0.78T$. From this figure one can see that $\tilde{k}_x(t)$ curves for the magnetic field strength smaller than $0.787T$ increase monotonically. This evidences that the system is in the magneto-Bloch regime, in which there is no pure oscillations of the momentum $\tilde{k}_x$. The green curve illustrates the case near the transition between the magneto-Bloch regime and the cyclotron-like regime. The inhibition of growth rate of $\tilde{k}_x$ with increase of $B$ appears due to enhancement of the cyclotron component in motion of electrons. Analysing the curves more carefully, we can see the slight fluctuations at the first several $ps$ before the curves approach to the linear asymptotes. Hence, they imply the co-effect of both the Bloch-like
and cyclotron-like mechanisms in the model. We reveal it in a zooming figure for $B = 7T$, as shown in figure 23. One can see two turning points in the curve.

Figure 24 shows the time dependence of the wave vector $\tilde{k}_x$ in cyclotron-like regime for the magnetic field strength ranging from 0.8T to 1.1T. From the discussion given in Chapter 4 we recall that the stronger magnetic field creates the higher barrier which prevents the $k_x$ from increasing to infinity, figure 17 (bottom left). Hence, the momentum $\tilde{k}_x$ is localized in one Brillouin zone. In figure 24 the amplitudes and oscillation periods of $\tilde{k}_x$ decrease with the increase of the magnetic field, which resembles the cyclotron oscillations.

From analysis of figure 22 and 24 we conclude that trajectories of the electron in k-space has two different regimes. The numerical results shown in figure 22 and 24 are in very good agreement with our analysis on the pendulum model in
FIG. 24. The trajectories of the wavepackets in $k$ space. In the cyclotron-like regime, $\tilde{k}_x$ oscillate with the amplitudes depending on the magnetic field applied.

equation\[92\] which is discussed in Chapter 4.

5.2. Trajectories in real space

In this section we determine the temporal behaviour of electron and visualize the trajectories in real space. The trajectories for different regimes are shown in figures 25 and 26 which are calculated by equation 135. Figure 25 shows the trajectories of the wavepackets in $x-y$ plane for the magneto-Bloch regime. The value of the magnetic field strength is chosen from 0.1T to 0.787T. As one can see, the transmission in $x$-direction is inhibited while the drift in $y$ direction is unbounded. Before the next scattering takes place, the amplitude of the oscillation in $x$-direction reduces exponentially on time, and the wavepackets approach to a steady position in $x$ axis of coordinate. The shifts before the oscillations
FIG. 25. The trajectory of the wavepackets in the magneto-Bloch regime in the real space. Before the scattering takes place, the wavepackets oscillate and tend to some static positions in x axis with pure shifts from their initial positions.

reflect the Hall effect and lead to the establishing of the polarized static electric field. These behaviors of the wavepackets agree with our qualitative analysis in Chapter 4.

Figure 26 shows the trajectories of the wavepackets in x − y plane for the cyclotron-like regime. In this case, the magnetic field is stronger so that the Bloch oscillation can not be initiated. The trajectories in the figure demonstrate clear characteristics of cyclotron-like behaviors and is agree with our discuss in Chapter 4.

To clarify the difference between the two regimes through the pendulum analogy, we plot the corresponding phase portraits on the phase plane (\( \frac{d\tilde{k}}{dt} \) vs \( \tilde{k} \)).

Figure 27 shows the phase portrait in magnetic-Bloch regime for the magnetic field \( B \) smaller than the critical field \( B_{crit} \). In the figure, \( \tilde{k}(t) \) can only
FIG. 26. The trajectory of the wavepackets in cyclotron-like regime in real space. The wavepackets drift in $x$ direction.

evolve along one of the phase trajectories before the scattering happens. In the magnetic-Bloch regime, all the phase trajectories are open to the right. Hence the point $(\tilde{k}_d, \tilde{k})$ can keep moving continuously to the right, and $\tilde{k}$ can reach the infinity large values. Recalling the pendulum analogy in Chapter 4, this regime is related to the rotary cases, in which the pendulum gains enough energy to cross the topmost point. Then the rotations continue and speed up. In such analogy, $\tilde{k}$ indicates the deflection angle. Therefore the values of $\tilde{k}$ increase continuously.

Figure 28 shows the phase portrait in cyclotron-like regime for the magnetic field stronger than the critical field strength. In the figure, some of the phase trajectories are closed, which implies the existence of the oscillations of $\tilde{k}_x(t)$. In the pendulum analogy, these closed phase trajectories are related to the oscillations of the pendulum, Figure 17(Bottom). Note that there are also phase trajectories with open end to the right. In these cases, $\tilde{k}_x$ increase continuous to the infinity.
and the wavepackets act similar to the cases of magneto-Bloch regime. We have seen that the wavepackets having characteristics of either magneto-Bloch regime or cyclotron-like regime co-exist in this case. The choice of either the behaviors depends on the initial conditions \((\frac{d\tilde{k}_x}{dt}, \tilde{k}_x)_{t=0}\).

From viewpoint of the pendulum model, at the transition point between two regimes, the oscillator remains at upper fixed point. In this case the potential energy arrives in maximal value. In order to obtain the critical \(B\) corresponding to the transition with a fixed electric field \(F\), we use the equation

\[
\ddot{\theta} = -\frac{\partial U(\theta)}{\partial \theta}.
\]  

(136)

By solving equation (136) and (89), we get the potential energy equation (137)
FIG. 28. Phase portrait for cyclotron-like regime. In the cyclotron-like regime, some of the phase trajectories are closed implying the corresponding $\tilde{k}_x$ are localized thus exhibit the cyclic motions. Here $\omega_y \tau = 10$ and $\omega_{cy} \tau = 13.2$.

\[ U(\theta) = -\omega_{cy} \tau \omega_{cx} \tau \cos \theta - \omega_y \tau \omega_{cy} \tau \theta. \]  \hspace{1cm} (137)

Considering the pendulum model, the transition point should occur when the potential energy of the initial condition $U(0)$ is equal to the potential energy $U_{Max}$ at the local maximum of curve, see figure 29. Corresponding to initial condition, we substitute $U(0) = U_{Max}$ to equation (137) and obtain

\[ -\omega_{cx} \tau = -\omega_{cx} \tau \cdot \cos[\pi - \arcsin(\frac{\omega_y \tau}{\omega_{cy} \tau})] - \omega_y \tau \cdot [\pi - \arcsin(\frac{\omega_y \tau}{\omega_{cy} \tau})]. \]  \hspace{1cm} (138)
FIG. 29. Potential energy curve which $F = 1\text{KV/cm}$, $B = 0.6T$ (blue), $0.78T$ (black) and $0.9T$ (purple). The red dash line indicates the value of $U(0)$.

We solve the equation (138) for $F = 1\text{KV/cm}$ and estimate the critical value $B = 0.78T$. Thus the analytical solution of transition point is excellent accordance with the result of numerical calculation, which we simulated in Section 5.1 and 5.2. The energy curve at the transition point is demonstrated in figure 29 (black curve), which is plotted for $F = 1\text{KV/cm}$ and $B = 0.78T$. The red dash line indicates the value of $U(0)$ which equals the maximum value in black curve. In cyclotron-like regime, the $U_{Max}$ is smaller than $U(0)$ (purple curve). It illustrates the electron has an oscillation motion in first miniband and cannot travel out, which corresponds to oscillation pendulum. In transition point,
$U_{Max} = U(0)$ as shown in figure 29. In magneto-Bloch regime, the maximum value of the potential energy lies above the red dash line (blue curve).

5.3. THz-emission spectroscopy—expected transients

In this section, we calculate the waveform of the THz radiation transient which are generated due to the coherent oscillations of the wavepackets. From the shape of the calculated waveform, we can examine the spectrum of the emitted THz radiation. The electric field strength of the radiated electromagnetic field is proportional to the second derivative of the oscillated wavepacket’s polarization

$$E_{THz,x} \propto \frac{d^2 P_x}{dt^2} \propto q \dot{v}_x.$$  \hspace{1cm} (139)

After taking into account the dephasing effects which are important in the light excitation experiments [96], $v_x$ can be written in the form

$$v_x = \frac{\Delta d}{2\hbar} \sin \left( \frac{p_x d}{\hbar} \right) e^{-t/\tau_d}. \hspace{1cm} (140)$$

Here exponent coefficient $e^{-t/\tau_d}$ reflects decay of the velocity in $x$-direction due to dephasing effect. Finally we get the equation of $\dot{v}_x$

$$\dot{v}_x = -\frac{\Delta d}{2\hbar} \left( \frac{1}{\tau_d} \cdot \exp \left( -\frac{1}{\tau_d} t \right) \cdot \sin (\tilde{k}_x) + \exp \left( -\frac{1}{\tau_d} t \right) \cos (\tilde{k}_x) \cdot (\omega_{cy} \tilde{k}_y) \right). \hspace{1cm} (141)$$

The shape of the waveforms of the transient behaviors are illustrated in figure 30. In the figure, the electric field is constant with magnitude $F = 1KV/cm$, while the magnetic field strength varying from $B = 0.05T$ to $1.5T$. The dephasing time $\tau_d$ is assumed to be close to 1ps [93 96]. The oscillations are clearly visible within a few picoseconds after the excitation of the coherent wavepackets and exponentially decay with the dephasing rate $1/\tau_d$. With the increasing of the magnetic
field, the amplitude of the oscillations increase and more periods of the oscillations become prominent. The two regimes of the oscillations are distinguishable through the different characteristics of the waveforms. The upper half of the panel shows the wavepackets oscillate in the cyclotron-like regime and the lower half of the panel shows the oscillations in the magneto-Bloch regime. Notably that in contrast to cyclotron-like regime, magneto-Bloch regime can demonstrate long living oscillations, see time interval 3-6 ps.

FIG. 30. THz transients for the constant electric field 1KV/cm and the magnetic field ranging from 0.05T to 1.5T with a step width 0.05T. The coherent time is assumed to be $\tau_d \approx 1 ps$. The curves are offset vertically for clarity. A few periods of vibrations are visible within the time of the scale of the coherence time. The transition is easy to distinguish by the different characteristics of the waveforms of the two regimes.

This could be understood from analysis of the wavepacket oscillations, illus-
trated in figure 31. In the magneto-Bloch regime, see figure 31(a), the amplitudes of the acceleration and the oscillation frequencies continuously increase with time and therefore less affected by exponent coefficient $e^{-t/\tau_d}$. In the cyclotron-like regime, see figure 31(c), the acceleration shows the periodic behaviors with constant amplitude. In the transition case, the acceleration goes to zero and the oscillations disappear, see figure 31(b).

5.4. Internal electro-optic sampling—expected transients

Another measurement techniques, which can be used to study the coherent Hall effect is time-resolved internal electro-optic sampling in transmission (TEOS) [99]. This measurement method can probe the polarization $P_x$ associated with the wavepackets oscillations in the $x$-direction [100–102]

\[ EO - signal \propto P_x \propto qx. \] (142)

From equation (142) we know that the polarization in $x$-direction is proportional to $x$. In order to be close to experimental reality, we consider that the damping equals 1ps [96]. Thus, we can calculate $x$ by solving the following differential equation:

\[ \dot{x} = \frac{\Delta d}{2\hbar} \exp\left(-\frac{t}{\tau}\right)\sin(\tilde{k}_x) \] (143)

Hence, we can simulate the TEOS signal by calculating $x$ numerically.

Figure 32 displays the numerically calculated TEOS signal transients for the constant electric field 1KV/cm. The values of the magnetic field strength range between 0.1T and 1.5T. The cycles of the oscillations are inhibited in the tran-
FIG. 31. The dependent $v_x$ closed to transition between magneto-Bloch regime and cyclotron-like regime. (a) This curve correspond to the magneto-Bloch regime close to the transition, $B=0.78\,\text{T}$. (c) This curve correspond to the cyclotron-like regime close to the transition, $B=0.79\,\text{T}$. (b) This curve represents the transition regime approximately.
FIG. 32. TEOS transients calculated for a constant electric bias field of 1 KV/cm and magnetic fields between 0.05 and 1.5 (0.05T steps). Each line represents the TEOS transient in different magnetic field. The curves are offset vertically for clarity. Damping with a time constant of 1ps is assumed.

Figure 33 shows the trajectories which are close to the transition point corresponding to the three regimes in figure 32 without damping. As shown in figure 33 (a) and (c), the motion of electron in x-direction is bounded in magneto-Bloch regime and unbounded in cyclotron-like regime. In transition point, the electron is resting in x-direction, corresponding to the critical point in pendulum model.
FIG. 33. The dependence $x$ closed to transition between magneto-Bloch regime and cyclotron-like regime. (a) This curve is in the magneto-Bloch regime close to the transition, $B=0.78$T. (c) This curve is in the cyclotron-like regime close to the transition, $B=0.79$T. (b) This curve represents the transition regime approximately.
5.5. Conclusion

In this chapter we study the coherent Hall effect in the extraordinary configuration of superlattice. We find out that there are two types motion of the charge carrier in different value of electric field and magnetic field. Although there are some similarities with the case of usual configuration of F and B \cite{96}, the extraordinary configuration brings new features, like we showed in section 5.1 and 5.2. The analysis and calculations in this thesis are primarily settled at the semiclassical approach. In the semiclassical approach, the charge carriers is treated classically but having wave-like spectrums of superlattice band structures. This approximation has been used widely and resulted in many successes in revealing the main characteristics of superlattices \cite{93, 95, 96}. We calculated the classical spatial trajectories of the semiparticles (charge carriers) in both the cyclotron-like and magneto-Bloch regimes which are very different. By mapping the semiclassical dynamics to a pendulum, the divergence between the two regimes are clearly shown in the phase portrait. The analytical calculation for the transition between the two regimes is also given. We also numerically calculated the expected THz transients signals. In the magneto-Bloch regime of the THz transients pictures, there is an unusual gap between the first peak of the signal and the group of concentrated oscillations. This phenomenon imply a different THz spectrum from the conventional configuration in which the electric field is in the superlattices growth direction. Finally we simulated the TEOS signals and predicted the experimental results. In TEOS, one can examine the build-in polarization field, which corresponding to the signals we plotted in figure \cite{32}.
6. Summary

Since they were proposed, the semiconductor superlattices have always been of a great research interest. The specific electronic and optical characteristics of the semiconductor superlattice make it one of the most popular among artificial materials in modern science and technology. However, the semiconductor superlattice still has a lot of unknown properties needed to be studied and utilized. In this thesis, we mainly focus on the effects of magnetic field on electron transport in semiconductor superlattices.

In Chapter 1, we discuss and review the current state of the art in physics of semiconductor superlattices. We also explain some basic definitions and derivations, which we need in this thesis and introduce some previous studies about superlattices briefly. Finally, we describe the models used in our studies.

In Chapter 2, we study the chaotic transport of electrons in a semiconductor superlattice subject to an electric and tilted magnetic fields. We describe the related dynamical system and classify possible dynamical regimes. It has been demonstrated that the magnetic field can extend the mean free path of electrons and make the electrons travel further in the superlattice axis before scattering. When the ratio \( r = \frac{\omega_B}{\omega_c} \) is an integer, the resonance of cyclotron and Bloch frequencies delocalizes the electron motion, which is demonstrated by calculating the electron drift velocity and by analysis of electron trajectories with the help of the Poincarè section method. It is shown also that these resonances produce additional peaks in the drift velocity curves. We then use these results in Chapter 3 while studying the absorption of an additional high-frequency electric field.

In Chapter 3 applying the semiclassical approach we analyse the absorption
and amplification of electromagnetic waves in a superlattice. We calculate the absorption coefficient and find out its remarkable dependence on the shape of the drift velocity–field characteristics. We demonstrate that the absorption coefficient becomes negative, when the curve has a negative slope. This suggests that a specific design of superlattice and appropriate application of electric and magnetic fields can promote amplification of very high-frequency (up to few THz) signals. We also show the strength and the direction of magnetic field provides an additional control of the amplification.

In Chapter 4, we numerically and analytically study the Hall effect in a semiconductor superlattice within an extraordinary configuration, which has an electric field along the superlattice layer direction and magnetic field along z-direction. We calculate both the dependence of Hall field and Hall current upon the current applied along the direction perpendicular to the superlattice axis. In contrast to the ordinary Hall configuration, where the Hall field is just proportional to the applied electric current, we find that in our case this dependence demonstrates a striking nonlinear and non-monotonic character. For small current, the Hall field decreases linearly with the current applied. However, above some critical value of the current the slope of the dependence changes its sign. At the same time, the related slope of the current–voltage characteristic has always the same sign. We attribute such unusual behaviour to the onset of Bloch-like oscillations caused by the Hall field. We reveal and analyse these novel dynamics regimes in Chapter 5.

In Chapter 5, we analyse in detail the ballistic electron trajectories in the extra-ordinary Hall configuration and compare them with ones in the traditional
configuration. These trajectories determine electromagnetic emission on the time scale comparable with or lower than the scattering time (coherent Hall effect). Here we find not only early observed transition from magneto-Bloch to cyclotron-like regimes but also reveal that for novel Bloch-like oscillations the dependence of electron momentum on time is not linear anymore. We demonstrate that in the extraordinary configuration the transition between the Bloch-like and cyclotron-like regimes is sharp, and that its observation is quite feasible.

The novel phenomena discussed in Chapter 3, 4 and 5 can be used in a number of applications, including amplification of sub-THz and THz signals, generation of THz waves and pulses [103] and also in the development of magnetoresistive sensors [104]. However, further studies on effects of temperature, spatial charge redistributions and comprehensive modeling of electric contacts would be very helpful here. Besides, in order to calculate simple, we use a simple representation of scattering time in this thesis. Actually, it cannot present the motion of the charge carriers accurately in superlattices. We will consider the all kinds of scattering events when study the motion of electron in future work. We will try to find a more accurate calculation method to calculate the drift velocity of electron under the influence of the magnetic field.


[42] M. F. Joubert, S. Guy and B. Jacquier. Model of the photon-avalanche effect,


[63] T. Hyart, K. N. Alekseev, and E. V. Thuneberg, Bloch gain in dc-ac-driven


