A probabilistic representation of the ground state expectation of fractional powers of the boson number operator

A Probabilistic Representation of the Ground State Expectation of Fractional Powers of the Boson Number Operator

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Abstract

We give a formula in terms of a joint Gibbs measure on Brownian paths and the measure of a random-time Poisson process of the ground state expectations of fractional (in fact, any real) powers of the boson number operator in the Nelson model. We use this representation to obtain tight two-sided bounds. As applications, we discuss the polaron and translation invariant Nelson models.

1 Introduction

1.1 Fractional boson number operator

Functional integration is a powerful tool in studying spectral properties of self-adjoint operators. In particular, quantum field theory provides a large number of interesting problems which can be addressed by using functional integration, see [BHLMS02] and [LHB11, Chapter 6]. In a more recent development we have extended this method to operators originating from relativistic quantum theory such as the Schrödinger operator $\sqrt{-\Delta + m^2} - m + V$, $m \geq 0$, fractional Schrödinger operators of the type $(-\Delta)^{\alpha/2} + V$, $0 < \alpha < 2$, and more generally, Bernstein functions of the Laplacian [HIL09]. While in the usual case of the Laplacian functional integration is performed over the paths of Brownian motion, in these cases we have α-stable processes and more general subordinate Brownian motion.

Inspired by this development, in this paper we are concerned with fractional and general non-integer powers

$$\mathsf{N}^k \text{ for } k \geq 0 \quad \text{and} \quad (\mathsf{N} + 1\!\!1)^k \text{ for } k < 0 \tag{1.1}$$

of the boson number operator N from a probabilistic point of view. We consider the ground state φ_{g} in Nelson's scalar quantum field model, and propose a representation of the ground state expectations $(\varphi_{\mathrm{g}}, \mathsf{N}^k \varphi_{\mathrm{g}})$, $k \geq 0$, and $(\varphi_{\mathrm{g}}, (\mathsf{N} + 1\!\!1)^k \varphi_{\mathrm{g}})$, $k < 0$, in terms of a Gibbs measure $\mathscr{N}$ on path space developed in [BHLMS02] and a random-time Poisson process $(N_t)_{t \geq 0}$. Note that estimating $(\varphi_{\mathrm{g}}, \mathsf{N}\varphi_{\mathrm{g}})$ is the cornerstone in the study the ground state properties, containing fundamental information on the behaviour of a particle interacting with the field. By our method presented below we are able to obtain similar estimates for both arbitrary integer and non-integer powers of the ground state expectations of the number operator. However, we note that while our initial purpose was to derive estimates, in fact we obtain equalities between functionals of a Fock space operator and expectations of the joint Gibbs-Poisson measure (see Theorem 2.3), and we believe that this method also has an interest in more general mathematics than just the confines of the specific model.

Let g be the coupling constant in the sum of the free and the interaction terms in the Nelson model. It is known that

$$(\varphi_{\mathrm{g}}, \mathsf{N}\varphi_{\mathrm{g}}) = O(g^2) \tag{1.2}$$

for large g, see for details (1.14) below. This can be derived from the so called pull-through formula and extended to $(\varphi_{\mathrm{g}}, \mathsf{N}^n \varphi_{\mathrm{g}})$ for $n \in \mathbb{N}$ by a simple combinatorial argument. We extend this to all powers $k \in \mathbb{R}$, and show that

$$(\varphi_{\mathrm{g}}, \mathsf{N}^k \varphi_{\mathrm{g}}) = O(g^{2k}), \quad k \geq 1 \tag{1.3}$$

in the strong coupling limit.

A basic idea is to take $k = m + \frac{\alpha}{2} \in \mathbb{R}$ with $m \in \mathbb{Z}$ and $0 \le \alpha < 2$, and write

$$\mathsf{N}^k = \mathsf{N}^m \mathsf{N}^{\frac{\alpha}{2}} = \mathsf{N}^m \int_0^\infty (1 - e^{-\beta \mathsf{N}}) \lambda(d\beta) \tag{1.4}$$

with a suitable Borel (in fact, Lévy) measure λ. We have, furthermore, that

$$\mathsf{N}^m = (-1)^m \frac{d^m}{d\beta^m} e^{-\beta \mathsf{N}} \lceil_{\beta=0}, \quad m > 0, \tag{1.5}$$

and

$$\mathsf{N}^m = \int_{[0,\infty)^m} e^{-(\beta_1 + \cdots + \beta_m)\mathsf{N}} \prod_{j=1}^m d\beta_j, \quad m < 0. \tag{1.6}$$

Combining (1.4), (1.5) and (1.6) with the path integral representation of $(\varphi_\mathrm{g}, e^{-\beta \mathsf{N}} \varphi_\mathrm{g})$ given in terms of a Gibbs measure $\mathscr{N}$, we derive the probabilistic representation of $(\varphi_\mathrm{g}, \mathsf{N}^k \varphi_\mathrm{g})$ for all $k \in \mathbb{R}$. Using this expression we obtain the asymptotic behaviour

$$C_1 \le \lim_{g \to \infty} \frac{(\varphi_\mathrm{g}, \mathsf{N}^k \varphi_\mathrm{g})}{g^{2k}} \le C_2, \quad k \ge 1,$$

with some constants $C_1, C_2 > 0$, as a corollary.

In the remainder of this section we recall the definition and some basic facts on the Nelson model. In Section 2 we state the main theorem and its proof. In Section 3 we discuss an application to the polaron model, which is related to the Nelson model in a specific sense.

1.2 Nelson model

For background material we refer to [LHB11]. We consider the Nelson model of an electrically charged spinless quantum particle coupled to a scalar boson field. The particle in assumed to be under an external potential V, and its Hamilton operator is described by the Schrödinger operator

$$H_\mathrm{p} = -\frac{1}{2}\Delta + V. \tag{1.7}$$

We choose V such that H_p has a unique strictly positive ground state $\varphi_\mathrm{p} \in L^2(\mathbb{R}^d)$.

To describe the scalar quantum field consider the boson Fock space $\mathscr{F} = \mathbb{C} \oplus \oplus_{n=1}^\infty \otimes_\mathrm{sym}^n L^2(\mathbb{R}^d)$ over $L^2(\mathbb{R}^d)$, where the subscript indicates symmetrized tensor product. The free field Hamilton operator is given by

$$H_\mathrm{f} = \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk, \tag{1.8}$$

where $a(k)$ and $a^*(k)$ are the boson annihilation and creation operators, respectively, satisfying $[a(k), a^*(k')] = \delta(k - k')$, and

$$\omega(k) = \sqrt{|k|^2 + \nu^2}, \quad \nu \geq 0 \tag{1.9}$$

is the dispersion relation, with boson mass ν. Finally, the Hamilton operator of the interaction term between particle and field is

$$\phi(x) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} \left(\frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} e^{-ik\cdot x} \otimes a^*(k) + \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} e^{+ik\cdot x} \otimes a(k) \right) dk, \tag{1.10}$$

where $\hat{\varphi}$ is the Fourier transform of a given function φ describing the charge distribution of the particle. We assume throughout this paper that $\varphi(x) \geq 0$. The charge distribution φ regularizes the particle from a point charge and imposes an ultraviolet cutoff making the interaction well defined. The Nelson Hamiltonian

$$H = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + g\phi \tag{1.11}$$

is defined on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F}$, with coupling constant $g \in \mathbb{R}$.

Under the assumption that $\sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^d)$ and $\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$ it can be proven that H is a self-adjoint operator on $D(H_{\mathrm{p}}) \cap D(H_{\mathrm{f}})$. Denote

$$W_\infty = \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk < \infty. \tag{1.12}$$

It is well known that H has a unique strictly positive ground state $\varphi_{\mathrm{g}} \in L^2(\mathbb{R}^d) \otimes \mathscr{F}$ for every $g \in \mathbb{R}$ whenever $\nu > 0$ [Ara01, BFS98, Ger00, HHS05, Sas05, Spo98], and it has no ground state in this space if $\nu = 0$ [DG04, Hir06, LMS02] unless (1.12) holds, imposing an infrared condition. Throughout this paper we assume that $\nu > 0$.

Let

$$\mathsf{N} = \int_{\mathbb{R}^d} a^*(k) a(k) dk \tag{1.13}$$

be the boson number operator acting in $\mathscr{F}$. The ground state expectation of N is very useful in studying the spectrum of the Nelson model and it is given by

$$(\varphi_{\mathrm{g}}, \mathsf{N}\varphi_{\mathrm{g}}) = \frac{g^2}{2} \int_{\mathbb{R}^d} \left\| (H - E + \omega(k))^{-1} \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} e^{-ik\cdot x} \varphi_{\mathrm{g}} \right\|_{\mathscr{H}}^2 dk, \tag{1.14}$$

where E is the infimum of the spectrum of H. The right hand side is bounded by (1.12). This is derived from the so called pull-through formula

$$a(k)\varphi_{\mathrm{g}} = \frac{g}{\sqrt{2}} (H - E + \omega(k))^{-1} e^{-ik\cdot x} \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} \varphi_{\mathrm{g}}.$$

In this paper our main objective is to estimate $(\varphi_{\mathrm{g}}, \mathsf{N}^k \varphi_{\mathrm{g}})$, for all $k \in \mathbb{R}$.

2 Main result and proofs

2.1 Main theorem

Consider the class of completely monotone functions

$$\mathscr{B} = \left\{ f \in C^\infty(\mathbb{R}^+) \,\middle|\, f(x) \geq 0,\, (-1)^n \frac{d^n f}{dx^n}(x) \leq 0,\, \forall n \in \mathbb{N} \right\}.$$

An element of $\mathscr{B}$ is called a Bernstein function. Bernstein functions are positive, increasing and concave. An example is $\Psi(u) = cu^{\frac{\alpha}{2}}$, $c \geq 0$, $0 < \alpha \leq 2$.

Let $\mathscr{L}$ be the set of Borel measures λ on $\mathbb{R} \setminus \{0\}$ such that

(1) $\lambda((-\infty, 0)) = 0$

(2) $\displaystyle\int_{\mathbb{R}\setminus\{0\}} (y \wedge 1)\lambda(dy) < \infty.$

For every Bernstein function $\Psi \in \mathscr{B}$ there exists a unique pair $(b, \lambda) \in \mathbb{R}^+ \times \mathscr{L}$ such that

$$\Psi(u) = bu + \int_0^\infty (1 - e^{-uy})\lambda(dy). \tag{2.1}$$

Conversely, the right hand side of (2.1) is an element of $\mathscr{B}$ for each pair $(b, \lambda) \in \mathbb{R}^+ \times \mathscr{L}$.

Instead of considering just $(\varphi_{\rm g}, \mathsf{N}^k \varphi_{\rm g})$ we study the more general functionals

$$(\varphi_{\rm g}, \mathsf{N}^m \Psi(\mathsf{N})\varphi_{\rm g}), \quad m \in \mathbb{N} \cup \{0\}, \quad \Psi \in \mathscr{B}_0 \tag{2.2}$$

and

$$(\varphi_{\rm g}, (\mathsf{N} + \mathbb{1})^{-m} \Psi(\mathsf{N} + \mathbb{1})\varphi_{\rm g}), \quad m \in \mathbb{N}, \quad \Psi \in \mathscr{B}_0. \tag{2.3}$$

Note that (2.2) and (2.3) include the cases $(\varphi_{\rm g}, \mathsf{N}^k \varphi_{\rm g})$, $k \geq 0$, and $(\varphi_{\rm g}, (\mathsf{N}+\mathbb{1})^k \varphi_{\rm g})$, $k < 0$, obtained for $\Psi(u) = u^{\frac{\alpha}{2}}$, $0 \leq \alpha < 2$. Since $u^m \Psi(u) = bu^{m+1} + u^m \int_0^\infty (1-e^{-uy})\lambda(dy)$, we can disregard the constant (drift) part and it suffices to work with the subclass

$$\mathscr{B}_0 = \{ f \in \mathscr{B} \,|\, b = 0 \}$$

and derive a formula for $\Psi \in \mathscr{B}_0$.

Define

$$\rho(\beta) = (\varphi_{\rm g}, e^{-\beta \mathsf{N}} \varphi_{\rm g}), \quad \beta \in \mathbb{C}. \tag{2.4}$$

In [LHB11, Chapter 6] it has been established that $\rho(\beta)$ is analytic and it is represented in terms of a Gibbs measure $\mathscr{N}$ on the path space $\mathscr{X} = C(\mathbb{R}; \mathbb{R}^d)$ of continuous functions. We give an outline of this relationship for the convenience of the reader.

Proposition 2.1 *There exist a probability measure $\mathscr{N}$ and a diffusion process $(X_t)_{t\in\mathbb{R}}$ on $(\mathscr{X}, \mathscr{G}, \mathscr{N})$ such that*

$$\rho(\beta) = \mathbb{E}_{\mathscr{N}}\left[e^{-g^2(1-e^{-\beta})W}\right], \quad \beta \in \mathbb{C}, \tag{2.5}$$

where

$$W = W(\omega) = \int_{-\infty}^0 ds \int_0^\infty \mathscr{W}(X_t(\omega) - X_s(\omega), t-s)dt, \quad \omega \in \mathscr{X},$$

and $\mathscr{W}(x,t)$ is a pair interaction potential given by

$$\mathscr{W}(x,t) = \frac{1}{2}\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} e^{-ik\cdot x} e^{-|t|\omega(k)} dk.$$

Proof: For a detailed proof we refer to [LHB11, Chapter 6]. Recall that $\varphi_{\rm p} > 0$ is the ground state of $H_{\rm p}$, and define the unitary operator $U : L^2(\mathbb{R}^d, \varphi_{\rm p}^2 dx) \to L^2(\mathbb{R}^d)$ by $f \to \varphi_{\rm p} f$. Then there exists a diffusion process $(X_t)_{t\in\mathbb{R}}$ on $(\mathscr{X}, \mathscr{G}, \mathscr{N}_0)$, whose generator is the self-adjoint operator $U^{-1}H_{\rm p}U$. The Nelson Hamiltonian H is transformed by $U^{-1} \otimes \mathbb{1}$ to the self-adjoint operator

$$U^{-1}H_{\rm p}U \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + g\phi$$

on $L^2(\mathbb{R}^d; \varphi_{\rm p}^2 dx) \otimes \mathscr{F}$. We denote it by the same H. Since the ground state of H is strictly positive, $(\varphi_{\rm g}, \mathbb{1} \otimes \Omega_{\mathscr{F}}) > 0$, where $\Omega_{\mathscr{F}} = (1, 0, 0, ...)$ denotes the Fock vacuum. Thus we have in the $T \to \infty$ limit

$$\varphi_{\rm g}^T = (\|e^{-TH}(\mathbb{1} \otimes \Omega_{\mathscr{F}})\|)^{-1} e^{-TH}(\mathbb{1} \otimes \Omega_{\mathscr{F}}) \to \varphi_{\rm g}$$

in strong sense, and we obtain that $(\varphi_{\rm g}, e^{-\beta \mathsf{N}}\varphi_{\rm g}) = \lim_{T\to\infty}(\varphi_{\rm g}^T, e^{-\beta\mathsf{N}}\varphi_{\rm g}^T)$ for $\beta > 0$. On the other hand, the path integral representation

$$\begin{aligned}&(\varphi_{\rm g}^T, e^{-\beta\mathsf{N}}\varphi_{\rm g}^T)\\ &= \frac{1}{Z_T}\int e^{-g^2\int_{-T}^0\int_0^T W(X_t(\omega)-X_s(\omega),t-s)(1-e^{-\beta})} e^{\frac{g^2}{2}\int_{-T}^T\int_{-T}^T W(X_t(\omega)-X_s(\omega),t-s)} d\mathscr{N}_0(\omega)\end{aligned} \tag{2.6}$$

holds, where

$$Z_T = \int e^{\frac{g^2}{2}\int_{-T}^T\int_{-T}^T W(X_t(\omega)-X_s(\omega),t-s)} d\mathscr{N}_0(\omega)$$

is the normalizing factor. We can prove that the family of probability measures

$$d\mathscr{N}_T = \frac{1}{Z_T} e^{\frac{g^2}{2}\int_{-T}^T\int_{-T}^T W(X_t - X_s, t-s)} d\mathscr{N}_0, \quad T \geq 0,$$

is tight, and the integrand $e^{-g^2\int_{-T}^0\int_0^T W(X_t-X_s,t-s)(1-e^{-\beta})}$ in (2.6) is uniformly convergent with respect to the paths ω as $T\to\infty$. Hence there exits a subsequence $(T'_n)_{n\in\mathbb{N}}$ such that $\mathscr{N}_{T'_n}$ weakly converges to a probability measure $\mathscr{N}$, and (2.5) is satisfied for $\beta>0$. This can be further extended to all $\beta\in\mathbb{C}$ by analytic continuation. $\mathscr{N}$ is a Gibbs measure on path space for potential V contained in $H_{\rm p}$ and the pair interaction $\mathscr{W}$ above. $\square$

Lemma 2.2 *For every path $\omega\in\mathscr{X}$ we have $0\le W(\omega)\le W_\infty$.*

Proof: The upper bound is straightforward. By Fourier transformation

$$\begin{aligned}\mathscr{W}(x-y,t) &= (\varphi(\cdot-x),\omega(-i\nabla)^{-1}e^{-|t|\omega(-i\nabla)}\varphi(\cdot-y))\\ &= \int_{|t|}^\infty(\varphi(\cdot-x),e^{-s\omega(-i\nabla)}\varphi(\cdot-y))ds, \qquad (2.7)\end{aligned}$$

for every $x,y\in\mathbb{R}^d$ and $t\in\mathbb{R}$, and where ω in the equality above is the dispersion relation. Since $e^{-s\omega(-i\nabla)}$ is positivity improving and $\varphi\ge 0$ by assumption, we conclude that $\mathscr{W}(x-y,t)\ge 0$. $\square$

Let $(N_t)_{t\ge0}$ be a Poisson process with intensity 1 on a probability space $(\Omega,\mathcal{F},P)$. Recall that

$$\mathbb{E}_P[e^{-uN_t}]=e^{t(e^{-u}-1)},\quad t\ge0, \qquad (2.8)$$

and write

$$\mathbb{E}=\mathbb{E}_{\mathscr{N}}\mathbb{E}_P.$$

The main result of this paper is as follows.

Theorem 2.3 *Let $m\in\mathbb{Z}$ and $\Psi\in\mathscr{B}_0$. Then*

$$m\ge1 \implies (\varphi_{\rm g},\mathsf{N}^m\Psi(\mathsf{N})\varphi_{\rm g})=\sum_{r=1}^m S(m,r)\mathbb{E}\left[\left(g^2W\right)^r\Psi(N_{g^2W}+r)\right] \qquad (2.9)$$

$$m=0 \implies (\varphi_{\rm g},\Psi(\mathsf{N})\varphi_{\rm g})=\mathbb{E}\left[\Psi(N_{g^2W})\right] \qquad (2.10)$$

$$m\le-1 \implies (\varphi_{\rm g},(\mathsf{N}+\mathbb{1})^m\Psi(\mathsf{N}+\mathbb{1})\varphi_{\rm g})=\mathbb{E}\left[(N_{g^2W}+1)^m\Psi(N_{g^2W}+1)\right] \qquad (2.11)$$

where $S(m,r)$ are the Stirling numbers of the second kind, and N_{g^2W} is N_t evaluated at the random time $t=g^2W$.

In order to avoid singularities, in Theorem 2.3 we replaced N by $\mathsf{N}+\mathbb{1}$ for the case $m<0$, however, any positive multiple of $\mathbb{1}$ can be used. In particular, we have the following formulae.

Corollary 2.4 *Let $m \in \mathbb{Z}$ and $0 \leq \alpha < 2$. Then*

$$m \geq 1 \implies (\varphi_{\rm g}, \mathsf{N}^{m+\frac{\alpha}{2}} \varphi_{\rm g}) = \sum_{r=1}^{m} S(m,r) \mathbb{E}\left[\left(g^2 W\right)^r (N_{g^2 W} + r)^{\frac{\alpha}{2}} \right] \quad (2.12)$$

$$m = 0 \implies (\varphi_{\rm g}, \mathsf{N}^{\frac{\alpha}{2}} \varphi_{\rm g}) = \mathbb{E}\left[(N_{g^2 W})^{\frac{\alpha}{2}} \right] \quad (2.13)$$

$$m \leq -1 \implies (\varphi_{\rm g}, (\mathsf{N} + \mathbb{1})^{m+\frac{\alpha}{2}} \varphi_{\rm g}) = \mathbb{E}\left[(N_{g^2 W} + 1)^{m+\frac{\alpha}{2}} \right]. \quad (2.14)$$

Moreover, for purely natural powers of the number operator we obtain the following expression.

Corollary 2.5 *For $\alpha = 0$ and $m \in \mathbb{N}$ we have*

$$(\varphi_{\rm g}, \mathsf{N}^m \varphi_{\rm g}) = \mathbb{E}_{\mathscr{N}} \mathbb{E}_{P^\omega}[X^m], \quad (2.15)$$

where X is a Poisson random variable on a probability space with random intensity $g^2 W(\omega)$, independent of $W(\omega)$, and whose distribution we denote by P^ω.

Proof: From (2.12) we have

$$(\varphi_{\rm g}, \mathsf{N}^m \varphi_{\rm g}) = \sum_{r=1}^{m} S(m,r) \mathbb{E}\left[\left(g^2 W\right)^r \right] = \mathbb{E}_{\mathscr{N}} \left[\sum_{r=1}^{m} S(m,r) \left(g^2 W\right)^r \right].$$

Let X be a Poisson random variable with intensity μ and probability distribution P^μ on a suitable probability space, and recall the formula

$$\mathbb{E}_{P^\mu}[X^m] = \sum_{r=1}^{m} S(m,r) \mu^r, \quad m \in \mathbb{N}.$$

Then the claim follows for $\mu = g^2 W(\omega)$ and $P^\mu = P^\omega$. □

2.2 Proof of Theorem 2.3

We first prove the case $m \geq 0$.

Proof of (2.9) and (2.10): It suffices to consider only $m \geq 1$. There exists a Lévy measure λ such that

$$\Psi(u) = \int_0^\infty (1 - e^{-uy}) \lambda(dy), \quad (2.16)$$

and

$$\mathsf{N}^m = (-1)^m \frac{d^m}{d\beta^m} e^{-\beta \mathsf{N}} \lceil_{\beta=0}. \quad (2.17)$$

By Proposition 2.1 and the combination of (2.16) and (2.17) we have

$$(\varphi_{\mathrm{g}}, \mathsf{N}^m \Psi(\mathsf{N})\varphi_{\mathrm{g}}) = (-1)^m \int_0^\infty (\partial^m \rho(0) - \partial^m \rho(\beta))\lambda(d\beta).$$

A direct calculation gives

$$\partial^m \rho(\beta) = (-1)^m \sum_{r=1}^m a_r(m) e^{-r\beta} \mathbb{E}_{\mathscr{N}} \left[\left(g^2 W\right)^r e^{-g^2 W(1-e^{-\beta})} \right],$$

where

$$a_r(m) = \frac{(-1)^r}{r!} \sum_{s=1}^r (-1)^s \binom{r}{s} s^m. \tag{2.18}$$

It is well-known [Lov07] that the sum in (2.18) can be expressed in terms of the Stirling numbers of the second kind $S(m,r)$, and thus

$$a_r(m) = \begin{cases} 0 & \text{if } \ 0 \leq m < r \\ S(m,r) & \text{if } \ m \geq r. \end{cases}$$

Note that $S(m,m) = 1$, for all $m \in \mathbb{N}$. This yields

$$(\varphi_{\mathrm{g}}, \mathsf{N}^m \Psi(\mathsf{N})\varphi_{\mathrm{g}}) = \mathbb{E}_{\mathscr{N}} \left[\int_0^\infty \sum_{r=1}^m S(m,r) \left(g^2 W\right)^r (1 - e^{-r\beta} e^{-g^2 W(1-e^{-\beta})})\lambda(d\beta) \right].$$

Using (2.8) gives

$$1 - e^{-r\beta} e^{-g^2 W(1-e^{-\beta})} = \mathbb{E}_P \left[1 - e^{-\beta(N_{g^2 W} + r)} \right].$$

Hence

$$(\varphi_{\mathrm{g}}, \mathsf{N}^m \Psi(\mathsf{N})\varphi_{\mathrm{g}}) = \mathbb{E}_{\mathscr{N}} \left[\sum_{r=1}^m S(m,r) \left(g^2 W\right)^r \mathbb{E}_P \left[\int_0^\infty (1 - e^{-\beta(N_{g^2 W} + r)})\lambda(d\beta) \right] \right],$$

and the theorem follows. ◻

Next we consider the case $m < 0$. The strategy is similar to the case of positive powers.

Proof of (2.11): By a combination of the Laplace transform

$$\int_{[0,\infty)^m} \prod_{j=1}^m d\beta_j e^{-\sum_{j=1}^m \beta_j (\mathsf{N}+\mathbb{1})} = (\mathsf{N} + \mathbb{1})^{-m} \tag{2.19}$$

and

$$\Psi(\mathsf{N}+\mathbb{1})=\int_0^\infty(1-e^{-\beta(\mathsf{N}+\mathbb{1})})\lambda(d\beta), \tag{2.20}$$

we obtain

$$\begin{aligned}&(\varphi_{\rm g},(\mathsf{N}+\mathbb{1})^{-m}\Psi(\mathsf{N}+\mathbb{1})\varphi_{\rm g})\\ &\quad=\int_{[0,\infty)^m}\prod_{j=1}^m d\beta_j\int_0^\infty\lambda(d\beta)(\varphi_{\rm g},(e^{-\sum_{j=1}^m\beta_j(\mathsf{N}+\mathbb{1})}-e^{-(\sum_{j=1}^m\beta_j+\beta)(\mathsf{N}+\mathbb{1})})\varphi_{\rm g}).\end{aligned}$$

By (2.21) it follows that

$$\begin{aligned}&(\varphi_{\rm g},(\mathsf{N}+\mathbb{1})^{-m}\Psi(\mathsf{N}+\mathbb{1})\varphi_{\rm g})\\ &=\int_{[0,\infty)^m}\prod_{j=1}^m d\beta_j\int_0^\infty\lambda(d\beta)e^{-\sum_{j=1}^m\beta_j}\mathbb{E}_{\mathscr{N}}\left[e^{-g^2W(1-e^{-\sum_{j=1}^m\beta_j})}-e^{-g^2W(1-e^{-\sum_{j=1}^m\beta_j+\beta})}e^{-\beta}\right].\end{aligned}$$

In terms of the Poisson process $(N_t)_{t\geq 0}$ we can rewrite as

$$\begin{aligned}&=\int_{[0,\infty)^m}\prod_{j=1}^m d\beta_j\int_0^\infty\lambda(d\beta)\mathbb{E}_{\mathscr{N}}\mathbb{E}_P\left[e^{-\sum_{j=1}^m\beta_j(N_{g^2W}+1)}-e^{-(\sum_{j=1}^m\beta_j+\beta)(N_{g^2W}+1)}\right]\\ &=\int_{[0,\infty)^m}\prod_{j=1}^m d\beta_j\int_0^\infty\lambda(d\beta)\mathbb{E}_{\mathscr{N}}\mathbb{E}_P\left[e^{-\sum_{j=1}^m\beta_j(N_{g^2W}+1)}(1-e^{-\beta(N_{g^2W}+1)})\right].\end{aligned}$$

Integrating with respect to $\lambda(d\beta)$ and then $\prod_j d\beta_j$, we finally obtain

$$\begin{aligned}&=\int_{[0,\infty)^m}\prod_{j=1}^m d\beta_j\mathbb{E}_{\mathscr{N}}\mathbb{E}_P\left[e^{-\sum_{j=1}^m\beta_j(N_{g^2W}+1)}\Psi(N_{g^2W}+1)\right]\\ &=\mathbb{E}\left[(N_{g^2W}+1)^{-m}\Psi(N_{g^2W}+1)\right].\end{aligned}$$

Hence the theorem follows. □

Remark 2.6 Since the coefficient $a_r(m)=1$ for $1=r=m$, we note that formula (1.14) can actually also be obtained from $(\varphi_{\rm g},\mathsf{N}\varphi_{\rm g})=\mathbb{E}[g^2W]$. We have

$$\mathbb{E}[g^2W]=\frac{g^2}{2}\int_{\mathbb{R}^d}dk\frac{|\hat{\varphi}(k)|^2}{\omega(k)}\int_{-\infty}^0dt\int_0^\infty e^{-|t-s|\omega(k)}\mathbb{E}[e^{-ik\cdot(X_s-X_t)}]ds. \tag{2.21}$$

Furthermore,

$$\mathbb{E}[e^{-ik\cdot(X_s-X_t)}]=\mathbb{E}_{\mathscr{N}}[e^{-ik\cdot(X_s-X_t)}]=(e^{-ik\cdot x}\varphi_{\rm g},e^{-|t-s|(H-E)}e^{-ik\cdot x}\varphi_{\rm g}).$$

Inserting this into (2.21) gives (1.14).

2.3 Lower and upper bounds

The above theorem allows to derive the asymptotic behaviour of the ground state expectation of non-integer powers of the boson number operator in the strong coupling limit.

Corollary 2.7 *Let $k = m + \frac{\alpha}{2} \geq 1$, $m \in \mathbb{N}$ and $0 \leq \alpha < 2$. Suppose that there exists $a > 0$ not depending on g, and $\mathbb{E}[W] \geq W_\infty - a > 0$. Then*

$$(W_\infty - a)^k \leq \lim_{g\to\infty} \frac{(\varphi_{\rm g}, \mathsf{N}^k \varphi_{\rm g})}{g^{2k}} \leq W_\infty^k. \tag{2.22}$$

Proof: By Jensen's inequality we have

$$\mathbb{E}_P \left[(N_{g^2W} + r)^{\frac{\alpha}{2}} \right] \leq (\mathbb{E}_P[N_{g^2W} + r])^{\frac{\alpha}{2}} = (g^2 W_\infty + r)^{\frac{\alpha}{2}}.$$

In particular, it follows that

$$(\varphi_{\rm g}, \mathsf{N}^{m+\frac{\alpha}{2}} \varphi_{\rm g}) \leq \sum_{r=1}^{m} S(m,r)(g^2 W_\infty + r)^{\frac{\alpha}{2}} (g^2 W_\infty)^r,$$

implying $\lim_{g\to\infty} (\varphi_{\rm g}, \mathsf{N}^{m+\frac{\alpha}{2}} \varphi_{\rm g})/g^{2m+\alpha} \leq S(m,m) W_\infty^k = W_\infty^k$. This gives the upper bound. Again, from Jensen's inequality it follows that

$$(\varphi_{\rm g}, \mathsf{N}^k \varphi_{\rm g}) \geq (\varphi_{\rm g}, \mathsf{N}\varphi_{\rm g})^k,$$

Since $(\varphi_{\rm g}, \mathsf{N}\varphi_{\rm g}) = g^2 \mathbb{E}_{\mathscr{N}}[W]$, the lower bound follows.

Remark 2.8 Since any Bernstein function Ψ is increasing and concave, by the Jensen inequality and Lemma 2.2 we obtain more generally that

$$\begin{aligned} (\varphi_{\rm g}, \Psi(\mathsf{N})\varphi_{\rm g}) &= \int \mathbb{E}_P[\Psi(N_{g^2W})] d\mathscr{N} \leq \int \Psi(\mathbb{E}_P[N_{g^2W}]) d\mathscr{N} \\ &= \int \Psi(g^2 W) d\mathscr{N} \leq \Psi(g^2 W_\infty). \end{aligned}$$

Remark 2.9 The assumption stated in Corollary 2.7 is fairly standard. Let $V = V_1 - V_2$, where $V_2 \geq 0$, $V_2 \in L^p(\mathbb{R}^d)$, $p = 1$ for $d = 1$, $p > d/2$ for $d \geq 2$, and $V_1 \in L^1_{\rm loc}(\mathbb{R}^d)$, $\inf_x V_1(x) > -\infty$. Denote $\Sigma = \liminf_x V(x)$, and assume

(1) $V_1(x) \to \infty$ as $|x| \to \infty$,

(2) $\Sigma > \max\{E, \inf_x V_1(x)\}$.

Under either (1) or (2) the ground state $\|\varphi_{\rm g}(x)\|$ decays exponentially, moreover

$$\||x|\varphi_{\rm g}(x)\|_{\mathscr{F}} \le C\|\varphi_{\rm g}\|_{\mathscr{H}}$$

with a suitable constant $C > 0$ independent of g (for details see [LHB11, Chapter 6]). Let $\sup_x \||x|\varphi_{\rm g}(x)\|_{\mathscr{F}} < C\|\varphi_{\rm g}\| < \infty$. Then it is also shown in the same reference that

$$\mathbb{E}[W] \ge \frac{1}{2}\int_{\mathbb{R}^d} \frac{\hat{\varphi}(k)^2}{\omega(k)^3}(1 - C|k|^2)dk. \tag{2.23}$$

(Remember that $\|\varphi_{\rm g}\| = 1$.) If the field mass $\nu > 0$ is sufficiently small, $d \le 3$ and $\hat{\varphi}$ is continuous in a neighborhood of $k = 0$, then the right hand side of (2.23) is strictly positive.

3 Applications

3.1 Polaron model

A related model is the polaron model obtained for dispersion relation $\omega(k) = 1$. The same result as in the Nelson model can be obtained by using a similar argument.

The Hamiltonian of the polaron is defined by

$$H_{\rm pol} = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{N} + g\phi, \tag{3.1}$$

where

$$\phi(x) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^d} \left(\frac{\hat{\varphi}(k)}{|k|} e^{-ik\cdot x} \otimes a^*(k) + \frac{\hat{\varphi}(-k)}{|k|} e^{+ik\cdot x} \otimes a(k) \right) dk. \tag{3.2}$$

If $\hat{\varphi}/|k| \in L^2(\mathbb{R}^d)$, then $H_{\rm pol}$ is self-adjoint on $D(H_{\rm p}) \cap D(\mathsf{N})$. We assume $\varphi(x) \ge 0$, and choose V such that $H_{\rm p}$ has a strictly positive ground state $\varphi_{\rm p}$. Also, we assume that

$$W_{\rm pol}^{\infty} = \frac{1}{2}\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{|k|^2} dk < \infty. \tag{3.3}$$

Note that whenever $d \ge 3$, there is no infrared divergence. Then under (3.3) $H_{\rm pol}$ has a unique ground state $\varphi_{\rm g} > 0$ for every $g \in \mathbb{R}$. In a similar way to Proposition 2.1 we obtain that there exist a probability measure $\mathscr{N}$ and a diffusion process $(X_t)_{t\in\mathbb{R}}$ on the probability space $(\mathscr{X}, \mathscr{G}, \mathscr{N})$ such that

$$(\varphi_{\rm g}, e^{-\beta \mathsf{N}}\varphi_{\rm g}) = \mathbb{E}_{\mathscr{N}}\left[e^{-g^2(1-e^{-\beta})W} \right], \quad \beta \in \mathbb{C}, \tag{3.4}$$

where

$$W_{\rm pol}(\omega) = \int_{-\infty}^0 ds \int_0^{\infty} \mathscr{W}_{\rm pol}(X_t(\omega) - X_s(\omega), t - s)dt$$

with pair interaction potential

$$\mathscr{W}_{\rm pol}(x,t) = \frac{1}{2} e^{-|t|} \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{|k|^2} e^{-ik\cdot x} dk.$$

In particular, for $\hat{\varphi}(k) = (2\pi)^{3/2}$ and $d = 3$ we have

$$W_{\rm pol}(\omega) = \frac{1}{8\pi} \int_{-\infty}^0 ds \int_0^\infty \frac{e^{-|t-s|}}{|X_t(\omega) - X_s(\omega)|} dt. \tag{3.5}$$

Lemma 3.1 $0 \le W_{\rm pol}(\omega) \le W_{\rm pol}^\infty$.

Proof: The upper bound is again straightforward. Fourier transformation gives

$$\mathscr{W}_{\rm pol}(x-y,t) = (\varphi(\cdot - x), \Delta^{-1} \varphi(\cdot - y)) e^{-|t|}.$$

Since $\Delta^{-1} = \int_0^\infty e^{\beta\Delta} d\beta$ and $e^{\beta\Delta}$ is positivity preserving for $\beta \ge 0$, the lemma follows as φ is positive by assumption. $\square$

Hence we have a similar result to Theorem 2.3.

Corollary 3.2 *Let $m \in \mathbb{Z}$ and $\Psi \in \mathscr{B}_0$. Then*

$$\begin{aligned}
m \ge 1 \quad &\Longrightarrow \quad (\varphi_{\rm g}, \mathsf{N}^m \Psi(\mathsf{N}) \varphi_{\rm g}) = \sum_{r=1}^m S(m,r) \mathbb{E}\left[\left(g^2 W_{\rm pol}\right)^r \Psi(N_{g^2 W_{\rm pol}} + r) \right] \\
m = 0 \quad &\Longrightarrow \quad (\varphi_{\rm g}, \Psi(\mathsf{N}) \varphi_{\rm g}) = \mathbb{E}\left[\Psi(N_{g^2 W_{\rm pol}}) \right] \\
m \le -1 \quad &\Longrightarrow \quad (\varphi_{\rm g}, (\mathsf{N} + \mathbb{1})^m \Psi(\mathsf{N} + \mathbb{1}) \varphi_{\rm g}) = \mathbb{E}\left[(N_{g^2 W_{\rm pol}} + 1)^m \Psi(N_{g^2 W_{\rm pol}} + 1) \right].
\end{aligned}$$

3.2 Nelson model with zero total momentum

Finally we consider the Nelson model H with zero external potential $V = 0$. In this case H is translation invariant and can be decomposed in terms of the spectrum of the total momentum

$$-i\nabla \otimes \mathbb{1} + \mathbb{1} \otimes P_{\rm f},$$

where $P_{\rm f} = \int_{\mathbb{R}^d} k a^*(k) a(k) dk$ is the field momentum operator. Thus the decomposition

$$H = \int_{\mathbb{R}^d}^\oplus H(P) dP, \quad \mathscr{H} = \int_{\mathbb{R}^d}^\oplus \mathscr{H}(P) dP \tag{3.6}$$

holds, where

$$H(P) = \frac{1}{2}(P - P_{\rm f})^2 + g\phi(0) + H_{\rm f} \tag{3.7}$$

and $\mathscr{H}(P) = \mathscr{F}$. As in the case of the Nelson model we assume that $\omega(k) = \sqrt{|k|^2 + \nu^2}$ with $\nu > 0$, and $\varphi(x) \geq 0$. It follows that

$$(F, e^{-tH(P)}G) = \mathbb{E}_{\mathcal{W}^0}\left[(F_0, e^{-g\phi_E(\int_0^t \tilde{\varphi}(\cdot - B_s)ds)} e^{i(P-P_{\rm f})\cdot B_t} G_t)_{\mathscr{F}}\right]. \tag{3.8}$$

Here $(B_t)_{t\in\mathbb{R}}$ is two-sided Brownian motion on $\mathbb{R}$, and $\mathcal{W}^0$ is Wiener measure starting from zero on $(\mathscr{X}, \mathcal{F})$, where $\mathscr{X} = C(\mathbb{R}; \mathbb{R}^d)$. Furthermore, $\tilde{\varphi}$ is the Fourier transform of $\hat{\varphi}/\sqrt{\omega}$, ϕ_E is the Euclidean scalar field, and F_0 and G_t are the projections at time zero and time t of the Euclidean scalar field of F and G, respectively. For further details we refer to [Hir07]. Since $e^{-iP_{\rm f}\cdot B_T}\Omega_{\mathscr{F}} = \Omega_{\mathscr{F}}$ for all $T \in \mathbb{R}$, taking $F = G = \Omega_{\mathscr{F}}$ we obtain that

$$(\Omega_{\mathscr{F}}, e^{-TH(P)}\Omega_{\mathscr{F}}) = \mathbb{E}_{\mathcal{W}^0}\left[e^{iP\cdot B_T} e^{g^2 \int_0^T ds \int_0^T dt \mathscr{W}(B_t - B_s, t-s)}\right]. \tag{3.9}$$

Now put $P = 0$. Then the phase $e^{iP\cdot B_T}$ in (3.8) becomes 1, and we see that $e^{-tH(0)}$ is positivity improving since $e^{-iP\cdot P_{\rm f}}$ is a shift operator. Thus the ground state of $H(0)$ is unique and strictly positive; we denote it by $\varphi_{\rm g}(0)$. We have then that

$$(\varphi_{\rm g}(0), e^{-\beta \mathsf{N}}\varphi_{\rm g}(0)) = \lim_{T\to\infty}(\varphi_{\rm g}^T(0), e^{-\beta \mathsf{N}}\varphi_{\rm g}^T(0))$$

for $\beta > 0$, where $\varphi_{\rm g}^T(0) = e^{-TH(0)}\Omega_{\mathscr{F}}/\|e^{-TH(0)}\Omega_{\mathscr{F}}\|$. Its path integral representation is

$$(\varphi_{\rm g}^T(0), e^{-\beta \mathsf{N}}\varphi_{\rm g}^T(0)) = \mathbb{E}_{\mu_T}\left[e^{-g^2(1-e^{-\beta})W_T}\right], \tag{3.10}$$

where

$$W_T = W_T(\omega) = \int_{-T}^0 dt \int_0^T \mathscr{W}(B_t(\omega) - B_s(\omega), t-s) ds \tag{3.11}$$

and

$$d\mu_T = \frac{1}{Z_T}\exp\left(\frac{g^2}{2}\int_{-T}^T dt \int_{-T}^T \mathscr{W}(B_t - B_s, t-s)ds\right) d\mathcal{W}^0. \tag{3.12}$$

Denote $\mathcal{F}_{[-T,T]} = \sigma(B_s, -T \leq s \leq T)$ for $T > 0$.

Proposition 3.3 *There exists a probability measure μ on $(\mathscr{X}, \mathcal{F})$ such that for every $\mathscr{F}_{[-T,T]}$-measurable function f, $T > 0$,*

$$\lim_{T\to\infty}\mathbb{E}_{\mu^T}[f] = \mathbb{E}_\mu[f]. \tag{3.13}$$

Proof: See [BS05] and [LHB11, Theorem 4.36]. □

Theorem 3.4 *Let $\beta \in \mathbb{C}$. Then*

$$(\varphi_{\mathrm{g}}(0), e^{-\beta \mathsf{N}} \varphi_{\mathrm{g}}(0)) = \mathbb{E}_\mu \left[e^{-g^2(1-e^{-\beta})W} \right]. \tag{3.14}$$

Proof: Suppose $\beta > 0$ and write $S_T = -g^2(1-e^{-\beta}) \int_{-T}^0 dt \int_0^T \mathscr{W}(B_t - B_s, t-s) ds$. Note again that $\lim_{T\to\infty} S_T = S_\infty$, uniformly in paths. For every $\varepsilon >$ there exists $T_0 > 0$ such that $|e^{S_T} - e^{S_\infty}| < \varepsilon$ for all $T > T_0$, uniformly in paths. Fix ε and T_0, and choose $t > T > T_0$. By telescoping

$$\begin{aligned} &\mathbb{E}_\mu[e^{S_\infty}] - \mathbb{E}_{\mu_T}[e^{S_T}] \\ &\quad = (\mathbb{E}_\mu[e^{S_\infty}] - \mathbb{E}_\mu[e^{S_t}]) + (\mathbb{E}_\mu[e^{S_t}] - \mathbb{E}_{\mu_T}[e^{S_t}]) + (\mathbb{E}_{\mu_T}[e^{S_t}] - \mathbb{E}_{\mu_T}[e^{S_T}]), \end{aligned}$$

we obtain

$$|\mathbb{E}_\mu[e^{S_\infty}] - \mathbb{E}_{\mu_T}[e^{S_T}]| \le \varepsilon + |\mathbb{E}_\mu[e^{S_t}] - \mathbb{E}_{\mu_T}[e^{S_t}]| + 2\varepsilon. \tag{3.15}$$

By Proposition 3.3, the second term at the right hand side of (3.15) converges to zero as $t \to \infty$. Thus the theorem follows for $\beta > 0$. By analytic continuation the theorem follows for all $\beta \in \mathbb{C}$. $\square$

Write $\mathbb{E} = \mathbb{E}_{\mathcal{W}^0}\mathbb{E}_P$. The next corollary is immediate.

Corollary 3.5 *Let $m \in \mathbb{Z}$ and $\Psi \in \mathscr{B}_0$. Then*

$$\begin{aligned} m \ge 1 \quad &\Longrightarrow \quad (\varphi_{\mathrm{g}}(0), \mathsf{N}^m \Psi(\mathsf{N}) \varphi_{\mathrm{g}}(0)) = \sum_{r=1}^m S(m,r) \mathbb{E}\left[\left(g^2 W\right)^r \Psi(N_{g^2W} + r)\right] \\ m = 0 \quad &\Longrightarrow \quad (\varphi_{\mathrm{g}}(0), \Psi(\mathsf{N}) \varphi_{\mathrm{g}}(0)) = \mathbb{E}\left[\Psi(N_{g^2W})\right] \\ m \le -1 \quad &\Longrightarrow \quad (\varphi_{\mathrm{g}}(0), (\mathsf{N}+\mathbb{1})^m \Psi(\mathsf{N}+\mathbb{1}) \varphi_{\mathrm{g}}(0)) = \mathbb{E}\left[(N_{g^2W}+1)^m \Psi(N_{g^2W}+1)\right]. \end{aligned}$$

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