Argument shift method and Manakov operators: applications to differential geometry

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Alexey Bolsinov
Loughborough University, UK

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What is it about?

Review on joint papers with V.Matveev, V.Kiosak, S.Rosemann, D.Tsonev and A.Konyaev

Around the following observation:

The curvature tensors of some interesting Riemannian metrics
coincide with
the Hamiltonians of multi-dimensional rigid bodies

Applications (for indefinite metrics):

- Obstructions to the existence of a projectively equivalent partner
- Pseudo-Riemannian analog of the Fubini theorem
- New class of holonomy groups
- New class of symmetric spaces
- Yano-Obata conjecture
- Local description of Bochner-flat Kähler metrics

Pre-history

Let $\mathfrak{g}$ be a semisimple Lie algebra, $R : \mathfrak{g}^* \simeq \mathfrak{g} \to \mathfrak{g}$ a symmetric linear operator.
Euler equations on $\mathfrak{g}^*$

$$\frac{dx}{dt} = [x, R(x)] \tag{1}$$

are Hamiltonian with $H = \frac{1}{2}\langle R(x), x\rangle$.
For which R, are the equations (1) integrable?

Definition

$R : \mathrm{so}(n) \to \mathrm{so}(n)$ is called a Manakov operator (with parameters A and B), if

$$[R(X), A] = [X, B] \quad \text{for all } X \in \mathrm{so}(g) \tag{2}$$

where A and B are some fixed symmetric matrices.

Theorem (Manakov, Mischenko, Fomenko)

Let R satisfy (2). Then

- (1) *can be rewritten as* $\frac{d}{dt}(X + \lambda A) = [X + \lambda A, R(X) + \lambda B]$;
- $\mathrm{Tr}(X + \lambda A)^k$ *are commuting first integrals of* (1);
- *if A is regular, then* (1) *are completely integrable.*

Properties of Manakov operators

1. A and B commute, moreover, B belongs to the centre of centraliser of A. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.
2. $R_0 = \frac{d}{dt}\Big|_{t=0} p(A + tX)$ satisfies (2). If A is regular, then R is unique, otherwise $R = R_0 + D$ where $D : \mathrm{so}(g) \to \mathfrak{g}_A = \{Y \in \mathrm{so}(g),\ AY = YA\}$ is arbitrary.
3. if $B = 0 = p_{\min}(A)$, then $R_0 = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX)$ still defines a non-trivial Manakov operator whose image is contained in $\mathfrak{g}_A$. Moreover, if for each eigenvalues of A there are at most 2 Jordan blocks, then the image R_0 coincides with $\mathfrak{g}_A$.
4. R_0 satisfies the Bianchi identity.
5. If in addition $p(A) = 0$, then R_0 satisfies the second Bianchi identity $[R_0(X), R_0(Y)] = R_0[R_0(X), Y]$.
6. Let R satisfy two identities $[R(X), A] = [X, B]$ and $[R(X), A'] = [X, B']$, where $A' \neq a\,A + b \cdot \mathrm{id}$. Then $R(X) = k \cdot X \mod \mathfrak{g}_A$. In particular, if A is regular, then $R = k \cdot \mathrm{id}$.
7. Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of A. Then $\frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j}$ are eigenvalues of R. Moreover, if A has a nontrivial Jordan λ_i-block, then $p'(\lambda_i)$ is an eigenvalue of R.

Riemann curvature tensor (quick reminder and "new" point of view)

Let ∇ be the Levi-Civita connection of a pseudo-Riemannian metric g.

Definition

The Riemann curvature tensor $R = (R^l_{ij\ k})$ is defined by (formula from a text-book):

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In other words, R can be understood as a map

$$R : (X,Y) \mapsto R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(TM).$$

Algebraic symmetries:

- $R(X,Y) = -R(X,Y)$, i.e., $R : \Lambda^2 V \to \mathrm{gl}(V)$, $V = T_x M$;
- $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$, i.e. $R(X,Y) \in \mathrm{so}(g)$;
- $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (Bianchi identity);
- $g(R(X,Y)Z,W) = -g(R(Z,W)X,Y)$.

Conclusion: $R : \mathrm{so}(g) \to \mathrm{so}(g)$ which is symmetric and satisfying Bianchi.

Easy observations:

- constant curvature $\Leftrightarrow$ $R = \mathrm{const} \cdot \mathrm{id}$
- Weyl tensor vanishes $\Leftrightarrow$ $R(X) = AX + XA$
  (cf., in rigid body dynamics: $M(\Omega) = J\Omega + \Omega J$)

Definition

g and $\bar{g}$ are projectively equivalent if they have the same (unparametrised) geodesics. Notation: $g \underset{\text{proj}}{\simeq} \bar{g}$.

Main equation: Let $A = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g$. Then $g \underset{\text{proj}}{\simeq} \bar{g}$ if and only if

$$\nabla_u A = \tfrac{1}{2}\big(u \otimes d\,\mathrm{tr}\,A + (u \otimes d\,\mathrm{tr}\,A)^*\big).$$

Theorem (B., Matveev)

Let $g \underset{\text{proj}}{\simeq} \bar{g}$. Then the Riemann curvature tensor of g is a Manakov operator:

$$[R(X), A] = [B, X] \quad \textit{for all } X \in \mathrm{so}(g), \textit{ where } B = \tfrac{1}{2}\nabla\big(\mathrm{grad}\ \mathrm{tr}\,A\big).$$

Proof.

Consider the compatibility condition for the main equation. □

Theorem (B., Matveev, Kiosak)

Let g, $\bar{g}$ and $\hat{g}$ be projectively equivalent. Assume that these metrics are linearly independent and g and $\hat{g}$ are strictly non-proportional, then g, $\bar{g}$ and $\hat{g}$ are metrics of constant sectional curvature.

Proof.

Apply Property 6. □

Definition

Let M be a smooth manifold endowed with an affine symmetric connection ∇. The holonomy group of ∇ is a subgroup $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(T_xM)$ that consists of the linear operators $A : T_xM \to T_xM$ being 'parallel transport transformations' along closed loops $\gamma(t)$ with $\gamma(0) = \gamma(1) = x$.

Problem. Given a subgroup $H \subset GL(n, \mathbb{R})$, can it be realised as the holonomy group for an appropriate symmetric connection on M^n?

Riemannian case and irreducible case: the problem is completely solved (Marcel Berger, D. V. Alekseevskii, R. Bryant, D. Joyce, L. Schwahhöfer, S. Merkulov).

Pseudo-Riemannian case: many fundamental results but still open (L. Bérard Bergery, A. Ikemakhen, C. Boubel, D. V. Alekseevskii, T. Leistner, A. Galaev).

Theorem (B., Tsonev)

For every g-symmetric operator $A : V \to V$, its centraliser in $\mathrm{SO}(g)$ (the identity connected component of)

$$G_A = \{Y \in \mathrm{SO}(g) \mid YA = AY\}$$

is a holonomy group for a certain (pseudo)-Riemannian metric.

Definition

A map $R : \Lambda^2 V \to \mathrm{gl}(V)$ is called a *formal curvature tensor* if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \qquad \text{for all } u, v, w \in V.$$

Definition

Let $\mathfrak{h} \subset \mathrm{gl}(V)$ be a Lie subalgebra. Consider the set of all formal curvature tensors $R : \Lambda^2 V \to \mathrm{gl}(V)$ such that $\mathrm{Im}\, R \subset \mathfrak{h}$:

$$\mathcal{R}(\mathfrak{h}) = \{R : \Lambda^2 V \to \mathfrak{h} \mid R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0,\ u, v, w \in V\}.$$

We say that $\mathfrak{h}$ is a Berger algebra if it is generated as a vector space by the images of the formal curvature tensors $R \in \mathcal{R}(\mathfrak{h})$, i.e.,

$$\mathfrak{h} = \mathrm{span}\{R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}),\ u, v \in V\}.$$

Berger test:

Let ∇ be a symmetric affine connection on TM. Then the Lie algebra $\mathfrak{hol}(\nabla)$ of its holonomy group $\mathrm{Hol}(\nabla)$ is Berger.

Definition

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$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \qquad \text{for all } u, v, w \in V,$$

where $u \wedge v = u \otimes g(v) - v \otimes g(u) \in \mathrm{so}(g)$.

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Berger test:

Let ∇ be a Levi-Civita connection on (M, g). Then the Lie algebra $\mathfrak{hol}(\nabla) \subset \mathrm{so}(g)$ of its holonomy group $\mathrm{Hol}(\nabla)$ is Berger.

Step one: Berger test for $\mathfrak{g}_A$ and Magic Formula 1

We have

$$\mathfrak{g}_A = \{X \in \mathrm{so}(g) \mid XA = AX\}$$

and we need to construct formal curvature tensors $R : \mathrm{so}(g) \to \mathrm{so}(g)$ whose images generate $\mathfrak{g}_A$.
Ideally, we want one single formal curvature tensor R such that $\mathrm{Im}\, R = \mathfrak{g}_A$.

Question: How to find R?

Answer: Apply Properties 3 and 4, i.e. define a linear mapping $R : \mathrm{so}(g) \to \mathrm{so}(g)$ by:

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX), \qquad (3)$$

where $p_{\min}(\lambda)$ is the minimal polynomial of A.

Conclusion: $\mathfrak{g}_A$ is Berger algebra.

Step two: Realisation and Magic Formula 2

We need to find an example of g such that $\mathfrak{hol}(\nabla) = \mathfrak{g}_A$. The idea is natural:

- set $A(x) = \text{const}$
- try to find the desired metric $g(x)$ in the form constant + quadratic:

$$g_{ij}(x) = g^0_{ij} + \sum \mathcal{B}_{ij,pq} x^p x^q. \qquad (4)$$

Question: How to find $\mathcal{B}$?
It is more convenient to work with "operators" rather than "forms":

$$\mathcal{B} = \sum \mathcal{C}_\alpha \otimes \mathcal{D}_\alpha \quad \longrightarrow \quad B = \sum C_\alpha \otimes D_\alpha,$$

where C_α and D_α are the g_0-symmetric operators corresponding to $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$.
In terms of B, the answer is amasingly simple $B = \frac{1}{2} R(\otimes)$, i.e.

$$R(X) = \frac{d}{dt}\Big|_{t=0} \, p_{\min}(A + tX) \quad \mapsto \quad B = \frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t \cdot \otimes),$$

Conclusion: The metric g defined by (4) satisfies two properties:
1) A is covariantly constant, i.e. $\mathfrak{hol}(\nabla) \subset \mathfrak{g}_A$ and
2) the curvature tensor at the origin is $R(X) = \frac{d}{dt}\big|_{t=0} \, p_{\min}(A + tX)$, and
therefore $\mathrm{Im}\, R = \mathfrak{g}_A \subset \mathfrak{hol}(\nabla)$ (hence solving the realisation problem)

Construction via $\mathbb{Z}_2$-graded Lie algebras

A homogeneous space G/H is (pseudo-)Riemannian symmetric if the corresponding Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ satisfy the following conditions:

- $\mathfrak{g} = \mathfrak{h} + V$ is a $\mathbb{Z}_2$-grading, i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, V] \subset V$ and $[V, V] \subset \mathfrak{h}$,
- V admits an $\mathfrak{h}$-invariant inner product.

In our situation, we take $R_0 : \mathrm{so}(g, V) \to \mathrm{so}(g, V)$ defined by
$R_0(X) = \frac{d}{dt}|_{t=0} p(A + tX)$ with $p(A) = 0$ and $X \in \mathrm{so}(g)$.
Then we simply set $\mathfrak{h} = \mathrm{Im}\, R_0$ and consider $\mathfrak{g} = \mathfrak{h} + V$. To complete the construction and get a $\mathbb{Z}_2$-grading on $\mathfrak{g}$, we need to define $[u, v] \in \mathfrak{h}$ for $u, v \in V$. The answer is given by the formal curvature tensor R_0:

$$[u, v] = R_0(u \wedge v).$$

The Jacobi identity for $\mathfrak{g}$ follows from the first and second Bianchi identities (Properties 4 and 5).

Conclusion: The decomposition $\mathfrak{g} = \mathfrak{h} + V$ defines a $\mathbb{Z}_2$-grading and therefore G/H is a symmetric (pseudo)-Riemannian space.

Kähler manifolds and c-projective equivalence

Observation 1. For Kähler manifolds, the curvature tensor can be understood as a linear map on the unitary Lie algebra

$$R : \mathrm{u}(g) \to \mathrm{u}(g)$$

Observation 2. The definition of Manakov operators still makes sense:

$$[R(X), A] = [X, B], \quad \text{for } X \in \mathrm{u}(g) \text{ and } A, B \text{ being } g\text{-Hermitian} \tag{5}$$

and Properties 1–7 have natural generalisations.

Definition

A curve $\gamma(t)$ on a Kähler manifold (M, g, J) is called *J-planar*, if

$$\nabla_\gamma \dot\gamma = \alpha\dot\gamma + \beta J\dot\gamma$$

where $\alpha, \beta \in \mathbb{R}$, and J is the complex structure on M. Two Kähler metrics g and $\bar g$ on a complex manifold (M, J) are called *c-projectively equivalent*, if they have the same J-planar curves.

Observation 3. Let g and $\bar g$ be c-projectively equivalent Kähler metrics. Then the Riemann curvature tensor of g is a Manakov operator in the sense of (5), where $A = \left(\frac{\det \bar g}{\det g}\right)^{\frac{1}{2(n+1)}} \bar g^{-1} g$ and $B = \frac{1}{2}\nabla(\mathrm{grad}\,\mathrm{tr}\,A)$.

Yano-Obata conjecture and Bochner-flat Kähler metrics of arbitrary signature

Definition

A vector field ξ on a Kähler manifold is called *c-projective*, if the flow of ξ preserves J-planar curves. A c-projective vector field is called *essential* if its flow changes the Levi-Civita connection.

Theorem (B., Matveev, Rosemann)

Let (M, g, J) be a closed connected Kähler manifold of arbitrary signature which admits an essential c-projective vector field. Then the manifold is isometric to $\mathbb{C}P^n$ with the Fubini-Study metric.

One of the ingredients of the proof is Property 7 for Jordan blocks.

Theorem (B., Matveev, Rosemann (in progress))

A local description of Bochner-flat Kähler metrics of arbitrary signature.

The proof uses a Kähler modification of the Magic formula and Kähler analogs of the pseudo-Riemannian symmetric spaces discussed above.

Thanks for your attention

Step two: Realisation

We need to find an example of g such that $\mathfrak{hol}(\nabla) = \mathfrak{g}_A$.

More specifically:
For a given operator $A : T_{x_0}M \to T_{x_0}M$, we need to find
a (pseudo)-Riemannian metric g on M and
a $(1,1)$-tensor field $A(x)$ (with the initial condition $A(x_0) = A$) such that

1. $\nabla A(x) = 0$;
2. $R(x_0)$ coincides with the formal curvature tensor R_{formal} just defined.

The idea is natural:

- set $A(x) = \mathrm{const}$
- try to find the desired metric $g(x)$ in the form:

$$\text{constant} + \text{quadratic}$$

i.e.,

$$g_{ij}(x) = g^0_{ij} + \sum \mathcal{B}_{ij,pq} x^p x^q \qquad (6)$$

where $\mathcal{B}$ satisfies obvious symmetry relations, namely, $\mathcal{B}_{ij,pq} = \mathcal{B}_{ji,pq}$ and $\mathcal{B}_{ij,pq} = \mathcal{B}_{ij,qp}$.

Magic Formula 2

Thus, we need to find $\mathcal{B}_{ij,pq}$ with the required properties. Such a tensor can be rewritten in the form $\mathcal{B} = \sum \mathcal{C}_\alpha \otimes \mathcal{D}_\alpha$, where $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$ are some symmetric forms. It is more convenient to work with "operators" rather than "forms":

$$\mathcal{B} = \sum \mathcal{C}_\alpha \otimes \mathcal{D}_\alpha \quad \longrightarrow \quad B = \sum C_\alpha \otimes D_\alpha,$$

where C_α and D_α are the g_0-symmetric operators corresponding to $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$. Then we can treat B as a linear map

$$B : \mathrm{gl}(V) \to \mathrm{gl}(V) \quad \text{defined by } B(X) = \sum C_\alpha X D_\alpha,$$

Question: How to find B?

Answer: Amasingly simple $B = \frac{1}{2} R(\otimes)$, i.e.

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(A + tX) \quad \mapsto \quad B = \frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t \cdot \otimes),$$

More precisely, if $p_{\min}(\lambda) = \sum_{m=0}^{n} a_m \lambda^m$ is the minimal polynomial of A, then

$$B = \frac{1}{2} \cdot \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} A^{m-1-j} \otimes A^j. \qquad (7)$$

Conclusion: This B solves the realisation problem.