Argument shift method and Manakov operators: applications to differential geometry

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Argument shift method and Manakov operators: applications to differential geometry

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Around the following observation:

The curvature tensors of some interesting Riemannian metrics coincide with the Hamiltonians of multi-dimensional rigid bodies

Applications (for indefinite metrics):

▶ Obstructions to the existence of a projectively equivalent partner
▶ Pseudo-Riemannian analog of the Fubini theorem
▶ New class of holonomy groups
▶ New class of symmetric spaces
▶ Yano-Obata conjecture
▶ Local description of Bochner-flat Kähler metrics
Let \( \mathfrak{g} \) be a semisimple Lie algebra, \( R : \mathfrak{g}^* \cong \mathfrak{g} \rightarrow \mathfrak{g} \) a symmetric linear operator. Euler equations on \( \mathfrak{g}^* \)

\[
\frac{dx}{dt} = [x, R(x)]
\]

(1)

are Hamiltonian with \( H = \frac{1}{2} \langle R(x), x \rangle \).

For which \( R \), are the equations (1) integrable?

**Definition**

\( R : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n) \) is called a Manakov operator (with parameters \( A \) and \( B \)), if

\[
[R(X), A] = [X, B] \quad \text{for all } X \in \mathfrak{so}(g)
\]

(2)

where \( A \) and \( B \) are some fixed symmetric matrices.

**Theorem (Manakov, Mischenko, Fomenko)**

Let \( R \) satisfy (2). Then

- (1) can be rewritten as \( \frac{d}{dt} (X + \lambda A) = [X + \lambda A, R(X) + \lambda B] \);
- \( \text{Tr}(X + \lambda A)^k \) are commuting first integrals of (1);
- if \( A \) is regular, then (1) are completely integrable.
Properties of Manakov operators

1. $A$ and $B$ commute, moreover, $B$ belongs to the centre of centraliser of $A$. In particular, $B = p(A)$, where $p(\cdot)$ is some polynomial.

2. $R_0 = \left. \frac{d}{dt} \right|_{t=0} p(A + tX)$ satisfies (2). If $A$ is regular, then $R$ is unique, otherwise $R = R_0 + D$ where $D : \text{so}(g) \to g_A = \{ Y \in \text{so}(g), \ AY = YA \}$ is arbitrary.

3. if $B = 0 = p_{\text{min}}(A)$, then $R_0 = \left. \frac{d}{dt} \right|_{t=0} p_{\text{min}}(A + tX)$ still defines a non-trivial Manakov operator whose image is contained in $g_A$. Moreover, if for each eigenvalues of $A$ there are at most 2 Jordan blocks, then the image $R_0$ coincides with $g_A$.

4. $R_0$ satisfies the Bianchi identity.

5. If in addition $p(A) = 0$, then $R_0$ satisfies the second Bianchi identity $[R_0(X), R_0(Y)] = R_0[R_0(X), Y]$.

6. Let $R$ satisfy two identities $[R(X), A] = [X, B]$ and $[R(X), A'] = [X, B']$, where $A' \neq aA + b \cdot \text{id}$. Then $R(X) = k \cdot X \mod g_A$. In particular, if $A$ is regular, then $R = k \cdot \text{id}$.

7. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $A$. Then $\frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j}$ are eigenvalues of $R$. Moreover, if $A$ has a nontrivial Jordan $\lambda_i$-block, then $p'(\lambda_i)$ is an eigenvalue of $R$. 
Riemann curvature tensor (quick reminder and “new” point of view)

Let $\nabla$ be the Levi-Civita connection of a pseudo-Riemannian metric $g$.

**Definition**
The Riemann curvature tensor $R = (R^l_{ij}{}^k)$ is defined by (formula from a text-book):

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$ 

In other words, $R$ can be understood as a map

$$R : (X, Y) \mapsto R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \text{End}(TM).$$

**Algebraic symmetries:**

- $R(X, Y) = -R(X, Y)$, i.e., $R : \Lambda^2 V \to gl(V), \ V = T_x M$;
- $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$, i.e. $R(X, Y) \in so(g)$;
- $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ (Bianchi identity);
- $g(R(X, Y)Z, W) = -g(R(Z, W)X, Y)$.

**Conclusion:** $R : so(g) \to so(g)$ which is symmetric and satisfying Bianchi.

**Easy observations:**

- constant curvature $\iff R = \text{const} \cdot \text{id}$
- Weyl tensor vanishes $\iff R(X) = AX +XA$
  (cf., in rigid body dynamics: $M(\Omega) = J\Omega + \Omega J$)
Projectively equivalent metrics

**Definition**

$g$ and $\bar{g}$ are projectively equivalent if they have the same (unparametrised) geodesics. Notation: $g \sim_{\text{proj}} \bar{g}$.

**Main equation:** Let $A = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g$. Then $g \sim_{\text{proj}} \bar{g}$ if and only if

$$\nabla_u A = \frac{1}{2} \left( u \otimes \text{d tr} A + (u \otimes d \text{tr} A)^\ast \right).$$

**Theorem (B., Matveev)**

Let $g \sim_{\text{proj}} \bar{g}$. Then the Riemann curvature tensor of $g$ is a Manakov operator:

$$[R(X), A] = [B, X] \quad \text{for all } X \in \text{so}(g), \text{ where } B = \frac{1}{2} \nabla(\text{grad tr } A).$$

**Proof.**

Consider the compatibility condition for the main equation.

**Theorem (B., Matveev, Kiosak)**

Let $g$, $\bar{g}$ and $\hat{g}$ be projectively equivalent. Assume that these metrics are linearly independent and $g$ and $\hat{g}$ are strictly non-proportional, then $g$, $\bar{g}$ and $\hat{g}$ are metrics of constant sectional curvature.

**Proof.**

Apply Property 6.
New class of holonomy groups in pseudo-Riemannian geometry

Definition
Let $M$ be a smooth manifold endowed with an affine symmetric connection $\nabla$. The holonomy group of $\nabla$ is a subgroup $\text{Hol}(\nabla) \subset \text{GL}(T_xM)$ that consists of the linear operators $A : T_xM \to T_xM$ being ‘parallel transport transformations’ along closed loops $\gamma(t)$ with $\gamma(0) = \gamma(1) = x$.

Problem. Given a subgroup $H \subset \text{GL}(n, \mathbb{R})$, can it be realised as the holonomy group for an appropriate symmetric connection on $M$?

Riemannian case and irreducible case: the problem is completely solved (Marcel Berger, D. V. Alekseevskii, R. Bryant, D. Joyce, L. Schwahhöfer, S. Merkulov).


Theorem (B., Tsonev)
For every $g$-symmetric operator $A : V \to V$, its centraliser in $\text{SO}(g)$ (the identity connected component of)

$$G_A = \{ Y \in \text{SO}(g) \mid YA = AY \}$$

is a holonomy group for a certain (pseudo)-Riemannian metric.
Definition
A map $R : \Lambda^2 V \to \text{gl}(V)$ is called a \textit{formal curvature tensor} if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \quad \text{for all } u, v, w \in V.$$ 

Definition
Let $\mathfrak{h} \subset \text{gl}(V)$ be a Lie subalgebra. Consider the set of all formal curvature tensors $R : \Lambda^2 V \to \text{gl}(V)$ such that $\text{Im } R \subset \mathfrak{h}$:

$$\mathcal{R}(\mathfrak{h}) = \{ R : \Lambda^2 V \to \mathfrak{h} \mid R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0, \ u, v, w \in V \}.$$ 

We say that $\mathfrak{h}$ is a \textit{Berger algebra} if it is generated as a vector space by the images of the formal curvature tensors $R \in \mathcal{R}(\mathfrak{h})$, i.e.,

$$\mathfrak{h} = \text{span}\{ R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}), \ u, v \in V \}.$$ 

Berger test:
\textit{Let } $\nabla$ \textit{be a symmetric affine connection on } $TM$. \textit{Then the Lie algebra } $\text{hol}(\nabla)$ \textit{of its holonomy group } $\text{Hol}(\nabla)$ \textit{is Berger.}
Definition
A map $R: \mathfrak{so}(g) \to \mathfrak{so}(g)$ is called a formal curvature tensor if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0$$

for all $u, v, w \in V$,

where $u \wedge v = u \otimes g(v) - v \otimes g(u) \in \mathfrak{so}(g)$.

Definition
Let $\mathfrak{h} \subset \mathfrak{so}(g)$ be a Lie subalgebra. Consider the set of all formal curvature tensors $R: \mathfrak{so}(g) \to \mathfrak{so}(g)$ such that $\text{Im} \, R \subset \mathfrak{h}$:

$$\mathcal{R}(\mathfrak{h}) = \{ R : \Lambda^2 V \to \mathfrak{h} \mid R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0, u, v, w \in V \}.$$

We say that $\mathfrak{h}$ is a Berger algebra if it is generated as a vector space by the images of the formal curvature tensors $R \in \mathcal{R}(\mathfrak{h})$, i.e.,

$$\mathfrak{h} = \text{span}\{ R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}), u, v \in V \}.$$

Berger test:
Let $\nabla$ be a Levi-Civita connection on $(M, g)$. Then the Lie algebra $\mathfrak{hol}(\nabla) \subset \mathfrak{so}(g)$ of its holonomy group $\text{Hol}(\nabla)$ is Berger.
Step one: Berger test for $\mathfrak{g}_A$ and Magic Formula 1

We have

$$\mathfrak{g}_A = \{X \in \text{so}(g) \midXA = AX\}$$

and we need to construct formal curvature tensors $R : \text{so}(g) \rightarrow \text{so}(g)$ whose images generate $\mathfrak{g}_A$.

Ideally, we want one single formal curvature tensor $R$ such that $\text{Im} R = \mathfrak{g}_A$.

**Question:** How to find $R$?

**Answer:** Apply Properties 3 and 4, i.e. define a linear mapping $R : \text{so}(g) \rightarrow \text{so}(g)$ by:

$$R(X) = \frac{d}{dt} \bigg|_{t=0} p_{\text{min}}(A + tX),$$

where $p_{\text{min}}(\lambda)$ is the minimal polynomial of $A$.

**Conclusion:** $\mathfrak{g}_A$ is Berger algebra.
Step two: Realisation and Magic Formula 2

We need to find an example of $g$ such that $\mathfrak{hol}(\nabla) = g_A$. The idea is natural:

- set $A(x) = \text{const}$
- try to find the desired metric $g(x)$ in the form constant + quadratic:

$$g_{ij}(x) = g^0_{ij} + \sum B_{ij, pq} x^p x^q.$$  \hspace{1cm} (4)

**Question**: How to find $B$?

It is more convenient to work with “operators” rather than “forms”:

$$B = \sum C_\alpha \otimes D_\alpha \quad \rightarrow \quad B = \sum C_\alpha \otimes D_\alpha,$$

where $C_\alpha$ and $D_\alpha$ are the $g_0$-symmetric operators corresponding to $C_\alpha$ and $D_\alpha$.

In terms of $B$, the answer is amazingly simple $B = \frac{1}{2} R(\otimes)$, i.e.

$$R(X) = \frac{d}{dt} \bigg|_{t=0} p_{\text{min}}(A + tX) \quad \leftrightarrow \quad B = \frac{1}{2} \cdot \frac{d}{dt} \bigg|_{t=0} p_{\text{min}}(L + t \cdot \otimes),$$

**Conclusion**: The metric $g$ defined by (4) satisfies two properties:

1) $A$ is covariantly constant, i.e. $\mathfrak{hol}(\nabla) \subset g_A$ and
2) the curvature tensor at the origin is $R(X) = \frac{d}{dt} \bigg|_{t=0} p_{\text{min}}(A + tX)$, and therefore $\text{Im} R = g_A \subset \mathfrak{hol}(\nabla)$ (hence solving the realisation problem)
A new (?) class of pseudo-Riemannian symmetric spaces

Construction via \( \mathbb{Z}_2 \)-graded Lie algebras

A homogeneous space \( G/H \) is (pseudo-)Riemannian symmetric if the corresponding Lie algebras \( \mathfrak{h} \subset \mathfrak{g} \) satisfy the following conditions:

- \( \mathfrak{g} = \mathfrak{h} + V \) is a \( \mathbb{Z}_2 \)-grading, i.e. \([\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, [\mathfrak{h}, V] \subset V \) and \([V, V] \subset \mathfrak{h}\),
- \( V \) admits an \( \mathfrak{h} \)-invariant inner product.

In our situation, we take \( R_0 : \text{so}(g, V) \to \text{so}(g, V) \) defined by

\[
R_0(X) = \frac{d}{dt} \bigg|_{t=0} p(A + tX) \quad \text{with} \quad p(A) = 0 \quad \text{and} \quad X \in \text{so}(g).
\]

Then we simply set \( \mathfrak{h} = \text{Im} \, R_0 \) and consider \( \mathfrak{g} = \mathfrak{h} + V \). To complete the construction and get a \( \mathbb{Z}_2 \)-grading on \( \mathfrak{g} \), we need to define \([u, v] \in \mathfrak{h}\) for \( u, v \in V \). The answer is given by the formal curvature tensor \( R_0 \):

\[
[u, v] = R_0(u \wedge v).
\]

The Jacobi identity for \( \mathfrak{g} \) follows from the first and second Bianchi identities (Properties 4 and 5).

Conclusion: The decomposition \( \mathfrak{g} = \mathfrak{h} + V \) defines a \( \mathbb{Z}_2 \)-grading and therefore \( G/H \) is a symmetric (pseudo)-Riemannian space.
Observation 1. For Kähler manifolds, the curvature tensor can be understood as a linear map on the unitary Lie algebra

\[ R : \mathfrak{u}(g) \to \mathfrak{u}(g) \]

Observation 2. The definition of Manakov operators still makes sense:

\[ [R(X), A] = [X, B], \quad \text{for } X \in \mathfrak{u}(g) \text{ and } A, B \text{ being } g\text{-Hermitian} \quad (5) \]

and Properties 1–7 have natural generalisations.

Definition

A curve \( \gamma(t) \) on a Kähler manifold \((M, g, J)\) is called \(J\text{-planar}\), if

\[ \nabla_{\gamma'} \dot{\gamma} = \alpha \dot{\gamma} + \beta \dot{J} \gamma \]

where \( \alpha, \beta \in \mathbb{R} \), and \( J \) is the complex structure on \( M \). Two Kähler metrics \( g \) and \( \tilde{g} \) on a complex manifold \((M, J)\) are called \textit{c-projectively equivalent}, if they have the same \( J\text{-planar} \) curves.

Observation 3. Let \( g \) and \( \tilde{g} \) be c-projectively equivalent Kähler metrics. Then the Riemann curvature tensor of \( g \) is a Manakov operator in the sense of (5), where

\[ A = \left( \frac{\det \tilde{g}}{\det g} \right)^{\frac{1}{2(n+1)}} \tilde{g}^{-1} g \quad \text{and} \quad B = \frac{1}{2} \nabla (\text{grad tr } A). \]
Yano-Obata conjecture and Bochner-flat Kähler metrics of arbitrary signature

Definition
A vector field $\xi$ on a Kähler manifold is called \textit{c-projective}, if the flow of $\xi$ preserves $J$-planar curves. A c-projective vector field is called \textit{essential} if its flow changes the Levi-Civita connection.

Theorem (B., Matveev, Rosemann)
Let $(M, g, J)$ be a closed connected Kähler manifold of arbitrary signature which admits an essential c-projective vector field. Then the manifold is isometric to $\mathbb{C}P^n$ with the Fubini-Study metric.

One of the ingredients of the proof is Property 7 for Jordan blocks.

Theorem (B., Matveev, Rosemann (in progress))
A local description of Bochner-flat Kähler metrics of arbitrary signature.
The proof uses a Kähler modification of the Magic formula and Kähler analogs of the pseudo-Riemannian symmetric spaces discussed above.
Thanks for your attention
We need to find an example of $g$ such that $\hhol(\nabla) = g_A$.

More specifically:
For a given operator $A : T_{x_0}M \to T_{x_0}M$, we need to find
a (pseudo)-Riemannian metric $g$ on $M$ and
a $(1,1)$-tensor field $A(x)$ (with the initial condition $A(x_0) = A$) such that
1. $\nabla A(x) = 0$;
2. $R(x_0)$ coincides with the formal curvature tensor $R_{\text{formal}}$ just defined.

The idea is natural:
- set $A(x) = \text{const}$
- try to find the desired metric $g(x)$ in the form:

\[ g_{ij}(x) = g_{ij}^0 + \sum B_{ij,pq} x^p x^q \]  

where $B$ satisfies obvious symmetry relations, namely, $B_{ij,pq} = B_{ji,pq}$ and $B_{ij,pq} = B_{ij,qp}$. 

i.e.,

\[ g_{ij}(x) = g_{ij}^0 + \sum B_{ij,pq} x^p x^q \]  

(6)
Thus, we need to find $B_{ij,pq}$ with the required properties. Such a tensor can be rewritten in the form $B = \sum C_\alpha \otimes D_\alpha$, where $C_\alpha$ and $D_\alpha$ are some symmetric forms. It is more convenient to work with “operators” rather than “forms”:

$$B = \sum C_\alpha \otimes D_\alpha \quad \rightarrow \quad B = \sum C_\alpha \otimes D_\alpha,$$

where $C_\alpha$ and $D_\alpha$ are the $g_0$-symmetric operators corresponding to $C_\alpha$ and $D_\alpha$. Then we can treat $B$ as a linear map

$$B : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V) \quad \text{defined by} \quad B(X) = \sum C_\alpha XD_\alpha,$$

Question: How to find $B$?

Answer: Amasingly simple $B = \frac{1}{2} R(\otimes)$, i.e.

$$R(X) = \frac{d}{dt} \bigg|_{t=0} p_{\min}(A + tX) \quad \Rightarrow \quad B = \frac{1}{2} \cdot \frac{d}{dt} \bigg|_{t=0} p_{\min}(L + t \cdot \otimes),$$

More precisely, if $p_{\min}(\lambda) = \sum_{m=0}^{n} a_m \lambda^m$ is the minimal polynomial of $A$, then

$$B = \frac{1}{2} \cdot \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} A^{m-1-j} \otimes A^j. \quad (7)$$

Conclusion: This $B$ solves the realisation problem.