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Cyclic codes and minimal strong Gröbner bases over a principal ideal ring

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Abstract

We characterise minimal strong Gröbner bases of $R[x]$, where $R$ is a commutative principal ideal ring and deduce a structure theorem for cyclic codes of arbitrary length over $R$. When $R$ is an Artinian chain ring with residue field $\overline{R}$ and $\gcd(\text{char}(\overline{R}), n) = 1$, we recover a theorem for cyclic codes of length $n$ over $R$ due to Calderbank and Sloane for $R = \mathbb{Z}/p^k\mathbb{Z}$.

1 Introduction

All rings in this paper are commutative. This work originates from two structure theorems: (i) for certain cyclic codes over $R = \mathbb{Z}/p^k\mathbb{Z}$, with $p$ a prime and $k$ an integer, $k \geq 2$, [5, Theorem 6] and (ii) for a minimal strong Gröbner basis (SGB) of an ideal of $D[x]$, $D$ a principal ideal domain, [9]. Intuitively, the first resembled a 'minimal SGB'. Since we had already developed a theory of SGB's over an principal ideal ring in [15], it was natural to ask whether (i) and (ii) have a common provenance. We confirm this and generalise (i) to a cyclic code of arbitrary length over a principal ideal ring.

In more detail, a cyclic code of length $n$ over a ring $R$ is an ideal of $R[x]/(x^n - 1)$. The structure theorem for cyclic codes over $R$ of [5] requires that $\gcd(p, n) = 1$ and the proofs used non-trivial results from Commutative Algebra on the ideal structure of $R[x]/(x^n - 1)$. A generalisation of [5, Theorem 6] to cyclic codes over an Artinian chain ring was given in [14]. We formalised the notion of a 'generating set in standard form', loc. cit., Definition 4.1 and showed that a cyclic code has a unique generating set in standard form, [14, Theorem 4.4]. See also [19, Theorem 3.9].

In addition, we recover the generating set in standard form of a cyclic code over an Artinian chain ring $R$ as a minimal SGB using [15]. This provides an alternative proof of [14, Theorem 4.4]. Moreover, a similar result holds for arbitrary $n$ (see Theorem 4.2 and condition (iv)) and also for
codes over a principal ideal ring (see Theorem 5.6).

We begin with some preliminaries on Artinian chain rings $R$ (e.g., Galois rings) and then characterise the structure of minimal SGB’s of $R[x]$; see Theorem 3.2. This result is similar to the principal ideal domain case of [9], recalled as Theorem 2.11; see also [18]. In Section 4, we show that if $p$ is the characteristic of the residue field of $R$ and $\gcd(p, n) = 1$, minimal SGB’s coincide with generating sets in standard form for cyclic codes over $R$. In Section 5, we generalise the structure theorems for minimal SGB’s mentioned above to a principal ideal ring. In the final section we discuss connections between minimal SGB’s over $R$ and the representation of a regular $f \in R[x]$ as $f = uf^*$ with $f^*$ monic and $u$ a unit in $R[x]$ of [10, Theorem XIII.6].

We have thus found a common background for the structure theorems of [1, 5, 9]. Some of the results of this paper appeared in [17]. We remark that Allan Steel has implemented an SGB algorithm in Version 2.8 of Magma [3] using Corollary 2.8, generalising Faugère’s algorithm [7] to Galois rings.

Related work for the special case of a Galois ring $A$ appears in [4], where an SGB is called a GB. Their approach depends on whether the elements of $A$ are represented additively or multiplicatively. On the other hand our notion of reduction is independent of how the elements of $A$ are represented and how the operations are performed in $A$, as needed for working over principal ideal rings in general.

More importantly, there is another strictly weaker notion of a (weak) GB over any ring, [1, Definition 4.1.13]. The key result [4, Theorem 2.5.10] depends on the characterisation of a (weak) GB (rather than an SGB) in terms of homogeneous syzygies of monomials in $R[x]$ given in [1, Theorem 4.2.3]. This means that [4, Theorem 2.5.10] only yields a (weak) GB and not necessarily an SGB as in [4, Definition 2.4.1]. It turns out a (weak) GB is an SGB over an Artinian chain ring, [15, Proposition 3.9], but this is point is not considered in [4].

Thus while one could potentially generalise parts of [4] to finite chain rings, we prefer to avoid circular arguments (i.e. appealing to [15, Proposition 3.9]), a ‘pre-selected division algorithm’ and homogeneous syzygies. For example, we need only specialise [15, Theorem 4.10] to the univariate case, as in Corollary 2.8 below. Finally, concerning the decoding application of [4], we note that a characterisation of the set of solutions of the key equation and a quadratic decoding algorithm for an alternate code over a finite chain ring appeared in [13]. We do not know if the decoding application in [4] runs in polynomial time.

## 2 Preliminaries

First some notation and known results on Artinian chain rings, SGB’s and minimal SGB’s.
2.1 Notation

Throughout this paper $R$ will denote a principal ideal ring which is not a field. We write the ideal of $R$ generated by $r_1, \ldots, r_m \in R$ as $\langle r_1, \ldots, r_m \rangle_R$. The ideal of $R[x]$ generated by $f_1, \ldots, f_m \in R[x]$ is written as $\langle f_1, \ldots, f_m \rangle$ and $\subseteq, \supseteq$ denotes strict inclusion. As usual, $f = \sum_{i=0}^{d} c_i x^i \in R[x]$ with $c_d \neq 0$ has degree $d = \deg(f)$; $\lt(f) = x^d$ is its leading term and $\lc(f) = c_d$ is its leading coefficient; we say that $f$ is monic if $\lc(f) = 1$. The leading monomial of $f$ is $\ln(f) = \lc(f) \lt(f)$ and we denote by $\cont(f)$ a content of $f$ i.e. a gcd of all its coefficients, which is well-defined up to a unit by [15, Lemma 4.3(ii)].

2.2 Artinian chain rings

We will need the following structure theorem:

**Theorem 2.1** ([20, Theorem 33, Section 15, Ch. 4]) A principal ideal ring is isomorphic to a finite direct product of principal ideal domains and Artinian chain rings.

Recall that a chain ring is a ring whose ideals are linearly ordered by inclusion, [6]. In this section, $R$ will denote an Artinian chain ring. The main properties of $R$ are:

**Proposition 2.2** $R$ is a local principal ideal ring with maximal ideal $J(R)$; the elements of $J(R)$ are nilpotent and the elements of $R \setminus J(R)$ are units.

Let $\gamma$ be a fixed generator of $J(R)$ and $\nu$ the nilpotency index of $\gamma$ i.e. the smallest positive integer for which $\gamma^\nu = 0$. (i) The distinct proper ideals of $R$ are $\langle \gamma^i \rangle_R, i = 1, \ldots, \nu - 1$. (ii) For any element $r \in R \setminus \{0\}$ there is a unique $i$ and a unit $u$ such that $r = u \gamma^i$, where $0 \leq i \leq \nu - 1$ and $u$ is unique modulo $\gamma^\nu$. (iii) $\Ann(\gamma^i) = \langle \gamma^{\nu - i} \rangle_R$.

It is not hard to see that a local principal ideal ring is a chain ring. Thus Artinian chain rings are precisely the Artinian local principal ideal rings.

From now on, $\gamma$ and $\nu$ will be as in Proposition 2.2. It follows that any $f \in R[x] \setminus \{0\}$ can be written as $\gamma^i g$ where $0 \leq i \leq \nu - 1$, $\deg(f) = \deg(g)$ and $\gamma /g$. The exponent $i$ is uniquely determined and $g$ is unique modulo $\gamma^\nu$.

For any $r \in R$, the canonical projection $\varphi_r : R \to R/\langle r \rangle_R$ induces a ring homomorphism $R[x] \to (R/\langle r \rangle_R)[x]$, which we also write as $\varphi_r$. Of course, $\varphi_\gamma$ projects $R$ onto its residue field $\overline{R} = R/J(R)$, and in this case we write $\overline{f}$ for $\varphi_\gamma(f)$.

The next theorem is stated for finite local rings in [10], but the proofs only use the fact that $R$ is local and that the maximal ideal is nilpotent and finitely generated; $R$ itself need not be finite. Recall that a polynomial in $R[x]$ is called regular if it is neither a unit nor a zero-divisor.
Theorem 2.3 ([10, Theorems XIII.2 and XIII.6]) Let \( f = \sum_{i=0}^{m} f_i x^i \in R[x] \setminus \{0\} \). Then:

(i) \( f \) is a zero-divisor iff \( \gamma_i | f_i \) for \( i = 0, \ldots, m \); (ii) \( f \) is a unit iff \( f_0 \) is a unit and \( \gamma_i | f_i \) for \( i = 1, \ldots, m \); (iii) If \( f \) is regular then there are \( \gamma^* u \in \mathbb{R}[x] \) such that \( f = u f^* \), \( u \) is a unit and \( f^* \) is monic.

The polynomials \( f^* \) and \( u \) in Theorem 2.3(iii) are constructed by Hensel lifting. We generalise the construction in Theorem 2.3(iii) to any polynomial in \( f \in \mathbb{R}[x] \setminus \{0\} \) by defining \( f^* = \gamma^i g^* \) where \( \gamma^i \in \text{cont}(f) \) and \( f = \gamma^i g \). It follows that there is a unit \( u \in \mathbb{R}[x] \) such that \( f = u f^* \). It is easy to show that \( f^* \) is unique in the sense that it satisfies the following property:

\[
\text{if } f = vh, v \text{ a unit in } \mathbb{R}[x] \text{ and } \text{lc}(h) = \gamma^i \in \text{cont}(f), \text{ then } h = f^*.
\]

(1)

Also, the unit \( u \) is unique modulo \( \gamma^u i \).

The following consequence of Property (1) will be used later.

**Lemma 2.4** Let \( f \in \mathbb{R}[x] \setminus \{0\} \) and \( \gamma^i \in \text{cont}(f) \). Then \( \deg(f^*) = \deg(\varphi_{\gamma_i+1}(f)) \).

**Proof.** Write \( f = \gamma^i g \). By definition, \( f^* = \gamma^i g^* \) and there is a unit \( u \in \mathbb{R}[x] \) such that \( f = \gamma^i ug^* \). Applying the homomorphism \( \varphi_{\gamma_i+1} \) we obtain \( \varphi_{\gamma_i+1}(f) = \varphi_{\gamma_i+1}(\gamma^i u) \varphi_{\gamma_i+1}(g^*) \). By Theorem 2.3(ii), \( \deg(\varphi_{\gamma_i+1}(ug^*)) = 0 \). Since \( g^* \) is monic, \( \deg(\varphi_{\gamma_i+1}(g^*)) = \deg(g^*) = \deg(f^*) \). Hence, \( \deg(\varphi_{\gamma_i+1}(f)) = \deg(f^*) \).

2.3 Strong Reduction and Strong Gröbner Bases

Let \( f, g, h \in \mathbb{R}[x] \). We write \( f \rightarrow_G h \) if \( f \) strongly reduces to \( h \) wrt. \( G \) in one step and also say that \( f \) is strongly reducible wrt. \( G \) (see [1, p. 252] for the definition of strong reduction). The reflexive and transitive closure of \( \rightarrow_G \) is denoted \( \rightarrow_G^* \). When \( f \rightarrow_G^* h \) we say that \( f \) strongly reduces to \( h \) wrt. \( G \). If \( h \) is not strongly reducible wrt. \( G \), then \( h \) is a remainder of \( f \) wrt. \( G \) (by strong reduction). The set of such remainders is SRem(\( f, G \)). We adopt the conventions \( 0 \rightarrow_G^* 0 \) and \( \text{SRem}(0, G) = \{0\} \) for any set \( G \). Note that for any polynomial \( f \) there is at least one remainder of \( f \) wrt. \( G \) (by strong reduction) and if \( f \rightarrow_G^* 0 \) then \( f \in \langle G \rangle \). As in the case of a field, we have:

**Theorem 2.5** Let \( I \) be a non-zero ideal of \( \mathbb{R}[x] \) and \( G \) a finite subset of \( I \setminus \{0\} \). The following assertions are equivalent: (i) any \( f \in I \) is strongly reducible wrt. \( G \); (ii) \( f \in I \) if and only if \( f \rightarrow_G^* 0 \); (iii) \( f \in I \) if and only if \( \text{SRem}(f, G) = \{0\} \).

Let \( I \) be a non-zero ideal of \( \mathbb{R}[x] \) and \( G \) a finite subset of \( I \setminus \{0\} \). Then \( G \) is a strong Gröbner basis (SGB) for \( I \) if it satisfies any of the conditions of Theorem 2.5. If \( G \) is an SGB for an ideal \( I \), then \( I = \langle G \rangle \). When we say '\( G \) is an SGB', we will mean \( G \) is an SGB for \( \langle G \rangle \). We will also appeal to:
PROPOSITION 2.6 ([15, Corollary 3.12, Proposition 4.2]) Let \( f \in R[x] \). Then \( f \) is an SGB if and only if \( f = rg \) for some \( r \in R \setminus \{0\} \) and \( g \in R[x] \) such that \( \kappa(g) \) is not a zero-divisor.

In [15], we characterised SGB’s for ideals of \( R[x_1, \ldots, x_n] \) in terms of S- and G-polynomials (see [2, Definition 10.9]) of pairs of polynomials and 'A-polynomials': an A-polynomial of \( f \) is any polynomial \( af \) where \( \text{Ann}(\kappa(f)) = \langle a \rangle \) \( n \) [15, Definition 4.9]. Sets of S-, G- and A-polynomials are denoted \( \text{Spol}(f_1, f_2) \), \( \text{Gpol}(f_1, f_2) \) and \( \text{Apol}(f) \) respectively.

We now restate [15, Corollaries 5.12 and 5.13]) for univariate polynomials:

**Corollary 2.7** A finite subset \( G \) of \( R[x]\setminus \{0\} \) is an SGB if and only if (A) for any \( g_1, g_2 \in G \) with \( g_1 \neq g_2 \), there is an \( h \in \text{Spol}(g_1, g_2) \) such that \( h \rightarrow \kappa^*_0 \); (B) for any \( g \in G \), there is an \( h \in \text{Apol}(g) \) such that \( h \rightarrow \kappa^*_0 \); (C) for any \( g_1, g_2 \in G \) with \( g_1 \neq g_2 \) there is an \( h \in \text{Gpol}(g_1, g_2) \) which is strongly reducible wrt. to \( G \).


**Corollary 2.8** (Cf. [4, Theorem 2.5.10]) Let \( R \) be an Artinian chain ring. A finite subset \( G \) of \( R[x]\setminus \{0\} \) is an SGB if and only if (A) for any \( g_1, g_2 \in G \) with \( g_1 \neq g_2 \), there is an \( h \in \text{Spol}(g_1, g_2) \) such that \( h \rightarrow \kappa^*_0 \) and (B) for any \( g \in G \), there is an \( h \in \text{Apol}(g) \) such that \( h \rightarrow \kappa^*_0 \).

### 2.4 Minimal SGB’s

If \( G \) is an SGB, then \( G \) is **minimal** if no proper subset of \( G \) is an SGB for \( \langle G \rangle \). One can easily see that an SGB \( G \) is minimal if for all distinct \( f, g \in G \) we have \( \text{lm}(f) \mid \text{lm}(g) \). Other properties of minimal SGB are described in [15, Section 7]. We recall some of these results for \( R[x] \):

**Corollary 2.9** Let \( G = \{g_0, \ldots, g_s\} \subset R[x] \) be an SGB. Then \( G \) is minimal if and only if for \( i = 0, \ldots, s - 1 \) (i) \( \langle \kappa(g_i) \rangle_R \supset \langle \kappa(g_{i+1}) \rangle_R \) and (ii) \( \deg(g_i) > \deg(g_{i+1}) \).

**Theorem 2.10** Let \( F = \{f_1, \ldots, f_k\} \) and \( G = \{g_1, \ldots, g_l\} \) be minimal SGB’s for an ideal \( I \) of \( R[x] \). Then \( k = l \) and there are units \( u_i \in R \) such that after a suitable renumbering \( \text{lm}(f_i) = u_i \text{lm}(g_i) \) for \( i = 1, \ldots, k \).

When \( R \) is a principal ideal domain, more is known about the structure of a minimal SGB. We recall a theorem based on [9]; see also [18]. Our formulation is close to the one in [1, Theorem 4.5.13 and Exercise 4.5.12].

**Theorem 2.11** Let \( D \) be a principal ideal domain which is not a field and let \( G \subset D[x] \setminus \{0\} \). Then \( G \) is a minimal SGB if and only if \( G = \{d_0g_0, \ldots, d_sg_s\} \) for some \( d_i \in D, g_i \in D[x] \) such that for \( 0 \leq i \leq s - 1 \), (i) \( \langle d_i \rangle_R \supset \langle d_{i+1} \rangle_R \); (ii) \( \kappa(g_i) = \kappa(g_{i+1}) \); (iii) \( \deg(g_i) > \deg(g_{i+1}) \) and (iv) \( d_{i+1}g_i \in \langle d_{i+1}g_{i+1}, \ldots, d_sg_s \rangle \). Moreover, \( d_0g_s = \gcd(d_0g_0, \ldots, d_sg_s) \).
3 Minimal SGB’s over an Artinian chain ring

Throughout this section, $R$ is an Artinian chain ring. The following result shows that all polynomials in a minimal SGB are of the form $vf^*$, $v$ a unit in $R$.

**Proposition 3.1** (i) Let $f \in R[x] \setminus \{0\}$. Any minimal SGB of $\langle f \rangle$ is equal to $\{vf^*\}$ for some unit $v \in R$. (ii) If $G$ is a minimal SGB, then any $f \in G$ is equal to $vf^*$ for some unit $v \in R$.

**Proof.** (i) This follows easily from Property (1) and Proposition 2.6. For (ii), let $f = vf^*$ where $v \in R[x]$ is a unit of minimal degree. It is enough to show that $\deg(f) = \deg(f^*)$. We know that $\deg(f) \geq \deg(f^*)$. Since $f^* = v^{-1}f \in \langle G \rangle$, $\ln(g)||\ln(f^*)$ for some $g \in G$. Hence if $\deg(f) > \deg(f^*)$, $\deg(f) > \deg(g)$ and $f \neq g$. This contradicts the minimality of $G$ since $\ln(g)||\ln(f^*)||\ln(f)$. Hence $\deg(f) = \deg(f^*)$ and $v \in R$. $\triangleright$

Thus any principal ideal of $R[x]$ admits an SGB consisting of a single element. This is no longer the case if $R$ is no longer an Artinian chain ring or the polynomials are no longer univariate; see [15, Examples 6.6, 6.12]. Corollary 2.9 can be improved, giving an analogue of Theorem 2.11:

**Theorem 3.2** Let $G \subset R[x] \setminus \{0\}$. Then $G$ is a minimal SGB if and only if $G = \{r_0g_0, \ldots, r_sg_s\}$ for some $s \leq \nu - 1$ where (i) $r_i = \gamma^{\beta_i}$ for $0 \leq \beta_i < \cdots < \beta_j \leq \nu - 1$; (ii) $\ln(g_i)$ is a unit in $R$ for $i = 0, \ldots, s$; (iii) $\deg(g_i) > \deg(g_{i+1})$ for $i = 0, \ldots, s - 1$ and (iv) $r_{i+1}g_i \in (r_{i+1}g_{i+1}, \ldots, r_sg_s)$ for $i = 0, \ldots, s - 1$.

**Proof.** Let $G = \{f_1, \ldots, f_s\}$ be a minimal SGB. By Corollary 2.9 we may assume $\deg(f_i) > \deg(f_{i+1})$ for $i = 0, \ldots, s - 1$. Define $j_i$ by $\gamma^{\beta_i} \in \cont(f_i)$ for $i = 0, \ldots, s$ and write $f_i = \gamma^{j_i}h_i$ with $h_i \in R[x]$. By Proposition 3.1(ii), there are units $v_i \in R$ such that $f_i = v_if_i^* = v_i\gamma^{j_i}h_i^*$. If we now put $r_i = \gamma^{j_i}$ and $g_i = v_ih_i^*$ for $i = 0, \ldots, s$, then (i)-(iii) are easily checked. To prove (iv), let $h = r_{i+1}g_i - r_{i+1}g_{i+1}x^{\deg(g_i)} - \deg(g_{i+1}) \in \langle G \rangle$. Since $h \rightarrow_{G}^0$ and $\deg(h) < \deg(g_i)$, only $r_{i+1}g_{i+1}, \ldots, r_sg_s$ can be used in the strong reduction, so $h \in (r_{i+1}g_{i+1}, \ldots, r_sg_s)$. Hence $r_{i+1}g_i \in (r_{i+1}g_{i+1}, \ldots, r_sg_s)$.

Conversely, assume that $G$ is as in the theorem and $0 \leq i \leq s$. We will prove by induction on $i$ that $G_i = \{r_ig_i, \ldots, r_sg_s\}$ is an SGB. The case $i = s$ follows from Proposition 2.6. Assume that $i < s$ and $G_{i+1}$ is an SGB. Firstly, Apol$(r_ig_i) = \{0\}$ since $\ln(g_i)$ is a unit. Now let $i \leq j < k \leq s$ and consider $h = r_kg_j - r_kg_kx^{\deg(g_j)} - \deg(g_{i+1}) \in \text{Spol}(r_jg_j, r_kg_k)$. We first show that $h \in \langle G_{i+1} \rangle$, which is clear if $i < j$. If $j = i$ then $r_{i+1}g_j \in \langle G_{i+1} \rangle$ by (iv) and $r_{i+1}g_k$, so $r_kg_j \in \langle G_{i+1} \rangle$ i.e. $h \in \langle G_{i+1} \rangle$. By the inductive hypothesis $h \rightarrow_{G_{i+1}}^0$ and therefore $h \rightarrow_{G_i}^0$. By Corollary 2.8, $G_i$ is an SGB as required. Thus $G = G_0$ is an SGB, and it is minimal by Corollary 2.9. $\triangleright$

Condition (iv) of Theorem 3.2 implies that $\overline{g}_s/\overline{g}_{s-1}/\cdots/\overline{g}_0$. It might be expected that $r_0g_s|r_ig_i$ for $i = 0, \ldots, s$ as in Theorem 2.11. However, this is in general false:
Example 3.3 Let \( R = \mathbb{Z}/8\mathbb{Z} \) and \( G = \{x^4 - 1, 2(x^2 + 1), 4(x - 1)\} \subset R[x] \). Putting \( r_0 = 1, \ g_0 = x^4 - 1, \ r_1 = 2, \ g_1 = x^2 - 3 \) and \( r_2 = 4, \ g_2 = x - 1 \), one easily sees that \( G \) is a minimal SGB by Theorem 3.2 and that \( r_1g_1 \) is not divisible by \( r_0g_2 \). Moreover, no other minimal SGB \( \{g_0, 2g_0', 4g_2'\} \) (by Theorem 2.10) for \( (G) \) has this property. Using Theorems 2.10 and 3.2 and the fact that \( 2g_1' \not| r_0 \) we see that, up to multiplication by units of \( R \) we can only have \( 2g_1' = 2g_1 \) or \( 2g_1' = 2g_1 + 4g_2 = 2x^2 + 4x + 6 \) and that \( 4g_2' = 4g_2 \) so \( g_2' = g_2 + 2a = x + 2a - 1 \) for some \( a \in R \).

Evaluating \( 2g_1' \) at \( x = 1, 3, 5, 7 \) shows that \( 2g_1' \) is not divisible by \( g_2' \).

It is clear that if \( G \) satisfies Theorem 3.2(i),(ii),(iii) and (iv)’ \( g_1 | \cdots | g_6 \) then \( G \) is a minimal SGB. Example 3.3 also shows that the converse is not true in general. It is however true under certain circumstances:

Theorem 3.4 Let \( I \) be an ideal of \( R[x] \). If there is a monic \( f \in I \) with \( \overline{f} \) square-free, then \( I \) has a minimal SGB \( G' = \{r_0g_0', \ldots, g_s'\} \) which satisfies Theorem 3.2(i)-(iii), (iv)’ above, \( j_0 = 0 \) and \( g_k' \not| f \).

Proof. Let \( G \) be a minimal SGB for \( I \) as in Theorem 3.2. As \( f \) is monic and \( f \not| r_0 \), \( j_0 = 0 \). By (iv), \( \frac{\overline{g}_i}{\overline{g}_i+1} \overline{g}_i \) for \( i = 0, \ldots, s - 1 \). Also \( \overline{g}_i \overline{f} \in \overline{I} = \langle \overline{g}_s \rangle \). Putting \( h_1 = f \overline{g}_0, \ h_i = \overline{g}_i \overline{g}_{i+1} \) for \( i = 0, \ldots, s - 1 \) and \( h_s = \overline{g}_s \), we have \( \overline{f} = h_1h_0 \cdots h_s \). Since \( \overline{f} \) is square-free, the factors \( h_i \) are pairwise coprime and Hensel lifting yields \( f = h_1', h_0', \cdots, h_s' \) with the \( h_i' \) monic, pairwise coprime and \( \overline{h}_i = h_i \) for \( -1 \leq i \leq s \). It is easy to check that \( g_k' | f \) and that \( G' \) satisfies (i)-(iv)’. Thus \( G' \) is a minimal SGB.

It remains to show that \( \langle G' \rangle = I \). To show that \( r_0g_i' \in I \) for \( i = 0, \ldots, s \) we will use a technique similar to that of [5, Corollary of Theorem 6]. Since \( \overline{g}_i = \overline{g}_i' \), \( g_i = g_i' + \gamma_i \) for some \( l_i \in R[x] \). It suffices to show that \( r_0\gamma_i l_i \in I \). We know that \( g_i' \not| f \), so \( f = v_i g_i' \) for some \( v_i \in R[x] \). Since \( \overline{f} = \overline{v_i} \overline{g}_i' = \overline{v_i} \overline{g}_i \) and \( \overline{f} \) is square-free, \( \overline{v_i} \) and \( \overline{g}_i \) are coprime. By [10, Theorem XIII.4] \( v_i \) and \( g_i \) are coprime in \( R[x] \) e.g. \( 1 = av_i + bg_i \) for some \( a, b \in R[x] \). Multiplying by \( r_0\gamma_i l_i \) gives

\[
r_0\gamma_i l_i = av_i(r_0\gamma_i l_i) + b(r_0\gamma_i l_i)g_i = av_i(r_0g_i' - g_i) + br_0\gamma_i l_i g_i = (av_i f + (br_0\gamma_i l_i - av_i)r_0g_i) \in I
\]

and so \( \langle G' \rangle ' \subset I \). For the reverse inclusion, suppose that \( h \in I \setminus \langle G' \rangle ' \) has minimal degree.

Since \( G \) is an SGB for \( I \), we have \( \text{lm}(r_jg_j) | \text{lm}(h) \) for some \( j \). But \( \text{lm}(r_jg_j') = \text{lm}(r_jg_j) \), so \( h \) is strongly reducible wrt. \( G' \), \( h \not| r_j g_j' \). Say then \( h = h_1 \in \langle G' \rangle ' \), \( h_1 \neq 0 \) (otherwise \( h \in I \)) and \( \text{deg}(h_1) < \text{deg}(h) \), for a contradiction.

\( \blacklozenge \)

Remarks 3.5 (i) The hypotheses of Theorem 3.4 can be relaxed to \( I \) having a minimal SGB

\( G = \{r_0g_0, \ldots, r_s g_s\} \) of Theorem 3.2 with \( r_0 = 1 \) and \( \overline{g}_i \overline{g}_{i+1} \) pairwise coprime for \( i = 0, \ldots, s - 1 \).

(ii) The minimal SGB of Theorem 3.2 is similar to the 'canonical generating system (CGS)' of an ideal of \( R[x] \), [11, Proposition 13], although GB's and cyclic codes were not mentioned in [11].
A CGS has been generalised to an ideal \( I \) of \( R[x_1, \ldots, x_n] \) for which \( R[x_1, \ldots, x_n]/I \) is finitely generated in [12]. Some connections with Corollary 2.8 are discussed in [12, Section 5].

4 Cyclic codes over a finite chain ring

We now consider cyclic codes of arbitrary length \( n \) over an Artinian chain ring \( R \). As usual, such codes correspond to ideals of \( R[x]/(x^n - 1) \). Let \( q: R[x] \to R[x]/(x^n - 1) \) be the quotient map. The following result is a straightforward generalisation of the corresponding result for fields (see [2, Theorem 9.19]).

**Proposition 4.1** Let \( I \) be an ideal of \( R[x] \) with \( x^n - 1 \in I \) and let \( G \) be an SGB for \( I \). Then for \( f \in R[x] \), \( q(f) \in q(I) \) if and only if \( f \mapsto 0 \).

Using Theorem 3.2 and Proposition 4.1 we obtain:

**Theorem 4.2** Let \( C \subset R[x]/(x^n - 1) \) be a non-zero cyclic code. There is an \( s \leq n - 1 \) and an \( G = \{r_0g_0, \ldots, r_sg_s\} \subset R[x] \) such that \( q(G) \) generates \( C \) and (i) \( r_i = \gamma_i^j \) for \( i = 0, \ldots, s \) and \( 0 \leq j_0 < \cdots < j_s \leq n - 1 \); (ii) \( \text{lc}(g_i) \) is a unit for \( i = 0, \ldots, s \); (iii) \( n > \deg(g_0) > \cdots > \deg(g_s) \) and (iv) \( r_{i+1}g_i \in \langle r_{i+1}g_{i+1}, \ldots, r_sg_s \rangle \) for \( i = 0, \ldots, s - 1 \).

Moreover \( r_0(x^n - 1) \mapsto 0 \) and if \( \deg(f) < n \) then \( q(f) \in C \) if and only if \( f \mapsto 0 \).

Note that the last property of the preceding theorem gives an error-detection algorithm for \( C \). Theorem 4.2 implies in particular that \( \gamma_1 \cdots \gamma_s | x^n - 1 \). Since \( x^n - 1 \) is square-free if and only if \( \gcd(\text{char}(R), n) = 1 \), Theorem 3.4 and Proposition 4.1 yield:

**Theorem 4.3** If \( \gcd(\text{char}(R), n) = 1 \), then Theorem 4.2 holds with property (iv) replaced by the stronger condition \( g_s | \cdots | g_0 | x^n - 1 \).

The restriction \( \gcd(\text{char}(R), n) = 1 \) is essential in Theorem 4.3 as Example 3.3 shows. The existence of a set of generators for a cyclic code as in Theorem 4.3 was proved in [5, Theorem 6] when \( R = \mathbb{Z}/p^k\mathbb{Z} \) and \( \gcd(p, n) = 1 \); see also [14, Theorem 3.17] and [8]. For negacyclic codes, constacyclic codes, or, more generally, codes which are ideals in \( R[x]/(g) \) for a given \( g \in R[x] \), we can obtain analogues of Theorem 4.2 by simply replacing \( x^n - 1 \) by \( g \). If \( \gamma \) is square-free, then we also obtain \( g_s | \cdots | g_0 | g \).

5 Minimal SGB’s over a principal ideal ring

We generalise Theorems 2.11 and 3.2 to a principal ideal ring using some technical results collected in Subsection 5.1.
5.1 Preliminaries

Suppose that $A = A_1 \times \cdots \times A_m$ is a direct product of rings. The projections $\pi_i : A \to A_i$ induce maps $\pi_i : A[x] \to A_i[x]$. It is straightforward to check that the induced map $\pi : A[x] \to A_1[x] \times \cdots \times A_m[x]$ given by $\pi(f) = (\pi_1(f), \ldots, \pi_m(f))$ and the map $\kappa : A_1[x] \times \cdots \times A_m[x] \to A[x]$, which collects coefficients of like terms, are mutually inverse ring homomorphisms.

**Definition 5.1** Let $G_i \subseteq A_i[x] \setminus \{0\}$ for $i = 1, 2$. Then $G_1 \uplus G_2$, the strong join of $G_1, G_2$ is the subset $G_1 \times \{0\} \cup \{0\} \times G_2 \cup \{(t_1 g_1, t_2 g_2) : g_i \in G_i, t_i = \text{lcm}(\text{lt}(g_1), \text{lt}(g_2))/\text{lt}(g_i)\}$ of $A_1[x] \times A_2[x]$.

It was shown in [16] that

**Theorem 5.2** Let $I$ be a non-zero ideal in $A[x]$ and $G_i \subseteq \pi_i(I) \setminus \{0\}$ for $i = 1, 2$. Then $\kappa(G_1 \uplus G_2)$ is an SGB for $I$ if and only if $G_i$ is an SGB for $\pi_i(I)$ for $i = 1, 2$.

We will use the following lemma:

**Lemma 5.3** Any non-zero ideal of $R[x]$ has an SGB $\{r_0 g_i, \ldots, r_s g_s\}$ with $r_i \in R, \text{lcm}(g_i) = r$ for $i = 0, \ldots, s$ and $r \in R$ is not a zero-divisor.

**Proof.** If $R$ is a principal ideal domain or an Artinian chain ring, the result follows by Theorem 2.11 and by Theorem 3.2, respectively. Suppose now that $R = R_1 \times R_2$ where $R_1, R_2$ are principal ideal rings such that the theorem holds in $R_1[x]$ and $R_2[x]$. We will show that the theorem holds for $R[x]$. Let $I$ be an ideal in $R[x]$. By hypothesis, for $l = 1, 2$ there are $r^{(l)} \in R_l$ which are not zero-divisors, $s_l \geq 0, r_i^{(l)} \in R_l, g_i^{(l)} \in R_l[x]$ with $\text{lcm}(g_i^{(l)}) = r^{(l)}$ for $i = 0, \ldots, s_l$ such that $G^{(l)} = \{r_0^{(l)} g_0^{(l)}, \ldots, r_{s_l}^{(l)} g_{s_l}^{(l)}\}$ is an SGB for $\pi_l(I)$. Let $G = \kappa(G^{(1)} \uplus G^{(2)})$. By Theorem 5.2, $G$ is an SGB for $I$. Let $s = |G| - 1$ and denote by $f_0, \ldots, f_s$ the elements of $G$. Let $r = \kappa(r^{(1)}, r^{(2)})$. Since neither $r^{(1)}$ nor $r^{(2)}$ are zero-divisors, $r$ is not a zero-divisor. For $k = 0, \ldots, s$, we will define $r_k \in R$ and $g_k \in R[x]$ such that $f_k = r_k g_k$ and $\text{lcm}(g_k) = r$. If $f_k = \kappa(r_i^{(1)} g_i^{(1)}, 0)$ for some $0 \leq i \leq s_1$, define $r_k = \kappa(r_i^{(1)}, 0)$ and $g_k = \kappa(g_i^{(1)}, r^{(2)} x^{\deg(g_i^{(1)})})$. If $f_k = \kappa(0, r_j^{(2)} g_j^{(2)})$ for some $0 \leq j \leq s_2$, define $r_k = \kappa(0, r_j^{(2)})$ and $g_k = \kappa(r^{(1)} x^{\deg(g_j^{(2)})}, g_j^{(2)})$. Finally, if

$$f_k = \kappa(r_i^{(1)} g_i^{(1)}, x^\max\{0, \deg(g_j^{(2)})\} s_i^{(1)} x^\max\{0, \deg(s_j^{(1)})\}, r_j^{(2)} g_j^{(2)}, x^\max\{0, \deg(s_j^{(2)})\})$$

for some $0 \leq i \leq s_1$ and $0 \leq j \leq s_2$, define $r_k = \kappa(r_i^{(1)}, r_j^{(2)})$ and

$$g_k = \kappa(g_i^{(1)} x^\max\{0, \deg(g_j^{(2)})\} s_i^{(1)} x^\max\{0, \deg(s_j^{(1)})\}, g_j^{(2)} x^\max\{0, \deg(s_j^{(2)})\})$$.

It is easy to verify now that $f_k = r_k g_k$ and $\text{lcm}(g_k) = r$ for $k = 1, \ldots, s$. The result now follows easily from Theorem 2.1.
5.2 Characterisation of minimal SGB over a principal ideal ring

We now generalize Theorems 2.11 and 3.2 to a principal ideal ring:

**Theorem 5.4** A finite set $G \subset R[x] \setminus \{0\}$ is a minimal SGB if and only if $G = \{r_0 g_0, \ldots, r_s g_s\}$ for some $r_i \in R$ and $g_i \in R[x]$ such that (i) $\langle r_i \rangle_R \supset \langle r_{i+1} \rangle_R$ for $i = 0, \ldots, s-1$; (ii) $\text{lc}(g_i) = r_i$ for $i = 0, \ldots, s$ and $r$ is not a zero-divisor; (iii) $\deg(g_i) > \deg(g_{i+1})$ for $i = 0, \ldots, s-1$ and (iv) $r_{i+1} g_i \in \langle r_{i+1} g_{i+1}, \ldots, g_s \rangle$ for $i = 0, \ldots, s-1$.

**Proof.** Let $G = \{f_0, \ldots, f_s\}$ with $\deg(f_i) > \deg(f_{i+1})$ for $i = 0, \ldots, s-1$ be a minimal SGB for $I = \langle G \rangle$. By Lemma 5.3 there are $r \in R$, $r$ not a zero-divisor, $s' \geq 0$, $r_i' \in R$, $g_i' \in R[x]$ with $\text{lc}(g_i') = r_i'$ for $i = 0, \ldots, s'$ such that $G' = \{r_0' g_0', \ldots, r_s' g_s'\}$ is an SGB for $I$. Without loss of generality, we may assume that $G'$ is minimal. By Theorem 2.10, $s' = s$. By Corollary 2.9, we may also assume that $\deg(g_i') > \deg(g_{i+1}')$ and $(r_i' \text{lc}(g_i'))_R \supset \langle r_{i+1}' \text{lc}(g_{i+1}') \rangle_R$ for $i = 0, \ldots, s-1$.

Since $\text{lc}(g_i') = r_i'$ for all $i$, $\langle r_i' \rangle_R \supset \langle r_{i+1}' \rangle_R$. By Theorem 2.10 again, there are units $u_i \in R$ such that $\text{lm}(f_i) = u_i \text{lm}(r_i' g_i') = u_i r_i' \text{lm}(g_i')$, for $i = 0, \ldots, s$. Now fix an $i$ with $0 \leq i \leq s$. Since $G'$ is an SGB for $I$ and $f_i \in I$, we have $f_i \rightarrow g_i'$. In this reduction only polynomials of degree at most $\deg(f_i) = \deg(g_i')$ can be used, so $f_i \in \langle r_i' g_i', \ldots, r_s' g_s' \rangle$. Since $r_i' | r_k'$ for all $0 \leq k \leq s$, we have $r_i | f_i$. So there is a $g_i \in R[x]$ such that $f_i = u_i r_i g_i$. Since $\text{lc}(f_i) = u_i r_i' \text{lc}(g_i')$ we can choose $\text{lc}(g_i)$ to be equal to $\text{lc}(g_i') = r$. Putting $r_i = u_i r_i'$, we have $f_i = r_i g_i$ and conditions (i)-(iii) are verified. Condition (iv) can be checked as in the proof of Theorem 3.2.

Conversely, assume that $G$ has the form $G = \{r_0 g_0, \ldots, r_s g_s\}$ with $r_i, g_i$ having the properties specified in the statement of the theorem. We will prove that $G$ is an SGB using Corollary 2.7. Conditions (A) and (B) follow by the same arguments as in the proof of Theorem 3.2. For condition (C), note that $r_i g_i \in \text{Gpd}(r_i g_i, r_j g_j)$ is obviously strongly reducible wrt $G$ for any $0 \leq i < j \leq s$. Hence $G$ is an SGB. The minimality of $G$ follows from Corollary 2.9.

If $G$ satisfies Theorem 5.4(i), (ii), (iii) and (iv) with $g_{i+1} | g_i$ for $i = 0, \ldots, s-1$ then $G$ is a minimal SGB. However, condition (iv)' is not a necessary condition, as Example 3.3 shows. We saw that when $R$ is an Artinian chain ring we have $\mathfrak{m}/\mathfrak{m}^{-1} \cdots \mathfrak{m}_0$. This weaker divisibility property is generalised below for principal ideal rings:

**Corollary 5.5** Let $G = \{r_0 g_0, \ldots, r_s g_s\}$ be a minimal SGB with $r_i, g_i$ as in Theorem 5.4. For $i = 0, \ldots, s$, let $a_i \in R$ be such that $a_i r_i = r_{i+1}$ and $\langle a_i \rangle_R = \langle (r_{i+1})_R : r_i \rangle$, with the convention $r_{s+1} = 0$. Then $\varphi_{a_j}(g_j) | \varphi_{a_i}(g_i)$ for all $0 \leq i < j \leq s$.

**Proof.** The existence of the $a_i$ follows by [15, Proposition 5.1]. A simple induction on $j - i$ shows that $r_j g_i \in \langle r_j g_j, \ldots, r_s g_s \rangle$ for all $0 \leq i < j \leq s$. (The base of the induction follows from Theorem 5.4(iv)). Hence there are $h_j, \ldots, h_s \in R[x]$ such that $r_j g_i = r_j g_j h_j + r_{j+1} g_{j+1} h_{j+1} + \cdots$
\( r_s g_s h_s \). This can be rewritten as \( r_j (g_i - g_j h_j) - r_{j+1} h = 0 \) with \( h = g_{j+1} h_{j+1} + a_{j+1} g_{j+2} h_{j+2} + \cdots + a_{j+1} \cdots a_1 g_s h_s \). Hence \( r_j (g_i - g_j h_j - a_j h) = 0 \) i.e. each coefficient of \( g_i - g_j h_j - a_j h \) is in \( \text{Ann}(r_j) = (\langle 0 \rangle_R : r_j) \subseteq (\langle r_{j+1} \rangle_R : r_j) = \langle a_j \rangle_R \). Hence \( \varphi_{a_j}(g_i - g_j h_j - a_j h) = \varphi_{a_j}(g_i - g_j h_j) = 0 \) i.e. \( \varphi_{a_j}(g_j) = \varphi_{a_j}(g_i) \).

Since Proposition 4.1 clearly applies to any ring, we deduce from Theorem 5.4:

**Theorem 5.6** Let \( C \subseteq \mathbb{Z}_{R[x]/\langle x^n - 1 \rangle} \) be a cyclic code over a principal ideal ring \( R \). There is a \( G = \{r_0 g_0, \ldots, r_s g_s\} \) such that \( q(G) \) generates \( C \) and \( r_i \in R \), \( g_i \in R[x] \) satisfy the properties (i)-(iv) in Theorem 5.4. Moreover, \( \deg(g_c) < n \), \( r_c(x^n - 1) \mapsto r_c^* 0 \) and for any \( f \in R[x] \) with \( \deg(f) < n \) we have \( q(f) \in C \) if and only if \( f \mapsto r_c^* 0 \).

6 Some algorithmic consequences

Throughout this section \( R \) will be an Artinian chain ring. Let \( f \in R[x] \setminus \{0\} \). We can compute \( f^* \) by Hensel lifting ([10, Theorem XIII.6]) or we can use Proposition 3.1(i) and compute a minimal SGB for \( \langle f \rangle \) via Algorithm SGB-FCR of [15, Subsection 6.2]; see also [16, Appendix].

We now compare their worst-case complexities. If \( n = \deg(f) \geq m = \deg(f^*) \) and \( d = n - m + 1 \), computing \( f^* \) by Hensel lifting has complexity \( O(\nu dm) \) since there are \( \nu \) lifting steps, each requiring at most \( dm \) operations. Computing an SGB of \( \langle f \rangle \) requires \( O(\nu^2 d^3 n) \) since at most \( nd \) new polynomials (of degree at least \( m \) and at most \( n \)) will be added to the basis and computing the remainder of an S-polynomial or an A-polynomial will take at most \( dn \) operations. It is worth noting that by Lemma 2.4 we can stop the algorithm as soon as we obtain a polynomial of degree \( \deg(\varphi_{\gamma^i}(f)) \) in the basis, where \( \gamma^i \in \text{cont}(f) \).

Thus the worst-case complexity of Hensel lifting is somewhat lower than that of SGB-FCR \( (\langle f \rangle) \). In practice however, the complexity of Hensel lifting varies little with the particular input polynomial, whereas the complexity of computing an SGB varies significantly and the worst-case behaviour is rarely achieved. Examples suggest that Algorithm SGB-FCR may be more efficient in general for computing \( f^* \).

Proposition 3.1(ii) yields a variant of Algorithm SGB-FCR for \( R[x] \):

**Algorithm 6.1** (SGB in \( R[x] \), \( R \) an Artinian chain ring, using the \( * \)-construction)

\[ G \leftarrow \text{SGB-FCR}^*(F) \]

**Input**: \( F \) a finite subset of \( R[x] \), where \( R \) is a computable Artinian chain ring.

**Output**: \( G \) an SGB for \( \langle F \rangle \).

**Note**: \( B \) is the set of pairs of polynomials in \( G \) whose S-polynomials still have to be computed.
\begin{verbatim}
begin G \leftarrow \{g^*|g \in F\}; B \leftarrow \{ \{f_1, f_2\}|f_1, f_2 \in G, f_1 \neq f_2\};
while B \neq \emptyset do
    select \{ f_1, f_2 \} from B
    B \leftarrow B \setminus \{ \{ f_1, f_2 \} \}
    compute h \in \text{Spol} (f_1, f_2)
    compute g \in \text{SRem}(h, G)
    if g \neq 0 then compute g^*; B \leftarrow B \cup \{ \{ g^*, f \}|f \in G\}; G \leftarrow G \cup \{ g^*\}; end if
end while
return(G)
end
\end{verbatim}

Note that \(g^*\) can be computed by Hensel lifting or via the original algorithm \textbf{SGB-FCR} \((\{g\})\), and that adding \(g^*\) rather than \(g\) to the basis is advantageous as \(\text{deg}(g^*) \leq \text{deg}(g)\) and \(\text{Im}(g^*) \leq \text{Im}(g)\).

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\section*{References}


