Negotiating between learner and mathematics: a conceptual framework to analyze teacher sensitivity toward constructivism in a mathematics classroom

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Negotiating Between Learner and Mathematics: A Conceptual Framework to Analyze Teacher Sensitivity Towards Constructivism in a Mathematics Classroom

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Side-header: Mathematics Education Experiments in Radical Constructivism

Problem – Constructivist teachers, particularly RC ones, who find themselves working within an educational system that adopts a realist epistemology, may find themselves at odds with their own beliefs when they catch themselves paying closer attention to the knowledge authorities intend them to teach rather than the knowledge being constructed by their learners. In the preliminary analysis of the mathematical learning of six low-performing Year 7 boys in a Maltese secondary school, whom one of us taught during the scholastic year 2014-15, we constructed a conceptual framework which would help us analyze the extent to which he managed to be sensitive to constructivism in a typical classroom setting.

Method – We describe the development of the framework M-N-L (Mathematics-Negotiation-Learner) as a viable analytical tool to search for significant moments in the lessons in which the teacher appeared to engage in what we define as “constructivist teaching” (CT) during mathematics lessons. The development of M-N-L is part of a research program investigating the way low-performing students make mathematical sense of new notation with the help of the software Grid Algebra.

Results – M-N-L was found to be an effective instrument which helped to determine the extent to which the teacher was sensitive to his own constructivist beliefs while trying to negotiate a balance between the mathematical concepts he was expected to teach and the conceptual constructions of his students.

Implications – The results of this study entail two major implications. One is that it is indeed possible for mathematics teachers to be sensitive to the individual constructions of their learners without losing sight of the concepts that society, represented by curriculum planners, deems necessary for students to learn. The other is that researchers in the field of education may find M-N-L a helpful tool to analyze CT during typical didactical situations established in classroom settings. In this paper M-N-L is presented as an analytical framework used in a particular setting: the teaching of mathematics to low-performing students with the help of computer software. In order to demonstrate the practicality and application of M-N-L, the report enters into the details of a mathematics lesson context. However, the basic tenets of M-N-L may be applied in the context of other types of lessons and in other subject areas.

Constructivist content: Radical constructivism as espoused by Ernst von Glasersfeld and as interpreted in the context of mathematics education by Leslie Steffe has played a
major role in devising M-N-L. The work of John Dewey, Martin Simon, Heinz Steinbring, and David Kolb inspired aspects of the framework as well.

Key Words – radical constructivism, mathematics teaching, mathematics learning, constructivist teaching, constructivist framework.

Whoever teaches learns in the act of teaching, and whoever learns teaches in the act of learning. – Paolo Freire 1998: 31

Introduction

1. There are at least four arguments researchers use to contest the existence of a kind of teaching which may be called “constructivist teaching” (CT):
   - Constructivism is a theory of learning, not of teaching (e.g., Simon 1994, Engström 2014);
   - Irrespective of the teaching approach, learners will come to know by actively constructing mental constructs for themselves (Simon 1994);
   - CT is mistakenly equated with progressive modes of teaching (e.g., Engström 2014);
   - A constructivist belief does not translate itself into a particular teaching method (Simon 1995).

2. We agree with all these arguments but we still claim that the notion of CT is viable if it is attributed to a teacher’s sensitivity to learners’ active and subjective construction of knowledge during the teaching-and-learning process. As part of a larger research project, we have developed an analytic framework (M-N-L) which may be used to investigate CT in a typical lesson within a school setting.

Key terms

3. The M-N-L framework and its application to analyze CT makes use of a number of key terms which need to be defined at the outset. These definitions are our own understandings of the meanings of these terms:

   a. “Mathematics” or “subject matter” is the consensual domain (in the sense of Maturana & Varela 1980; Glasersfeld 1991a) of mathematical concepts and skills existing among persons who have come to internalize them through personal experiences, and who are capable of utilizing them to make sense of further “similar” experiences and to use them as a scaffold to deepen their understanding of those concepts and skills and to learn others. Such “mathematics” is agreed upon and communicated within mathematical communities (such as in a mathematics classroom) by building a consensus in certain aspects of the subjective realities of the persons belonging to those communities.
b. “Teacher” is the person who, besides being knowledgeable in subject matter, is also knowledgeable in the art of helping other persons (students) to construct meanings, and establish a consensual domain with them in such a way that they can communicate each other’s knowledge through interaction.

c. “Interaction” is the act of re-presenting (Glasersfeld 1991c) internalized concepts through various external expressions (such as utterances, body movements, diagrams, and symbols) which are interpreted by other persons in relation to their experiential world. The persons taking part in an interaction, such as the teacher and the students, are “first-order observers” of their own experiences made up of their own re-presentations and of the occurrences that follow those re-presentations.

d. “Learner” may be taken to be synonymous with “student”, but while a “student” is such because he or she attends a lesson (usually in a school setting), a “learner” is such because he or she learns something from that lesson. Only the learners themselves have “control” over and first-order knowledge of their own learning and so, while the M-N-L framework assigns the status of “learner” to each and every student, actions done by the teacher are discussed in relation to “students”. This is a subtle, but important change of terms to indicate that the teacher strives for learning but the act of learning is up to the student.

e. “Second-order observer”, a term first used by Humberto Maturana (1978), is a person who “observes the occasions of observing of the first-order observer” (Steffe & Wiegel 1996: 483). As observers of the interactions between teacher and students, we regard ourselves as second-order observers, where we can only hypothesize about the mental states of the teacher and the students with reference to our (mathematical) consensual domain. Being the teacher-researcher, one of us took on the roles of both first- and second-order observer, the former as teacher during the lessons and the latter as researcher during the data analysis. Since much of what is reported here is presented from his perspective as a teacher-researcher, the analysis is mostly an expression of self-reflexivity.

f. “Mathematics of the students” (MOS) is the “mathematics” inside students’ minds and hence no one, except the learners themselves, can have access to it. A teacher may form second-order experiential models (Steffe et al. 1983) of MOS by observing students’ external actions, compare them to other “similar” actions, including his or her own, and hypothesize about the mental state of the students. By engaging students in mathematical discussion and activities, the teacher establishes viable models of students’ syntactical and lexical mathematical meanings (Thompson 2013).

g. “Mathematics for the students” (MFS) is the mathematics the teacher intends to teach to a particular group of students. It includes the teacher’s models of “mathematics” of other students “like” the current students he or she has gained through teaching experiences and also the teacher’s “mathematics” he or she would not attribute to the current students. The teacher’s formation of MFS is the
settlement of the continual perturbations created by tensions arising from these two kinds of “mathematics” and based on his or her interpretation of MOS.

h. “Curriculum”, as used in this report, is the program of teaching and learning a body of subject content determined by curriculum planners in educational authority positions and made obligatory in schools. In educational contexts that adopt a realist epistemology, such as that of the participants of this study, the curriculum is handed down from authorities as a body of a priori knowledge. RC teachers, however, believe that no such thing exists and that the curricular topics they are required to teach are actually knowledge they (the teachers) have construed for themselves. In particular, “teachers’ mathematics” is the “mathematics” (as defined above) teachers have constructed from their experiences as learners, combined and enriched with MOS and MFS they continually assimilate from their experiences as teachers.

**Description of the M-N-L framework**

4. The Mathematics-Negotiation-Learner (M-N-L) framework proposes a theory about what CT might mean by focusing on the role of the teacher in the educative process and portrays him or her as a negotiator between the learner and the subject content (hence the dashes in M-N-L). This does not mean we are presenting a curriculum-versus-learner construct. Rather, we propose that just as “two points define a straight line” (Dewey 1902: 16) so does the “mathematics” of the teacher and that of the student (MOS) define the course of teaching. We view the teacher as a guide who walks with students in their journey towards the formation of a consensual domain that contains the teacher’s “mathematics”, which forms part of the curriculum and is designated by the teacher as “mathematics for the (current) students” (MFS), but which also contains MOS. An important derivative of this process is that the teacher’s “mathematics” is deepened by his or her reflections on MOS. The constructivist teacher learns in the process of teaching and thus, the consensual domain between teacher and learner (and, possibly, other learners in the classroom) is not only a construction of the learner but also of the teacher. Figure 1 illustrates the dynamics of such a negotiation, where “negotiation” is demonstrated by a left-right arrow connecting “mathematics” and learner.

![Figure 1: The Mathematics-Negotiation-Learner framework.](image-url)
We use the metaphor of a two-way road to explain the process of negotiation, with two stages going one way and two stages going the other way. The following is a description of the stages starting from the upper left-hand arrow going from “mathematics” to learner:

(a) Building on knowledge of MOS of previous similar students, the teacher anticipates possible didactic processes that can enable the current students to conceptualize and internalize MFS, a “subset” of the teacher’s mathematics. Martin Simon (1995) calls this a hypothetical learning trajectory, since the teacher has no way of knowing the learning processes in advance.

(b) The teacher interacts with the students according to his or her anticipations of the didactic processes. The interaction can take many forms, including:

- Questioning the students to stimulate discussion and to create second-order models of their thinking processes and of the MOS;
- Answering students’ questions and elaborating on their statements to encourage students to think more deeply about MOS;
- Introducing and setting goal-oriented activities (through demonstrations and discussions), intended to help students gain access to MFS and guiding the students in those activities;
- Giving feedback to students about the outcome of set activities.

Besides the more progressive goal-oriented activities, our understanding of “interaction” thus includes teacher exposition (plenary approach), which a second-order observer might consider “traditional” and “non-constructivist.” What constructivist teachers bring to the table that would distinguish them from non-constructivist ones is that their actions are intended to form second-order experiential models of their students’ thinking. This may be observed (by second-order observers) through teacher expressions such as “Tell me what you think about this”, or “Ask me something about this”, even in “traditional” plenary teacher expositions.

(c) The “Learner” section of Figure 1 shows how this interaction helps students experience some form of mathematical phenomenon that the teacher encourages them to reflect upon (“Now, why do you think that happens?”), sometimes with the help of further experimentation on that experience. Students are bound to make abstract conceptualizations which result from this interplay between experience and reflection, a process by which they become “learners.” This is reminiscent of David Kolb’s (1984) Experiential Learning construct but the emphasis here is on what constructivist teachers do with the manifestations of learners’ mathematics.

(d) The first arrow moving from learner to “mathematics” illustrates the result of the constructivist teacher’s first-order observation of students’ representations of their knowledge, which they do as a result of reflecting upon
their experiences. The teacher uses this observation to create second-order models of MOS.

(e) Without abandoning the curriculum or his or her “mathematics” the teacher uses these models of MOS to review MFS (Steffe 1991). “Review” is a rather mild term which we use to denote the teacher’s entry into the "mathematics" perturbation that follows.

(f) The “Mathematics” section of Figure 1 signifies the perturbation that the teacher goes through when faced with the tension created between his or her original MFS and the models of MOS. Sometimes this perturbation is settled easily by simply assimilating MOS with the intended MFS. At other times the teacher needs to adapt his or her MFS in order to cater for MOS (accommodation). The teacher’s equilibration (Piaget 1985) of MFS, which forms part of his or her “mathematics”, renders teachers learners themselves.

6. The structure of M-N-L may give the impression of a cycle of stages occurring in a neat order throughout the lesson such as Simon’s (1995) model of teaching from a constructivist perspective or Heinz Steinbring’s (1998) construct of teaching and learning processes. We propose that the stages of the M-N-L framework are only indicative of the order in which they occur and the two ways in which “mathematics” is connected to the learner. In the course of a lesson a teacher might skip a stage or revert to a preceding stage if he or she feels the need to focus on one aspect of the negotiation dynamics.

7. Something which distinguishes M-N-L from both Simon’s (1995) and Steinbring’s (1998) models is that the teacher-learner interaction does not only help the teacher form hypothetical learning trajectories (Simon 1995) or inform the teacher about the appropriateness of further learning offers (Steinbring 1998), but also enriches the teacher’s mathematical content knowledge. This is where we disagree with Steinbring’s (1998) claim that the act of mathematics teaching can be regarded as an autonomous system. While learners, even though they may be better or worse off with the interventions of the teacher, construct mathematical ideas in a relatively autonomous way, teachers can never regard their actions in the classroom as being independent of the learners. What M-N-L proposes is quite the contrary. CT is almost totally dependent on students’ feedback. Even if autonomy of teaching approach is granted by school authorities, teachers engaging in CT need to allow the learning environment they set up to be affected and, to a certain extent, determined by what they learn from their students.

8. We propose that a teacher engaging in CT will show sensitivity to constructivist notions when he or she is observed to try to negotiate a “road” that connects the “mathematics” of the teacher to that of the learner. All this would not be possible without the teacher’s sensitivity to students’ mathematical understandings (Jaworski 2012). Consequently, we regard the teacher’s lack of sensitivity or deliberate disregard of this sensitivity to be a roadblock in one of the two paths between “mathematics” and learner: the teacher either fails to let his or her mathematics be influenced by knowledge about learners’ “mathematics” or fails to interact with the learners in a way that encourages active participation. Before demonstrating how second-order observers may use M-N-L to
investigate CT in a typical mathematics lesson, we will now describe the classroom context and participants of the present research study.

The research context and participants

9. The group of participants consisted of six Year 7 boys (pseudonyms: Dwayne, Tony, Omar, Jordan, and Joseph, being 11-year-olds and Dan, a 12-year-old) attending a single-sex secondary school where the teacher-researcher had been working for 18 years as a full-time mathematics teacher\(^1\). In the subjects of Mathematics, Maltese (mother language), and English (second language) students in this school were divided into three groups according to their performance in these particular subjects. The participants formed the lowest-performing group in Year 7 mathematics. This performance was measured from their scores in a national benchmark examination at the end of Year 6. In the mathematics examination, the participants’ scores ranged between 1 and 3 standard deviations below the national mean.

10. A substantial part of the research project concerned the use of software called Grid Algebra\(^2\) (GA) which is a grid-based computer environment built on the multiplication table grid. GA was used due to its potential to help students conceptualise the meaning signified by standard mathematical notation, something which we regard as crucial in students’ introduction to formal algebra. Although the focus here is not GA, an explanation of some of its features is necessary to understand the protocols of lesson episodes included later on. Figures 2–4 show GA computer interfaces. The first, Figure 2, shows a blank interface with cells representing multiples of 1-6. An integer may be chosen from a number menu and inserted anywhere on the grid, as long as it is a multiple of the number showing at the beginning of a row.

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\(^1\) The first author was the teacher-researcher in this study up till the scholastic year 2014-2015. For the sake of readability, he will be mostly referred to as “the teacher”.

\(^2\) This software was developed by Dave Hewitt. Readers are invited to view short demonstrations of how it works by visiting https://www.youtube.com/watch?v=HmVjprJWInM and the subsequent links (Grid Algebra 1-4).
11. Once a number is placed inside a cell it determines what numbers could be entered in the other cells. As shown in Figure 3, entering “5” in the second column of the first row means that the number that could be entered in the second column of the third row has to be “15” (5 times 3) and the one in the second column of the sixth row has to be “30” (5 times 6). An important feature of GA is that the cells (containing numbers or expressions) may be moved around the grid. When a cell is dragged onto another, the number or expression in the original position changes according to the mathematical operation associated with that movement:

- right movement corresponds to addition;
- left movement corresponds to subtraction;
- downward movement corresponds to multiplication;
- upward movement corresponds to division.
12. Figure 4 shows the result of such movements. The new position of a cell does not show the single numeric “answer” of the operation, but the “answer” is shown by GA as an expression showing standard mathematical notation. So, for example, dragging a cell containing “15” in the third row to the sixth row (corresponding to a multiplication by 2) will result in the expression 2(15) or 15×2 (depending upon a chosen setting). Successive cell movements produce more and more complex expressions. There is also the possibility to assign a letter to a cell. As shown later on, a letter may represent a variable (a general number) or an unknown (a specific number) depending on the situation. Moving a cell with a letter results in expressions similar to those obtained when moving a cell with a number, the only difference being that the resulting expressions would be formal-algebraic rather than just numeric. Moreover, when a cell is the destination of more than one journey, it stands for more than one expression. When this happens the software allows the user to see the equality of the two expressions, such as 2(5+1)=10+2. This feature was used to help students understand that (a) an “answer” is not necessarily a single number (it can be an expression) and (b) the number on the right of the equals sign is not necessarily the result of the calculation on the left (in the above equality, 10 is not the result of 2(5+1)).

![Figure 4: Movements of GA cells to new positions to obtain new expressions.](image)

**Data collection and analysis**

13. The research extended over the scholastic year 2014–2015, where students were given twenty double lessons (80 min each) using GA. These lessons formed part of the curricular lessons for that particular scholastic year. The rest of the lessons did not form part of the research data, although the teacher-researcher (acting only as a teacher in these other lessons) may have used observations from these lessons to serve him for building second-order models of MOS in the research lessons.

14. In the research lessons data was collected through video recordings of the lessons, computer-screen recordings of students’ work, students’ pen-and-paper work, and video-
recorded interviews (five per student throughout the year). The first part of a typical lesson consisted of a class discussion about the topic of the lesson which included plenary demonstrations of GA activities on an interactive whiteboard by the teacher which he usually made with the participation of the students themselves. The second part consisted of students working in pairs on their computers where the teacher went around and assisted the students where necessary.

15. Without assuming any particular teaching method, the teacher-researcher wanted to answer this question: How can GA be used as a tool for constructivist teaching? This question was divided into two more specific subsidiary questions:

- What are the significant moments in the lessons where the teacher can be observed trying to teach in a way that is sympathetic to constructivist notions of learning?
- What are the significant moments in the lessons where the teacher can be observed to be less sympathetic to constructivist notions of learning?

These questions were answered through the formation of categories and themes with reference to the M-N-L framework.

16. An important aspect of the whole research project was that the lessons that provided the data were not a teaching experiment in the classical sense (such as Steffe 1991). The teacher-researcher did not try out something new (he had been using GA for some years with other groups), the lessons formed part of the yearly scholastic scheme of work for that particular group, and none of the class was singled out or excluded. The research was principally an exercise of reflection about the teacher’s teaching and about the students’ learning with the help of insights from relevant literature. The lessons themselves, the teaching interventions, the students’ actions, and the overall dynamics of the teaching episodes served as a stimulus for the generation of data, pertaining to the issue of CT, for this reflection and analysis.

Evidence from preliminary data analysis

17. Preliminary data analysis of the lessons led to the emergence of two central themes in relation to CT:

A. The shifts of focus during CT in order to negotiate relationships between the learners and “mathematics.”

B. The teacher’s loss of sensitivity towards constructivist notions when one of the negotiation paths on the “road” between learners and “mathematics” was blocked.

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3 Only preliminary analysis of the lesson videos and the computer-screen-capture videos will be used for this report.
A. Teacher as negotiator between learner and mathematics

18. This theme was developed from four categories which correspond to shifts in the teacher’s focus as he was trying to establish one of the two paths on the road that connected his “mathematics” with that of the learners:

   I. from the learner (experience and reflection) to the negotiation process (teacher’s formation of models of MOS and review of MFS);

   II. from the negotiation process (as in I) to “mathematics” (association/adaptation of mathematical content intended to be taught);

   III. from “mathematics” (as in II) to the negotiation process (teacher’s anticipation of possible didactic process and interaction with students);

   IV. from the negotiation process (as in III) to the learner (as in I).

19. The order of these shifts of focus are according to the M-N-L framework but there is no starting or finishing stage. In a lesson, the teacher may start with IV and move on to I, II, and III, or start with III, IV, and so on. Moreover, the number of times the teacher shifts his or her focus in a lesson is indefinite. In the following four episodes, we attempt to demonstrate moments in different lessons when the teacher seemed to make such shifts of focus in order to engage in CT. This is done with the help of protocols of lesson conversations with reference to preceding or subsequent moments in the lesson. Protocols are accompanied by diagrams showing the GA-screen interface at the moment of the protocol. Some of the episodes are goal-oriented moments. Others are situations where the teacher and the students were engaged in whole-class discussion.

Shift I: Learner-to-Negotiation

20. In the Learner-to-Negotiation shift, the teacher observes the way that students are dealing with a particular experience and the way they express their reflections upon that experience, and uses this observation to form second-order models of MOS, which he or she may later use to review and possibly modify MFS. In one of the lessons the students were involved in a GA task where a letter was given in a random place on the grid and a random empty cell was highlighted. Their goal was to use an “expression calculator” to formulate a mathematical expression involving that letter that could fit in the highlighted cell. The students were expected to construct three different algebraic expressions for that particular cell before GA presented them with a new problem. Figure 5 shows a screen-capture video still of the computer being shared by Omar and Jordan, who were taking turns in trying out the problems.
21. The “mathematics” required to solve this problem was primarily the knowledge of the operations that correspond to particular, ordered, horizontal or vertical movements along the multiplication grid. This activity was intended to familiarize students with the standard notational convention that designates a particular order of operations for particular mathematical expression. For example, $2(d+1)$ corresponds to the addition of 1 to “d” followed by a multiplication by 2. Traditionally, students have been offered mnemonics such as PEMDAS to remember the convention of the order of operations signified by a mathematical expression (in this case, parenthesis needs to be worked out before multiplication). GA contains inbuilt tasks where students are expected to construct an expression that corresponds to a particular order of operations (movements along the grid) such as the task shown above. It also contains inverse tasks, where students are expected to show the order of operations for a given expression.

22. At this particular moment it was Omar’s turn to try a problem and he had already got one correct answer out of the expected three. Analysis of the video shows that he got to the expression $2(d+1)$ by imagining the cell containing “d” to move one cell to the right (plus 1) and then move from the one-times table to the two-times table (times 2). The circled numbers (1, 2, 3) are superimposed on the video still to show where Omar was pointing during Protocol 1. Just before the episode, Omar had tried out the second algebraic expression for the highlighted cell but got a “no-entry” signal (Figure 5), meaning that his expression was wrong and was not being accepted in the cell. At that moment the teacher was moving around offering his assistance to the students and he turned his attention to Omar, who seemed to be frustrated.

**Protocol 1: Omar’s mistake**

PB: (Notices that Omar got frustrated) What’s the matter there, mate? […]
Omar: I, actually, I did the “d”, I will go here...
PB: Show me with the mouse […] you said, “d” will go there (position 1), good.
Omar: I’ll move over here (position 1), go down here (position 2),...
PB: Good.
Omar: And then I go over here (position 3)
PB: Good. But… Tell me how much plus you do from here (position 2).
23. At this moment the teacher shifted his focus to establish a model of Omar’s MOS. Further discussion suggests that the teacher noticed that from position 2 to position 3 Omar was adding in steps of 1, rather than 2, as expected. Omar himself realized he was making another mistake, that he was not including the designated cell in his counting (he had written +3 instead of +4). The ensuing dialogue suggests that the teacher was comparing Omar’s MOS with teacher’s own “mathematics” (that moving in the two-times table meant adding twos) and helped Omar to realize that in order to move four cells to the right he had to finish his expression with +8 (not +4).

Shift II: Negotiation-to-Mathematics

24. This transition refers to the several occasions where constructivist teachers act upon their model of MOS to review their MFS by assimilating/accommodating it in their “mathematics” schemas. The following episode is not taken from a goal-oriented section of a lesson but from a class discussion where the teacher was checking out whether students were prepared to learn about the concept of a variable and about the distinction between a variable and an unknown.

25. As previously explained (Figure 3), when a number is inserted in the GA grid it predetermines the numbers that could be placed in the other cells. Placing a letter in such a grid would mean that the letter represents a constant number, an unknown. For example, placing “n” in the second-row, second-column cell in the grid of Figure 3 would mean that “n” represents an unknown (n=10). Prior to the episode presented here, at the beginning of the lesson, the students had been working on possible values that a letter in a GA cell could signify if other numbers were already on the grid. The teacher wanted to see what they thought if a letter was placed in an empty grid. On the interactive whiteboard, he inserted “k” inside a random cell, as shown in Figure 6, and asked the students what they thought about it.

![Figure 6: Reproduction of interactive whiteboard showing a letter in the grid.](image)

**Protocol 2: The letter symbolizing a variable**

PB: What number is k symbolizing?
(Joseph and Dwayne put up their hands eagerly)
Dwayne: (Shaking his finger) Sir. Sir. I know. I know.
PB: Be careful! Be careful and think a bit before replying.
Tony: (Without raising his hands) Any number.
(At the same time that Tony spoke PB was giving permission to Dwayne to speak)
PB: (Gesturing towards Dwayne) Come, let’s see.
Dwayne: Any number (here he is joined by Joseph and they talk together) that lies in the two-times table. (Looking at each other) Jinx!...

26. The ensuing discussion suggests the teacher was trying to develop a model of MOS by asking the students to give examples of what “k” might symbolize. When it was established that the students agreed that “k” could stand for any multiple of 2, the teacher decided to introduce the term “variable” and tried to demonstrate the notion of a variable by scrolling along the numbers menu provided by GA. When negative numbers started appearing, Jordan said, “Eh! There would be the negative numbers!” The tone in Jordan’s exclamation seemed to be interpreted by the teacher as meaning that Jordan (and possibly other students) did not consider negative integers to be multiples of two like positive integers (which is what multiplication tables show). Comparing to his “mathematics” that if a variable is a multiple of 2, then that variable could also take on negative values, the teacher set out to ask students what would happen if one skipped backwards on the two-times table and continued skipping beyond 2 and 0. This developed into a discussion about negative multiples of 2, and it was established that the two-times table actually stretched indefinitely on both sides of the number line.

27. The Negotiation-to-Mathematics shift was done twice here by the teacher. The first shift occurred from developing a model of the students’ concept of variable to introducing the term “variable” and discussing it with them in terms of multiples of 2. As soon as the teacher started offering a learning experience by scrolling the numbers menu, Jordan made a comment that the teacher seems to have interpreted as a combination of surprise and realization. The teacher assimilated this model of MOS to his original “mathematics” of a variable and set out to discuss why varying multiples of 2 were not limited to positive integers.

Shift III: Mathematics-to-Negotiation

28. Every time the teacher-researcher introduced something new about GA, his approach was to project some image from GA on the interactive whiteboard and start a discussion about what the students were observing on the screen. This involved at least two anticipations of the didactic process. The first was his anticipation of possible didactic processes through which he might set out to present MFS to the students. Some decisions about MFS were planned before the onset of the lesson according to models the teacher had previously built of the current students’ “mathematics” and about experiential models of MOS of past students who were “similar” in age, grade, and performance level to the current students (the settlement of the “mathematics” perturbation). Some decisions about MFS had to be done in the course of a lesson. The other anticipation was about the possible, viable interactions that could enable those didactic processes.

29. The following episode is taken from the lesson in which the students experienced the GA software for the very first time. The teacher started off by showing them just one row of cells with numbers (Figure 7). The “mathematics” the teacher intended to discuss with the students was that the first row of numbers (1, 2, 3, 4, 5) formed, among other things, the first row of the multiplication tables (the one-times table). The teacher’s goal of the first lesson was to familiarise the students with GA and he decided the best way to do this
was to present the GA grid as a snapshot of the multiplication table grid. Past experience with “similar” students showed that starting with many rows made it hard for students to focus on individual rows and so the decision for MFS at the start of the first lesson was to show only the first row, interact with them (in this case through demonstration and discussion) and discuss what they thought about those numbers by asking questions that deliberately helped students fall back on their experience of such a number sequence.

Protocol 3 follows right after the teacher ran the GA software and set it with the first row of cells filled with numbers, as reproduced in Figure 7.

![Figure 7: GA’s first row of cells with numbers.](image)

**Protocol 3: Making a learning offer**

PB: First of all, are you noticing what these are?
Dwayne: Yes.
PB: What are those that you have in the whole row?
Dwayne: It is a row with numbers.
PB: Okay, tell me where you’ve seen it (the row) before? Have you ever seen it somewhere else? Not just a row of numbers. Mention where you’ve seen it before.
Omar: In a grid.
PB: In a grid of what?
Omar: Of the numbers. When you have all the numbers.
PB: When you have all the numbers. The number grid.
Omar: When you, when you learn to say the tables and the numbers.
...
PB: Stay on what Omar said… He said, “I have seen these before in a number grid, when learning the tables.” So that (pointing to the first row), don’t you think that it is something…? (Dan raised his hand, PB nods towards him).
Dan: The one-times table.

30. Probing questions (anticipation and interaction) usually led to a reflection beyond what was immediately apparent. It is interesting to note how these students built on one another’s statements, each time taking the discussion to a higher mathematical level. For Dwayne it was just a row with numbers, further probing helped Omar “realize” that these numbers were found in the number grid he had used to learn the multiplication tables, and further interaction led Dan to state that what they were seeing was the one-times table. The students’ “mathematics” progressed from “a row of numbers”, to a set taken from the table of multiples, to a particular multiple set – the multiples of 1. The teacher built on this model of MOS to introduce, in this row, large numbers that are not usually seen in multiplication tables (but are still multiples of 1). The lesson then progressed with a discussion about the second row and the relation between the numbers in the first row and those in the second row.

**Shift IV: Negotiation-to-Learner**

31. During this transition, the teacher’s focus turned to what the learners were experiencing and reflecting as a result of his interactions. This shift was at once interesting and demanding for the teacher because the diverse reflections of individual students could,
at times, prove to be overwhelming. Inevitably, this made the teacher negotiate from the students back to his “mathematics” to strengthen or renew the pathways on the “road” between his “mathematics” and that of his students.

32. In the following episode, taken from a lesson where students were quite familiar with GA, the teacher used one aspect of GA to help students focus on the meaning of the equals sign. The “mathematics” that the teacher was aiming to teach here was that besides following a calculation and preceding an answer, the equals sign also showed equality, which could resemble a balance between the left- and the right-hand sides of an equation. The teacher was aware that this was still a restricted concept of the equals sign but he intended to use it to help students start expanding their meaning of equality. At this point in the discussion, the teacher was using the metaphor of the see-saw to help students consider that balance could only be kept if each side of the see-saw had equal weight. Dan raised his hand to speak…

**Protocol 4: Equality of weights on a barbell**

Dan: Like those (meaning weightlifters), don’t they lift that iron bar like that (stands up and takes a weightlifter pose with his hands up)?
PB: (Nodding) The iron bar. Well done!
Dan: If on this side (gestures towards his left hand) you have much more, then this side (left) will topple (makes a toppling motion – Figure 8) and he won’t be able to keep the balance.
PB: Well done. Well done. We have another example of “equals”…

![Figure 8: Dan’s weightlifting analogy.](image)

33. The teacher proceeded to elaborate the discussion about the equals sign using Dan’s weightlifting analogy. We make two observations here. The first is that Dan seemed to find a viable analogy between “equals” and the balance of weights on a weightlifter’s bar, an example of how a mathematical concept may be re-presented in terms of an experiential

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4There are at least eight different mathematical uses of the equals sign (see, for example, Usiskin 1988 and Jones & Pratt 2012).
occurrence. Furthermore, Dan acknowledged the conditions necessary for a balance by showing what would happen if these conditions were not met: a toppling of the barbell.

34. The second observation is that the teacher claims he had never used this analogy before and Dan’s contribution to the discussion enriched the teacher’s “mathematics” because it added to his repertoire of balance metaphors that could be used to help other students become acquainted with the balance notion of the equals sign. His use of the seesaw metaphor (right before the episode) was an attempt to use an example from the probable experiential realities of the current students since experience of past students had taught him that the long-standing metaphor of a balance scales had become almost obsolete in the daily experiences of today’s students due to the current rarity of such an instrument.

35. In the following section we show how M-N-L was found to be helpful in identifying moments in the lesson where the teacher appeared to lose his sensitivity to constructivist notions during the lessons.

B. Teacher as a barrier to negotiation

36. Building second-order models of MOS requires purposeful observation of the interaction of the students with the learning experience. Such an observation is only possible when the teacher is sensitive to constructivist notions of learning. Data revealed that even a self-claimed RC teacher could create barriers to the negotiation of teacher and learner knowledge when he lost or ignored this sensitivity. Figure 9 summarizes how the M-N-L framework helped to identify two ways in which the teacher seemed to fail to engage in CT by barring one of the two negotiation paths on the mathematics-learner “road”:

(a) from the learner side, or
(b) from the mathematics side,

where the black line symbolizes the “roadblock” in that path.

![Figure 9: Roadblocks in the mathematics-learner negotiation paths.](image)

37. There were instances in some lessons where the teacher was observed not to elaborate on a learner’s response or deliberately stopping the discussion. We regard this as a barrier
in the negotiation road from learner to mathematics (Figure 9a), where the teacher seemed to inhibit learners from teaching him something about their “mathematics”. The following protocol is taken from an episode in the first lesson, where the teacher was showing the students what numbers could be allowed to exist in the first row of the GA grid, by scrolling the numbers menu.

**Protocol 5: Stopping a discussion**

PB: …Over there we are going a bit further away as well. (PB scrolls backwards a bit and showing 0 and some negative numbers and scrolling quickly back to 1). I am interested from one.

Joseph: Or from minus one.

PB: From one. But I’m interested from one. Do not take notice that we did, that there was zero as well. When the time comes we will do that as well. (PB continues with what he was originally doing).

38. The teacher said that, at that moment, he was aware of missing the opportunity to learn about Joseph’s mathematics, and possibly to review his own intended MFS. At that particular moment he refused to “try and build up a model of the particular student’s own thinking” (Glasersfeld 1991b: 178) and therefore, from an RC point of view, he could not attempt to strengthen or modify Joseph’s conceptual structures. The teacher also missed the opportunity to foster the student’s motivation (Steffe 1991) to talk and possibly learn about zero and negative numbers. This momentary roadblock between learner and negotiation originated from the nature of the comment of the learner (Joseph) which, at the moment, the teacher felt as threatening the direction he intended for that part of the lesson: a focus on the first few natural numbers and to acknowledge them as the one-times table. The teacher was aware of this failure on his part and at a later stage in the lesson, when the issue of negative numbers cropped up again, he dedicated some time to talk about the need for negative numbers (such as underground car parks or levels shown inside an elevator).

39. The M-N-L framework helped to identify a similar but distinct type of barrier in the mathematics-learner link. It originated when the teacher was so focused on his “mathematics” that he forgot about the learner altogether. We regard this as a block from the “mathematics” side of the negotiation road (Figure 9b) because the obstruction was caused by the teacher’s missing the opportunity to interact with a learner due to a “single-mindedness” on the teacher’s “mathematics”, rather than perceiving a “threat” in the learner’s response (as in the first barrier). The following episode shows such an instance in one of the lessons.

40. The protocol is taken from a lesson where the teacher was discussing with the learners what happens when a number in a cell is dragged downwards to another row. At this moment (Figure 10), two students Dwayne and Tony were assisting the teacher in demonstrating what happens when a number in the first row is dragged to the third row (multiplied by 3) and the resulting expression is again dragged to the sixth row (multiplied by 2).
Figure 10: Teacher blocking the negotiation by insisting on his “mathematics”.

Protocol 6: Too much focus on “mathematics”

PB: So. Let’s go. (pointing to the cell with a “10” and then to the cell in the same column, row 3) If I lower this here what will it become?
Dwayne: Te…, it becomes…ten times three.
PB: (PB moves the “10” onto the cell in row 3 and GA transforms it into $10 \times 3$). Good. And if the “$10 \times 3$”, you put it here (pointing first on the cell with “$10 \times 3$” and then to the cell in the same column, row 6), what does it become? (pointing to the numbers on the left) Look from here, look at the side (Jordan raised his finger and wiggled it)…Three times how much so that… Look from here.
Dwayne: Ten times two,…times…one.
PB: So, “$10 \times 3$” is going to stay there.
Dwayne: Aha (agrees)
PB: So “$10 \times 3$” is going to appear for sure.
Dwayne: Aha (agrees).
PB: Now it (making a gesture like grabbing something), something is going to happen to it.
Dwayne: It is going to change.
PB: It is going to change and something will be added to it (meaning to the expression). But it will stay there, the “$10 \times 3$”.
Dwayne: Aha (agrees). Ten times two…
PB: Ten times two! Why are you saying ten times two? Ten times three is what you have…

41. Hindsight and second-order observation allow us to suggest that Dwayne could have been saying “ten times two” because he was thinking about the multiplication required for a multiple of 3 to become a multiple of 6 ("jumping" from row 3 to row 6). The “correct” expression was $2(10 \times 3)$ which is actually equivalent to “ten times two times three”. The teacher, however, did exactly what Dewey (1902) warned against: he focused too much on the content he intended to teach and forgot about the conceptual needs of the student. The
teacher was observed to *insist* on his “mathematics” even by his (apparent) disregard of Jordan’s wiggling finger. If Jordan had been allowed to contribute, the teacher could have observed how he interacted with Dwayne and used that observation to create a model of Dwayne’s and Jordan’s MOS.

**Conclusion**

42. The few lesson snippets presented in this paper were used to demonstrate how the M-N-L framework was found to be viable in the analysis of a computer-aided lesson to identify instances where a teacher was sensitive to constructivist notions of learning and also instances where he failed to be. Without prescribing a particular teaching method, the framework places the teacher as a negotiator between the subject matter and the learner and suggests that the main task of constructivist teachers is to find ways of building roads between their subject content knowledge (in this case mathematics) and that of their students. In order to do so they must

a. maintain a sensitivity, flexibility, and openness to learn about the mental constructs and reasoning of their students,

b. learn about, review, and possibly adapt the subject matter they intend to teach,

c. anticipate and plan for purposeful interactions with the students, and

d. provide students with opportunities to experience and reflect upon phenomena presented by the topics of the subject matter.

43. These four elements show that deciding whether one’s teaching may be called CT is not a simple dichotomous matter of determining whether the teacher takes or does not take into account learners’ mental processes. As we hope to have demonstrated, teachers who strive to establish CT in their daily lessons also need to take into consideration the subject content that they are *expected* to help their students gain access to and associate it with the students’ experiential realities through ongoing interactions with the students themselves. The M-N-L framework shows the complexity that such a task entails in the context of mathematics teaching and learning. In order to engage in CT, teachers have to be flexible and responsive enough to shift their focus continuously during the lesson such that none of the above four elements is neglected.

44. Although the M-N-L framework has been devised from and for the analysis of a mathematics lesson, we believe it can be found viable by teachers and educational researchers in other disciplines, especially in situations where the curriculum is developed by persons other than the class teacher and where it specifies a well-defined set of topics to be taught for specific levels in schools. For instance, a similar S-N-L framework for the investigation of CT in science lessons may help teachers reflect on their sensitivity towards their students’ construction of ideas about physical phenomena and the functional relationships and regularities (Peschl 2001) among those observed phenomena. Physical
phenomena and the relationships between them may be perceived by science teachers as viable models to help students learn how scientists predict and control certain aspects of our experiential realities. A similar parallel framework, *L-N-L*, may be found viable to analyze CT in the teaching of languages, where teachers learn from students’ ways of coordinating sound- or word-images and representations of their experiences that are compatible with the coordinations constructed by other speakers of the language (Glasersfeld 1995). The discipline or subject matter may be different but the basic tenet of CT, to learn from and create models of students’ ways of coming to know, is the same. Moreover, in school contexts where curricula are predetermined by educational authorities adopting a realist epistemology, and where teachers need to fit in the teaching of the subject content in the number of contact hours they have with their students in the span of a scholastic year, the (Subject)-Negotiation-Learner construct may be found effective to reflect upon and analyze teaching episodes with regard to the extent to which constructivist teachers *still* manage to engage in CT by negotiating a two-way road between the specific subject and the learner.

45. Teacher apprehensions to implement CT as analyzed by M-N-L, such as attending to one student’s conceptual operations at the expense of other students in the class, or giving each student time to express their thoughts knowing that time for the lessons is limited, was beyond the scope of this paper. Nonetheless, research which analyzes the extent to which constructivist beliefs are implemented in lessons may also find the framework useful to investigate such difficulties, which may well be the reason why constructivist teachers may sometimes fail to act on their epistemological beliefs in their lessons.

References


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