Vanishing theorems for linearly obstructed divisors

VANISHING THEOREMS FOR LINEARLY OBSTRUCTED DIVISORS

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Dedicated to the memory of Anthony Geramita

ABSTRACT. We study divisors on the blow-up of $\mathbb{P}^n$ at points in general position that are non-special with respect to the notion of linear speciality introduced in [6]. We describe the cohomology groups of their strict transforms via the blow-up of the space along their linear base locus. We extend the result to non-effective divisors that sit in a small region outside the effective cone. As an application, we describe linear systems of divisors in $\mathbb{P}^n$ blown-up at points in star configuration and their strict transforms via the blow-up of the linear base locus.

1. INTRODUCTION

The motivation for studying vanishing theorems of divisors comes from Birational Geometry (Mori's Minimal Model Program, see [21]) and Commutative Algebra (higher order embeddings of projective varieties, see [2]). In particular, vanishing theorems have applications to positivity properties of divisors such as global generation, very ampleness and, more in general, k-very ampleness properties.

We denote by $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ passing through a collection of s points in general position with multiplicities at least $m_1, \ldots, m_s \geq 0$ respectively. The *(affine) virtual dimension* of $\mathcal{L}$ is denoted by

$$\operatorname{vdim}(\mathcal{L}) = \binom{n+d}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n},$$

and the *expected dimension* of $\mathcal{L}$ is defined to be $\operatorname{edim}(\mathcal{L}) = \max(\operatorname{vdim}(\mathcal{L}), 0)$. The problem of computing the dimension of such linear systems is often referred to as *(polynomial) interpolation problem in* $\mathbb{P}^n$ (see e.g. [9] for an account).

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If D is the strict transform of a general divisor in $\mathcal{L}$ in the blow-up X of $\mathbb{P}^n$ at the s points,

$$D := dH - \sum_{i=1}^{s} m_i E_i, \tag{1.1}$$

then $\operatorname{vdim}(D) := \operatorname{vdim}(\mathcal{L})$ equals $\chi(X, \mathcal{O}_X(D))$, the Euler characteristic of the sheaf on X associated with D, while $\dim(\mathcal{L})$ is the number of global section of $\mathcal{O}_X(D)$, namely the dimension of the space $H^0(X, \mathcal{O}_X(D))$. Using the terminology of the interpolation problem, we will refer to D as a divisor of degree d *interpolating* s general points with assigned multiplicities $m_1, \ldots, m_s$.

The inequality $\dim(\mathcal{L}) \geq \operatorname{edim}(\mathcal{L})$ is always satisfied. However, if the conditions imposed by the assigned multiple points are not linearly independent, then the actual dimension of $\mathcal{L}$ is strictly greater than the expected one: in that case we say that $\mathcal{L}$ (or D) is *special*. Otherwise, whenever the actual and the expected dimension coincide we say that $\mathcal{L}$ is *non-special*. The *speciality* of $\mathcal{L}$ (or D) is defined to be the difference $\dim(\mathcal{L}) - \operatorname{edim}(\mathcal{L})$.

In the last century the problem of computing the dimension (or, equivalently, computing the speciality) of linear systems was studied with different techniques by many people. In the planar case, the Segre-Harbourne-Gimigliano-Hirschowitz conjectures predicts all special linear systems. This famous conjecture gives information about the Mori cone of X, $\overline{\mathrm{NE}}(X)$, together with its dual, the nef cone of X. For example, one of its implications is the so called (-1)-Curves Conjecture, see [10, Conjecture 3.2.1] and [13, Conjecture 1.1]. This consists of a geometric description of the Mori cone of X: while the K-negative part of $\overline{\mathrm{NE}}(X)$, namely the set of classes intersecting negatively the canonical divisor, is known to be generated by classes of (-1)-curves, in the case $s \geq 10$ the K-non-negative part would be a region with circular portions of boundary.

The degeneration technique introduced by Ciliberto and Miranda (see e.g. [9, 10]) is a successful method in the study of interpolation problems. However, in spite of many partial results, both conjectures are still open in general.

In the case of $\mathbb{P}^3$, there is an analogous conjectural classification of special linear systems formulated by Laface and Ugaglia (see e.g. [23]).

Due to its complexity and mysterious geometry, the simple question of predicting and computing dimensions of such vector spaces is not even conjectured when n is four or higher. In the case of $\mathbb{P}^n$ general results are rare and few things are known. The well-known Alexander-Hirschowitz Theorem states that a linear system in $\mathbb{P}^n$ with an arbitrary number of double points in general position is non-special except in a list of exceptional cases in small degree (see e.g. [1, 5, 26] for more details). For higher multiplicities, the only general result known so far is a complete cohomological classification of the speciality of *only linearly obstructed* effective divisors, proved by Brambilla and the two authors of this manuscript in [6] (see also [8]). One of the goals of this paper is to extend such a classification to the non-effective case.

In order to classify the special divisors, one has to understand first what are the *obstructions*, namely what are the varieties that whenever contained with multiplicity in the base locus of a given divisor force $\mathcal{L}$ to be special. In [3, 4] these varieties are named *special effect varieties*. Few examples of obstructions were classified before [6]. The only examples known were (-1)-curves in $\mathbb{P}^2$ and $\mathbb{P}^3$ (see [23]) and those appering in the list of exceptions from the Alexander-Hirschowitz Theorem. Theorem 1.4 below (that was proved in [6, Theorem 4.6]) and Corollary 5.3 show that, for any effective divisor, *linear cycles* of arbitrary dimension are always obstructions.

In the direction of extending the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture to $\mathbb{P}^n$ and possibly to other rational projective varieties, we pose the same natural and general question as in [6].

Question 1.1 ([6, Question 1.1]). *Consider any non-empty linear system $\mathcal{L}$ in $\mathbb{P}^n$. Let $\widetilde{\mathcal{X}}$ be the smooth composition of blow-ups of $\mathbb{P}^n$ along the (strict transforms of the) cycles of the base locus of $\mathcal{L}$, ordered in increasing dimension. We denote by D a general divisor of the linear system $\mathcal{L}$, and by $\widetilde{\mathcal{D}}$ the strict transform of D in $\widetilde{\mathcal{X}}$. Does $h^i(\widetilde{\mathcal{X}}, \mathcal{O}_{\widetilde{\mathcal{X}}}(\widetilde{\mathcal{D}}))$ vanish for all $i \geq 1$?*

We remark that $\widetilde{\mathcal{D}}$ is obtained by suitably blowing-up the whole base locus of D and subtracting the fixed hypersurfaces. Precisely, since these divisorial components of the base locus split off the system, taking proper transform under blow-up is equivalent to the deletion of these divisorial components.

For linear divisorial components more details are presented in Section 2.

In general Question 1.1 is difficult to answer since it requires first to describe the base locus of a linear system and second to compute the cohomology of the strict transform after blowing it up.

An affirmative answer to Question 1.1 implies that $\dim(\mathcal{L}) = \chi(\widetilde{\mathcal{X}}, \mathcal{O}_{\widetilde{\mathcal{X}}}(\widetilde{\mathcal{D}}))$, translating the classical dimensionality problem for linear systems into a Riemann-Roch formula for divisors living in subsequently blown-up spaces.

We denote by $\tilde{D}$ the strict transform of D in $\tilde{X}$, the blow-up of X along the linear cycles of the base locus of D. A precise definition is given below in (1.4) with $r = n-1$. We remark that $\tilde{D}$ is different from $\widetilde{\mathcal{D}}$ that is introduced in Question 1.1: the second one denotes the strict transform of D in $\widetilde{\mathcal{X}}$, the blow-up of X along all -linear and non linear - cycles of the base locus of D.

Due to the combinatorial and geometrical complexity of this problem so far we only understand properties of divisors $\tilde{D}$ and we will present them in detail.

We mention that for $s \leq n+2$ the divisors $\tilde{D}$ obtained by subsequent blow-up of the linear base locus (described in details in Sections 5 and 6) are divisors in $\overline{\mathcal{M}}_{0,n+3}$, the moduli space of stable rational curves with $n+3$ marked points, see [20]. Therefore understanding their cohomological description can be used in the study of positivity properties such as the effective cone and the ample cone of $\overline{\mathcal{M}}_{0,n+3}$ (see also [6, Section 6.3]).

In the article [6], the authors introduced a new notion of expected dimension for linear systems, that takes into account the linear obstructions and extends the notion of virtual dimension, namely the *linear virtual dimension*. In this paper, we will use $\operatorname{ldim}(\mathcal{L})$ to denote the (affine) linear virtual dimension, instead of the (projective) expected linear dimension as used in [6]. Given two linear systems $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$ and $\mathcal{L}_{n,d}(m'_1, \ldots, m'_s)$ with the same degree, we write $\mathcal{L} \prec_s \mathcal{L}'$ if $m_i \geq m'_i$ for all $i \in \{1, \ldots, s\}$.

Definition 1.2 ([6, Definition 3.2]). *Given a linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$, for any integer $-1 \leq r \leq s-1$ and for any multi-index $I(r) = \{i_1, \ldots, i_{r+1}\} \subseteq$*

$\{1,\dots,s\}$, *define the integer*

$$k_{I(r)} := \max(m_{i_1} + \cdots + m_{i_{r+1}} - rd, 0). \tag{1.2}$$

The (affine) linear virtual dimension *of* $\mathcal{L}$ *(or of* D*), denoted by* $\mathrm{ldim}(\mathcal{L})$*, is the number*

$$\sum_{r=-1}^{s-1} \sum_{I(r)\subseteq\{1,\dots,s\}} (-1)^{r+1} \binom{n + k_{I(r)} - r - 1}{n}, \tag{1.3}$$

where we set $I(-1) = \emptyset$*. The* (affine) linear expected dimension *of* $\mathcal{L}$ *is defined as follows: it is* 0 *if* $\mathcal{L} \prec_s \mathcal{L}'$ *and* $\mathrm{ldim}(\mathcal{L}') \leq 0$*, otherwise it is the maximum between* $\mathrm{ldim}(\mathcal{L})$ *and* 0.

We remark that this notion is well-defined not only for all effective divisors but also for non-effective ones. We will study this type of divisors in Sections 5 and 6.

In this light, asking whether the dimension of a given linear system equals its linear expected dimension can be thought of as a refinement of the classical question of asking whether the dimension equals the expected dimension. If the answer to this question is affirmative, then $\mathcal{L}$ (or D) is said to be a *only linearly obstructed*. Obviously, non-special linear systems are always only linearly obstructed.

There exist linear systems that are *linearly obstructed* without being *only linearly obstructed*. For instance $\mathcal{L}_{4,10}(6^7)$ contains all lines L_{ij}, $i,j \in \{1,\dots,7\}$ with multiplicity two in its base locus as well as the rational normal curve through the seven points, see [6, Example 6.2] for more details.

Connections between $\mathcal{L}$ being only linearly obstructed and the Fröberg-Iarrobino Conjecture (see [8]), describing the Hilbert series of an ideal generated by s forms, can be found in [6, Section 6]. This reveals the importance of the notion of linear speciality, that was achieved and developed independently from both the geometric and the algebraic setting.

Linear systems with an arbitrary number of points and with bounded sum of the multiplicities were classified in [6], for $n \geq 1$, $d \geq 2$, by proving that they are only linearly obstructed.

Theorem 1.3 ([6, Theorem 5.3])**.** *All non-empty linear systems of the form* $\mathcal{L} = \mathcal{L}_{n,d}(m_1,\dots,m_s)$ *with* $s \leq n+2$ *base points are only linearly obstructed. Moreover, if* $s \geq n+3$ *and*

$$\textstyle\sum_{i=1}^s m_i \leq nd + \min(n - s(d), s - n - 2), \quad 1 \leq m_i \leq d, \tag{4.1}$$

where $s(d) \geq 0$ *is the number of points of multiplicity* d*, then* $\mathcal{L}$ *is non-empty and only linearly obstructed.*

The new perspective introduced in [6] is built upon the cohomological study of the strict transforms of effective and only linearly obstructed divisors. More precisely, the strict transforms are taken after subsequently blowing-up their linear base locus, first the lines, then the planes, etc. Moreover, in [6] a complete classification was given for effective divisors interpolating $s \leq n+2$ general points with assigned multiplicities, in which range the effective cone was known (see for example [7]).

For every effective divisor D, let $D_{(r)}$ denote the strict transform of D in the space $X^n_{(r)}$ obtained as a sequence of blow-ups of $\mathbb{P}^n$ along the linear base locus of D up to dimension r, with $r \leq n-1$ (we refer to Section 2 for details about this construction):

$$D_{(r)} := D - \sum_{\rho=1}^{r} \sum_{I(\rho)\subseteq\{1,\ldots,s\}} k_{I(\rho)} E_{I(\rho)}, \tag{1.4}$$

where $E_{I(\rho)}$ denotes the (strict transform of the) exceptional divisor of the linear subspace of $\mathbb{P}^n$ of dimension ρ spanned by the points parametrised by the multi-index $I(\rho)$ and $k_{I(\rho)}$ is the non-negative integer introduced in (1.2), which is the multiplicity with which the aforementioned subspace is contained in the base locus; this will be proved in Section 4.1. Let $\bar{r}$ be the maximum dimension of the linear base locus; we will set

$$\tilde{D} := D_{(\bar{r})}. \tag{1.5}$$

To simplify notation here and throughout the paper we will also abbreviate $h^i(X^n_{(r)}, \mathcal{O}_{X^n_{(r)}}(D_{(r)}))$ by $h^i(D_{(r)})$.

Theorem 1.4 ([6, Theorem 4.6])**.** *Given integers $d, m_1, \ldots, m_s$, consider the divisor (1.1). If $s \leq n+2$ and D is effective, the following statements hold.*

(a) $h^0(D) = \operatorname{ldim}(D)$ and $h^i(\tilde{D}) = 0$ for every $i \geq 1$.

(b) For any $0 \leq r \leq n-1$, $h^i(D_{(r)}) = 0$ for every $i \geq 1$ and $i \neq r+1$, while

$$h^{r+1}(D_{(r)}) = \sum_{\rho=r+1}^{s-1} \sum_{I(\rho)\subseteq\{1,\ldots,s\}} (-1)^{\rho-r-1} \binom{n + k_{I(\rho)} - \rho - 1}{n}.$$

The goal of this paper is to show that the same type of results as in Theorem 1.4 holds for larger classes of divisors, such as effective divisors with arbitrary number of general base points and non-effective divisors. The definition of strict transform after blowing-up the linear base locus is formally extended to the non-effective case in Section 5.1.

We also extend the formula in Theorem 1.4 to the case of any effective divisor, not necessarily only linearly obstructed, interpolating an arbitrary collection of general multiple points.

Theorem 1.5. *Given integers $d, m_1, \ldots, m_s$, consider the divisor D of the form (1.1). If D is effective, then for any $0 \leq r \leq n-1$ we have*

$$\begin{aligned} h^{r+1}(D_{(r)}) = &\sum_{\rho=r+1}^{s-1} \sum_{I(\rho)\subseteq\{1,\ldots,s\}} (-1)^{\rho-r-1} \binom{n + k_{I(\rho)} - \rho - 1}{n} \\ &+ \sum_{\rho=r+1}^{n} (-1)^{\rho-r-1} h^\rho(\tilde{D}). \end{aligned}$$

In particular,

$$h^0(D) = \operatorname{ldim}(D) + \sum_{\rho=1}^{n} (-1)^{\rho+1} h^\rho(\tilde{D}).$$

Moreover, if $h^i(\tilde{D}) = 0$, for all $i \geq 1$, then $h^i(D_{(r)}) = 0$ for all $i \neq r+1$.

This result is part of Theorem 5.2 that will be proved in Section 6 in a more general setting. The geometric interpretation is that for any effective divisor D, every ρ-dimensional linear cycle $L_{I(\rho)}$, for which $k_{I(\rho)} \geq 1$ and $\rho \geq r+1$, gives a contribution with alternating sign, $(-1)^{r-\rho+1}$, equal to

$$\binom{n+k_{I(\rho)}-\rho-1}{n} \tag{1.6}$$

to $h^{r+1}(D_{(r)})$ and to the formula for $\mathrm{ldim}(D)$ (cfr. Theorem 5.2 and Corollary 5.3). Moreover, such a contribution is zero when $k_{I(\rho)} \leq \rho$.

The main result of this paper is a complete cohomological description of $D_{(r)}$ in the following cases, where we set

$$b := b(D) = \sum_{i=1}^{s} m_i - nd. \tag{1.7}$$

Theorem 1.6. *Fix $d, m_1, \ldots, m_s$. Statements (a) and (b) of Theorem 1.4 hold for all divisors D of the form (1.1) with $m_i \leq d+1$ under the following hypothesis: $s \leq n+1$ and $b \leq n$, or $s = n+2$ and $b \leq 1$, or $s \geq n+3$ and $b \leq \min(n-s(d), s-n-2)$.*

Moreover, if $s \leq n+1$ then $h^i(\tilde{D}) = 0$, for all $i \leq n-1$, and $h^n(\tilde{D}) = \binom{b-1}{n}$ for $b \geq n+1$ and zero otherwise.

The theorem summarises the results contained in Theorem 4.1, Theorem 5.4, Theorem 5.11 and Theorem 5.12. This result shows that it makes sense to extend Question 1.1 to non-effective divisors in a small region outside the effective cone with a correct definition of $D_{(r)}$. In order to study classical interpolation problems, a crucial step is the study of non-effective divisors. More precisely, whenever a linear cycle is contained in the base locus of a divisor D , the normal bundle of its exceptional divisor after blow-up is given by a non-effective divisor (see Lemma 2.8), and the cohomology groups of its multiples produce a contributions to the speciality of the form (1.6).

From Theorems 1.4 and 1.6, for any effective divisor with $s \leq n+2$ and any only linearly obstructed divisor satisfying the bound (4.1), one obtains $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{D})) = \mathrm{ldim}(D)$. This gives a strong interpretation of the notion of linear expected dimension that, not only represents a dimension count for the linear system $\mathcal{L}$, but also computes the Euler characteristic of the sheaf $\mathcal{O}_{\tilde{X}}(\tilde{D})$. As a corollary of Theorem 1.6 we extend this Riemann-Roch formula to larger classes of divisors obtaining interesting combinatorial identities. In particular, toric divisors sitting on the facets of the effective cone have Euler characteristic equal to one.

This paper is organised as follows. In Section 2 we introduce the general construction and notation.

In Section 3 we provide a cohomological classification of a class of interesting divisors, namely integer multiples of *standard Cremona transformations* of the hyperplane classes.

In Section 4 we first give an explicit description of the linear base locus of any divisor D, Proposition 4.2. In Theorem 4.1 we show that linear cycles are the only obstructions for divisors with $s \leq n+2$ or satisfying (4.1).

In Section 5 we give vanishing theorems for the cohomology groups of divisors D with $s \leq n+2$ and with multiplicities bounded above by $d+1$, since in this range ldim is well-defined, see Theorem 5.4 and Theorem 5.11. In Theorem 5.12 we extend the result to the case of non-effective divisors with $s \geq n+3$ points with multiplicities satisfying the bound (4.1).

Section 6 is dedicated to the proofs of the results stated in Section 5 .

In Section 7 we use the vanishing theorems from Section 5 to study linear systems with points in special position. Theorem 7.3 computes the dimensions of a class of linear systems in $\mathbb{P}^n$ interpolating *star configurations* of points with higher multiplicities.

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2. Blowing-up: construction and notation

In this section we recall the main construction that was partially presented in [6, Sect. 4.1].

Let $\mathcal{I}$ be a set of subsets of $\{1,\ldots,s\}$. For every integer $0\le r\le \min(n,s)-1$, we denote by $I(r)=\{i_1,\ldots,i_{r+1}\}\in\mathcal{I}$ a *multi-index* of length $|I(r)|=r+1$. Let us also introduce the notation

$$\begin{aligned}\mathcal{I}(r) &:= \{I(\rho)\in\mathcal{I} : 0\le\rho\le r\};\\ \mathcal{I}(r)_j &:= \{I(\rho)\in\mathcal{I}(r)\setminus\mathcal{I}(0) : j\in I(\rho)\};\\ \mathcal{I}(r)_{j_1,j_2} &:= \{I(\rho)\in\mathcal{I}(r)\setminus\mathcal{I}(1) : j_1,j_2\in I(\rho)\}.\end{aligned}\tag{2.1}$$

Let $p_1,\ldots,p_s$ be general points in $\mathbb{P}^n$ and, for every $I(r)\in\mathcal{I}$, let $L_{I(r)}\cong\mathbb{P}^r$ denote the r-dimensional linear subspace spanned by the points $\{p_j :\ j\in I(r)\}$, which we will refer to as a *linear r-cycle*. Notice that $L_{I(0)}=p_j$ is a point. An arbitrary multi-index will be denoted by I without specifying its cardinality.

We will assume that $\mathcal{I}$ satisfies the following properties:

(I) $\{j\}\in\mathcal{I}$, for all $j\in\{1,\ldots,s\}$;
(II) if $I\subset J$ and $J\in\mathcal{I}$, then $I\in\mathcal{I}$.

Let $\Lambda=\Lambda(\mathcal{I})\subset\mathbb{P}^n$ be the subspace arrangement corresponding to $\mathcal{I}$, i.e. the (finite) union of the linear cycles L_I with $I\in\mathcal{I}$. Let $\bar r$ be the largest dimension of a linear cycle in Λ, i.e. $\bar r=\max_{I\in\mathcal{I}}(|I|)-1$. Write $\Lambda=\Lambda_{(1)}+\cdots+\Lambda_{(\bar r)}$, where $\Lambda_{(r)}=\cup_{I(r)\in\mathcal{I}}L_{I(r)}$.

Assume moreover that $\mathcal{I}$ satisfies the following condition

(III) if $I,J\in\mathcal{I}$, then $L_I\cap L_J=L_{I\cap J}$.

We denote by $\pi_{(0)}:X^n_{(0)}\to\mathbb{P}^n$ the blow-up of $\mathbb{P}^n$ at $p_1,\ldots,p_s$, with $E_1,\ldots,E_s$ exceptional divisors. Let us also consider the sequence of blow-up maps

$$X^n_{(n-1)}\xrightarrow{\pi_{(n-1)}}\cdots\xrightarrow{\pi_{(2)}}X^n_{(1)}\xrightarrow{\pi_{(1)}}X^n_{(0)},$$

where $X^n_{(r)}\xrightarrow{\pi_{(r)}}X^n_{(r-1)}$ is the blow-up of $X^n_{(r-1)}$ along the strict transform of $\Lambda_{(r)}\subset\mathbb{P}^n$, via $\pi_{(r-1)}\circ\cdots\circ\pi_{(0)}$. For any $I(r)\in\mathcal{I}$, we denote by $E_{I(r)}$ the exceptional divisor of the cycle $L_{I(r)}$ in $X^n_{(r)}$. Notice that conditions (I), (II) and (III) ensures that, for every r, the sum of the strict transforms of the exceptional divisors of $\pi_{(r)}$ is simple normal crossing. We will write, abusing notation, H for the strict transform in $X^n_{(r)}$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ and $E_{I(\rho)}$, for $I(\rho)\in\mathcal{I}(r-1)$, for the strict transform in $X^n_{(r)}$ of the exceptional divisor $E_{I(\rho)}$ in $X^n_{(\rho)}$, respectively.

Remark 2.1. Notice that condition (III) is obviously satisfied when $s\leq n+1$. In particular if $\mathcal{I}$ is the set of al subsets of $\{1,\ldots,n+1\}$, then each map $\pi_{(r)}$ is the blow-up of the strict transforms of all coordinate r-dimensional subspaces of $\mathbb{P}^n$. These are toric morphisms, for every r. In particular the defining lattice polytope of $X_{(n-2)}$ is the so called *permutohedron* studied by Kapranov in [19]. The space $X_{(n-2)}$ coincides with the Losev-Manin moduli space studied in [25].

Remark 2.2. Notice that the map $\pi_{(n-1)}:X^n_{(n-1)}\longrightarrow X^n_{(n-2)}$ is an isomorphism. In particular, $\operatorname{Pic}(X^n_{(n-1)})\cong(\pi_{(n-1)})^*\operatorname{Pic}(X^n_{(n-2)})$. Thus, in our notation, for every $I(n-1)\in\mathcal{I}$ we have

$$E_{I(n-1)}=H-\sum_{\substack{I(\rho)\in\mathcal{I}(n-2),\\ I(\rho)\subset I(n-1)}}E_{I(\rho)}\in\operatorname{Pic}(X^n_{(n-2)}). \tag{2.2}$$

In other terms $E_{I(n-1)}$ denotes the strict transform on $X^n_{(n-2)}$ of the hyperplane

$$H_{I(n-1)}:=H-\sum_{i\in I(n-1)}E_i\in\operatorname{Pic}(X^n_{(0)}). \tag{2.3}$$

For this reason we will sometimes write

$$E_{I(n-1)}=\tilde{H}_{I(n-1)}\in\operatorname{Pic}(X^n_{(n-2)}). \tag{2.4}$$

We will use the first notation if we want to stress its nature as exceptional divisor, whereas we will use the second notation when we want to consider it as the strict transform of a hyperplane.

Finally, notice that $E_{I(n-1)}\cong X^{n-1}_{(n-3)}$, the blow-up of $\mathbb{P}^{n-1}$ along all coordinate subspaces, in increasing dimension, i.e. the $(n-1)-$dimensional toric variety discussed in Remark 2.1.

The Picard group of $X^n_{(r)}$ is

$$\operatorname{Pic}(X^n_{(r)})=\langle H,E_{I(\rho)}:I(\rho)\in\mathcal{I}(r-1)\rangle.$$

Remark 2.3. For $r=1,\ldots,n-1$ and F a divisor on $X^n_{(r-1)}$, for any $i\geq 0$, we have

$$h^i(X^n_{(r)},(\pi_{(r)})^*F)=h^i(X^n_{(r-1)},F).$$

It follows from Zariski connectedness theorem and by the projection formula (see for instance [18] or [24, Lemma 1.3] for a more detailed proof.).

2.1. **The geometry of the divisor E_j in $X^n_{(r)}$.** Let $r\geq 1$. Consider the composition of blow-ups $\pi_{(r,0)}:=\pi_{(r)}\circ\cdots\circ\pi_{(1)}:X^n_{(r)}\to X^n_{(0)}$. By abuse of notation we will denote by E_j the strict transform $(\pi_{(r,0)})^{-1}_*E_j\in\operatorname{Pic}(X^n_{(r)})$ of the exceptional divisor $E_j\in\operatorname{Pic}(X^n_{(0)})$ of the point $p_j\in\mathbb{P}^n$.

For every multi-index $I(\rho)\in\mathcal{I}(r)_j$, let $E_{I(\rho)}\in\operatorname{Pic}(X^n_{(r)})$ be the strict transform of the exceptional divisor in $X^n_{(\rho)}$ of $L_{I(\rho)}$, and set $e_{I(\rho)|j}:=E_{I(\rho)}|_{E_j}$. Moreover, let h be the hyperplane class of E_j.

Lemma 2.4. *In the above notation, a basis for the Picard group of E_j is given by h and $e_{I(\rho)|j}$, for all $I(\rho)\in\mathcal{I}(r)_j$.*

In particular, we have the isomorphism $Pic(E_j)\cong Pic(X^{n-1}_{(r-1)})$.

Rephrasing Lemma 2.4, the exceptional divisor $E_j \in \operatorname{Pic}(X^n_{(r)})$ is isomorphic to a blown-up $\mathbb{P}^{n-1}$ along linear $(\rho-1)$-cycles, $\rho \le r$, spanned by subsets of a collection of $s-1$ general points. These $s-1$ points correspond to the lines $L_{I(1)}$, with $I(1) = \{j, l\}$, for all indices $l \in \{1, \ldots, \hat{j}, \ldots, s\}$. Similarly, the linear $(\rho-1)$-cycles blown-up in E_j correspond to the linear ρ-cycles $L_{I(\rho)}$ of $\mathbb{P}^n$ satisfying the condition that $j \in I(\rho)$.

2.2. **The geometry of the divisor $E_{I(\rho)}$ in $X^n_{(r)}$.** Let $I = I(\rho) \in \mathcal{I}(r)$ be any multi-index. Notice that if $\rho = 0$, $E_{I(\rho)}$ is the exceptional divisor of a point that was already described in Section 2.1. Consider the composition of blow-ups $\pi_{(r,\rho)} := \pi_{(r)} \circ \cdots \circ \pi_{(\rho+1)} : X^n_{(r)} \to X^n_{(\rho)}$. By abuse of notation we will denote by E_I the strict transform via $\pi_{(r,\rho)}$ in $X^n_{(r)}$ of the exceptional divisor $E_I \in \operatorname{Pic}(X^n_{(\rho)})$ of a linear ρ-cycle $L_I \subset \mathbb{P}^n$: $E_I \in \operatorname{Pic}(X^n_{(r)})$ is a Cartesian product that we are now going to describe.

Consider first the case $\rho = r$. We have the following isomorphism: $E_I \cong X^r_{(r-2)} \times \mathbb{P}^{n-r-1}$; we refer to [6, Section 4] for details. The Picard group of the first factor is generated by $\langle h, e_{I(t)} : I(t) \subset I, I(t) \in \mathcal{I}(r-2)\rangle$,

where $E_{I(t)}|_{E_I} =: e_{I(t)} \boxtimes 0$, while the Picard group of the second factor is generated by the hyperplane class.

Assume now that $0 \le \rho < r$. Notice first of all that the restriction $E_{I(t)}|_{E_I}$ of $X^n_{(r)}$ is zero on both factors unless one of the following containment relations is satisfied: $I \subset I(t)$ or $I(t) \subset I$. We denote by h_b and h_f the hyperplane classes of the two factors respectively. Moreover we introduce divisors $e_{I(t)}$ on the first factor and $e_{I(t)|I}$ on the second factor according to the following intersection table:

$$\begin{aligned} H|_{E_I} &=: h_b \boxtimes 0; \\ E_{I(t)}|_{E_I} &=: e_{I(t)} \boxtimes 0, \text{ for all } I(t) \subset I, \ t \ge 0; \\ E_{I(t)}|_{E_I} &=: 0 \boxtimes e_{I(t)|I}, \text{ for all } I \subset I(t), \ t \le r. \end{aligned} \tag{2.5}$$

Notice that if $\rho = 0$, the first factor of $E_{I(\rho)}$ is a point and we have $h_f = h$ in the notation of Section 2.1.

Remark 2.5. If $t = \rho - 1$, and $I(t) \subset I$, i.e. $L_{I(\rho-1)} \subset \mathbb{P}^n$ is a hyperplane of $L_I \subset \mathbb{P}^n$, using (2.2) we obtain the following equality:

$$E_{I(\rho-1)}|_{E_I} = \Big(H - \sum E_{I(\tau)}\Big)|_{E_I} = \Big(h - \sum e_{I(\tau)}\Big) \boxtimes 0,$$

where the sums range over the multi-indices $I(\tau) \subset I$, $I(\tau) \in \mathcal{I}(\rho-2)$. Accordingly, $e_{I(\rho-1)} = h_b - \sum e_{I(\tau)}$. A similar argument holds for divisors on the second factor, when $t = n-1$ and $I \subset I(t)$.

Lemma 2.6. *In the above notation, assume $\mathcal{I}$ is a set of multi-indices satisfying conditions* (I),(II) *and* (III). *We have*

$$E_I \cong X^{\rho}_{(\rho-2)} \times X^{n-\rho-1}_{(r-\rho-1)}.$$

Moreover, bases for the Picard groups of the two factors of the product E_I, for $I \in \mathcal{I}$ of length $\rho+1$, are given respectively by

$$\begin{aligned} &\langle h_b, \, e_{I(t)} : I(t) \subset I, I(t) \in \mathcal{I}(\rho-2)\rangle; \\ &\langle h_f, \, e_{I(t)|I(\rho)} : I \subset I(t), \ I(t) \in \mathcal{I} \setminus \mathcal{I}(\rho)\rangle. \end{aligned}$$

The element $e_{I(t)}$ represents the (strict transform of the) exceptional divisor in $X^{\rho}_{(\rho-2)}$ of a linear $t-$cycle of $\mathbb{P}^{\rho}$ and $e_{I(t)|I(\rho)}$ represents the (strict transform of the) exceptional divisor in $X^{n-\rho-1}_{(r-\rho-1)}$ of a linear $(t-\rho-1)-$cycle of $\mathbb{P}^{n-\rho-1}$, spanned by $|I(t)\setminus I(\rho)|$ points.

Remark 2.7. In the above notation, using the assumptions (I),(II) and (III), one can see that there are exactly $\rho+1$ exceptional divisors of the form $e_{I(0)}$ on the first factor $X^{\rho}_{(\rho-2)}$ of E_I. They can be thought as the exceptional divisors of the $\rho+1$ coordinate points of $\mathbb{P}^{\rho}$. In the same way the $e_{I(t)}$'s, for $t\geq 1$, represent the (strict transforms of the) exceptional divisors of $t-$dimensional coordinate subspaces. Hence $X^{\rho}_{(\rho-2)}$ is the $\rho-$dimensional toric variety discussed in Remark 2.1.

We now give a characterisation of the normal bundle of the exceptional divisor E_I in the space $X^n_{(r)}$. To this purpose, we introduce the following divisor on $X^{\rho}_{(\rho-2)}$:

$$\mathrm{Cr}_{\rho}(h_b) := \rho h_b - \sum_{\substack{I(t)\subset I\\ I(t)\in\mathcal{I}(\rho-2)}} (\rho-t-1)e_{I(t)}. \tag{2.6}$$

We will give a detailed cohomological description of such a divisor in Section 3.

Lemma 2.8. *In the notation above, we have*

$$-E_I|_{E_I} = \mathrm{Cr}_{\rho}(h_b)\boxtimes h_f.$$

Proof. The proof follows from the computation of the conormal bundle of the first factor of E_I:

$$N^*_{X^{\rho}_{(\rho-2)}|X^n_{(r)}} = \mathcal{O}_{X^{\rho}_{(\rho-2)}}(\mathrm{Cr}_{\rho}(h_b)).$$

See [6, Lemma 4.3] for details. □

2.3. **Restriction exact sequences.** Given a divisor $F\in\mathrm{Pic}(X^n_{(r)})$, we will consider three types of restriction exact sequences.

Restriction to the exceptional divisor of a point. In the same notation as above, let E_1 be the strict transform in $X^n_{(r)}$ of the exceptional divisor of the point $p_1\in\mathbb{P}^n$. We will call *sequence of type (A)* the following restriction sequence:

$$0\longrightarrow F-E_1\longrightarrow F\longrightarrow F|_{E_1}\longrightarrow 0. \tag{A}$$

Restriction to the exceptional divisor of a linear cycle. Fix integers $1\leq\rho\leq r\leq n-2$. In the same notation as above, let $E_{I(\rho)}$ be the strict transform in $X^n_{(r)}$ of the exceptional divisor of the linear subspace of $\mathbb{P}^n$ spanned by the points parametrised by $I(\rho)$. We call *sequence of type (B)* the following restriction sequence:

$$0\longrightarrow F-E_{I(\rho)}\longrightarrow F\longrightarrow F|_{E_{I(\rho)}}\longrightarrow 0. \tag{B}$$

Restriction to hyperplanes. Let now F be a divisor on $X^n_{(n-1)}\cong X^n_{(n-2)}$ and let $I(n-1)$ be the index set parametrising a hyperplane of $\mathbb{P}^n$ spanned by n points. Let $\tilde{H}_{I(n-1)}$ be the strict transform in $X^n_{(n-2)}$ of such a hyperplane, cfr. (2.2) and (2.4). We will call *sequence of type (C)* the following restriction sequence:

$$0\longrightarrow F-\tilde{H}_{I(n-1)}\longrightarrow F\longrightarrow F|_{\tilde{H}_{I(n-1)}}\longrightarrow 0. \tag{C}$$

For each of the above sequences, it will be possible to study the restricted divisor by means of the intersection table (2.5).

3. Standard Cremona transformations of hyperplane classes

We recall that the *standard Cremona transformation* of $\mathbb{P}^n$, based at the $n+1$ coordinate points, is the birational transformation defined by the following rational map:

$$\mathrm{Cr} : (x_0 : \cdots : x_n) \to (x_0^{-1} : \ldots , x_n^{-1}),$$

where $x_0, \ldots, x_n$ are homogeneous coordinates of $\mathbb{P}^n$. This map induces an action on the Picard group of $\mathbb{P}^n$ blown-up at s points, $X^n_{(0)}$. Without loss of generality we may assume that an effective divisor D of the form (1.1) is based on the $n+1$ coordinate points and other general points of the projective space and we label their corresponding exceptional divisors by $E_1, \ldots, E_{n+1}, E_{n+2}, \ldots, E_s$. The Cremona action on the divisor D is described by the following rule (see e.g. [14]). Set

$$D = dH - \sum_{i=1}^{s} m_i E_i, \quad c := (n-1)d - \sum_{i=1}^{n+1} m_i.$$

Then

$$\mathrm{Cr}(D) = (d+c)H - \sum_{i=1}^{n+1} (m_i + c)E_i - \sum_{i=n+2}^{s} m_i E_i.$$

In the case $n = 3$, Cr is often called the *cubo-cubic* Cremona transformation, see for instance [23].

From now on throughout this section, $\mathcal{I}$ will denote the set of all subsets of $\{1, \ldots, n+1\}$. Notice that $\mathcal{I}$ satisfies conditions (I), (II) and (III) of Section 2, cfr. Remark 2.1. Moreover $X^n_{(n-1)} \cong X^n_{(n-2)}$ will be the blow-up of $\mathbb{P}^n$ along all linear subspaces parametrised by elements of $\mathcal{I}(n-2)$, i.e. along all coordinate subspaces of $\mathbb{P}^n$, in increasing dimension.

The divisors (2.6), that naturally arise in the blowing-up construction, are the strict transforms in $X^n_{(n-1)}$ via the standard Cremona transformations of the strict transform H of the hyperplane class $\mathcal{O}_{\mathbb{P}^n}(1)$, where we abbreviate the notation for the strict transform in $X^n_{(n-1)}$ of $\mathrm{Cr}(H)$ by

$$\mathrm{Cr}_n(H) = nH - \sum_{I(\rho) \in \mathcal{I}(n-2)} (n - \rho - 1) E_{I(\rho)}. \tag{3.1}$$

This is the divisor (2.6) obtained by replacing ρ by n, t by ρ, I by $\{1, \ldots, n+1\}$ and h_b by H.

In this section we compute all cohomologies of any multiple of Cremona transformations of hyperplane classes. In particular we show that they have the same cohomological behaviour as the same multiples of the hyperplane classes.

Recall that the canonical divisor of the blown-up projective space $X^n_{(n-1)}$ is

$$K_{X^n_{(n-1)}} = -(n+1)H + \sum_{I(\rho) \in \mathcal{I}(n-2)} (n - \rho - 1) E_{I(\rho)}$$

and notice that

$$K_{X^n_{(n-1)}} + \mathrm{Cr}_n(H) = -H.$$

Theorem 3.1. *For any integer a, we have that $h^i(X^n_{(n-1)}, \mathcal{O}_{X^n_{(n-1)}}(a\mathrm{Cr}_n(H))) = h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a))$.*

In particular, we have that $h^i(X^n_{(n-1)}, \mathcal{O}_{X^n_{(n-1)}}(-\alpha\mathrm{Cr}_n(H))) = 0$*, for* $1 \leq \alpha \leq n$*,* $i \geq 0$*.*

Proof. If $a = 0$ the statement is obvious.

Assume that $a \geq 1$. It is known that (see for example [15, Theorem 3]) $h^0(a\mathrm{Cr}_n(H)) = h^0(aH)$. Moreover, the effective divisor $a\mathrm{Cr}_n(H)$ on $X^n_{(n-1)}$ is not obstructed, that is $h^i(a\mathrm{Cr}_n(H)) = 0$, for all $i \geq 1$, see [6, Theorem 4.6].

Assume now that $a \leq -1$ and denote $a = -\alpha$, where α is a positive integer.

Case $\alpha \geq n+1$. Recall that $h^i(\mathbb{P}^n, \mathcal{O}(-\alpha)) = 0$ for $i \neq n$ and that, by Serre duality, $h^n(\mathbb{P}^n, \mathcal{O}(-\alpha)) = h^0(\mathbb{P}^n, \mathcal{O}(\alpha - n - 1)) > 0$. Notice that, on the blown-up space $X^n_{(n-1)}$, we have the following:

$$\begin{aligned} h^i(-\alpha\mathrm{Cr}_n(H))) &= h^{n-i}((\alpha-1)\mathrm{Cr}_n(H) - H) \\ &= h^{n-i}\left([(\alpha-1)n-1]H - (\alpha-1)\sum_{I(\rho)\in\mathcal{I}(n-2)}(n-\rho-1)E_{I(\rho)}\right), \end{aligned}$$

where the first equality follows from Serre duality and the second equality is just the expanded form of $(\alpha-1)\mathrm{Cr}_n(H) - H$. We claim that the cohomologies vanish for all $i \neq n$. To show this, we notice that the divisor

$$((\alpha-1)n-1)H - (\alpha-1)(n-1)\sum_{i=1}^{n+1} E_i$$

on $X^n_{(0)}$ is effective and that each cycle $L_{I(\rho)}$ is contained in its base locus with multiplicity $k_{I(\rho)} = (\alpha-1)(n-\rho-1)+\rho$, by [6, Lemma 2.1]. Each integer $k_{I(\rho)}$ differs from the coefficient of $E_{I(\rho)}$ in the above expression by ρ. Hence the vanishing of the i−th cohomology group, for all $i \neq n$, follows by [6, Theorem 4.6]. If $i = n$, notice that

$$h^n\left(X^n_{(n-1)}, -\alpha\mathrm{Cr}_n(H)\right) = h^0\left(X^n_{(0)}, ((\alpha-1)n-1)H - (\alpha-1)\sum_{j=0}^{n+1}(n-1)E_j\right).$$

We compute the number of global sections by preforming a standard Cremona transformation in $\mathbb{P}^n$, which preserves that number:

$$\begin{aligned} h^n\left(X^n_{(n-1)}, a\mathrm{Cr}_n(H)\right) &= h^0\left(X^n_{(0)}, (\alpha-n-1))H + (n-1)\sum_{j=0}^{n+1} E_j\right) \\ &= h^0\left(X^n_{(0)}, (\alpha-n-1)H\right). \end{aligned}$$

Case $1 \leq \alpha \leq n$. We prove the statement by induction on n and α.

The base steps $n = 2$, $\alpha = 1, 2$ are easily verified by means of Serre duality. Indeed, as the canonical divisor of $X^2_{(0)}$ is $K = -3H + E_1 + E_2 + E_3$, we have $h^i(-\mathrm{Cr}_2(H)) = h^{2-i}(-H) = 0$ and $h^i(-2\mathrm{Cr}_2(H)) = h^{2-i}(H - E_1 - E_2 - E_3) = 0$.

Fix $n \geq 3$ and recall from (2.1) that $\mathcal{I}(n-2)_1$ denotes the set of all subsets of $\{1, \ldots, n\}$ of cardinality at most $n-1$ containing $\{1\}$ as a proper subset. Using

(2.2), we compute the following equality:

$$
\begin{aligned}
-\sum_{1\in I(n-1)} E_{I(n-1)} =& -nH + \sum_{I(\rho)\in\mathcal{I}(n-2)_1}(n-\rho)E_{I(\rho)}\\
&+\sum_{I(\rho)\in\mathcal{I}(n-2)\setminus\mathcal{I}(n-2)_1}(n-\rho-1)E_{I(\rho)}\\
=& -\mathrm{Cr}_n(H)+E_1+\sum_{I(\rho)\in\mathcal{I}(n-2)_1}E_{I(\rho)}.
\end{aligned}
$$

For $1\le r\le n-1$, and $I=I(r)$ we recall from (2.5) that $E_{I(\rho)}|_{E_I}=e_{I(\rho)}\boxtimes 0$ if $I(\rho)\subset I$ and that $E_{I(\rho)}|_{E_I}=0\boxtimes *$ if $I(\rho)\not\subseteq I$, where we use $*$ to denote the appropriate divisor, as we are only interested in the first factor. Hence, the above computation and Remark 2.5 show that

$$
-\sum_{1\in I(r-1)} E_{I(r-1)}|_{E_I} = \left(-\mathrm{Cr}_r(h)+e_1+\sum_{I(\rho)\in\mathcal{I}(r-2)_1}e_{I(\rho)}\right)\boxtimes *.
$$

Therefore,

$$
\left(-E_1-\sum_{I(\rho)\in\mathcal{I}(r-1)_1}E_{I(\rho)}\right)|_{E_I} = -\mathrm{Cr}_r(h)\boxtimes *. \tag{3.2}
$$

Now, for every integer r such that $0\le r\le n-1$, we will consider the ordered set of all multi-indices of length $r+1$ that contain $\{1\}$. We will denote it by $\{I(r)_0,\dots,I(r)_{s_r}\}\subset\mathcal{I}(n-1)_1$, where s_r+1 is its cardinality. Notice that for $r=0$ we have $s_0=0$ and the only set that we consider is the singleton $\{1\}$. Let $\prec$ be the lexicographical order on $\mathcal{I}(n-1)_1$ defined as follows: $I(r')_{j'}\prec I(r)_j$ if and only if $r'<r$ or $r'=r$ and $j'<j$.

In the space $X^n_{(n-1)}$ we consider the divisors $F(r,j)$ defined by recursion starting from $F:=0$ as follows:

$$
\begin{aligned}
F(0,0) &= F-E_1,\\
F(r,0) &= F(r-1,s_{r-1})-E_{I(r)_0},\ 1\le r\le n-1,\\
F(r,j) &= F(r,j-1)-E_{I(r)_j},\ 1\le j\le s_r.
\end{aligned}
$$

Notice that the last divisor is $F(n-1,s_{n-1})=-\mathrm{Cr}_n(H)$.

The divisor $F(0,0)$ is obtained from the divisor F as the kernel of an exact sequence of type (*A*). For $1\le r\le n-2$ each divisor $F(r,j)$ is obtained from the previous one as kernel of a sequence of type (*B*) and for $r=n-1$ from a sequence of type (*C*), cfr. Section 2.3.

Precisely, we consider the following exact sequences of sheaves, performed following the order $\prec$ on $\mathcal{I}(n-1)_1$:

$$
\begin{aligned}
&0\to F(0,0)\to F\to F|_{E_1}\to 0,\\
&0\to F(r,0)\to F(r-1,s_{r-1})\to F(r-1,s_{r-1})|_{E_{I(r)_1}}\to 0,\quad 1\le r\le n-1\\
&0\to F(r,j)\to F(r,j-1)\to F(r,j-1)|_{E_{I(r)_j}}\to 0,\quad 1\le j\le s_r.
\end{aligned}
$$

The divisor $-E_1$ has vanishing cohomologies, for all $n\ge 1$. Moreover, in all sequences the restricted divisor is of the form (3.2) and has therefore vanishing cohomologies, by induction on n, using the Kunneth formula for the cohomology of factors. This implies that $H^i(-\mathrm{Cr}_n(H)) = 0$, $i \geq 0$, concluding the proof of the statement in the case $\alpha = 1$.

We are left to prove the vanishing for $-\alpha\mathrm{Cr}_n(H)$, $2 \leq \alpha \leq n$. For $\alpha \geq 2$, we assume the statement true for $\alpha - 1$. We apply the recursive restriction procedure as above. Setting $F := -(\alpha-1)\mathrm{Cr}_n(H)$, we get $F(n-1, s_{n-1}) = -\alpha\mathrm{Cr}_n(H)$.

We first consider the first sequence, the restriction to E_1. Notice that the space $E_1 \cong X^{n-1}_{(n-2)}$ is the $(n-1)-$dimensional toric variety described in Remark 2.1. Now, we first notice that $-\mathrm{Cr}_n(H)|_{E_1} = -\mathrm{Cr}_{n-1}(h)$ on $E_1 \cong X^{n-1}_{(n-2)}$. Indeed,

$$-nH+(n-\rho-1)\sum_{I(\rho)\in\mathcal{I}(n-2)} E_{I(\rho)}|_{E_1} = -(n-1)h+(n-\rho-1)\sum_{I(\rho)\in\mathcal{I}(n-2)\setminus\mathcal{I}(1)} e_{I(\rho)|1}.$$

Therefore, the restricted divisor of the first sequence, that is $-(\alpha-1)\mathrm{Cr}_n(H)|_{E_1} = -(\alpha-1)\mathrm{Cr}_{n-1}(h)$ on E_1, has vanishing cohomologies by induction on n, for all $\alpha \leq n-1$.

Moreover, a computation similar to that preceding (3.2) shows that, for all $I = I(r)_j$, $-\mathrm{Cr}_n(H)|_{E_I} = 0 \boxtimes *$. Therefore, for every pair (r, j) the restriction of the corresponding sequence is

$$\left(-(\alpha-1)\mathrm{Cr}_n(H) - E_1 - \sum_{\substack{I(\rho)\prec I\\ I(\rho)\in\mathcal{I}(n-1)_1}} E_{I(\rho)}\right)|_{E_I},$$

that equals the divisor on the right hand side of (3.2). It has vanishing cohomologies by the argument above (case $\alpha = 1$). This concludes the proof. □

4. Vanishing theorems for effective only linearly obstructed divisors

Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ be a linear system of hypersurfaces of degree d of $\mathbb{P}^n$ interpolating s points with assigned multiplicities m_i's. Elements of $\mathcal{L}$ are in bijection with divisors on $X^n_{(0)}$ of the form (1.1),

$$D := dH - \sum_{i=1}^{s} m_i E_i.$$

Assume that the following bound is satisfied:

$$\sum_{i=1}^{s} m_i - nd \leq \min(n - s(d), s-n-2), \quad 1 \leq m_i \leq d, \tag{4.1}$$

where the integer $s(d)$ denotes the number of indices i such that $m_i = d$.

When a given list of integers $(d, m_1, \ldots, m_s)$ satisfies condition (4.1), if D is the divisor defined by these integers as in (1.1), we will say that D *satisfies condition* (4.1).

Condition (4.1) is sufficient condition for D to be effective (and for $\mathcal{L}$ to be non-empty), see Theorem 1.3.

In [6, Section 5] the dimensions of all linear systems $\mathcal{L}$ in $\mathbb{P}^n$ (equivalently the number of basis elements of global sections of the line bundles associated to the divisors D) that satisfy the bound (4.1) are given, see also Theorem 1.3. In this section we compute the dimension of all higher cohomology groups of their strict transforms in the blown-up spaces $X_{(r)}$, using the notation introduced in Section 2.

Theorem 4.1. *Statements (a) and (b) of Theorem 1.4 hold for divisors satisfying condition (4.1).*

This result implies that such linear systems are only linearly obstructed. Moreover this (partially) answers Question 1.1 for divisors satisfying (4.1).

This section is organised as follows. In Section 4.1 we prove a base locus lemma, Proposition 4.2, that computes the exact multiplicity of containment of a linear cycle in the base locus of a linear system $\mathcal{L}$. Section 4.2 contains the proof of Theorem 4.1.

4.1. **Linear base locus lemma.** Let $\mathcal{I}$ be the set of all subsets of $\{1,\ldots,s\}$.

For all $I(r)=\{i_1,\ldots,i_{r+1}\}\in\mathcal{I}$ with $0\leq r\leq \min(n,s)-1$, we introduce the integers (cfr. Definition (1.2)):

$$K_{I(r)}=K_{i_1,\ldots,i_{r+1}}:=\sum_{i\in I(r)}m_i-rd;$$
$$k_{I(r)}=k_{i_1,\ldots,i_{r+1}}:=\max(K_{I(r)},0).$$

Moreover we introduce the integer

$$\bar{r}=\bar{r}(\mathcal{L})=\bar{r}(D):=\max(\rho|K_{I(\rho)}>0),$$

and the following subset of $\mathcal{I}$:

$$\mathcal{I}^{>}=\mathcal{I}^{>}(\mathcal{L})=\mathcal{I}^{>}(D):=\{I(r)\subseteq\{1,\ldots,s\}:0\leq r\leq n-1,K_{I(r)}>0\}. \tag{4.2}$$

Proposition 4.2. *Let $\mathcal{L}=\mathcal{L}_{n,d}(m_1,...,m_s)$ be a non-empty linear system. Let $I(r)\in\mathcal{I}^{>}$. The linear cycle $L_{I(r)}$ spanned by the points parametrised by $I(r)$ is contained with multiplicity $k_{I(r)}$ in the base locus of $\mathcal{L}$.*

We can rephrase Proposition 4.2 in the setting of the blow-up $X^n_{(0)}$ by saying that the strict transform of $L_{I(r)}$ is contained with multiplicity $k_{I(r)}$ in the base locus of D.

Proof. Let $1\leq r\leq n-1$ and let $I(r)$ be a multi-index with $k_{I(r)}=K_{I(r)}\geq 0$ and let $\tilde{k}_{I(r)}\geq 0$ be the multiplicity with which $L_{I(r)}$ is contained in the base locus of $\mathcal{L}$. By Bézout's theorem one has $\tilde{k}_{I(r)}\geq k_{I(r)}$, see [6, Lemma 2.1] for details. We introduce the following notation:

$$R=R(\mathcal{L},I(r)):=\max(\rho\leq n-1|K_{I(\rho)}\geq 0,I(r)\subseteq I(\rho)).$$

If $r=n-1(=R)$ then the claim is true by [7, Lemma 4.4]. For $r\leq n-2$ we consider separately the following cases:

(i) $r=R=\bar{r}$,
(ii) $r=R<\bar{r}$,
(iii) $r<R$.

Case (i). We prove the statement by backward induction on R. Precisely, given $R\leq n-2$, we assume that for every non-empty linear system $\mathcal{M}$ in $\mathbb{P}^n$ such that $R(\mathcal{M},I(r))=R+1$ and for every multi-index $I(R+1)$ with $K_{I(R+1)}\geq 0$, the cycle $L_{I(R+1)}$ is contained in the base locus of $\mathcal{M}$ with multiplicity $K_{I(R+1)}$, and we prove the statement for $\mathcal{L}$ and $I(r)$ with $R(\mathcal{L},I(r))=R$. Let $I(R)=$

$\{i_1,\ldots,i_{R+1}\}$ be a multi-index with $K_{I(R)}\geq 0$, and consider the inclusions $I(R)\subset J=\{i_1,\ldots,i_{R+1},i_{R+2}\}$ and $J\subset \bar{J}=\{i_1,\ldots,i_{R+1},i_{R+2},\ldots,i_{\min(s,n)}\}$. Because $K_J\leq 0$, then by induction the cycle L_J is not in the base locus of $\mathcal{L}$. Let $H_{\bar{J}}:=H-\sum_{i\in\bar{J}}E_i$ denote the strict transform on $X^n_{(0)}$ of the hyperplane of $\mathbb{P}^n$ spanned by the points indexed by elements of $\bar{J}$. We introduce the following divisor:

$$D':=D+(-K_J)H_{\bar{J}}.$$

Notice that D' is an effective divisor on $X^n_{(0)}$, of the form (1.1). We will denote by m'_i and d' its coefficients: $d':=d-K_J$, $m'_i:=m_i-K_J$ for $i\in\bar{J}$ and $m'_i:=m_i$ for $i\notin\bar{J}$. By construction,

$$K'_J:=\sum_{i\in J}m'_i-(R+1)d'=0.$$

Moreover, $K'_{I(R)}=K_{I(R)}+(-K_J)\geq K_{I(R)}\geq 0$, so that $L_{I(R)}$ is contained in the base locus of D' with multiplicity at least $K'_{I(R)}$.

Assume that $\tilde{k}_{I(R)}\geq K_{I(R)}+1$. The multiplicity of containment of the linear cycle $L_{I(R)}$ in the base locus of D' is

$$\tilde{k}'_{I(R)}\geq\tilde{k}_{I(R)}+(-K_J)\geq K_{I(R)}+1+(-K_J)=K'_{I(R)}+1.$$

For any point $p\in L_{I(R)}$, we compute the multiplicity $\tilde{k}'_{p,p_{R+2}}$ with which the line spanned by the points p and $p_{i_{R+2}}$ is contained in the base locus of D':

$$\tilde{k}'_{p,p_{R+2}}\geq\tilde{k}'_{I(R)}+m'_{R+2}-d'\geq K'_{I(R)}+1+m'_{R+2}-d'=K'_{I(R+1)}+1=1.$$

This shows that such a line is in the base locus of D' for any $p\in L_{I(R)}$. Letting p vary in $L_{I(R)}$, we obtain that L_J is in the base locus of D'. This gives a contradiction.

Case (ii). We know by the previous case that for all cycles $L_{I(\bar{r})}$, the multiplicity of containment is given by $k_{I(\bar{r})}$. For smaller cycles, $I(R)$, with $1\leq r=R<\bar{r}$ and $K_{I(R)}\geq 0$ we run induction on $\bar{r}-R$ and the same argument used for Case (ii) applies.

Case (iii). We assume that the statement holds for $I(R)$, with $I(r)\subseteq I(R)$ and such that $K_{I(R)}\geq 0$ and we show that it holds for $I(r)$. Namely, assuming $\tilde{k}_{I(R)}=k_{I(R)}$, for all $I(R)$, we prove that $\tilde{k}_{I(r)}=k_{I(r)}$. Notice that $0\leq K_{I(R)}\leq K_{I(r)}$, since $m_i\leq d$. Therefore, all linear subspaces $L_{I(\rho)}$ of $L_{I(R)}$ are contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity at least $K_{I(\rho)}\geq 0$, by [6, Lemma 2.1]. In particular, $L_{I(r)}$ is contained at least $K_{I(r)}$ times.

Assume now by contradiction that $L_{I(r)}$ is contained in $\mathrm{Bs}(\mathcal{L})$ with multiplicity at least $1+K_{I(r)}$. We know that the linear cycle $L_{I(R)\setminus I(r)}$ is contained in the base locus with multiplicity at least $K_{I(R)\setminus I(r)}\geq 0$. For any point p in the cycle $L_{I(r)}$ and p' in $L_{I(R)\setminus I(r)}$ we obtain that the line spanned by p and p' is contained in the base locus with multiplicity at least $1+K_{I(r)}+K_{I(R)\setminus I(r)}-d=1+K_{I(R)}$. This gives a contradiction. $\square$

Remark 4.3. We must mention that the first part of the proof of Proposition 4.2, Case (i), was established in [6, Proposition 2.5], but we include it here also for the sake of completeness.

4.2. **Vanishing theorems.** In this section we prove Theorem 4.1. The proof will be based on induction on $n \geq 1$ and on the multiplicities. The case $n = 1$ is trivial. The case $s \leq n+2$, namely $m_i = 0$, for $i \geq n+3$, is solved in [6, Ch. 4].

As in Section 4.1, let $\mathcal{I}$ be the set of all subsets of $\{1, \ldots, s\}$ and let $\mathcal{I}^>$ be the subset of $\mathcal{I}$ introduced in (4.2). For all $1 \leq r \leq n-1$, let $\mathcal{I}(r) \subset \mathcal{I}$ be the set of multi-indices of $\mathcal{I}$ of length at most $r+1$, as in (2.1).

Lemma 4.4. *In the above notation, if D satisfies (4.1), then the set $\mathcal{I}^>(D)$ satisfies conditions (I), (II) and (III) of Section 2.*

Proof. Condition (I) follows by the definition.

Since $m_i \leq d$, for all i, then $K_I \geq K_J$ for $I \subset J$. Hence (II) is satisfied.

If $|I \cup J| \leq n+1$ then (III) follows easily, because one may assume that L_I and L_J are coordinate subspaces. Assume that $I \cap J = \emptyset$ and $|I \cup J| = n+2$. The inequalities (4.1) and $m_i \geq 1$ for all i, imply that $K_I + K_J = \sum_{i \in I \cup J} m_i - nd \leq 0$, hence at most one among I and J is in $\mathcal{I}^>$. This proves (III) in this case. If $I \cap J \neq \emptyset$ and $|I \cup J| = n+2$, we conclude by just noticing that $K_I + K_J \leq K_I + K_{J \setminus I} \leq 0$. We leave it to the reader to verify that condition (III) holds also for $|I \cup J| \geq n+3$. $\square$

In the notation of Section 2, for every $1 \leq r \leq n-1$, let $X^n_{(r)}$ be the blow-up of $X^n_{(r-1)}$ along the union of the strict transforms of the r-cycles $L_{I(r)}$, $I(r) \in \mathcal{I}^>$. The total transform of $D_{(r-1)} \subset X^n_{(r-1)}$ is

$$(\pi_{(r)})^* D_{(r-1)} = dH - \sum_{I(\rho) \in \mathcal{I}(r-1)} k_{I(\rho)} E_{I(\rho)}, \tag{4.3}$$

while the strict transform of $D_{(r-1)}$ (cfr. (1.4)) is

$$\begin{aligned} D_{(r)} &= dH - \sum_{I(\rho) \in \mathcal{I}(r-1)} k_{I(\rho)} E_{I(\rho)} - \sum_{I(r) \in \mathcal{I}} k_{I(r)} E_{I(r)} \\ &= dH - \sum_{I(\rho) \in \mathcal{I}(r)} k_{I(\rho)} E_{I(\rho)}. \end{aligned} \tag{4.4}$$

4.2.1. *Induction on the sum of the multiplicities.* Let D be as in (1.1). Modulo reordering the indices $\{1, \ldots, s\}$ if necessary, we may assume that $m_1 \geq 1$. We introduce the following divisors on $X^n_{(0)}$:

$$D' := D + E_1 = dH - (m_1 - 1)E_1 - \sum_{i=2}^{s} m_i E_i.$$

It corresponds to the linear system of hypersurfaces of $\mathbb{P}^n$ denoted by $\mathcal{L}' := \mathcal{L}_{n,d}(m_1 - 1, m_2, \ldots, m_s)$.

Remark 4.5. Notice that $\mathcal{I}^>(D') \subset \mathcal{I}^>(D)$. In particular the linear base locus of D' is contained in that of D. Precisely, the cycles L_I with $I \notin \mathcal{I}(n-1)_1$ have the same multiplicity of containment in both base loci, while the cycles L_I with $I \in \mathcal{I}(n-1)_1$ are contained with multiplicity one more in D, see Proposition 4.2.

The following provides the induction step on the integer $b = \sum_{i=1}^{s} m_i - nd$, that was defined in (1.7).

Lemma 4.6. *If D satisfies (4.1), then so does D'. In particular the set $\mathcal{I}^>(D')$ satisfies condition (I), (II) and (III) of Section 2.*

Proof. It is a trivial computation that $b(D') = b(D) - 1$. The second statement follows from Lemma 4.4. $\square$

For all $I \in \mathcal{I}(r)_1$ (cfr. definition (2.1)), set $k'_I := k_I(D') = \max(K_I - 1, 0)$. We have

$$D'_{(r)} = dH - (m_1 - 1)E_1 - \sum_{i=2}^{s} m_i E_i - \sum_{I \in \mathcal{I}(r)_1} k'_I E_I - \sum_{I \in \mathcal{I}(r) \setminus \mathcal{I}(r)_1} k_I E_I. \tag{4.5}$$

Using the notation

$$\mathcal{I}(r)_1^{>} := \mathcal{I}^{>} \cap \mathcal{I}(r)_1 = \{I(\rho) : 1 \leq \rho \leq r, 1 \in I(\rho), K_{I(\rho)} > 0\},$$

we can write

$$D'_{(r)} = D_{(r)} + \sum_{I \in \mathcal{I}(r)_1^{>}} E_I.$$

Remark 4.7. For every $1 \leq r \leq n-1$, the natural space where $D'_{(r)}$ lives is the subsequent blow-up of $\mathbb{P}^n$ along the linear cycles parametrised by $\mathcal{I}^{>}(D')$, ordered in increasing dimension. We can consider the total transform of $D'_{(r)}$ in the subsequent blow-up of the latter space along the linear cycles parametrised by $\mathcal{I}^{>}(D) \setminus \mathcal{I}^{>}(D')$, in increasing dimension: this space is denoted by $X^n_{(r)}$. The dimensions of the cohomology groups of $D'_{(r)}$ and of its total transform are equal. Therefore, abusing notation, in this section and throughout this paper, we will use the same symbol for $D'_{(r)}$ and for its total transform in $X^n_{(r)}$ and we will consider the cohomology groups $H^i(X^n_{(r)}, \mathcal{O}_{X^n_{(r)}}(D'_{(r)}))$.

4.2.2. *Induction on n.* Let us assume, without loss of generality, that $d \geq m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$. In order to employ induction on n we want to restrict the divisor $D'_{(r)}$ to the strict transform of the exceptional divisor E_1, that is isomorphic to the blown-up space $X^{n-1}_{(r-1)}$. Such a restricted divisor will also satisfy condition (4.1).

Precisely, for every $1 \leq r \leq n-1$, in the space $X^n_{(r)}$ we will use the following restriction sequence of type (*A*):

$$0 \longrightarrow D'_{(r)} - E_1 \longrightarrow D'_{(r)} \longrightarrow D'_{(r)}|_{E_1} \longrightarrow 0. \tag{4.6}$$

of which now we give a detailed description. By abuse of notation, write E_1 for the strict transform $(\pi_{(r,0)})_*^{-1} E_1$ of the exceptional divisor in $X^n_{(0)}$ of the point p_1. Recall from Section 2.1 that $E_1 \cong X^{n-1}_{(r-1)}$ has Picard group generated by the hyperplane class h and by the exceptional classes $e_{I(\rho)|1} = E_{I(\rho)}|_{E_1}$, for $I(\rho) \in \mathcal{I}(r)_1$. The divisor $D'_{(r)}$ restricts to E_1 as

$$D'_{(r)}|_{E_1} = (m_1 - 1)h - \sum_{I \in \mathcal{I}(r)_1} k'_I e_{I|1},$$

where $k'_I := \max(K_I - 1, 0)$, as in Section 4.2.1.

Remark 4.8. Notice that for $r = 1$, $E_1 \cong X^{n-1}_{(0)}$ is the space $\mathbb{P}^{n-1}$ blown-up at a collection of points in general position. Therefore $G := D'_{(r)}|_{E_1}$ can be seen as a divisor of the form (1.1) given by the integers $(m_1 - 1, k'_{12}, \ldots, k'_{1s})$. Moreover one can check that $G_{(r-1)} = D'_{(r)}|_{E_1}$, for every $1 \leq r \leq n-1$.

The following provides the induction step on n; in fact $D'_{(1)}|_{E_1}$ is a divisor of the form (1.1) in a blown-up $\mathbb{P}^{n-1}$ in points in general position.

Lemma 4.9. *If D satisfies (4.1), then so does $D'_{(1)}|_{E_1}$.*

Even though the same argument appeared in the proof of [6, Lemma 5.7], we include it here for the sake of completeness.

Proof. Set $\bar{s} := \#\mathcal{I}(1)_1^{>}$ be the number of lines contained in the base locus of D passing through the first point p_1. Consider the divisor

$$D'_{(1)}|_{E_1} = (m_1 - 1)h - \sum_{i=2}^{\bar{s}} k'_{1i} e_{1i|1}$$

in $E_1 \cong X^{n-1}_{(0)}$. Notice that $k'_{1i} = m_1 + m_i - d - 1 \leq m_1 - 1$, as $m_i \leq d$, for all $i = 2, \ldots, \bar{s}$. If $\bar{s} \leq n-1$, the first inequality of (4.1) is trivially satisfied by $D'_{(1)}|_{E_1}$. When $\bar{s} \geq n$, we conclude by computing

$$\begin{aligned}\sum_{i=2}^{\bar{s}} k'_{1i} &= \sum_{i=2}^{\bar{s}} m_i + \bar{s}(m_1 - d - 1)\\ &\leq \sum_{i=1}^{s} m_i - m_1 - (s - \bar{s} - 1) + \bar{s}(m_1 - d - 1)\\ &\leq nd + s - n - m_1 - (s - 1) + \bar{s}(m_1 - d)\\ &\leq (n-1)(m_1 - 1).\end{aligned}$$

□

4.2.3. *Global sections in the exact sequences (4.6).* Let us denote by

$$l(D, r) := \sum_{I(\rho) \in \mathcal{I} \setminus \mathcal{I}(r-1)} (-1)^{\rho - r - 1} \binom{n + k_{I(\rho)} - \rho - 1}{n}, \tag{4.7}$$

the integer that appears in Theorem 1.4 (b). It is the alternating sum of the contributions to the formula for the linear virtual dimension of $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$, given by the multiple base cycles of dimension at least $r + 1$. Notice that $l(D, 0) = \mathrm{ldim}(\mathcal{L})$, see Definition 1.2.

Lemma 4.10. *In the above notation, the following equality holds:*

$$l(D, r) = l(D', r) - l(D'_{(1)}|_{E_1}, r - 1).$$

Proof. It follows from the equality of Newton binomials $\binom{\alpha}{\beta} = \binom{\alpha-1}{\beta} + \binom{\alpha-1}{\beta-1}$, for $\alpha, \beta \geq 1$. □

Proposition 4.11. *Assume that a divisor D of the form (1.1) satisfies condition (4.1) and fix $1 \leq r \leq n - 1$. The following sequence of global sections is exact:*

$$0 \longrightarrow H^0(D'_{(r)} - E_1) \longrightarrow H^0(D'_{(r)}) \longrightarrow H^0(D'_{(r)}|_{E_1}) \longrightarrow 0.$$

Proof. It is enough to show the statement for $r = 1$, because the number of global sections of $D_{(1)}$ and $D'_{(1)}$ (and of $D_{(1)}|_{E_1}$) are preserved when taking strict transforms in the spaces $X^n_{(r)}$ ($E_1 \cong X^{n-1}_{(r-1)}$ respectively).

Notice that $h^0(D'_{(1)} - E_1) = h^0(D' - E_1)$. Since $D' - E_1 = D$ by definition, we conclude that $h^0(D'_{(1)} - E_1) = h^0(D)$.

Moreover, notice that $D_{(1)}|_{E_1}$ is a divisor on $E_1 \cong X^{n-1}_{(0)}$, the $(n-1)$-dimensional projective space blown-up at points.

One can verify that $\mathrm{ldim}(D) = \mathrm{ldim}(D') - \mathrm{ldim}(D'_{(1)}|_{E_1})$, using Lemma 4.10. Furthermore, by Theorem 1.3 all three divisors are effective and only linearly obstructed, as they satisfy (4.1) by Lemma 4.6 and Lemma 4.9. Hence $h^0(D) = h^0(D') - h^0(D'_{(1)}|_{E_1})$. □

4.2.4. *The cohomologies of the kernel of the sequences* (4.6). We now compute all cohomology groups of the divisors in the sequences (4.6), for every r, and in particular we obtain that all cohomology groups but the $(r+1)-$st vanish.

Proposition 4.12. *Assume that D satisfies conditions* (4.1) *and fix $1 \leq r \leq n-1$. Assume that statement* (b) *of Theorem 1.4 holds for the pairs (D', r) and $(D'_{(1)}|_{E_1}, r-1)$. Then statement* (b) *of Theorem 1.4 holds for the pair $(D'_{(r)} - E_1, r)$.*

Proof. For r such that $1 \leq r \leq \min(\bar{r}, n-1)$, in the space $X^n_{(r)}$ we consider sequence (4.6). By assumption we have

- $h^i(D'_{(r)}|_{E_1}) = 0$, for all $i \neq 0, r$ and $h^r(D'_{(r)}|_{E_1}) = l(D'_{(r)}|_{E_1}, r-1)$;
- $h^{i+1}(D'_{(r)}) = 0$, for all $i \neq 0, r$ and $h^{r+1}(D'_{(r)}) = l(D', r)$.

Therefore, the long exact sequence in cohomology associated with sequence (4.6) splits into the fundamental sequences

$$0 \to H^0(D'_{(r)} - E_1) \to H^0(D'_{(r)}) \to H^0(D'_{(r)}|_{E_1}) \to H^1(D'_{(r)} - E_1) \to 0, \tag{4.8}$$

and

$$0 \to H^r(D'_{(r)}|_{E_1}) \to H^{r+1}(D'_{(r)} - E_1) \to H^{r+1}(D'_{(r)}) \to 0. \tag{4.9}$$

Using Proposition 4.11, we obtain $h^1(D'_{(r)} - E_1) = 0$ from (4.8). Moreover

$$h^{r+1}(D'_{(r)} - E_1) = h^r(D'_{(r)}|_{E_1}) + h^{r+1}(D'_{(r)}) = l(D, r). \tag{4.10}$$

The first equality is obtained by means of the sequence (4.9), the second equality follows from Lemma 4.10. In particular, $h^i(D'_{(r)} - E_1) = 0$, for all $i \neq 0, r+1$ and this concludes the proof. □

4.2.5. *The cohomology of $D_{(r)}$.* Now we deduce, from the above results, the cohomologies of $D_{(r)}$ for all $r \leq n-1$.

Proposition 4.13. *Assume that D is effective. For all $0 \leq r \leq n-1$ and $i \geq 0$ we have*

$$h^i(D'_{(r)} - E_1) = h^i(D_{(r)}).$$

Proof. If $r = 0$ the statement is obvious.

Fix $1 \leq \rho \leq r \leq n-1$. For every ρ, let $\{I(\rho)_0, \ldots, I(\rho)_{s_\rho}\}$ be the set of multi-indices of length $\rho+1$ that contain $\{1\}$. We use the notation $\mathcal{I}(\rho)_1^{>} = \{I \in \mathcal{I}(\rho)_1 : K_I > 0\}$ of Section 4.2.1.

Let $\prec$ be the total order on $\mathcal{I}(\rho)_1^{>}$ inherited by the lexicographical order on $\mathcal{I}(n-1)_1$, that was introduced in the proof of Theorem 3.1. Precisely $I(\rho')_{j'} \prec I(\rho)_j$ if and only if $\rho' < \rho$ or $\rho' = \rho$ and $j' < j$.

In the space $X^n_{(r)}$, for every pair (ρ, j), $1 \leq \rho \leq r$ and $0 \leq j \leq s_\rho$, for which $I(\rho)_j \in \mathcal{I}(\rho)^>_1$, we consider the divisor

$$F(\rho, j) = D'_{(r)} - E_1 - \sum_{\substack{I(\rho')_{j'} \prec I(\rho)_j \\ I(\rho')_{j'} \in \mathcal{I}(\rho)^>_1}} E_{I(\rho')_{j'}} - E_{I(\rho)_j}, \tag{4.11}$$

that we recursively define, using the same idea as in the proof of Theorem 3.1, using sequences of type (A) and (B), by the following rule:

$$\begin{aligned} &F(0,0) = D'_{(r)} - E_1, \\ &F(\rho, 0) = F(\rho-1, s_{\rho-1}) - E_{I(\rho)_0},\ 1 \leq \rho \leq r, \\ &F(\rho, j) = F(\rho, j-1) - E_{I(\rho)_j},\ 1 \leq j \leq s_\rho. \end{aligned} \tag{4.12}$$

The divisor $F(0,0)$ is obtained from the divisor $D'_{(r)}$ as the kernel of an exact sequence of type (A), precisely sequence (4.6). For $1 \leq \rho \leq n-2$ (resp. $\rho = n-1$) each divisor $F(\rho, j)$ is obtained from the previous one as kernel of a sequence of type (B) (resp. (C)), cfr Section 2.3. Notice that the last divisor obtained is $F(r, s_r) = D_{(r)}$.

We claim that, for all pairs (ρ, j) such that $I(\rho)_j \in \mathcal{I}(\rho)^>_1$, if $\rho \leq n-2$ (resp. $\rho = n-1$), the first factor of the restriction (resp. the restriction) of $F(\rho, j-1)$ to $E_{I(\rho)_j}$ in the sequence of type (B) (resp. (C)) is $-\mathrm{Cr}_\rho(h)$. Cfr. (2.6) for a definition of the divisor class $\mathrm{Cr}_\rho(h)$. The latter has vanishing cohomologies by Theorem 3.1. Hence, by the Kunneth formula, we conclude that the restriction itself has vanishing cohomologies.

In order to prove the claim, recall that the first factor of $E_{I(\rho)_j}$ is isomorphic to $X^\rho_{(\rho-2)}$, the toric variety obtained from $\mathbb{P}^\rho$ by blowing-up all coordinate points, lines etc, cfr Remark 2.1. Using (4.5), (4.11) and the third line in (4.12), we can expand the expression of $F(\rho, j-1)$:

$$\begin{aligned} F(\rho, j-1) = dH &- \sum_{I(t) \in \mathcal{I}(\rho-1)} k_{I(t)} E_{I(t)} - \sum_{1 \leq l \leq j-1} k_{I(\rho)_l} E_{I(\rho)_l} - (k_{I(\rho)_j} - 1) E_{I(\rho)_j} \\ &- \sum_{j < l \leq s_\rho} (k_{I(\rho)_l} - 1) E_{I(\rho)_l} - \sum_{I(t) \in \mathcal{I}(n-1)^>_1 \setminus \mathcal{I}(\rho)} (k_{I(t)} - 1) E_{I(t)}. \end{aligned}$$

Using the intersection table (2.5), we see that the second summation in the first line above, as well as the whole second line, restricts to 0 on the first factor of $E_{I(\rho)_j}$. Therefore, the restriction of $F(\rho, j-1) + (k_{I(\rho)_j} - 1)E_{I(\rho)_j}$ to $E_{I(\rho)_j}$ is the following product

$$\left(dh - \sum_{\substack{I(t) \subset I(\rho)_j \\ I(t) \in \mathcal{I}(\rho-1)}} k_{I(t)} e_{I(t)} \right) \boxtimes * = \left(-k_{I(\rho)_j} \mathrm{Cr}_\rho(h) \right) \boxtimes *, \tag{4.13}$$

where $*$ denotes the appropriate divisor. In order to prove the equality in (4.13), first notice that, for every $I(\rho-1) \in \mathcal{I}$, $I(\rho-1) \subset I(\rho)_j$, we have (cfr. Remark 2.5):

$$e_{I(\rho-1)} = h - \sum_{\substack{0 \leq t < \rho-1 \\ I(t) \subset I(\rho-1)}} e_{I(t)}.$$

Therefore, assuming without loss of generality that $I(\rho)_j = \{1, \dots, \rho+1\}$, one can see that the coefficient of h is

$$
\begin{aligned}
d - \sum_{\substack{I(\rho-1)\subset I(\rho)_j \\ I(\rho-1)\in\mathcal{I}(\rho-1)}} k_{I(\rho-1)} &= d - \sum_{i=1}^{\rho+1}(m_1 + \cdots + \check{m}_i + \cdots + m_{\rho+1} - (\rho-1)d)\\
&= d - \rho\sum_{i=1}^{\rho+1} m_i + (\rho+1)(\rho-1)d\\
&= -\rho k_{I(\rho)_j}.
\end{aligned}
$$

We leave to the reader to verify that also the coefficients of the exceptional divisors in the left hand side and right hand side of (4.13) coincide. Finally, the self-intersection $E_{I(\rho)_j}|_{E(\rho)_j}$ is $-\mathrm{Cr}_\rho(h)$ on the first factor, see Lemma 2.8. This concludes the proof.

□

Proof of Theorem 4.1. We first prove part (b); part (a) follows.

The proof is by induction on n, the dimension of the ambient space, and on $b = b(D)$. If $n = 1$ the statement is trivially true, as well as if $b = -nd$, i.e. if $m_i = 0$ for all $i = 1, \dots, s$.

Fix the pair (n, b) and assume by induction that the statement is true for all divisors in the $(n-1)$-dimensional space satisfying the bound (4.1), and for all divisors with the values $(n, b-1)$. In particular, the statement is true for the divisors $D' = D + E_1$ and $D'_{(1)}|_{E_1}$. By Lemma 4.6 and Lemma 4.9, both D' and $D'_{(1)}|_{E_1}$ satisfy condition (4.1). Finally, for all $1 \le r \le n-1$, the conclusion follows from Propositions 4.11, 4.12 and 4.13.

□

5. Non-effective divisors on blown-up projective space

For an arbitrary number of general points s, let D be a divisor on $X^n_{(0)}$ of the form (1.1):

$$D = dH - \sum_{i=1}^{s} m_i E_i,$$

with $m_i \le d+1$ and $d, m_i \ge 0$. If $m_i = d+1$, for some i, then D is obviously not effective, namely it has no non-zero global sections. However, under some further assumptions on the integer $b = \sum_{i=1}^{s} m_i - nd$, introduced in (1.7), we will see in this and in the next section that the same vanishing theorems as in the effective case hold for the cohomology groups of the strict transforms of D in $X^n_{(r)}$.

5.1. **Formal definition of linear base locus for non-effective divisors.** With the aim of studying the cohomology of the strict transforms on $X^n_{(r)}$ of non-effective divisors, we generalise the notion of linear base locus.

In order to do so, we introduce the following notation, that generalises (4.2):

$$\mathcal{J}^{>} := \mathcal{J}^{>}(D) := \{I(r) \subseteq \{1, \dots, s\} : 0 \le r \le n-1, K_J > 0, \forall J \subseteq I(r)\}, \tag{5.1}$$

where the integer K_J is defined, given $(d, m_1, \dots, m_s)$, as in Section 4.1. We have the following inclusion of sets: $\mathcal{J}^{>}(D) \subseteq \mathcal{I}^{>}(D)$.

For a non-effective divisor, whenever $I(r) \in \mathcal{J}^{>}$, we will say that the cycle $L_{I(r)}$ is *contained in the base locus of* D. Moreover we will call *strict transform* of D the divisor $D_{(r)}$ (and $\tilde{D} = D_{(\bar{r})}$) defined as the formal sum of divisors given by

$$D_{(r)} := dH - \sum_{\substack{I(\rho)\in\mathcal{J}^{>}\\ 0\leq\rho\leq r}} k_{I(\rho)}E_{I(\rho)}, \tag{5.2}$$

where

$$\bar{r} = \bar{r}(D) := \max\{r : I(r) \in \mathcal{J}^{>}\}.$$

Remark 5.1. The above definitions formally extend the notion of linear base locus and of strict transform, after blowing-up such base locus, from effective divisors to non-effective ones. When D is effective, we have $m_i \leq d$. Therefore the condition $K_J \geq K_{I(r)}$ is obviously satisfied for every $J \subseteq I(r)$. This implies the following equality of sets: $\mathcal{J}^{>}(D) = \mathcal{I}^{>}(D)$. In particular (4.4) and (5.2) coincide.

Moreover each exceptional divisor $E_{I(\rho)}$ in $X_{(r)}$ will be a product and the first factor will be the toric variety described in Remark 2.1, cfr Remark 2.7

5.2. **Main theorems.** We will prove the following formula for the dimension of the $(r+1)-$st cohomology group of $D_{(r)}$ in terms of the speciality of $\tilde{D}$. Recall from (4.7) that $l(D, r+1)$ denotes the alternating sum of the integers (1.6) that compute the contribution of the linear r-cycles $L_{I(r)}$ of dimension at least $r+1$, contained in the base locus of D with multiplcity $k_{I(r)}$, to its speciality.

Theorem 5.2. *Let D in $X^n_{(0)}$, as in (1.1), be any effective divisor or a non-effective divisor with $m_i \leq d+1$. Assume that $\mathcal{J}^{>}(D)$ satisfies conditions* (I), (II) *and* (III) *of Section 2. Then the following holds.*

(1) *For any $0 \leq r \leq n-1$, we have*

$$h^{r+1}(D_{(r)}) = l(D, r+1) + \sum_{\rho=r+1}^{n} (-1)^{\rho-r-1} h^{\rho}(\tilde{D}).$$

Moreover, if $h^i(\tilde{D}) = 0$, for all $i \geq 1$, then $h^i(D_{(r)}) = 0$ for all $i \neq r+1$.

(2) *For any integer $l_{I(r)}$ with $0 \leq l_{I(r)} \leq \min(r, k_{I(r)})$, we have*

$$h^i(D_{(r)}) = h^i(D_{(r)} + \sum_{I(r)} l_{I(r)}E_{I(r)}), \quad \text{for } i \geq 0.$$

Observe first that for $r = -1$ the binomial sum $l(D, 0)$ defined in (4.7) becomes $\operatorname{ldim}(D)$, the linear virtual dimension of D introduced in Definition 1.2. Moreover, setting $D_{(-1)} := D_{(0)} = D$ for $r = -1$ the theorem reads

Corollary 5.3. *For any D in $X^n_{(0)}$ effective divisor or non-effective divisor with $m_i \leq d+1$ and $\mathcal{J}^{>}$ satisfying conditions* (I), (II) *and* (III) *from Section 2, then*

$$h^0(D) = \operatorname{ldim}(D) + \sum_{\rho=1}^{n} (-1)^{\rho} h^{\rho}(\tilde{D}).$$

5.3. **Vanishing theorems for toric divisors.** If $s \leq n+1$, the blow-up $X^n_{(0)}$ of $\mathbb{P}^n$ at s points in general position, that we can think of as coordinate points of $\mathbb{P}^n$, is a toric variety. All the blow-ups $X^n_{(r)}$ of $X^n_{(0)}$ in this case will also be toric varieties, cfr Remark 2.1. We will call *toric divisor* a divisor D on $X^n_{(0)}$ of the form (1.1), not necessarily effective, with $s \leq n+1$. This section is devoted to the study of the cohomology of the strict transforms $D_{(r)}$ of D, for D a toric divisor.

Theorem 5.4. *If D is any toric divisor with $m_i \leq d+1$, then $h^0(D) = \operatorname{ldim}(D)$, $h^n(\tilde{D}) = \binom{b-1}{n}$ and $h^i(\tilde{D}) = 0$ for every $1 \leq i \leq n-1$.*

Remark 5.5. If D is an effective toric divisor (then $m_i \leq d$ and $b(D) \leq 0$), it is $h^n(\tilde{D}) = 0$. In this case Theorem 5.4 is just a particular case of [6, Theorem 4.6]. However, for all non-effective toric divisors with $m_i \leq d+1$ this result is new. Theorem 5.4 suggests that the *virtual* n-dimensional cycle $L_{1,\ldots,n+1} \cong \mathbb{P}^n$ should be considered in the dimension count. More precisely, this virtual cycle is detected by the $n-$th cohomology group; its contribution depends on its virtual multiplicity $k_{1,\ldots,n+1}$ that becomes nothing else than $b(D)$. Moreover, for non-effective toric divisors with $m_1 = d+1$ or $b(D) \leq 0$, Theorem 5.4 implies that $\operatorname{ldim}(\mathcal{L}) = 0$ where r runs from -1 to n (see also Corollary 5.13).

5.3.1. *Chambers of non-effective toric divisors with vanishing theorems.* We recall that the effective cone $\operatorname{Eff}(X)$ of $X^n_{(0)}$, for $s \leq n+1$, is given by the inequalities

$$d \geq 0, \quad b \leq 0, \quad m_i \leq d, \quad \forall i \in \{1, \ldots, s\}. \tag{5.3}$$

See e.g. [6, Lemma 2.2]. Moreover, for any effective toric divisor, the strict transform $\tilde{D}$ in $X^n_{(n-1)}$ has vanishing cohomologies, see Theorem 1.4.

The following inequalities define chambers of $\mathrm{N}^1(X) \setminus \operatorname{Eff}(X)$ such that if D lies in those chambers, then $\tilde{D}$ has vanishing theorems, by Theorem 5.4 (cfr. also Theorem 1.6):

$$s \leq n+1: \quad 1 \leq b \leq n, \quad m_j \leq d+1; \tag{5.4a}$$

$$s \leq n+1: \quad b \leq 0, \quad m_1 = d+1, \ m_j \leq d+1. \tag{5.4b}$$

Remark 5.6. In the above notation, if D satisfies (5.4), then the set $\mathcal{J}^{>}(D)$ satisfies conditions (I), (II) and (III) of Section 2.

Theorems 5.2 and 5.4 have the following consequence, for divisors in the chambers (5.4).

Corollary 5.7. *Statements (a) and (b) of Theorem 1.4 hold for non-effective toric divisors satisfying the inequalities (5.4).*

5.3.2. *Strict transforms and Cremona transformations.* In this section we show that the strict transform $\tilde{D}$ of a non-effective toric divisor equals a negative multiple of the strict transform of the standard Cremona transformation of the hyperplane class of $\mathbb{P}^n$. Section 3 was dedicated to a cohomological descriptions of the latter.

Proposition 5.8. *Let D be a toric divisor with $d \geq 1$, $0 \leq m_i \leq d+1$.*

1. *If $0 \leq m_i \leq d$ and $b = 0$, then $D_{(n-1)} = 0$.*
2. *If either $b \geq n$ or $m_i \leq d$ for all i and $b \geq 1$, then $D_{(n-1)} = -b\mathrm{Cr}_n(H)$.*

Proof. Notice in the first case that $b = 0$ for $s \leq n$, can occur only for $m_i = d$, for all $i \in \{1, \ldots, n\}$. In this case, the strict transform is obviously $D_{(n-1)} = 0$, because it is obtained by subtracting the strict transform of the hyperplane through the n points d times. This observation also implies in the second case that the hypothesis forces $s = n + 1$. This allows us to restrict in both cases to $s = n + 1$ for the rest of the proof.

Case (1) if $s = n + 1$ with $b = 0$ and $m_j \leq d$, for all j, then all hyperplanes spanned by sets of n points are contained in the base locus of D with (exact) multiplicity

$$k_{1,\ldots,\check{j},\ldots,n+1} = K_{1,\ldots,\check{j},\ldots,n+1} = \sum_{i=1}^{n+1} m_i - m_j - (n-1)d = d - m_j \geq 0.$$

Similarly, for Case (2), if $s = n + 1$ and $m_j \leq d + 1$ for all j, the assumption $b \geq 1$ implies that

$$k_{1,\ldots,\check{j},\ldots,n+1} = K_{1,\ldots,\check{j},\ldots,n+1} = \sum_{i=1}^{n+1} m_i - m_j - (n-1)d = b + d - m_j \geq 0.$$

Moreover, if $m_i \leq d$ and $K_{I(r)} > 0$, then for any subset $I(\rho) \subset I(r)$ we have $K_{I(\rho)} > 0$. The same holds for $m_i \leq d+1$ and $b \geq n$, indeed these two assumptions together imply that $K_{I(r)} \geq r \geq 0$, for every $I(r)$ with $1 \leq r \leq n-1$. Therefore, also in this case, all hyperplanes spanned by sets of n points are contained in the base locus of D with (exact) multiplicity $k_{I(n-1)}$.

In all above cases, the strict transform $D_{(n-1)}$ of D is

$$dH - \sum_{I(\rho)\in\mathcal{I}(n-2)} k_{I(\rho)} E_{I(\rho)} - \sum_{\substack{I(\rho)\in\mathcal{J}^{>} \\ 0\leq\rho\leq n-2}} k_{I(\rho)} E_{I(\rho)} - \sum_{i=1}^{n+1} k_{I(n-1)} E_{I(n-1)},$$

where $E_{I(n-1)}$ is the strict transform in $X^n_{(n-1)}$ of the hyperplane of $\mathbb{P}^n$ passing through the n points parametrised by $I(n-1)$, see Remark 2.1. One can verify that the coefficients of the hyperplane class H and the coefficient of the exceptional divisors $E_{I(\rho)}$ in the expression for $D_{(n-1)}$ are respectively

$$d - \sum_i (b - m_i + d) = -\, nb,$$
$$k_{I(\rho)} - \sum_{i\notin I(\rho)} (b - m_i + d) = -\,(n - \rho - 1)b.$$

This concludes the proof. □

Remark 5.9. Proposition 5.8 provides another proof that the effective cone of $X^n_{(0)}$, the space blown-up at $s \leq n+1$ points, is described by the inequalities (5.3).

5.4. **Vanishing theorems for non-effective divisors with $s = n+2$ points or more.** We recall here that in the case $s = n+2$, the space $X^n_{(n-2)}$ was identified by Kapranov [20] with the moduli space $\overline{\mathcal{M}}_{0,n+3}$ of stable rational curves with $n+3$ marked points. There are chambers of divisors with $s = n+2$ for which the strict transform $\tilde{D}$ in $X^n_{(n-1)}$ has vanishing theorems.

5.4.1. *Chambers of non-effective divisors on $\overline{\mathcal{M}}_{0,n+3}$ with vanishing theorems.* We recall that the effective cone $\mathrm{Eff}(X)$ of $X_{(0)}$, with $s = n+2$, is formed by the divisors of the form (1.1) that satisfy the following inequalities:

$$d \geq 0, \quad b \leq 0, \quad m_i \leq d, \quad b \leq m_i, \quad \forall i \in \{1, \ldots, n+2\}, \tag{5.5}$$

where $b := b(D) = \sum_{i=1}^{n+2} m_i - nd$, as defined in (1.7). See e.g. [6, Lemma 2.2]. Moreover, for any effective divisor D satisfying (5.5), the strict transform $\tilde{D}$ has vanishing cohomologies, see Theorem 1.4.

In this section we extend the vanishing theorems to divisors sitting in a "small" region outside the effective cone of $X := X^n_{(0)}$, the blow-up of $\mathbb{P}^n$ at $n+2$ points in general position.

For a divisor D of the form (1.1), we consider the following chambers of $\mathrm{N}^1(X) \setminus \mathrm{Eff}(X)$:

$$s = n+2: \quad b = 1, \quad m_j \leq d+1; \tag{5.6a}$$

$$s = n+2: \quad b \leq 0, \quad m_1 = d+1, \; m_j \leq d+1. \tag{5.6b}$$

Remark 5.10. In the above notation, if D satisfies (5.6), then the set $\mathcal{J}^{>}(D)$ satisfies conditions (I),(II) and (III) of Section 2.

Theorem 5.11. *Statements (a) and (b) of Theorem 1.4 hold for non-effective divisors with $s = n+2$ points satisfying the inequalities (5.6).*

We will also generalise Theorem 4.1 to non-effective divisors with an arbitrary number of points in a small region around the effective cone, namely for $m_i \leq d+1$. Recall the bound (4.1) of Theorem 1.3 that was proved in [6] to be, together with $m_i \leq d$, sufficient condition for a divisor with $s \geq n+3$ to be effective.

Theorem 5.12. *Statements (a) and (b) of Theorem 1.4 hold for non-effective divisors with $s \geq n+3$ points such that $m_i \leq d+1$ and $b \leq \min(n - s(d), s-n-2)$.*

5.5. **Euler characteristic and linear virtual dimension.** In this section we compute the Euler characteristic of the strict transforms of non-effective divisors D and compare them with their linear expected dimension.

Corollary 5.13. *Let D be a divisor with $m_i \leq d+1$. The following holds.*

(1) If D is toric and effective, then

$$\chi(\tilde{D}) = \mathrm{ldim}(D) \geq 1.$$

In particular if $b = 0$, then

$$\chi(\tilde{D}) = \mathrm{ldim}(D) = 1.$$

(2) If D is toric and non-effective, then

$$\mathrm{ldim}(D) = 0, \quad \chi(\tilde{D}) = (-1)^n h^n(\tilde{D}) = (-1)^n \binom{b-1}{n}.$$

In particular if $b \leq n$ then

$$\chi(\tilde{D}) = h^n(\tilde{D}) = \mathrm{ldim}(D) = 0.$$

(3) If D is non-effective such that $s \leq n+1$ and $b \leq n$, or $s = n+2$ and $b \leq 1$, or $s \geq n+3$ and $b \leq s-n-2$, then

$$\chi(\tilde{D}) = \mathrm{ldim}(D) = 0.$$

Proof. Part (1) follows from Proposition 5.8 and [6, Theorem 4.6]. Moreover, part (2) and (3) follow from Theorem 5.4 if $s \leq n+1$, Theorem 5.11 if $s = n+2$ and Theorem 5.12 for $s \geq n+3$, in fact $h^0(D) = \operatorname{ldim}(D) = 0$. □

In particular all non-effective divisors satisfying the conditions of Corollary 5.13(3) are only linearly obstructed and their strict transform $\tilde{D}$ is not linearly obstructed.

We conclude the section by noticing that if one moves further away from the effective cone, by choosing for instance $m_1 = d+2$ or higher, then the strict transform after blowing-up all linear subspaces contained in the base locus may still be linearly obstructed. We illustrate some instances where this happens.

Example 5.14. *Let $D = 3H - 5E_1 - mE_2$ in $X^3_{(0)}$ be a divisor with $h^0(D) = 0$.*

- *If $m = 3$, we have $\chi(D_{(1)}) = -5$. Hence $h^1(D_{(1)}) \geq 5$.*
- *If $m = 4$, we have $\chi(D_{(1)}) = 0$. Is $h^1(D_{(1)}) = h^2(D_{(1)}) = 0$?*
- *If $m = 5$, we have $\chi(D_{(1)}) = 6$. Hence $h^2(D_{(1)}) \geq 6$.*

6. Proofs of the result of Section 5

In this section we prove all results stated in Section 5. More precisely this section is organised as follows. We prove Theorem 5.4, Theorem 5.11 and Theorem 5.12 for the planar case in Section 6.1. This case is the base step for an induction argument on n. In Section 6.2 we will construct the induction procedure, that will follow the line of Section 4.2. We will complete the proof of Theorem 5.4, Theorem 5.2, Theorem 5.11 and Theorem 5.12 for dimension $n \geq 3$ in Sections 6.3, 6.4, 6.5 and 6.6 respectively.

6.1. The planar case.

Proposition 6.1. *Let D be a non-effective divisor on $X^2_{(0)}$ with $m_i \leq d+1$ such that $b \leq 2$ if $s \leq 3$, $b \leq 1$ for $s = 4$ and $b \leq \min(n - s(d), s-4)$ for $s \geq 5$. Then $h^i(\tilde{D}) = 0$, for $i = 0,1,2$.*

Proof. If $s = 1$ the statement is trivial. Therefore we will assume that $s \geq 2$. Without loss of generality, we will also assume $m_1 \geq \cdots \geq m_s \geq 1$.

Notice that in the case $s \leq 3$ (resp. $s = 4$), the assumption that D is not effective implies that (5.4) (resp (5.6)) is satisfied. Moreover if $s \geq 5$, the same assumption implies that $m_1 = d+1$.

Assume $s \leq 3$ and $b = 2$. In this case the strict transform of D is $\tilde{D} = -2\mathrm{Cr}_2(H)$ and $h^i(\tilde{D}) = 0$ for $i = 0,1,2$ by Theorem 3.1.

We will assume from now on that $b \leq 1$ for $s \leq 4$ and $b \leq s-4$ for $s \geq 5$.

Case (1) $m_1 = d+1$. We have $k_{1i} = 1 + m_i \geq 1$, for $2 \leq i \leq s$. The condition $b = (d+1) + m_2 + \cdots + m_s - 2d \leq 1$ implies that $k_{ij} \leq 0$, if $1 \notin \{i,j\}$. We can write

$$\tilde{D} = D_{(1)} := D - \sum_{i=2}^{s} k_{1i}(H - E_1 - E_i) = (-b-s+2)H - (-b-s+3)E_1 + \sum_{i=2}^{s} E_i.$$

By Serre duality and the fact that $b \leq 1$, we obtain

$$h^2(\tilde{D}) = h^0((b+s-5)H - (b+s-4)E_1) = 0.$$

Moreover, using $\chi(\tilde{D}) = 0$ and $h^0(\tilde{D}) = 0$, we conclude that also $h^1(\tilde{D}) = 0$.

Case (2) $m_1 \leq d$. In this case $s \leq 4$ and the non-effectivity assumption implies that $b = 1$.

If $s = 3$, then $m_1 + m_2 + m_3 - 2d = 1$ so all three lines are in the base locus of D. We leave to the reader to check that $\tilde{D} = -2H + E_1 + E_2 + E_3$. The vanishing of $h^i(\tilde{D})$, for $i = 0, 1, 2$, follows from Theorem 3.1.

Assume $s = 4$ and $m_1 = d$. We have $k_{1i} = m_i$ and we conclude $h^1(\tilde{D}) = h^2(\tilde{D}) = 0$, since the strict transform is

$$\tilde{D} = -H + E_1.$$

Case $s = 4$ and $m_1 < d$. We must have $k_{12} = m_1 + m_2 - d > 0$ (otherwise $K_{34} \leq K_{12} \leq 0$ would be in contradiction with $b \geq 1$). From $K_{ij} + K_{pq} = 1$, we obtain that $k_{34} = 0$ and that the lines L_{ij} and L_{pq} can not be both contained in the base locus of D, therefore either $k_{23} = 1$ or $k_{14} = 1$. Now, observe that by assumption we have $k_{13} \geq k_{23}$ and $k_{13} \geq k_{14}$. We have the following two cases.

Case (i). The lines L_{12}, L_{14}, L_{13} are in the base locus and

$$\tilde{D} = [4d-(3m_1+m_2+m_3+m_4)]H-[3d-(2m_1+m_2+m_3+m_4)]E_1-\sum_{i=2}^{4}(d-m_1)E_i.$$

Case (ii). The lines L_{12}, L_{23}, L_{13} are in the base locus and

$$\tilde{D} = 2(m_4 - 1)H - \sum_{i=1}^{3}(m_4 - 1)E_i - m_4 E_4.$$

We claim $h^i(\tilde{D}) = 0, i \geq 0$. Indeed, in both cases $\tilde{D}$ has the property that $k_{12}(\tilde{D}) = 0$ and $k_{34}(\tilde{D}) = 1$, therefore $h^i(\tilde{D}) = h^i(\tilde{D} - L_{34})$. Furthermore, the divisor $\tilde{D} - L_{34}$ has now $k_{12} = 1$ and $k_{34} = 0$, therefore $h^i(\tilde{D} - L_{34}) = h^i(\tilde{D} - L_{34} - L_{12})$. We continue to eliminate simple obstructions until the residue becomes a toric divisor with positive coefficients of the form $H - E_1 - E_2 - E_3$ that has all vanishing theorems. □

Proposition 6.2. *Statements* (*a*) *and* (*b*) *of Theorem 1.4 hold for non-effective divisors in* $\mathbb{P}^2$ *satisfying the inequalities* (5.4) *if* $s \leq 3$, (5.6) *if* $s = 4$ *and* $b \leq \min(n - s(d), s - 4)$ *if* $s \geq 5$.

Proof. For non-effective divisors in $\mathbb{P}^2$, we reduce the proof to the vanishing theorems for $\tilde{D}$. Indeed, for any line L_{ij} through two points p_i and p_j with corresponding $k_{ij} := -D \cdot (H - E_i - E_j) \geq 1$, we have $D|_{L_{ij}} = \mathcal{O}_{\mathbb{P}^1}(-k_{ij})$.

For $\tilde{D} = D - \sum_{1 \leq i \leq j \leq 4} k_{ij}(H - E_i - E_j)$, the Riemann-Roch Theorem implies

$$\chi(\tilde{D}) = \sum_{k_{i,j} \geq 1} \binom{k_{ij}}{2} + \chi(D). \tag{6.1}$$

Furthermore, since D has positive coefficients it follows that $h^2(D) = 0$ while $h^0(D) = h^0(\tilde{D})$. The formula (6.1) implies that

$$h^1(D) = \sum_{k_{ij} \geq 1} \binom{k_{ij}}{2} + h^1(\tilde{D}) - h^2(\tilde{D}),$$

$$\chi(D) = h^0(D) - h^1(D) = \binom{d+2}{2} - \sum_{i=1}^{4} \binom{m_i + 1}{2}.$$

Finally, the above equalities imply

$$h^0(D) = \mathrm{ldim}(D) + h^1(\tilde{D}) - h^2(\tilde{D}).$$

We conclude using Proposition 6.1.

□

Corollary 6.3. *Theorem 5.4, Theorem 5.11 and Theorem 5.12 hold for $n = 2$.*

6.2. **Case $n \geq 3$: an induction procedure.** We will prove the statements of Section 5 by induction on n and on b.

As in Section 4.2.1, let us introduce the following divisor

$$D' := D + E_1.$$

We will consider D' and its strict transforms $D'_{(r)}$ as living in $X^n_{(r)}$, the space blown-up along the base linear cycles of D, parametrised by the set $\mathcal{J}^>(D)$, cfr. Remark 4.7.

Remark 6.4. Since $m_i \leq d+1$, the divisor $D'_{(1)}|_{E_1}$ in $E_1 \cong X^{n-1}_{(0)}$ (introduced in (4.5)) has $k'_{1i} := k_{1i}(D') = \max(K_{1i} - 1, 0) \leq m_1$, for all $2 \leq i \leq s$, namely the multiplicities of the points $e_{1i|1}$ do not exceed the multiplicity $m_1 - 1$ by more than one.

Lemma 6.5. *In the above notation, assume D satisfies (5.4) if $s = n+1$, or $m_1 = d+1$ if $s \geq n+2$. Then $b(D'_{(1)}|_{E_1}) = b(D') = b - 1$.*

Proof. Assume first that $m_1 = d+1$. Notice that $k'_{1i} = m_1 + m_i - d - 1 = m_i \geq 1$, for all $2 \leq i \leq s$. Therefore, the restricted divisor $D'_{(1)}|_{E_1}$ is of the form

$$dh - \sum_{i=2}^{s} m_i e_{1i|1},$$

and one computes

$$b(D'_{(1)}|_{E_1}) = \sum_{i=2}^{s} m_i - (n-1)d = \sum_{i=1}^{s} m_i - nd - m_1 + d = b - 1.$$

Assume now that $s \leq n+1$ and that $m_i \leq d$, for all $1 \leq i \leq s$. In this case $b \geq 1$. Fix an index $i \geq 2$ and write

$$b - 1 = \sum_{j=2,\ j \neq i}^{s} m_j - (n-1)d + (m_1 + m_i - d - 1).$$

As $b \geq 1$, $k'_{1i} = m_1 + m_i - d - 1 \geq 0$. Therefore, the divisor $D'_{(1)}|_{E_1}$ is of the form

$$(m_1 - 1)h - \sum_{i=2}^{s} k'_{1i} e_{1i|1},$$

with possibly some of the k'_{1i} being zero. One computes

$$b(D'|_{E_1}) = \sum_{i=2}^{s} (m_1 + m_i - d - 1) - (n-1)(m_1 - 1) = b - 1.$$

□

In the conditions of Lemma 6.5, it is clear that the map

$$H^0(D'_{(r)}) \to H^0(D'_{(r)}|_{E_1})$$

between global sections is injective, as the kernel $D'_{(r)} - E_1$ has no non-zero sections. We prove in the next proposition that it is in fact an isomorphism.

Proposition 6.6. *Assume D satisfies the same assumptions as Lemma 6.5. Then $h^0(D'_{(1)}) = h^0(D'_{(1)}|_{E_1})$.*

Proof. Notice that $h^0(D') = h^0(D'_{(1)})$. We first consider the case (5.4a) with $m_i \leq d$. Notice that $D'_{(1)}|_{E_1}$ has base points with multiplicity bounded by the degree, that is $k'_{1i} \leq m_1 - 1$, for all $2 \leq i \leq s$. If $2 \leq b \leq n$, then, using Lemma 6.5, we get that $b(D') = b(D'_{(1)}|_{E_1}) \geq 1$, therefore both divisors D' and $D'_{(1)}|_{E_1}$ have no non-zero global sections. Indeed, they both lie outside the effective cones of the respective spaces, described in (5.3). If $b = 1$, then, using Lemma 6.5, we get that $b(D') = b(D'_{(1)}|_{E_1}) = 0$, so the two divisors are effective. Using Proposition 5.8, we obtain that they both have only one non-zero global section, namely $h^0 = 1$.

We now consider the case in which $m_1 = d + 1$. Notice that $k'_{1i} = m_i$, for $2 \leq i \leq s$. Assume first that $m_2 = d + 1$. Then D' has a point, E_2, with multiplicity larger than the degree, that is $m_2 = d + 1$; moreover $D'_{(1)}|_{E_1}$ has a point, $e_{12|1}$, with multiplicity larger than the degree, that is $k'_{12} = d+1$. Therefore, none of them has non-trivial global sections. If, instead, $m_i \leq d$, for all $2 \leq i \leq s$, then both D' and $D'_{(1)}|_{E_1}$ have points of multiplicity bounded by the degree. By the trivial observation that D' has a point of multiplicity d and $s - 1$ points of multiplicity respectively $m_2, \ldots, m_s$ and $D'_{(1)}|_{E_1}$ has $s - 1$ points of multiplicity respectively $m_2, \ldots, m_s$, we conclude that their spaces of global section have the same dimension, see for instance [6, Lemma 5.1]. □

Proposition 6.7. *Assume D satisfies the same assumptions as Lemma 6.5 and let $1 \leq r \leq n - 1$.*

Assume that statement (a) of Theorem 1.4 holds for D' and $D'_{(1)}|_{E_1}$ and that statement (b) of Theorem 1.4 holds for the pairs (D', r) and $(D'_{(1)}|_{E_1}, r - 1)$. Then statements (a) and (b) hold for (D, r).

Proof. We prove the statement in two steps. Following the idea of the proofs of Proposition 4.12 and Proposition 4.13, we first show that the statement holds for $D'_{(r)} - E_1$ and then that $h^i(D_{(r)}) = h^i(D'_{(r)} - E_1)$, for all $i \geq 0$.

To prove the first part, we consider the sequences in cohomology associated with the short exact sequence (4.6), as in the proof of Proposition 4.12. Since $h^0(D'_{(r)} - E_1) = 0$, using (4.8), Proposition 6.6 and the assumption $h^1(D'_{(r)}) = 0$ implies that $h^1(D'_{(r)} - E_1) = 0$. Moreover, we can compute the other cohomologies of $D'_{(r)} - E_1$ exploiting those of $D'_{(r)}$ and $D'_{(r)}|_{E_1}$ using the long exact sequence in cohomology associated. In fact if $i \neq 0, r + 1$, then $h^i(D'_{(r)} - E_1) = 0$ because $h^{i-1}(D'_{(r)}|_{E_1}) = 0$ and $h^i(D'_{(r)}) = 0$ by induction on n and b, respectively. If $i = r + 1$ we conclude using (4.9).

To prove the second part, we argue as in the proof of Proposition 4.13. If $r = 0$ the statement is obvious. Fix $1 \leq \rho \leq r$. Similarly to Section 4.2.1 we introduce the sets

$$\mathcal{J}(\rho)^{>} := \mathcal{J}^{>} \cap \mathcal{I}(r)_1,$$

the set of multi-indices that parametrise cycles through p_1 that are in the base locus of D.

In the space $X^n_{(r)}$, for every pair (ρ, j), $1 \leq \rho \leq r$ and $0 \leq j \leq s_\rho$, for which $I(\rho)_j \in \mathcal{J}(\rho)^{>}_1$, we consider the divisor $F(\rho, j)$ defined in (4.11), using sequences of type (B). We claim that for any pair (ρ, j) such that $I(\rho)_j \in \mathcal{J}(\rho)^{>}_1$, the divisor $F(\rho, j)$ restricts, on the first factor of $E_{I(\rho)_j}$ which is isomorphic to $X^\rho_{(\rho-1)}$, to $-\mathrm{Cr}_\rho(h)$. The latter has vanishing cohomology groups, by Theorem 3.1. Then, by means of the Kunneth formula, we can conclude that the restriction itself has vanishing cohomologies. □

6.3. **The toric case.** In this section we complete the proof of Theorem 5.4, for $n \geq 3$. The case $n = 2$ was covered in Section 6.1.

Proof of Theorem 5.4. In this proof D will be a toric divisor. If D is effective, i.e. when $b \leq 0$ and $m_i \leq d$ (cfr. (5.3), the vanishing theorems for the higher cohomology groups of the strict transforms of D in $X_{(r)}$ were established in Theorem 1.4.

Assume $\bar{r} = n - 1$ and consider the following two independent cases:

(1) $b \geq n$ or $b \geq 1$ and $m_i \leq d$ for all $1 \leq i \leq n+1$.
(2) $b \leq n - 1$ and $m_1 = d + 1$.

Case (1). By Proposition 5.8 part (2), we conclude that $\tilde{D} = -b\mathrm{Cr}(H)$. In Theorem 3.1 we proved that for all $i \neq n$ then $h^i(\tilde{D}) = h^i(\mathcal{O}(-b\mathrm{Cr}_n(H))) = 0$, while

$$h^n(\tilde{D}) = h^n(\mathcal{O}(-b\mathrm{Cr}_n(H))) = h^n(\mathbb{P}^n, \mathcal{O}(-b)) = \binom{b-(n+1)+n}{n}.$$

In particular, if $b = n$, we obtain $h^i(\tilde{D}) = 0, \forall i \geq 0$.

Case (2). We assume that $1 \leq b \leq n-1$. In this case, following the notation of Section 6.2, we have, by Lemma 6.5 that $b(D'_{(1)}|_{E_1}) = b(D') = b - 1$. Therefore by Proposition 5.8, we have that the two divisors have strict transforms equal to $-(b-1)\mathrm{Cr}_{n-1}(h) \in \mathrm{Pic}(E_1) \cong \mathrm{Pic}(X^{n-1}_{(n-3)})$ and $-(b-1)\mathrm{Cr}_n(H) \in \mathrm{Pic}(X^n_{(n-2)})$, hence both have vanishing cohomologies. By means of the long exact sequence in cohomology associated to the sequence of type (A), used in (4.9) with $r = n-1$, we obtain $H^i(\tilde{D}) = 0$, for every $i \geq 0$. Using the formulas (4.10), we can deduce the cohomologies of the strict transforms $D_{(r)}$, for $1 \leq r \leq n-2$.

Assume $\bar{r} \leq n-2$. The proof is by induction on b and n. If $n = 2$ the statement is covered in Section 6.1. The case $b = -nd$, i.e. if $m_i = 0$ for all $i = 1, \ldots, s$ is trivial.

For the pair (n, b), with $n \geq 3$, and $m_1 \geq 1$, we will assume the statement to be true for $(n-1, b)$ and $(n, b-1)$. We can can conclude using Proposition 6.7 that provides the induction step, that $h^i(D_{(\bar{r})}) = 0$, for all $i \geq 0$. Using the formulas (4.10), we can deduce the cohomologies of the strict transforms $D_{(r)}$, for $1 \leq r \leq \bar{r} - 1$.

□

6.4. **Proof of Theorem 5.2.**

Proof of Theorem 5.2. Let D be a divisor of the form (1.1). Assume that $k_{I(r)} = k_{I(r)}(D) \geq 1$ for some linear cycle of dimension r, $L_{I(r)}$, spanned by a subset $I(r)$ on $r+1$ points.

Consider the toric divisor

$$G = dh - \sum_{i \in I(r)} m_i e_i \tag{6.2}$$

on $X^r_{(r-2)}$. Notice that $k_{I(r)} = b(G)$ and $m_i \leq d+1$.

We have that the restriction of $D_{(r-1)}$ to the exceptional divisor is of the form

$$D_{(r-1)}|_{E_{I(r)}} = \tilde{G} \boxtimes 0,$$

where $\tilde{G}$ represents the strict transform of the divisor G defined in (6.2), cfr. (2.5).

By applying Theorem 5.4 to the toric divisor G in $X^r_{(0)}$ with $m_i \leq d+1$ and $b(G) = k_{I(r)}$, we obtain

$$h^i(E_{I(r)}, D_{(r-1)}|_{E_{I(r)}}) = h^i(X^r_{(r-2)}, \tilde{G}) = h^i(X^r_{(0)}, -k_{I(r)}h). \tag{6.3}$$

Furthermore, since conditions (I),(II) and (III) of Section 2 are satisfied and using (6.3), the result contained in [6, Proposition 4.10] implies that the following formulae also hold

$$h^{r+1}(D_{(r)}) = h^{r+1}(D_{(r+1)}) - h^{r+2}(D_{(r+1)}) + \sum_{k_{I(r)\geq 1}} \binom{n + k_{I(r)} - (r+1) - 1}{n}, \tag{6.4}$$

while, for all $i \leq r+1$, we have

$$h^i(D_{(r+1)}) = h^i(D_{(r+2)}). \tag{6.5}$$

We recall if $r = n-1$, i.e. I is a subset with $|I| = n$, the (6.4) and (6.5) hold for the divisor $E_I = \tilde{H}_I$, a blown-up hyperplane along the cycles spanned by the subsets of $I \in \mathcal{J}^>$ in $X^{n-1}_{(n-3)}$. Notice that for each multi-index $I \in \mathcal{J}^>$ of cardinality n and H_I hyperplane spanned by the points parametrised by I, with $k_I \geq 1$, then on $X^n_{(n-2)}$ we have the following divisor:

$$\tilde{D} = D_{(n-2)} - \sum_{I = I(n-1) \in \mathcal{J}^>} k_I H_I.$$

We iterate the formula (6.4) for $h^{\rho+1}(D_{(\rho)})$ with $\rho \geq r+1$. More explicitly, (6.5) allows us to substitute $h^{\rho+1}(D_{(\rho+1)}) = h^{\rho+1}(D_{(\rho+2)}) = \cdots = h^{\rho+1}(\tilde{D})$ and apply again equation (6.4) for $h^{\rho+2}(D_{(\rho+1)})$. By iteration of (6.4) for $\rho \geq r+1$ we obtain that $h^{r+1}(D_{(r)})$ equals the sum of $\sum_{\rho=r+1}^{n} (-1)^{\rho-r-1} h^{\rho}(\tilde{D})$ and $l(D, r+1)$, see (4.7). This concludes the proof of part (1).

To prove part (2), observe that $D_{(r)}$ is obtained from $D_{(r-1)}$ by subtracting $k_{I(r)}$ times the exceptional divisors $E_{I(r)}$, for all $I(r)$ on the space $X^n_{(r)}$. By the above argument we obtain that the restricted divisor, for $l = 0, \ldots, k_{I(r)} - 1$, satisfies

$$h^i(E_{I(r)}, D_{(r-1)} - (k_{I(r)} - l)E_{I(r)}|_{E_{I(r)}}) = h^i(\mathbb{P}^r, -lh).$$

The right-hand side cohomology group is zero, for all $0 \leq l \leq \min(r, k_{I(r)})$. $\square$

6.5. **The case $s = n + 2$.**

Remark 6.8. We say that a divisor of the form (1.4) has *non-negative coefficients* if and only if the degree d and the multiplicities m_i and $k_{I(r)}$ are non-negative. We notice that for any effective divisor D, the strict transform $\tilde{D}$ has non-negative coefficients. On the other hand, non-effective divisors behave differently. More precisely, if D is non-effective, the strict transform $\tilde{D}$ may have both non-negative and positive coefficients.

Proof of Theorem 5.11. As the case $n = 2$ was worked out in Section 6.1. For $n \geq 3$ by Remark 5.6 and Theorem 5.2 we can reduce the proof of Theorem 5.11 for a divisor D to the proof of vanishing theorems for its strict transform $\tilde{D}$.

We will prove the vanishing theorems for $\tilde{D}$ by induction on n, based on the planar case.

Without loss of generality we can assume $m_1 \geq \cdots \geq m_{n+2} \geq 1$.

We split the proof in two parts corresponding to the two cases $m_1 = d + 1$ and $m_1 \leq d$.

Case (1) $m_1 = d+1$. The proof is by induction on b and n. If $n = 2$ the statement is proved in Section 6.1, while if $b = -nd$, i.e. if $m_i = 0$ for all $i = 1, \dots, s$, the statement is trivial.

For the pair (n, b), with $n \geq 3$, and $m_1 \geq 1$, we will assume the statement to be true for $(n-1, b)$ and $(n, b-1)$. We can can conclude using Proposition 6.7 that provides the induction step.

Case (2) $m_1 \leq d$. We claim first that $\tilde{D}$ has always non-negative coefficients except in the family of examples contained in Example 6.9.

Example 6.9. *Let* $D := dH - \sum_{i=1}^{n-1} dE_i - m_n E_n - m_{n+1}E_{n+1} - m_{n+2}E_{n+2}$ *with* $m_n + m_{n+1} + m_{n+2} = d+1$. *Notice that only hyperplanes* H_I*, with* $I = \{1, \dots, n-1, n\}$, $I = \{1, \dots, n-1, n+1\}$ *or* $I = \{1, \dots, n-1, n+1\}$*, split off* D *with multiplicity* m_n, m_{n+1} *and* m_{n+2} *respectively. This implies that* $h^i(\tilde{D}) = 0$ *for all* $i \geq 0$ *where* $\tilde{D} = -H + E_1 + \ldots + E_{n-1} + \sum_{\substack{I \subset \{1,\dots,n-1\} \\ 2 \leq |I| \leq n-1}} E_I$.

To prove this claim, we will first show that conditions $b \leq 1$ and $m_i \leq d$ imply that only hyperplanes H_I with $I = \{1, \dots, \check{i}, \dots, n \check{+} 2\}$ or $I = \{1, \dots, n-1, n+2\}$ can be fixed part of D. Indeed, assume by contradiction that for some $I = \{1, \dots, \check{i}, \dots, \check{j}, \dots n+2\}$ with $j \leq n+1$ and $\{i, j\} \neq \{n, n+1\}$, the hyperplane H_I is in the base locus of D, i.e. $k_I \geq 1$. This implies that the line spanned by the points of $J = \{i, j\}$ is not fixed part of D since $K_J \leq 0$ for $I \coprod J = \{1, \dots, n+2\}$. By the fact that multiplicities are arranged in a decreasing order we obtain

$$d \geq m_i + m_j \geq m_{i+1} + m_{j+1}.$$

This contradicts the formal definition of base locus of non-effective divisors, see (5.2) and (5.1), since $K_I \geq 1$ and $K_J \leq 0$, for $\{i+1, j+1\} \subset I$.

To prove that $\tilde{D}$ has non-negative coefficients except in the cases discussed in Example 6.9 we consider the divisor $D' = D + E_{n+2}$. We notice that D' is an effective divisor since $m_i \leq d$ and $b(D') = b(D) - 1 \leq 0$, therefore $\tilde{D}'$ has non-negative coefficients. It is easy to see that $k_I(D') = k_I(D)$ whenever $I = \{1, \dots, \check{i}, \dots, n+1, n \check{+} 2\}$, while $k_I(D') = k_I(D) - 1$ if $I = \{1, \dots, n-1, \check{n}, n \check{+} 1, n+2\}$. Conclude that

$$\tilde{D} = \tilde{D}' - \tilde{H}_{1,\dots,n-1,\check{n},n\check{+}1,n+2} - \sum_{\substack{I\in\mathcal{J}^>, n+2\in I\\ 2\leq|I|\leq n-1}} E_I. \tag{6.6}$$

Also observe that

$$K_{1,\dots,n-1,\check{n},n\check{+}1,n+2}(D) = m_1 + \cdots + m_{n-1} + m_{n+2} - (n-1)d \leq m_{n+2}.$$

Equality occurs if and only if $m_1 = \cdots = m_{n-1} = d$, i.e. D is a divisor as in Example 6.9. Moreover, $K_{1,\dots,n-1,\check{n},n\check{+}1,n+2}(D) = m_{n+2} - 1$ if and only if $m_1 = \cdots = m_{n-2} = d$ and $m_{n-1} = d-1$. We leave it to the reader to check that $\tilde{D}$ has positive coefficients as well. Therefore we conclude

$$K_{1,\dots,n-1,\check{n},n\check{+}1,n+2}(D') + 1 = K_{1,\dots,n-1,\check{n},n\check{+}1,n+2}(D) \leq m_{n+2} - 2. \tag{6.7}$$

Now $\tilde{D}'$ is an effective divisor with $m'_{n+2} = m_{n+2} - 1 - K_{1,\dots,n-1\check{n},n\check{+}1,n+2}(D') \geq 2$. We conclude that the coefficient of the hyperplane class H, which is the degree of D', is at least 2, therefore equation (6.6) implies $\tilde{D}$ has positive coefficients.

We are going to prove now that vanishing theorems hold for divisors $\tilde{D}$, under the hypotheses (5.6).

The idea of the proof is as follows. Starting from a divisor, that is of the form $\tilde{D}$ defined in (5.2), using sequences of type (C) we decrease by one degree and n multiplicities by passing from $\tilde{D}$ to the divisor in the kernel of the sequence. In general, the kernel divisor will not be a strict transform of the form (5.2), since it can acquire simple linear base locus. We further use sequences of type (B) to eliminate the simple base locus and take the strict transform of the kernel divisor. See Section 2.3 for a definition of sequences of type (B) and (C).

The proof is based on induction on the dimension, n (as the restricted divisor lives in a an $(n-1)-$dimensional space) and on induction on the degree d and the multiplicities m_i (as the kernel divisor has lower coefficients).

We recall that $b = K_I + K_{I^c} \leq 1$ for any subset $I \subset \{1,\dots,n+2\}$, where $I^c := \{1,\dots,n+2\} \setminus I$. Therefore, K_I can be both positive or zero for $|I| = 2$. Moreover, we recall that the multiplicities have been arranged in decreasing order from the beginning. Furthermore, since $m_i \leq d+1$ and $b \leq 1$, we can have at most $n-1$ multiplicities equal to $d+1$. Let us assume, without loss of generality, that $J = \{1,\dots,n\}$. Fix the hyperplane spanned by the points parametrised by J, H_J, and let $\tilde{H}_J$ denote its strict transform, cfr. Remark 2.2. We use first a sequence of type (C) for the divisor $\tilde{D}$:

$$0 \to F := \tilde{D} - \tilde{H}_J \to \tilde{D} \to \tilde{D}|_{\tilde{H}_J} \to 0.$$

An easy computation shows that the kernel F could have simple linear base locus, i.e. $K_I(F) = 1$, for some index sets I with $\{n+1,n+2\} \subset I$. Using the notation (2.1), let $\mathcal{J}(n-1)^{>}_{n+1,n+2}$ be the set of index sets parametrising these cycles. We will use restriction sequences of type (B) to remove the simple linear base locus of F of dimension at most $n-1$, in increasing dimension, starting from the exceptional divisor of the line spanned by the last two points,

$$0 \to F - E_{n+1,n+2} \to F \to F|_{E_{n+1,n+2}} \to 0,$$

and then continuing with the cycles of the set $\mathcal{J}(n-1)^{>}_{n+1,n+2}$.

An algorithm that does this can be constructed following the same idea as in the proofs of Proposition 3.1 and 4.13. Precisely, let $\prec$ be the total order on the index sets of $\mathcal{J}(n-1)^{>}_{n+1,n+2}$ inherited from the lexicographical order on the set of all index sets of $\{1,\dots,n+2\}$. For $2 \le r \le n-1$, write s_r for the cardinality of the set of index sets of $\mathcal{J}(n-1)^{>}_{n+1,n+2}$ of length $r+1$. We recursively define:

$$\begin{aligned} F(0,0) &= F - E_{n+1,n+2},\\ F(r,0) &= F(r-1,s_{r-1}) - E_{I(r)_0},\ 2 \le \rho \le n-1,\\ F(r,j) &= F(r,j-1) - E_{I(r)_j},\ 1 \le j \le s_r. \end{aligned}$$

The output of the algorithm is the divisor

$$F(n-1,s_{n-1}) = \tilde{F} := F - \sum_{I \in \mathcal{J}(n-1)^{>}_{n+1,n+2}} E_I.$$

As in the proof of Proposition 4.13, the fist factor of every restriction is

$$-\mathrm{Cr}_r(h) \boxtimes *,$$

that has vanishing cohomologies. This implies that $h^i(\tilde{F}) = h^i(F)$. Notice that $b(\tilde{F}) = b(F) = b(F) = b \le 1$, so the kernel $\tilde{F}$ is now in the starting assumption (5.6). The induction argument on the degree and multiplicities applies to the kernel divisor of (C). Eventually the kernel divisor will become toric and by using Theorem 5.4, we know it has vanishing cohomologies.

In order to conclude the proof, we analyse the restricted divisor $\tilde{D}|_{\tilde{H}_J}$ of the sequence of type (C) and prove by induction on n that it has vanishing theorems. We denote by e' the trace of the cycle $E_{n+1,n+2}$ on the hyperplane H_J, $e' := E_{n+1,n+2}|_{\tilde{H}_J}$. We have

$$\tilde{D}|_{\tilde{H}_J} = dh - \sum_{i=1}^{n} m_i e_i - k_{n+1,n+2}e' - \sum_{\substack{k_I>0\\ |I|\le n-2, I\subset J}} k_I e_I - \sum_{\substack{k_M>0\\ |M|=n-1, M\subset J}} k_M \tilde{h}_M,$$

where for every subset $M \subset J$, $|M| = n-1$ then $\tilde{h}_M$ is the restriction $E_M|_{\tilde{H}_J}$ and E_M is the exceptional divisor of a codimension-2 cycle in the base locus of D. Note that $\tilde{h}_M$ is (the strict transform of) a the hyperplane of H_J passing through the points of J parametrised by M, as in Remark 2.3. We denote by G the divisor

$$G := dh - \sum_{i=1}^{n} m_i e_i - k_{n+1,n+2}e'.$$

We leave to the reader to check that the restriction is a strict transform

$$\tilde{D}|_{\tilde{H}_J} = \tilde{G}.$$

If $k_{n+1,n+2} = 0$, the trace $\tilde{D}|_{\tilde{H}_J}$ is toric. In this case $b(G) = K_J \le K_J + K_{n+1,n+2} = b(D) \le 1$ and Theorem 5.4 applies to the divisor G, proving the vanishing theorems for $\tilde{G}$.

If $k_{n+1,n+2} \ge 1$, the restriction $\tilde{D}|_{\tilde{H}_J} = \tilde{G}$ satisfies inequalities (5.6) with $b(F) = b(D) \le 1$. Since $\tilde{G}$ is a divisor in the blown-up $\mathbb{P}^{n-1}$ at $s = n+1$ points, we can prove by induction on n that the vanishing theorems hold for $\tilde{G}$. Proposition 6.2 in fact treats the first step of the induction on n, namely the case $n = 2$. We use induction on degree and multiplicities to conclude the vanishing of $\tilde{D}$. $\square$

6.6. **Case** $s \geq n+3$**.** In this section we prove the result for the case of arbitrary number of points satisfying $b \leq \min(n-s(d), s-n-2)$.

Proof of Theorem 5.12. As in Theorem 5.11 we observe that using Remark 5.6 and Theorem 5.2 we reduce the proof of Theorem 5.11 for a divisor D to the proof of vanishing theorems for its strict transform $\tilde{D}$.

We assume $m_1 \geq \cdots \geq m_s$. The effective case, namely $m_i \leq d$, was proved in Theorem 4.1. We can assume $m_1 = d+1$. Notice that $K_{2,\ldots,n+1} = b(D) - m_1 - \sum_{i \geq n+2} m_i + d \leq s-n-2-m_1-(s-n-1)+d \leq 0$, the inequality follows from $b \leq s-n-2$, $m_i \geq 1$ and $m_1 = d+1$. This implies that the hyperplane spanned by $\{2,\ldots,n+1\}$ is not contained in the base locus of D. Furthermore, no hyperplane spanned by $I(n-1) \subset \{2,\ldots,s\}$ is.

Notice that the inequality $b \leq s-n-2$ implies $K_I + K_J = \sum_{i \in I \cup J} m_i - nd \leq 0$ for any disjoint sets I, J with $|I|+|J| = n+2$. Therefore, $\mathcal{I}$, the set of all multi-indices in $\{1,\ldots,s\}$, satisfies conditions (I) and (III) of Section 2.

We conclude that $h^i(\tilde{D}) = 0$ using induction on n (the case $n=2$ was covered in proposition 6.2), and the induction procedure of Section 6.2.

□

7. Vanishing theorems for points in star configuration in $\mathbb{P}^n$

As an application of the results proved in the previous sections, we compute the number of global sections and prove vanishing theorems for the cohomology groups of the strict transforms along the linear base locus of some families of divisors interpolating points in *star configuration* in $\mathbb{P}^n$.

A star configuration of points is a collection of points satisfying some particular geometric relation. They have been object of study in many papers lately, see [16, 17] and references therein.

Given l hyperplanes in $\mathbb{P}^n$ that meet properly, i.e. not three of them intersecting along a $\mathbb{P}^{n-2}$, not four of them intersecting along a $\mathbb{P}^{n-3}$ etc, a *star configuration* of *dimension r subspaces* is the set given by the $\binom{l}{n-r}$ linear subspaces of dimension r in $\mathbb{P}^n$ formed by taking all possible intersections of $n-r$ among the l hyperplanes.

In [17, Theorem 3.2], the authors compute the Hilbert function of the ideals of star configurations of dimension r subspaces of multiplicity two. This provides a complete classification of linear systems in $\mathbb{P}^n$ interpolating such a scheme. In Theorem 7.3 we compute the number of global sections of a class of linear systems in $\mathbb{P}^n$ interpolating star configurations of points obtained by $l = n+2$ hyperplanes with higher multiplicities.

Remark 7.1. When $l = n+2$, star configurations of dimension r subspaces in $\mathbb{P}^n$ are obtained as follows. Let us embed $\mathbb{P}^n \hookrightarrow H \subset \mathbb{P}^{n+1}$ and denote by $p_1,\ldots,p_{n+2}$ a general collection of points of $\mathbb{P}^{n+1}$, that we may think of as the coordinate points, with respect to which H is a general hyperplane. The family of points $q_{ij} := L_{ij} \cap H \in H$, for all lines $L_{ij} = \langle p_i, p_j \rangle \subset \mathbb{P}^{n+1}$, forms a star configuration of points in H. Indeed, denoting by H_l the hyperplane spanned by all p_i's with $i \neq l$ for all $1 \leq l \leq n+2$, we can write $q_{ij} = H \cap \bigcap_{l \neq i,j} H_l$. Similarly, the family of r-linear subspaces $\lambda_I := L_I \cap H$, for all multi-indices $I = \{i_1,\ldots,i_{r+2}\} \subset \{1,\ldots,n+2\}$, is a star configuration of dimension r subspaces in H.

We now study effective divisors in $\mathbb{P}^n$ interpolating star configurations of points. Set $Y_{(0)} = Y^n_{(0)}$ to be the blow-up of $\mathbb{P}^n$ at the star configuration of points given as intersections of $\binom{n+2}{n}$ hyperplanes. Adopting the same notation of Remark 7.1, we call q_{ij}, $1 \leq i < j \leq n+2$ such points. Let us denote by h the hyperplane class and by e_{ij} the exceptional divisors.

Remark 7.2. As in Remark 7.1, let us embed $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ and, as in Section 2, let $X_{(1)} = X_{(1)}^{n+1}$ denote the blow-up of $\mathbb{P}^{n+1}$ at general points $p_1, \dots, p_{n+2}$ and, subsequently, along the lines spanned by those points, with exceptional divisors E_{ij}, $1 \leq i < j \leq n+1$. By writing $h = H|_H$ and $e_{ij} = E_{ij}|_H$, we obtain the following isomorphism $Y_{(0)} \cong X_{(1)}|_H$.

Given integers $d \geq m_1, \dots, m_{n+2} \geq 0$, we use the notation $k_{ij} = \max(m_i + m_j - d, 0)$. Assume

$$\sum_{i=1}^{n+2} m_i \leq (n+1)d, \tag{7.1}$$

and consider the following divisor on $Y_{(0)}$:

$$\Delta := dh - \sum_{1 \leq i < j \leq n+2} k_{ij} e_{ij}.$$

We prove that it is only linearly obstructed, with linear base locus supported on the star configurations of linear subspaces λ_I defined above (Remark 7.1) and that its subsequent strict transforms after blowing-up the dimension r star configuration have vanishing theorems.

Let $\mathcal{I}$ be the set of all multi-indices in $\{1, \dots, n+2\}$, and for each multi-index $I(r) \subset \mathcal{I}$ of cardinality $r+1$, we use the notation (1.2). For increasing r, let $Y_{(r)} = Y_{(r)}^n$ denote the blow-up of $Y_{(r-1)}$ along the strict transform of the star configuration of r-subspaces $\lambda_{I(r)}$ in H and let $\Delta_{(r)}$ be the strict transform of Δ.

Theorem 7.3. *In the above notation, assume that* (7.1) *is satisfied. then we have*

$$h^0(\Delta) = \binom{n+d}{n} + \sum_{I(r) \in \mathcal{I}, r \geq 1} (-1)^r \binom{n + k_{I(r)} - r}{n}.$$

Moreover, for all $1 \leq r \leq n-1$, $h^i(\Delta_{(r)}) = 0$, *for all* $i \neq 0, r+1$, *and*

$$h^{r+1}(\Delta_{(r)}) = \sum_{I(\rho) \in \mathcal{I}, \rho \geq r+2} (-1)^\rho \binom{n + k_{I(\rho)} - \rho}{n}.$$

In particular, $h^i(\tilde{\Delta}) = 0$, *for all* $i > 0$.

Proof. Recall the inclusion $H \subset \mathbb{P}^{n+1}$ and let $X_{(r+1)} := X_{(r+1)}^{n+1}$ be the blow-up of $\mathbb{P}^{n+1}$ first at the points $p_1, \dots, p_{n+2}$ and then along the linear subspaces L_I of dimension bounded above by $r+1$, in increasing dimension. We have the following inclusion $Y_{(r)} \subset X_{(r+1)}$.

Let us consider the following divisors on $X_{(0)}$

$$D = dH - \sum_{i=2}^{n+2} m_i E_i, \quad D' = D - H.$$

Notice that, since (7.1) is satisfied, then D is effective (cfr. (5.3)) and D' is either effective or satisfies condition (5.4a).

Abusing notation, denote by H the strict transform of $H \cong \mathbb{P}^n \subset \mathbb{P}^{n+1}$ and notice that the restriction $D_{(r+1)}|_H$ belongs to the linear system associated with $\Delta_{(r)}$:

$$D_{(r+1)}|_H \in |\Delta_{(r)}|.$$

Consider the following restriction sequence

$$0 \to D_{(r+1)} - H \to D_{(r+1)} \to D_{(r+1)}|_H \to 0. \tag{7.2}$$

We claim that the strict transforms of the linear ρ-cycles contained in base locus of $D_{(r+1)} - H$ have multiplicity bounded above by ρ, for all $\rho \leq r+1$. Hence, by Theorem 5.2 (2), they do not provide linear obstruction to such a divisor, namely $h^i(D_{(r+1)} - H) = h^i(D'_{(r+1)}) = 0$, for all $i \neq 0, r+2$. Moreover the following holds: $h^0(D_{(r-1)} - H) = h^0(D'_{(r-1)}) = h^0(D')$ and $h^0(D_{(r-1)}) = h^0(D)$. By Theorem 1.4 we have that $h^i(D_{(r+1)}) = 0$, for all $i \neq 0, r+2$. Therefore, we conclude that

$$h^0(\Delta_{(r)}) = h^0(D) - h^0(D'),$$
$$h^{r+1}(\Delta_{(r)}) = h^{r+2}(D'_{(r+1)}) - h^{r+2}(D_{(r+1)}),$$

and that all higher cohomology groups vanish, by means of the long exact sequence in cohomology associated with (7.2).

In order to conclude, we are left to prove the claim. Notice that the multiplicity of containment of $L_{I(\rho)}$ in the base locus of D' is $k'_{I(\rho)} := \max(\sum_{i\in I(\rho)} m_i - \rho(d-1), 0)$. Moreover, in $D_{(r+1)} - H$ the strict transform of the exceptional divisor of $L_{I(\rho)}$ has been removed $k_{I(\rho)} := \max(\sum_{i\in I(\rho)} m_i - \rho d, 0)$ times. To conclude, it is enough to observe that $k'_{I(\rho)} - k_{I(\rho)} \leq \rho$. □

Corollary 7.4. *The linear subspace λ_I is contained with multiplicity k_I in the base locus of $\Delta_{(r)}$.*

Proof. Let D be as in the proof of Theorem 7.3. The linear base locus of $\Delta_{(r)}$ is the intersection with the hyperplane H of the linear base locus of $D_{(r+1)}$, described in Proposition 4.2, and in particular it is supported at the dimension ρ star configurations, with $\rho \geq r+1$. □

An interpretation of the above corollary is that the only obstructions are the linear subspaces λ_I.

Moreover, this suggests a definition of virtual linear dimension for divisors interpolating points in star configuration in $\mathbb{P}^n$, that generalises the notion of virtual linear dimension for divisors interpolating points in general position that was introduced in [6] and that has been extensively studied throughout this paper. While in the general case the linear obstructions are the linear subspaces spanned by the points, in this case they are given by the star configurations of linear subspaces.

Remark 7.5. One may study general hyperplane sections of effective linearly obstructed divisors D in $\mathbb{P}^{n+1}$ interpolating arbitrary numbers of points in general position with bounds (4.1) or (5.6). Using Theorem 4.1 or Theorem 1.4 one analyses the divisor D and using Corollary 5.12 or Theorem 5.11 respectively one analyses the kernel divisor. The resulting restricted divisor $\Delta = D_{(1)}|_H$ in $Y^n_{(0)}$ interpolates points in special configuration and is linearly obstructed. Moreover, a cohomological description such as the one established in Theorem 7.3 can be obtained for such divisors.

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