Brittle mixed-mode cracks between linear elastic layers

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Additional Information:

- A Doctoral Thesis. Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University.

Metadata Record: https://dspace.lboro.ac.uk/2134/24177

Publisher: © Joseph D. Wood

Rights: This work is made available according to the conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) licence. Full details of this licence are available at: https://creativecommons.org/licenses/by-nc-nd/4.0/

Please cite the published version.
Brittle Mixed-Mode Cracks Between Linear Elastic Layers

by

Joseph D. Wood

DOCTORAL THESIS

Submitted in partial fulfilment of the requirements for the award of

Doctor of Philosophy

of Loughborough University

November 2016

© by Joseph D. Wood 2016
Dedicated to my parents
David and Kathleen Wood
Abstract

Original analytical theories are developed for partitioning mixed-mode fractures on rigid interfaces in laminated orthotropic double cantilever beams (DCBs) based on 2D elasticity by using some novel methods. Note that although the DCB represents a simplified case, it provides a deep understanding and predictive capability for real applications and does not restrict the analysis to a simple class of fracture problems. The developed theories are generally applicable to so-called 1D fracture consisting of opening (mode I) and shearing (mode II) action only with no tearing (mode III) action, for example, straight edge cracks, circular blisters in plates and shells, etc. A salient point of the methods is to first derive one loading condition that causes one pure fracture mode. It is conveniently called the first pure mode. Then, all other pure fracture modes can be determined by using this pure mode and the property of orthogonality between pure mode I modes and pure mode II modes. Finally, these 2D-elasticity-based pure modes are used to partition mixed-mode fractures into contributions from the mode I and mode II fracture modes by considering a mixed-mode fracture as the superposition of pure mode I and mode II fractures. The partition is made in terms of the energy release rate (ERR) or the stress intensity factor (SIF).

An analytical partition theory is developed first for a DCB composed of two identical linear elastic layers. The first pure mode is obtained by introducing correction factors into the beam-theory-based mechanical conditions. The property of orthogonality is then used to determine all other pure modes in the absence of through-thickness-shear forces. To accommodate through-thickness shear forces, first two pure through-thickness-shear-force pure modes (one pure mode I and one pure mode II) are discovered by extending a Timoshenko beam partition theory from Wang and Harvey\textsuperscript{1–3}. Partition of mixed-mode fractures under pure through-thickness shear forces is then achieved by using these two pure modes in conjunction with two thickness-ratio-dependent correction factors: (1) a shear correction factor, and (2) a pure-mode-II ERR correction factor. Both correction factors closely follow a normal distribution around a symmetric DCB geometry. The property of orthogonality between all pure mode I and all pure mode II fracture modes is then used to complete the mixed-mode fracture partition theory for a DCB with bending moments, axial forces and through-thickness shear forces.
Fracture on bimaterial interfaces is an important consideration in the design and application of composite materials and structures. It has, however, proved an extremely challenging problem for many decades to obtain an analytical solution for the complex SIFs and the crack extension size-dependent ERRs, based on 2D elasticity. Such an analytical solution for a brittle interfacial crack between two dissimilar elastic layers is obtained in two stages. In the first stage the bimaterial DCB is under tip bending moments and axial forces and has a mismatch in Young’s modulus; however, the Poisson’s ratios of the top and bottom layers are the same. The solution is achieved by developing two types of pure fracture modes and two powerful mathematical techniques. The two types of pure fracture modes are a SIF-type and a load-type. The two mathematical techniques are a shifting technique and an orthogonal pure mode technique. In the second stage, the theory is extended to accommodate a Poisson’s ratio mismatch. Equivalent material properties are derived for each layer, namely, an equivalent elastic modulus and an equivalent Poisson’s ratio, such that both the total ERR and the bimaterial mismatch coefficient are maintained in an alternative equivalent case. Cases for which no analytical solution for the SIFs and ERRs currently exist can therefore be “transformed” into cases for which the analytical solution does exist. It is now possible to use a completely analytical 2D-elasticity-based theory to calculate the complex SIFs and crack extension size-dependent ERRs.

The original partition theories presented have been validated by comparison with numerical simulations. Excellent agreement has been observed. Moreover, one partition theory is further extended to consider the blister test and the adhesion energy of mono- and multi-layered graphene membranes on a silicon oxide substrate. Use of the partition theory presented in this work allows the correct critical mode I and mode II adhesion energy to be obtained and all the experimentally observed behaviour is explained.
Acknowledgements

I am extremely grateful for the guidance, help and support that I have received from my supervisors Dr. Christopher Harvey, Dr. Simon Wang and Dr. Andrew Watson throughout my PhD. Additionally, I am very thankful for their contributions to my conference funding, which enabled me to present my research at two international conferences in Portugal.

Also, I would like to express my gratitude to Loughborough University and more specifically the Department of Aeronautical and Automotive Engineering for the funding to complete my PhD.

Finally, I would like to thank my family for their encouragement and support during my studies.
**List of Publications**

**Journal papers**


**Conference presentations**


## Contents

Abstract iii

Acknowledgements v

List of Publications vi

List of Figures x

List of Tables xiv

Symbols xv

Abbreviations xvi

### Chapter 1: Interfacial cracks

1.1. Introduction ........................................................................................................... 1

1.2. Linear elastic fracture mechanics ........................................................................ 7

1.2.1. Griffith theory ................................................................................................. 7

1.2.2. Energy release rate (ERR) ............................................................................. 10

1.2.3. Stress intensity factor (SIF) ........................................................................... 12

1.3. Analytical theories ............................................................................................. 13

1.4. Numerical methods ......................................................................................... 20

1.4.1. Virtual crack closure technique (VCCT) ..................................................... 20

1.4.2. Interface modelling ......................................................................................... 21

1.4.3. Interfacial crack between dissimilar materials ............................................. 22

1.5. Experimental ..................................................................................................... 26

1.5.1. Double cantilever beam (DCB) test ............................................................. 27

1.5.2. End-notched flexure (ENF) test ..................................................................... 28

1.5.3. Mixed-mode bending (MMB) test ............................................................... 28

1.5.4. Mixed-mode fracture investigations ............................................................. 30

1.6. Conclusion ........................................................................................................ 36

### Chapter 2: Orthotropic laminated beams with bending moments and axial forces

2.1. Introduction ..................................................................................................... 39

2.2. Review of Wang and Harvey work ................................................................... 41

2.3. Analytical development .................................................................................. 46

2.3.1. 2D elasticity partition theory ....................................................................... 46

2.3.2. 2D elasticity bending moment and axial force pure modes ....................... 48

2.4. 2D elasticity mixed-mode partition theories .................................................. 55

2.4.1. Suo and Hutchinson ...................................................................................... 55

2.4.2. Wang and Harvey approximate rule 1 ......................................................... 56

2.4.3. Wang and Harvey approximate rule 2 ......................................................... 57

2.4.4. Luo and Tong ............................................................................................... 57

2.5. Comparisons with Suo and Hutchinson’s partition theory ............................... 58

2.5.1. DCB with tip bending moments only ........................................................ 59

2.5.2. DCB with tip bending moments and axial forces ....................................... 64

2.6. Conclusion ..................................................................................................... 67
Chapter 3: Orthotropic laminated beams with bending moments, axial forces and through-thickness shear forces 69
3.1. Introduction .......................................................................................................... 69
3.2. Analytical development ....................................................................................... 71
  3.2.1. Timoshenko partition theory for crack tip through-thickness shear forces only ................................................................. 72
  3.2.2. FEM procedure .............................................................................................. 73
  3.2.3. Comparison of Timoshenko beam partition theory and 2D FEM results ..... 76
  3.2.4. 2D elasticity partition theory for crack tip through-thickness shear forces only................................................................................................................................. 78
  3.2.5. 2D elasticity crack tip through-thickness shear force pure modes ......... 79
  3.2.6. Thickness-ratio-dependent shear correction factor ....................................... 82
  3.2.7. Pure-mode-II ERR correction factor ........................................................... 84
  3.2.8. 2D elasticity partition theory for general loads ............................................. 86
3.3. Numerical verification ....................................................................................... 89
  3.3.1. Crack tip through-thickness shear forces only .............................................. 94
  3.3.2. Bending moments, axial forces and through-thickness shear forces ............. 96
3.4. Conclusion ........................................................................................................... 98

Chapter 4: Dissimilar laminated beams with bending moments and axial forces 101
4.1. Introduction ........................................................................................................ 101
4.2. Analytical development ..................................................................................... 103
  4.2.1. Interfacial stresses ahead of the crack tip .................................................... 105
  4.2.2. Relative interfacial stresses behind the crack tip ......................................... 106
  4.2.3. Partitioning the ERR $G$ using pure modes in terms of $K_I$ and $K_{II}$ .......... 107
  4.2.4. Partitioning the ERR $G$ using pure modes in terms of crack tip loads ...... 115
  4.2.5. Determining the ERRs, $G_I$ and $G_{II}$ ............................................................ 119
4.3. Numerical verification ....................................................................................... 127
  4.3.1. Shifting technique ........................................................................................ 129
  4.3.2. Calculating the complete set of orthogonal pure modes ................................ 130
  4.3.3. Calculating and choosing the SIFs .............................................................. 135
  4.3.4. Calculating the ERR partitions .................................................................... 145
4.4. Conclusion ......................................................................................................... 149

Chapter 5: Effect of Poisson’s ratio mismatch on dissimilar laminated beams 152
5.1. Introduction ........................................................................................................ 152
5.2. Analytical development ..................................................................................... 154
  5.2.1. Interfacial stresses ahead of the crack tip .................................................... 154
  5.2.2. Total energy release rate ............................................................................. 155
  5.2.3. Equivalent bimaterial properties ................................................................. 156
5.3. Numerical verification ....................................................................................... 158
  5.3.1. Bending moments only .............................................................................. 160
  5.3.2. Bending moments and axial forces .............................................................. 164
5.4. Conclusion ......................................................................................................... 167
List of Figures

Figure 1.1: Fracture modes ................................................................. 3
Figure 1.2: A DCB ........................................................................... 5
Figure 1.3: A circular blister ............................................................. 5
Figure 1.4: Modelling a crack as a flat plane and tip as a straight line ............................................................................... 13
Figure 1.5: The VCCT using QUAD4 elements .................................. 20
Figure 1.6: Interface modelling using springs in the intact section .......... 22
Figure 1.7: Modelling an interfacial crack ......................................... 23
Figure 1.8: The pure mode I DCB test .............................................. 27
Figure 1.9: The pure mode II ENF test ............................................ 28
Figure 1.10: The mixed-mode I/II MMB test ................................. 29
Figure 1.11: Mixed-mode failure criteria examples ............................ 32
Figure 1.12: Replicated results for a FRMM test comparing mixed-mode partition theories and a linear failure criterion for epoxy-matrix/carbon-fibre composite specimens ........................................ 33
Figure 2.1: A laminated composite DCB. (a) General description. (b) Details of the \( \Delta t \) -length crack influence region ................................................. 42
Figure 2.2: Variation of the correction factor \( \gamma \) with \( \gamma \) ................................................... 53
Figure 2.3: Comparisons between various partition theories when \( M_{1B} = 1 \text{Nmm} \) and \( M_{2B} = N_{1B} = N_{2B} = 0 \) ................................................................. 60
Figure 2.4: Magnitude of \( G_1/G \) error of various partition theories relative to Suo and Hutchinson’s theory with \( 0.01 \leq \gamma \leq 100 \), \( -20 \leq M_{2B}/M_{1B} \leq 20 \), \( N_{1B} = N_{2B} = 0 \) and \( M_{1B} = 1 \text{Nmm} \) .................................................................................. 63
Figure 2.5: Magnitude of \( G_1/G \) error of various partition theories relative to Suo and Hutchinson’s theory with \( 0.01 \leq \gamma \leq 100 \), \( -20 \leq N_{1B}/M_{1B} \leq 0 \), \( N_{2B} = M_{2B} = 0 \) and \( M_{1B} = 1 \text{Nmm} \) .................................................................................. 66
Figure 3.1: A laminated DCB. (a) General description. (b) Details local to the crack tip. ................................................................. 71
Figure 3.2: Non-uniform FEM mesh centred on crack tip for a symmetric DCB i.e. \( \gamma = 1 \) ......................................................................................... 75
Figure 3.3: Application of loading conditions using point forces for a uniform mesh in the FEM model for the upper or lower beam \( (i = 1, 2 \text{ respectively}) \) .............................................. 75
Figure 3.4: Variation of the 2D-elasticity-theory-based pure mode II \( \beta_{P-2D} \) for through-thickness shear forces only from Eq. (3.15), an approximate method and 2D FEM simulations with respect to \( \gamma \) ..................................................................................... 81
Figure 3.5: Variation of the shear correction factor \( \kappa(\gamma) \) from Eq. (3.16) and 2D FEM simulations with respect to the thickness ratio \( \gamma \) ..................................................................................... 83
Figure 3.6: Variation of the pure-mode-II ERR correction factor \( c(\gamma) \) from Eq. (3.17) and the 2D FEM simulations with respect to the thickness ratio \( \gamma \) ........................................... 85
Figure 3.7: Comparison of the present analytical theory and the 2D FEM for the total ERR \( G \) and the ERR partition \( G_I/G \) for variable \( \gamma \) and loading conditions with \( \nu = 0.29 \) ..................................................................................... 92
Figure 3.8: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for variable $\gamma$ and loading conditions with $\nu = 0.1$. ................................................................. 93

Figure 3.9: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for $\log_{10}(1/\gamma) = -0.7$ and variable $P_{2B}/P_{1B}$ with $\nu = 0.29$. ................................................................. 95

Figure 3.10: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for $\log_{10}(1/\gamma) = 0.8$ and variable $P_{1B}/M_{1B}$ with $\nu = 0.29$. ................................................................. 97

Figure 3.11: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for $\log_{10}(1/\gamma) = 0.9$ and variable $N_{1Be}/P_{1B}$ with $\nu = 0.29$. ................................................................. 98

Figure 4.1: A bimaterial DCB. (a) General description. (b) Interfacial stresses and crack tip forces.............................................................. 103

Figure 4.2: Variation of the pure mode I conditions $\theta_K$ and $\theta'_{K}$ and the approximate pure mode I condition $\tilde{\theta}_K$ with respect to the crack extension size $\Delta a$ for different values of the bimaterial constant $\varepsilon$ with $\nu_1 = \nu_2 = 0.29$. ........................................... 113

Figure 4.3: Variation of the pure mode I conditions $\theta_1$ and $\theta'_1$ with respect to the crack extension size $\Delta a$ for different values of the bimaterial mismatch constant $\varepsilon$ with $\nu_1 = \nu_2 = 0.29$. ........................................... 117

Figure 4.4: FEM data for ERR partition $G_i/G$ based on the crack extension size $\Delta a = 0.05$ mm with $M_{2B}/M_{1B} = 0$ and $\nu_1 = \nu_2 = 0.29$. ................................. 120

Figure 4.5: Selection of the correct $\beta_1$ solution................................................................. 122

Figure 4.6: Comparison of the shifting technique (line) and FEM data (markers) for the ERR partition $G_i/G$ at crack extension size $\Delta a = 0.05$ mm with $M_{2B}/M_{1B} = 0$ and $\nu = 0.29$ ........................................... 130

Figure 4.7: Comparison of the present analytical theory (lines) and FEM data (markers) for the ERR partition $G_i/G$ at crack extension size $\Delta a = 0.05$ mm with $M_{2B}/M_{1B} = -1$ and $\nu = 0.29$ ........................................... 131

Figure 4.8: Comparison of the present analytical theory and FEM data for the ERR partition $G_i/G$ for variable $\gamma$, $\eta$ and $\nu$ at crack extension size $\Delta a = 0.05$ mm with $M_{2B}/M_{1B} = 0$ and $M_{2B}/M_{1B} = -1$....................................................... 133

Figure 4.9: Comparison of the present analytical theory (lines) and FEM data (markers) for the ERR partition $G_i/G$ at crack extension size $\Delta a = 0.05$ mm with $N_{1B}/M_{1B} = 10$ mm$^{-1}$. ................................................................. 134

Figure 4.10: Comparison of the present analytical theory (lines) and FEM data (markers) for the ERR partition $G_i/G$ at crack extension size $\Delta a = 0.05$ mm with $N_{2B}/M_{1B} = 10$ ................................................................. 135

Figure 4.11: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_i$ at crack extension size $\Delta a = 0.05$ mm with $M_{2B}/M_{1B} = 0$ ....................................................... 137
Figure 4.12: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{II}$ at crack extension size $\Delta a = 0.05 \text{ mm}$ with $M_{1b}/M_{1b} = 0$ ........................................ 138
Figure 4.13: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{I}$ at crack extension size $\Delta a = 0.05 \text{ mm}$ with $M_{2b}/M_{1b} = -1$ ........................................ 139
Figure 4.14: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{II}$ at crack extension size $\Delta a = 0.05 \text{ mm}$ with $M_{1b}/M_{1b} = -1$ ........................................ 140
Figure 4.15: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{I}$ at crack extension size $\Delta a = 0.05 \text{ mm}$ with $N_{1b}/M_{1b} = 10 \text{ mm}^{-1}$ ................................... 141
Figure 4.16: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{II}$ at crack extension size $\Delta a = 0.05 \text{ mm}$ with $N_{2b}/M_{1b} = 10 \text{ mm}^{-1}$ ................................... 142
Figure 4.17: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{I}$ at crack extension size $\Delta a = 0.05 \text{ mm}$ with $N_{2b}/M_{1b} = 10 \text{ mm}^{-1}$ ................................... 143
Figure 4.18: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{II}$ at crack extension size $\Delta a = 0.05 \text{ mm}$ with $M_{2b}/M_{1b} = 0$ .......................................................... 144

Figure 4.19: Comparison of the present analytical theory (lines) and FEM data (markers) for the ERR partition $G_{i}/G$ at crack extension size $\Delta a = 0.01 \text{ mm}$ with $M_{2b}/M_{1b} = 0$ .......................................................... 145
Figure 4.20: Comparison of the present analytical theory and FEM data for the ERR partition $G_{i}/G$ for variable $\gamma , \eta$ and $M_{2b}/M_{1b}$ at crack extension sizes $\Delta a = 0.01 \text{ mm}$ and $\Delta a = 0.1 \text{ mm}$ .......................................................... 146
Figure 4.21: Comparison of the present analytical theory and FEM data for the ERR partitioning $G_{i}/G$ for variable $\gamma , \eta , N_{1b}/M_{1b}$ and $N_{2b}/M_{1b}$ at crack extension size $\Delta a = 0.1 \text{ mm}$ .......................................................... 147

Figure 5.1: A bimaterial DCB. (a) General description. (b) Interfacial stresses and crack tip forces.......................................................... 154
Figure 5.2: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_{i}/G$ for variable $\eta , N$ and $M_{2b}/M_{1b}$ with $\gamma = 1$ and $\Delta a = 0.01 \text{ mm}$ under the plane stress condition.......................... 162
Figure 5.3: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_{i}/G$ for variable $\eta , N$ and $M_{2b}/M_{1b}$ with $\gamma = 1$ and $\Delta a = 0.01 \text{ mm}$ under the plane strain condition.......................... 163
Figure 5.4: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_{i}/G$ for variable $\eta , N$ and $N_{1b}/M_{1b}$ with $\gamma = 1$ and $\Delta a = 0.01 \text{ mm}$ under the plane stress condition.......................... 164
Figure 5.5: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for variable $\eta$, $N$ and $N_{1B}/M_{1B}$ with $\gamma = 1$ and $\delta r = 0.01$ mm under the plane strain condition............................................. 166
Figure 6.1: A laminated DCB. (a) General description. (b) Details local to the crack tip. ............................................................................................................................... 171
Figure 6.2: A membrane blister. ............................................................................................................................... 174
Figure 6.3: Blister test interface crack. (a) Thin layer on a thick substrate. (b) Effective crack tip forces and bending moments................................................................. 174
Figure 6.4: Adhesion energy for mono- and multi-layered graphene membranes from the work104 ............................................................................................................ 187
Figure A.1: Convergence study for the total ERR $G$ and mode partition $G_i/G$ for an orthotropic DCB with $\gamma = 10$, $\eta = 1$ and $-10 \leq P_{2B}/P_{1B} \leq 10$ .................................................. 197
Figure A.2: Convergence study for the total ERR $G$ and mode partition $G_i/G$ for an orthotropic DCB with $\gamma = 1$, $\eta = 1$ and $-10 \leq P_{2B}/P_{1B} \leq 10$ ................................................................. 197
Figure A.3: Convergence study for the total ERR $G$ and mode partition $G_i/G$ for an orthotropic DCB with $\gamma = 10$, $\eta = 1$ and $-10 \leq P_{2B}/P_{1B}[\text{mm}^{-1}] \leq 10$ .......................... 198
Figure A.4: Convergence study for the total ERR $G$ and mode partition $G_i/G$ for an orthotropic DCB with $\gamma = 1$, $\eta = 1$ and $-10 \leq P_{2B}/P_{1B}[\text{mm}^{-1}] \leq 10$ .............................. 198
Figure A.5: Convergence study for the total ERR $G$ for a bimaterial DCB with $\gamma = 10$, $\eta = 1/100$ and $-20 \leq M_{2B}/M_{1B} \leq 20$ ................................................................. 200
Figure A.6: Convergence study for the total ERR $G$ for a bimaterial DCB with $\gamma = 1$, $\eta = 100$ and $-20 \leq M_{2B}/M_{1B} \leq 20$ ............................................................................................ 200
List of Tables

Table 3.1: Comparison of 2D FEM against Timoshenko beam partition theory for $\gamma = 1$.
................................................................................................................................. 78
Table 3.2: Effect of changing the Young’s modulus $E$ on the pure mode II $\beta_{p,2D}$ ..... 82
Table 3.3: Effect of changing the Young’s modulus $E$ on the shear correction factor $\kappa(\gamma)$ ........................................................................................................................ 84
Table 3.4: Effect of changing the Young’s modulus $E$ on the pure-mode-II ERR correction factor $c(\gamma)$ .................................................................................................................. 86
Table 6.1: Adhesion toughness of monolayer graphene membranes. ......................... 182
Table 6.2: Adhesion toughness of two-layer graphene membranes. ......................... 183
Table 6.3: Adhesion toughness of three-layer graphene membranes. ....................... 184
Table 6.4: Adhesion toughness of four-layer graphene membranes. ....................... 184
Table 6.5: Adhesion toughness of five-layer graphene membranes. ....................... 185
Table 6.6: Average adhesion toughness of multilayer graphene membranes ............ 185
Table B.1: Adhesion toughness of monolayer graphene membranes ...................... 202
Table B.2: Adhesion toughness of two-layer graphene membranes ...................... 203
Table B.3: Adhesion toughness of three-layer graphene membranes ...................... 204
Table B.4: Adhesion toughness of four-layer graphene membranes ...................... 204
Table B.5: Adhesion toughness of five-layer graphene membranes ...................... 205
Table B.6: Adhesion toughness of monolayer graphene membranes ...................... 206
Table B.7: Adhesion toughness of two-layer graphene membranes ...................... 207
Table B.8: Adhesion toughness of three-layer graphene membranes ...................... 208
Table B.9: Adhesion toughness of four-layer graphene membranes ...................... 208
Table B.10: Adhesion toughness of five-layer graphene membranes ..................... 209
### Symbols

- $a$\hspace{1cm}crack length in a DCB [mm]
- $A_1, A_2, A$\hspace{1cm}cross sectional areas of upper lower and intact beams [mm$^2$]
- $b$\hspace{1cm}width of a DCB [mm]
- $c(\gamma)$\hspace{1cm}$\gamma$- dependent correction factor for ERR due to $\beta_{P,2D}$ mode II, $G_{\beta_{P,2D}}$
- $c_\theta, c_\beta$\hspace{1cm}2D elasticity correction factors for $\theta$ mode I and $\beta$ mode II
- $D_n, D_s$\hspace{1cm}relative interfacial opening displacement and shear displacement [mm]
- $E$\hspace{1cm}Young’s modulus [N/mm$^2$]
- $E_L, E_T$\hspace{1cm}in-plane Young’s moduli in the longitudinal and transverse directions [N/mm$^2$]
- $E_1, E_2$\hspace{1cm}Young’s modulus of upper and lower beams [N/mm$^2$]
- $\bar{E}$\hspace{1cm}effective Young’s modulus [N/mm$^2$]
- $\bar{E}_i$\hspace{1cm}effective Young’s modulus of the upper beam [N/mm$^2$]
- $\tilde{\bar{E}}_i$\hspace{1cm}equivalent Young’s modulus of the upper beam [N/mm$^2$]
- $G_{\theta, \beta}$\hspace{1cm}$\theta$ mode I and $\beta$ mode II ERRs [J/mm$^2$]
- $G_{\theta_{T}, \beta_{T}}$\hspace{1cm}ERRs due to $\theta_{T}$ mode I and $\beta_{T}$ mode II in Timoshenko beam theory [J/mm$^2$]
- $G_{\theta_{p,T}, \beta_{p,T}}$\hspace{1cm}ERRs due to $\theta_{p,T}$ mode I and $\beta_{p,T}$ mode II in Timoshenko beam theory [J/mm$^2$]
- $G_{\theta_{p,2D}, \beta_{p,2D}}$\hspace{1cm}ERRs due to $\theta_{p,2D}$ mode I and $\beta_{p,2D}$ mode II in 2D elasticity theory [J/mm$^2$]
- $G_{\theta_{1,2D}, \beta_{1,2D}}$\hspace{1cm}ERRs due to $\theta_{1,2D}$ mode I and $\beta_{1,2D}$ mode II in 2D elasticity theory [J/mm$^2$]
- $h_1, h_2, h$\hspace{1cm}thicknesses of upper, lower and intact beams [mm]
- $I_1, I_2, I$\hspace{1cm}second moments of area of upper, lower and intact beams [mm$^4$]
- $k$\hspace{1cm}Kolosov constant
- $k_s$\hspace{1cm}interface spring stiffness [N/mm]
- $K_1, K_2$\hspace{1cm}real and imaginary parts of the complex stress intensity factor [N/mm$^{3/2}$]
- $L$\hspace{1cm}uncracked length of DCB [mm]
- $M_{1, 2}$\hspace{1cm}DCB tip bending moments on upper and lower beams [Nmm]
- $M_{1B, 2B, B}$\hspace{1cm}crack tip bending moments on upper, lower and intact beams [Nmm]
- $N_1, N_2$\hspace{1cm}DCB tip axial forces on upper and lower beams [N]
- $N_{1B, 2B, B}$\hspace{1cm}crack tip axial forces on upper, lower and intact beams [N]
- $N_{1Be}$\hspace{1cm}crack tip effective axial force on upper beam [N]
- $P_{1, 2}$\hspace{1cm}DCB tip shear forces on upper and lower beams [N]
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{ib}$, $P_{ob}$</td>
<td>crack tip shear forces on upper and lower beams [N]</td>
</tr>
<tr>
<td>$r$</td>
<td>radius coordinate centred on crack tip [mm]</td>
</tr>
<tr>
<td>$\beta_i$, $\beta'_i$</td>
<td>load-type pure mode II modes (with $i = 1,2,3,4$)</td>
</tr>
<tr>
<td>$\beta_K$, $\beta'_K$</td>
<td>SIF-type pure mode II modes</td>
</tr>
<tr>
<td>$\bar{\beta}_K$, $\bar{\theta}_K$</td>
<td>approximate SIF-type pure mode II and pure mode I modes</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>thickness ratio, $\gamma = h_2/h_1$</td>
</tr>
<tr>
<td>$\delta a$</td>
<td>crack extension size [mm]</td>
</tr>
<tr>
<td>$\Delta a$</td>
<td>crack influence length in a DCB</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Young’s modulus ratio, $\eta = E_2/E_1$</td>
</tr>
<tr>
<td>$\bar{\eta}$</td>
<td>effective Young’s modulus ratio</td>
</tr>
<tr>
<td>$\bar{\eta}$</td>
<td>equivalent Young’s modulus ratio</td>
</tr>
<tr>
<td>$\theta_i$, $\theta'_i$</td>
<td>load-type pure mode I modes (with $i = 1,2,3,4$)</td>
</tr>
<tr>
<td>$\theta_{1,2D}$, $\theta_{3,2D}$</td>
<td>pure mode I and II (with $i = 1, 2, 3, 4$) in 2D elasticity theory</td>
</tr>
<tr>
<td>$\theta_K$, $\theta'_K$</td>
<td>SIF-type pure mode I modes</td>
</tr>
<tr>
<td>$\theta_{P,T}$, $\beta_{P,T}$</td>
<td>shear force only pure mode I and II in Timoshenko beam theory</td>
</tr>
<tr>
<td>$\theta_{P,2D}$, $\beta_{P,2D}$</td>
<td>shear force only pure mode I and II in 2D elasticity theory</td>
</tr>
<tr>
<td>$\kappa(\gamma)$</td>
<td>$\gamma$-dependent through-thickness shear correction factor</td>
</tr>
<tr>
<td>$\mu_{LZ}$</td>
<td>through-thickness shear modulus [N/mm$^2$]</td>
</tr>
<tr>
<td>$\mu_1$, $\mu_2$</td>
<td>shear modulus of upper and lower beams [N/mm$^2$]</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>$\nu_{LT}$, $\nu_{TL}$</td>
<td>in-plane Poisson’s ratios</td>
</tr>
<tr>
<td>$\nu_1$, $\nu_2$</td>
<td>Poisson’s ratio of upper and lower beams</td>
</tr>
<tr>
<td>$\bar{\nu}$</td>
<td>equivalent Poisson’s ratio, $\nu_1 = \nu_2 = \bar{\nu}$</td>
</tr>
<tr>
<td>$N$</td>
<td>ratio of Poisson’s ratios, $N = \nu_2/\nu_1$</td>
</tr>
<tr>
<td>$\sigma_n$, $\tau_s$</td>
<td>interfacial opening stress and shear stress [N/mm$^2$]</td>
</tr>
</tbody>
</table>

**Abbreviations**

- DCB: double cantilever beam
- ELS: end-loaded split
- ENF: end-notched flexure
- ERR: energy release rate
- FEM: finite element method
- FRMM: fixed ratio mixed-mode
- MMB: mixed-mode bending
- QUAD4: four noded quadrilateral
- VCCT: virtual crack closure technique
- SIF: stress intensity factor
- VRMM: variable ratio mixed-mode
Chapter 1: Interfacial cracks

1.1. Introduction

Fibre reinforced polymer composite materials possess many advantages over conventional metallic materials, a few of which are their high specific strength and specific stiffness, and their excellent fatigue strength and corrosion resistance. The various complex failure modes of fibre reinforced composites materials are being increasingly understood and this has led to them being more widely used in high-performance, safety-critical structures. One example is the new Boeing 787 aircraft, for which 50% of the weight of the primary structure is fibre reinforced polymer composite material, of which the main type is carbon fibre/epoxy matrix laminated composite. There are various concerns, however, about the widespread use of composite materials in these applications: Cracks such as delaminations are often not visible (unlike in metals); the initiation and propagation of cracks is difficult to predict; propagation can often occur unstably; and this can result in catastrophic structural failure. To alleviate these concerns, one solution is to over-design composite structures which results in the full weight-saving potential not being realised. Another solution is to better understand the mechanics of cracking in order to more accurately predict the phenomenon. This thesis focuses on interfacial cracks since the interfaces between the constituent materials of a composite typically represent a weakness along which cracks can initiate and propagate.

A composite material is constructed of two or more constituent materials that are combined in order to create a new material that usually has more desirable properties than the individual elements alone. The constituent materials consist of a matrix and reinforcement. The role of the matrix material is to hold the reinforcement in place, therefore providing support and protection whilst acting as a means of transferring the applied load to the reinforcement. Whereas, the primary role of the reinforcement is to provide the composite’s desired mechanical properties, ensuring the applied load can be sustained. By carefully selecting the matrix and reinforcement, it is possible to tailor the composite material to a given situation. Composite materials are not limited to man-made structures and some of the most common examples exist in nature, such as bamboo and bone.
Fibre reinforced composites involve implanting fibres in a matrix material. The ratio of fibre volume to composite volume is known as the fibre-volume fraction. It is usually possible to characterise the type of fibre as being either continuous or short. The main focus of this thesis is on continuous fibre reinforced laminated composite materials. Fibre reinforced laminated composites consist of one or many layers known as laminae or plies. If the fibres inside a ply are arranged in the same direction, it is known as unidirectional. It is possible to create a woven fabric by weaving the fibres within a ply in at least two directions. Fibre reinforced laminated composites usually involve stacking a number of differently orientated plies together in order to meet the structural needs in a given direction. Some typical examples of fibre reinforced composite materials are squash rackets, fishing rods, skis, golf clubs, bicycles, aircraft fuselage and wing panels, car body kits, etc.

Fibre reinforced laminated composite materials can fail in a number of failure modes. These include and are not limited to matrix failure (tensile, compressive or shear), fibre failure (tensile or compressive) and interfacial failure (matrix-fibre debonding and delamination). Furthermore, it is possible for one failure mode to interact with or initiate others, meaning that failure could be due to some combination of the above modes. Some factors that affect the failure mode of a fibre reinforced laminated composite include the properties of the constituents, the fibre-volume fraction, the orientation of the individual plies, the stress state and loading conditions, environmental conditions, etc.

Delamination occurs due to the separation of adjacent plies in a laminated composite material, usually due to the applied loading conditions or a flaw can be introduced in the composite manufacture process. Delamination is the most predominant and severe failure mechanism for a composite material and results in a rapid loss in the structures ability to support the required load. As it is possible for other failure modes to initiate delamination in the fibre reinforce composite material, it is of fundamental importance that research is conducted into the failure mode. Only after gaining a strong understanding in the most severe failure mode, delamination, can the effects of the interactions between failure modes be examined and therefore the full potential of composite materials be realised. It is for this reason that the delamination failure mode will be the focus of this thesis.
Crack propagation in brittle isotropic homogeneous materials has been shown to follow the “criterion of local symmetry\(^7,8\)”\(^\text{\textsuperscript{7,8}}\). That is, the crack will follow a path in order to maintain conditions of pure mode I opening at the crack tip, as seen in Fig. 1.1a. A mixed-mode fracture in an isotropic homogeneous material will generally kink by a certain angle in order to ensure that the crack tip remains under pure mode I opening conditions.

![Fracture modes](image)

**Figure 1.1: Fracture modes.**

When considering crack propagation in a layered material, for example in a fibre reinforced composite, as the interface acts as a plane of weakness between two adjacent materials, the crack is usually restricted to the interface and it is not possible to kink in order to maintain mode I opening conditions. In such a scenario, the crack will propagate as a mixed-mode, with some combination of mode I opening (Fig. 1.1a), mode II shearing (Fig. 1.1b) and mode III tearing (Fig. 1.1c) action. It is also possible for the interfacial crack to propagate under pure mode I, pure mode II or pure mode III conditions. Therefore, a layered materials ability to support an applied load is dependent on the fracture toughness of the interface.

Fracture toughness, also known as the critical energy release rate (ERR), is a measure of a materials ability to prevent the propagation of a crack. It is vitally important in the design of high integrity structures that the fracture toughness can be calculated. For crack propagation under pure mode conditions, it is generally the case that a material will exhibit different fracture toughness’s depending on the mode of crack propagation and these can be obtained through experimental testing. For mixed-mode crack propagation the fracture toughness is no longer a completely intrinsic material property and depends on the fracture mode partition, meaning it is load-dependent. In such a situation, it is essential to be able to partition a mixed-mode fracture to identify the contributions that each fracture mode has on the overall crack.
propagation and therefore the fracture toughness can be predicted using a failure criterion. However, for mixed-mode fracture it is only possible to identify the total ERR from experimental results. After which a mixed-mode partition theory is required to obtain the individual mode components. There currently exist a number of different methods to partition mixed-mode fractures, which are either based on energy considerations or details local to the crack tip. It is still an unanswered question as to which of the current mixed-mode partition theories provides the correct partition of the fracture toughness. Furthermore, it is unknown as to which partition theory is applicable to certain material configurations and loading conditions for example fatigue or thermal loading.

To date, experimental validation of the mixed-mode partition theories has been performed on macroscopic scale engineering structures, such as unidirectional and multidirectional laminated composite materials. From which it has been identified that the approach based on energy considerations offers more consistent results. It is believed that this is due to the damage on the macroscale developing over the whole region that is mechanically influenced by the crack tip. It is thought that when considering materials where damage occurs on a much smaller scale ahead of the crack tip, for example thin films (microscale), partition theories based on details local to the crack tip will provide a more accurate partition of the fracture toughness. This is due to the fact that details at the crack tip will dominate the damage zone. As a consequence it is extremely important to develop mixed-mode partition theories that will partition the fracture toughness based on details at the crack tip and this is the motivation for this thesis.

Although in most applications, crack propagation in structures is three dimensional, meaning the crack will propagate under a pure or combination of all three modes, it is possible to model many applications as one dimensional, meaning crack propagation will occur in a straight line. As mode III tearing is usually negligible for in-plane bending moments and forces, modelling crack propagation as one dimensional enables a simplification of the analysis as the crack will propagate under mode I opening and/or mode II shearing only. The double cantilever beam (DCB) is the fundamental case for one dimensional fracture and by studying it we gain deep understanding and predictive capability for real life applications. Fig. 1.2 shows how an interfacial crack can be
modelled using the DCB. The DCB has tip bending moments $M_1$ and $M_2$, axial forces $N_1$ and $N_2$ and shear forces $P_1$ and $P_2$. The crack has length $a$ and the tip is located at B.

![Figure 1.2: A DCB.](image)

A circular blister is an example of one dimensional fracture and is shown in Fig. 1.3. The DCB in Fig. 1.2 can be used to model the cross section of the blister and therefore investigate an interfacial crack between the membrane and substrate. Some further examples of where the DCB can be used to model one dimensional fracture include separation of stiffeners and skins in stiffened panels, fracture in straight and curved laminated composite beams, thermal barrier coating cracking in gas turbine engines, needle puncture of a biological cell, etc.

![Figure 1.3: A circular blister.](image)

When considering pure mode I or pure mode II fracture, one dimensional fracture offers a means of calculating the fracture toughness in experimental tests, such as the double cantilever beam and end-notched flexure tests. It is also used in the case of
mixed-mode fracture in order to examine fracture propagation criteria by partitioning the total ERR.

Therefore the primary goals of this thesis are to: (1) derive a completely new analytical mixed-mode partition theory based on 2D elasticity for an orthotropic DCB with general loading conditions and (2) analytically obtain the complex stress intensity factors and crack extension size-dependent mode partitions of the energy release rate (ERR) for a bimaterial DCB with a mismatch in the elastic modulus as well as the Poisson’s ratio, under tip bending moments and axial forces.

The work in this thesis has been documented to the scientific community on several occasions\textsuperscript{16–20}.

In the remainder of this chapter a review of the relevant background theory into linear elastic fracture mechanics will be presented. After which a literature review will be performed into the area of mixed-mode partition theories. The main partition theories and methods for calculating the total ERR and then partitioning it into its individual mode components will be reviewed. The majority of theories can be classified as being analytical, numerical or experimental, with some falling under more than one category.

The other chapters in this thesis are as follows:

**Chapter 2** – A mixed-mode partition theory based on 2D elasticity for an orthotropic DCB under tip bending moments and axial forces is given. The theory is validated by comparison with the current most accurate 2D elasticity partition theory\textsuperscript{21}.

**Chapter 3** – The mixed-mode partition theory for an orthotropic DCB is extended to include crack tip through-thickness shear forces. The theory is validated using 2D FEM simulations.

**Chapter 4** – An analytical method to obtain the complex stress intensity factors and crack extension size-dependent partitions of the ERR for a bimaterial DCB with a mismatch in the elastic modulus but with the same Poisson’s ratio is given. The theory is validated using 2D FEM simulations.

**Chapter 5** – The work in Chapter 4 is extended to include a mismatch in the Poisson’s ratio as well as the elastic modulus. The theory is validated using 2D FEM simulations.

**Chapter 6** – Using previously published experimental results the work in Chapter 3 and Chapter 4 is applied to the blister test for interface fracture toughness.
Chapter 7 – Finally conclusions on the work presented in this thesis are given and any areas of further work are stated.

1.2. Linear elastic fracture mechanics

This section presents the background theory for linear elastic fracture mechanics (LEFM). The main references for this section are\textsuperscript{22–25}.

1.2.1. Griffith theory

It is possible to use the linear elasticity theory of strength to determine when a material will fracture under an applied load. The theory states that once the maximum stress within a body is equal to the strength of the material, the body will fail. There are, however, a few potential problems with the linear elasticity theory of strength. One of which is that elastic conditions are assumed throughout the material; however, this is rarely a correct assumption when dealing with fracture. Another problem with the theory of elasticity is that it cannot be used to determine the strength of a material; therefore, experimental procedures are required to obtain this material data. Experimentally determining the strength of a material can be problematic due to the assumption that the structure of each test specimen is perfect, implying a uniform material strength, independent of the material sample. In reality a perfect structure is an idealisation and microstructural flaws will exist in the specimen. As a flaw has the effect of locally increasing the applied stress as a result of stress concentrations, each sample will fail at a different strength. The degree at which the flaw will locally increase the applied stress within the material is dependent on the shape of the flaw present; however, this is generally unknown. The maximum stress in a body can be determined by solving the boundary value problem in elasticity, however, it is only possible to solve a few situations analytically; therefore approximate analytical solutions or numerical techniques such as the FEM are usually required. Therefore, it is clearly very difficult to identify the maximum stress within a body using linear elasticity theory and to obtain the material strength through experimental procedures.

One well-known boundary value problem which has been solved analytically is the stress concentration caused by a circular hole in an infinite sheet by Timoshenko and Goodier\textsuperscript{26}. It was discovered that the maximum stress experienced was located at the
edge of the hole and was equal to three times the value of the applied stress. As previously mentioned, this method requires the use of experimental results to obtain the strength of the material. A theoretical material strength can be predicted using atomistic simulations to determine the stress that is required to break the materials atomic bonds. From which, it has been concluded that the theoretical strength is approximately 10% of the materials Young’s modulus. However, when these theoretical results are compared to that obtained experimentally for a bulk sample of glass with a Young’s modulus \( \sim 70\text{GPa} \), the experimental strength is approximately two orders of magnitude lower than the theoretical strength. The discrepancy between the experimental and theoretical strengths and the fact that the experimental strength varies from sample to sample can be answered by assuming that the material strength is dependent on flaws present in the test specimens and these are not accounted for in the theoretical simulations.

Inglis\(^\text{27}\) formulated a means of approximating the effect of a flaw in a test specimen. Considering an infinite sheet under an applied stress, \( \sigma \), with an elliptic hole present, Inglis was able to calculate the maximum stress, \( \sigma_{\text{max}} \), in the material as

\[
\frac{\sigma_{\text{max}}}{\sigma} = 1 + \frac{2a}{b}
\]

(1.1)

where \( a \) and \( b \) are the two semi-axes of the ellipse. The radius of curvature at the tip of the ellipse can be obtained using \( \rho = b^2/a \). Using the dimensions of an atom \( (a = 10^{-6}\text{m} \text{ and } \rho = 10^{-10}\text{m}) \), Inglis was then able to consider a deep, sharp flaw in the infinite sheet, giving \( \sigma_{\text{max}} \approx 200\sigma \) and explaining why the experimental results are two orders of magnitude smaller than that from the simulations. Although it is possible to obtain the strength of a material using either atomistic simulation or Inglis’ formulation, both methods have their disadvantages. For example atomistic simulations involve very high computational power and Inglis’ formulation requires the size of the flaw to be known.

Griffith\(^\text{28}\) looked at the fracture of glass and determined a means of relating the atomic process of fracture to that on the macroscopic scale. By assuming a piece of glass is never completely perfect, Griffith\(^\text{28}\) hypothesised that small cracks exist in the specimens, which cause stress concentrations and lead to failure. Unfortunately, due to the nonlinear nature of crack propagation on the atomic scale and the fact that it differs
depending on the material under consideration and crack tip shape, this is hard to quantify. Nevertheless, Griffith\textsuperscript{28} was able to derive a theory of fracture based on the conservation of energy, therefore avoiding the nonlinear behaviour associated with the crack tip.

Griffith\textsuperscript{28} introduced the concept of surface energy which is defined as the difference in energy per unit area between the atoms at the surface of the specimen compared to the atoms in the bulk of the material. Consider a cracked specimen loaded with a prescribed displacement. As the displacement is applied and then the loading grips are held fixed, no work is done on the specimen during crack growth. This means that the total energy of the system is equal to the elastic energy of the specimen plus the surface energy in the faces of the crack. As the crack grows, the elastic energy is reduced but as crack growth means the surface area increases, then the surface energy is increased. The crack will propagate if the total energy of the specimen, that is the elastic energy and surface energy decreases.

Instead of solving a complex boundary value problem, Griffith\textsuperscript{28} noted that a crack can be considered as an ellipse when \( b/a \to 0 \). This meant it was possible to use Inglis’s\textsuperscript{27} linear elastic solution for an ellipse in an infinite sheet subjected to a stress \( \sigma \), to determine the decrease in elastic energy relative to an un-cracked infinite sheet when a crack of size \( 2a \) is formed. A full solution for the decrease in elastic energy between the crack and un-cracked sheet can be found in Timoshenko and Goodier\textsuperscript{26}, however, for simplicity only the result is stated here

\[
\pi \frac{\sigma^2 a^2}{E} \tag{1.2}
\]

Relative to the un-cracked sheet, the combined surface and elastic energy is

\[
\Gamma = 4a\gamma - \pi \frac{\sigma^2 a^2}{E} \tag{1.3}
\]

where \( \gamma \) is the surface energy. From Eq. (1.3) it is clear that as the crack propagates, the surface energy of the specimen will increase and the elastic energy will decrease. It is possible to obtain a critical crack length by setting \( d\Gamma/da = 0 \). This means that if the crack length, \( a \), is greater than the critical crack length, \( a_c \), then the crack will propagate.
From which, it is possible to identify the main conclusion from Griffith’s theory

\[ \sigma_c = \frac{2\gamma E}{\pi a} \]  

(1.5)

Using data obtained experimentally, Griffith was able to show that \( \sigma_c \sqrt{a} \) is a constant, therefore confirming the hypothesis regarding flaws in materials.

### 1.2.2. Energy release rate (ERR)

The previously mentioned Griffith theory is only valid when considering materials that can be idealised as being completely brittle, such as glass. When a ductile material, for example steel, is considered, a large amount of plastic deformation accompanies fracture. This means that the surface energy predicted is much greater than expected and therefore, Griffith’s theory breaks down. Irwin stated that for a ductile material, as plastic deformation occurs during fracture, the majority of the energy is dissipated through plastic flow at the crack tip and not absorbed by the creation of new surfaces. The reduction in total potential energy per unit area of crack growth is known as the critical ERR, \( G_c \). Griffith stated that the critical ERR is equal to twice the value of the surface energy due to the fact that two new surfaces are created in the propagation of a crack.

\[ G_c = 2\gamma \]  

(1.6)

Irwin’s modification of the Griffith theory for ductile materials takes the form

\[ G_c = 2\gamma + G_p \]  

(1.7)

where \( G_p \) is the plastic energy dissipation per unit area of crack growth. Therefore, Eqs. (1.4) and (1.5) can be rewritten as

\[ a_c = \frac{G_c E}{\pi \sigma^2} \]  

(1.8)

\[ \sigma_c = \sqrt{\frac{G_c E}{2\pi a}} \]  

(1.9)
The critical ERR can be thought of as the “resistance” to the extension of the crack. Irwin also discovered that if the size of the plastic deformation zone at the crack tip is small when compared to the whole crack, it is possible to use a purely elastic theory.

It is now convenient to derive a quantity that is the “driving force” for crack growth, known as the ERR, $G$. The ERR is defined as the energy dissipated when the crack grows per unit area. As the crack grows, energy is dissipated meaning there is a decrease in the total potential energy. For a small increase in crack area, the change in total potential energy $\Pi$ is equal to

$$d\Pi = dU - dW$$  \hspace{1cm} (1.10)

where $U$ is the strain energy stored in the specimen and $W$ is the work done by external forces. Therefore, the energy that is dissipated when the crack grows by a unit area is given by

$$G = -\frac{d\Pi}{dS}$$  \hspace{1cm} (1.11)

where $S$ is the crack area. It is now possible to combine Eqs. (1.10) and (1.11) to obtain the ERR

$$G = \frac{dW}{dS} - \frac{dU}{dS}$$  \hspace{1cm} (1.12)

When testing a specimen, the displacements are usually held constant while allowing the crack to grow. In this situation, no work is done on the specimen during crack growth, therefore

$$G = -\frac{\partial U}{\partial S}$$  \hspace{1cm} (1.13)

In Eq. (1.13), the partial derivative signs are used to signify that the displacements are held constant while the crack is allowed to grow. On the other hand, if the applied loads are fixed then external work is now done on the system during crack growth. When the crack extends, the work done on the system by the force and the change in strain energy are given by

$$dW = P\,du \quad \text{and} \quad dU = \frac{1}{2}P\,du$$  \hspace{1cm} (1.14)
Chapter 1: Interfacial cracks

where $P$ is the applied load and $u$ is the displacement. Substituting Eq. (1.14) in to Eq. (1.12) gives the ERR for fixed loads as

$$G = + \frac{\partial U}{\partial S}$$  \hspace{1cm} (1.15)

In Eq. (1.15), the partial derivative signs are used to signify that the applied loads are held constant while the crack is allowed to grow. It can be seen that the only difference on the ERR between applying fixed displacements and fixed loads is the sign change. This is important as although it is the preferred method to apply fixed displacements in experimental testing, when using FEM it is usually preferred to apply fixed loads.

### 1.2.3. Stress intensity factor (SIF)

Stress intensity factors offer another means of characterising the stress field at the crack tip when using linear elasticity theory. This model uses the fact that the stress field around the crack tip is singular. Although the singular field results from the idealised model, Irwin and others made the singular field a fundamental component of fracture mechanics. Irwin\textsuperscript{29} used the stress intensity factor to define the stress field around the crack tip as

$$\sigma_y(r, \theta) = \frac{K}{\sqrt{2 \pi r}} f_y(\theta)$$  \hspace{1cm} (1.16)

After which, Irwin\textsuperscript{30} provided a way in which to partition the stress intensity factor at the crack tip to consider the contribution that each of the different modes had on fracture.

$$K_{I,II,III} = \lim_{r \to 0} \sqrt{2\pi r} (\sigma_{yy}, \tau_{xy}, \tau_{yz})$$  \hspace{1cm} (1.17)

Irwin established the relationship between the mode I, II and III ERRs $G$ and their corresponding SIFs as follows

$$G_I = \frac{K_I^2}{E}$$  \hspace{1cm} (1.18)

$$G_{II} = \frac{K_{II}^2}{E}$$  \hspace{1cm} (1.19)
\[ G_{III} = \frac{K^2_{III}}{2\mu} \]  

where \( \mu \) is the shear modulus and \( \bar{E} \) is the equivalent Young’s modulus and is selected to be \( \bar{E} = \frac{E}{(1-\nu^2)} \) for plane strain or \( \bar{E} = E \) for plane stress.

### 1.3. Analytical theories

It has been identified by Williams\(^{31}\) that when a crack propagates in an isotropic homogeneous material, a singularity is present at the crack tip. By modelling the crack as a flat plane and the tip as a straight line (see Fig. 1.4), Williams\(^{31}\) was able to solve a linear elastic eigenvalue problem which enabled the stress field around the crack tip to be characterised as being square-root singular \( \left( r^{1/2} \right) \), where \( r \) is the radial distance from the crack tip and \( \psi \) the angular coordinate. When a crack propagates along an interface between two dissimilar materials however, there are some well-known issues with the linear elastic solution.

![Crack](image)

**Figure 1.4:** Modelling a crack as a flat plane and tip as a straight line.

The first of which was discovered by Williams\(^{32}\) when extending the previous linear elastic solution for a crack in an isotropic homogeneous material to the bimaterial case. Williams\(^{32}\) showed that the stress field around the crack tip singularity is still square-root singular, however it now also contains a new component which causes the stress and displacement fields to oscillate near to the crack tip. The work of Williams is often referred to as the oscillatory model and has been verified\(^{33-35}\), proving that the oscillatory characteristics exist as the distance from the crack tip tends to zero. England\(^{36}\) continued to work with the oscillatory stress field for a crack between two dissimilar materials and revealed that the solution was physically unacceptable as it
Chapter 1: Interfacial cracks

predicted a small region near the crack tip where the top and bottom surfaces of the crack would wrinkle up and then overlap.

In an attempt to remove the inadmissible oscillatory singularities and the resultant material interpenetration associated with the oscillatory model\textsuperscript{32}, Comninou\textsuperscript{37–39} modelled the cracks with a small frictionless contact zone at the tip. It is for this reason that the work of Comninou\textsuperscript{37–39} is known as the contact model. Comninou\textsuperscript{37–39} was then able to obtain a new stress intensity factor for an interface crack between two dissimilar materials under tensile, shearing and a combination of both these loads. Following the work of Comninou\textsuperscript{37–39}, Gautesen and Dundurs\textsuperscript{40,41} and Gautesen\textsuperscript{42} developed analytical theories in order to obtain the size of the contact zone at the crack tip and interface tractions.

It appears that one of the main questions in LEFM for interfacial cracks between dissimilar materials is which of these models best represents reality. One argument for the use of the oscillatory model\textsuperscript{32} was given by England\textsuperscript{36}, where it was concluded that as the size of the region where the oscillatory model predicts material interpenetration is small, the formulation could be used as an approximation away from the crack tip. Rice\textsuperscript{43} supported the findings of England\textsuperscript{36} stating that the crack surface interpenetration of the Williams\textsuperscript{32} solution is not a reason to discard it. Rice\textsuperscript{43} argued that although the predicted surface interpenetration means that the solutions are physically incorrect, on the scale of the contact zone, it still provides a means of describing the near-tip field when it is much smaller than the crack length. The oscillatory and contact models have been compared\textsuperscript{44} and the near tip stress fields are almost identical except for the very small oscillatory region near to the crack tip. These reasons may partially explain why the oscillatory model appears to be more commonly accepted by researchers.

Another potential problem affecting the use of LEFM in cracks propagating between two dissimilar materials is caused by the mismatch in adjacent material properties, such as the elastic modulus, which can result in the crack advancing under mixed-mode conditions. Despite these issues, researchers have still pursued solutions to partition mixed-mode fractures using the oscillatory model\textsuperscript{32} and the work of Suo and Hutchinson\textsuperscript{21} is currently considered the most accurate partition theory for 2D elasticity. One result of the use of the oscillatory model\textsuperscript{32} is that the stress intensity factor has a complex form\textsuperscript{43,45}, due to the crack-tip stress field being square-root singular and the
previously mentioned issue with surface interpenetration is ignored. The complex stress intensity factor takes the form

\[ K = K_I + iK_{II} \]  

(1.21)

where \( K_I \) and \( K_{II} \) are the corresponding mode I and mode II stress intensity factors, respectively, and \( i = \sqrt{-1} \).

Suo and Hutchinson’s\(^{21}\) partition theory for mixed-mode fracture between two homogeneous elastic layers makes use of classical plate theory to model the elastic layers above and below the crack. Schapery and Davidson\(^{46}\) also used classical plate theory to partition the total ERR using force and moment resultants at the crack tip. However, Schapery and Davidson\(^{46}\) identified that classical plate theory alone does not offer enough information to partition a mixed-mode fracture. Therefore the results for a continuum analysis are required to complete the partition of the complex stress intensity factor. Unlike Schapery and Davidson\(^{46}\) who used the FEM to obtain a “mode mix parameter” to partition the total ERR, Suo and Hutchinson\(^{21}\) used integral equations to obtain a “single real scalar function”. Therefore both methods\(^{21,46}\) are semi-analytical approaches and use details at the crack tip in order to obtain the mode mixity. An approximation for the “single real scalar function” has been presented by Suo\(^{47}\) for the isotropic homogeneous case, meaning that a completely analytical partition is possible based on the thickness ratio.

Now for some noteworthy extensions to the work of Suo and Hutchinson\(^{21}\), the first of which was by Wang and Qiao\(^{48}\) who considered the effect of shear deformation on the cracked laminate by replacing the use of classical plate theory with first-order shear deformable plate theory. The J-integral\(^{49}\) was used in order to obtain the total ERR and then the mode mixity was obtained using the same approach as Suo and Hutchinson\(^{21}\) through the use of the complex stress intensity factor. The J-integral was derived by Rice\(^{49}\) and involves the calculation of a path independent contour integral around the crack tip to determine the total ERR. Wang and Qiao\(^{48}\) then identified through comparison with FEM that the inclusion of shear deformation increases the accuracy of the ERR and phase angle calculations.

At the same time however, Li et al.\(^{50}\) also extended the work of Suo and Hutchinson to account for transverse shear loading and came to the conclusion that a higher-order
beam theory could not be used to account for shear deformation as such an approach neglects the contribution of the deformation local to the crack tip. This contradicts the work of Wang and Qiao. Li et al. sought a full elastic solution for the shear deformation using the FEM and then combined this with Suo and Hutchinson’s partition theory for bending moments and axial forces to provide the mode partition for layered materials under general loading conditions. Another extension was performed by Sheinman and Kardomateas and meant that the work could now be applied to delamination between layers in a generally non-homogeneous laminated composite material. Obviously as these are all extensions to the work of Suo and Hutchinson, the same limitations apply, that being the use of tabulated data to obtain the value for the “single real scalar function” and the crack tip field being singular.

The use of beam and plate theories to partition crack propagation under mixed-mode conditions is a very popular method as it offers the advantages of avoiding the previously mentioned problems with LEFM and simplifies the problem, especially when delamination occurs in laminated composite materials. The first occurrence of this method is from Williams where an isotropic homogeneous cracked laminate was modelled as a pair of beams exposed to tip bending moments and axial forces. Williams was able to calculate the total ERR of the cracked laminate using the values of the applied bending moments and axial forces. Mixed-mode fracture was then discussed, however not pursued. Instead the ERR was split into its pure mode I and pure mode II components. It was suggested that a pure mode I fracture is achieved when the bending moments at the crack tip are equal and opposite. A pure mode II fracture is achieved when the curvature of the upper and lower beams are the same.

However, there appear to be some issues regarding the pure modes suggested by the Williams theory. For example Shim and Hong reviewed the work of Williams and identified that for a pure mode I condition, the specimen geometry as well as the applied moments must be symmetric. Using FEM with the virtual crack closure technique (VCCT) (see Section 1.4.1), Shim and Hong modelled a DCB with equal and opposite tip bending moments to study the effects of increasing the thickness ratio of the upper and lower beams. When the thickness ratio was increased from the case of even beam thicknesses, a mode II component of the ERR was induced in the beam, therefore
proving Williams\textsuperscript{52} wrong. This issue with the Williams\textsuperscript{52} pure mode I is well supported\textsuperscript{1,46,54}.

An orthogonal pure mode methodology has been developed by Wang and Harvey\textsuperscript{1,55} in order to partition the total ERR into its mode I and II components. By identifying the orthogonal pure modes that correspond to pure fracture modes in classical and first-order shear deformable beam theories the ERR is partitioned. By considering an isotropic DCB with a rigid interface, two sets of orthogonal pure modes are expected, with each containing a pure mode I and pure mode II condition. The pure mode I conditions are obtained by setting the crack tip relative shearing displacement or crack tip shearing force to zero. The pure mode II conditions are obtained by setting the crack tip opening force or crack tip relative opening displacement to zero. It is identified that when using first-order shear deformable beam theory the two pure mode I conditions coincide with each other and likewise so do the pure mode II conditions. However, as the pure modes do not coincide for classical beam theory, this results in an energy transfer between the modes and therefore changes the ERR partition. Results for the partition theory were compared to FEM simulations established using classical and first-order shear deformable beam theories and the VCCT has been utilised in order to obtain the mode I and mode II components of the ERR by extracting forces from normal and shear point springs at the crack tip. The partition theory has been extended to laminated composite beams with rigid interfaces\textsuperscript{2} and layered isotropic DCBs with non-rigid cohesive interfaces\textsuperscript{56}.

Wang and Harvey\textsuperscript{1} also presented an approximate partition rule for 2D elasticity. Using the partitioned ERR results from the classical and first-order shear deformable beam theories, Wang and Harvey state that these represent the upper and lower bounds of the 2D elasticity partition, therefore the average of these results should agree well with 2D FEM simulations. When comparing the approximate 2D elasticity partition rule to plane stress 2D FEM simulations and Suo and Hutchinson’s (1990) theory\textsuperscript{21}, good agreement is seen. The approximate 2D elasticity partition rule was then modified by Harvey and Wang\textsuperscript{2} as a result of a negative partition of the ERR for certain loading scenarios. By including new pure modes which are incorporated when the previous rule gives a negative partition, the accuracy of the approximate partition rule can be increased.
Now for two similar approaches to partition the total ERR based on global energy considerations which utilise interface models between the upper and lower beams. The first of which by Bruno and Greco\textsuperscript{54,57–59}, where classical and first-order shear deformable plate theories are used to model the delamination between two homogeneous orthotropic layers connected via a linear interface model. The second method, proposed by Luo and Tong\textsuperscript{60,61} where classical and first-order shear deformable beam theories were used to model a crack propagating in an adhesive layer between two beams. Unlike Bruno and Greco\textsuperscript{54,57–59} who partitioned the ERR by taking the limit of the interface stiffness as it approached infinity, Luo and Tong\textsuperscript{60,61} obtained completely analytical closed form solutions by taking the limits as the adhesive thickness approached zero. When the mode mixity results for isotropic beams with bending moments only were compared to each other it was identified that they were almost identical. These results were also compared to Suo and Hutchinson’s partition theory and there was a high error. As previously mentioned, this is due to the fact that Bruno and Greco’s and Luo and Tong’s partition theories are based on global energy considerations whereas Suo and Hutchinson’s theory requires a singular field at the crack tip. Therefore, this provides the question as to which method of partitioning the ERR provides the more accurate results. In order to determine this, experimental results should be used with a mixed-mode failure criterion to verify the methods. Readers are directed to Section 1.5.4.

Luo and Tong\textsuperscript{61} also compared the results for two equal beam thickness bimaterial DCBs with equal and opposite tip shear forces to Li et al.’s\textsuperscript{50} previously mentioned extension to Suo and Hutchinson’s\textsuperscript{21} theory for shear forces. When comparing the normalised total ERR and the phase angle vs. the normalised crack length, the maximum errors are given as 7.2% and 5.7%, respectively. There is mostly good agreement between the results, which questions Li et al.’s statement about the use of higher-order beam theories to account for shear deformation not being applicable. However, as only one loading condition has been considered for two equal beam thickness bimaterial DCBs, no firm conclusion can be drawn.

In an attempt to create a completely analytical theory to partition the ERR using classical plate theory, Luo and Tong\textsuperscript{62} combined the “local” approach\textsuperscript{21,46} based on the forces or singular field present at the crack tip with the “global” approach\textsuperscript{52} based on
energy considerations, to create a new theory known as the “global-local” method. Initially considering a cracked laminate with bending moments only, a new mode partition formulation was developed which enabled the partition of the total ERR using a global approach.

Schapery and Davidson’s partition theory\(^{46}\) was then utilised to obtain the total ERR for a cracked laminate with bending moments and axial forces. As stated earlier, this partition theory requires the results from a continuum analysis in order to complete the mode partition. Luo and Tong\(^{62}\) were able to use the previously derived partition theory for a cracked laminate with bending moments only, to complete the mode partition and therefore obtain a completely new analytical closed form solution. The effects of shear forces were added using Schapery and Davidson’s\(^{46}\) modified VCCT for use with classical plate theory. To validate the new “global-local” partition theory, Luo and Tong\(^{62}\) compared results with Suo and Hutchinson’s\(^{21}\) crack tip singular field based partition theory, Schapery and Davidson’s\(^{46}\) crack tip element model and the FEM using the VCCT. It was found that the mode partitions from the “global-local” approach agreed well with the other methods.

Finally, more recently Valvo\(^{63}\) gave completely analytical formulations to partition the total ERR for laminated beams with general stacking sequences and through-the-width delaminations under bending moments, axial forces and shear forces using classical lamination theory with first-order shear deformable beam theory. By deriving new quantities which are defined as the relative displacements per unit increase in crack length, it was possible to partition the total ERR by modifying the VCCT. Excellent agreement has been obtained for the total ERR when compared to Suo and Hutchinson’s\(^{21}\) partition theory for homogeneous and bimaterial beams under bending moments and axial forces. There are however some differences when the ERR partitions are compared. It is stated that a limitation of the model and reason for the differences in ERR partitions could be because in an attempt to keep the model as uncomplicated as possible, some effects that would contribute to the ERR partitions have been ignored, for example the effect of the Poisson’s ratio and root rotations.
1.4. Numerical methods

1.4.1. Virtual crack closure technique (VCCT)

As well as analytical theories, numerical methods have also been developed to partition the total ERR for mixed-mode crack propagation. One of the most popular methods appears to be the VCCT; therefore a brief history of the development of this method will now be conducted. Initially Irwin\textsuperscript{30} stated that the energy absorbed by allowing a crack to extend by an infinitesimal amount $\delta a$ is equivalent to the work that would be used when closing the crack back to its original length. The work of Irwin\textsuperscript{30} is known as the crack closure technique or crack closure integral. Therefore, this led to the displacements being used before and after an infinitesimal crack extension (two simulations) to determine the ERR. The work of Irwin in the field of fracture mechanics has been reviewed\textsuperscript{64}.

A modification of the crack closure technique has been presented by Rybicki and Kanninen\textsuperscript{65}, which enabled the ERR to be obtained and partitioned from one FEM simulation. Known as the VCCT, the method assumes that for an infinitesimal crack extension, the relative displacements behind the new crack tip are approximately equal to that behind the original crack tip.

![Figure 1.5: The VCCT using QUAD4 elements.](image-url)
An example of the VCCT is now given. Consider Fig. 1.5 where a crack is modelled using QUAD4 elements. The crack of length \( a \) is allowed to extend by a distance \( \delta a \), meaning that the new crack length is \( a + \delta a \). The opening displacements of the upper and lower beam behind the new crack tip are given by \( w_u \) and \( w_l \), respectively. The shearing displacements of the upper and lower beam behind the new crack tip are given by \( u_u \) and \( u_l \), respectively. The opening and shearing forces at the new crack tip are given by \( F_z \) and \( F_u \), respectively. The work \( W \) that is required to close the crack back to length \( a \) is given as

\[
W = \frac{1}{2} F_z (w_u - w_l) + \frac{1}{2} F_u (u_u - u_l)
\]  
(1.22)

As the ERR is defined as the energy that is dissipated when the crack grows per unit area, i.e. \( G = \frac{W}{\Delta a} \), it is possible to calculate the mode I and II components of the ERR as follows

\[
G_I = \frac{F_z (w_u - w_l)}{2b \delta a}
\]  
(1.23)

\[
G_{II} = \frac{F_u (u_u - u_l)}{2b \delta a}
\]  
(1.24)

where \( b \) is the thickness of the QUAD4 elements. The VCCT was then extended by Raju\(^66\) to calculate the ERRs when using higher order and singular finite elements. A review of the VCCT is given by Krueger\(^67\).

1.4.2. Interface modelling

To model interfacial fracture in the FEM, an interface model is required to account for the intact section of the interface, i.e. at and in front of the crack tip. This section, to the right of the crack tip in Fig. 1.5, is where collocated nodes of the upper and lower beam are rigidly connected together. A simple and popular method for connecting collocated nodes is to use point springs\(^25,68\).

As the work in this thesis uses 2D FEM simulations, only two springs are required for each of the collocated nodes in the intact region. The first of which, known as a mode I spring, is orientated in the normal direction to the interface and restricts opening
along the interface. The second spring, known as a mode II spring, is orientated in the tangential direction to the interface and restricts shearing along the interface.

An example of how the mode I and II springs are used to connect the nodes at the interface of the upper and lower beam at and ahead of the crack tip can be seen in Fig. 1.6. To provide a rigid interface it is necessary to specify a large spring stiffness, $k_s$, in order to prevent the beams separating from each other. Furthermore, as the nodes are collocated, the springs are implemented in a zero thickness region. Using interface springs at and ahead of the crack tip means that forces can easily be extracted and used to identify the mode I and II components of the ERR by directly using them with the VCCT.

![Figure 1.6: Interface modelling using springs in the intact section.](image)

### 1.4.3. Interfacial crack between dissimilar materials

When a crack between two dissimilar materials is numerically modelled, the issues with LEFM are still present and the oscillatory component of the displacement and stress fields leads to non-convergence of the ERR partitions. Therefore this section will now discuss the methods that have been commonly used in order to overcome these issues and obtain the ERR partitions.

Raju, Crews Jr. and Aminpour\textsuperscript{69} used a quasi-three dimensional finite element analysis with the previously mentioned VCCT to investigate the non-convergence of the ERR partitions experienced for a bimaterial interfacial crack. Using isotropic material properties above and below the crack, two models for a laminate with an edge delamination were analysed. The first of which had material properties meaning that the imaginary part of the stress field existed and in the second model, the material properties were carefully selected so that the imaginary part of the stress field was removed. When the results were analysed it was identified that as the size of the mesh was decreased around the crack tip the total ERR converged to a well-defined value for
both models. However, when looking at the ERR partitions there was non-convergence for the model with the imaginary part of the stress field. As the individual partitions converged for the model with no imaginary part of the stress field, it was concluded that the non-convergence was due to the imaginary part of the singularity causing oscillations in the stress field at the crack tip as the mesh size was decreased.

To overcome the issue of non-convergence, it was stated that in reality a laminate will not have a “bare interface” between plies (Fig. 1.7a) but will instead have a very thin resin layer (Fig. 1.7b). Therefore, if the crack is modelled in the thin resin layer, as it is an isotropic homogeneous material, the imaginary part of the stress singularity will be removed. Raju et al.\textsuperscript{69} modelled the composite laminate with an edge delamination using orthotropic material properties above and below the crack. It was determined that using the “thin resin layer” model meant that the total ERR and its mode partitions converged as the mesh density at the crack tip increased.

![Figure 1.7: Modelling an interfacial crack](image)

Therefore the non-convergence of the ERR partitions can be removed by carefully selecting the material properties to remove the imaginary part of the stress field or by using the “thin resin layer” model. However, it is very inconvenient to have to select the materials for a given application so that the imaginary part of the stress field is removed. Some disadvantages of the resin interface model have been given\textsuperscript{70,71}. These include the fact that the resin interface is difficult to model in the FEM and requires a high mesh density. Also the partition will become dependent on the elastic properties and thickness of the resin layer.
Davidson, Hu and Schapery\textsuperscript{70} extended the crack-tip element model\textsuperscript{46} to consider a crack between dissimilar materials. The crack-tip element model is a semi-analytical approach which uses classical plate theory to determine the total ERR and mode partitions from the force and moment resultants at the crack tip. As plate theory doesn’t provide enough information to partition the total ERR alone, results for a continuum analysis are required. Previously Schapery and Davidson\textsuperscript{46} used FEM to complete the mode partitioning, however a modification to the method is required for the bimaterial interfacial crack due to the oscillatory behaviour of the singularity. Therefore Davidson et al.\textsuperscript{70} used a method that was initially given by He and Hutchinson\textsuperscript{72} and is known as the “$\beta = 0$ approach”. In this method, the FEM was still used; however the Poisson’s ratio of either the material above or below the crack tip was changed to remove the imaginary part of the stress field and the oscillatory characteristics. As the oscillatory characteristics had been removed it was possible to complete the ERR partition using the crack-tip element model. Although the crack tip element model still requires the use of the FEM, it is far less computationally expensive as it is only necessary to perform an analysis on the crack tip geometry. As the “$\beta = 0$ approach” is established on the crack tip singularity being characterised as square-root singular, the same limitations as that from Suo and Hutchinson\textsuperscript{21} are applicable.

Continuing to work with the crack-tip element model, Davidson, Hu and Yan\textsuperscript{71} assessed the validity of the “$\beta = 0$ approach” for an edge delamination in a laminated composite material. Results for the “$\beta = 0$ approach” were compared to “thin resin layer” model\textsuperscript{69} and it was identified that both methods predicted the same ERR mode partitions. As previously mentioned the crack-tip element model has the advantage of being computationally efficient whereas the “thin resin layer” requires a high mesh density. This leads to the conclusion that the crack-tip element model with the “$\beta = 0$ approach” is a better solution to the bimaterial interfacial crack than the “thin resin layer” model.

The interface crack between dissimilar materials was also analysed by Sun and Jih\textsuperscript{73} using both an analytical and numerical approach. Firstly, using the VCCT analytical expressions were obtained for the total and mode partitions of the ERR based on the mode I and II stress intensity factors (SIFs). From which it could be seen that the individual mode partitions were dependent on the crack extension size. Although the
total ERR was well defined, the mode partitions showed non-convergence as the crack extension size was decreased. As the region of the violent oscillations was small it was found that if the oscillatory characteristics of the mode partitions were ignored then both the mode I and mode II component of the ERR would contribute to half of the total ERR. This behaviour was also seen through the use of the FEM for a centre crack between two dissimilar materials under normal stresses at the boundary. As the crack extension size was decreased the mode partitions converged to half of the total ERR before the oscillations started.

The analytical formulations for the total ERR and mode partitions were then used by Sun and Qian. By allowing a finite crack length extension in the VCCT, it was possible to evaluate the crack closures and therefore obtain the finite extension based mode I and II ERRs using the FEM. After which, the analytical formulation could be used with the finite extension ERRs to obtain the mode I and II SIFs. As the analytical formulation gives more than one solution for each of the SIFs, conditions are given, which are based on the FEM crack tip displacements in order to pick the correct SIFs. It is important to note that the SIFs given in the work are different to that of Suo and Hutchinson; however a relationship between the two has been given. Results for the stress intensity factors calculated from the finite extension method were compared to that of Rice and Sih for a centre crack in an infinite bimaterial and showed excellent agreement. A new procedure is also presented, known as the “displacement ratio method”. In which the SIFs can be obtained using the ratio of crack surface displacements obtained from FEM and the total ERR. The “displacement ratio method” was shown to give very good results.

Morioka and Sun also used the near tip field equations from Sun and Jih to present a new method to obtain the SIFs, known as the “projection method”. By plotting the SIFs for a bimaterial interfacial crack in the region where they are well-defined, away from the crack tip, it is possible to then use these results and project backwards into the very small oscillatory region close to the crack tip to obtain an estimate of the SIFs at the crack tip. To validate the new method the SIFs were also calculated using the finite crack extension and “displacement ratio” methods. It was identified that when the bimaterial material properties were isotropic, very accurate results were obtained. Unfortunately, when the bimaterial material properties were changed to be
orthotropic the accuracy was greatly reduced. The new “projection method” also has the disadvantage of requiring a high mesh density around the crack tip to correctly model the stress distributions.

Finally Zou et al.\textsuperscript{75} were able to remove the stress singularity and oscillatory behaviour associated with LEFM by splitting a cracked laminate up into a number of sublaminates. Using first-order shear deformable laminate theory to model the individual sublaminates, it was identified that although the stress singularity was removed, it is still represented by stress resultant discontinuities across the crack tip. With the oscillatory behaviour eliminated, it is then possible to use the FEM with the VCCT in order to obtain the total ERR and mode partitions. The work of Raju et al.\textsuperscript{69} was then reconsidered and the “bare interface” model for a laminate with an edge delamination was reanalysed using the sublamine model and compared to the results from the “thin resin layer” model\textsuperscript{69}. It was identified that the results from the sublamine model\textsuperscript{75} had good agreement with the “thin resin layer” model for the total ERR and the mode partitions.

A disadvantage of the sublamine model is that the VCCT requires the FEM mesh to be orthogonal to the crack tip and uniform, however, this requirement has been eliminated by Zou et al.\textsuperscript{76}. Therefore after the stress resultant discontinuities and relative displacements are obtained from the FEM, the total ERR can be partitioned into its individual mode components. Although the sublamine model requires the use of the FEM, an analytical solution using two sublaminates for a one-dimensional isotropic homogeneous DCB is presented by Zou et al.\textsuperscript{76} and was compared to Suo and Hutchinson’s\textsuperscript{21} partition theory. Good agreement for the total ERR is achieved, however as the thickness ratio of the beams is increased the individual mode partition error increased. Zou et al.\textsuperscript{75,76} showed that by increasing the number of sublaminates, the accuracy of the model increases and four sublaminates provided good results for all thickness ratios when compared to Suo and Hutchinson\textsuperscript{21}. However due to complexity the analytical solution is restricted to two sublaminates.

1.5. Experimental

It is of crucial importance to identify the fracture toughness of a material in order to design safe structures. A number of experimental test procedures have been established
and improved over the years, meaning that an accurate value of the fracture toughness can be obtained. In this section the three main types of testing methods to calculate the fracture toughness will be reviewed, consisting of the DCB test for pure mode I fracture, the end-notched flexure (ENF) test for pure mode II fracture and finally the mixed-mode bending (MMB) test for mixed-mode fracture. After which, as an interfacial crack is usually restricted to the interface, meaning that it will propagate under mixed-mode conditions the review will then switch focus to mixed-mode fracture investigations. In particular the review will concentrate on the previously mentioned partition theories to identify which method provides more accurate results. If the reader is after a more in-depth review of the experimental test procedures then they are directed to the work\textsuperscript{77–79}.

1.5.1. Double cantilever beam (DCB) test

The DCB test, as seen in Fig. 1.8, is the standardised methodology\textsuperscript{80} in order to obtain the pure mode I fracture toughness of a laminated fibre-reinforced composite material. By recording the applied load and opening displacement the compliance calibration method or modified beam theory can be used to calculate the energy release rate when the delamination grows. The test must be performed on a constant thickness laminate and in order to initiate delamination a non-adhesive insert is positioned at the centre of one end of the specimen before the curing process. Loading blocks are attached to the same end of the specimen as that of the delamination and are used in order to apply a given load or opening displacement. There are however some limitations to the DCB test, the first of which is that the test specimen must be a
unidirectional composite laminate with a brittle and tough single-phase polymer matrix. Another limitation is with the size of the testing specimen where the length must be at least 125mm and have a width 20–25mm. The specimen must contain an even number of UD plies with a total laminate thickness of 3–5mm. Martin showed that a laminate with 24 plies enables acceptable calculation of the mode I fracture toughness without having to provide modifications for geometric non-linearity.

1.5.2. End-notched flexure (ENF) test

![Figure 1.9: The pure mode II ENF test.](image)

The ENF test, as seen in Fig. 1.9, is the standardised methodology in order to obtain the pure mode II fracture toughness of a laminated fibre-reinforced composite. Similar to the DCB test, the test specimen must be a constant thickness UD laminate and a non-adhesive insert is again positioned at one end on the mid-plane to initiate delamination. The specimen is then placed on top of two roller supports and then three-point bending is initiated under fixed displacement by loading the specimen from the centre. By recording the applied force and load displacement, the mode II fracture toughness can then obtained using the compliance calibration method for specimens with the non-adhesive insert crack or a pre-cracked specimen. Restrictions on the size of the specimen are given: the total length must be greater than 160mm with a crack length greater than 45mm, a width of 19–26mm and thickness of 3.4–4.7mm. A disadvantage of the ENF test is that the crack propagation is unstable.

1.5.3. Mixed-mode bending (MMB) test

The previously mentioned tests for laminated composite materials are both used in order to calculate the pure mode fracture toughness. However, it is well known that the
fracture toughness of a material is not a completely intrinsic material property, meaning that it depends on the loading conditions as well as the material. As it is possible for an interfacial crack to propagate under mixed-mode conditions it is necessary to devise a testing procedure in order to investigate the fracture toughness for a specimen experiencing mixed-mode fracture.

Reeder and Crews Jr.\textsuperscript{82} combined the previously mentioned pure mode I DCB test and pure mode II ENF in order to create a mixed-mode fracture test, known as the MMB test. Using a hinge and loading lever, it is possible to apply the opening load from the DCB test to a mid-span loaded ENF specimen using a single load, therefore producing mixed-mode conditions at the crack tip. By moving the loading lever position the ratio of the mode I and II ERR can be adjusted. A major problem with the original MMB test was identified by Reeder and Crews Jr.\textsuperscript{83}. This problem was also shown in the work\textsuperscript{11} where the lever rotation and crack-size-dependent bending moments meant that non-linearity’s were induced in the specimen, which led to errors of up to 30\% in the calculation of mode I and II ERRs. Reeder and Crews Jr.\textsuperscript{83} removed this issue with non-linear behaviour by modifying the original apparatus to apply the load through the use of a saddle and bearing arrangement.

![Figure 1.10: The mixed-mode I/II MMB test.](image)

The MMB test has been standardised\textsuperscript{84} to obtain the fracture toughness for symmetric UD laminated fibre-reinforced composite materials with brittle and tough single-phase polymer matrices under mixed-mode conditions. Fig. 1.10 shows the MMB test apparatus. By recording the applied load and opening displacement, the fracture toughness and mode mixity can be obtained using the critical values.
1.5.4. Mixed-mode fracture investigations

Finally the literature review will look at mixed-mode fracture investigations. It is well known that experimental tests can only provide the critical energy release rate for a specimen. After which the experimental results are used in combination with a mixed-mode partition theory to obtain the contribution that each mode has on the fracture toughness. As previously mentioned, there are two main methods to partition a mixed-mode fracture, the first of which is based on energy considerations and the second uses quantities at the crack tip. In order to identify which method provides the correct partition of the fracture toughness, the experimental results must be analysed using each partition theory and then the assessed using a failure criterion. Therefore this section will particularly focus on the performance of some of the previously mentioned partition theories when partitioning experimental results and predicting the fracture toughness.

Hashemi, Kinloch and Williams\textsuperscript{9} used experimental tests on laminated composite materials to compare the Williams\textsuperscript{52} partition theory based on energy considerations to that of Suo and Hutchinson\textsuperscript{21} where the crack tip stress field is assumed to be singular. The previously mentioned pure mode I DCB test and a different pure mode II test known as the end-loaded split (ELS) test were modified to use specimens with asymmetric delaminations; meaning fracture was mixed-mode. For symmetric specimens, the crack will propagate under pure mode conditions and the partition theories predict the same ERR partitions. However, this is not the case with the asymmetric specimens and therefore mixed-mode conditions are present.

Hashemi et al.\textsuperscript{9} then obtained mixed-mode experimental results from two other tests, known as the variable ratio mixed-mode (VRMM) test and fixed ratio mixed-mode (FRMM) test. In both of these tests the specimen considered is symmetric, therefore both partition theories predicted the same ERR partition. From which it was possible to plot the mode I ERR vs. the mode II ERR and therefore obtain the material failure locus. As a failure locus is an intrinsic material property, results from the FRMM test should be on the same failure locus as the VRMM. Therefore, results were superimposed onto each other and it was clear that the partition theories predicted very different results. For the Williams\textsuperscript{52} partition theory approximately the same failure locus was obtained for both tests, however results from the Suo and Hutchinson\textsuperscript{21}
partition theory differed substantially between tests. Hashemi et al.\textsuperscript{9} concluded that the Williams\textsuperscript{52} partition theory is the more accurate of the two, however it has previously been stated in Section 1.2 that there are errors with Williams’ theory.

In the experiments by Hashemi et al.\textsuperscript{9} the specimens underwent a mode I precrack in order to start the delamination. A precrack is introduced into specimens to prevent the delamination growth initiating from a blunt crack which is a result of using a non-adhesive insert, as this can affect the fracture toughness. Therefore, a mode I precrack means the specimen is loaded under pure mode I conditions in order to begin the delamination from a more realistic representation of the crack tip. Similarly a mode II precrack means the specimen is loaded under pure mode II conditions before the experimental test. The effects of precracking have been studied by Carlsson et al.\textsuperscript{85,86} and have shown that a mode I precrack lowers the critical ERR when compared to a mode II precrack. Therefore, it is unclear as to what effect the mode I precrack would have on the mixed-mode fracture toughness.

Charalambides et al.\textsuperscript{10} also compared partition results from the Williams\textsuperscript{52} theory and the Suo and Hutchinson\textsuperscript{21} theory. After identifying that the results from the Williams theory were in much better agreement with experimental results, a new failure criterion was proposed for mixed-mode crack propagation. Again, the test specimens used are asymmetric, meaning the upper and lower beams have a different thickness, which questions the validity of Williams’s partition theory. The failure criterion assumes that when the specimen is loaded with a mixed-mode, an additional mode I component will be induced and this is equal to the failure value. The failure criterion requires the determination of three parameters from experimental observations, these being the induced mode I component, phase angle due to the elastic modulus mismatch across a bimaterial interface and the slope of the fracture surface roughness. The failure criterion gives excellent agreement with the experimental data when comparing the failure locus obtained.

Following the work of Hashemi et al.\textsuperscript{9}, Kinloch et al.\textsuperscript{11} compared the failure loci produced from different fracture tests (DCB, ELS, MMB and FRMM) again by partitioning the results with the Williams\textsuperscript{52} theory and the Suo and Hutchinson\textsuperscript{21} theory. Using results from the FRMM test, the accuracy of the partition theories was assessed and identified that the Williams theory had excellent agreement, whereas the Suo and
Hutchinson partition theory followed no form of failure locus. The failure criterion proposed by Charalambides et al.\textsuperscript{10} was found to be in good agreement with the failure loci obtained from the previously mentioned mixed-mode tests.

A review of mixed-mode failure criteria has been given by Reeder\textsuperscript{87}. In which four simple failure criteria are presented and these can be seen on Fig. 1.11, where $G_I$ and $G_{II}$ are the mode I and II components of the ERR respectively and $G$ the total ERR. Critical values of which are denoted by a subscript c, i.e. $G_{IC}$, $G_{IIC}$ and $G_c$. The first failure criterion, known as the “$G_I$ criterion” assumes that only the mode I component of the ERR $G_I$ causes crack propagation, therefore $G_{Ic} = G_{IC}$ and $G_{IIC} = \infty$. The second failure criterion, known as the “$G_{II}$ criterion” is the opposite of the first, meaning that only the mode II component of the ERR $G_{II}$ causes crack propagation, therefore $G_{IIc} = G_{IIC}$ and $G_{IC} = \infty$. The third failure criterion, known as the “total ERR $G$ criterion” assumes that a crack will propagate when the total ERR $G$ reaches a critical value i.e. $G_{Ic} + G_{IIc} = G_c$. This means that the “total ERR $G$ criterion” doesn’t take the mode mixity into account. As the values of $G_{IC}$ and $G_{IIC}$ are usually different for most
Chapter 1: Interfacial cracks

materials, the fourth failure criterion normalises the “total ERR $G$ criterion” using the critical values giving the “linear criterion” as

$$\frac{G_f}{G_{lc}} + \frac{G_{II}}{G_{IIC}} = 1$$

(1.25)

and is one of the most used in literature.

Harvey and Wang\textsuperscript{3} used experimental results obtained from literature for a crack propagating in a composite material to perform an analysis of some of the previously mentioned partition theories. These included the Williams\textsuperscript{52} Euler beam based theory, the Suo and Hutchinson\textsuperscript{21} partition theory based on the stress field being characterised as square root singular and finally the Wang and Harvey\textsuperscript{1,2,55} Euler and Timoshenko beam partition theories and the averaged rule for 2D elasticity by averaging the results. For more information see Section 1.3.

![Figure 1.12: Replicated results\textsuperscript{3} for a FRMM test comparing mixed-mode partition theories and a linear failure criterion for epoxy-matrix/carbon-fibre composite specimens.](image-url)

Figure 1.12: Replicated results\textsuperscript{3} for a FRMM test comparing mixed-mode partition theories and a linear failure criterion for epoxy-matrix/carbon-fibre composite specimens.
Using a linear failure criterion in order to assess the theories for a fixed-ratio mixed-mode test the results have been replicated in Fig. 1.12. From which the following observations can be made: (1) the partition theories based on details at the crack tip (Suo and Hutchinson\textsuperscript{21} and Wang and Harvey\textsuperscript{1,2,55} averaged rule) and the Wang and Harvey\textsuperscript{1,2,55} Timoshenko beam partition theory showed poor performance when assessed against the linear failure criteria, (2) the Wang and Harvey\textsuperscript{1,2,55} Euler beam partition theory was able to best describe the experimental results with the linear fit closest to the linear failure locus. It is stated that the superior performance of the theory when compared to the others is due to the fact that the ERR partition takes into account the whole region that is mechanically influenced and not just details at the crack tip. (3) The Williams\textsuperscript{52} Euler beam based theory also showed good agreement with the linear failure locus, this is due to it being founded on the same fundamentals as the Wang and Harvey Euler beam theory.

Some limitations of Suo and Hutchinson’s\textsuperscript{21} singular-field-based partition theory and reasons for the poor performance when using experimental results are given by Charalambides et al.\textsuperscript{10} and include the method being heavily reliant on the presence of the square-root-singularity at the crack-tip, which was not present in most of the test specimens considered. It is also stated that when the conditions for the singular field are not present, it is “far from clear” whether the use of square-root singular field partition theories are appropriate as they require the singular field to dominate the damage zone and therefore the failure mechanism. Davidson et al.\textsuperscript{13} also identified for a bimaterial interface crack that square root singular field based partition theories showed poor correlation with experimental results due to the lack of a dominant singular field at the crack tip.

Further comparisons of the singular field based partition theory\textsuperscript{46,70} and the Williams\textsuperscript{52} partition theory were performed by Davidson et al.\textsuperscript{14}. By performing experimental tests which included the DCB, ENF and MMB tests on specimens with a central delamination, graphs of the mode mix versus the critical ERR were obtained. Using these graphs, the partition theories were assessed against the criteria that specimens calculated to be at the same mode mix must display the same critical ERR. Experimental tests were then performed on laminates with an offset delamination, i.e. asymmetric specimens and results of the mode mix versus critical ERR were
superimposed onto the central delamination results. Davidson et al.\cite{14} came to the conclusion that neither the Williams nor the singular field based partition theories were able to provide accurate results for the composite material (C12K/R6376 graphite/epoxy).

A new definition of the mode mix parameter\cite{70} was also used by Davidson et al.\cite{14} which removed the classical singular field previously associated with their crack tip element\cite{46,70} model. The experiments were then reanalysed using the non-singular field definition of the mode mix parameter with the crack tip element and it was discovered that the specimens at the same mode mixity displayed the same critical ERR i.e. “toughness was found to be a single-valued function of mode mix”. The findings of Davidson et al.\cite{14} were also confirmed by Davidson et al.\cite{15} for a different composite material (T800H/3900-2 graphite/epoxy). When central delaminations are analysed with the non-singular field definition of the mode mix parameter\cite{70} and the crack tip element model the results are equal to using the singular field definition of the mode mix parameter\cite{46,70}, as in both cases the mode mix parameter equals zero.

Davidson et al.\cite{13} continued to work with the singular and non-singular field partition theories using experimental test results and judged the theories against the criteria that if the same mode mix is calculated for different specimens then they should display the same fracture toughness. For multidirectional laminated composites tested, as the delamination occurred between dissimilar materials, the mixed-mode partition theories had to eliminate the oscillatory singularity at the crack tip. Thus the three singular field based partition theories included the finite crack extension method\cite{44} and the “$\beta = 0$ approach”\cite{72}. Due to previously published results showing that the resin interface method\cite{69} produced similar mode partitions to that using the “$\beta = 0$ approach”, it was excluded from the investigation. The non-singular-field-based approach was the crack tip element model with the non-singular definition of the mode mix parameter\cite{46,70}. The mode mix parameter was determined through the use of experimental results.

Davidson et al.\cite{13} determined that all models based on the singular field approach considered for partitioning the ERR showed poor performance when predicting delamination growth. The main reason for this was due to the fact that the laminates considered had a small ply thickness and this parameter is used to scale the size of the singular field at the crack tip. This means that if the singular field does not dominate the
damage zone at the crack tip, singular field based partition theories are not applicable. On the other hand, the crack tip element using the non-singular field approach produced very accurate results for predicting the delamination for a variety of layups and loading configurations.

Following the excellent results obtained from the non-singular-field partition theory proposed by Davidson et al.\textsuperscript{70}, Harvey et al.\textsuperscript{12} compared its performance to the Wang and Harvey\textsuperscript{1–3,55} Euler beam partition theory and FEM based on 2D elasticity using previously published experimental results\textsuperscript{13,15}. From which it had already been identified that the singular field based partition theories showed poor performance and Davidson et al.’s non-singular field theory showed very accurate results for predicting delamination. A linear failure locus was used to assess the partition theories and it was identified that Wang and Harvey’s Euler beam partition theory showed very good agreement to that of Davidson et al.’s for unidirectional specimens. When considering multidirectional specimens however, the Wang and Harvey Euler beam partition theory increases the accuracy of the interfacial fracture toughness predictions. As expected the FEM results based on the singular field being present at the crack tip showed poor performance.

\section*{1.6. Conclusion}

The literature review has looked at the analytical, numerical and experimental methods used to partition a mixed-mode fracture and obtain the fracture toughness of a specimen. It appears that most techniques used will fall under two categories, the first of which based on the ERR and does not consider the form of the near-crack-tip stresses and strains but only the energy, with the main theories in this category being given by Williams\textsuperscript{52}, Wang and Harvey\textsuperscript{1–3,12,25}, Bruno and Greco\textsuperscript{54,57–59} and finally Luo and Tong\textsuperscript{60–62}. The second category is a local approach where quantities at the crack tip are considered for example the Davidson et al.\textsuperscript{46,70} crack tip element model or the Suo and Hutchinson\textsuperscript{21} theory both of which are based on a square-root singular stress field.

For interfacial cracks in isotropic homogeneous materials it has been determined that a singularity is present at the crack tip and the stress field can be characterised as being square-root singular\textsuperscript{31}. However, when the interfacial crack is between two dissimilar materials, the singularity is still square-root singular but also contains a new component.
which causes the stress and displacement fields to oscillate near to the crack tip. It has been determined that the oscillatory part of the stress field causes non-convergence of the mode I and II components of the ERRs. Therefore many numerical methods have been pursued to partition the ERR for an interfacial crack between two dissimilar materials and these include: (1) modelling the crack in a “thin resin layer” between the plies, (2) the \( \beta = 0 \) approach” by modifying the Poisson’s ratio above or below the crack tip to eliminate the imaginary component of the stress field and thus the oscillatory behaviour, (3) setting a finite crack extension in the VCCT enabling the calculation of the finite crack extension mode I and II components of the ERR, etc. It is identified that the first two methods provide the same partition. All numerical methods have the disadvantages of being computationally expensive and time consuming to implement when compared to analytical approaches.

Unfortunately for experimental tests only the total ERR can be obtained and therefore the use of one of the partition theories is still required in order to obtain the mode components. From which many investigations on unidirectional and multidirectional laminated composite materials have been performed in order to compare the partition theories. It has been identified that partition theories based on energy considerations provide much more accurate results compared to the local approaches based on the square-root singular stress field. From the literature it is thought that this is due to the damage in specimens being large when compared to the singular field. The singular-based partition theories are based on the requirement that the stress field is singular around the crack tip; therefore if the damage occurs out of the singular field the partition theories will provide poor results, i.e. the singular field must dominate the damage zone.

From the literature review it has been discovered that the current 2D-elasticity-based partition theory is actually a semi-analytical approach and requires the use of tabulated data to complete the mode partition. Also there currently is no analytical method for obtaining the crack-extension-size-dependent ERR components and the stress intensity factors for an interfacial crack between dissimilar materials. Therefore, the aims of this thesis are to: (1) derive a mixed-mode partition theory based on 2D elasticity for a DCB with bending moments, axial forces and shear forces, (2) analytically obtain the crack-extension-size-dependent ERR components for an interfacial crack between two
dissimilar materials and then obtain the mode I and II SIFs and (3) validate the new theories using current analytical theories, numerical simulations and existing experimental results.

To achieve the previously set aims, the task will be split up into chapters in order to show a step-by-step account of the methods used to derive the theories. Firstly, in Chapter 2 a new completely analytical 2D elasticity partition theory will be derived for an orthotropic laminated DCB with crack tip bending moments and axial forces. This partition theory has the advantage of being completely analytical as the current most accurate 2D elasticity partition theory is a semi-analytical approach. It is thought that the method in this section will have a stronger capability for solving more complex mixed-mode partition problems.

In Chapter 3, the partition theory for an orthotropic laminated DCB with crack tip bending moments and axial forces will be extended to include through-thickness shear forces.

In Chapter 4, focus will shift to a bimaterial DCB with different elastic properties across the interface but with the same Poisson’s ratio and a completely analytical partition theory will be derived to obtain the mode I and II ERRs and SIFs. The bimaterial DCB will have tip bending moments and axial forces.

In Chapter 5, the bimaterial DCB partition theory will be extended to the case where there is a mismatch in the Poisson’s ratio as well as the elastic properties across the interface.

The partition theories in all chapters will be validated against results obtained from 2D FEM simulations using the VCCT. However, it is also essential to validate the theories using experimental results. Therefore, in Chapter 6, the mixed-mode partition theory given in Chapter 3 will be applied to the blister test for interfacial fracture toughness using previously published experimental results. Finally conclusions will be drawn in Chapter 7.
Chapter 2: Orthotropic laminated beams with bending moments and axial forces

2.1. Introduction

When considering one-dimensional fracture, the DCB is the fundamental case for in-depth study as it simplifies the analysis but crucially keeps the important mechanics of the problem. Studying the DCB means it is possible to gain a deep understanding and predictive capability as many real life occurrences of fracture can be considered as being one-dimensional. Some common examples of fracture that can be modelled as one dimensional include separation of stiffeners and skins in stiffened panels, fracture in straight and curved laminated composite beams, thermal barrier coating cracking in gas turbine engines, needle puncture of a biological cell, straight edge cracks, circular blisters in plates and shells, etc. Fig. 1.3 has previously shown how the DCB can be used to model the cross section of a circular blister and therefore investigate an interfacial crack between a membrane blister and substrate. Similarly, the cross sections of other one dimensional fractures can also be modelled using the DCB.

It is possible to characterise many of the fibre reinforced laminated composite materials currently in use for high performance, safety-critical applications for example in the aerospace industry as being orthotropic. By orthotropic it means that the material has three mutually orthogonal planes of material symmetry. In such applications, it is extremely important to be able to predict the fracture toughness as an interfacial crack can lead to catastrophic failure of the structure. Therefore, the model developed in this chapter will focus on orthotropic laminated beams, meaning that it can be directly applied to real world applications. In order to develop a model to obtain the total ERR and mode partitions, initially the focus will be on the fundamental case of a DCB with tip bending moments and axial forces. Only after the fundamental case has been considered and understood is it then possible to increase the complexity of the model by including crack tip through-thickness shear forces. Therefore, Chapter 3 will present this extension to a DCB with crack tip bending moments, axial forces and through-thickness shear forces.

Partition theories based on the DCB have received a lot of attention in the literature as they offer a means of obtaining the fracture toughness in pure mode tests such as the
DCB or ENF tests. When the crack is propagating under mixed-mode conditions a partition theory is required to obtain the mode components of the ERR and then can be used with a failure criterion.

It is currently an unanswered question as to which method of partitioning a mixed-mode fracture offers the most accurate results. This could be based on the ERR and energy considerations or a local approach where quantities at the crack tip are considered. From experimental test results on macroscopic scale engineering structures\textsuperscript{3,9–15} such as unidirectional and multidirectional laminated composite materials it appears that the approach based on energy considerations offers more consistent results. This is due to the damage occurring over the whole region that is mechanically affected by the crack tip, which is much larger than the singular field. It is thought that when looking at the microscale for example in thin films, as the damage now occurs on a much smaller scale, the singular field can dominate the damage zone and a 2D-elasticity-based approach considering quantities at the crack tip will give more accurate results.

It is well known that for a crack propagating in an isotropic homogeneous material, a singularity is present at the crack tip and the 2D elasticity stress field can be characterised as being square-root singular\textsuperscript{31}. When partitioning mixed-mode fractures based on 2D elasticity, Suo and Hutchinson’s\textsuperscript{21} partition theory for a crack between two homogeneous elastic layers is currently considered to be the most accurate. Using classical plate theory to model the elastic layers it has previously been shown that it doesn’t provide enough information to partition a mixed-mode fracture\textsuperscript{46}. This means the results from a continuum analysis are required in order to complete the partition. Unlike Schapery and Davidson\textsuperscript{46} who used the FEM to complete the partition, Suo and Hutchinson\textsuperscript{21} used integral equations. An approximation has been given\textsuperscript{47} which allows a completely analytical solution for the isotropic homogeneous case based on the beam thickness ratio. However, this conventional 2D-elasticity-based partition theory often has limitations in dealing with more complex problems, for example, in the bimaterial case where the partition relies on extensively tabulated numerical results over a finite range of geometries and material properties.

A completely analytical method using classical plate theory was developed by Luo and Tong\textsuperscript{62}. Following the same approach as Suo and Hutchinson\textsuperscript{21}, instead of solving
integral equations to complete the continuum analysis and obtain the partition, Luo and Tong\textsuperscript{62} derived a new partition theory based on the energy considerations of a cracked laminate with bending moments only. From which they were able to use the results to solve the continuum analysis and therefore partition the ERR for a cracked laminate with bending moments and axial forces using their analytical “global-local” method.

An approximate rule for 2D elasticity has been given by Wang and Harvey\textsuperscript{1,2} based on their mixed-mode partition theory that uses orthogonal pure modes derived from considering classical and first-order shear deformable beam theories. Stating that the ERR partitions from the classical and first-order shear deformable beam theories represent the upper and lower bounds of the 2D elasticity solution, the average of the results was taken. The approximate rule has been modified to increase the accuracy by deriving new pure modes when the rule gives a negative partition for the ERR.

The work in this chapter revisits the partitioning of mixed-mode fractures in laminated orthotropic composite DCBs with rigid interfaces by taking 2D elasticity into consideration in a novel way by introducing correction factors into the beam theory based mechanical conditions. The present work aims to develop a novel and powerful method to calculate ERR partitions with the same level of accuracy as in the work of Suo and Hutchinson\textsuperscript{21}. Furthermore, it aims for the method to have a stronger capability for solving more complex mixed-mode partition problems for example considering a bimaterial DCB.

The structure of the chapter is as follows. A review of the work by Wang and Harvey\textsuperscript{1–3,12,25} is given in Section 2.2. In Section 2.3 the novel method to account for 2D elasticity is developed. The theory is validated against current analytical mixed-mode partition theories, in particular Suo and Hutchinson’s\textsuperscript{21} partition theory as it is regarded as the most accurate when considering 2D elasticity. Therefore the partition theories are given in Section 2.4 and the validation is in Section 2.5. Finally conclusions are made in Section 2.6.

2.2. Review of Wang and Harvey work

As the work in this chapter uses the same orthogonal pure fracture mode methodology as Wang and Harvey\textsuperscript{1–3,12,25}, initially a review of the relevant work will be
produced. After which a novel method to apply the theory to account for 2D elasticity is developed.

![Diagram of a laminated composite DCB](image)

**Figure 2.1:** A laminated composite DCB. (a) General description. (b) Details of the $\Delta a$-length crack influence region.

Fig. 2.1a shows a laminated composite DCB with its geometry and tip bending moments $M_1$ and $M_2$ and axial forces $N_1$ and $N_2$. The crack tip is located at point B and the influence region extends a distance $\Delta a$ ahead of the crack tip. The coordinate system is located at the crack tip, with the x-direction positive to the left and y-direction positive vertically. Fig. 2.1b shows how the tip bending moments and axial forces transfer to the crack tip, where a subscript B denotes a crack tip quantity. Furthermore, Fig 2.1b shows the sign convention that has been used for the interface normal stress $\sigma_n$ and shear stress $\tau_s$ and does not take into account the actual distribution of these stresses. Beyond point A to the left, it is assumed that the interface normal stress $\sigma_n$ and shear stress $\tau_s$ become zero.
To calculate the total ERR $G$ of the DCB, first consider the total strain energy $U$ calculated using the energies from the individual load inputs giving

$$U = \frac{1}{2E} \left[ \frac{M_{1B}^2 a}{I_1} + \frac{M_{2B}^2 a}{I_2} + \frac{N_{1B}^2 a}{A_1} + \frac{N_{2B}^2 a}{A_2} + \frac{M_{LB}^2 L}{I} + \frac{N_{LB}^2 L}{A} \right]$$  \hspace{1cm} (2.1)

where

$$M_B = M_{1B} + M_{2B} + \frac{h_1 N_{2B} - h_2 N_{1B}}{2}$$  \hspace{1cm} (2.2)

and

$$N_B = N_{1B} + N_{2B}$$  \hspace{1cm} (2.3)

where $a$ is the crack length, $h$ is the beam thickness, $A$ is the cross sectional area and $I$ the second moment of area. The upper beam is denoted by a subscript 1, the lower beam by a subscript 2 and no subscript for the intact section. For an orthotropic material $E = E_L$ for plane stress or $E = E_L / \left(1 - (\nu_{LT} \nu_{TL})\right)$ for plane strain, with $E_L$ being the Young’s modulus in the longitudinal direction, which is the x-direction in Fig. 2.1, and $\nu_{LT}$ and $\nu_{TL}$ being the in-plane Poisson’s ratios. $M_{1B}$ and $M_{2B}$ are the two bending moments at the crack tip B, and $N_{1B}$ and $N_{2B}$ are the axial forces at the crack tip B. As only a mismatch in axial stresses will contribute to the ERR, it is convenient to define an effective axial force given by

$$N_{1Be} = N_{1B} - \frac{N_{2B}}{\gamma}$$  \hspace{1cm} (2.4)

Therefore, the total ERR of the DCB can be calculated using

$$G = \frac{1}{b} \frac{\partial U}{\partial a} = \frac{1}{2Eb} \left[ \frac{M_{1B}^2}{I_1} + \frac{M_{2B}^2}{I_2} - \frac{1}{I} \left( M_{1B} + M_{2B} - \frac{h_1 N_{1Be}}{2} \right)^2 + \left( \frac{1}{A_1} - \frac{1}{A} \right) N_{1Be}^2 \right]$$  \hspace{1cm} (2.5)

where $\gamma = h_2 / h_1$ and $b$ is the width of the beam. It is also possible to write the total ERR $G$ in terms of the coefficient matrix $[C]$ as follows:

$$G = \{M_{1B} \quad M_{2B} \quad N_{1Be}\}^\top [C] \{M_{1B} \quad M_{2B} \quad N_{1Be}\}$$  \hspace{1cm} (2.6)

where
It can be seen from Eqs. (2.5)–(2.7) that the total ERR is of quadratic form in terms of the crack-tip loads $M_{1B}$, $M_{2B}$ and $N_{1Be}$ with the coefficient matrix. It is also worth noting that the total ERR is the same for both Euler and Timoshenko beam theories and 2D elasticity theory. When partitioning the ERR based on these different theories the individual mode components will differ depending on the theory being used.

Wang and Harvey\textsuperscript{1–3,12,25} assume that there generally exists two sets of orthogonal pure modes for rigid interface fracture in DCBs, meaning that it is possible to partition the total ERR in Eq. (2.5) into its mode I and mode II components using Euler beam theory as follows:

$$G_{IE} = c_{IE} \left( M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1Be}}{\beta_2} \right) \left( M_{1B} - \frac{M_{2B}}{\beta'_1} - \frac{N_{1Be}}{\beta'_2} \right)$$ \hspace{1cm} (2.8)

$$G_{IIIE} = c_{IIIE} \left( M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1Be}}{\theta_2} \right) \left( M_{1B} - \frac{M_{2B}}{\theta'_1} - \frac{N_{1Be}}{\theta'_2} \right)$$ \hspace{1cm} (2.9)

where $c_{IE}$ and $c_{IIIE}$ are two constants, and $(\theta_1, \beta_1)$ and $(\theta'_1, \beta'_1)$ with $i = 1, 2$ represent the first and second sets of orthogonal pure modes respectively.

$$c_{IE} = G_{\theta_1} \left[ 1 - \frac{\theta_1}{\beta_1} \right] \left[ 1 - \frac{\beta_1}{\beta'_1} \right]$$ \hspace{1cm} (2.10)

$$c_{IIIE} = G_{\beta_1} \left[ 1 - \frac{\beta_1}{\theta_1} \right] \left[ 1 - \frac{\theta_1}{\theta'_1} \right]$$ \hspace{1cm} (2.11)

$$G_{\theta_1} = \frac{1}{2bE} \left( \frac{1}{I_1} + \frac{\theta_1^2}{I_2} - \frac{(1 + \theta_1)^2}{I} \right)$$ \hspace{1cm} (2.12)

$$G_{\beta_1} = \frac{1}{2bE} \left( \frac{1}{I_1} + \frac{\beta_1^2}{I_2} - \frac{(1 + \beta_1)^2}{I} \right)$$ \hspace{1cm} (2.13)
Therefore a pure mode I fracture will result in $G_{IE} = 0$, for example $M_{2iB} = \theta_i M_{1iB}$ with $N_{1iBe} = 0$ will give $G_{IE} = 0$ in Eq. (2.9), therefore a pure mode I condition is denoted by $\theta_i$ or $\theta'_i$ with $i = 1, 2$. Similarly, a pure mode II fracture will result in $G_{IE} = 0$, for example $M_{2iB} = \beta_i' M_{1iB}$ with $N_{1iBe} = 0$ will give $G_{IE} = 0$ in Eq. (2.8), therefore a pure mode II condition is denoted by $\beta_i$ or $\beta'_i$ with $i = 1, 2$. It can be seen that in Euler beam theory with rigid interfaces, the two sets of orthogonal pure modes do not coincide with each other and that this results in “stealthy” interactions which change the ERR partitions $G_i$ and $G_{ii}$ but do not change the total $G$. The first set of orthogonal pure modes corresponds to a pure mode I condition of the relative shearing displacement at the crack tip equalling zero and a pure mode II condition of the opening force at the crack tip equalling zero and are given as:

$$\theta_i = -\gamma^2$$

$$\theta'_i = \frac{-6}{h_i}$$

$$\beta_i = \frac{\gamma^2(3+\gamma)}{1+3\gamma}$$

$$\beta'_i = \begin{cases} 
\frac{2(3+\gamma)}{h_i(\gamma-1)} & \text{if } \gamma \neq 1 \\
\infty & \text{if } \gamma = 1
\end{cases}$$

The second set of orthogonal pure modes corresponds to a pure mode I condition of the crack tip shear force equalling zero and a pure mode II condition of the crack tip relative opening displacement equalling zero and are given as:

$$\theta'_i = -1$$

$$\theta''_i = -\frac{6(1+\gamma)}{h_i(1+\gamma^3)}$$

$$\beta'_i = \gamma^3$$

$$\beta''_i = \infty$$
Now when the total ERR in Eq. (2.5) is partitioned into its mode I and II components using Timoshenko beam theory with either rigid or non-rigid interfaces, the two sets of pure modes coincide on the first set of pure modes from Euler beam theory i.e. \((\theta_i, \beta_i) = (\theta'_i, \beta'_i)\) with \(i = 1, 2\), resulting in no stealthy interactions. This means that Eqs. (2.8)–(2.11) become:

\[
G_{IT} = c_{IT} \left( M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1Be}}{\beta_2} \right)^2
\]

(2.22)

\[
G_{IT} = c_{IT} \left( M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1Be}}{\theta_2} \right)^2
\]

(2.23)

\[
c_{IT} = G_{\theta_1} \left( 1 - \frac{\theta_1}{\beta_1} \right)^2
\]

(2.24)

\[
c_{IT} = G_{\beta_1} \left( 1 - \frac{\beta_1}{\theta_1} \right)^2
\]

(2.25)

### 2.3. Analytical development

#### 2.3.1. 2D elasticity partition theory

Now the same hypothesis as Wang and Harvey\(^1\text{–}^3,12,25\) will be used, namely that there generally exist two sets of orthogonal pure fracture modes for rigid interface fracture in DCBs. However as the pure modes are now based on 2D elasticity, the remainder of the work in this chapter represents an original contribution to the field of fracture mechanics. The total ERR \(G\) in Eq. (2.5) can be partitioned based on 2D elasticity theory as

\[
G_I = c_I \left( M_{1B} - \frac{M_{2B}}{\beta_{1-2D}} - \frac{N_{1Be}}{\beta_{2-2D}} \right) \left( M_{1B} - \frac{M_{2B}}{\beta'_{1-2D}} - \frac{N_{1Be}}{\beta'_{2-2D}} \right)
\]

(2.26)

\[
G_{II} = c_{II} \left( M_{1B} - \frac{M_{2B}}{\theta_{1-2D}} - \frac{N_{1Be}}{\theta_{2-2D}} \right) \left( M_{1B} - \frac{M_{2B}}{\theta'_{1-2D}} - \frac{N_{1Be}}{\theta'_{2-2D}} \right)
\]

(2.27)

where \(c_I\) and \(c_{II}\) are two constants, and \((\theta_{i-2D}, \beta_{i-2D})\) and \((\theta'_{i-2D}, \beta'_{i-2D})\) with \(i = 1, 2\) represent the first and second sets of orthogonal pure modes. The subscript 2D denotes
that the pure modes are based on 2D elasticity theory. For example, when
\[ M_{2B} = \theta_{1-2D} M_{1B} \] and \( N_{1Be} = 0 \), the pure mode I mode occurs as the relative shearing displacement just behind the crack tip is zero. This pure mode I is denoted \( \theta_{1-2D} \). Its orthogonal pure mode II is \( \beta_{1-2D} \) which corresponds to zero crack tip opening force.

Here, the ‘orthogonal’ means

\[ \{ 1 \quad \theta_{1-2D} \quad 0 \}^T C \{ 1 \quad \beta_{1-2D} \quad 0 \}^T = 0 \]  

(2.28)

For simplicity it is possible to rewrite Eq. (2.28) as \( \theta_{1-2D} = \text{orthogonal} (\beta_{1-2D}) \).

Similarly, when \( M_{2B} = \theta'_{1-2D} M_{1B} \) and \( N_{1Be} = 0 \), the pure mode I mode occurs as the crack tip shearing force is zero. This pure mode I is denoted \( \theta'_{1-2D} \). Its orthogonal pure mode II is \( \beta'_{1-2D} \) which corresponds to zero crack tip opening displacement.

Similar to the partition theory using Timoshenko beam theory seen in Section 2.2, it has been identified by Suo and Hutchinson\(^{21}\) that the two sets of orthogonal pure modes also coincide in 2D elasticity theory for rigid interfaces, i.e.

\[ (\theta_{1-2D}, \beta_{1-2D}) = (\theta'_{1-2D}, \beta'_{1-2D}) \] with \( i = 1, 2 \). Therefore, Eqs. (2.26) and (2.27) become

here for laminated orthotropic composite DCBs with rigid interfaces in 2D elasticity,

\[ G_I = c_i \left( M_{1B} - \frac{M_{2B}}{\beta_{1-2D}} - \frac{N_{1Be}}{\beta_{2-2D}} \right)^2 \]  

(2.29)

\[ G_{II} = c_{II} \left( M_{1B} - \frac{M_{2B}}{\theta_{1-2D}} - \frac{N_{1Be}}{\theta_{2-2D}} \right)^2 \]  

(2.30)

where

\[ c_i = G_{\theta_{1-2D}} \left( 1 - \frac{\theta_{1-2D}}{\beta_{1-2D}} \right)^2, \quad c_{II} = G_{\beta_{1-2D}} \left( 1 - \frac{\beta_{1-2D}}{\theta_{1-2D}} \right)^2 \]  

(2.31)

\[ G_{\theta_{1-2D}} = \frac{1}{2bE} \left( \frac{1 + \theta_{1-2D}^2 - (1 + \theta_{1-2D})^2}{I_1} \right), \quad G_{\beta_{1-2D}} = \frac{1}{2bE} \left( \frac{1 + \beta_{1-2D}^2 - (1 + \beta_{1-2D})^2}{I_2} \right) \]  

(2.32)

Now, the key task is to determine the orthogonal pure mode set \( (\theta_{1-2D}, \beta_{1-2D}) \) with \( i = 1, 2 \). At this point, it is important to note that the orthogonal property demonstrated
in Eq. (2.28) exists between any pair of pure modes in the pure mode set \((\theta_{1-2D}, \beta_{1-2D})\) with \(i = 1, 2\). That is,

\[
\theta_{1-2D} = \text{orthogonal}(\beta_{1-2D} \text{ and } \beta_{2-2D}) \tag{2.33}
\]

\[
\theta_{2-2D} = \text{orthogonal}(\beta_{1-2D} \text{ and } \beta_{2-2D}) \tag{2.34}
\]

As long as one pure mode is known, say \(\theta_{1-2D}\), the others can be obtained by using the orthogonal property. This knowledge provides a powerful methodology to find all the pure modes and to partition mixed-modes, which will be used in the following development.

2.3.2. 2D elasticity bending moment and axial force pure modes

The central task of the present work is to determine the pure mode set \((\theta_{1-2D}, \beta_{1-2D})\), after which it is possible to use the orthogonality condition between pure modes in order to obtain the other pure modes and therefore partition the ERR based on 2D elasticity.

To determine the pure mode I \(\theta_{1-2D}\) only the simplest loading case is required: a DCB with two crack tip bending moments \(M_{1b}\) and \(M_{2b}\). As previously mentioned, when Euler beam theory is used there are two sets of orthogonal pure modes, with the pure mode I modes in Eqs. (2.14), (2.15), (2.18) and (2.19) and the pure mode II modes in Eqs. (2.16), (2.17), (2.20) and (2.21). When Timoshenko beam theory is used, the two sets of pure modes coincide on the first set in Eqs. (2.14)–(2.17). That is \((\theta_{1-T}, \beta_{1-T}) = (\theta'_{1-T}, \beta'_{1-T}) = (\theta_1, \beta_1)\) where the subscript \(T\) denotes for Timoshenko beam theory. This is due to the introduction of the drastic uniform through-thickness shearing strain by the Timoshenko beam theory. It is therefore reasonable to anticipate that the pure mode set \((\theta_{1-2D}, \beta_{1-2D})\) in 2D elasticity theory will be bounded by the \((\theta_1, \beta_1)\) set and the \((\theta', \beta')\) set because 2D elasticity theory more appropriately considers the mechanics at the crack tip. That is

\[
\theta_1 < \theta_{1-2D} < \theta' \quad \text{and} \quad \beta_1 < \beta_{1-2D} < \beta'	ag{2.35}
\]

To determine \(\theta_{1-2D}\) and \(\beta_{1-2D}\), the beam-theory-based mechanical conditions for \(\theta_1\) and \(\beta_1\) pure modes are re-examined.
Consider the DCB in Fig. 2.1a with tip bending moments only, the resultant moment \( M_n \) about point A due to \( \sigma_n \) within the \( \Delta a \) region in Fig. 2.1b is given by Wang and Harvey\(^1\,^2\) as

\[
M_n = b \int_0^{\Delta a} \sigma_n \, dx = b \int_0^{\Delta a} \sigma_n (\Delta a - x) \, dx = M_{nn} + F_n \Delta a \tag{2.36}
\]

where \( M_{nn} \) is the resultant moment produced by the interface normal stress \( \sigma_n \) about point B and given by

\[
M_{nn} = -b \int_0^{\Delta a} x \sigma_n \, dx \tag{2.37}
\]

and \( F_n \) is the resultant normal force due to the interface normal stress \( \sigma_n \) and given by

\[
F_n = -b \int_0^{\Delta a} \sigma_n \, dx \tag{2.38}
\]

which is zero since there are no crack tip shear forces. At point A, there is a continuity of curvature condition. By considering this condition, \( M_{nn} \) is calculated to be

\[
M_{nn} = \frac{1}{h_1 + h_2} \left\{ h_2 \left[ M_{1B} - \frac{I_1}{I} (M_{1B} + M_{2B}) \right] - h_1 \left[ M_{2B} - \frac{I_2}{I} (M_{1B} + M_{2B}) \right] \right\} \tag{2.39}
\]

It is important to note that Eqs. (2.36)–(2.39) remain the same regardless of whether Euler or Timoshenko beam theory or 2D elasticity theory is used because no deformation is considered within the \( \Delta a \) region. In both the Euler and Timoshenko beam theories, the pure mode II condition \( M_{2B} = \beta_1 M_{1B} \) is obtained by setting the resultant moment about the crack tip at point B due to the normal stress \( \sigma_n \) in the \( \Delta a \) region to zero, i.e. \( M_{nn} = 0 \), which produces

\[
\frac{M_{1B}}{I_1} - \frac{M_{1B} + M_{2B}}{I} = \gamma^2 \left( \frac{M_{2B}}{I_2} - \frac{M_{1B} + M_{2B}}{I} \right) \tag{2.40}
\]

It can be shown that the physical meaning of this condition in the Euler and Timoshenko beam theories is uniformly zero normal stress within the \( \Delta a \) region. Obviously, in 2D elasticity theory, this condition does not produce the same stress distribution and therefore does not represent the pure mode II condition. This is due to the fact that the \( \Delta a \) region is under consideration and the 2D stress field will give a
more realistic representation of the actual mechanics at the crack tip affecting the intact section of the beam. Therefore, to account for the more realistic stress distribution given by the 2D elasticity solution, Eq. (2.40) must be corrected before it is applicable and can then be used to identify the pure mode II condition. It can be seen that Eq. (2.40) is in the following form

\[ w_{1B}' - w_A' = \gamma^2 \left( w_{2B}' - w_A' \right) \]  
(2.41)

The left hand side of the Eq. (2.41) gives the difference between the curvature at the crack tip for the upper beam \( w_{1B}' \) and the curvature at point A \( w_A' \), which is related to the difference between the curvature at the crack tip for the lower beam \( w_{2B}' \) and the curvature at point A \( w_A' \) by the right hand side of Eq. (2.41). It is known that the stress field and thus curvatures of the beams will be different when considering 2D elasticity theory. Therefore, a correction factor \( c_\beta(\gamma) \) is introduced which will account for this difference in curvatures, meaning the result will represent the curvatures of the beam for the 2D elasticity solution. Furthermore, as the crack tip influence region \( \Delta a \) is located in the intact section of the beam, it is anticipated that the curvature of the intact beam will provide the greatest difference when comparing results from the beam theory and 2D elasticity solutions. Therefore, the correction factor \( c_\beta(\gamma) \) is introduced to the curvature of the intact section of the beam at point A \( w_A' \) as follows

\[ \frac{M_{1B}}{I_1} - c_\beta \frac{M_{1B} + M_{2B}}{I} = \gamma^2 \left( \frac{M_{2B}'}{I_2} - c_\beta \frac{M_{1B} + M_{2B}}{I} \right) \]  
(2.42)

Therefore the pure mode II mode \( \beta_{1-2D} \) in 2D elasticity theory can be obtained by substituting \( M_{2B} = \beta_{1-2D} M_{1B} \) into Eq. (2.42) and rearranging

\[ \beta_{1-2D} = \gamma \left[ \frac{(1 + \gamma)^2 + c_\beta (\gamma - 1)}{(1 + \gamma)^2 - c_\beta (\gamma - 1)} \right] \]  
(2.43)

It is worth noting that from Eqs. (2.42) and (2.43) that when \( \gamma = 1 \), \( \beta_{1-2D} = \beta_1 = \beta'_1 = 1 \) and \( c_\beta(1) \) can take any value. However, when \( \gamma \neq 1 \) it is reasonable to assume that \( \beta_{1-2D} \) is bounded by \( \beta_1 \) and \( \beta'_1 \), i.e. \( \beta_1 < \beta_{1-2D} < \beta'_1 \) when \( \gamma > 1 \), and \( \beta'_1 < \beta_{1-2D} < \beta_1 \).
when \( \gamma < 1 \), as mentioned earlier. The bounds of the correction factor \( c_\beta(\gamma) \) are therefore found to be

\[
1 < c_\beta(\gamma) < \left[ \frac{(1+\gamma)^3}{1+\gamma^3} \right] \quad (2.44)
\]

Note that \( c_\beta(\gamma) = c_\beta(1/\gamma) \) because \( \beta_{1\sim2D}(\gamma) = 1/\beta_{1\sim2D}(1/\gamma) \) due to mechanical symmetry.

Now it is necessary to perform the same analysis in order to obtain the pure mode I condition \( \theta_{1\sim2D} \) in terms of the pure mode I correction factor \( c_\theta(\gamma) \). Therefore in both Euler and Timoshenko beam theories the pure mode I condition \( M_{2B} = \theta_iM_{1B} \) is obtained by considering the relative shearing displacement \( D_{sh} \) at a small distance \( \delta u \) behind the crack tip giving

\[
D_{sh} = \frac{1}{2E} \left( \frac{h_1M_{1B}}{I_1} + \frac{h_2M_{2B}}{I_2} \right) \delta u \quad (2.45)
\]

Therefore the pure mode I condition \( M_{2B} = \theta_iM_{1B} \) is obtained by zeroing the relative shearing displacement at the crack tip i.e. \( D_{sh} = 0 \), giving

\[
\frac{M_{1B}}{I_1} = -\gamma \left( \frac{M_{2B}}{I_2} \right) \quad (2.46)
\]

Now, similarly to the pure mode II case, a correction factor must be introduced into Eq. (2.46) so that it represents the pure mode I condition in 2D elasticity theory. However, unlike the pure mode II case, Eq. (2.46) does not take into account the intact section of the beam and only considers the curvatures at the crack tip for the top \( w_{1B}^\prime \) and bottom \( w_{2B}^\prime \) beam. It is essential that the curvature of the intact section of the beam \( w_{1B}^\prime \) at point A is considered as the crack tip influence region is located in this section of the beam. Again, it is expected that this region will produce the greatest difference between the stress distributions predicted using beam theory and 2D elasticity theory. In order to solve this problem, the curvature of the intact beam at point A \( w_{1B}^\prime \) is added to Eq. (2.46) as well as the 2D elasticity correction factor \( c_\gamma(\gamma) \) giving
\[
\frac{M_{1B}}{I_1} + c_\theta \left( \frac{M_{1B} + M_{2B}}{I} \right) = -\gamma \left[ \frac{M_{2B}}{I_2} + c_\theta \left( \frac{M_{1B} + M_{2B}}{I} \right) \right]
\]

(2.47)

Therefore the pure mode I mode \( \theta_{1-2D} \) in 2D elasticity theory can be obtained by substituting \( M_{2B} = \theta_{1-2D} M_{1B} \) into Eq. (2.47) and rearranging

\[
\theta_{1-2D} = -\gamma^2 \left[ \frac{(1 + \gamma)^2 + c_\theta}{(1 + \gamma)^2 + c_\theta \gamma^2} \right]
\]

(2.48)

Similarly, it is worth noting from Eq. (2.48) that when \( \gamma = 1 \), \( \theta_{1-2D} = \theta_1 = \theta_1' = -1 \) and \( c_\theta(1) \) can take any value. However when \( \gamma \neq 1 \), it is reasonable to assume that \( \theta_{1-2D} \) is bounded by \( \theta_1 \) and \( \theta_1' \), i.e. \( \theta_1 < \theta_{1-2D} < \theta_1' \) when \( \gamma > 1 \), and \( \theta_1' < \theta_{1-2D} < \theta_1 \) when \( \gamma < 1 \), as mentioned earlier. The bounds of the correction factor \( c_\theta(\gamma) \) are therefore found to be

\[
0 < c_\theta(\gamma) < \left[ \frac{\gamma}{c_\theta(\gamma)} = \infty \right]
\]

(2.49)

Also note that \( c_\theta(\gamma) = c_\theta(1/\gamma) \) because \( \theta_{1-2D}(\gamma) = 1/\theta_{1-2D}(1/\gamma) \) due to mechanical symmetry.

By making use of the orthogonal property between the 2D elasticity pure modes \( \theta_{1-2D} \) and \( \beta_{1-2D} \), it is possible to obtain a relationship between the two correction factors \( c_\theta(\gamma) \) and \( c_\beta(\gamma) \). The relationships are

\[
c_\theta(\gamma) = \frac{(1 - c_\theta)(1 + \gamma)^3}{(1 + \gamma^3) c_\beta - (1 + \gamma)^3}
\]

(2.50)

and

\[
c_\beta(\gamma) = \frac{(1 + c_\theta)(1 + \gamma)^3}{(1 + \gamma^3) c_\beta + (1 + \gamma)^3}
\]

(2.51)

At this point, it is helpful to pay attention to the values of \( c_\theta(1) \) and \( c_\beta(1) \). It is seen from Eqs. (2.42) and (2.47) that the two correction terms represent the contributions of the curvature of the intact beam to those of the upper and lower beams at the crack tip. Therefore for \( \gamma = 1 \), it is reasonable to assume that the corrected curvatures are those obtained by averaging those of the intact beam and two separated beams. This argument
gives \( c_\beta(1) = 1 \) which then gives \( c_\beta(1) = 1.6 \) from Eq. (2.51). It is worth noting that the work of Luo and Tong\(^6\) effectively uses the constant \( c_\beta = 1 \).

It is seen from Eq. (2.51) that when \( \gamma \) tends to infinity or to zero, the correction factor \( c_\beta(\gamma) \) tends to a constant unit value, \( \overline{\gamma} = 1 \). Similarly, it is expected that when \( \gamma \) tends to infinity or to zero, the correction factor \( c_\theta(\gamma) \) also tends to a constant value, \( \overline{\gamma} \).

The value of \( \overline{\gamma} \) now needs to be determined. When \( \gamma \) tends to infinity or zero, Eq. (2.51) gives

\[
\lim_{\gamma \to \infty} \left( \frac{dc_\beta(\gamma)}{d\gamma} \right) = 0 \quad \text{and} \quad \lim_{\gamma \to 0} \left( \frac{dc_\beta(\gamma)}{d\gamma} \right) = \frac{3\overline{\gamma}}{1 + \overline{\gamma}}
\]  

(2.52)

Note that the second equation in Eq. (2.52) can also be obtained from Eq. (2.50). The variation of the correction factor \( c_\beta \) against \( \gamma \) is now plotted and can be seen in Fig. 2.2. The thick dashed line in Fig. 2.2 shows the trend of the correction factor \( c_\beta \) against \( \gamma \), tending to the correct gradients when \( \gamma \) tends to infinity or to zero, and passing through \( c_\beta(1) = 1.6 \).

![Figure 2.2: Variation of the correction factor \( c_\beta(\gamma) \) with \( \gamma \).](image)

From Fig. 2.2, the following approximate assumption can be made

\[
\left( \frac{dc_\beta}{d\gamma} \right)_{\gamma \to 0} \approx \frac{c_\beta(1)}{1}
\]

(2.53)
Therefore, equating this result to the second equation in Eq. (2.52) gives
\[
\bar{\varepsilon}_\theta \approx \frac{8}{7} \quad (2.54)
\]
The accuracy of Eq. (2.54) will be examined in the next section by comparing results to other current 2D elasticity partition theories. An even more accurate value of \( \bar{\varepsilon}_\theta \) is found to be
\[
\bar{\varepsilon}_\theta \approx \frac{6}{5} \quad (2.55)
\]
Information about how this improved value of \( \bar{\varepsilon}_\theta \) has been obtained is given in Section 2.5.1. The variation of the correction factor \( c_\theta(\gamma) \) can then be expressed as
\[
c_\theta(\gamma) = \bar{\varepsilon}_\theta \left[ \varphi^{(0)} - \varphi^{(\gamma)} \right] \quad (2.56)
\]
Eq. (2.56) is a choice that has been obtained by inspection because when \( \gamma = 1 \), \( c_\theta = 1 \) is recovered, and when \( \gamma \to \infty \) or \( \gamma \to 0 \), \( c_\theta = \bar{\varepsilon}_\theta \) is recovered. Using Eqs. (2.48) and (2.56), the pure mode I \( \theta_{1-2D} \) mode is obtained. Then \( \theta_{2-2D} \), \( \beta_{1-2D} \) and \( \beta_{2-2D} \) are determined using the orthogonal property, that is
\[
\theta_{2-2D}(\gamma) = \text{orthogonal}(\beta_{1-2D}) = \text{orthogonal}(\beta_{2-2D}) \quad (2.57)
\]
\[
\beta_{1-2D}(\gamma) = \text{orthogonal}(\theta_{1-2D}) = \text{orthogonal}(\theta_{2-2D}) \quad (2.58)
\]
\[
\beta_{2-2D}(\gamma) = \text{orthogonal}(\theta_{1-2D}) = \text{orthogonal}(\theta_{2-2D}) \quad (2.59)
\]
The 2D elasticity pure modes for bending moments and axial forces are given as
\[
\theta_{1-2D} = -\gamma^2 - \frac{\bar{\varepsilon}_\theta^2 \gamma^2 (1 - \gamma^2)}{\bar{\varepsilon}_\theta^2 \gamma^2 + \varphi(1 + \gamma)^2} \quad (2.60)
\]
\[
\beta_{1-2D} = \frac{\gamma^2 (3 + \gamma)}{1 + 3\gamma} - \frac{3\bar{\varepsilon}_\theta^2 \gamma^2 (1 - \gamma^2)}{(1 + 3\gamma) \left[ \bar{\varepsilon}_\theta^2 + \varphi(1 + 3\gamma) \right]} \quad (2.61)
\]
\[
\theta_{2-2D} = -\frac{6}{h_i} \frac{6\bar{\varepsilon}_\theta^2 \gamma (1 - \gamma)}{h_i \left[ \bar{\varepsilon}_\theta^2 (1 - \gamma + \gamma^2) + \varphi(1 + \gamma)^2 \right]} \quad (2.62)
\]
\[
\beta_{2-2D} = \begin{cases} 
2(3 + \gamma) + \frac{2\pi^2 \gamma}{h_1(\gamma - 1)} & \text{if } \gamma \neq 1 \\
\infty & \text{if } \gamma = 1
\end{cases}
\] (2.63)

where
\[
\varphi = \tilde{c}_{\gamma}^{(1+\gamma)/[1+\gamma]} \] (2.64)

Therefore as the 2D elasticity pure modes have been obtained it is now possible to partition the total ERR using Eqs. (2.29)–(2.32). Note that in Eqs. (2.60)–(2.63), the first term is the corresponding pure mode from Timoshenko beam theory given in Eqs. (2.14)–(2.17) and the second term is the correction applied for 2D elasticity.

### 2.4. 2D elasticity mixed-mode partition theories

In order to validate the present 2D elasticity partition theory for bending moments and axial forces, the results will be compared to Suo and Hutchinson’s\(^{21}\) partition theory as it is currently considered to be the most accurate when considering 2D elasticity. This is due to the fact that the theory is established using the 2D elasticity stress field that has been identified as being square-root singular by Williams\(^{31}\). Furthermore, the partition is calculated using the complex stress intensity factor\(^{43,45}\). A comparison of other 2D-elasticity-based partition theories will also be conducted. Therefore, in this section the partition theories will be presented with their notations adapted to follow the same definitions given in this thesis.

#### 2.4.1. Suo and Hutchinson

Suo and Hutchinson’s\(^{21}\) semi-analytical partition theory is for mixed-mode fracture between two homogeneous elastic layers under bending moments and axial forces. The complex stress intensity factors\(^{43,45}\) as a result of the oscillatory model\(^{31,32}\) are used to obtain the mode mixity and can be seen below

\[
K_{I-SH} = \frac{N}{\sqrt{2h_U}} \cos(\omega) + \frac{M}{\sqrt{2h_V}} \sin(\omega + \phi) \] (2.65)

\[
K_{II-SH} = \frac{N}{\sqrt{2h_U}} \sin(\omega) - \frac{M}{\sqrt{2h_V}} \cos(\omega + \phi) \] (2.66)
where $M$ and $N$ are the loading parameters given by

$$
M = \frac{M_1}{b} - \frac{C_1}{b} \left( M_1 + M_2 + \frac{h_1}{2} (N_2 - \gamma N_1) \right)
$$

(2.67)

$$
N = -\frac{N_1}{b} + \frac{C_1}{b} (N_1 + N_2) - C_2 \left[ M_1 + M_2 + \frac{h_1}{2} (N_2 - \gamma N_1) \right]/bh_1
$$

(2.68)

where

$$
C_1 = \frac{1}{1 + \gamma}, \quad C_2 = \frac{6\gamma}{(1 + \gamma)^3}, \quad C_3 = \frac{1}{(1 + \gamma)^3}
$$

(2.69)

Finally $U$, $V$ and $\phi$ are obtained using

$$
U = \frac{\gamma^3}{\gamma^3 + 4\gamma^2 + 6\gamma + 3}, \quad V = \frac{\gamma^3}{12(1 + \gamma^3)}, \quad \sin(\phi) = \frac{6(1 + \gamma)}{\gamma^3}
$$

(2.70)

The partition theory is semi-analytical as it requires the determination of a “single real scalar function” $\omega$ using a continuum analysis, meaning that the use of tabulated data is required to complete the mode partition. However, an approximation to $\omega$ has been given by Suo\textsuperscript{47} as

$$
\omega = 52.1^\circ - 3^\circ/\gamma
$$

(2.71)

which allows a completely analytical solution for the isotropic homogeneous case. Therefore the ERR components can be calculated from Suo and Hutchinson’s\textsuperscript{21} partition theory as

$$
G_{I-SH} = \frac{K_{I-SH}^2}{E} \quad \text{and} \quad G_{II-SH} = \frac{K_{II-SH}^2}{E}
$$

(2.72)

2.4.2. Wang and Harvey approximate rule 1

Wang and Harvey’s approximate rule \textsuperscript{1,2} is based on the Euler and Timoshenko partition theories given in Section 2.2. Stating that the results for the Euler and Timoshenko beam theories should act as the upper and lower bounds of the 2D elasticity solution, the average of these results should agree well with the 2D elasticity partition theory as follows
2.4.3. Wang and Harvey approximate rule 2

Wang and Harvey’s approximate rule 2\(^2\) is a modification of the approximate rule 1 as a result of a negative partition of the ERR for certain loading conditions. By deriving new pure modes which are included when the previous rule gives a negative partition, the accuracy of the approximate rule 1 should be improved.

By considering the loading condition \( M_{1B} = 1 \) and \( N_{1Be} = 0 \), it is possible to obtain \( \beta_{1-WH2} \) by solving

\[
\frac{d\left(G_{1-WH1}/G\right)}{d\left(M_{2B}\right)} = 0
\]

for \( M_{2B} \). It is also possible to obtain \( \theta_{1-WH2} \) by solving

\[
\frac{d\left(G_{1-WH1}/G\right)}{d\left(M_{2B}\right)} = 0
\]

for \( M_{2B} \). After which the second loading condition considered is \( M_{1B} = 1 \) and \( M_{2B} = 0 \). Then it is possible to obtain \( \beta_{2-WH2} \) by solving

\[
\frac{d\left(G_{1-WH1}/G\right)}{d\left(N_{1Be}\right)} = 0
\]

for \( N_{1Be} \). It is also possible to obtain \( \theta_{2-WH2} \) by solving

\[
\frac{d\left(G_{1-WH1}/G\right)}{d\left(N_{1Be}\right)} = 0
\]

for \( N_{1Be} \). It is now possible to partition the ERR using Eqs. (2.22)–(2.25) and the new pure modes derived using the conditions above.

2.4.4. Luo and Tong

The Luo and Tong\(^6\) “global-local” method is completely analytical and can be used to partition the ERR of a cracked laminate under bending moments and axial forces by
means of classical plate theory. Where Suo and Hutchinson\textsuperscript{21} needed to solve a continuum analysis to obtain the “single real scalar function”, Luo and Tong\textsuperscript{62} obtained an analytical solution using energy considerations for a cracked laminate with bending moments only to complete the mode partition. The pure modes obtained by Luo and Tong are

\begin{equation}
\theta_{1-LT} = -\frac{\gamma^2(\gamma^2 + 2\gamma + 2)}{2\gamma^2 + 2\gamma + 1} \quad \text{and} \quad \theta_{2-LT} = -\frac{6(\gamma^2 + 2\gamma + 2)}{h_1(2\gamma^2 + \gamma + 2)}
\end{equation}

\begin{equation}
\beta_{1-LT} = \frac{\gamma^2(2\gamma + 3)}{(3\gamma + 2)} \quad \text{and} \quad \beta_{2-LT} = -\frac{4\gamma + 6}{h_1(1 - \gamma)}
\end{equation}

Therefore the total ERR can be partitioned using

\begin{equation}
G_{I-LT} = c_{I-LT} \left( M_{1B} - \frac{M_{2B}}{\beta_{1-LT}} - \frac{N_{1Be}}{\beta_{2-LT}} \right)^2
\end{equation}

\begin{equation}
G_{II-LT} = c_{II-LT} \left( M_{1B} - \frac{M_{2B}}{\theta_{1-LT}} - \frac{N_{1Be}}{\theta_{2-LT}} \right)^2
\end{equation}

where

\begin{equation}
G_{\theta_{1-LT}} = \frac{1}{2Eb} \left[ \frac{1}{I_1} + \frac{\theta_{1-LT}^2}{I_2} - \frac{(1 + \theta_{1-LT})^2}{I} \right]
\end{equation}

\begin{equation}
G_{\beta_{1-LT}} = \frac{1}{2Eb} \left[ \frac{1}{I_1} + \frac{\beta_{1-LT}^2}{I_2} - \frac{(1 + \beta_{1-LT})^2}{I} \right]
\end{equation}

\begin{equation}
c_{I-LT} = \frac{G_{\theta_{1-LT}}}{\left( 1 - \frac{\theta_{1-LT}^2}{\beta_{1-LT}^2} \right)^2} \quad \text{and} \quad c_{II-LT} = \frac{G_{\beta_{1-LT}}}{\left( 1 - \frac{\beta_{1-LT}}{\theta_{1-LT}} \right)^2}
\end{equation}

2.5. Comparisons with Suo and Hutchinson’s partition theory

It is now possible to validate the present theory by comparing it to Suo and Hutchinson’s\textsuperscript{21} partition theory. A number of loading conditions and thickness ratios will be considered in order to determine the accuracy of the new theory. Hutchinson and Suo\textsuperscript{88} showed that the ERR components of an orthotropic material are essentially the
same as their isotropic counterparts, except using the longitudinal tensile modulus. For convenience, therefore, only isotropic material constants are used for all loading conditions and thickness ratios.

2.5.1. DCB with tip bending moments only

The first test performed in order to validate the new 2D elasticity partition theory involved a laminated unidirectional composite DCB with crack tip bending moments only. Only a bending moment on the top beam was considered by setting $M_{1B} = 1 \text{Nmm}$ and $M_{2B} = N_{1B} = N_{2B} = 0$, meaning that the bending moment ratio $M_{2B}/M_{1B} = 0$. This loading ratio was selected so that an accurate value for the pure mode $\theta_{1-2D}$ could be obtained by calibrating the value of the constant $\bar{c}_0$. The thickness ratio $\gamma$ was set to vary from 0.01 to 100, ensuring a wide range of values commonly seen in engineering applications was considered. The six previously mentioned partition theories were then used to calculate the mode partition $G_I/G$, i.e. the mode I component of the ERR $G_I$ divided by the total ERR $G$, for the given loading condition and range of thickness ratios and the results were plotted in Fig. 2.3. The partition theories evaluated are Suo and Hutchinson’s partition theory, Wang and Harvey’s approximate rule $1^{1,2}$, Harvey and Wang’s approximate rule $2^2$, the present theory with $\bar{c}_0 = 8/7$ and $\bar{c}_0 = 6/5$ and Luo and Tong’s partition theory.

It can be seen from Fig. 2.3, that the present theory with $\bar{c}_0 = 8/7$ is almost identical to Suo and Hutchinson’s partition theory when $0.1 \leq \gamma \leq 10$ and is slightly different when $\gamma < 0.1$ or $\gamma > 10$, giving a root mean square error of 0.009. In an attempt to increase the accuracy of the present theory, the value of $\bar{c}_0$ was adjusted to minimise the root mean square error between the present theory and Suo and Hutchinson’s partition theory$^{21}$. Using this method means that the pure mode I $\theta_{1-2D}$ is as accurate as possible. This is essential as all other pure modes are obtained using the pure mode I $\theta_{1-2D}$ and the orthogonality condition, meaning the model should give a high level of accuracy for other loading conditions.
Figure 2.3: Comparisons between various partition theories when $M_{1B} = 1 \text{ Nmm}$ and $M_{2B} = N_{1B} = N_{2B} = 0$.

It is identified that the minimum root mean square error possible is 0.004 and achieved using the present theory with $\overline{c}_\theta = 6/5$. As a result the present theory with $\overline{c}_\theta = 6/5$ is almost identical to Suo and Hutchinson’s partition theory over the entire range of $\gamma$ under consideration, i.e. $0.01 \leq \gamma \leq 100$, which represents most engineering applications.

When considering the other partition theories it is seen that the modification of Wang and Harvey’s approximate rule $1^{1,2}$ to consider new pure modes as a result of a negative ERR partition (Harvey and Wang’s approximate rule $2^2$) increases the accuracy of the model, however both show poor agreement when compared to Suo and Hutchinson’s partition theory. Finally Luo and Tong’s partition theory provides excellent agreement when compared to Suo and Hutchinson’s partition theory when $-0.5 \leq \log(1/\gamma) \leq 0.5$ but the accuracy of the model decreases outside of this range.
It is now vital to assess the accuracy of the present theory for other bending moment ratios. In Fig. 2.4 the same partition theories as Fig. 2.3 are compared, again for a laminated unidirectional composite DCB, however now 5 individual contour plots are used to show the error of the partition theory relative to Suo and Hutchinson’s theory\textsuperscript{21}. The bending moment ratio is now varied in the range $-20 \leq M_{2B}/M_{1B} \leq 20$ along the $x$-axis with $M_{1B} = 1$ Nmm and $N_{1B} = N_{2B} = 0$. This bending moment ratio was selected to show the models applicability to a wide range of practically useful loading conditions that would be expected in engineering applications. In reality the bending moment would be greater than $M_{1B} = 1$ Nmm, however as the bending moment ratio is of interest, increasing $M_{1B}$ will result in scaling the results and not affect the error between the theories, therefore $M_{1B} = 1$ Nmm was selected for convenience. The thickness ratio $\gamma$ was varied in the range $0.01 \leq \gamma \leq 100$ and plotted on the $y$-axis using a logarithmic scale for clarity. These are the only two parameters which affect the partition $G_i/G$. The partition error is obtained by first calculating $G_i/G$ from the respective theory and then subtracting the result for $G_i/G$ from Suo and Hutchinson’s theory\textsuperscript{21}. By taking the magnitude of the result, the contour plot offers a means of displaying the error by associating a different shade of colour to a certain error. In this case dark blue corresponds to zero error and red the maximum error which has been capped to 5% in order to make clearer comparisons as only one partition theory has an error greater than this and it is Wang and Harvey’s approximate rule\textsuperscript{1,2}; its maximum error is 10% and is in Fig. 2.4a. By identifying the colour that corresponds to the given bending moment ratio and thickness ratio under consideration, the error for the theory relative to Suo and Hutchinson’s\textsuperscript{21} can be obtained using the colour bar in the bottom right of Fig. 2.4.

Note that Fig. 2.3 represents a section through Fig. 2.4 at $M_{2B}/M_{1B} = 0$. The present theory with $\bar{c}_p = 8/7$ is given in Fig. 2.4c and it is almost identical to Suo and Hutchinson’s\textsuperscript{21} theory for the entire range of bending moment ratios considered when $0.1 \leq \gamma \leq 10$ and is slightly different when $\gamma < 0.1$ or $\gamma > 10$, giving a maximum error of 1.6%. These areas of increased error can be seen by again considering Fig. 2.3 and noting that the present theory with $\bar{c}_p = 8/7$ begins to separate from Suo and
Hutchinson’s\textsuperscript{21} theory when $\gamma < 0.1$ or $\gamma > 10$. Fig. 2.4d shows the error when the value of $\bar{c}_p$ is modified to $\bar{c}_p = 6/5$ and the theory is almost identical to Suo and Hutchinson’s\textsuperscript{21} theory over the entire range of $\gamma$ and $M_{2b}/M_{1b}$ under consideration due to the dark blue nature of the plot showing a maximum of error of 0.7%. This is due to the fact that when $\bar{c}_p = 6/5$, the correction factor provides an improved representation of the beam curvatures when considering 2D elasticity meaning that Eq. (2.47) represents the pure mode I condition based on 2D elasticity and therefore provides a more accurate pure mode I $\theta_{l-2D}$. Although the result for $\bar{c}_p = 8/7$ shows excellent agreement with Suo and Hutchinson’s theory\textsuperscript{21}, it is essential that the pure mode I $\theta_{l-2D}$ is as accurate as possible. Even with a small error in $\theta_{l-2D}$ as the other pure modes are obtained using this pure mode and the orthogonality condition then the errors can be compounded for other loading conditions such as the inclusion of axial forces.

When comparing the error for Wang and Harvey’s approximate rule $1^{1,2}$ and Harvey and Wang’s approximate rule $2^{2}$ relative to Suo and Hutchinson’s theory\textsuperscript{21} in Figs. 2.4a and 2.4b respectively, it is seen that the modification given by Harvey and Wang’s approximate rule $2^{2}$ increases the accuracy of the theory, decreasing the maximum error from 10% to 2.6%. Finally, Luo and Tong’s\textsuperscript{62} partition theory gives excellent agreement in the region $-1 \leq \log(1/\gamma) \leq 1$, however the accuracy of the theory decreases rapidly for higher and lower thickness ratios i.e. $\gamma > 10$ and $\gamma < 0.1$, giving a maximum error of 4.9%. 
Figure 2.4: Magnitude of $G_1/G$ error of various partition theories relative to Suo and Hutchinson’s theory with $0.01 \leq \gamma \leq 100$, $-20 \leq M_{2B}/M_{1B} \leq 20$,

\[ N_{1B} = N_{2B} = 0 \text{ and } M_{1B} = 1 \text{ Nmm}. \]
2.5.2. DCB with tip bending moments and axial forces

The second test performed to validate the new partition theory can be seen in Fig. 2.5 and considered both the use of bending moments and axial forces by varying the loading ratio \( \frac{N_{1B}}{M_{1B}} \) in the range \(-20 \leq \frac{N_{1B}}{M_{1B}} \leq 20\) with \(M_{1B} = 1\, \text{Nmm}\) and \(M_{2B} = N_{2B} = 0\). The same five partition theories are again compared against that of Suo and Hutchinson’s\(^{21}\) theory for a laminated unidirectional composite DCB. Again 5 individual contour plots are used with the loading ratio varied along the x-axis and the thickness ratio \(\gamma\) varied in the range \(0.01 \leq \gamma \leq 100\) and plotted on the y-axis using a logarithmic scale for clarity. Both the loading ratio and thickness ratio ranges were selected to show the models suitability for typical values that would be expected in engineering applications. These are the only two parameters which affect the partition \(G_{I}/G\). The partition error is obtained by first calculating \(G_{I}/G\) from the respective theory and then subtracting the result for \(G_{I}/G\) from Suo and Hutchinson’s theory\(^{21}\). By taking the magnitude of the result, the contour plot offers a means of displaying the error by associating a different shade of colour to a certain error. Similar to Fig. 2.4, dark blue corresponds to zero error and red the maximum error, which has been capped at 5% in order to make a clear comparison. Note that only the Wang and Harvey approximate rule\(^{1,2}\) in Fig. 2.5a has an error greater than 5%; its maximum error is 16%. By identifying the colour that corresponds to the given loading ratio and thickness ratio under consideration, the error for the theory relative to Suo and Hutchinson’s\(^{21}\) can be obtained using the colour bar in the bottom right of Fig. 2.5.

The present theory with \(\overline{\theta}_{\phi} = 8/7\) is given in Fig. 2.5c and it is almost identical to Suo and Hutchinson’s\(^{21}\) theory over the entire loading ratio \(\frac{N_{1B}}{M_{1B}}\) under consideration when \(0.1 \leq \gamma \leq 10\) and is slightly different when \(\gamma < 0.1\) or \(\gamma > 10\), giving a maximum error of 1.6%. Fig 2.5d shows the error when the values of \(\overline{\theta}_{\phi}\) is modified to \(\overline{\theta}_{\phi} = 6/5\) and the theory is almost identical to Suo and Hutchinson’s\(^{21}\) theory over the entire range of \(\gamma\) and \(\frac{N_{1B}}{M_{1B}}\) under consideration due to the dark blue nature of the plot showing a maximum of error of 0.7%. Again, this is due to the same reasons for the increased accuracy of the theory in Fig. 2.4d, namely that the increased accuracy of the pure mode I \(\theta_{1-2D}\), means that the accuracy of the axial force pure modes are increased.
Similar to Fig. 2.4, the modification of Wang and Harvey’s approximate rule 1\textsuperscript{1,2} seen in Fig. 2.5b (Harvey and Wang’s approximate rule 2\textsuperscript{2}) increases the accuracy of the model as the error between the theory and Suo and Hutchinson’s theory is reduced from 16\% to 2.6\%. Furthermore, Luo and Tong’s\textsuperscript{62} partition theory also gives excellent agreement in the same region as Fig. 2.4e i.e. \(-1 \leq \log(1/\gamma) \leq 1\), however the accuracy of the theory decreases rapidly for higher and lower thickness ratios i.e. \(\gamma > 10\) and \(\gamma < 0.1\), giving a maximum error of 4.9\%. 
Figure 2.5: Magnitude of $G_i/G$ error of various partition theories relative to Suo and Hutchinson’s theory with $0.01 \leq \gamma \leq 100$, $-20 \leq N_{1B}/M_{1B} \text{[mm}^{-1}] \leq 20$, $N_{2B} = M_{2B} = 0$ and $M_{1B} = 1 \text{Nmm}$.
2.6. Conclusion

This chapter has presented the development and validation for a new 2D-elasticity-based partition theory for mixed-mode fracture in laminated orthotropic composite DCBs with rigid interfaces under tip bending moments and axial forces.

Building on previous work by Wang and Harvey\textsuperscript{1–3,12,25} where Euler and Timoshenko beam theories have been used to determine orthogonal pure fracture modes, 2D elasticity theory is taken into account by introducing correction factors into the beam-theory-based mechanical conditions. As the crack tip and thus the influence region is located in the intact part of the DCB the correction factors have been introduced to the curvature of this section. Pure mode I and II conditions in 2D elasticity theory are obtained based on the 2D elasticity theory correction factors and the thickness ratio $\gamma$ of the DCB.

To determine the correction factors, the pure modes produced by Wang and Harvey\textsuperscript{1–3,12,25} when considering Euler and Timoshenko beam theories are used and it is assumed that these values act as the upper and lower limits of the 2D elasticity pure modes $\theta_{\text{2D}}$ and $\beta_{\text{2D}}$. From which it is possible to obtain the bounds of the two correction factors $c_{\theta}(\gamma)$ and $c_{\beta}(\gamma)$. The theory of orthogonality between pure modes is utilised in order to determine a relationship between the two correction factors. Finally taking the limits of the correction factors enables a formula for $c_{\theta}(\gamma)$ to be obtained by inspection which gives the correct value of $c_{\theta}(\gamma)$ for certain values of $\gamma$. Therefore, once $c_{\theta}(\gamma)$ is determined, it is possible to determine the other correction factor $c_{\beta}(\gamma)$, meaning the 2D-elasticity-based pure modes can then be used to partition a mixed-mode fracture.

The new 2D elasticity partition theory has been validated by comparison with Suo and Hutchinson’s\textsuperscript{21} partition theory, currently the most accurate when considering 2D elasticity theory. However, Suo and Hutchinson’s\textsuperscript{21} partition theory relies on an approximation for an isotropic DCB and extensively tabulated numerical results over a finite range of geometries and material configurations for a bimaterial DCB. Therefore it is thought that the current method developed in this chapter will have a stronger capability for solving more complex mixed-mode partition problems, for example the inclusion of crack tip through-thickness shear forces and the bimaterial case. This is due to the fact that the beam theory correction factor method presented here offers a
completely analytical means of obtaining the mode partition, meaning that once the bimaterial DCB correction factors are obtained it is anticipated the theory will partition a mixed-mode fracture for any combination of loading conditions, material properties and geometry.

The tests performed to validate the present theory include a DCB with tip bending moments only and a DCB with both tip bending moments and axial forces. It has been concluded that the present theory with $\tau_a = 6/5$ provides almost identical results to Suo and Hutchinson’s\textsuperscript{21} partition theory over the entire range of loading conditions and values of the thickness ratio $\gamma$. It is believed that this is a result of the correction factor correctly modifying the beam theory curvatures meaning they are more representative of the expected result based on 2D elasticity. Therefore, the pure fracture modes can be more accurately obtained, decreasing the error of the current theory.

The work in this chapter has been published in Harvey et al.\textsuperscript{16}. The theory presented in this section is for a laminated orthotropic composite DCB with rigid interfaces under tip bending moments and axial forces. In reality however, crack tip through-thickness shear forces will also be present. Therefore it is vital that the theory in this chapter for the fundamental case is extended to consider the increased complexity of crack tip through-thickness shear forces as well as bending moments and axial forces.
Chapter 3: Orthotropic laminated beams with bending moments, axial forces and through-thickness shear forces

3.1. Introduction

Chapter 2 has presented an analytical method for partitioning mixed-mode fractures in orthotropic laminated DCBs with rigid interfaces by taking 2D elasticity into consideration in a novel way by introducing correction factors into the beam theory based mechanical conditions. The DCBs are under crack tip bending moments and axial forces. This chapter extends the work to consider crack tip through-thickness shear forces which commonly occur in practice.

Some of the previous studies on the topic include the extension by Wang and Qiao of the conventional 2D elasticity-based partition theory of Suo and Hutchinson to take into account shear deformation by using first-order shear deformable plate theory for a bimaterial interfacial crack. To obtain the total ERR the J-integral was used and then related to the individual stress intensity factors by introducing an unknown parameter which, as the solution to an integral equation, must be determined from tabulated numerical data from a limited range of geometries and material configurations. At the same time however, Li et al. also extended the work of Suo and Hutchinson to consider the effects of transverse shear loading on a crack between a layered, isotropic, linear elastic material. It was concluded that it is not possible to use a higher-order beam theory to determine the shear component of the ERR as such an approach neglects the contribution of the deformation local to the crack-tip, which plays a crucial role. Therefore, this contradicts the work of Wang and Qiao. Consequently a full elastic solution was required by Li et al. and the FEM was used. Results were then combined with Suo and Hutchinson’s partition theory for bending moments and axial forces to provide the ERR and mode partition for layered materials under general loading conditions. A disadvantage of this method, again, is its reliance on tabulated data to obtain the mode mixity.

Other analytical work that has used first-order shear deformable beam theories to account for shear deformation, however are based on energy considerations are given by Wang and Harvey, Bruno and Greco, Luo and Tong and Valvo. Wang and Harvey identified that for an isotropic DCB with rigid interfaces, crack
tip through-thickness shear forces only contribute to the mode I component of the ERR. Furthermore, Luo and Tong\textsuperscript{62} added shear effects to their previously mentioned “global-local” approach using Schapery and Davidson’s\textsuperscript{46,70} modified VCCT for use with classical plate theory. Therefore, it is clearly evident from the conflicting information provided by analytical theories as to the effect of through-thickness shear forces on the fracture-mode partition of a DCB that it still remains a crucial area of research.

A numerical study on the effects of transverse shear for an orthotropic beam with a crack present was given by Lu et al.\textsuperscript{89} where the FEM was used. The analysis was simplified by using orthotropic rescaling so only three non-dimensional parameters were needed. Using the J-integral\textsuperscript{49} to obtain the total ERR, it was possible to use the crack tip displacements obtained from the FEM to partition the total ERR into its individual mode I and II components.

The work of Wang and Harvey\textsuperscript{1–3,12,25} based on classical beam theory has given excellent prediction of mixed-mode fracture toughness for delamination in generally laminated composite beams\textsuperscript{3,12,90} when compared to experimental test data from some independent comprehensive testing\textsuperscript{9–11,13–15,79,91–93}. In comparison, mixed-mode partition theories based on 2D elasticity\textsuperscript{16,21} have shown poor correlation with experimental results. It is still; however, an unanswered question as to which mixed-mode partition theory provides the most accurate results for brittle fracture subjected to other loading conditions, such as fatigue or thermal loading. Therefore, it is essential to develop mixed-mode partition theories based on 2D elasticity in order to provide a comprehensive set of tools for the understanding of interfacial fractures. This is the motivation for the present work.

The structure of this chapter is as follows. In Section 3.2 the previous mixed-mode partition theory for orthotropic beams with tip bending moments and axial forces is extended to include crack tip through-thickness shear forces. Section 3.3 presents the validation of the new theory against 2D FEM simulations for a number of loading conditions. Finally, conclusions are given in Section 3.4.
3.2. Analytical development

This section will now present the analytical development for the extension of the previous 2D elasticity partition theory for a DCB with tip bending moments and axial forces in Chapter 3 to now include through-thickness shear forces.

![Figure 3.1: A laminated DCB. (a) General description. (b) Details local to the crack tip.](image)

Fig. 3.1a shows a laminated DCB with its geometry and tip bending moments $M_1$ and $M_2$, axial forces $N_1$ and $N_2$, and through-thickness shear forces $P_1$ and $P_2$. The crack tip is located at point B. The coordinate system is located at the crack tip, with the x-direction positive to the left and y-direction positive vertically. Fig. 3.1b shows the internal loads acting at the crack tip where a subscript B denotes a crack tip quantity and the sign convention of the interface normal stress $\sigma_n$ and shear stress $\tau_s$ and does not take into account the actual distribution of these stresses.
3.2.1. Timoshenko partition theory for crack tip through-thickness shear forces only

Based on the work of Wang and Harvey\(^1\)\(^–\)\(^3\),\(^12\),\(^25\), the total ERR \(G_T\) under crack tip through-thickness shear forces, \(P_{1b}\) and \(P_{2b}\), can be calculated using Timoshenko beam theory as

\[
G_T = \frac{1}{2b^2h_1\kappa \mu_{Lz}} \left( P_{1b}^2 + \frac{P_{2b}^2}{\gamma} - \frac{(P_{1b} + P_{2b})^2}{1 + \gamma} \right) = \left[ P_{1b} \quad P_{2b} \right] C_T \left[ \begin{array}{l} P_{1b} \\ P_{2b} \end{array} \right]^T
\]

(3.1)

where

\[
C_T = \frac{1}{2b^2h_1\kappa \mu_{Lz}(1 + \gamma)} \begin{bmatrix}
\gamma & -1 \\
-1 & \gamma^{-1}
\end{bmatrix}
\]

(3.2)

and \(\gamma = h_2/h_1\) is the thickness ratio, \(b\) is the width of the beam and \(\mu_{Lz}\) is the through-thickness shear modulus with the correction factor \(\kappa\), usually taken to be \(5/6\) for a rectangular cross section. In the derivation of first-order shear deformable beam theory, the shear stress is assumed to be constant through the beam thickness. However, as this is not the case, in order to take this variation of shear stress into account the shear correction factor \(\kappa\) is introduced and determined based on the beams cross section\(^94\).

Now using an orthogonal pure mode methodology the total ERR \(G_T\) in Eq. (3.1) can be partitioned using Timoshenko beam theory as follows

\[
G_{T-T} = c_{I-T} \left( P_{1b} - \frac{P_{2b}}{\beta_{P,T}} \right)^2
\]

(3.3)

\[
G_{B-T} = c_{B-T} \left( P_{1b} - \frac{P_{2b}}{\theta_{P,T}} \right)^2
\]

(3.4)

where the ratios \((\theta_{P,T}, \beta_{P,T}) = (-1, \gamma)\) represent the set of orthogonal pure mode I and II conditions. The subscript \(P-T\) denotes that the pure modes are for the through-thickness shear forces at the crack tip, \(P_{1b}\) and \(P_{2b}\), and are based on Timoshenko beam theory. That is, when \(P_{2b} = \theta_{P,T} P_{1b}\) pure mode I occurs. Its orthogonal pure mode II is \(\beta_{P,T} P_{1b}\). Similarly, when \(P_{2b} = \beta_{P,T} P_{1b}\) the pure mode II mode occurs. Here, the ‘orthogonal’ means
Chapter 3: Orthotropic laminated beams with bending moments, axial forces and through-thickness shear forces

\[ \begin{bmatrix} 1 & \beta_{p-T} \end{bmatrix} \begin{bmatrix} C_T \end{bmatrix} \begin{bmatrix} 1 & \theta_{p-T} \end{bmatrix}^T = 0 \]  

(3.5)

The coefficients \( c_{I,T} \) and \( c_{II,T} \) are currently unknown and can be determined by considering the pure mode I and pure mode II conditions. Consider the pure mode I condition \( P_{2B} = \theta_{p-T} P_{1B} \) and equate Eqs. (3.1) and (3.3), then rearrange for \( c_{I,T} \). Similarly, consider the pure mode II conditions \( P_{2B} = \beta_{p-T} P_{1B} \) and equate Eqs. (3.1) and (3.4), then rearrange for \( c_{II,T} \). This gives

\[
c_{I,T} = G_{\theta_{p-T}} \left( 1 - \frac{\theta_{p-T}}{\beta_{p-T}} \right)^{-2} \quad \text{and} \quad c_{II,T} = G_{\beta_{p-T}} \left( 1 - \frac{\beta_{p-T}}{\theta_{p-T}} \right)^{-2}
\]

(3.6)

\[
G_{\theta_{p-T}} = \frac{1}{2b^2 h_1 k \mu_{LZ}} \left( 1 + \frac{\theta_{p-T}^2}{\gamma} - \frac{(1 + \theta_{p-T})^2}{1 + \gamma} \right)
\]

(3.7)

\[
G_{\beta_{p-T}} = \frac{1}{2b^2 h_1 k \mu_{LZ}} \left( 1 + \frac{\beta_{p-T}^2}{\gamma} - \frac{(1 + \beta_{p-T})^2}{1 + \gamma} \right)
\]

(3.8)

It is worth noting that \( G_{II,T} = 0 \) as \( G_{\beta_{p-T}} = 0 \) with \( \beta_{p-T} = \gamma \). That is, the two crack tip through-thickness shear forces \( P_{1B} \) and \( P_{2B} \), only produce mode I ERR within the context of the Timoshenko beam theory. Therefore, the Timoshenko beam partition theory will now be compared to 2D FEM results in order to assess the accuracy of the model.

3.2.2. FEM procedure

Before comparing the results for the ERR from the Timoshenko beam partition theory to the 2D FEM using the VCCT it is essential that the FEM procedure is stated. The numerical simulations have been carried out on the DCB shown in Fig. 3.1a using MSC/NASTRAN.

Hutchinson and Suo have previously shown that the ERR components of an orthotropic material are essentially the same as their isotropic counterparts, except using the longitudinal tensile modulus. For convenience, therefore, isotropic material constants were used, as follows and apply for all loading conditions and thickness ratios
considered in this chapter: The Young’s modulus $E = 1000 \text{ N/mm}^2$, the Poisson’s ratio \( \nu = 0.29 \), and the shear modulus $\mu = E/[2(1+\nu)]$. The Poisson’s ratio was selected as Ref.\textsuperscript{95} states that for most isotropic materials it is in the range $0.25 \leq \nu \leq 0.33$, meaning the middle value has been selected.

The DCB geometry consisted of the uncracked length $L = 100 \text{ mm}$, the cracked length $a = 10 \text{ mm}$, the width $b = 10 \text{ mm}$ and the minimum beam thickness $h_{\text{min}} = 1 \text{ mm}$. The thickness of the upper and lower beams were therefore dependent on the thickness ratio $\gamma = h_2/h_1$, with $h_1 = h_{\text{min}}$ and $h_2 = \gamma h_{\text{min}}$ if $\gamma > 1$, and with $h_1 = h_{\text{min}}/\gamma$ and $h_2 = h_{\text{min}}$ if $\gamma < 1$.

The FEM meshing procedure will now be summarised. Since very fine meshes were required at the crack tip in order to obtain converged numerical solutions, non-uniform meshes were used in order to avoid excessive computation. In the x-direction 2000 square elements of size $0.001 \times 0.001$ were centred on the crack tip and 100 square elements of the same size were centred on the crack tip in the y-direction. This can be seen in Fig. 3.2 where this area of high element density is represented by the solid black region around the crack tip. This element size was found to provide mesh independence and was used for all simulations in this section. A convergence study for this section can be found in the Appendix A. Beyond the region of uniform element size surrounding the crack tip, elements were allowed to grow at a constant rate of 1.1 in both the x- and y-directions up to maximum sizes of 1.0 and 0.1 respectively, after which they remained at these maximum sizes. Very small adjustments were made to the element size growth rate, or to the maximum element size, as appropriate, to satisfy the boundary geometry. As the total length of the DCB was a constant the number of elements in the x-direction remained constant and the number in the y-direction was dependent on the thickness ratio $\gamma$. This gave a maximum mesh density of $2233 \times 285$ when $\gamma = 10$ or $\gamma = 0.1$ and a minimum mesh density of $2233 \times 196$ when $\gamma = 1$. 
Fig. 3.2 shows how the application of positive bending moments, axial forces and shear forces were applied to the FEM model with a uniform mesh. Axial forces, $N_1$ and $N_2$, and shear forces, $P_1$ and $P_2$, were applied as point forces to the tips of the upper and lower beams respectively. For a non-uniform mesh, it is essential to uniformly-distribute the axial and shear forces by the element area. Bending moments, $M_1$ and $M_2$, were applied as equal and opposite axial forces to the top and bottom corners of each of the upper and lower beam tips respectively.

The upper and lower beams were modelled using quadrilateral plane-strain shell elements. A rigid interface between the upper and lower beams was modelled by ‘connecting’ the translational degrees of freedom of co-located interface nodes on the upper and lower beams using multi-point constraints; however, at the crack tip, instead of rigidly connecting the crack tip nodes, the interface was modelled with normal and
shear point springs. The stiffness of both of the springs was selected using 
\( k_s = \frac{E_{CT}A_{CT}}{L} \) where \( A_{CT} \) is the element area at the crack tip calculated as \( A_{CT} = b p \) where \( b \) is the DCB width and \( p \) the element pitch. The spring length was selected to be a unit value, however in reality the interface has zero thickness. Finally, 
\( E_{CT} = 10^{10} \text{ N/mm}^2 \), which is the Young’s modulus of the interface at the crack tip. Therefore, the springs stiffness \( k_s \) was sufficiently high in comparison to \( E \) to simulate brittle interfacial cracking without introducing excessive numerical error and calculated for the above case as \( k_s = 1 \times 10^8 \text{ N/mm} \). As a DCB is being modelled, the boundary conditions were the same as that for a cantilever beam, therefore at the fixed support the translational and rotational degrees of freedom were restricted. The ERRs were calculated using the VCCT and the forces from the crack tip springs\(^{68,96}\). Contact between the upper and lower surfaces of the crack was not considered.

3.2.3. Comparison of Timoshenko beam partition theory and 2D FEM results

A comparison of results between the Timoshenko beam partition theory and that obtained using 2D FEM simulations will now be given. The material properties given in Section 3.2.2 were used and \( \gamma = 1 \). The loading condition selected involved varying \( P_{2B} \) in the range \(-10,000 \leq P_{2B} [\text{N}] \leq 10,000\) and keeping \( P_{1B} \) constant as \( P_{1B} = 1000 \text{ N} \) with \( M_{1B} = M_{2B} = N_{1B} = N_{2B} = 0 \). These values were selected to give a wide range of values which would commonly be seen in real life engineering applications of the DCB. To achieve this loading condition using the FEM, DCB tip through-thickness shear forces were applied as \( P_1 = P_{1B} \) and \( P_2 = P_{2B} \). To eliminate the crack tip bending moments produced by applying the through-thickness shear forces to the tip of the DCB in the FEM model, tip bending moments were also applied as \( M_1 = -P_1 a \) and \( M_2 = -P_2 a \). The results for this test are given in Table 3.1.

From Table 3.1 the following points can be identified. The first of which is when comparing values of total ERR obtained using the FEM simulations \( G \) and Timoshenko beam partition theory \( G_T \). It can be seen that the Timoshenko beam partition theory under predicts the value of the total ERR for all values of the loading ratio \( P_{2B}/P_{1B} \). For
example, when $P_{2B}/P_{1B} = -10$, the total ERR produced from the FEM $G = 1578.8\, J/mm^2$ is 68% greater than that calculated using the Timoshenko beam partition theory $G_T = 939.5\, J/mm^2$. It is believed that this is due to the constant through-thickness shear correction factor $\kappa = 5/6$ no longer being valid for the 2D elasticity solution as an improved representation of the shear stress through the beam thickness is obtained.

Another reason for this difference between the total ERR obtained in Table 3.1 could be due to the Timoshenko beam partition theory calculating $G_{II-T} = 0$ for all values of the loading ratio $P_{2B}/P_{1B}$, whereas the 2D FEM results calculated a mode II component of the ERR $G_{II}$ for all values of the loading ratio $P_{2B}/P_{1B}$ except for the pure mode I condition when $P_{2B}/P_{1B} = -1$.

Therefore, in order to provide the most accurate comparison between the theory and FEM results the pure mode I condition will be now considered i.e. $P_{2B}/P_{1B} = -1$. From which the FEM gives $G_I = 49.8\, J/mm^2$, whereas the Timoshenko beam partition theory gives $G_{I-T} = 30.96\, J/mm^2$, meaning the FEM result is 61% greater. Similarly to the case for the total ERR, the mode I component of the ERR is also under predicted by the Timoshenko beam partition theory $G_{I-T}$ when compared to the 2D FEM result $G_I$ for all values of the loading ratio. This reinforces the previous statement about the constant through-thickness shear correction factor $\kappa = 5/6$ no longer being valid for the 2D elasticity solution.
Table 3.1: Comparison of 2D FEM against Timoshenko beam partition theory for \( \gamma = 1 \).

<table>
<thead>
<tr>
<th>( \frac{P_{1B}}{P_{2B}} )</th>
<th>2D FEM ( J/\text{mm}^2 )</th>
<th>Timoshenko beam theory ( J/\text{mm}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{1B}/P_{2B} )</td>
<td>( G_I )</td>
<td>( G_{II} )</td>
</tr>
<tr>
<td>-10</td>
<td>1505.50</td>
<td>73.32</td>
</tr>
<tr>
<td>-9</td>
<td>1244.60</td>
<td>57.88</td>
</tr>
<tr>
<td>-8</td>
<td>1007.82</td>
<td>44.35</td>
</tr>
<tr>
<td>-7</td>
<td>796.31</td>
<td>32.59</td>
</tr>
<tr>
<td>-6</td>
<td>609.70</td>
<td>22.63</td>
</tr>
<tr>
<td>-5</td>
<td>447.95</td>
<td>14.48</td>
</tr>
<tr>
<td>-4</td>
<td>311.17</td>
<td>8.14</td>
</tr>
<tr>
<td>-3</td>
<td>199.11</td>
<td>3.62</td>
</tr>
<tr>
<td>-2</td>
<td>112.01</td>
<td>0.91</td>
</tr>
<tr>
<td>-1</td>
<td>49.79</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>12.45</td>
<td>0.91</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3.62</td>
</tr>
<tr>
<td>2</td>
<td>12.43</td>
<td>8.15</td>
</tr>
<tr>
<td>3</td>
<td>49.74</td>
<td>14.49</td>
</tr>
<tr>
<td>4</td>
<td>111.93</td>
<td>22.64</td>
</tr>
<tr>
<td>5</td>
<td>199.00</td>
<td>32.54</td>
</tr>
<tr>
<td>6</td>
<td>310.97</td>
<td>44.37</td>
</tr>
<tr>
<td>7</td>
<td>447.81</td>
<td>57.95</td>
</tr>
<tr>
<td>8</td>
<td>609.54</td>
<td>73.34</td>
</tr>
<tr>
<td>9</td>
<td>796.14</td>
<td>90.54</td>
</tr>
<tr>
<td>10</td>
<td>1007.65</td>
<td>109.56</td>
</tr>
</tbody>
</table>

3.2.4. 2D elasticity partition theory for crack tip through-thickness shear forces only

Therefore the Timoshenko beam partition theory will now be modified in order to make it applicable to 2D elasticity. It has previously been identified from Table 3.1 that when considering 2D elasticity theory for two crack tip through-thickness shear forces \( P_{1B} \) and \( P_{2B} \), unlike the Timoshenko beam partition theory, they contribute to both the mode I and mode II ERR. Furthermore, the total ERR \( G \) was found to be different from \( G_T \) in Eq. (3.1) as the constant through-thickness shear correction factor \( \kappa = 5/6 \) is no longer valid. Moreover, the orthogonal pure mode set \( (\Theta_{p-T}, \beta_{p-T}) = (-1, \gamma) \) will also
change to be \((\theta_{p,2D}, \beta_{p,2D})\). Therefore the ERR partitions based on 2D elasticity, \(G_I\) and \(G_H\), can be written as

\[
G_I = c_{IP}\left(P_{1B} - \frac{P_{2B}}{\beta_{p,2D}}\right)^2
\]  
(3.9)

\[
G_H = c_{II}\left(P_{1B} - \frac{P_{2B}}{\theta_{p,2D}}\right)^2
\]  
(3.10)

\[
c_{IP} = G_{\theta_{p,2D}} \left(1 - \frac{\theta_{p,2D}}{\beta_{p,2D}}\right)^2
\]  
(3.11)

and

\[
c_{II} = G_{\beta_{p,2D}} \left(1 - \frac{\beta_{p,2D}}{\theta_{p,2D}}\right)^2
\]

\[
G_{\theta_{p,2D}} = \frac{1}{2b^2hE\kappa(\gamma)}\left(1 + \frac{\theta_{p,2D}^2}{\gamma}\right)
\]  
(3.12)

\[
G_{\beta_{p,2D}} = \frac{1}{2b^2hE\kappa(\gamma)}\left(1 + \frac{\beta_{p,2D}^2}{\gamma} - \frac{(1 + \beta_{p,2D})^2}{1 + \gamma}c(\gamma)\right)
\]  
(3.13)

and

\[\overline{E} = E_L\]  
for plane stress or

\[\overline{E} = E_L/(1 - \nu_{LT}\nu_{TL})\]  
for plane strain, with \(E_L\) being the Young’s modulus in the longitudinal direction, and \(\nu_{LT}\) and \(\nu_{TL}\) being the in-plane Poisson’s ratios. It is worthwhile to note that the influence of the material properties on the ERR is collectively shown by one effective property \(\overline{E}\). Notice that in Eqs. (3.12) and (3.13) the shear correction factor \(\kappa(\gamma)\) is now dependent on the thickness ratio \(\gamma\).

Furthermore, a pure-mode-II ERR correction factor \(c(\gamma)\) has been introduced in Eq. (3.13). Therefore it is now essential to determine the crack tip through-thickness pure modes \((\theta_{p,2D}, \beta_{p,2D})\), the shear correction factor \(\kappa(\gamma)\) and the pure mode II correction factor \(c(\gamma)\) using the FEM in order to partition the total ERR based on 2D elasticity.

### 3.2.5. 2D elasticity crack tip through-thickness shear force pure modes

To determine the crack tip through-thickness shear force pure modes \((\theta_{p,2D}, \beta_{p,2D})\) the FEM is used with the material and geometric properties stated in Section 3.2.2. By determining the loading conditions that produce the pure mode I condition i.e. \(G_H = 0\)
and pure mode II condition i.e. $G_t = 0$ for different values of the thickness ratio $\gamma$, it is possible to obtain an expression for the crack tip through-thickness pure modes. Considering the pure mode I condition $\theta_{p,2D}$, it is identified that

$$
\theta_{p,2D} = -1
$$

(3.14)

Note that the $\theta_{p,2D}$ pure mode remains the same as $\theta_{p,T} = -1$. Furthermore, this is the reason why a pure-mode-I ERR correction factor is not required in Eq. (3.12) as the pure mode cancels out the correction factor and therefore reduces the equation. Now consider the pure mode II condition $\beta_{p,2D}$ and the variation with respect to the thickness ratio $\gamma$ can be seen in Fig. 3.4. The thickness ratio is varied along the x-axis using a logarithmic scale for clarity and varied in the range $-2 \leq \log_{10}(1/\gamma) \leq 2$ with the pure mode II condition $\beta_{p,2D}$ on the y-axis again using a logarithmic scale as the values for very large and very small thickness ratios skew the data. Therefore Fig. 3.4 shows the values of $\beta_{p,2D}$ i.e. the ratio of crack tip through-thickness shear forces $P_{2B}/P_{1B}$ required for a given thickness ratio that will return $G_t = 0$. 
Chapter 3: Orthotropic laminated beams with bending moments, axial forces and through-thickness shear forces

Figure 3.4: Variation of the 2D-elasticity-theory-based pure mode II $\beta_{p,2D}$ for through-thickness shear forces only from Eq. (3.15), an approximate method and 2D FEM simulations with respect to $\gamma$.

From the results in Fig. 3.4 it is possible to obtain the following analytical expression for the pure mode II $\beta_{p,2D}$ by fitting a line to the FEM results and Eq. (3.15) has also been plotted on the figure.

$$\beta_{p,2D} = \gamma \exp(-1.986\text{atanh}(0.5635\gamma_i)) \quad (3.15)$$

where $\gamma_i = \log_{10}(1/\gamma)$. Note that the $\beta_{p,2D}$ is different to that in Timoshenko beam theory $\beta_{p,T} = \gamma$ and now has an exponential multiplier. This is due to the 2D elasticity solution providing a more accurate model of the stress field at the crack tip when compared to the beam theory approximations. An accurate approximation for pure-mode-II $\beta_{p,2D}$ is also given on the graph which is $\beta_{p,2D} \approx \gamma^{1.5}$ if $\gamma \approx 1$. At $\gamma = 3$ or $\gamma = 1/3$ it approximates $\beta_{p,2D}$ to within about 1% of the actual value. Inside this range the accuracy is much higher; however, outside it the accuracy decreases rapidly and Eq. (3.15) should be used instead. It is easily shown that $\beta_{p,2D}(1/\gamma) = 1/\beta_{p,2D}(\gamma)$ as required.
by physical symmetry of the DCB. Unfortunately, the pure-mode-II \( \beta_{p,2D} \) mode, is only valid in the range where \(-1.7 \leq \gamma \leq 1.7\). It was not possible to determine \( \beta_{p,2D} \) using the FEM outside this range due to the very large meshes needed to model the DCB and it is not clear as to whether \( \beta_{p,2D} \) will continue to follow the trend given by Eq. (3.15). It is vitally important to see what effect changing the material properties have on the pure mode \( \beta_{p,2D} \), therefore the value of the Young’s modulus \( E \) has been adjusted and results for the pure mode II \( \beta_{p,2D} \) are given for this new value of Young’s modulus \( E \) in Table 3.2. It can be seen that changing the Young’s modulus has no effect on the pure mode II \( \beta_{p,2D} \), i.e. it is independent of material properties and only depends on the DCB geometry.

Table 3.2: Effect of changing the Young’s modulus \( E \) on the pure mode II \( \beta_{p,2D} \).

<table>
<thead>
<tr>
<th>Young’s modulus, ( E ) [J/mm(^2)]</th>
<th>( \log_{10}(1/\gamma) )</th>
<th>Pure mode II, ( \beta_{p,2D} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.1</td>
<td>1.41</td>
<td></td>
</tr>
<tr>
<td>-0.3</td>
<td>2.81</td>
<td></td>
</tr>
<tr>
<td>-0.5</td>
<td>5.69</td>
<td></td>
</tr>
<tr>
<td>-1.0</td>
<td>36.43</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.1</td>
<td>1.41</td>
<td></td>
</tr>
<tr>
<td>-0.3</td>
<td>2.81</td>
<td></td>
</tr>
<tr>
<td>-0.5</td>
<td>5.69</td>
<td></td>
</tr>
<tr>
<td>-1.0</td>
<td>36.43</td>
<td></td>
</tr>
<tr>
<td>10000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.1</td>
<td>1.48</td>
<td></td>
</tr>
<tr>
<td>-0.3</td>
<td>2.81</td>
<td></td>
</tr>
<tr>
<td>-0.5</td>
<td>5.69</td>
<td></td>
</tr>
<tr>
<td>-1.0</td>
<td>36.43</td>
<td></td>
</tr>
</tbody>
</table>

3.2.6. *Thickness-ratio-dependent shear correction factor*

To determine the shear correction factor \( \kappa(\gamma) \) the pure mode I condition \( P_{2B} = \theta_{p,2D} P_{1B} \) i.e. \( P_{2B} = -P_{1B} \) is implemented in the FEM with the material and geometric properties stated in Section 3.2.2. By varying the thickness ratio \( \gamma \) it is possible to numerically determine the mode I component of the ERR \( G_i \). From which it is then possible to use Eqs. (3.9), (3.11) and (3.12) with the numerically determined
value of $G_i$ to obtain the shear correction factor $\kappa(\gamma)$. Fig. 3.5 shows how the shear correction factor $\kappa(\gamma)$ varies with the thickness ratio $\gamma$. The thickness ratio is varied along the $x$-axis using a logarithmic scale for clarity and varied in the range $-2 \leq \log_{10}(1/\gamma) \leq 2$ with the shear correction factor $\kappa(\gamma)$ on the $y$-axis.

Figure 3.5: Variation of the shear correction factor $\kappa(\gamma)$ from Eq. (3.16) and 2D FEM simulations with respect to the thickness ratio $\gamma$.

From the results in Fig. 3.5 it is possible to obtain the following analytical expression for the shear correction factor $\kappa(\gamma)$ by fitting a line to the FEM results and Eq. (3.16) has also been plotted on the figure.

$$\kappa(\gamma) = 0.1355 + 0.0477 \exp(-1.39\gamma_i^2)$$  (3.16)

It is simple to prove that $\kappa(1/\gamma) = \kappa(\gamma)$. It is interesting to see that $\kappa(\gamma)$ has a perfect normal distribution form with respect to $\gamma_i$. Again, as the pure mode II $\beta_{p,2D}$ is only known in the range $-1.7 \leq \gamma_i \leq 1.7$, it is not possible to determine the shear correction factor outside of this range. It is vitally important to see what effect changing the
material properties have on the shear correction $\kappa(\gamma)$, therefore the value of the Young’s modulus $E$ has been adjusted and results for the shear correction factor $\kappa(\gamma)$ are given for this new value of Young’s modulus $E$ in Table 3.3. It can be seen that changing the Young’s modulus has no effect on the shear correction factor $\kappa(\gamma)$, i.e. it is independent of material properties and only depends on the DCB geometry.

Table 3.3: Effect of changing the Young’s modulus $E$ on the shear correction factor $\kappa(\gamma)$.

<table>
<thead>
<tr>
<th>Young’s modulus, $E$ [J/mm$^2$]</th>
<th>$\log_{10}(1/\gamma)$</th>
<th>Shear correction factor, $\kappa(\gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>-0.1</td>
<td>0.183</td>
</tr>
<tr>
<td></td>
<td>-0.3</td>
<td>0.178</td>
</tr>
<tr>
<td></td>
<td>-0.5</td>
<td>0.169</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>0.148</td>
</tr>
<tr>
<td>2000</td>
<td>-0.1</td>
<td>0.183</td>
</tr>
<tr>
<td></td>
<td>-0.3</td>
<td>0.178</td>
</tr>
<tr>
<td></td>
<td>-0.5</td>
<td>0.169</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>0.148</td>
</tr>
<tr>
<td>10000</td>
<td>-0.1</td>
<td>0.183</td>
</tr>
<tr>
<td></td>
<td>-0.3</td>
<td>0.178</td>
</tr>
<tr>
<td></td>
<td>-0.5</td>
<td>0.169</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>0.148</td>
</tr>
</tbody>
</table>

3.2.7. Pure-mode-II ERR correction factor

To determine the pure-mode-II ERR correction factor $c(\gamma)$, the pure mode II condition $P_{2\theta} = \beta_{P_{2\theta}} P_{1\theta}$ is implemented in the FEM with the material and geometric properties stated in Section 3.2.2. By varying the thickness ratio $\gamma$ it is possible to numerically determine the mode II component of the ERR $G_{II}$. From which it is then possible to use Eqs. (3.10), (3.11) and (3.13) with the numerically determined value of $G_{II}$ and $\kappa(\gamma)$ from Eq. (3.16) to obtain the pure-mode-II ERR correction factor $c(\gamma)$. Fig. 3.6 shows how the pure-mode-II correction factor $c(\gamma)$ varies with the thickness ratio $\gamma$. The thickness ratio is varied along the x-axis using a logarithmic scale for
clarity and varied in the range $-2 \leq \log_{10}(1/\gamma) \leq 2$ with the pure-mode-II ERR correction factor $c(\gamma)$ on the $y$-axis.

![Graph showing variation of the pure-mode-II ERR correction factor](image)

Figure 3.6: Variation of the pure-mode-II ERR correction factor $c(\gamma)$ from Eq. (3.17) and the 2D FEM simulations with respect to the thickness ratio $\gamma$.

From the results in Fig. 3.6 it is possible to obtain the following analytical expression for the pure-mode-II ERR correction factor $c(\gamma)$ and Eq. (3.17) has been plotted on the figure.

$$
c(\gamma) = \frac{(1 + \beta_{p.2D}^2 / \gamma)(1 + \gamma)}{(1 + \beta_{p.2D})^2} C_F
$$

(3.17)

$$
C_F = 1 - 0.071 \exp(-3.28\gamma_i^2)
$$

(3.18)

It is simple to prove that $c(1/\gamma) = c(\gamma)$, as required by the physical condition $G_{p.2D}(1/\gamma) = \left(\beta_{p.2D}(\gamma)\right)^2 G_{p.2D}(\gamma)$. Due to the limitations on the pure mode II $\beta_{p.2D}$ it is not possible to determine the pure-mode-II ERR correction factor $c(\gamma)$ outside of the
range \(-1.7 \leq \gamma, \leq 1.7\). Finally, it is important to identify whether the pure-mode-II ERR correction factor \(c(\gamma)\) is independent of material properties. Therefore, the values of the Young’s modulus \(E\) has been adjusted and results for the pure-mode-II ERR correction factor \(c(\gamma)\) are given for this new value of Young’s modulus \(E\) in Table 3.4. In which three values of the Young’s modulus are tested for different values of the thickness ratio \(\gamma\) and it is identified that the Young’s modulus has no effect on the pure-mode-II ERR correction factor \(c(\gamma)\), i.e. it is independent of material properties and only depends on the DCB geometry.

Table 3.4: Effect of changing the Young’s modulus \(E\) on the pure-mode-II ERR correction factor \(c(\gamma)\).

<table>
<thead>
<tr>
<th>Young’s modulus, (E) [J/mm(^2)]</th>
<th>(\log_{10}(1/\gamma))</th>
<th>Pure-mode-II ERR Correction Factor, (c(\gamma))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>-0.1</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>-0.3</td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>-0.5</td>
<td>1.01</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>1.04</td>
</tr>
<tr>
<td>2000</td>
<td>-0.1</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>-0.3</td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>-0.5</td>
<td>1.01</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>1.04</td>
</tr>
<tr>
<td>10000</td>
<td>-0.1</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>-0.3</td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>-0.5</td>
<td>1.01</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>1.04</td>
</tr>
</tbody>
</table>

3.2.8. 2D elasticity partition theory for general loads

Table 3.2–3.4 have shown that the pure modes \((\theta_{p,2D}, \beta_{p,2D})\), shear correction factor \(\kappa(\gamma)\) and pure-mode-II ERR correction factor \(c(\gamma)\) are independent on material properties. Therefore, the ERR can now be partitioned for any value of the crack tip through-thickness shear forces, any configuration of orthotropic material properties and the thickness ratio \(\gamma\) in the range \(-1.7 \leq \gamma, \leq 1.7\). The theory can now be incorporated
Chapter 3: Orthotropic laminated beams with bending moments, axial forces and through-thickness shear forces

with the previous work in Chapter 2 to partition the total ERR under any combination of bending moments, axial forces and through-thickness shear forces, for any material properties when the thickness ratio $\gamma$ is in the range $-1.7 \leq \gamma \leq 1.7$. Obviously when considering bending moments and/or axial forces only, the constraint on the thickness ratio is removed and the partition theory will provide accurate results for any combination of loading, orthotropic material properties and beam thicknesses.

Therefore the ERR partitions for bending moments, axial forces and crack tip through-thickness shear forces become

$$G_I = c_I \left( M_{1B} - \frac{M_{2B}}{\beta_{1,2D}} - \frac{N_{1Be}}{\beta_{2,2D}} - \frac{P_{1B}}{\beta_{3,2D}} - \frac{P_{2B}}{\beta_{4,2D}} \right)^2$$

(3.19)

$$G_{II} = c_{II} \left( M_{1B} - \frac{M_{2B}}{\theta_{1,2D}} - \frac{N_{1Be}}{\theta_{2,2D}} - \frac{P_{1B}}{\theta_{3,2D}} - \frac{P_{2B}}{\theta_{4,2D}} \right)^2$$

(3.20)

where

$$c_I = G_{\theta_{2D}} \left( 1 - \frac{\theta_{1,2D}}{\beta_{1,2D}} \right)^2 \quad \text{and} \quad c_{II} = G_{\beta_{2D}} \left( 1 - \frac{\beta_{1,2D}}{\theta_{1,2D}} \right)^2$$

(3.21)

$$G_{\theta_{2D}} = \frac{6}{b^2 h_1^2 E} \left( 1 + \frac{\theta_{1,2D}^2}{\gamma^3} - \frac{(1 + \theta_{1,2D})^2}{(1 + \gamma)^3} \right)$$

(3.22)

$$G_{\beta_{2D}} = \frac{6}{b^2 h_1^2 E} \left( 1 + \frac{\beta_{1,2D}^2}{\gamma^3} - \frac{(1 + \beta_{1,2D})^2}{(1 + \gamma)^3} \right)$$

(3.23)

where $N_{1Be} = N_{1B} - N_{2B}/\gamma$. The pure modes $\theta_{1,2D}$ and $\beta_{1,2D}$, have been determined in the previous chapter by introducing correction factors into the beam-theory-based pure-mode-I and pure-mode-II mechanical conditions. Then the pure-mode-I $\theta_{1,2D}$ mode and the pure-mode-II $\beta_{2,2D}$ mode were obtained by using the orthogonality condition that exists between pure modes. The previously determined pure modes are now repeated below

$$\theta_{1,2D} = -\gamma^2 - \frac{\theta_{1,2D}^2 \gamma^2 (1 - \gamma^2)}{\theta_{1,2D}^2 \gamma^2 + \phi (1 + \gamma)^2}$$

(3.24)
\[
\beta_{1,2D} = \frac{\gamma^2(3 + \gamma)}{1 + 3\gamma} - \frac{3\lambda^2\gamma^2(1 - \gamma^2)}{(1 + 3\gamma)[\lambda^2 + \phi(1 + 3\gamma)]} \quad (3.25)
\]

\[
\theta_{2,2D} = -\frac{6}{h_1} - \frac{6\lambda^2\gamma(1 - \gamma)}{h_1[\lambda^2(1 - \gamma + \gamma^2) + \phi(1 + \gamma)^2]} \quad (3.26)
\]

\[
\beta_{2,2D} = \begin{cases} 
\frac{2(3 + \gamma)}{h_1(\gamma - 1)} + \frac{2\lambda^2\gamma}{h_1\phi(\gamma - 1)} & \text{if } \gamma \neq 1 \\
\infty & \text{if } \gamma = 1 
\end{cases} \quad (3.27)
\]

where

\[
\overline{\lambda} = 6/5 \quad \text{and} \quad \phi = \overline{\lambda}^{[\gamma + (1 + \gamma)]/2} \quad (3.28)
\]

In Eqs. (3.24)–(3.27), the first term is the corresponding pure mode from Timoshenko beam theory and the second term is the correction applied for 2D elasticity.

It is now necessary to obtain the pure-mode-I modes, \(\theta_{3,2D}\) and \(\theta_{4,2D}\), and the pure-mode-II modes, \(\beta_{3,2D}\) and \(\beta_{4,2D}\) in order to partition the total ERR \(G\) for general loading conditions. Consider the pure-mode-I \(\theta_{P,2D}\) mode where \(P_{2\theta} = \theta_{P,2D}P_{1\theta}\) and equating the ERR \(G\) in Eqs. (3.9) and (3.19) gives

\[
\beta_{3,2D} = \frac{\beta_{1,2D}(1 + \beta_{P,2D})}{\beta_{P,2D}(\theta_{1,2D} - \beta_{1,2D}(G_{\theta_{3,2D}}/G_{\theta_{2,2D}})^{1/2})} \quad (3.29)
\]

Note that in the case of Timoshenko beams, \(\beta_{3,T} = -(3 + \gamma)(3\kappa \mu / E)^{1/2}/[h_1(1 + \gamma)]\) where \(\kappa = 5/6\). Now consider the pure-mode-II \(\beta_{P,2D}\) mode where \(P_{2\beta} = \beta_{P,2D}P_{1\beta}\) and equating Eqs. (3.9) and (3.19) gives

\[
\beta_{4,2D} = -\beta_{P,2D}\beta_{3,2D} \quad (3.30)
\]

Note that in the case of Timoshenko beams, \(\beta_{4,T} = -\gamma \beta_{3,T}\).

Similarly, consider the pure-mode-II \(\beta_{P,2D}\) mode where \(P_{2\beta} = \beta_{P,2D}P_{1\beta}\) and equating Eqs. (3.10) and (3.20) gives
Finally, consider the pure-mode-I $\theta_{p,2D}$ mode where $P_{2B} = \theta_{p,2D} P_{1B}$ and equating Eqs. (3.10) and (3.20) gives

$$\theta_{4,2D} = -\theta_{p,2D} \theta_{3,2D}$$

(3.32)

Note that $\theta_{3-T} = \theta_{4-T} = -\infty$. Again, it is worth noting that in the context of 2D elasticity, the influence of the material properties on the ERR is collectively shown by one effective property $\bar{E}$ and that the pure modes are affected only by the geometry. This is in agreement with Hutchinson and Suo.

Therefore, as all pure modes are now known, it is possible to partition the ERR into its individual mode components for any combination of bending moments, axial forces and shear forces by using Eqs. (3.19) and (3.20). It is thought that the present analytical theory will be a valuable tool in the design phase of composite materials as it will drastically reduce the amount of time required to analyse a configuration when compared to the FEM. Furthermore, the FEM is computationally expensive and can be prone to errors if not operated by an experienced engineer.

### 3.3. Numerical verification

To verify the present analytical partition theory, 2D FEM simulations were carried out on the DCB shown in Fig. 3.1a using MSC/NASTRAN. The FEM model is the same as that presented in Section 3.2.2 and the material properties are the same unless explicitly stated, therefore readers are directed to Section 3.2.2 for further details. The present analytical theory has previously been verified for combinations of DCB tip bending moments and axial forces against Suo and Hutchinson’s 2D-elasticity-based semi-analytical partition theory in Chapter 2. Note that although Suo and Hutchinson’s theory is regarded as the most accurate for bending moments and axial forces, the method that has been developed in Chapter 2 has a stronger capability for solving more complex mixed-mode partition problems, for example, in the bimaterial case and in this work which accounts for shear forces.
To verify the present analytical theory, the thickness ratio $\gamma$ was varied in the range $1/10 \leq \gamma \leq 10$ under three different sets of loading conditions. In the first loading condition, seen in Figs. 3.7a and 3.7b, there were only through-thickness shear forces at the crack tip and $P_{2B}$ was varied in the range $-10,000 \leq P_{2B} [N] \leq 10,000$ with $P_{1B} = 1000$ N. In the second loading condition, seen in Figs. 3.7c and 3.7d, there was a combination of through-thickness shear forces and bending moments at the crack tip and $P_{1B}$ was varied in the range $-10,000 \leq P_{1B} [N] \leq 10,000$ with $M_{1B} = 1000$ Nmm. In the third loading condition, seen in Fig. 3.7e and 3.7f, there were through-thickness shear forces and axial forces at the crack tip and $N_{1Be}$ was varied in the range $-10,000 \leq N_{1Be} [N] \leq 10,000$ with $P_{1B} = 1000$ N, where $N_{1Be} = N_{1B} - N_{2B}/\gamma$. These loading conditions were selected to show the models applicability to a wide range of values that would typically be expected in real life engineering applications of the DCB.

Fig. 3.7 shows 6 individual contour plots giving the relative error between the theory denoted by a subscript “th” and the 2D FEM simulations denoted by a subscript “FEM”. On each contour plot the loading ratio was varied along the x-axis and the thickness ratio $\gamma$ along the y-axis in the range $1/10 \leq \gamma \leq 10$ using a logarithmic scale for clarity. In Figs. 3.7a, 3.7c and 3.7e the total ERR $G$ of the new analytical partition theory was assessed by dividing the theory $G_{th}$ by the FEM result $G_{FEM}$ and then subtracting the result from 1. By taking the magnitude of the result, the error obtained was then plotted on the contour plot by associating the error to a given colour. Therefore to read the contour plot, identify the colour for the given loading condition and thickness ratio and use the colour bar at the bottom of the figure to obtain the associated error. In this case no error is associated a dark blue colour and the maximum error a dark red. In Figs. 3.7b, 3.7d and 3.7f the partition $G_{th}/G$ of the new analytical theory was assessed by subtracting the FEM result ($G_{th}/G)_{FEM}$ from the theory ($G_{th}/G)_{th}$ and then taking the magnitude of the error. The error was then plotted on the contour plot.

To further validate the analytical model, Fig. 3.7 has been replicated in Fig. 3.8, however the value of the Poisson’s ratio has been adjusted from $\nu = 0.29$ to an extreme value $\nu = 0.1$ to show that the present analytical theory works for other values of the
material properties and isn’t limited to the loading ratios, material properties and DCB geometry given in this section. It is assumed if the model can provide accurate results by considering such an extreme value of the Poisson’s ratio then it should provide accurate results for more realistic values.
Figure 3.7: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for variable $\gamma$ and loading conditions with $\nu = 0.29$. 
Figure 3.8: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for variable $\gamma$ and loading conditions with $\nu = 0.1$. 
3.3.1. Crack tip through-thickness shear forces only

This section considers the first loading condition in which there were only through-thickness shear forces at the crack tip and \( P_{2B} \) was varied in the range \(-10,000 \leq P_{2B} [N] \leq 10,000\) with \( P_{1B} \) held at a constant value \( P_{1B} = 1000 \) N. To achieve this loading condition, DCB tip through-thickness shear forces were applied as \( P_1 = P_{1B} \) and \( P_2 = P_{2B} \). To avoid bending moments, \( M_{1B} \) and \( M_{2B} \), at the crack tip, bending moments were also applied at the DCB tip as \( M_1 = -P_1a \) and \( M_2 = -P_2a \).

Fig. 3.7a shows the difference between the total ERRs \( G \) from the present analytical theory and from the 2D FEM for each value of \( \gamma \) and \( P_{2B}/P_{1B} \). Fig. 3.7b shows the differences between the ERR partitions \( G_i/G \) from the present analytical theory and from the 2D FEM. There is excellent agreement between the present analytical theory and the 2D FEM results for both the total ERR \( G \) and the ERR partition \( G_i/G \) for the majority of \( \gamma \) and \( P_{2B}/P_{1B} \) values considered, seen on the figure by the mostly dark blue appearance meaning there is close to zero error. There are however some regions of slightly increased error on both Figs. 3.7a and 3.7b. One possible reason for this could be due to the fact that although the shear correction factor \( \kappa(\gamma) \) and pure-mode-II ERR correction factor \( c(\gamma) \) have been obtained using the FEM there are however inaccuracies in the analytical expressions. An example of this can be seen by considering the area of maximum error in Fig. 3.7a, \( P_{2B}/P_{1B} = 10 \) and \( \log_{10}(1/\gamma) = -0.8 \).

From Fig. 3.6 for the pure-mode-II ERR correction factor \( c(\gamma) \), the value of \( \log_{10}(1/\gamma) = -0.8 \) corresponds to a peak in the correction factor where there is an increased error in the analytical expression. If this was the cause of the error, as the correction factor is symmetric, then it is expected that the error should also be present when \( \log_{10}(1/\gamma) = 0.8 \), however this is not the case.

Another reason could be due to the small values of the total ERR \( G \) in these regions in combination with previous statement about the errors in the analytical expressions for the correction factors, therefore magnifying the apparent error between the present analytical theory and 2D FEM simulations. To examine this further, Fig. 3.9 compares
the absolute values of $G$ and $G_t/G$ from the present analytical theory and the FEM for the cross-sections through Figs. 3.7a and 3.7b where $\log_{10}(l/\gamma) = -0.7$. It is seen that there is excellent agreement for the whole range of $P_{2B}/P_{1B}$ and that $G$ becomes small close to $P_{2B}/P_{1B} = 10$ which is where the error increases in Figs. 3.7a and 3.7b. Therefore, the increased error is due to this reason.

Figure 3.9: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_t/G$ for $\log_{10}(l/\gamma) = -0.7$ and variable $P_{2B}/P_{1B}$ with $\nu = 0.29$.

Note that that changing the value of the Poisson’s ratio from $\nu = 0.29$ in Fig. 3.7 to $\nu = 0.1$ in Fig. 3.8 doesn’t change the accuracy of the analytical model for all loading ratios considered and the results are identical. This means that the comments regarding Fig. 3.7 are applicable to Fig. 3.8 and no further analysis is required.
3.3.2. Bending moments, axial forces and through-thickness shear forces

In the second loading condition, there was a combination of through-thickness shear forces and bending moments at the crack tip and \( P_{ib} \) was varied in the range \(-10,000 \leq P_{ib} [N] \leq 10,000 \) with \( M_{ib} = 1000 \text{ Nmm} \). To achieve this loading condition, a DCB tip through-thickness shear force and a DCB tip bending moment were applied as \( P_i = P_{ib} \) and \( M_i = 1000 - P_i a \) respectively.

Fig. 3.7c shows the differences between the total ERRs \( G \) from the present analytical theory and from the 2D FEM for each value of \( \gamma \) and \( P_{ib}/M_{ib} \). Fig. 3.7d shows the differences between the ERR partitions \( G_i/G \) from the present analytical theory and 2D FEM. Again, there is excellent agreement between the present theory and 2D FEM results for both the total ERR \( G \) and the ERR partition \( G_i/G \) for the majority of \( \gamma \) and \( P_{ib}/M_{ib} \) values considered, seen on the figure by the mostly dark blue appearance relating to zero error. It is worth noting that the maximum error in Fig. 3.7c has been capped to 0.15 in order to make clearer comparisons between the present analytical theory and the 2D FEM. Similar to 3.7a and 3.7b, the areas of increased error in Figs. 3.7c and 3.7d are also due to the small values of the ERR \( G \) in these regions magnifying the errors in the analytical theory. Fig. 3.10 compares the absolute values of \( G \) and \( G_i/G \) from the present analytical theory and the FEM for the cross sections through Figs. 3.7c and 3.7d where \( \log_{10}(l/\gamma) = 0.8 \). It is seen that there is excellent agreement for the whole range of \( P_{ib}/M_{ib} \) and that \( G \) becomes small close to \( P_{ib}/M_{ib} = -1 \) which is where the error increases in Figs. 3.7c and 3.7d.
Chapter 3: Orthotropic laminated beams with bending moments, axial forces and through-thickness shear forces

Figure 3.10: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for $\log_{10}(1/\gamma) = 0.8$ and variable $P_{1B}/M_{1B}$ with $\nu = 0.29$.

Finally, the third loading condition considered, in which there were through-thickness shear forces and axial forces at the crack tip and $N_{1Be}$ was varied in the range $-10,000 \leq N_{1Be} [N] \leq 10,000$ with $P_{1B} = 1000 \text{ N}$. To achieve this loading condition, a DCB tip through-thickness shear force, a DCB tip bending moment, and a DCB tip axial force were applied as $P_1 = P_{1B}$, $M_1 = -P_1 a$ and $N_1 = N_{1Be}$ respectively.

Figs. 3.7e shows the differences between the total ERRs $G$ from the present analytical theory and from the 2D FEM for each value of $\gamma$ and $N_{1Be}/P_{1B}$. Figs. 3.7f shows the differences between the ERR partitions $G_i/G$ from the present analytical theory and from the 2D FEM. Again, there is excellent agreement between the present analytical theory and 2D FEM results for both the total ERR $G$ and the ERR partition $G_i/G$ for the majority of $\gamma$ and $N_{1Be}/P_{1B}$ considered. The maximum error in both Figs. 3.7e and 3.7f has again been capped to 0.15 in order to make clearer comparisons.
between the present analytical theory and the 2D FEM. The areas of increased error in Figs. 3.7e and 3.7f are again due to the small values of the ERR $G$ in these regions, which amplify the errors in the analytical expression for the correction factors. Fig. 3.11 compares the absolute values of $G$ and $G_i/G$ from the present analytical theory and the FEM for the cross-sections through Figs. 3.7e and 3.7f where $\log_{10}(1/\gamma) = 0.9$. It is seen that there is excellent agreement for the whole range of $N_{1Be}/P_{1B}$ and that $G$ becomes small close to $N_{1Be}/P_{1B} = -1$ which is where the error increases in Figs. 3.7e and 3.7f.

![Figure 3.11: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for $\log_{10}(1/\gamma) = 0.9$ and variable $N_{1Be}/P_{1B}$ with $\nu = 0.29$.](image)

3.4. Conclusion

This chapter has presented an extension to the work in Chapter 2, where a new 2D-elasticity-based partition theory for mixed-mode fracture in laminated orthotropic DCBs with rigid interfaces under tip bending moments and axial forces has been derived and
validated\textsuperscript{16}. The extension now means that the partition theory is valid for crack tip bending moments, axial forces and through-thickness shear forces. The theory is not limited to the loading ratios, material properties and DCB geometry shown here and can be used for any combination of loads, orthotropic material properties and DCB geometry with thickness ratio in the range $-1.7 \leq \gamma_i \leq 1.7$ where $\gamma_i = \log_{10}(l/\gamma)$.

Using the same orthogonal pure mode methodology as Wang and Harvey\textsuperscript{1–3,12,25}, 2D elasticity pure modes have been obtained for a DCB with crack tip through-thickness shear forces only. It is identified that the pure mode I condition in 2D elasticity $\theta_{P,2D}$ is the same as that when using Timoshenko beam theory $\theta_{P,T}$. However, the pure mode II condition is different and an expression for this pure mode has been obtained by fitting a curve to results produced from the FEM. Unfortunately, due to the high mesh densities needed and thus computational cost required to obtain the pure mode II, the theory presented here is only valid in the range $-1.7 \leq \gamma_i \leq 1.7$ where $\gamma_i = \log_{10}(l/\gamma)$. Timoshenko beam theory also shows that for crack tip through-thickness shear forces only acting on the DCB, the mode II component of the ERR is zero, meaning that they only contribute to the mode I component of the ERR. However, the FEM with the VCCT has shown that the crack tip through-thickness shear forces contribute to both $G_I$ and $G_{II}$. Furthermore, the total ERR $G$ is different when comparing results from Timoshenko beam theory to 2D elasticity theory as Timoshenko beam theory assumes a constant through-thickness shear correction factor, which is not the case for 2D elasticity.

As a result of these findings, the Timoshenko beam partition theory has been modified to account for 2D elasticity by introducing a thickness-ratio-dependent shear correction factor $\kappa(\gamma)$ and a pure-mode-II ERR correction factor $c(\gamma)$. Both correction factors are obtained using the FEM and then curves have again been fitted to the results. It is interesting to note that the correction factors closely follow elegant normal distributions around a symmetric DCB geometry and are independent of the material properties. Once the shear correction factor and pure-mode-II ERR correction factor are obtained, the total ERR $G$ can be partitioned using the 2D-elasticity-based crack tip
through-thickness pure modes. Finally, the work in this chapter is incorporated with that from Chapter 2 to partition the total ERR for general loading conditions.

The new 2D elasticity partition theory has been validated by comparison with results obtained from the FEM using the VCCT. Three loading conditions have been considered, the first of which being crack tip through-thickness shear forces only, the second being a combination of bending moments and shear forces and finally axial forces and shear forces. Excellent agreement has been observed for all thickness ratios between the present theory and 2D FEM, particularly when the total ERR $G$ is not close to zero. Finally, the effect of the Poisson’s ratio on the theory has been examined and it is seen that changing the Poisson’s ratio doesn’t affect the accuracy of the new 2D elasticity partition theory.

Therefore, the work in this chapter now offers a means of calculating the 2D-elasticity-based ERR partition for an orthotropic laminated DCB under any combination of bending moments, axial forces and shear forces. The work has been published in Wood et al.19
Chapter 4: Dissimilar laminated beams with bending moments and axial forces

4.1. Introduction

Bimaterials are commonly found in both natural and artificial products. Examples include skin and tissue in biological bodies, painted metal in vehicle bodies, and thermal barrier coatings in gas turbine engines, among many others. Fracture on bimaterial interfaces remains a very important and challenging mechanics problem today.

In the pioneering work by Williams\textsuperscript{32} it was discovered that the elastic field around the crack tip on a bimaterial interface showed oscillatory characteristics as the distance from the crack tip tends to zero. Due to this the work of Williams\textsuperscript{32} is often called the oscillatory model. Subsequently, Erdogan\textsuperscript{33,34} and Rice and Sih\textsuperscript{35} verified the presence of these oscillatory singularities and England\textsuperscript{36} identified that the solution was physically unacceptable as it predicted that the upper and lower surfaces of the crack would wrinkle up and overlap near the crack tip. To remove the oscillatory characteristics and material interpenetration, Comninou\textsuperscript{37–39} assumed that there was a small frictionless contact zone near the crack tip. The work of Comninou\textsuperscript{37–39} is therefore usually called the contact model. Using the contact model, Gautesen and Dundurs\textsuperscript{40,41} and Gautesen\textsuperscript{42} were able to develop analytical theories to determine physical quantities such as the contact zone size and the interface tractions.

It is still an unanswered question as to which model more closely represents the reality. Some detailed comparisons of near crack tip stress, contact zone size and oscillation zone size between the two models have been given\textsuperscript{44}. One argument for the use of the oscillatory model is that it does capture the essential stress state near the crack tip when the contact zone size is much smaller than the crack length\textsuperscript{36,43,44}. This may partially explain why the oscillatory model appears to be more commonly accepted among researchers.

One consequence of the oscillatory model however, is that the SIF is of complex form, of which various forms have been given\textsuperscript{35,44,45}. The complex SIF gives rise to fundamental differences between cracks on bimaterial interfaces and interfacial cracks between similar materials (which possess a real SIF), and presents two major challenges that must be solved in order to obtain analytical solutions for $K_I$ and $K_{II}$: (1) In the
case of interfacial cracks between similar materials, the ERRs, $G_I$ and $G_{II}$, are each related to the corresponding SIFs, $K_I$ or $K_{II}$; however, in the case of cracks on bimaterial interfaces, $G_I$ and $G_{II}$ are each coupled with both $K_I$ and $K_{II}$ together. The first challenge is to reveal the mechanical meaning of this coupling. (2) In the case of interfacial cracks between similar materials, both the total ERR $G$ and its partition into $G_I$ and $G_{II}$ are independent of the crack extension size or the FEM mesh size; however, in the case of cracks on bimaterial interfaces, the individual ERRs, $G_I$ and $G_{II}$, vary with crack extension size or FEM mesh size, although the total ERR $G$ remains constant. The second challenge is to accurately determine $G_I$ and $G_{II}$ analytically for a certain crack extension size.

These two challenges have been preventing researchers from obtaining analytical solutions for $G_I$, $G_{II}$, $K_I$ and $K_{II}$ for decades. Using numerical methods it has been identified that the imaginary part of the stress field is the cause of the non-convergence of the ERR partitions. Numerical methods to remove the imaginary part of the stress field and therefore obtain converged ERR partitions include modelling the interface crack in an isotropic homogeneous thin resin layer, carefully selecting the material properties, and modifying the Poisson’s ratio of the material above or below the interface. Sun and Qian identified that by setting a finite crack extension in the VCCT, it is possible to obtain finite extension based mode I and II ERRs using the FEM and then use crack tip displacements to obtain the SIFs. Other numerical methods are given.

The work of Suo and Hutchinson relies on numerically determining inconvenient discrete parameters and then tabulating the results in order to calculate $K_I$ and $K_{II}$. It is still widely used to study bimaterial interfacial fractures because of a lack of better alternatives. Also, some studies simply ignore the material mismatch altogether to avoid using these inconvenient discrete parameters. An improvement over the discrete parameters in Suo and Hutchinson’s work has been made by using continuous parameter curves obtained by interpolating FEM results. The applicability of these curves is, however, limited by the range of thickness ratios and restricted loading conditions.
This work aims to present a complete analytical solution to the problem. To address the first of the challenges mentioned above, the coupling between the ERRs, $G_I$ and $G_{II}$, and the SIFs, $K_I$ and $K_{II}$, is studied by using an orthogonal pure mode methodology\textsuperscript{1–3,12,25} and the fundamental mechanical meaning of the coupling is revealed. The second challenge is then overcome by using two powerful mathematical techniques: The first technique is developed in this work and is called the shifting technique; the second technique again makes use of the orthogonal pure mode methodology\textsuperscript{1–3,12,25}. Accurate analytical solutions are achieved for the crack extension size-independent SIFs, $K_I$ and $K_{II}$, and the crack extension size-dependent ERRs, $G_I$ and $G_{II}$.

The structure of the chapter is as follows. In Section 4.2, an analytical method for obtaining the complex SIFs and the crack extension size-dependent ERRs for brittle interfacial cracking between two dissimilar elastic layers is developed. Section 4.3 presents the validation of the new theory against 2D FEM results. Finally, conclusions are made in Section 4.4.

4.2. Analytical development

![Figure 4.1: A bimaterial DCB. (a) General description. (b) Interfacial stresses and crack tip forces.](image-url)
Fig. 4.1a shows a bimaterial DCB with its material properties, geometry and loading conditions. The DCB has tip bending moments, \( M_1 \) and \( M_2 \), and axial forces \( N_1 \) and \( N_2 \). The Young’s modulus, shear modulus and Poisson’s ratio of beam \( i \) are denoted by \( E_i \), \( \mu_i \) and \( \nu_i \) respectively (with \( i = 1,2 \)). Based on the work of Wang and Harvey\(^{1-3,12,25}\), the total ERR \( G \) can be calculated as follows

\[
G = \frac{1}{2E_i b^2} \begin{bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B} \end{bmatrix}^T \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B} \end{bmatrix} = \frac{1}{2E_i b^2} \begin{bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B} \end{bmatrix}^T [C] \begin{bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B} \end{bmatrix} \tag{4.1}
\]

where \([C]\) is the coefficient matrix, given by

\[
C_{11} = \frac{12\eta \gamma^3 + 4\gamma^2 + 6\gamma + 3}{h_i^3 C} \tag{4.2}
\]

\[
C_{12} = -\frac{12(\eta \gamma + 1)}{h_i^3 C} \tag{4.3}
\]

\[
C_{13} = \frac{6\eta \gamma (\gamma + 1)}{h_i^3 C} \tag{4.4}
\]

\[
C_{14} = -\frac{6(\gamma + 1)}{h_i^2 C} \tag{4.5}
\]

\[
C_{22} = \frac{12(3\eta \gamma^3 + 6\eta \gamma^2 + 4\eta \gamma + 1)}{\eta h_i^3 \gamma^3 C} \tag{4.6}
\]

\[
C_{23} = \frac{6\eta \gamma (\gamma + 1)}{h_i^2 C} \tag{4.7}
\]

\[
C_{24} = -\frac{6(\gamma + 1)}{h_i^2 C} \tag{4.8}
\]

\[
C_{33} = \frac{\eta \gamma (\eta \gamma^3 + 1)}{h_i C} \tag{4.9}
\]

\[
C_{34} = -\frac{\eta \gamma^3 + 1}{h_i C} \tag{4.10}
\]
Chapter 4: Dissimilar laminated beams with bending moments and axial forces

\[ C_{44} = \frac{\eta \gamma^3 + 1}{\eta h_1 C} \]  

(4.11)

\[ \bar{C} = \eta^2 \gamma^4 + 4 \eta \gamma^3 + 6 \eta \gamma^2 + 4 \eta \gamma + 1 \]  

(4.12)

and \( \bar{E}_1 \) is the effective Young’s modulus of the top beam. For plane stress then \( \bar{E}_1 = E_1 \) and for plane strain then \( \bar{E}_1 = E_1 / \left( 1 - \nu_1^2 \right) \). The crack tip bending moments are \( M_{1B} \) and \( M_{2B} \) and axial forces \( N_{1B} \) and \( N_{2B} \).

### 4.2.1. Interfacial stresses ahead of the crack tip

The interfacial opening stress and shear stress ahead of the crack tip, \( \sigma_n \) and \( \tau_s \), for the bimaterial DCB in Fig. 4.1a, can be expressed in a combined complex form as

\[ \sigma_n + i \tau_s = \frac{(K_I + i K_{II})}{\sqrt{2 \pi r}} \]  

(4.13)

or in individual real form as

\[ \sigma_n = \frac{1}{\sqrt{2 \pi r}} \{ K_I \cos[\varepsilon \ln(r)] - K_{II} \sin[\varepsilon \ln(r)] \} \]  

(4.14)

\[ \tau_s = \frac{1}{\sqrt{2 \pi r}} \{ K_I \sin[\varepsilon \ln(r)] + K_{II} \cos[\varepsilon \ln(r)] \} \]  

(4.15)

where \( r \) is the radius coordinate centred on the crack tip and \( K_I \) and \( K_{II} \) are the mode I and II SIFs, respectively. The signs of \( \sigma_n \) and \( \tau_s \) are positive in the directions shown in Fig. 4.1b. In Eqs. (4.13)–(4.15), the bimaterial constant \( \varepsilon \) is defined as

\[ \varepsilon = \frac{1}{2 \pi} \ln \left[ \left( \frac{k_1}{\mu_1} + \frac{1}{\mu_2} \right) \left( \frac{k_2}{\mu_2} + \frac{1}{\mu_1} \right)^{-1} \right] \]  

(4.16)

where the Kolosov constant \( k_i \) (with \( i = 1, 2 \)) is defined as \( k_i = 3 - 4 \nu_i \) for plane strain and \( k_i = (3 - \nu_i) / (1 + \nu_i) \) for plane stress. It is easy to verify that when \( \nu_1 = \nu_2 \) then \( \varepsilon(\eta^{-1}) = -\varepsilon(\eta) \) where \( \eta = E_2 / E_1 \) is the Young’s modulus ratio.
4.2.2. Relative interfacial stresses behind the crack tip

From the work\textsuperscript{44}, the relative opening displacement behind the crack tip $D_n$ and the relative interfacial shear displacement behind the crack tip $D_s$, of the upper beam 1 with respect to the lower beam 2, can be expressed in individual real form as

$$D_n = \frac{\sqrt{2r}}{4(1+4\varepsilon^2)} \pi \left[ \frac{k_1+1}{\mu_1} + \frac{k_2+1}{\mu_2} \right] \{K_{I-S}H_1 - K_{II-S}H_2\} \quad (4.17)$$

$$D_s = \frac{\sqrt{2r}}{4(1+4\varepsilon^2)} \pi \left[ \frac{k_1+1}{\mu_1} + \frac{k_2+1}{\mu_2} \right] \{K_{I-S}H_2 + K_{II-S}H_1\} \quad (4.18)$$

where

$$H_1 = \cos \left( \varepsilon \ln \left( \frac{r}{a} \right) \right) + 2\varepsilon \sin \left( \varepsilon \ln \left( \frac{r}{a} \right) \right) \quad (4.19)$$

$$H_2 = \sin \left( \varepsilon \ln \left( \frac{r}{a} \right) \right) - 2\varepsilon \cos \left( \varepsilon \ln \left( \frac{r}{a} \right) \right) \quad (4.20)$$

and $a$ is the crack length. It is possible to relate the SIFs, $K_{I-S}$ and $K_{II-S}$, given by the work of Sun and Qian\textsuperscript{44}, to the SIFs, $K_I$ and $K_{II}$, defined by Suo and Hutchinson\textsuperscript{21} using

$$K_{I-S} = \frac{1}{\cosh(\pi\varepsilon)} \left[ K_I \cos(\varepsilon \ln(a)) - K_{II} \sin(\varepsilon \ln(a)) \right] \quad (4.21)$$

$$K_{II-S} = \frac{1}{\cosh(\pi\varepsilon)} \left[ K_I \sin(\varepsilon \ln(a)) + K_{II} \cos(\varepsilon \ln(a)) \right] \quad (4.22)$$

Therefore, the relative opening displacement behind the crack tip $D_n$ and the relative interfacial shear displacement behind the crack tip $D_s$, of the upper beam 1 with respect to the lower beam 2 become

$$D_n = D \cos(\xi) \sqrt{2\alpha} \{K_I \cos[\varepsilon \ln(r) - \xi] - K_{II} \sin[\varepsilon \ln(r) - \xi]\}$$

$$= D \cos(\xi) \sqrt{2\alpha} \cos[\varepsilon \ln(r) - \xi] \left[ K_I - \bar{B}^{-1}_K K_{II} \right] \quad (4.23)$$

$$D_s = D \cos(\xi) \sqrt{2\alpha} \{K_I \sin[\varepsilon \ln(r) - \xi] + K_{II} \cos[\varepsilon \ln(r) - \xi]\}$$

$$= D \cos(\xi) \sqrt{2\alpha} \sin[\varepsilon \ln(r) - \xi] \left[ K_I - \bar{\partial}^{-1}_K K_{II} \right] \quad (4.24)$$
where the signs of \( D_s \) and \( D_t \) are consistent with the sign of the interfacial stresses shown in Fig. 4.1b, and

\[
D = \frac{(C_1 + C_2)}{4\pi \cosh(\pi\epsilon)}
\]  

(4.25)

\[
\tilde{\theta}_k = -\tan[\epsilon \ln(r) - \xi] \quad \text{and} \quad \tilde{\beta}_k = \frac{1}{\tan[\epsilon \ln(r) - \xi]}
\]

(4.26)

with

\[
C_i = \frac{1 + k_i}{\mu_i} \quad \text{with} \quad i = 1, 2
\]

(4.27)

\[
\cos(\xi) = \frac{1}{(1 + 4\epsilon^2)^{\frac{1}{2}}} \quad \text{and} \quad \sin(\xi) = \frac{2\epsilon}{(1 + 4\epsilon^2)^{\frac{1}{2}}}
\]

(4.28)

Note that \( \tilde{\theta}_k \tilde{\beta}_k = -1 \) due to pure mode orthogonality which is discussed later in Section 4.2.3. Also note that \( \tilde{\theta}_k (-\epsilon) = -\tilde{\theta}_k (\epsilon) \) and \( \tilde{\beta}_k (-\epsilon) = -\tilde{\beta}_k (\epsilon) \) due to physical symmetry, which become \( \tilde{\theta}_k (\eta^{-1}) = -\tilde{\theta}_k (\eta) \) and \( \tilde{\beta}_k (\eta^{-1}) = -\tilde{\beta}_k (\eta) \) when \( \nu_1 = \nu_2 \). It is seen from Eqs. (4.14), (4.15), (4.23) and (4.24) that the interfacial stresses and the relative interfacial displacements are out of phase by \( \xi \) because of the bimaterial mismatch constant \( \epsilon \). The profound mechanical meaning of this phase difference will be shown in the next section for ERR partitions.

### 4.2.3. Partitioning the ERR \( G \) using pure modes in terms of \( K_j \) and \( K_{\|} \)

The relationships between the ERRs, \( G_j \) and \( G_{\|} \), and the SIFs, \( K_j \) and \( K_{\|} \), are traditionally obtained by using the VCCT. The following relationships were originally derived in the work\(^{44,73}\) using the VCCT, but they are written here in a different form and also use the SIFs from Eq. (4.13) for more convenient calculations

\[
G = G_j + G_{\|} = \frac{D\pi}{4\cosh(\pi\epsilon)} \left(K_j^2 + K_{\|}^2\right)
\]

(4.29)

\[
G_a = G_j - G_{\|} = D|B|\cos^2(\xi) \left(1 + \frac{\alpha}{a}\right)[(K_j^2 - K_{\|}^2)\cos(\beta) + 2K_j K_{\|} \sin(\beta)]
\]

(4.30)
The individual ERRs, $G_I$ and $G_{II}$, can be written as

$$G_I = \frac{G}{2} + \frac{D|B|\cos^2(\xi)}{2} \left[1 + \frac{\delta a}{a}\right] \left[ (K_I^2 - K_{II}^2) \cos(\bar{\rho}) + 2K_I K_{II} \sin(\bar{\rho}) \right]$$

(4.31)

$$G_{II} = \frac{G}{2} - \frac{D|B|\cos^2(\xi)}{2} \left[1 + \frac{\delta a}{a}\right] \left[ (K_I^2 - K_{II}^2) \cos(\bar{\rho}) + 2K_I K_{II} \sin(\bar{\rho}) \right]$$

(4.32)

where

$$B = 0.5\pi^{0.5} (0.5 + i\varepsilon) \frac{\Gamma(0.5 - i\varepsilon)}{\Gamma(1 - i\varepsilon)} = B_{re} + iB_{im}$$

(4.33)

$$\bar{\rho} = \rho - 2\varepsilon \ln \left(\frac{\delta a}{2}\right)$$

(4.34)

$$\cos(\rho) = \frac{B_{re}}{|B|} \quad \text{and} \quad \sin(\rho) = \frac{B_{im}}{|B|}$$

(4.35)

The notation $|B|$ represents the complex modulus of $B$. The gamma function is represented by $\Gamma$, and the crack extension size by $\delta a$. Eq. (4.29) shows that the total ERR $G$ is independent of $\delta a$, but Eqs. (4.30)–(4.32) show that the individual ERRs, $G_I$ and $G_{II}$, are dependent on $\delta a$. By solving Eqs. (4.29) and (4.30) together, $K_I^2$ can be expressed as

$$K_I^2 = \frac{2\cosh(\pi \varepsilon)}{D\pi} \left[ G + \cos(\bar{\rho})C_d G_d \pm \sin(\bar{\rho}) \left( G^2 - C_d^2 G_d^2 \right)^{1/2} \right]$$

(4.36)

where

$$C_d = \frac{\pi}{4|B|\cosh(\pi \varepsilon)\cos^2(\xi)|1 + \delta a/a|}$$

(4.37)

Note that $C_d$ is close to 1. From Eq. (4.36) two values of $K_I^2$ are given if $\varepsilon \neq 0$, denoted here by $(K_I^2)_h$ and $(K_I^2)_l$. The corresponding two values of $K_{II}^2$ are $(K_{II}^2)_h$ and $(K_{II}^2)_l$ respectively and they can be obtained by using Eq. (4.29). The four pairs of solutions for $K_I$ and $K_{II}$, denoted by $K_{I-1}$ to $K_{I-4}$ and $K_{II-1}$ to $K_{II-4}$ respectively, can be found by using Eq. (4.30).
From Eq. (4.30), we have

\[ K_I K_{II} = \frac{G_d}{2D}B \cos^2(\varepsilon) \sin(\bar{\psi}) \left( 1 + \frac{\tilde{\alpha}t}{a} \right) - \frac{\left( K_I^2 - K_{II}^2 \right)}{2 \tan(\bar{\psi})} \]  

(4.38)

which has two values, \((K_I K_{II})_1\) and \((K_I K_{II})_2\), based on the two pairs of \(K_I^2\) and \(K_{II}^2\).

If \((K_I K_{II})_1 > 0\) then

\[ K_{I-1} = +\sqrt{(K_I^2)_1}, \quad K_{II-1} = +\sqrt{(K_{II}^2)_1}, \quad K_{I-2} = -\sqrt{(K_I^2)_1}, \quad K_{II-2} = -\sqrt{(K_{II}^2)_1} \]

(4.39)

otherwise if \((K_I K_{II})_1 < 0\)

\[ K_{I-1} = +\sqrt{(K_I^2)_1}, \quad K_{II-1} = -\sqrt{(K_{II}^2)_1}, \quad K_{I-2} = -\sqrt{(K_I^2)_1}, \quad K_{II-2} = +\sqrt{(K_{II}^2)_1} \]

(4.40)

Similarly, if \((K_I K_{II})_2 > 0\) then

\[ K_{I-3} = +\sqrt{(K_I^2)_2}, \quad K_{II-3} = +\sqrt{(K_{II}^2)_2}, \quad K_{I-4} = -\sqrt{(K_I^2)_2}, \quad K_{II-4} = -\sqrt{(K_{II}^2)_2} \]

(4.41)

Otherwise if \((K_I K_{II})_2 < 0\) then

\[ K_{I-3} = +\sqrt{(K_I^2)_2}, \quad K_{II-3} = -\sqrt{(K_{II}^2)_2}, \quad K_{I-4} = -\sqrt{(K_I^2)_2}, \quad K_{II-4} = +\sqrt{(K_{II}^2)_2} \]

(4.42)

Note that if \(\varepsilon = 0\) then \(\bar{\psi} = 0\) and the four pairs of solutions for \(K_I\) and \(K_{II}\) can be easily calculated using Eqs. (4.29) and (4.36). In all cases, however, there will only be one mechanically admissible pair for a given loading condition, from among the complete set of mathematical solutions for \(K_I\) and \(K_{II}\). Previously FEM simulations have been used to determine this pair\(^{44}\). In this work, a method is devised to guide the selection of the correct pair by purely analytical means.

In the case of interfacial cracks between similar materials, that is, when the material mismatch coefficient \(\varepsilon = 0\), the ERRs, \(G_I\) and \(G_{II}\), are related to the corresponding SIFs, \(K_I\) or \(K_{II}\); however, in the case of cracks on bimaterial interfaces, that is, when \(\varepsilon \neq 0\), \(G_I\) and \(G_{II}\) are each coupled with both \(K_I\) and \(K_{II}\) together, as shown by Eqs. (4.31) and (4.32). What is the mechanical meaning of the coupling? In Wang and Harvey’s\(^{1-3,12,25}\) previous work, a powerful orthogonal pure mode technique has been developed for partitioning mixed-mode fractures. Orthogonal pure modes are derived in
terms of the applied crack tip forces and moments. Here, it is expected that pure modes also exist in terms of the SIFs, $K_I$ and $K_{II}$, because SIFs can be considered as an alternative form of load. Based on this mechanical understanding, and by writing the pure modes in terms of the SIFs, $K_I$ and $K_{II}$, the total ERR $G$ in Eq. (4.29), is partitioned as

$$G_I = \frac{D\pi}{4\cosh(\pi\varepsilon)(1 + \beta\varepsilon^{-1})} \left( K_I - \beta\varepsilon^{-1}K_{II} \right) \left( K_I - \beta\varepsilon^{-1}K_{II} \right)$$

$$G_{II} = \frac{D\pi}{4\cosh(\pi\varepsilon)(1 + \theta\varepsilon^{-1})} \left( K_I - \theta\varepsilon^{-1}K_{II} \right) \left( K_I - \theta\varepsilon^{-1}K_{II} \right)$$

From Eq. (4.44) it is seen that when $K_{II} = \theta\varepsilon K_I$ then $G_{II} = 0$. The relationship, $K_{II} = \theta\varepsilon K_I$, produced a pure mode I $\theta\varepsilon$ fracture. Also, from Eq. (4.44), when $K_{II} = \theta'\varepsilon K_I$ then $G_{II} = 0$. The relationship $K_{II} = \theta'\varepsilon K_I$ produces a pure mode I $\theta'\varepsilon$ fracture. The physical meaning of these two pure mode I modes are zero effective relative crack tip shear displacement and zero effective crack tip shear force respectively. When the bimaterial mismatch constant $\varepsilon \neq 0$, it is seen from Eqs. (4.14), (4.15), (4.23) and (4.24) that the variations of interfacial stresses and the relative interfacial displacements are out of phase by $\xi$. This causes $\theta\varepsilon \neq \theta'\varepsilon$ and leads to two pure mode I modes.

Similarly, from Eq. (4.43) it is seen that when $K_{II} = \beta\varepsilon K_I$ then $G_I = 0$. The relationship, $K_{II} = \beta\varepsilon K_I$, produces a pure mode II $\beta\varepsilon$ fracture. Also, from Eq. (4.43), when $K_{II} = \beta'\varepsilon K_I$ then $G_I = 0$. The relationship $K_{II} = \beta'\varepsilon K_I$ produces a pure mode II $\beta'\varepsilon$ fracture. The physical meanings of these two pure mode II modes are zero effective crack tip opening force and zero effective crack tip relative opening displacement respectively. Generally, $\beta\varepsilon \neq \beta'\varepsilon$ and there are two pure mode II modes.

These four pure modes, $\theta\varepsilon$, $\theta'\varepsilon$, $\beta\varepsilon$ and $\beta'\varepsilon$, form two sets of orthogonal pure modes. The first set is $(\theta\varepsilon, \beta\varepsilon)$, and the second set is $(\theta'\varepsilon, \beta'\varepsilon)$. Taking the $(\theta\varepsilon, \beta\varepsilon)$ set as an example, here, orthogonal means that

$$\begin{pmatrix} 1 & \theta\varepsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \beta\varepsilon \end{pmatrix} = 0$$

(4.45)
where the square matrix is the coefficient matrix of the total ERR $G$, given in Eq. (4.29). Obviously, $\theta_k^*\beta_k = -1$. Similarly, $\theta_k^*\beta_k' = -1$. One important consequence of the existence of two sets of pure modes is that negative $G_j$ or $G_j'$ can occur (similarly, so can $G_j > G$ or $G_j' > G$) as seen from Eqs. (4.43) and (4.44). Note that the total ERR $G$ is still non-negative-definite. The situation here is very similar to Wang and Harvey’s\textsuperscript{1–3,12,25} Euler beam partition theory for mixed-mode fractures, in which there are also two sets of pure modes, but which are caused by the ‘global’ nature of the partition (i.e. the Euler beam partition is equivalent to the elasticity-based partition calculated over the entire region that is mechanically affected by the crack tip). Here, however, the two sets of pure modes are caused by the out-of-phase oscillation of the relative displacements and stresses near to the crack tip, which are “local” in nature. By letting $G_{j'} = 0$ and $G_j = 0$ in Eqs. (4.32) and (4.31), respectively, the pure modes are found to be

$$\theta_k = \begin{cases} \frac{\sin(\rho) + \sqrt{1 - C_d^2}}{C_d + \cos(\rho)} & \text{if } \varepsilon < 0 \\ \frac{\sin(\rho) - \sqrt{1 - C_d^2}}{C_d + \cos(\rho)} & \text{if } \varepsilon > 0 \end{cases}$$

(4.46)

$$\theta_k' = \begin{cases} \frac{-\sin(\rho) - \sqrt{1 - C_d^2}}{C_d - \cos(\rho)} & \text{if } \varepsilon < 0 \\ \frac{-\sin(\rho) + \sqrt{1 - C_d^2}}{C_d - \cos(\rho)} & \text{if } \varepsilon > 0 \end{cases}$$

(4.47)

$$\beta_k = \begin{cases} \frac{-\sin(\rho) + \sqrt{1 - C_d^2}}{C_d - \cos(\rho)} & \text{if } \varepsilon < 0 \\ \frac{-\sin(\rho) - \sqrt{1 - C_d^2}}{C_d - \cos(\rho)} & \text{if } \varepsilon > 0 \end{cases}$$

(4.48)

$$\beta_k' = \begin{cases} \frac{-\sin(\rho) - \sqrt{1 - C_d^2}}{C_d - \cos(\rho)} & \text{if } \varepsilon < 0 \\ \frac{-\sin(\rho) + \sqrt{1 - C_d^2}}{C_d - \cos(\rho)} & \text{if } \varepsilon > 0 \end{cases}$$

(4.49)

Note that $\theta_k(-\varepsilon) = -\theta_k(\varepsilon)$, $\theta_k'(-\varepsilon) = -\theta_k'(\varepsilon)$, $\beta_k(-\varepsilon) = -\beta_k(\varepsilon)$ and $\beta_k'(-\varepsilon) = -\beta_k'(\varepsilon)$ due to physical symmetry, which become $\theta_k(\eta) = -\theta_k(\eta^{-1})$, $\theta_k'(\eta) = -\theta_k'(\eta^{-1})$, $\beta_k(\eta^{-1}) = -\beta_k(\eta)$ and $\beta_k'(\eta^{-1}) = -\beta_k'(\eta)$ when $\nu_1 = \nu_2$.

It is now clear that the mechanical meaning of the coupling between the ERRs, $G_j$ and $G_{j'}$, and the SIFs, $K_j$ and $K_{j'}$, is the existence of two sets of orthogonal pure
modes, \((\theta_k, \beta_k)\) and \((\theta'_k, \beta'_k)\). Based on this understanding and the partition given by Eqs. (4.43) and (4.44), a method is now devised to guide the selection of the mechanically admissible SIF pair, \(K_i\) and \(K_{ii}\) by purely analytical means. The idea comes from the fact that when the bimaterial mismatch constant \(\varepsilon\) is not large, the two pure mode I modes \(\theta_k\) and \(\theta'_k\), are close to each other. This is also true for the two pure mode II modes, \(\beta_k\) and \(\beta'_k\). It is therefore reasonable to expect that the middle values between these two pure modes are good approximations. Using Eq. (4.24) and the condition that \(D_s(\bar{\alpha}_1) = 0\) gives the approximate pure mode I condition relationship,

\[
K_{ii} = \tilde{\theta}_k K_i
\] (4.50)

The variation of \(\theta_k\), \(\theta'_k\) and \(\tilde{\theta}_k\) with respect to \(\bar{\alpha}_1\) is shown in Fig. (4.2) for different values of the bimaterial mismatch constant \(\varepsilon\). As expected, \(\tilde{\theta}_k\) is close to \(\theta_k\) for all values of \(\bar{\alpha}_1\) and \(\varepsilon\), and is between \(\theta_k\) and \(\theta'_k\), which demonstrates that \(\tilde{\theta}_k\) is a good approximation.
Chapter 4: Dissimilar laminated beams with bending moments and axial forces

Figure 4.2: Variation of the pure mode I conditions $\theta_K$ and $\theta'_K$ and the approximate pure mode I condition $\tilde{\theta}_K$ with respect to the crack extension size $\delta a$ for different values of the bimaterial constant $\varepsilon$ with $\nu_1 = \nu_2 = 0.29$.

Similarly, using Eq. (4.23) and the condition $D_s(\delta a) = 0$ gives the approximate pure mode II relationship,

$$K_{II} = \tilde{\beta}'_K K_I$$

(4.51)

Note that in Eq. (4.51), $\tilde{\beta}'_K$ is used in place of the $\tilde{\beta}_K$ in Eq. (4.23) for consistency with the discussion above regarding, Eqs. (4.43) and (4.44), where $\theta_K$ and $\beta'_K$ give the values of $K_{II}/K_I$ required for zero effective relative displacements. Here $\tilde{\theta}_K$ and $\tilde{\beta}'_K$ give the values of $K_{II}/K_I$ required for zero relative displacements at $r = \delta a$. It is easy to show that $\tilde{\theta}_K$ and $\tilde{\beta}'_K$ are orthogonal to each other, that is,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\theta}_K \\ \tilde{\beta}'_K \end{bmatrix} = 0$$

(4.52)
Physically though, $\tilde{\theta}_k$ is orthogonal to $\tilde{\beta}_k$, as indicated by Eq. (4.45) and also as explained in the work of Wang and Harvey\textsuperscript{1–3,12,25}. Therefore, Eq. (4.52) implies that $\tilde{\beta}_k' = \tilde{\beta}_k$ and consequently that $\tilde{\theta}_k' = \tilde{\theta}_k$. It is easy to validate this from Eq. (4.13) by showing that $F_n(\tilde{\alpha}_t) = \int_0^{\tilde{\alpha}_t} \sigma_n \, dr = 0$ when $D_n(\tilde{\alpha}_t) = 0$ and that $F_s(\tilde{\alpha}_t) = \int_0^{\tilde{\alpha}_t} \tau_s \, dr = 0$ when $D_s(\tilde{\alpha}_t) = 0$. Therefore, the approximate pure mode sets, $(\tilde{\theta}_k, \tilde{\beta}_k)$ and $(\tilde{\theta}_k', \tilde{\beta}_k')$, coincide with each other and $\tilde{\theta}_k \tilde{\beta}_k = -1$ from Eq. (4.52). An approximate partition is then obtained as

\[
G_I = \frac{D \pi}{4 \cosh(\pi \varepsilon)} \left( K_I - \tilde{\beta}_k^{-1} K_{II} \right)^2
\]

\[
G_{II} = \frac{D \pi}{4 \cosh(\pi \varepsilon)} \left( K_I - \tilde{\theta}_k^{-1} K_{II} \right)^2
\]

From Eq. (4.53),

\[
K_I - \tilde{\beta}_k^{-1} K_{II} = \pm \sqrt{G_I} \sqrt{1 + \tilde{\beta}_k^{-2}}
\]

where

\[
\overline{G}_I = \frac{4 \cosh(\pi \varepsilon) G_I}{D \pi}
\]

Eq. (4.23) shows that $\text{sgn}(K_I - \tilde{\beta}_k^{-1} K_{II}) = \text{sgn}[D_n / \cos(\varepsilon \ln(\tilde{\alpha}_t) - \varepsilon)] \equiv S_n$. That is,

\[
K_I - \tilde{\beta}_k^{-1} K_{II} = S_n \sqrt{\overline{G}_I} \sqrt{1 + \tilde{\beta}_k^{-2}}
\]

It is seen that as long as the sign of $D_n$ is known, then the sign of $(K_I - \tilde{\beta}_k^{-1} K_{II})$ in Eq. (4.55) is also known. Note that $\text{sgn}(D_n)$ will be determined in the next section. Similarly, from Eq. (4.54),

\[
K_I - \tilde{\theta}_k^{-1} K_{II} = \pm \sqrt{G_{II}} \sqrt{1 + \tilde{\theta}_k^{-2}}
\]

where

\[
\overline{G}_{II} = \frac{4 \cosh(\pi \varepsilon) G_{II}}{D \pi}
\]
Eq. (4.24) shows that 
\[ \text{sgn}(K_I - \tilde{\theta}_K^{-1}K_H) = \text{sgn}[D_s / \sin(\varepsilon \ln(\tilde{\alpha}) - \xi)] = S_s. \]
That is,
\[ K_I - \tilde{\theta}_K^{-1}K_H = S_s \sqrt{G_H} \sqrt{1 + \tilde{\theta}_K^{-2}}. \] (4.60)

It is seen that as long as the sign of \( D_s \) is known, then the sign of \( (K_I - \tilde{\theta}_K^{-1}K_H) \) in Eq. (4.58) is known. Note that \( \text{sgn}(D_s) \) will be determined in the next section. Finally, when the ERRs, \( G_I \) and \( G_H \), are known, a unique pair of approximate SIFs, \( K_I \) and \( K_H \), can be determined from Eqs. (4.57) and (4.60), as follows

\[ K_I = \begin{cases} \left( S_s \sqrt{G_I} \sqrt{1 + \tilde{\theta}_K^{-2}} + \tilde{\theta}_K^{-2} S_s \sqrt{G_H} \sqrt{1 + \tilde{\beta}_K^{-2}} \right) / \left(1 + \tilde{\theta}_K^{-2}\right) & \text{if } \tilde{\theta}_K \neq 0 \text{ and } \tilde{\beta}_K \neq 0 \\ S_s \sqrt{G_I} & \text{if } \tilde{\theta}_K = 0 \\ S_s \sqrt{G_H} & \text{if } \tilde{\beta}_K = 0 \end{cases} \] (4.61)

\[ K_H = \begin{cases} \left( S_s \sqrt{G_I} \sqrt{1 + \tilde{\theta}_K^{-2}} - S_s \sqrt{G_H} \sqrt{1 + \tilde{\beta}_K^{-2}} \right) / \left(\tilde{\theta}_K - \tilde{\beta}_K\right) & \text{if } \tilde{\theta}_K \neq 0 \text{ and } \tilde{\beta}_K \neq 0 \\ -S_s \sqrt{G_H} & \text{if } \tilde{\theta}_K = 0 \\ -S_s \sqrt{G_I} & \text{if } \tilde{\beta}_K = 0 \end{cases} \] (4.62)

This approximate pair of SIFs, \( K_I \) and \( K_H \), from Eqs. (4.61) and (4.62), should be used to guide the choice of the one admissible pair of SIFs, \( K_I \) and \( K_H \), from Eqs. (4.39)–(4.42).

It is now concluded that the first challenge, stated in the Section 4.1, has been overcome. Now, in the following development, the aim is to overcome the second challenge.

4.2.4. Partitioning the ERR \( G \) using pure modes in terms of crack tip loads

As seen earlier, the SIFs for rigid bimaterial interfaces being complex implies that the interfacial stresses ahead of the crack tip are out of phase with the relative interfacial displacements behind the crack tip. Therefore, two sets of orthogonal pure modes must exist. Based on the work of Wang and Harvey\(^{1-3,12,25}\), the ERR partitions must be in the form,
Chapter 4: Dissimilar laminated beams with bending moments and axial forces

\[ G_I = c_I \left( \frac{M_{1B}}{\beta_i} - \frac{N_{1B}}{\beta_2} - \frac{N_{2B}}{\beta_3} \right) \left( \frac{M_{1B}}{\beta'_i} - \frac{N_{1B}}{\beta'_2} - \frac{N_{2B}}{\beta'_3} \right) \]  \hspace{1cm} (4.63)

\[ G_H = c_H \left( \frac{M_{1B}}{\theta_i} - \frac{N_{1B}}{\theta_2} - \frac{N_{2B}}{\theta_3} \right) \left( \frac{M_{1B}}{\theta'_i} - \frac{N_{1B}}{\theta'_2} - \frac{N_{2B}}{\theta'_3} \right) \]  \hspace{1cm} (4.64)

where

\[ c_I = G_{\theta_i} \left[ \left( 1 - \frac{\theta_i}{\beta_i} \right) \left( 1 - \frac{\theta_i}{\beta'_i} \right) \right]^{-1} \]  \hspace{1cm} (4.65)

\[ c_H = G_{\theta'_i} \left[ \left( 1 - \frac{\beta_i}{\theta_i} \right) \left( 1 - \frac{\beta_i}{\theta'_i} \right) \right]^{-1} \]  \hspace{1cm} (4.66)

\[ G_{\theta_i} = 6 \left[ \theta_i^2 \left( 3 \eta \gamma^3 + 6 \eta \gamma^2 + 4 \eta \gamma + 1 \right) - \theta_i \left( 2 \eta \gamma^4 + 2 \eta \gamma^3 \right) + \left( \eta \gamma^7 + 4 \eta \gamma^6 + 6 \eta \gamma^5 + 3 \eta \gamma^4 \right) \right] \]  \hspace{1cm} (4.67)

\[ G_{\theta'_i} = 6 \left[ \beta_i^2 \left( 3 \eta \gamma^3 + 6 \eta \gamma^2 + 4 \eta \gamma + 1 \right) - \beta_i \left( 2 \eta \gamma^4 + 2 \eta \gamma^3 \right) + \left( \eta \gamma^7 + 4 \eta \gamma^6 + 6 \eta \gamma^5 + 3 \eta \gamma^4 \right) \right] \]  \hspace{1cm} (4.68)

The two sets of orthogonal pure modes are given by \((\theta_i, \beta_i)\) and \((\theta'_i, \beta'_i)\) (with \(i = 1, 2, 3\)). Note that both sets of modes depend on the crack extension size \(\delta a\). Fig. 4.3 shows the variation of \(\theta_i\) and \(\theta'_i\), as determined from 2D FEM simulations using interfacial point springs\(^{68,96,103}\), with respect to \(\delta a\) for different values of the bimaterial mismatch constant \(\varepsilon\).
Figure 4.3: Variation of the pure mode I conditions $\theta$ and $\theta'$ with respect to the crack extension size $\delta a$ for different values of the bimaterial mismatch constant $\varepsilon$ with $\nu = 0.29$.

Three stages are seen. In the first stage, $\delta a$ is large and $\theta$ and $\theta'$ are separated from each other for all the values of $\varepsilon$. This is caused by the global nature of the partition when $\delta a$ is large. This behaviour is described by Wang and Harvey's Euler beam partition theory for mixed-mode fractures. In the second stage, $\delta a$ is small and $\theta$ and $\theta'$ approach to each other for all the values of $\varepsilon$. This is due to the diminishing global nature of the partition as $\delta a$ reduces in size. In all cases, $\theta$ and $\theta'$ are coincident at approximately $\delta a = 0.05$ mm. Note from Fig. 4.2 that the coincidence of $\theta$ and $\theta'$ does not result in the coincidence of $\theta_K$ and $\theta'_K$. It is also worth noting that after the coincidence of $\theta$ and $\theta'$, (1) for $\varepsilon = 0$ they remain coincident and converged, and (2) for $\varepsilon \neq 0$ they remain coincident for a certain range of $\delta a$ but not converged. In the third stage, $\delta a$ is extremely small and $\theta$ and $\theta'$ remain converged for the case where
\[ c = 0 \) but otherwise diverge away from each other. This is due to the oscillation of the interfacial stresses and relative interfacial displacements near to the crack tip. In the second stage, where the two sets of pure modes, \((\theta_i, \beta_i)\) and \((\theta'_i, \beta'_i)\) (with \(i = 1, 2, 3\)), coincide with each other, the partitions become

\[
G_I = c_I \left( \frac{M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1B}}{\beta_2} - \frac{N_{2B}}{\beta_3}}{\beta_1} \right)^2 \tag{4.69}
\]

\[
G_{II} = c_{II} \left( \frac{M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1B}}{\theta_2} - \frac{N_{2B}}{\theta_3}}{\theta_1} \right)^2 \tag{4.70}
\]

From which the signs of the relative interfacial opening and shear displacements, \(D_n\) and \(D_s\), are then obtained as

\[
\text{sgn}(D_n) = \begin{cases} 
-\text{sgn} \left( \frac{M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1B}}{\beta_2} - \frac{N_{2B}}{\beta_3}}{\beta_1} \right) & \text{if } \beta_1 < 0 \text{ and } \eta > 1 \\
+\text{sgn} \left( \frac{M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1B}}{\beta_2} - \frac{N_{2B}}{\beta_3}}{\beta_1} \right) & \text{otherwise}
\end{cases} \tag{4.71}
\]

\[
\text{sgn}(D_s) = \begin{cases} 
+\text{sgn} \left( \frac{M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1B}}{\theta_2} - \frac{N_{2B}}{\theta_3}}{\theta_1} \right) & \text{if } \theta_1 > 0 \text{ and } \eta < 1 \\
-\text{sgn} \left( \frac{M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1B}}{\theta_2} - \frac{N_{2B}}{\theta_3}}{\theta_1} \right) & \text{otherwise}
\end{cases} \tag{4.72}
\]

Note that in most cases, \(\theta_i\) and \(\beta_i\) are negative and positive respectively, in which the second solutions in Eqs. (4.71) and (4.72) are the correct choices. For some extreme cases however, namely where \(\eta << 1\) and \(\gamma >> 1\), and where \(\eta >> 1\) and \(\gamma << 1\), \(\theta_i\) and \(\beta_i\) can become positive and negative respectively, and the first solutions in Eqs. (4.71) and (4.72) become the correct choices. A more detailed explanation will be given in Section 4.2.5. Also note the reversal of the sign in Eq. (4.72), which is due to the different directions of \(D_s\) in Fig. 4.1b and the work\(^{1-3,12,25}\). Eqs. (4.71) and (4.72) now allow the evaluation of \(S_n\) and \(S_s\) in Eqs. (4.57) and (4.60).
4.2.5. Determining the ERRs, $G_i$ and $G_{II}$

It is seen either from Eq. (4.36), or from Eqs. (4.61) and (4.62), that $K_i$ and $K_{II}$ can be determined if $G_i$ and $G_{II}$ are known first. A powerful methodology has been developed by Wang and Harvey\textsuperscript{1–3,12,25} for mixed-mode partitions based on orthogonal pure modes. Once one pure mode has been found–by analytical, numerical or experimental means–the other pure modes can be determined analytically by using the orthogonality property between them. The ERRs, $G_i$ and $G_{II}$, for a general loading condition can then be calculated by using these pure modes. Readers are referred to Refs. \textsuperscript{1–3,12,25} for a detailed description of the methodology, which is also used here.

The methodology starts by considering the bimaterial DCB shown in Fig. 4.1a but with only the two crack tip bending moments, $M_{1B}$ and $M_{2B}$, and no other loads. The aim is to find the pure mode I relationship $M_{2B} = \theta_i M_{1B}$. Note that $\theta_i$ must be a function of the thickness ratio $\gamma$, the modulus ratio $\eta$ and the Poisson’s ratio $\nu$. In many engineering contexts, the Poisson’s ratios of the top and bottom layers, $\nu_1$ and $\nu_2$, are close to each other\textsuperscript{88}, that is, $\nu_1 \approx \nu_2$. Therefore in the following, $\nu_1 = \nu_2 = \nu$ is assumed. Also it is worth noting that in many of these cases $\nu \approx 1/3$, although the assumption is not required in this work\textsuperscript{88,95}. Furthermore, for a given geometry and loading condition, the total ERR $G$ and bimaterial mismatch coefficient $\varepsilon$ are the major factors that affect the ERR partitions. Therefore, equivalent material properties can be derived for each of the layers, namely, an equivalent elastic modulus and an equivalent Poisson’s ratio, such that both the total ERR $G$ and the bimaterial mismatch coefficient $\varepsilon$ are maintained in an alternative equivalent case. It is anticipated that this will allow cases where $\nu_1 \neq \nu_2$ to be considered using the theory presented in this chapter. More details on this can be found in Chapter 5, where the theory has been extended to account for cases when $\nu_1 \neq \nu_2$.

Due to the violent oscillation of interfacial stresses near to the crack tip in Eqs. (4.13)–(4.15), the value of $\theta_i$ is crack extension size-dependent or FEM mesh size-dependent. In the following, an analytical $\theta_i$ is found for a crack extension size of $\Delta \alpha = 0.05\text{mm}$, at which it is seen from Fig. 4.3 that $\theta_i$ and $\theta_i'$ approximately coincide. This is achieved by considering the ERR partition $G_i/G$ based on the crack extension
size $\delta a = 0.05$ mm with loading condition $M_{2B}/M_{1B} = 0$ and $M_{1B} = 1000$ Nmm versus the modulus ratio $\eta$ for different thickness ratios $\gamma$ and Poisson’s ratios $\nu$. Results obtained from 2D FEM simulations are presented for these loading conditions in Fig. 4.4 for $\nu = 0.29$. A crack extension size of $\delta a = 0.05$ mm corresponded to a maximum mesh density of $1270 \times 146$ when $\gamma = 10$ or $\gamma = 0.1$ and a minimum mesh density of $1270 \times 40$ when $\gamma = 1$.

![Figure 4.4: FEM data for ERR partition $G_f/G$ based on the crack extension size $\delta a = 0.05$ mm with $M_{2B}/M_{1B} = 0$ and $\nu_1 = \nu_2 = 0.29$.](image)

Similar graphs can be obtained for different Poisson’s ratios. The results from the Wang and Harvey$^{1-3,12,25}$ Timoshenko beam partition theory are also presented for the $\gamma = 1$ case, denoted by the thick black line which is labelled $(G_{1T}/G)_{\eta=1,\eta}$. It is interesting to note that the 2D FEM simulation results for this $\gamma = 1$ case approximately coincide with the results from the Wang and Harvey$^{1-3,12,25}$ Timoshenko beam partition theory. Furthermore, Fig. 4.4 shows that when $M_{2B}/M_{1B} = 0$, the 2D FEM partition
results correspond to a non-uniform vertical shift of the Timoshenko beam partition results with $\gamma = 1$. These observations lead to the determination of the pure mode I $\theta_1(\gamma, \eta, \nu)$ mode by means of a shifting technique. From Fig. 4.4, it is seen that when $\gamma < 1$ each of the curves are easily distinguishable, whereas when $\gamma > 1$ the curves are closely grouped together. It is anticipated that this tight grouping would lead to inaccuracies and high sensitivity in the shifting technique. Therefore, the shifting technique is developed for $\gamma \leq 1$. By considering physical symmetry, the method can be used for $\gamma > 1$.

By considering Fig. 4.4, the partition $(G_1 / G)_{\gamma, \eta}$ for the loading cases $M_{2B} / M_{1B} = 0$, based on a non-uniform vertical shift $S(\gamma, \eta, \nu)$, can be written as

$$
\left( \frac{G_1}{G} \right)_{\gamma, \eta, \nu} = \left( \frac{G_{II}}{G} \right)_{\gamma=1, \eta} + S(\gamma, \eta, \nu)
$$

(4.73)

where $\left( G_{II} / G \right)_{\gamma=1, \eta}$ is the partition of ERR from Timoshenko beam theory$^{1-3,12,25}$ with $\gamma = 1$ and variable $\eta$, and is given by

$$
\left( \frac{G_{II}}{G} \right)_{\gamma=1, \eta} = \frac{(7 + \eta)^2}{4(1 + \eta)(13 + \eta)}
$$

(4.74)

Since $(G_1 / G)_{\gamma, \eta, \nu}$ obtained from Eq. (4.73) in terms of the shift $S$ must be the same as $(G_1 / G)_{\gamma, \eta, \nu}$ given by Eq. (4.69) with $M_{2B} / M_{1B} = 0$, which is in terms of $\theta_1$, then

$$
\left( \frac{G_{II}}{G} \right)_{\gamma=1, \eta} + S = c_i M_{1B}^2
$$

(4.75)

where $G = C_1 M_{1B}^2 / (2E_1 b^2)$, as given by Eq. (4.1) with $M_{2B} = N_{1B} = N_{2B} = 0$, and $c_i$ is given by Eq. (4.65) with $\beta'_i = \beta_i$ (since $\delta = 0.05$ mm). Note that $\beta_i$ is orthogonal to $\theta_i$ and that therefore $\beta_i = -(C_{1i} + C_{12i})/(C_{12} + C_{22i})$ with $C_{ij}$ given in Eqs. (4.2), (4.3) and (4.6). The resulting relationship between $\theta_i$ and $S$ is then obtained as

$$
\frac{C_{11}}{2E_1 b^2} \left[ \left( \frac{G_{II}}{G} \right)_{\gamma=1, \eta} + S \right] = G_\theta_1 \left[ \frac{C_{11} + C_{12i} \theta_i}{C_{22i} \theta_i^2 + 2C_{12i} \theta_i + C_{11}} \right]^2
$$

(4.76)
where \( G_{\theta_i} \) is given by Eq. (4.67). Eq. (4.76) can now be solved for \( \theta_i \) in terms of the shift \( S \), which gives two possible solutions. Using the orthogonality condition between \( \theta_i \) and \( \beta_i \), there are also two corresponding solutions for \( \beta_i \). The algebraic expression for each solution is too long to be presented here, however Eq. (4.76) is easy to solve for numerical values of \( \gamma \), \( \eta \) and \( S \). To proceed, it is essential that the correct solutions for \( \theta_i \) and \( \beta_i \) are determined. Therefore, Fig. 4.5 shows a flow chart of how to select the correct sign for \( \beta_i \) and then the corresponding value of \( \theta_i \) can be selected.

![Flowchart](image)

Figure 4.5: Selection of the correct \( \beta_i \) solution.
Chapter 4: Dissimilar laminated beams with bending moments and axial forces

The procedure in Fig. 4.5 to determine the correct sign for $\beta_i$ will now be discussed. Initially, by considering Fig. 4.4, it is seen that for some values of $\gamma$, there exists a value of $\eta$, denoted here by $\eta_{\beta}$, at which the partition $G_i/G$ reaches zero, that is, when $\eta = \eta_{\beta}$ then $\beta_1 = 0$. Therefore the first step is to identify whether $\eta_{\beta}$ exists by solving Eq. (4.73) with $(G_i/G)_{\gamma, \eta, \nu} = 0$ where $S(\gamma, \eta, \nu)$ is the non-uniform vertical shift and will be determined shortly. Therefore, if $\eta_{\beta}$ does not exist, then the positive solution of $\beta_i$ must be selected by using the corresponding solution of $\theta_i$ from Eq. (4.76) which gives a positive $\beta_i$ value through the orthogonality condition.

If $\eta_{\beta}$ does exist then if the solutions of $\beta_i$ from Eq. (4.76) have opposite signs then if $\eta > \eta_{\beta}$, or $\log_{10} \left( \frac{1}{\eta} \right) < \log_{10} \left( \frac{1}{\eta_{\beta}} \right)$, select the negative solution. For $\eta < \eta_{\beta}$, or $\log_{10} \left( \frac{1}{\eta} \right) > \log_{10} \left( \frac{1}{\eta_{\beta}} \right)$, choose the positive solution.

The above procedure for choosing the correct solution for $\beta_i$ works very well when the two solutions have opposite signs, which is usually the case. However, when $\gamma$ becomes close to 1, sometimes the two solutions for $\beta_i$ have the same sign. When this happens, it is still easy to determine the correct solution because when $\gamma \approx 1$ then $\theta_i \approx -\eta \gamma^2$ and $\beta_i \approx \eta \gamma^2 \left( \eta \gamma^2 + 4 \gamma + 3 \right) \left( 3 \eta \gamma^2 + 4 \eta \gamma + 1 \right)$, which are the values based on Timoshenko beam theory. The pair of solutions which are closest to these values are the correct ones.

Now more detailed explanations for Eqs. (4.71) and (4.72) can be given. Consider Eq. (4.71) with $M_{1B} > 0$ and $M_{2B} = N_{1B} = N_{2B} = 0$. Since $D_n$ is positive when $\eta < \eta_{\beta}$, or $\log_{10} \left( \frac{1}{\eta} \right) > \log_{10} \left( \frac{1}{\eta_{\beta}} \right)$, and $D_n = 0$ when $\eta = \eta_{\beta}$, therefore when $\eta > \eta_{\beta}$, or $\log_{10} \left( \frac{1}{\eta} \right) < \log_{10} \left( \frac{1}{\eta_{\beta}} \right)$, then $D_n$ must be negative. A similar argument can be used for Eq. (4.72).

It now remains to find the non-uniform vertical shift $S$ for $\gamma \leq 1$. The shift $S$ is assumed to be in the following form

$$S(\gamma, \eta, \nu) = S_0(\gamma, \nu) + S_1(\gamma, \nu) \log_{10} \left( \frac{1}{\eta} \right) + S_2(\gamma, \nu) \left[ \log_{10} \left( \frac{1}{\eta} \right) \right]^2 \quad (4.77)$$
The coefficients, $S_0$, $S_1$ and $S_2$, are determined empirically by considering $\eta = 1, 1/10, 1/100, 10, 100$ in Eqs. (4.73) and (4.77). First consider $\eta = 1$, which means that Eq. (4.77) reduces to $S(\gamma, \eta = 1, \nu) = S_0(\gamma, \nu)$, therefore Eq. (4.73) can be rearranged to give an expression for $S_0(\gamma, \nu)$ as

$$S_0(\nu) = \left( \frac{G_I}{G} \right)_{\gamma, \eta = 1} - \left( \frac{G_{II}}{G} \right)_{\gamma = 1, \eta = 1} \tag{4.78}$$

Similarly follow the same procedure with $\eta = 1/10$ giving

$$S_1(\nu) + S_2(\nu) = \left( \frac{G_I}{G} \right)_{\gamma, \eta = 1/10, \nu} - \left( \frac{G_{II}}{G} \right)_{\gamma = 1, \eta = 1/10} - S_0 \tag{4.79}$$

Next use $\eta = 1/100$ giving

$$S_1(\nu) + 2S_2(\nu) = \frac{1}{2} \left[ \left( \frac{G_I}{G} \right)_{\gamma, \eta = 1/100, \nu} - \left( \frac{G_{II}}{G} \right)_{\gamma = 1, \eta = 1/100} - S_0 \right] \tag{4.80}$$

Using $\eta = 10$ gives

$$S_2(\nu) - S_1(\nu) = \left( \frac{G_I}{G} \right)_{\gamma, \eta = 10, \nu} - \left( \frac{G_{II}}{G} \right)_{\gamma = 1, \eta = 10} - S_0 \tag{4.81}$$

Finally $\eta = 100$ gives

$$2S_2(\nu) - S_1(\nu) = \frac{1}{2} \left[ \left( \frac{G_I}{G} \right)_{\gamma, \eta = 100, \nu} - \left( \frac{G_{II}}{G} \right)_{\gamma = 1, \eta = 100} - S_0 \right] \tag{4.82}$$

Chapter 2 has presented an accurate expression for $(G_I/G)_{\gamma, \eta = 1}$ in Eq. (4.78), which has been shown to be independent of $\nu$. From 2D FEM simulations with the loading condition $M_{2B}/M_{1B} = 0$ values for $(G_I/G)_{\gamma, \eta = 1/10, \nu}$, $(G_I/G)_{\gamma, \eta = 1/100, \nu}$, $(G_I/G)_{\gamma, \eta = 10, \nu}$ and $(G_I/G)_{\gamma, \eta = 100, \nu}$ for different values of $\gamma$ and $\nu$ can be obtained. When used in conjunction with Eqs. (4.78)–(4.82), the following accurate empirical formulae are obtained for $S_1$ and $S_2$ with $\gamma \leq 1$

$$S_1 = S_{11}(\nu) \log_{10}(1/\gamma) + S_{12}(\nu) \left[ \log_{10}(1/\gamma) \right]^2 + S_{13}(\nu) \log_{10}(1/\gamma) + S_{10} \tag{4.83}$$
\[
S_2 = S_{23} (v) \left[ \log_{10} (1/\gamma) \right]^3 + S_{22} (v) \left[ \log_{10} (1/\gamma) \right]^2 + S_{21} (v) \log_{10} (1/\gamma) + S_{20} \tag{4.84}
\]

where \( S_{ij} \) in the plane strain condition are given by

\[
S_{13} = \begin{cases} 
-0.179784 \nu^2 + 0.150620 \nu - 0.045250 & \text{if } \eta < 1 \\
-1.707125 \nu^2 + 0.098035 \nu + 0.327941 & \text{if } \eta > 1 
\end{cases}
\tag{4.85}
\]

\[
S_{12} = \begin{cases} 
0.487995 \nu^2 - 0.208470 \nu + 0.024941 & \text{if } \eta < 1 \\
3.145748 \nu^2 - 0.592065 \nu - 0.653980 & \text{if } \eta > 1 
\end{cases}
\tag{4.86}
\]

\[
S_{11} = \begin{cases} 
-0.404560 \nu^2 - 0.109477 \nu + 0.084873 & \text{if } \eta < 1 \\
-1.510591 \nu^2 + 0.668378 \nu + 0.394354 & \text{if } \eta > 1 
\end{cases}
\tag{4.87}
\]

\[
S_{10} = \begin{cases} 
-1.339729 \nu^2 - 0.112935 \nu + 0.164729 & \text{if } \eta < 1 \\
-1.387396 \nu^2 - 0.281472 \nu + 0.219649 & \text{if } \eta > 1 
\end{cases}
\tag{4.88}
\]

\[
S_{23} = \begin{cases} 
0.055138 \nu^2 - 0.054976 \nu + 0.010606 & \text{if } \eta < 1 \\
-1.184839 \nu^2 + 0.480577 \nu + 0.139204 & \text{if } \eta > 1 
\end{cases}
\tag{4.89}
\]

\[
S_{22} = \begin{cases} 
-0.154013 \nu^2 + 0.107145 \nu - 0.004940 & \text{if } \eta < 1 \\
1.591445 \nu^2 - 0.995923 \nu - 0.201936 & \text{if } \eta > 1 
\end{cases}
\tag{4.90}
\]

\[
S_{21} = \begin{cases} 
0.134731 \nu^2 - 0.038117 \nu - 0.016633 & \text{if } \eta < 1 \\
-0.307568 \nu^2 + 0.335473 \nu + 0.139806 & \text{if } \eta > 1 
\end{cases}
\tag{4.91}
\]

\[
S_{20} = \begin{cases} 
0.359714 \nu^2 + 0.087096 \nu - 0.063181 & \text{if } \eta < 1 \\
-0.381264 \nu^2 - 0.090766 \nu + 0.066329 & \text{if } \eta > 1 
\end{cases}
\tag{4.92}
\]

and \( S_{ij} \) in the plane stress condition are given by

\[
S_{13} = \begin{cases} 
-0.047964 \nu^2 + 0.091462 \nu - 0.045410 & \text{if } \eta < 1 \\
-0.316860 \nu^2 - 0.142473 \nu + 0.333225 & \text{if } \eta > 1 
\end{cases}
\tag{4.93}
\]

\[
S_{12} = \begin{cases} 
0.086063 \nu^2 - 0.084216 \nu + 0.024542 & \text{if } \eta < 1 \\
0.701381 \nu^2 - 0.074332 \nu - 0.666427 & \text{if } \eta > 1 
\end{cases}
\tag{4.94}
\]

\[
S_{11} = \begin{cases} 
0.021738 \nu^2 - 0.170242 \nu + 0.085179 & \text{if } \eta < 1 \\
-0.465320 \nu^2 + 0.358274 \nu + 0.402455 & \text{if } \eta > 1 
\end{cases}
\tag{4.95}
\]
\[ S_{10} = \begin{cases} -0.088802\nu^2 - 0.327669\nu + 0.170305 & \text{if } \eta < 1 \\ -0.018717\nu^2 - 0.486112\nu + 0.224870 & \text{if } \eta > 1 \end{cases} \] (4.96)

\[ S_{23} = \begin{cases} 0.007841\nu^2 - 0.032222\nu + 0.010789 & \text{if } \eta < 1 \\ -0.388178\nu^2 + 0.264900\nu + 0.144129 & \text{if } \eta > 1 \end{cases} \] (4.97)

\[ S_{22} = \begin{cases} -0.028062\nu^2 + 0.057052\nu - 0.004924 & \text{if } \eta < 1 \\ 0.627350\nu^2 - 0.649216\nu - 0.210480 & \text{if } \eta > 1 \end{cases} \] (4.98)

\[ S_{21} = \begin{cases} 0.018448\nu^2 - 0.006505\nu - 0.016887 & \text{if } \eta < 1 \\ -0.153962\nu^2 + 0.244120\nu + 0.142364 & \text{if } \eta > 1 \end{cases} \] (4.99)

\[ S_{20} = \begin{cases} -0.004950\nu^2 + 0.139969\nu - 0.064582 & \text{if } \eta < 1 \\ 0.005332\nu^2 - 0.147687\nu + 0.067805 & \text{if } \eta > 1 \end{cases} \] (4.100)

When the crack extension size \( \delta a = 0.05 \text{ mm} \) then, as has been seen in Fig. 4.3, \( \theta'_i \approx \theta_i \) and \( \beta'_i \approx \beta_i \). Therefore, negative \( G_i \) or \( G_{ii} \) does not occur (similarly, neither does \( G_i > G \) or \( G_{ii} > G \)). The shift \( S \), given by Eq. (4.77), must therefore not result in \( G_i/G < 0 \) or \( G_i/G > 1 \). Due to small numerical inaccuracies, the shift must be capped in some cases to prevent this from happening. From Eq. (4.73),

\[ S = \begin{cases} 1 - \left( \frac{G_{ii}}{G} \right)_{\gamma=1,\eta} & \text{if } S > 1 - \left( \frac{G_{ii}}{G} \right)_{\gamma=1,\eta} \\ - \left( \frac{G_{ii}}{G} \right)_{\gamma=1,\eta} & \text{if } S < - \left( \frac{G_{ii}}{G} \right)_{\gamma=1,\eta} \end{cases} \] (4.101)

It is now possible to determine the pure mode I \( \theta_i(\gamma,\eta,\nu) \) when \( \gamma > 1 \). Using the symmetry of the beam, it is a requirement that

\[ \theta_i(\gamma,\eta,\nu) = \theta_i^{-1}(\gamma^{-1},\eta^{-1},\nu) \] (4.102)

Therefore, when \( \gamma > 1 \), \( \theta_i(\gamma,\eta,\nu) \) can be found by first finding \( \theta_i(\gamma^{-1},\eta^{-1},\nu) \) with the previously derived method and then using Eq. (4.102).

The pure mode I \( \theta_i \) modes and the pure mode II \( \beta_i \) modes (with \( i = 1, 2, 3 \)), which are required in Eqs. (4.69) and (4.70), are determined using the orthogonal
methodology\textsuperscript{1–3,12,25}. The pure mode II $\beta_i$ mode is orthogonal to the pure mode I $\theta_i$ mode. This is written as

$$\beta_i = \text{orthogonal}(\theta_i)$$

which is equivalent to

$$0 = [1 \theta_i 0 0][C][1 \beta_i 0 0]^T$$

where $[C]$ is given in Eqs. (4.1)–(4.12). Therefore, if $\theta_i$ is known, then $\beta_i$ can easily be determined. Similarly,

$$\theta_2 = \text{orthogonal}(\beta_i) \text{ or } [1 0 \theta_2 0][C][1 \beta_i 0 0]^T = 0$$

$$\theta_3 = \text{orthogonal}(\beta_i) \text{ or } [1 0 \theta_3 0][C][1 \beta_i 0 0]^T = 0$$

$$\beta_2 = \text{orthogonal}(\theta_i) \text{ or } [1 0 \beta_2 0][C][1 \theta_i 0 0]^T = 0$$

$$\beta_3 = \text{orthogonal}(\theta_i) \text{ or } [1 0 \beta_3 0][C][1 \theta_i 0 0]^T = 0$$

The ERRs, $G_i$ and $G_{ii}$, at a crack extension size of $\delta a = 0.05$ mm can now be obtained from Eqs. (4.69) and (4.70). The various pairs of mathematically admissible SIFs, $K_i$ and $K_{ii}$, can be obtained from Eqs. (4.36)–(4.42). The only mechanically admissible pair is chosen by using Eqs. (4.61) and (4.62) as a guide. The ERR partitions, $G_i$ and $G_{ii}$, for all crack extension sizes, $\delta a$, can then be calculated from Eqs. (4.31) and (4.32) or from Eqs. (4.43) and (4.44).

4.3. Numerical verification

To verify the present analytical theory and the supporting mathematical techniques that have previously been developed for brittle interfacial cracking between two dissimilar elastic layers, a thorough program of 2D FEM simulations was carried out parametrically on the DCB shown in Fig. 4.1a using MSC/NASTRAN. The Young’s modulus ratio $\eta = E_2/E_1$ was varied in the range $1/100 \leq \eta \leq 100$; the Poisson’s ratio $\nu_1 = \nu_2 = \nu$ was varied in the range $0 \leq \nu < 0.5$; the thickness ratio $\gamma = h_2/h_1$ was varied in the range $1/10 \leq \gamma \leq 10$; and the DCB tip loads, $M_2$, $N_i$ and $N_2$, were varied in the range $-20000 \leq M_2 \leq 212000$ Nmm and $N_i \leq 20000$ Nmm with $M_1 = 1000$ Nmm.
this way the entire practically useful domain of cracking between bimaterial layers was considered as these are typical values that would be expected to be seen in real world engineering applications of the DCB. Furthermore, the theory is applicable to other loading conditions, DCB geometry and material properties outside of the values given here, however due to the time constraints and computational cost involved it was not practical to obtain results from further FEM simulations.

The top and bottom layers in the DCB were modelled using quadrilateral plane-strain shell elements with a thickness of \( b = 10 \text{ mm} \) and isotropic material properties within each layer. The minimum Young’s modulus \( E_{\text{min}} = 1000 \text{ N/mm}^2 \). The Young’s modulus of the top and bottom layers therefore varied with \( E_1 = E_{\text{min}} \) and \( E_2 = \eta E_{\text{min}} \) respectively if \( \eta > 1 \), and with \( E_1 = E_{\text{min}} / \eta \) and \( E_2 = E_{\text{min}} \) if \( \eta < 1 \). The shear modulus was calculated as \( \mu_i = E_i / [2(1+\nu_i)] \) with \( i = 1,2 \). The minimum beam thickness \( h_{\text{min}} = 1 \). The thickness of the top and bottom layers therefore varied with \( h_1 = h_{\text{min}} \) and \( h_2 = \gamma h_{\text{min}} \) respectively if \( \gamma > 1 \), and with \( h_1 = h_{\text{min}} / \gamma \) and \( h_2 = h_{\text{min}} \) if \( \gamma < 1 \). The uncracked length of the DCB \( L = 100 \text{ mm} \) and the cracked length \( a = 10 \text{ mm} \). The interface between the top and bottom layers was modelled with normal and shear point springs with a stiffness of \( k_s = 10^{11} \text{ N/mm} \), which was sufficiently high in comparison to \( E_1 \) and \( E_2 \) to simulate brittle interfacial cracking without introducing excessive numerical error. Contact between the upper and lower surfaces of the crack was not considered.

Generally, the desired crack extension size \( \delta a \) (over which the ERR was calculated) determined the size of the elements surrounding the crack tip, rather than the requirement for mesh independence. Since \( \delta a \) is typically a small quantity (\( \delta a \ll a \)), non-uniform meshes were used in order to avoid excessive computation. Up to 2000 square elements of size \( p \times p \) were centred on the crack tip in the x-direction, and up to 100 square elements were centred on the crack tip in the y-direction. Unless noted otherwise, in this section \( p = 0.01 \) is used. The crack extension size \( \delta a \) is specified for each set of simulations. Beyond the region of uniform element size surrounding the crack tip, elements were allowed to grow at a constant rate of 1.1 in both the x- and y-directions up to a maximum size of 1.0 and 0.1 respectively, after which they remained at this maximum size. Very small adjustments were made to the element size growth
rate where necessary to satisfy the boundary geometry. As the total length of the DCB is unchanged in the simulations the number of elements in the x-direction was a constant for a given value of $p$ and the number in the y-direction depended on the thickness ratio $\gamma$. Therefore when $p = 0.01$ the maximum mesh density was $2128 \times 229$ when $\gamma = 10$ or $\gamma = 0.1$ and the minimum mesh density was $2128 \times 138$ when $\gamma = 1$. A convergence study for this section can be found in the Appendix A. Axial forces, $N_1$ and $N_2$, were applied as point forces to the ends of the top and bottom layers respectively and were uniformly-distributed by area. Bending moments, $M_1$ and $M_2$, were applied as equal and opposite axial forces in the top- and bottom-right corners of the top and bottom layers respectively. As this is the same force application as Chapter 3, the reader is directed to Fig. 3.3 to see the application of forces on the upper or lower beam. As the interface is rigid, the ERRs were calculated using the VCCT and interfacial point springs and as many spring pairs as exist inside the specified crack extension size $\delta a$. For example, if the element size at the crack tip is $p = 0.01$ and the ERRs are to be calculated for a crack extension size of $\delta a = 0.05$ mm, then five normal and shear springs must be used in the VCCT calculation for $G_I$ and $G_{II}$ respectively.

4.3.1. Shifting technique

The shifting technique is based on the observations that when $\delta a = 0.05$ mm and $M_{2B}/M_{1B} = 0$, (1) the ERR partitions from FEM simulations with $\gamma = 1$ and $\nu = 0.29$ are very close to the results from the Wang and Harvey’s Timoshenko beam partition theory for the entire range of the modulus ratio $1/100 \leq \eta \leq 100$ with $\gamma = 1$; and (2) the FEM partition results for the entire ranges of $1/100 \leq \gamma \leq 10$, $0 \leq \nu \leq 0.5$ and $1/100 \leq \eta \leq 100$ correspond to a non-uniform shift along the $G_I/G_{II}$-axis of the Timoshenko beam partition results with $\gamma = 1$.

Fig. 4.6 confirms that the shifting technique accurately predicts the ERR partitions from FEM simulations with $\delta a = 0.05$ mm, $M_{2B}/M_{1B} = 0$, $M_{1B} = 1000$ Nmm and $\nu = 0.29$. A crack extension size of $\delta a = 0.05$ mm corresponded to a maximum mesh density of $1270 \times 146$ when $\gamma = 0.1$ and a minimum mesh density of $1270 \times 40$ when
\( \gamma = 1 \). In the figure, there is a marker and a line for each value of \( \gamma \). The marker indicates the value from the FEM simulation, and the line indicates the analytical value, which is based on the shifting technique. Results are only shown for \( \gamma \leq 1 \) since the shift has only been determined for this region. There is excellent agreement between the value of \( G_i/G \) determined from the shifting technique and that determined from the FEM for all values of \( \gamma \) and \( \eta \). The shifting technique will be further tested in Section 4.3.2 with different values of the Poisson’s ratio \( \nu \).

Figure 4.6: Comparison of the shifting technique (line) and FEM data (markers) for the ERR partition \( G_i/G \) at crack extension size \( \Delta a = 0.05 \text{ mm} \) with \( M_{2B}/M_{1B} = 0 \) and \( \nu = 0.29 \).

4.3.2. Calculating the complete set of orthogonal pure modes

The shifting technique, verified in Section 4.3.1, allows pure mode I \( \theta_1 \) mode to be determined directly for given values of \( \gamma \), \( \eta \) and \( \nu \). It is then easy to determine the complete set of pure modes by making use of the orthogonality condition. If the values of \( \theta_1 \) and its orthogonal pure modes, \( \beta_1 \), \( \theta_2 \), \( \beta_2 \), \( \theta_3 \) and \( \beta_3 \), are correct, then \( G_i \) and
$G_{II}$ can be calculated analytically for any combination of $M_{1B}$, $M_{2B}$, $N_{1B}$ and $N_{2B}$ with $\ddot{a} = 0.05$ mm, and they will agree well with those values calculated from FEM simulations. If the pure modes are not accurately calculated, then it will not be possible to get agreement between the present analytical theory and the FEM for other loading conditions besides $M_{2B}/M_{1B} = 0$. The purpose of this section is to assess the accuracy of the pure modes, $\theta_1$, $\beta_1$, $\theta_2$, $\beta_2$, $\theta_3$ and $\beta_3$. Note that no SIFs are involved in this section.

Fig. 4.7 compares $G_{II}$ from the present analytical theory (lines) with that from the FEM (markers) for $M_{2B}/M_{1B} = -1$ and $\nu = 0.29$ for different values of $\gamma$ and $\eta$. For cases where $\gamma > 1$, $\theta_1$ and $\beta_1$ are determined by making use of physical symmetry, i.e. Eq. (4.102). Fig. 4.7 shows excellent agreement.
between the present analytical theory and the FEM data. The small “dips” in the present
analytical theory at $\log_{10}(l/\gamma) = \pm 1$ and $\log_{10}(l/\eta) \approx \mp 1.5$ can be explained by
considering Fig. 4.6. When $\log_{10}(l/\gamma) = 1$ and $\log_{10}(l/\eta) \approx -1.5$, the ERR partition
$G_1/G$ becomes close to zero and therefore $\beta_1$ also becomes close to zero. Any
numerical inaccuracies in the empirical shifting technique, from which $\theta_1$ and the
orthogonal $\beta_1$ are calculated, therefore become magnified and this makes it more
difficult to calculate $\theta_1$ and $\beta_1$ with the same degree of accuracy. Furthermore, recall
that when $\delta a = 0.05$ mm then $\theta'_1 \approx \theta_1$ and $\beta'_1 \approx \beta_1$. Because $\theta_1$ and $\theta'_1$ do not coincide
exactly when $\delta a = 0.05$ mm (and neither do $\beta_1$ and $\beta'_1$), it is possible for $G_1/G$ to
become just less than 0, or just greater than 1. The present analytical theory cannot
allow this, so the amount of shift is capped in some cases to prevent this from
happening. For $\log_{10}(l/\gamma) = 1$, capping is employed when $\log_{10}(l/\eta) \leq -1.5$. The effect
of this is the small dip in the analytical theory in Fig. 4.7 at $\log_{10}(l/\gamma) = 1$ and
$\log_{10}(l/\eta) = -1.5$, and also the small dip at $\log_{10}(l/\gamma) = -1$ and $\log_{10}(l/\eta) = 1.5$ due to
physical symmetry.
Figure 4.8: Comparison of the present analytical theory and FEM data for the ERR partition $G_i/G$ for variable $\gamma$, $\eta$ and $\nu$ at crack extension size $\delta a = 0.05$ mm with $M_{2B}/M_{1B} = 0$ and $M_{2B}/M_{1B} = -1$.

Fig. 4.8a shows the effect of Poisson’s ratio on the shifting technique and Fig. 4.8b shows its ability to determine the pure modes. The figure shows that the shifting technique works extremely well for $\nu \geq 0.29$, however, for smaller values of $\nu$, the adverse effect of capping on the shift increases for the same ranges of $\gamma$ and $\eta$ noted above and becomes severe for very small values of $\nu$. As $\nu$ increases above 0.3, the capping rapidly disappears entirely and this explains the excellent performance in this region. Although a Poisson’s ratio value of $\nu = 0$ is not practically possible, it has been
included to give the reader confidence in the analytical theories ability to produce accurate results for other Poisson’s ratio values, even extreme ones. Overall, it is concluded that the shifting technique works very well and that $\theta_1$ and $\beta_1$ can be calculated with high accuracy when $\delta a = 0.05 \text{ mm}$ over the entire ranges of $1/100 \leq \eta \leq 100$, $0 \leq \nu < 0.5$ and $1/10 \leq \gamma \leq 10$, except for the small “corner” locations which represent very thin stiff layer spalling. Therefore, to avoid repetitive verification in the following tests, a fixed Poisson’s ratio of $\nu = 0.29$ is used, as Ref. 95 states that for most isotropic materials it is in that range $0.25 \leq \nu \leq 0.33$, meaning the middle value has been selected.

![Figure 4.9: Comparison of the present analytical theory (lines) and FEM data (markers) for the ERR partition $G_1/G$ at crack extension size $\delta a = 0.05 \text{ mm}$ with $N_{1b}/M_{1b} = 10 \text{ mm}^{-1}$.](image_url)

Figs. 4.9 and 4.10 assess the accuracy of the $\theta_2$ and $\theta_3$ values respectively (and the orthogonal values of $\beta_2$ and $\beta_3$), obtained from the use of the shifting technique and the
orthogonality condition. The loading case in Fig. 4.9 is $N_{1B}/M_{1B} = 10 \text{ mm}^{-1}$ and $M_{1B} = 1000 \text{ Nmm}$ with $M_{2B} = N_{2B} = 0$, and the loading case in Fig. 4.10 is $N_{2B}/M_{1B} = 10 \text{ mm}^{-1}$ and $M_{1B} = 1000 \text{ Nmm}$ with $M_{2B} = N_{1B} = 0$. The agreement between the present analytical theory and the FEM is excellent everywhere, but slightly less good for the same values of $\gamma$ and $\eta$ noted previously for Fig. 4.7. Overall, it is concluded that $\theta_2$, $\beta_2$, $\theta_3$ and $\beta_3$ can also be calculated with high accuracy when $\delta a = 0.05 \text{ mm}$ by using the shifting technique and the orthogonality condition.

Figure 4.10: Comparison of the present analytical theory (lines) and FEM data (markers) for the ERR partition $G_i/G$ at crack extension size $\delta a = 0.05 \text{ mm}$ with $N_{2B}/M_{1B} = 10$.

4.3.3. Calculating and choosing the SIFs

Section 4.3.2 confirms that the ERR components, $G_i$ and $G_{II}$, can be accurately determined for any combination of $M_{1B}$, $M_{2B}$, $N_{1B}$ and $N_{2B}$ with $\delta a = 0.05 \text{ mm}$ and
for any given values of $\gamma$ and $\eta$. Sun and Qian\textsuperscript{44} and the present work have shown that for given values of $G_I$ and $G_{II}$, there are four pairs of solutions for $K_I$ and $K_{II}$, of which only one is mechanically admissible. In the work of Sun and Qian\textsuperscript{44}, FEM simulations were used to determine the correct pair. In Section 4.2.3, a method was devised to guide the selection of the correct pair by purely analytical means, in which an approximate pair of SIFs, $K_I$ and $K_{II}$, is calculated by partitioning the total ERR $G$ using approximate SIF-based orthogonal pure modes, $\tilde{\theta}_K$ and $\tilde{\beta}_K$. The purpose of this section is to verify that (1) the approximate pair of SIFs can correctly and robustly guide the choice of the one physically correct accurate pair of SIFs from among the four that are provided mathematically, and that (2) the chosen SIFs are accurate.

Figs. 4.11 and 4.12 compare $K_I$ and $K_{II}$ respectively for $M_{2B}/M_{1B} = 0$ and $M_{1B} = 1000$ Nmm with $\delta a = 0.05$ mm for different values of $\gamma$ and $\eta$, as obtained from the FEM (markers), the present analytical theory (lines), and the approximate analytical method (lines with small markers). Note that these figures are counterparts to Fig. 4.6. The FEM and the present analytical theory each provide $G_I$ and $G_{II}$ for $\delta a = 0.05$ mm (as in Section 4.3.2). The SIFs are then calculated by using the relationships between $G_I$, $G_{II}$ and $K_I$, $K_{II}$ seen in Section 4.2.3. In both cases, only the pair of SIFs that is closest to the approximate pair of SIFs is shown. For all values of $\gamma$ and $\eta$, excellent agreement is observed between the SIFs from the present analytical theory and those from the FEM. Furthermore, very close agreement is also observed between these SIFs and the approximate ones. Small discrepancies are seen, but this is of course expected due to it being an approximate method. For example, the small jump seen in Figs. 4.11 and 4.12 for $\log_{10}(1/\gamma) = -0.8$ at $\log_{10}(1/\eta) = 1.5$ is due to the approximate stress intensity factors in Eqs. (4.61) and (4.62) becoming closer to a different solution of the stress intensity factors and therefore the solution switches. The important observation given by Figs. 4.11 and 4.12 is that the approximate SIFs are close enough to one of the four mathematical pairs of SIFs to accurately guide the correct choice.
Figure 4.11: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_I$ at crack extension size $\delta a = 0.05$ mm with $M_{2B}/M_{1B} = 0$. 

$K_I \left[ \text{N/mm}^3 \right]$ vs. $\log_{10}(1/\eta)$.
Figure 4.12: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{II}$ at crack extension size $\delta a = 0.05$ mm with $M_{2B}/M_{1B} = 0$.

Figs. 4.13 and 4.14, as the counterpart to Fig. 4.7, compare $K_I$ and $K_{II}$ respectively for $M_{2B}/M_{1B} = -1$ and $M_{1B} = 1000$ Nmm with $\delta a = 0.05$ mm for different values of $\gamma$ and $\eta$. Figs. 4.15 and 4.16, as the counterpart to Fig. 4.9, compare $K_I$ and $K_{II}$ respectively for $N_{iB}/M_{1B} = 10$ mm$^{-1}$ and $M_{1B} = 1000$ Nmm with $\delta a = 0.05$ mm. Figs. 4.17 and 4.18, as the counterpart to Fig. 4.10, compare $K_I$ and $K_{II}$ respectively for $N_{2B}/M_{1B} = 10$ mm$^{-1}$ and $M_{1B} = 1000$ Nmm with $\delta a = 0.05$ mm. In all these figures, markers represent data obtained from the FEM, lines represent the present analytical theory, and lines with small markers represent the approximate analytical method. As before, (1) excellent agreement is observed between the SIFs from the present analytical theory and those from the FEM, and (2) very close agreement is also observed between these SIFs and the approximate ones. Again the jumps in Figs. 4.13–4.18 are due to the
approximate stress intensity factors becoming closer to a different solution for the stress intensity factor and therefore the solution switches.

![Graph](image_url)

**Figure 4.13**: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_I$ at the crack extension size $\Delta a = 0.05$ mm with $M_{2\beta}/M_{1\beta} = -1$. 
Figure 4.14: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{II}$ at crack extension size $\hat{a} = 0.05$ mm with $M_{2\theta}/M_{1\theta} = -1$. 
Figure 4.15: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_I$ at crack extension size $\delta a = 0.05 \text{ mm}$ with $N_{1B}/M_{1B} = 10 \text{ mm}^{-1}$.
Figure 4.16: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{II}$ at crack extension size $\Delta a = 0.05 \text{ mm}$ with $N_{1B}/M_{1B} = 10 \text{ mm}^{-1}$. 
Figure 4.17: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_I$ at crack extension size $\delta a = 0.05$ mm with $N_{2B}/M_{1B} = 10$ mm$^{-1}$. 
Figure 4.18: Comparison of the present analytical theory (lines), approximate analytical method (lines with small markers) and FEM data (markers) for the SIF $K_{II}$ at crack extension size $\delta \alpha = 0.05 \text{ mm}$ with $N_{2B}/M_{1B} = 10 \text{ mm}^{-1}$.

To further verify that the approximate SIFs can guide the choice of the correct pair of SIFs and to confirm the accuracy of the chosen SIFs, Fig. 4.19 compares the ERR partitions $G_I/G$ from the present analytical theory (lines) and from the FEM (markers) at a different crack extension size of $\delta \alpha = 0.01 \text{ mm}$. The analytical $G_I/G$ is calculated using the relationships between $G_I, G_{II}$ and $K_I, K_{II}$ in Section 4.2.3. In the present analytical theory, $K_I$ and $K_{II}$ are calculated at a crack extension size of $\delta \alpha = 0.05 \text{ mm}$, that is, by using the ERRs, $G_I$ and $G_{II}$, at $\delta \alpha = 0.05 \text{ mm}$. If they are correctly chosen and accurately calculated, the analytical $G_I/G$ at $\delta \alpha = 0.01 \text{ mm}$ should be in excellent agreement with the $G_I/G$ from FEM simulations at $\delta \alpha = 0.01 \text{ mm}$. Fig. 4.19 shows that this is indeed the case. Note that the effect of capping is again evident at $\log_{10}(1/\eta) = \pm 1$ and $\log_{10}(1/\eta) \approx \pm 1.5$. 

$\log_{10}(1/\eta) = -0.8, -0.4, -0.2, 0.2, 0.4, 0.6, 0.8$
Overall, it can be concluded that the present analytical theory can robustly choose the physically correct pair of SIFs from among the four that are provided mathematically. Also the present analytical theory accurately calculates the SIFs. Excellent agreement is seen between the present analytical theory and the FEM data for almost the entire domain of $\gamma$ and $\eta$.

4.3.4. Calculating the ERR partitions

So far in this numerical verification, Sections 4.3.1 to 4.3.3 combined have shown that the present analytical theory can accurately calculate the SIFs, $K_I$ and $K_{II}$, for any combination of $M_{1B}$, $M_{2B}$, $N_{1B}$ and $N_{2B}$, and for any given values of $\gamma$ and $\eta$. To reinforce this verification, more extensive comparisons between the ERR partitions $G_{I}/G$ from the present analytical theory and from the FEM are presented in this
section, with the aim of showing that the present analytical theory accurately determines the crack size-dependent ERRs, $G_I$ and $G_{II}$.

Figs. 4.20 and 4.21 show the difference between the values of $G_I/G$ obtained from 2D FEM simulations and the values predicted by the present analytical theory over the entire practically useful domain of cracking between bimaterial layers. The error is obtained by subtracting the partition $G_I/G$ from the FEM simulations away from $G_I/G$ using the present analytical theory. By taking the modulus of the error, the result is plotted on a contour plot by associating a colour to the given error. In this case dark blue is associated to zero error and dark red to the maximum error.

The Young’s modulus ratio $\eta$ was varied in the range $100 \leq \eta \leq 100$; the thickness ratio $\gamma$ was varied in the range $10 \leq \gamma \leq 10$; and the DCB tip loads, $M_2$, $N_1$ and $N_2$, were varied in the range $-20000 \leq M_2 [\text{Nmm}], N_1[N], N_2[N] \leq 20000$ with $M_1 = 1000 \text{Nmm}$. Fig. 4.20 considers the variation of the crack tip bending moment ratio, $M_{2B}/M_{1B}$ with $M_{1B} = 1000 \text{Nmm}$ at crack extension sizes of $\delta \alpha = 0.01 \text{mm}$ and $\delta \alpha = 0.1 \text{mm}$. Note that the sizes of the elements at the crack tip are $p = 0.01$ and $p = 0.1$ respectively. The mesh densities when $p = 0.01$ have previously been given. When $p = 0.1$ the maximum mesh density was $1100 \times 110$ when $\gamma = 10$ or $\gamma = 0.1$ and the minimum mesh density was $1100 \times 20$ when $\gamma = 1$. Fig. 4.21 considers the variation of $N_{1B}/M_{1B}$ and $N_{2B}/M_{1B}$ with $M_{1B} = 1000 \text{Nmm}$ and $\delta \alpha = 0.1 \text{mm}$. In all cases in Figs. 4.20 and 4.21, if $10 \leq \eta \leq 10$ then the maximum difference between $G_I/G$ from the present analytical theory and $G_I/G$ from the 2D FEM is 0.03 across the whole range of $\gamma$ and all the loading conditions. This is extremely close agreement. Outside of this range of $\eta$, the same level of agreement is mainly achieved except for a very small “corner” location in some loading conditions. The reasons for this are the same as those given above for Figs. 4.7 to 4.10 in Section 4.3.2.

Overall, it can be concluded that the present analytical theory is able to calculate the ERR partitions, $G_I$ and $G_{II}$, to a very high level of accuracy in relation to the FEM over the entire practically useful domain of cracking between bimaterial layers.
Figure 4.20: Comparison of the present analytical theory and FEM data for the ERR partition $G_i/G$ for variable $\gamma$, $\eta$ and $M_{2B}/M_{1B}$ at crack extension sizes $\Delta\alpha = 0.01\,\text{mm}$ and $\Delta\alpha = 0.1\,\text{mm}$.
Figure 4.21: Comparison of the present analytical theory and FEM data for the ERR partitioning $G_f/G$ for variable $\gamma$, $\eta$, $N_{1B}/M_{1B}$ and $N_{2B}/M_{1B}$ at crack extension size $\delta a = 0.1 \text{mm}$.
4.4. Conclusion

This chapter has presented the development and validation for a completely analytical method to obtain the complex SIFs and the crack extension size-dependent ERRs, based on 2D elasticity, for brittle interfacial cracking between two dissimilar elastic layers. The solution has been achieved by developing two types of pure fracture modes and two powerful mathematical techniques. The two types of pure fracture modes are a SIF type and a load type. The two mathematical techniques are a shifting technique and an orthogonal pure mode technique.

It has been identified that the mismatch in material properties between the two layers causes the existence of two distinct sets of orthogonal pure modes, \((\theta_k, \beta_k)\) and \((\theta'_k, \beta'_k)\), and two sets of coincident orthogonal approximate pure modes, \((\tilde{\theta}_k, \tilde{\beta}_k)\) and \((\tilde{\theta}'_k, \tilde{\beta}'_k)\), which are in terms of the SIFs, \(K_I\) and \(K_{II}\), and which are crack extension size-dependent or FEM mesh size-dependent. The total ERR \(G\) can be partitioned by using these pure modes. In general, for cracks on a bimaterial interface, there are four pairs of mathematically admissible SIFs, \(K_I\) and \(K_{II}\), for a given loading condition. Only one pair, however, is mechanically admissible and it has been analytically determined.

A brittle interface causes the existence of two distinct sets of orthogonal pure modes, \((\theta_i, \beta_i)\) and \((\theta'_i, \beta'_i)\) (with \(i = 1, 2, 3\)), which are in terms of the crack tip loads. In the case of interfacial cracks between similar materials, these two sets of pure modes approach to each other and remain converged with the diminishing global effect as the crack extension size or FEM mesh size decreases. In the case of cracks on bimaterial interfaces, although these two sets of pure modes also approach to each other and become coincident as the crack extension size or FEM mesh size decreases, they do not converge and are crack extension size-dependent or FEM mesh size-dependent. Furthermore, they separate again for very small crack extension sizes or FEM mesh sizes.

At a crack extension size \(a = 0.05\) mm, a thickness ratio \(\gamma = 1\) and Poisson’s ratio \(\nu = 0.29\), the two distinct sets of orthogonal pure modes, \((\theta_i, \beta_i)\) and \((\theta'_i, \beta'_i)\) (with \(i = 1, 2, 3\)), approximately coincide with each other and are also approximately equal to the pure modes based on Timoshenko beam theory for the entire modulus ratio range.
A shifting technique has been developed and used in conjunction with an orthogonal pure mode methodology to determine the pure modes for Poisson’s ratio in the range $0 \leq \nu \leq 0.5$, Young’s modulus ratio in the range $1/100 \leq \eta \leq 100$, and thickness ratio in the range $1/10 \leq \gamma \leq 10$. Consequently, the SIFs, $K_I$ and $K_{II}$, and the crack extension size-dependent ERRs, $G_I$ and $G_{II}$, are analytically determined.

To validate the new theory a thorough program of parametric 2D FEM simulations on a bimaterial DCB have been carried out using MSC/NASTRAN. It has been discovered that:

1. The shifting technique, which has been developed for $\gamma \leq 1$, is able to accurately give the ERR partition $G_I/G$ when $\delta a = 0.05$ mm and $M_{2B}/M_{1B} = 0$.

2. The complete set of orthogonal pure modes, $\theta_1$, $\beta_1$, $\theta_2$, $\beta_2$, $\theta_3$ and $\beta_3$, can be calculated with high accuracy when $\delta a = 0.05$ mm by using the shifting technique and the orthogonality relationship that exists between them. This allows the present analytical theory to accurately calculate the ERR partition $G_I/G$ for any combination of crack tip bending moments and axial forces and any given values of $\gamma$, $\eta$ and $\nu$, when $\delta a = 0.05$ mm.

3. The present analytical theory accurately calculates the SIFs. The approximate SIFs, which are calculated by partitioning the total ERR $G$ using approximate SIF-based orthogonal pure modes, $\tilde{\theta}_K$ and $\tilde{\beta}_K$, are in excellent agreement with the numerically accurate ones. Furthermore, the approximate pair of SIFs are sufficiently close to the one physically correct pair of SIFs to allow the correct pair to be chosen from among the four that are provided mathematically.

4. Conclusions (1) to (3) combined allow the ERR components, $G_I$ and $G_{II}$, to be calculated for any crack extension size $\delta a$ with a very high level of accuracy over the entire practically useful domain of cracking between bimaterial layers.

The work in this chapter has been published in Harvey et al.\textsuperscript{17,18}. A limitation of the theory presented in this chapter is that although there is a mismatch in the Young’s modulus of the beams, unfortunately it requires that the Poisson’s ratios of the beams are the same, i.e. $\nu_1 = \nu_2 = \nu$. As most bimaterials will have a mismatch in the Poisson’s ratio, as well as the Young’s modulus, it is essential that the theory in this chapter is...
extended to the case when $E_1 \neq E_2$ and $\nu_1 \neq \nu_2$. Such an extension to the work is presented in Chapter 5.
Chapter 5: Effect of Poisson’s ratio mismatch on dissimilar laminated beams

5.1. Introduction

It is well known from the work of Williams \(^{32}\) that the SIF for a brittle interfacial crack between two dissimilar elastic layers is of complex form, that is, \(K = K_I + iK_{II}\). The complex SIF indicates oscillatory singularities in the elastic field around the crack tip. This was shown in 1959 and since then one of the major challenges in the field of fracture mechanics has been to analytically obtain the SIFs, \(K_I\) and \(K_{II}\), and the crack extension size-dependent ERR components, \(G_I\) and \(G_{II}\).

In Chapter 4 a completely analytical theory \(^{17}\) has been given to calculate \(K_I\), \(K_{II}\), \(G_I\) and \(G_{II}\) for a brittle interfacial crack between two elastic materials with an elastic modulus mismatch but with equal Poisson’s ratios, that is, with \(E_1 \neq E_2\) and \(\nu_1 = \nu_2\), under bending moments and axial forces (subscripts 1 and 2 represent the upper and lower layers respectively). The theory was also extensively verified \(^{18}\). The work makes use of the analytical formulations for the complex SIFs based on the finite crack extension size ERR partitions \(^{44,73}\). It is observed that for the loading condition \(M_{2B}/M_{1B} = 0\), when the crack extension size \(\delta a = 0.05\) mm and the thickness ratio \(\gamma = 1\) the 2D FEM results are equal to a Timoshenko beam partition theory from Wang and Harvey \(^{1-3,12,25}\). Furthermore, for the cases when \(\gamma \neq 1\), the 2D FEM results can be obtained using a non-uniform vertical shift of the Timoshenko beam partition theory results when \(\gamma = 1\). This enables the calculation of 2D elasticity pure modes using the shifting technique in conjunction with an orthogonal pure mode methodology. Therefore, the total ERR \(G\) can be partitioned for any combination of bending moments and axial forces with \(\delta a = 0.05\) mm. Using approximate SIFs to determine which solution of the SIFs is physically admissible, it is then possible to obtain the crack extension size-dependent ERRs, \(G_I\) and \(G_{II}\), analytically for any crack extension size.

A limitation of the theory in Chapter 5 is that the Poisson’s ratio of each layer must be the same, that is, \(\nu_1 = \nu_2\). In applications of layered material systems, for example in
thermal barrier coatings in gas turbine engines or in surface coatings to protect against corrosion, friction and wear, it is typical, however, to have a mismatch in the Poisson’s ratio as well as in the elastic modulus. It is therefore important that the theory in Chapter 4 is extended to accommodate Poisson’s ratio mismatch in addition to the existing capability for elastic modulus mismatch, that is, to accommodate both $E_1 \neq E_2$ and $\nu_1 \neq \nu_2$. This chapter reports such an extension.

To achieve this extension, it is noted that for a given geometry and loading condition, the total ERR and the bimaterial mismatch coefficient (the oscillation index of the interfacial stresses) are the two main factors affecting the partitions of ERR. Based on this, the approach has been to derive equivalent material properties for each layer, namely, an equivalent elastic modulus and an equivalent Poisson’s ratio, such that both the total ERR and the bimaterial mismatch coefficient are maintained in an alternative case. Cases for which no analytical solution for the SIFs and ERRs currently exist can therefore be “transformed” into other cases for which the analytical solution does exist.

The structure of this chapter is as follows. In Section 5.2 the previous analytical partition theory to obtain the complex SIFs and crack extension size-dependent ERRs for brittle interfacial cracking between two dissimilar elastic layers with $E_1 \neq E_2$ and $\nu_1 = \nu_2$ is extended to account for a Poisson’s ratio mismatch as well as a Young’s modulus mismatch. Section 5.3 presents the validation of the extension by comparing results to that obtained from 2D FEM simulations. Finally, conclusions are given in Section 5.4. Note that this chapter is a supplement to the work previously given in Chapter 4. Due to the complexity of Chapter 4, only the new analytical development is presented here with accompanying background only where necessary. The reader is directed to Chapter 4 for further information and full details. A review of the literature can also be found in Chapter 4.
5.2. Analytical development

Fig. 5.1a shows a bimaterial DCB with its geometry, tip bending moments, $M_1$ and $M_2$, and tip axial forces, $N_1$ and $N_2$. The Young’s modulus, shear modulus and Poisson’s ratio of beam $i$ are denoted by $E_i$, $\mu_i$ and $\nu_i$ respectively (with $i=1,2$). The interfacial opening stress and shear stress ahead of the crack tip, $\sigma_n$ and $\tau_s$, can be expressed in combined complex form as

$$\sigma_n + i\tau_s = \left(K_i + iK_{II}\right)\frac{\varepsilon}{\sqrt{2\pi r}}$$

or in individual real form as

$$\sigma_n = \frac{1}{\sqrt{2\pi r}}\{K_i \cos[\varepsilon \ln(r)] - K_{II} \sin[\varepsilon \ln(r)]\}$$

$$\tau_s = \frac{1}{\sqrt{2\pi r}}\{K_i \sin[\varepsilon \ln(r)] + K_{II} \cos[\varepsilon \ln(r)]\}$$

5.2.1. Interfacial stresses ahead of the crack tip

Fig. 5.1a shows a bimaterial DCB with its geometry, tip bending moments, $M_1$ and $M_2$, and tip axial forces, $N_1$ and $N_2$. The Young’s modulus, shear modulus and Poisson’s ratio of beam $i$ are denoted by $E_i$, $\mu_i$ and $\nu_i$ respectively (with $i=1,2$). The interfacial opening stress and shear stress ahead of the crack tip, $\sigma_n$ and $\tau_s$, can be expressed in combined complex form as

$$\sigma_n + i\tau_s = \left(K_i + iK_{II}\right)\frac{\varepsilon}{\sqrt{2\pi r}}$$

or in individual real form as

$$\sigma_n = \frac{1}{\sqrt{2\pi r}}\{K_i \cos[\varepsilon \ln(r)] - K_{II} \sin[\varepsilon \ln(r)]\}$$

$$\tau_s = \frac{1}{\sqrt{2\pi r}}\{K_i \sin[\varepsilon \ln(r)] + K_{II} \cos[\varepsilon \ln(r)]\}$$
where \( r \) is the radius coordinate centred on the crack tip and \( K_i \) and \( K_{ii} \) are the real and imaginary parts of the complex SIF. The signs of \( \sigma_\sigma \) and \( \tau_\tau \) are positive in the directions shown in Fig. 5.1b. In Eqs. (5.1)–(5.3), the bimaterial mismatch coefficient \( \varepsilon \) is defined as

\[
\varepsilon = \frac{1}{2\pi} \ln \left[ \left( \frac{k_i + \frac{1}{\mu_2}}{\mu_1} \right) \left( \frac{k_2 + \frac{1}{\mu_1}}{\mu_2} \right)^{-1} \right]
\]  

(5.4)

where the Kolosov constant \( iik \) (with \( 2,1 = i \)) is defined as

\[
iik = \nu_4 - \nu_3
\]

for plane strain and as \( iik = (\nu_2 - \nu_1)/(1 + \nu_1) \) for plane stress. By introducing the Young’s modulus ratio, \( \eta = E_2/E_1 \), then \( \varepsilon \) becomes

\[
\varepsilon = \begin{cases} 
\frac{1}{2\pi} \ln \left[ \frac{3\eta + \nu_2 - \eta\nu_1 + 1}{\eta - \nu_2 + \eta\nu_1 + 3} \right] & \text{for plane stress} \\
\frac{1}{2\pi} \ln \left[ \frac{-4\eta\nu_1^2 + 3\eta + \nu_2 - \eta\nu_1 + 1}{-4\nu_2^2 + \eta - \nu_2 + \eta\nu_1 + 3} \right] & \text{for plane strain}
\end{cases}
\]

(5.5)

5.2.2. Total energy release rate

In Chapter 4 and from the work of Wang and Harvey\(^1\)–\(^3\),\(^12\),\(^25\), and with reference to Fig. 5.1b, the total ERR \( G \) of a bimaterial DCB with two crack tip bending moments, \( M_{1B} \) and \( M_{2B} \), and two crack tip axial forces, \( N_{1B} \) and \( N_{2B} \), is given by

\[
G = \frac{1}{2E_b b^2 C} \begin{bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B} \end{bmatrix}^T \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B} \end{bmatrix} = \frac{1}{2E_b b^2 C} \begin{bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B} \end{bmatrix}^T \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B} \end{bmatrix}
\]

(5.6)

where

\[
\overline{C} = \eta^2 \gamma^4 + 4\eta\gamma^3 + 6\eta^2\gamma^2 + 4\eta\gamma + 1
\]

(5.7)

and where \( \overline{E}_1 \) is the effective Young’s modulus of the upper beam and \( \overline{\eta} \) is the effective Young’s modulus ratio. For plane stress \( \overline{E}_1 = E_1 \) and \( \overline{\eta} = \eta \); for plane strain \( \overline{E}_1 = E_1/(1 - \nu_1^2) \) and \( \overline{\eta} = \eta (1 - \nu_1^2)/(1 - \nu_2^2) \). The coefficient matrix \( [C] \) is given by
\[ C_{11} = 12\eta \gamma (\eta \gamma^3 + 4\gamma^2 + 6\gamma + 3)/h_1^3 \] (5.8)

\[ C_{12} = -12(\eta \gamma + 1)/h_1^3 \] (5.9)

\[ C_{13} = 6\eta \gamma (\gamma + 1)/h_1^2 \] (5.10)

\[ C_{14} = -6(\gamma + 1)/h_1^2 \] (5.11)

\[ C_{22} = 12\left(3\eta \gamma^3 + 6\eta \gamma^2 + 4\eta \gamma + 1\right)/(\eta h_1^3 \gamma^3) \] (5.12)

\[ C_{23} = 6\eta \gamma (\gamma + 1)/h_1^2 \] (5.13)

\[ C_{24} = -6(\gamma + 1)/h_1^2 \] (5.14)

\[ C_{33} = \eta \gamma (\eta \gamma^3 + 1)/h_1 \] (5.15)

\[ C_{34} = -(\eta \gamma^3 + 1)/h_1 \] (5.16)

\[ C_{44} = (\eta \gamma^3 + 1)/(\eta h_1 \gamma) \] (5.17)

### 5.2.3. Equivalent bimaterial properties

From the work in Chapter 4, it has been observed that for a given geometry and loading condition, the bimaterial mismatch coefficient \( \epsilon \) and the total ERR \( G \) are the two main factors affecting the partitions of ERR, \( G_i \) and \( G_{ij} \). Eqs. (5.5) and (5.6) show that \( \epsilon \) and \( G \) depend on \( E_1, \eta, \nu_1 \) and \( \nu_2 \). It is therefore proposed that a given real case with \( E_1, E_2 = \eta E_1, \nu_1 \) and \( \nu_2 \) can be replaced by an equivalent case with \( \tilde{E}_1, \tilde{E}_2 = \eta \tilde{E}_1 \) and \( \tilde{\nu}_1 = \tilde{\nu}_2 = \tilde{\nu} \) that maintains \( \epsilon \) and \( G \) with similar partitions of the ERR, \( G_i \) and \( G_{ij} \). Such behaviour would be advantageous to transform cases for which no analytical solution for the SIFs and ERRs currently exist into other cases for which the analytical solution does exist, such as for those in Chapter 4. To maintain the same \( \epsilon \), the equivalent Poisson’s ratio \( \tilde{\nu} \) is obtained by using Eq. (5.5) and equating the real case with \( E_1, E_2 = \eta E_1, \nu_1 \) and \( \nu_2 \) to the equivalent case with \( \tilde{E}_1, \tilde{E}_2 = \eta \tilde{E}_1 \) and \( \tilde{\nu}_1 = \tilde{\nu}_2 = \tilde{\nu} \), and rearranging for \( \tilde{\nu} \), giving
It is seen from Eq. (5.6) that under plane stress conditions the total ERR $G$ is independent of the Poisson’s ratios. Therefore, changing the values of the Poisson’s ratios only affects the value of the bimaterial mismatch coefficient $\varepsilon$ and hence the partitions of the ERR, $G_I$ and $G_{II}$. For this case, as the total ERR $G$ is maintained regardless of $\nu_1$ and $\nu_2$, only an equivalent Poisson’s ratio $\tilde{\nu}$ is needed and it is possible to set $\tilde{\eta} = \eta$, in which case Eq. (5.18) for plane stress reduces to

$$\tilde{\nu} = \frac{\eta \nu_1 - \nu_2}{(\eta - 1)} \quad (5.19)$$

In Eq. (5.19), if $\eta \rightarrow 1$ then $\tilde{\nu} \rightarrow \infty$, which is unacceptable behaviour for $\tilde{\nu}$. Therefore, when $\eta$ is close to 1, setting $\tilde{\eta} = \eta$ is no longer suitable and Eq. (5.18) must be used instead with an alternative equivalent $\tilde{\eta}$.

Using an alternative equivalent $\tilde{\eta}$ affects the total ERR $G$ under both plane stress and plane strain conditions. Therefore, to maintain $G$, $E_i$ must also be replaced by $\tilde{E}_i$, which represents the equivalent effective Young’s modulus of the upper beam. Let $[\tilde{C}]$ and $\tilde{C}$ denote the $[C]$ and $C$ in Eqs. (5.6) and (5.7) respectively with the substitution $\tilde{\eta} = \eta$. To maintain the same $G$, $\tilde{E}_i$ is obtained by using Eq. (5.6) and equating the real case to the equivalent case, and rearranging for $\tilde{E}_i$, giving

$$\tilde{E}_i = \frac{E_i C}{\tilde{C}} \begin{bmatrix} M_{1B} & M_{2B} & N_{1B} & N_{2B} \end{bmatrix} \tilde{C}^{-1} \begin{bmatrix} M_{1B} & M_{2B} & N_{1B} & N_{2B} \end{bmatrix} \tilde{E}_i = \tilde{E}_i \left(1 - \tilde{\nu}^2\right) \quad (5.20)$$

Then, $E_i$ is replaced by $\tilde{E}_i$ where for plane stress $\tilde{E}_i = \tilde{E}_i$ and for plane strain $\tilde{E}_i = \tilde{E}_i \left(1 - \tilde{\nu}^2\right)$. Note that for plane stress cases with $\tilde{\eta} = \eta$, Eq. (5.20) reduces to $\tilde{E}_i = E_i$. 

$$\tilde{\nu} = \begin{cases} \frac{\nu_1 + \tilde{\eta} \nu_1 - 2}{\tilde{\eta} - 1} - \frac{(\tilde{\eta} + 1)(\nu_1 + \nu_2 - 2)}{(\eta + 1)(\eta - 1)} & \text{for plane stress} \\ \frac{2\eta - 2\tilde{\eta} + \nu_2 - \eta \nu_1 + \nu_2 - 3\eta \nu_1^2 + 3\tilde{\eta} \nu_2^2 + \nu_2^2 - \eta \tilde{\eta} \nu_1 - \eta \tilde{\eta} \nu_1^2}{3\eta - 3\tilde{\eta} + \nu_2 - \eta \tilde{\eta} - \eta \nu_1 + \nu_2 - 4\eta \nu_1^2 + 4\tilde{\eta} \nu_2^2 - \eta \tilde{\eta} \nu_1 + 1} & \text{for plane strain} \end{cases} \quad (5.18)$$
The method above derives formulae for $\tilde{v}$ and $\tilde{E}_1$, which are dependent on the initial selection of $\tilde{\eta}$. Any consistent combination of $\tilde{E}_1$, $\tilde{\eta}$, $\tilde{v}$ will maintain both $G$ and $\varepsilon$ in an alternative equivalent case with $E_1 \neq E_2$ and $\nu_1 = \nu_2$. The following recommends which combinations give the most accurate partitions of ERR, $G_i$ and $G_{II}$. The general principle in the following is to minimize the difference between the real material properties and the equivalent material properties while still achieving $\tilde{\nu}_1 = \tilde{\nu}_2 = \tilde{\nu}$.

Based on the FEM results in Section 5.3, for plane stress conditions, using $\tilde{\eta} = \eta$ provides accurate results for almost the whole range of $\eta$; however, when $-0.1 < \log_{10}(1/\eta) < 0.1$, since $\tilde{v} \to \infty$ as $\eta \to 1$, it has been identified that using $\tilde{\eta} = 1.1$ and $\tilde{E}_1$ as given by Eq. (5.20) instead works well throughout this range.

For plane strain conditions, selecting the equivalent material properties, $\tilde{E}_1$, $\tilde{\eta}$ and $\tilde{v}$, is more involved as $\tilde{E}_1$ and $\tilde{v}$ are very sensitive to the chosen value of $\tilde{\eta}$. Initially the value of $\log_{10}(1/\tilde{\eta})$ is varied by increments of 0.1 in the range $-2 \leq \log_{10}(1/\tilde{\eta}) \leq 2$. If the corresponding value of $\tilde{v}$ is in the range of physically admissible Poisson’s ratios, that is, $0 < \tilde{v} < 0.5$, then the values are saved. If only one value of $\tilde{v}$ is in this range then it is selected with the corresponding value of $\tilde{\eta}$; however, if multiple values of $\tilde{v}$ obey this condition, then the ones which minimize the arithmetic difference between $\eta$ and $\tilde{\eta}$ are selected. Finally, the value of $\tilde{E}_1$ can be calculated using $\tilde{E}_1$ from Eq. (5.20).

5.3. Numerical verification

A method has been described in Section 5.2 for reducing cases of bimaterial interfacial cracking with $E_1 \neq E_2$ and $\nu_1 \neq \nu_2$ to equivalent cases with $E_1 \neq E_2$ and $\nu_1 = \nu_2$. The ERRs, $G_i$ and $G_{II}$, can then be calculated by using the analytical mixed-mode partition theory in Chapter 4 for brittle interfacial cracks between two elastic materials with $E_1 \neq E_2$ and $\nu_1 = \nu_2$. In order to verify this approach, a series of 2D FEM simulations were conducted using MSC/NASTRAN on the DCB shown in Fig. 5.1a with a range of values of $E_1$, $E_2$, $\nu_1$ and $\nu_2$, from which the ERRs, $G_i$ and $G_{II}$,
were calculated. The verification was then performed by comparing values of the total ERR $G$ and the ERR partition $G_t / G$ from the FEM and the analytical theory.

In the FEM simulations, the thickness ratio $\gamma = h_2 / h_1$ was kept at a constant value of $\gamma = 1$; the Young’s modulus ratio $\eta = E_2 / E_1$ was varied in the range $1/100 \leq \eta \leq 100$; the ratio of Poisson’s ratios $N = \nu_2 / \nu_1$ was varied in the range $-0.7 \leq \log_{10}(1/N) \leq 0.7$; and the DCB tip loads were varied in the range $-10,000 \leq M_2 [\text{Nm}], N_1 [\text{N}] \leq 10,000$ with $M_1 = 1000 \text{ Nmm}$. Note that it was not necessary to vary the thickness ratio $\gamma$. The effect of the through-thickness location of the crack on the ERRs and SIFs has already been thoroughly dealt with in Chapter 4. This chapter adds additional material mismatch capability, and further consideration of the thickness ratio $\gamma$ is therefore not needed. It can, however, be easily shown that the same conclusions apply if $\gamma \neq 1$. Therefore, the entire practically useful domain of cracking between bimaterial layers was considered. The upper and lower layers of the DCB were modelled using quadrilateral plane stress or plane strain shell elements with a thickness of $b = 10 \text{ mm}$ and isotropic material properties within each beam. The thicknesses of the upper and lower layers were equal with $h_1 = h_2 = 1 \text{ mm}$. The minimum Young’s modulus was $E_{\text{min}} = 1000 \text{ N/mm}^2$. If the modulus ratio $\eta > 1$, then the Young’s modulus of the upper and lower layers was selected to be $E_1 = E_{\text{min}}$ and $E_2 = \eta E_{\text{min}}$ respectively, otherwise $E_2 = E_{\text{min}}$ and $E_1 = E_{\text{min}} / \eta$. The Poisson’s ratios were controlled by specifying a mean value of $\bar{\nu} = 0.29$ such that $\nu = (\nu_1 + \nu_2) / 2$. The Poisson’s ratio of the upper beam and lower layers were then determined as $\nu_1 = 2\bar{\nu} / (1 + N)$ and $\nu_2 = N \nu_1$ for the upper and lower layers respectively. This provided an even spread of the Poisson’s ratios to be considered while still keeping the maximum value below 0.5. Note that a value of $N = 1$ corresponds to no Poisson’s ratio mismatch or $\nu_1 = \nu_2$, as previously considered in Chapter 4. The shear modulus was calculated using $\mu_i = E_i / [2(1 + \nu_i)]$ with $i = 1, 2$ for the upper and lower beams, respectively. The uncracked length of the DCB was $L = 100 \text{ mm}$ and the cracked length was $a = 10 \text{ mm}$.

The partitions of ERR, $G_t$ and $G_H$, depend on the crack extension size $\varepsilon \alpha$. The analytical partition theory in Chapter 4 accommodates any value of $\varepsilon \alpha$ by determining
Chapter 5: Effect of Poisson’s ratio mismatch on dissimilar laminated beams

$G_I$ and $G_{II}$ for $\Delta a = 0.05 \text{ mm}$, from which the SIFs, $K_I$ and $K_{II}$, are determined. With knowledge of $K_I$ and $K_{II}$, $G_I$ and $G_{II}$ can be determined for any value of $\Delta a$. In this work, therefore, the selection of $\Delta a$ is somewhat arbitrary; however, if $\Delta a \neq 0.05 \text{ mm}$ then the verification is even more rigorous due to the extra steps in the analytical calculation, the necessary accurate calculation of the SIFs as part of the process, and the opportunity for compounding inaccuracy. A crack extension size $\Delta a = 0.01 \text{ mm}$ was therefore selected in order to calculate the ERR. The choice of $\Delta a$ determined the size of the elements surrounding the crack tip. The FEM mesh procedure and load application was the same as that in Chapter 3, Section 3.2.2, therefore to avoid repetition the reader is directed here for further details. As the element size at the crack tip was $0.01 \times 0.01$ this gave a constant mesh density of $2128 \times 138$ for all tests as $\gamma = 1$. Furthermore, this element size corresponded to a spring stiffness at the crack tip of $k_s = 10^6 \text{ N/mm}$. This meant that the spring stiffness $k_s$ was sufficiently high with respect to $E_1$ and $E_2$ to simulate brittle interfacial cracking without introducing excessive numerical error. Because the interface was rigid, the ERRs were calculated using the VCCT. Contact between the upper and lower surfaces of the crack was not considered.

5.3.1. Bending moments only

The DCB was subjected to tip bending moments in order to vary the crack tip bending moment on the lower beam $M_{2B}$ in the range $-10,000 \leq M_{2B} [\text{Nmm}] \leq 10,000$ while keeping the crack tip bending moment on the upper beam constant at $M_{1B} = 1000 \text{ Nmm}$. Results from the plane stress condition are shown in Fig. 5.2 and results from the plane strain condition are shown in Fig. 5.3. Figs. 5.2a and 5.3a show the difference between the total ERR $G$ from the present theory $G_{th}$ and from the FEM $G_{FEM}$, defined as $|1 - G_{th}/G_{FEM}|$. Figs. 5.2b and 5.3b show the difference between the ERR partition $G_I/G$ from the present theory $(G_I/G)_{th}$ and from the FEM $(G_I/G)_{FEM}$, defined as $|(G_I/G)_{th} - (G_I/G)_{FEM}|$. Note that, as described above, the present theory combines the partition theory in Chapter 4 with the method in Section 5.2 for
transforming cases with Poisson’s ratio mismatch into alternative cases with no Poisson’s ratio mismatch.

It is seen from Figs. 5.2a (plane stress) and 5.3a (plane strain) that there is virtually exact agreement over the whole domain between the present theory and the FEM when considering total ERR $G$. Then, from Figs. 5.2b (plane stress) and 5.3b (plane strain), there is generally excellent agreement between the theory and the FEM when considering the ERR partition $G_i/G$. In both cases, the majority of the theoretical results are within about 4% of that obtained from the FEM. From Fig. 5.2b (plane stress), the maximum error between the ERR partitions is 10.5% and located at $\log_{10}(l/\eta) = 1.7$, $\log_{10}(l/N) = 0.7$ and $M_{2b}/M_{1b} = 0$. (The error colour bar has been capped at 0.10 for clear presentation). For Fig. 5.3b (plane strain) the maximum error between the ERR partitions is 36.8%, located at $\log_{10}(l/\eta) = -1.2$, $\log_{10}(l/N) = 0.7$ and $M_{2b}/M_{1b} = 5$, and rapidly diminishes.

As explained in Section 5.2.3, an equivalent Poisson’s ratio $\tilde{\nu}$, an equivalent Young’s modulus ratio $\tilde{\eta}$, and an equivalent Young’s modulus of the upper beam $\tilde{E}_i$ are all needed in order to find a suitable equivalent bimaterial case under plane strain conditions. For the selected value of $\tilde{\eta}$, two approximations are needed to find $\tilde{\nu}$ and $\tilde{E}_i$ using Eqs. (5.18) and (5.20) respectively. Furthermore, $\tilde{\nu}$ and $\tilde{E}_i$ are very sensitive to the chosen value of $\tilde{\eta}$. In contrast, only $\tilde{\nu}$ is required for plane stress conditions (unless $\eta \to 1$), which requires only one approximation using Eq. (5.18) with $\tilde{\eta} = \eta$ being maintained. The increased maximum error in the plane strain results is attributed to the compounding of error from the two approximations $\tilde{\nu}$ and $\tilde{E}_i$, while the plane stress results agree with the FEM results more closely due to there being only one approximation of $\tilde{\nu}$. 
Figure 5.2: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for variable $\eta$, $N$ and $M_{2B}/M_{1B}$ with $\gamma = 1$ and $\delta a = 0.01$ mm under the plane stress condition.
Figure 5.3: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_{ij}/G$ for variable $\eta$, $N$ and $M_{2B}/M_{1B}$ with $\gamma = 1$ and $\delta a = 0.01$ mm under the plane strain condition.
5.3.2. Bending moments and axial forces

The DCB was also subjected to tip axial forces and bending moments in order to vary the crack tip axial force on the upper beam $N_{1B}$ in the range $-10,000 \leq N_{1B} [\text{N}] \leq 10,000$ while keeping the crack tip bending moment on the upper beam constant at $M_{1B} = 1000 \text{ N}$. Results from the plane stress condition are shown in Fig. 5.4 and results from the plane strain condition are shown in Fig. 5.5. Figs. 5.4a and 5.5a show the difference between the total ERR $G$ from the present theory $G_{\text{th}}$ and from the FEM $G_{\text{FEM}}$, defined as $|1 - G_{\text{th}}/G_{\text{FEM}}|$. Figs. 5.4b and 5.5b show the difference between the ERR partition $G_i / G$ from the present theory $(G_i / G)_{\text{th}}$ and from the FEM $(G_i / G)_{\text{FEM}}$, defined as $|(G_i / G)_{\text{th}} - (G_i / G)_{\text{FEM}}|$.

Again it is seen from Figs. 5.4a (plane stress) and 5.5a (plane strain) that there is virtually exact agreement over the whole domain between the present theory and the FEM when considering the total ERR $G$. Then, from Figs. 5.4b (plane stress) and 5.5b (plane strain), there is also excellent agreement between the theory and the FEM when considering the ERR partition $G_i / G$. In both cases, the majority of the theoretical results are again within about 4% of that obtained from the FEM. For Fig. 5.4b (plane stress) the maximum error between the ERR partitions is 16.3%, located at $\log_{10}(l/\eta) = 1.6$, $\log_{10}(l/N) = 0.4$ and $N_{1B}/M_{1B} = -10$, and rapidly diminishes. For Fig. 5.5b (plane strain) the maximum error between the ERR partitions is 36.3%, located at $\log_{10}(l/\eta) = 1.2$, $\log_{10}(l/N) = -0.7$ and $N_{1B}/M_{1B} = 10$, and rapidly diminishes.
Figure 5.4: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_i/G$ for variable $\eta$, $N$ and $N_{1B}/M_{1B}$ with $\gamma = 1$ and $\delta a = 0.01$ mm under the plane stress condition.
Figure 5.5: Comparison of the present analytical theory and the 2D FEM for the total ERR $G$ and the ERR partition $G_t/G$ for variable $\eta$, $N$ and $N_{1B}/M_{1B}$ with $\gamma = 1$ and $\delta a = 0.01$ mm under the plane strain condition.
5.4. Conclusion

This chapter has presented an extension to the work in Chapter 4, where an analytical partition theory to obtain the complex SIFs and crack extension size-dependent ERRs for brittle interfacial cracking between two dissimilar elastic layers has been presented. The extension now accommodates a Poisson’s ratio mismatch between the upper and lower beams, in addition to the existing capability for elastic modulus mismatch.

It has been identified that for a given geometry and loading condition, the total ERR and bimaterial mismatch coefficient are the two main factors that affect the ERR partitions. Based on this, equivalent material properties are derived for each layer, namely, an equivalent elastic modulus and an equivalent Poisson’s ratio, such that both the total ERR and the bimaterial mismatch coefficient are maintained in an alternative equivalent case. For the plane stress condition, it has been shown that the total ERR is unaffected by the choice of the Poisson’s ratio of the beams, therefore only an effective Poisson’s ratio is required. Once the equivalent material properties have been obtained for the case when $\nu_1 \neq \nu_2$, it is possible to use the partition theory from Chapter 4, when $\nu_1 = \nu_2$ with the equivalent material properties in order to obtain the complex SIF and crack extension size-dependent ERRs.

To validate the extension to the partition theory in Chapter 5, results for the total ERR $G$ and the ERR partition $G_1/G$ were compared to those obtained from the 2D FEM. Excellent agreement is observed for both cases of plane stress and plane strain and under a variety of loading conditions. Under crack tip bending moments and axial forces, it is now possible to calculate the crack extension size-dependent ERRs, $G_1$ and $G_\mu$, for a brittle interfacial crack between two dissimilar elastic layers with a Poisson’s ratio mismatch as well as a Young’s modulus mismatch.

It should be remembered that this work represents an approximate method. It will be useful for researchers and engineers to quickly obtain predictions of the fracture mode partition without full FEM simulations. Despite it being an approximate method, in the majority of cases the partition can be predicted to within 4% of the FEM result. If improved accuracy is required, then it may be that only a full FEM simulation can provide this. To be confident of avoiding any of the localised areas of increased error, it
is suggested to be cautious when dealing with extreme cases of Poisson’s ratio mismatch, for example \( \log_{10}(t/N) \approx \pm 0.7 \).

The work in this chapter has been published in Wood et al.\textsuperscript{20}. 
Chapter 6: Application of theory using previously published experimental results

6.1. Introduction

It is now possible to apply the partition theory derived in Chapter 3 to the blister test for interface fracture toughness using previously published experimental results by Koenig et al.\textsuperscript{104}. As previously mentioned in Section 1.5, 2D-elasticity-based partition theories provide poor results when considering unidirectional and multidirectional composite materials. It is believed that this is due to the fact that in these brittle materials, damage occurs over the whole region that is mechanically influenced by the crack tip and this zone is much larger than the singular field. A requirement for singular-field-based theories is that the singular field must dominate the damage zone, as this is not the case for these materials the partition theories cannot provide an accurate partition of the total ERR. It is thought, however, that on the microscale, when damage is much smaller, a 2D-elasticity-based partition theory will provide accurate results for example in thin films. Therefore using previously published experimental results for the adhesion energy of graphene membranes\textsuperscript{104}, it is possible to assess the theory from Chapter 3’s ability to explain the experimental findings and therefore obtain the critical mode I and II fracture toughnesses.

Analytical work on the blister test has been given by Jensen\textsuperscript{105,106} where the work of Suo and Hutchinson\textsuperscript{21} was modified for use with the blister test to determine the interface fracture toughness and expressions are obtained for a thin membrane blister on an elastic substrate. By assuming that the substrate is infinitely thick, meaning that it is completely rigid when compared to the film, it is possible to obtain an expression for the total ERR that is independent of the substrate material properties. Formulas are also given for the crack tip bending moment and axial force when at the membrane limit, i.e. the membrane thickness tends to zero. The work of Jensen\textsuperscript{105,106} isn’t completely analytical and relies on a numerical analysis to obtain a Poisson’s ratio dependent parameter. The total ERR in the work of Jensen for both a pressure loaded and point loaded blister has been numerically verified using nonlinear FEM and the VCCT\textsuperscript{107}.

Experimental work on the subject is given by Koenig et al.\textsuperscript{104} where a pressurised blister test for interface fracture toughness was used in order to obtain the adhesion
energy of mono- and multi-layered graphene membranes on a silicon oxide substrate. Initially 1 to 5 layers of graphene were fabricated over a 5 µm well in the silicon substrate. As this was performed at ambient conditions, the graphene layer was flat and the internal pressure was atmospheric. In order to create a pressure difference the specimens were placed in a pressure chamber containing nitrogen until the internal pressure equilibrated by diffusion through the silicon oxide substrate. After which, when the specimens were removed from the pressure chamber, as the internal pressure was now greater than atmospheric, the membrane blister was formed. Through the use of an atomic force microscope, it was possible to obtain the maximum central deflection $\delta$ and radius $R_B$ for different pressure values. By increasing the pressure, it was identified that as well as increased deflection of the blisters, delamination occurred and this increased the value of $R_B$. From which it was possible to obtain the adhesion energy for the specimen. When comparing the adhesion energy of mono-layered blisters to multi-layered it was seen that there was large decrease in the adhesion energy of the multi-layered specimens. The average adhesion energy of a monolayer graphene blister was reported as $G = 0.45 \pm 0.02 \text{J/m}^2$ whereas the multi-layered specimens have an average adhesion energy of $G = 0.31 \pm 0.03 \text{J/m}^2$. From the work it is not clear as to why this drop in the adhesion energy occurs.

Unlike the work of Koenig et al. where the blister was under a pressure load, Zong et al.\textsuperscript{108} modelled nanoparticles at the interface as a point load. Again a silicon substrate was used and the graphene blister had approximately 5 layers. It is stated that the silicon substrate showed minor deformation and was assumed to be completely rigid, with the graphene membrane accounting for the majority of the elastic deformation. The adhesion energy of the point loaded graphene blister was reported as being $G = 0.151 \pm 0.028 \text{J/m}^2$. Other work on graphene blisters has been performed\textsuperscript{109–113}.

However, it is observed that the current mechanical models\textsuperscript{104,108–113} do not consider the fracture mode mixity and the sliding effect in the determination of the adhesion energy for multi-layered graphene membranes using the blister test. This has caused a lot of confusion when interpreting the adhesion energy. The present work is triggered by the large reduction of the adhesion energy from monolayer and multilayer graphene. 
membranes in the work\textsuperscript{104}. A mechanical model is developed to give a complete calculation and correct interpretation of the adhesion energy.

The structure of this chapter is as follows. In Section 6.2 a summary of the previously derived partition theory for an orthotropic laminated DCB with crack tip bending moments, axial forces and through-thickness shear forces from Chapter 3 is given. Section 6.3 presents the analytical development so that the partition theory can be applied to the blister test for interfacial fracture toughness. Using previously published experimental results in Section 6.4 the theory is assessed and experimental findings are justified. After which, the critical mode I and II fracture toughnesses are obtained and then conclusions are given in Section 6.5.

6.2. Orthotropic laminated DCB with general loading

Figure 6.1: A laminated DCB. (a) General description. (b) Details local to the crack tip.

Figure 6.1a shows a DCB with its geometry and loading conditions, which consist of tip bending moments $M_1$ and $M_2$, axial forces $N_1$ and $N_2$, and through thickness shear forces $P_1$ and $P_2$. Figure 6.1b shows the internal loads at the crack tip and the sign convention of the interface normal stress $\sigma_n$ and shear stress $\tau_s$. From the work\textsuperscript{16,19} in
Chapter 6: Application of theory using previously published experimental results

Chapter’s 2 and 3 the total ERR $G$ of the DCB with crack tip bending moments, axial forces and through-thickness shear forces can be partitioned into its mode I and mode II components $G_1$ and $G_{II}$ using an orthogonal pure mode methodology\textsuperscript{1–3,12} as follows

$$
G_1 = c_1 \left( M_{1B} - M_{2B} \frac{N_{1Bc}}{\beta_{1-2D}} - P_{1B} \frac{N_{1Bc}}{\beta_{3-2D}} - P_{2B} \frac{P_{2B}}{\beta_{4-2D}} \right)^2
$$

(6.1)

$$
G_{II} = c_{II} \left( M_{1B} - \frac{M_{2B}}{\theta_{1-2D}} - \frac{N_{1Bc}}{\theta_{2-2D}} - \frac{P_{1B}}{\theta_{3-2D}} - \frac{P_{2B}}{\theta_{4-2D}} \right)^2
$$

(6.2)

where $\theta_{i-2D}$ and $\beta_{i-2D}$ (with $i = 1, 2, 3, 4$) represent the orthogonal 2D-elasticity-based pure mode I and II conditions. The pure mode I and II conditions are as follows

$$
\theta_{i-2D} = -\gamma^2 - \frac{\bar{e}_i}{\nu} \gamma^2 \left( 1 - \gamma^2 \right)
$$

(6.3)

$$
\beta_{1-2D} = \frac{\gamma^2(3 + \gamma)}{1 + 3\gamma} - \frac{3\bar{e}_1}{(1 + 3\gamma)\left( \bar{e}_1 + \alpha(1 + \gamma) \right)}
$$

(6.4)

$$
\theta_{2-2D} = -\frac{6}{h} \left[ \frac{6\bar{e}_2}{h_1} \frac{\gamma(1 - \gamma)}{(1 - \gamma + \gamma^2) + \alpha(1 + \gamma)^2} \right]
$$

(6.5)

$$
\beta_{2-2D} = \begin{cases} \frac{2(3 + \gamma)}{h_1(\gamma - 1)} + \frac{2\bar{e}_2}{h_1\alpha(\gamma - 1)} & \text{if } \gamma \neq 1 \\ \infty & \text{if } \gamma = 1 \end{cases}
$$

(6.6)

$$
\theta_{3-2D} = \frac{\theta_{1-2D}(1 + \beta_{P-2D})}{(\beta_{1-2D} - \theta_{1-2D})^2 \left( G_{\beta_{P-2D}} / G_{\beta_{3-2D}} \right)^2}
$$

(6.7)

$$
\beta_{3-2D} = \frac{\beta_{1-2D}(1 + \beta_{P-2D})}{\beta_{P-2D}(\theta_{1-2D} - \beta_{1-2D})^2 \left( G_{\beta_{P-2D}} / G_{\beta_{3-2D}} \right)^2}
$$

(6.8)

$$
\theta_{4-2D} = -\theta_{P-2D} \theta_{3-2D}
$$

(6.9)

$$
\beta_{4-2D} = -\beta_{P-2D} \beta_{3-2D}
$$

(6.10)
where \( N_{1b} = N_{1b} - N_{2b} / \gamma \) and \( \gamma \) is the thickness ratio of the beams \( \gamma = h_2 / h_1 \) . Also \( \tilde{c}_o \approx 6/5 \) and \( \alpha = c_o \) with \( e = \left[ (1 + \gamma) / (1 + \gamma^2) \right]^{1/2} \). The pure modes \( \theta_{P,2D} \) and \( \beta_{P,2D} \) for through-thickness shear forces only \( P_{1b} \) and \( P_{2b} \) acting at the crack tip (i.e. \( M_{1b} = M_{2b} = N_{1b} = N_{2b} = 0 \)) are given as

\[
(\theta_{P,2D}, \beta_{P,2D}) = (-1, \gamma \exp(-1.986 \tanh(0.5635 \gamma_i)))
\]

(6.11)

where \( \gamma_i = \log_{10}(1/\gamma) \). The remainder of the parameters are now given

\[
c_i = G_{\theta_{i,2d}} \left( 1 - \theta_{i,2d}^2 \right)^{-2} \quad \text{and} \quad c_{\mu} = G_{\beta_{i,2d}} \left( 1 - \beta_{i,2d}^2 \right)^{-2}
\]

(6.12)

\[
G_{\theta_{i,2d}} = \frac{6}{b^2 h_1 E} \left( 1 + \frac{\theta_{i,2d}^2}{\gamma^3} - \frac{(1 + \theta_{i,2d})^2}{(1 + \gamma)^3} \right)
\]

(6.13)

\[
G_{\beta_{i,2d}} = \frac{6}{b^2 h_1 E} \left( 1 + \frac{\beta_{i,2d}^2}{\gamma^3} - \frac{(1 + \beta_{i,2d})^2}{(1 + \gamma)^3} \right)
\]

(6.14)

\[
G_{\theta_{P,2d}} = \frac{1}{2b^2 h_1 E \kappa(\gamma)} \left( 1 + \frac{\theta_{P,2d}^2}{\gamma} \right)
\]

(6.15)

\[
G_{\beta_{P,2d}} = \frac{1}{2b^2 h_1 E \kappa(\gamma)} \left( 1 + \frac{\beta_{P,2d}^2}{\gamma} - \frac{(1 + \beta_{P,2d})^2}{1 + \gamma} c(\gamma) \right)
\]

(6.16)

where \( b \) is the width of the beam, \( E = E \) for plane stress or \( E = E / (1 - \nu^2) \) for plane strain with \( E \) being the Young’s modulus of the beam and \( \nu \) the Poisson’s ratio. The two thickness ratio \( \gamma \)-dependent correction factors, \( \kappa(\gamma) \) the through-thickness shear correction factor and \( c(\gamma) \) the pure-mode-II ERR correction factor can be calculated using the following expressions

\[
\kappa(\gamma) = 0.1355 + 0.0477 \exp(-1.39 \gamma_i^2)
\]

(6.17)

\[
c(\gamma) = \frac{\left( 1 + \beta_{P,2d}^2 / \gamma \right)(1 + \gamma)}{(1 + \beta_{P,2d})^2} C_f
\]

(6.18)

\[
C_f = 1 - 0.071 \exp(-3.28 \gamma_i^2)
\]

(6.19)
6.3. Analytical development

6.3.1. Blister test pressure load

Now the previously mentioned partition theory from Chapter’s 2 and 3 is modified in order to be applied to the blister test for interface fracture toughness between a thin layered material and a substrate. Fig. 6.2 schematically shows a circular blister test to determine the adhesion toughness of mono- and multi-layered graphene membranes under pressure loading. The blister has a radius $R_B$, membrane thickness $h$ and is under pressure load $p$.

Figure 6.3: Blister test interface crack. (a) Thin layer on a thick substrate. (b) Effective crack tip forces and bending moments.
Fig. 6.3a shows how the blister test interfacial crack can be modelled between the thin graphene layer and thick substrate as a DCB. Figure 6.3b shows the effective forces and bending moment at the blister interface crack tip. From the work\textsuperscript{105,106}, at the membrane limit, the effective bending moment and crack tip forces are given as

\[ M = \frac{nt}{4} \left( \frac{nEtp^2R_B^2}{3(1-\nu_1^2)} \right)^{1/3} \]  \hspace{1cm} (6.20)

\[ N = \left( nEtp^2R_B^2 \right)^{2/3} \varphi(v_1) \]  \hspace{1cm} (6.21)

\[ P = \frac{1}{2} pR_B \]  \hspace{1cm} (6.22)

where the subscript \( B \) represents the crack tip at radius \( R_B \), \( n \) the number of layers of graphene and \( E_i \) and \( t \) the membrane properties given as the Young’s modulus and thickness of a monolayer graphene membrane, respectively. The membranes are under pressure \( p \). The membrane Poisson’s ratio \( \nu \) dependent parameter is given as\textsuperscript{105,106}

\[ \varphi(v_1) = \frac{(1.078 + 0.636\nu_1)^{2/3}}{2(1-\nu_1^2)^{1/3}} \]  \hspace{1cm} (6.23)

Following the work of Jensen\textsuperscript{105,106} by assuming that the substrate is infinitely thick when compared to the membrane, i.e. \( \gamma = h_2/h_1 \to \infty \), then the present work does not consider the mismatch in material properties between the membrane and substrate and the blister interface crack can be modelled as an isotropic material. As the blister is circular, the width of the beam is set to the unit value i.e. \( b=1 \). Therefore, it is possible to modify Eqs. (6.1) and (6.2) for the blister test, giving

\[ G_I = c_I \left( \frac{M}{\beta_{2D}} - \frac{N}{\beta_{3D}} - \frac{P}{\beta_{3D}} \right)^2 \]  \hspace{1cm} (6.24)

\[ G_{II} = c_{II} \left( \frac{M}{\theta_{2D}} - \frac{N}{\theta_{3D}} - \frac{P}{\theta_{3D}} \right)^2 \]  \hspace{1cm} (6.25)
Chapter 6: Application of theory using previously published experimental results

It is now essential to obtain the pure modes $\theta_{2-2D}$, $\beta_{2-2D}$, $\theta_{3-2D}$ and $\beta_{3-2D}$ and the parameters $c_t$ and $c_{II}$ when the thickness ratio $\gamma$ tends to infinity, i.e. $\gamma = h_2/h_1 = \infty$. After which it will be possible to partition the total ERR $G$.

From Chapter 2, it is seen that the total ERR $G$ of an isotropic DCB with tip bending moments and axial forces only is given by Eq. (2.5) as

$$G = \frac{1}{2E_b} \left[ \frac{M_{1B}^2}{I_1} + \frac{M_{2B}^2}{I_2} - \frac{1}{I} \left( M_{1B} + M_{2B} - \frac{h_2 N_{1Be}}{2} \right)^2 + \frac{1}{A} \right] N_{1Be}^2$$

(6.26)

where $E = E$ for plane stress or $E = E/(1 - \nu^2)$ for plane strain, with $E$ being the Young’s modulus, and $\nu$ the Poisson’s ratio. $M_{1B}$ and $M_{2B}$ are the two bending moments at the crack tip B, and $N_{1B}$ and $N_{2B}$ are the axial forces at the crack tip B. Other symbols have their conventional meanings.

It is possible to apply Eq. (6.26) to the blister in Fig. 6.3, by setting $b = 1$, $h_1 = h = nt$ and $E = E_1$. Considering an effective bending moment only at the blister interface crack tip, meaning that $M_{1B} = M_{Be}$ and $M_{2B} = N_{1Be} = 0$, the total ERR $G$ becomes

$$G = \frac{1}{2E_1} \left[ \frac{M_{Be}^2}{I_1} \right] = \frac{6}{E_1(nt)^3} M_{Be}^2$$

(6.27)

Also from Chapter 2, it is seen in Fig. 2.3 that for an isotropic material when $\gamma \to \infty$

$$\frac{G_I}{G} = 0.6227 \quad \text{and} \quad \frac{G_{II}}{G} = 0.3773$$

(6.28)

Using the Eqs. (6.24) and (6.25) and the results in Eq (6.28), the mode I and II components for the effective crack tip bending moment become

$$c_t M_{Be}^2 = 0.6227G \quad \text{and} \quad c_{II} M_{Be}^2 = 0.3773G$$

(6.29)

Therefore

$$c_t = 0.6227 \frac{6}{E_1(nt)^3} \quad \text{and} \quad c_{II} = 0.3773 \frac{6}{E_1(nt)^3}$$

(6.30)

From the work$^{90}$, or using Eqs. (6.5) and (6.6) the pure modes $\theta_{2-2D}$ and $\beta_{2-2D}$ when $\gamma \to \infty$ become
To obtain the pure modes $\theta_{3,2D}$ and $\beta_{3,2D}$ in Eqs. (6.7) and (6.8), they are first rearranged into a more convenient form, giving

$$\theta_{3,2D} = \frac{(1 + \beta_{P,2D})}{(\beta_{1,2D}/\theta_{1,2D} - 1)(G_{\beta_{P,2D}}/G_{\gamma_{\beta,2D}})^2}$$

$$\beta_{3,2D} = \frac{(1/\beta_{P,2D} + 1)}{(\theta_{1,2D}/\beta_{1,2D} - 1)(G_{\gamma_{\beta,2D}}/G_{\gamma_{\gamma,2D}})^2}$$

From which it is possible to use Eqs. (6.12), (6.13), (6.14) and (6.30), to determine the ratio of pure modes $\theta_{1,2D}/\beta_{1,2D}$ as

$$\frac{\theta_{1,2D}}{\beta_{1,2D}} = -0.6059$$

The thickness ratio $\gamma$-dependent through-thickness shear correction factor in Eq. (6.17) is now $\kappa(x) = 0.1355$, $C_f = 1$ from Eq. (6.19) and $1/\beta_{P,2D} = 0$, therefore

$$G_{\theta_{P,2D}} = \frac{1}{2h_E E_1} 0.1355$$

$$G_{\beta_{P,2D}} = 0$$

$$G_{\gamma_{\beta,2D}} = \frac{6}{h_E E_1} -1.0659$$

$$G_{\gamma_{\gamma,2D}} = \frac{6}{h_E E_1} 2.6505$$

Finally, the pure modes $\theta_{3,2D}$ and $\beta_{3,2D}$ are obtained from Eqs. (6.32) and (6.33) as

$$(\theta_{3,2D}, \beta_{3,2D}) = \left(\infty, -\frac{1.0063}{h_i}\right)$$
Note that as $\theta_{1,2D} = \infty$, the effective crack tip through-thickness shear force $P_{be}$ only contributes to the mode I component of the ERR $G_I$. Therefore it is possible to combine Eqs. (6.24), (6.25), (6.30), (6.31) and (6.39) under plane strain conditions, to give

$$G_I = \frac{6M_{be}^2}{E_i(nt)^3} \left(1 - \nu^2 \right) \left(1 - \frac{\lambda (\nu)}{4.45 + \lambda} \right)^2 0.6227 \tag{6.40}$$

$$G_{II} = \frac{6M_{be}^2}{E_i(nt)^3} \left(1 - \nu^2 \right) \left(1 + \frac{\chi(\nu)}{2.697} \right)^2 0.3773 \tag{6.41}$$

where

$$\chi(\nu) = \frac{N_{be}nt}{M_{be}} = 4 \left[3 - (1 - \nu^2) \rho^3 (\nu) \right]^{1/2} \tag{6.42}$$

and

$$\lambda = - \frac{P_{be}}{M_{be}\beta_{3,2D}} \tag{6.43}$$

Note that more consideration will be given to $\lambda$ in Eq. (6.43) shortly. Substituting Eq. (6.23) into Eqs. (6.40) and (6.41) gives

$$G_I = \frac{6M_{be}^2}{E_i(nt)^3} \left(1 - \nu^2 \right) \left(0.758 - 0.143\nu + \lambda \right)^2 0.6227 \tag{6.44}$$

$$G_{II} = \frac{6M_{be}^2}{E_i(nt)^3} \left(1 - \nu^2 \right) \left(1.400 + 0.236\nu \right)^2 0.3773 \tag{6.45}$$

It is possible to obtain the mode mixity $\rho = G_I / G_{II}$ as

$$\rho = \frac{1}{0.6059} \left( \frac{0.758 - 0.143\nu + \lambda}{1.400 + 0.236\nu} \right)^2 \tag{6.46}$$

Now consider $\lambda$ in Eq. (6.43). The latest numerical work has shown that the total ERR $G = G_I + G_{II}$ is accurately calculated from Eqs. (6.44) and (6.45) when $\lambda = 0$. This proves that the crack tip through-thickness shear force has no contribution to the total ERR in the membrane limit as the through-thickness shear force disappears. Therefore, the shear force will not contribute to the total ERR in the case of monolayer...
graphene membranes. However, in the case of multilayer graphene membranes, the sliding between graphene layers activates the action of the existing shear force in Eq. (6.22). This action is introduced by modifying the \( \lambda \) parameter giving

\[
\lambda = 3.442 \left(1 - \nu^2\right) \phi(\nu)^{1/2} \left( \frac{pR_B}{nEt} \right)^{1/3} S(n) = \bar{S}(n)
\] (6.47)

where \( S(n) \) is the so-called sliding factor assumed as

\[
S(n) = \left(1 - e^{-1\times n}\right)
\] (6.48)

Eq. (6.48) has been selected as it satisfies the condition that \( S(1) = 0 \) and \( S(\infty) = 1 \). The total ERR \( G \) can then be written in terms of the sliding component \( G_s \) and \( G_j \), the contribution from the crack tip bending moment \( M_{bc} \) in Eq. (6.20) and in-plane force \( N_{bc} \) in Eq. (6.21) as

\[
G = G_j + G_s = G_j (1 + \eta)
\] (6.49)

where \( G_j \) can be calculated as\(^{105,106}\)

\[
G_j = \phi(\nu) \left( \frac{p^4 R_B^4}{nEt} \right)^{1/3} = \phi(\nu) \frac{nEt}{f^4(\nu)} \left( \frac{\delta}{R_B} \right)^4 = \phi(\nu) \frac{p\delta}{f(\nu)}
\] (6.50)

The subscript \( J \) indicates Jensen’s work\(^{105,106}\). The parameter \( \phi \) is

\[
\phi(\nu) = \left( \frac{1}{8\nu} + \frac{(1 - \nu^2)\phi^2}{2} \right)
\] (6.51)

The deflection \( \delta \) at the centre of the blister is

\[
\delta = f(\nu) \left( \frac{pR_B^4}{nEt} \right)^{1/3}
\] (6.52)

with

\[
f(\nu) = 0.9635 \left( \frac{3(1 - \nu)}{7 - \nu} \right)^{1/3}
\] (6.53)
The factor 0.9635 in Eq. (6.53) has been introduced to the work\textsuperscript{114} as the equation \( f(\nu) = \left(3(1-\nu)/(7-\nu)\right)^{1/3} \) is an approximation. Jensen\textsuperscript{105} gives a benchmark value for Eq. (6.53) as \( f(1/3) = 0.645 \). The ratio \( \eta = G_j/G_s \) is

\[
\eta = \frac{2.491\lambda(\lambda + 1.516 - 0.286\nu)}{4.390 + 0.457\nu + 0.135\nu^2}
\] (6.54)

The parameter \( \lambda \) in Eq. (6.47) can have the following alternative expressions by using Eq. (6.52)

\[
\lambda = \phi(\nu) \left( \frac{pR_b}{nEt} \right)^{1/3} = \phi(\nu) \left( \frac{1}{f(\nu)} \right) \left( \frac{\delta}{R_b} \right) = \phi(\nu) \left( \frac{p\delta}{f(\nu)nEt} \right)^{1/4}
\] (6.55)

where

\[
\phi(\nu) = 3.442(1-\nu^2)^{3/2}
\] (6.56)

### 6.3.2 Blister test point load

The mechanical model for a point loaded blister\textsuperscript{105,106} is very much the same as the model for the pressure load developed in Section 6.2.1 and some essential formulas are recorded here. The Poisson’s ratio dependent parameter \( \phi(\nu) \) can be obtained by fitting a curve to the data in Fig. 15 from the work\textsuperscript{105} as

\[
\phi(\nu) = 0.418\nu^3 - 0.006\nu^2 + 0.25\nu + 0.422
\] (6.57)

and \( f(\nu) \) now becomes

\[
f(\nu) = \frac{1}{2\phi(\nu)} + 2\varphi^2(\nu)(1-\nu^2)
\] (6.58)

Then by replacing the pressure load \( p \) with \( P/\pi R_b^2 \) in Eqs. (6.50) and (6.55) where \( P \) represents the point load

\[
G_j = \phi(\nu) \left( \frac{P^4}{\pi^4 R_b^4 nEt} \right)^{1/3} = \phi(\nu) \left( \frac{nEt}{f^4(\nu)} \right) \left( \frac{\delta}{R_b} \right)^4 = \phi(\nu) \frac{P\delta}{\pi R_b^2 f(\nu)nEt}
\] (6.59)

\[
\lambda = \phi(\nu) \left( \frac{P}{\pi R_b nEt} \right)^{1/3} = \phi(\nu) \left( \frac{1}{f(\nu)} \right) \left( \frac{\delta}{R_b} \right) = \phi(\nu) \left( \frac{P\delta}{\pi R_b^2 f(\nu)nEt} \right)^{1/4}
\] (6.60)
with $\phi(\nu)$ and $\bar{\phi}(\nu)$ calculated using Eqs. (6.51) and (6.56) respectively.

6.4. Experimental application

6.4.1. Blister test pressure load

The work$^{104}$ finds the value of $Et = 347 \text{ N/m}$ with $E \approx 1 \text{ TPa}$. Taking the Poisson’s ratio $\nu = 0.16$ as in the work$^{104}$, $\phi(0.16) = 0.3099$, $f(0.16) = 0.6907$, $\phi(0.16) = 0.4502$ and $\bar{\phi}(0.16) = 1.891$. Then, some essential equations above become

$$G_j = 0.4502 \left( \frac{p^4 R^2}{n Et} \right)^{1/3} = 1.978 n Et \left( \frac{\delta}{R} \right)^4 = 0.652 p \delta$$  \hspace{1cm} (6.61)

$$G_S = \eta G_j = 0.5577 \lambda (1.470 + \lambda) G_j$$  \hspace{1cm} (6.62)

$$G_{II} = 0.6988 G_j$$  \hspace{1cm} (6.63)

$$\rho = \frac{G_I}{G_{II}} = 0.7984 (0.7351 + \lambda)^2$$  \hspace{1cm} (6.64)

$$\bar{\lambda} = 1.891 \left( \frac{p R_B}{n Et} \right)^{1/3} = 2.738 \frac{\delta}{R_B} = 2.074 \left( \frac{p \delta}{n Et} \right)^{1/4}$$  \hspace{1cm} (6.65)

Note that in the work$^{104}$ $G_j = 0.655 p \delta$, which is very close to Eq. (6.61) calculated above. In the following, the pressure $p$, the centre deflection $\delta$ and the radius $R_B$ of the multilayer graphene membrane blisters are measured from the Figure S4, S2 and S3 in work$^{104}$, respectively. Therefore, in the following tables $p$, $\delta$ and $R_B$ are obtained from the experimental results of the work$^{104}$. The results are recorded in Table 6.1–6.5 for the mono-, two-, three-, four- and five-layer graphene membrane blisters respectively. To keep consistency with the work$^{104}$, results are calculated using the pressure $p$ and centre deflection $\delta$ meaning that $G_j = 0.652 p \delta$ and $\bar{\lambda} = 2.074 (p \delta / n Et)^{1/4}$ have been used.
Table 6.1: Adhesion toughness of monolayer graphene membranes.

<table>
<thead>
<tr>
<th>The Work\textsuperscript{104}</th>
<th>Analytical Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p (\text{MPa}) )</td>
<td>( \delta (\mu m) )</td>
</tr>
<tr>
<td>( p_o = 3.18 \text{ MPa} )</td>
<td></td>
</tr>
<tr>
<td>1.709</td>
<td>0.363</td>
</tr>
<tr>
<td>1.514</td>
<td>0.396</td>
</tr>
<tr>
<td>1.267</td>
<td>0.463</td>
</tr>
<tr>
<td>1.096</td>
<td>0.496</td>
</tr>
<tr>
<td>Group Average</td>
<td>0.383</td>
</tr>
<tr>
<td>( p_o = 3.55 \text{ MPa} )</td>
<td></td>
</tr>
<tr>
<td>1.648</td>
<td>0.405</td>
</tr>
<tr>
<td>1.429</td>
<td>0.456</td>
</tr>
<tr>
<td>1.242</td>
<td>0.493</td>
</tr>
<tr>
<td>Group Average</td>
<td>0.420</td>
</tr>
<tr>
<td>( p_o = 3.95 \text{ MPa} )</td>
<td></td>
</tr>
<tr>
<td>1.632</td>
<td>0.437</td>
</tr>
<tr>
<td>1.547</td>
<td>0.466</td>
</tr>
<tr>
<td>1.320</td>
<td>0.509</td>
</tr>
<tr>
<td>Group Average</td>
<td>0.458</td>
</tr>
<tr>
<td>( p_o = 4.10 \text{ MPa} )</td>
<td></td>
</tr>
<tr>
<td>1.494</td>
<td>0.475</td>
</tr>
<tr>
<td>1.429</td>
<td>0.502</td>
</tr>
<tr>
<td>1.255</td>
<td>0.514</td>
</tr>
<tr>
<td>Group Average</td>
<td>0.450</td>
</tr>
<tr>
<td>Total Average</td>
<td>0.424</td>
</tr>
<tr>
<td>The Work\textsuperscript{104}</td>
<td>0.450</td>
</tr>
</tbody>
</table>
Table 6.2: Adhesion toughness of two-layer graphene membranes.

<table>
<thead>
<tr>
<th>$p$ (MPa)</th>
<th>$\delta$ (µm)</th>
<th>$\mu_1$ (µm)</th>
<th>$\lambda$</th>
<th>$G_a$ (J/m²)</th>
<th>$G_a$ (J/m²)</th>
<th>$\rho = G_a/G_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_e = 3.25$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.684</td>
<td>0.288</td>
<td>2.401</td>
<td>0.213</td>
<td>0.316</td>
<td>0.380</td>
<td>0.718</td>
</tr>
<tr>
<td>1.471</td>
<td>0.319</td>
<td>2.573</td>
<td>0.211</td>
<td>0.306</td>
<td>0.367</td>
<td>0.715</td>
</tr>
<tr>
<td>1.284</td>
<td>0.345</td>
<td>2.738</td>
<td>0.208</td>
<td>0.289</td>
<td>0.345</td>
<td>0.711</td>
</tr>
<tr>
<td><strong>Group Average</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total Average</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_e = 3.67$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.380</td>
<td>0.341</td>
<td>2.830</td>
<td>0.212</td>
<td>0.307</td>
<td>0.367</td>
<td>0.715</td>
</tr>
<tr>
<td>1.189</td>
<td>0.376</td>
<td>2.978</td>
<td>0.209</td>
<td>0.291</td>
<td>0.348</td>
<td>0.711</td>
</tr>
<tr>
<td>1.085</td>
<td>0.407</td>
<td>3.146</td>
<td>0.208</td>
<td>0.288</td>
<td>0.344</td>
<td>0.711</td>
</tr>
<tr>
<td><strong>Group Average</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total Average</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_e = 4.35$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.076</td>
<td>0.456</td>
<td>3.322</td>
<td>0.214</td>
<td>0.320</td>
<td>0.384</td>
<td>0.719</td>
</tr>
<tr>
<td>0.901</td>
<td>0.542</td>
<td>3.467</td>
<td>0.214</td>
<td>0.318</td>
<td>0.382</td>
<td>0.718</td>
</tr>
<tr>
<td>0.756</td>
<td>0.583</td>
<td>3.679</td>
<td>0.208</td>
<td>0.287</td>
<td>0.343</td>
<td>0.710</td>
</tr>
<tr>
<td><strong>Group Average</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total Average</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Work\textsuperscript{104}
Table 6.3: Adhesion toughness of three-layer graphene membranes.

<table>
<thead>
<tr>
<th>$p$ (MPa)</th>
<th>$\delta$ (µm)</th>
<th>$R_e$ (µm)</th>
<th>$\lambda$</th>
<th>$G_J (J/m^2) (p\delta)$</th>
<th>$G_G (J/m^2)$</th>
<th>$p = G_J / G_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_e = 3.25$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.623</td>
<td>0.280</td>
<td>2.467</td>
<td>0.259</td>
<td>0.296</td>
<td>0.370</td>
<td>0.789</td>
</tr>
<tr>
<td>1.376</td>
<td>0.339</td>
<td>2.615</td>
<td>0.261</td>
<td>0.304</td>
<td>0.380</td>
<td>0.792</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td>0.300</td>
<td>0.375</td>
<td>0.791</td>
<td></td>
</tr>
<tr>
<td>$p_e = 3.67$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.425</td>
<td>0.334</td>
<td>2.862</td>
<td>0.262</td>
<td>0.310</td>
<td>0.388</td>
<td>0.794</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td>0.310</td>
<td>0.388</td>
<td>0.794</td>
<td></td>
</tr>
<tr>
<td>$p_e = 4.35$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.210</td>
<td>0.411</td>
<td>3.286</td>
<td>0.265</td>
<td>0.325</td>
<td>0.408</td>
<td>0.799</td>
</tr>
<tr>
<td>1.020</td>
<td>0.478</td>
<td>3.405</td>
<td>0.264</td>
<td>0.318</td>
<td>0.399</td>
<td>0.797</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td>0.321</td>
<td>0.404</td>
<td>0.798</td>
<td></td>
</tr>
<tr>
<td>Total Average</td>
<td></td>
<td></td>
<td>0.311</td>
<td>0.389</td>
<td>0.794</td>
<td></td>
</tr>
<tr>
<td>The Work$^{104}$</td>
<td></td>
<td></td>
<td>0.431</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6.4: Adhesion toughness of four-layer graphene membranes.

<table>
<thead>
<tr>
<th>$p$ (MPa)</th>
<th>$\delta$ (µm)</th>
<th>$R_e$ (µm)</th>
<th>$\lambda$</th>
<th>$G_J (J/m^2) (p\delta)$</th>
<th>$G_G (J/m^2)$</th>
<th>$p = G_J / G_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_e = 3.25$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.535</td>
<td>0.265</td>
<td>2.664</td>
<td>0.258</td>
<td>0.265</td>
<td>0.331</td>
<td>0.787</td>
</tr>
<tr>
<td>1.420</td>
<td>0.271</td>
<td>2.845</td>
<td>0.254</td>
<td>0.251</td>
<td>0.313</td>
<td>0.782</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td>0.258</td>
<td>0.322</td>
<td>0.785</td>
<td></td>
</tr>
<tr>
<td>$p_e = 3.67$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.407</td>
<td>0.319</td>
<td>2.998</td>
<td>0.264</td>
<td>0.293</td>
<td>0.368</td>
<td>0.797</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td>0.293</td>
<td>0.368</td>
<td>0.797</td>
<td></td>
</tr>
<tr>
<td>$p_e = 4.35$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.118</td>
<td>0.414</td>
<td>3.513</td>
<td>0.266</td>
<td>0.302</td>
<td>0.380</td>
<td>0.801</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td>0.302</td>
<td>0.380</td>
<td>0.801</td>
<td></td>
</tr>
<tr>
<td>Total Average</td>
<td></td>
<td></td>
<td>0.278</td>
<td>0.348</td>
<td>0.792</td>
<td></td>
</tr>
<tr>
<td>The Work$^{104}$</td>
<td></td>
<td></td>
<td>0.431</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 6.5: Adhesion toughness of five-layer graphene membranes.

<table>
<thead>
<tr>
<th>The Work$^{104}$</th>
<th>Analytical Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$ (MPa)</td>
<td>$\delta$ (µm)</td>
</tr>
<tr>
<td>$p_e = 3.25$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.700</td>
<td>0.244</td>
</tr>
<tr>
<td>1.621</td>
<td>0.252</td>
</tr>
<tr>
<td>1.417</td>
<td>0.305</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td>$p_e = 3.67$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.596</td>
<td>0.276</td>
</tr>
<tr>
<td>1.517</td>
<td>0.289</td>
</tr>
<tr>
<td>1.430</td>
<td>0.306</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td>$p_e = 4.35$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.297</td>
<td>0.376</td>
</tr>
<tr>
<td>1.181</td>
<td>0.384</td>
</tr>
<tr>
<td>1.056</td>
<td>0.436</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td>Total Average</td>
<td></td>
</tr>
<tr>
<td>The Work$^{104}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.6: Average adhesion toughness of multilayer graphene membranes.

<table>
<thead>
<tr>
<th>Graphene membranes</th>
<th>$G_f$ (J/m$^2$)</th>
<th>$G_f$ (J/m$^2$)</th>
<th>$\rho = G_f/G_{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
<td>Work$^{104}$</td>
<td>Present</td>
</tr>
<tr>
<td>Monolayer</td>
<td>0.424</td>
<td>0.450</td>
<td>0.424</td>
</tr>
<tr>
<td>Multilayer</td>
<td>0.295</td>
<td>0.310</td>
<td>0.364</td>
</tr>
</tbody>
</table>

For the purpose of completeness and comparison, $\lambda = 1.891(pR_g/nE)^{1/3}$ and $\lambda = 2.738\delta/R_g$ have also been used and results can be seen in Appendix B in Table B.1–B.5 and Table B.6–B.10, respectively. There is generally good agreement between the results for the different calculation of in $\lambda$ Eq. (6.65).

The average adhesion energies are $G = 0.424, 0.362, 0.389, 0.348$ and $0.359$ J/m$^2$ for the mono-, two-, three-, four- and five-layer graphene membrane blisters, respectively, which correspond to the following mode mixities $\rho = G_f/G_{\alpha} = 0.431$, …
0.714, 0.794, 0.792 and 0.786. The large increase of the mode mixity occurs between monolayer and two layer graphene membrane blisters, which results in a big drop in adhesion energies. The adhesion energies have no significant changes afterwards as there are no significant mode mixity changes. An overall average adhesion energy of multilayer graphene membrane blisters is $G = 0.364 \text{ J/m}^2$ with $ho = G_I/G_{II} = 0.764$. These results are shown in Table 6.6.

A linear failure criterion has previously shown excellent predictive capabilities to obtain the fracture toughness for low adhesion energies$^{3,12}$. From which it is possible to estimate the mode I and II critical ERRs as follows

$$\frac{G_I}{G_{ic}} + \frac{G_{II}}{G_{IIc}} = 1$$ (6.66)

Therefore, using the results from Table 6.6 gives

$$G_{IIc} = \left(\rho_1 - \rho_m\right) \left(\frac{G_{II}}{G_{II}} - \frac{G_{II}}{G_{II}}\right)^{-1} = 0.683 \text{ J/m}^2$$ (6.67)

and

$$G_{ic} = \rho_1 G_{IIc} \left(\frac{G_{IIc}}{G_{II}} - 1\right)^{-1} = 0.226 \text{ J/m}^2$$ (6.68)

where $G_{ic}$ and $G_{IIc}$ are the mode I and mode II critical ERRs respectively, $G_{II}$ is the mode II ERR calculated using Eq. (6.63) and $\rho = G_I/G_{II}$ is the mode mixity where a subscript 1 denotes monolayer graphene and $m$ denotes multi-layered graphene.

Fig. 6.4 shows the adhesion energies for mono- and multi-layered graphene membranes. Along the x-axis is the specimen number and then the total ERR is given on the y-axis. The solid black line represents the mean energy release rate of the monolayer results and the dashed line is the mean energy release rate of the multi-layered results. The brown lines show the mode I and II critical ERRs calculated using Eqs. (6.67) and (6.68).
6.4.2. Blister test point load

Taking the Poisson’s ratio to be 0.16 as in the work\textsuperscript{104}, $\varphi(0.16) = 0.4636$, $f(0.16) = 1.4974$, $\phi(0.16) = 0.3743$, $\overline{\phi}(0.16) = 2.3134$ and $\chi(0.16) = 2.1588$. Then some essential equations become

\begin{equation}
G_j = 0.3743 \left( \frac{P^4}{\pi^3 R_h^2 nEt} \right)^{1/3} = 0.07445 nEt \left( \frac{\delta}{R_h} \right)^4 = 0.2500 \frac{P \delta}{\pi R_h^2} \tag{6.69}
\end{equation}

\begin{equation}
\bar{\lambda} = 2.3134 \left( \frac{P}{\pi R_h nEt} \right)^{1/3} = 1.5449 \frac{\delta}{R_h} = 2.0913 \left( \frac{P \delta}{\pi R_h^2 nEt} \right)^{1/4} \tag{6.70}
\end{equation}

\begin{equation}
G_S = \eta G_j = 0.4486 \lambda \left( 1.0298 + \lambda \right) G_j \tag{6.71}
\end{equation}

\begin{equation}
G_H = 0.8811 G_j \tag{6.72}
\end{equation}

\begin{equation}
\rho = \frac{G_L}{G_H} = 0.5091 \left( 0.5149 + \lambda \right)^2 \tag{6.73}
\end{equation}
From Eq. (6.73) it can be seen that \( \rho = 0.135 \) for monolayer graphene in the point loading condition, which is much smaller than that calculated for the pressure loading condition \( \rho = 0.431 \). The adhesion energy for the monolayer graphene in the point loading can be estimated using results from the pressure loading case in Eqs. (6.68) and (6.67) giving \( G_{ic} = 0.226 \text{J/m}^2 \), \( G_{iic} = 0.683 \text{J/m}^2 \) and a linear failure criterion, therefore \( G = 0.550 \text{J/m}^2 \). Obviously, it is larger than \( G = 0.424 \text{J/m}^2 \) in the pressure loading case.

The adhesion energy for multilayer graphene under point loading can be estimated in a similar way with experimental results in work \(^{108}\). The work \(^{108}\) reports the measurement of adhesion energy of five layer graphene membrane blisters. Unlike the work \(^{104}\) where the blister is under a pressure load, the work \(^{108}\) uses nanoparticles acting as a point load. An average blister possesses a radius \( R_B \) in the range 250–300 nm and height \( \delta \) in the range 50–70 nm. The work \(^{108}\) used the formula \( G = 0.0625nEt(\delta/R_B)^4 \) with \( E = 0.5 \text{TPa} \) and \( nt = 1.7 \text{nm} \). It is seen that the value of \( E \) is half of that in the work \(^{104}\) and \( n \approx 5 \). The adhesion energy is reported to be \( G = 0.151 \text{J/m}^2 \) meaning that \( \delta/R_B = 0.2309 \). Therefore from Eq. (6.69), \( G_j = 0.180 \text{J/m}^2 \). The numerical results in recent work \(^{107}\) show that Eq. (6.69) gives accurate calculations. When using \( E = 1.0 \text{TPa} \) as in the work \(^{104}\) \( G_j = 0.360 \text{J/m}^2 \), which is much larger than \( G_j = 0.288 \text{J/m}^2 \) for the pressure load in Table 6.5.

To explain the reason, again consider the mode mixity. Using Eqs (6.70) and (6.71) for the point loaded blister with \( \delta/R_B = 0.2309 \), \( E = 1.0 \text{TPa} \), \( nt = 1.7 \text{nm} \) and \( n = 5 \), the total ERR \( G \) can be obtained as \( G = 0.438 \text{J/m}^2 \) which is very close to \( G = 0.424 \text{J/m}^2 \) for the monolayer graphene membrane blister under a pressure load from Table 6.6. Using Eq. (6.73) the mode mixity is calculated to be \( \rho = 0.381 \) which is very close to \( \rho = 0.431 \) for the monolayer graphene membrane blister under a pressure load from Table 6.6. It can be seen that the mode mixity plays a key role in determining the adhesion energy. When using \( G_{ic} = 0.226 \text{J/m}^2 \), \( G_{iic} = 0.683 \text{J/m}^2 \), \( \rho = 0.381 \) and a linear failure criterion, \( G \) is found to be \( G = 0.438 \text{J/m}^2 \). It is seen that the accuracy of \( G_{ic} = 0.226 \text{J/m}^2 \), \( G_{iic} = 0.683 \text{J/m}^2 \) and the linear failure criterion are excellent.
6.5. Conclusion

This chapter has presented an application of the partition theory for an orthotropic DCB with general loading conditions given in Chapter 3. The partition theory has been adapted to consider the blister test for interface fracture toughness and it is now possible to obtain the mode I and II components of the ERR.

It has been identified that in the previous mechanical models for the blister test, the fracture mode mixity and the sliding effect in multi-layered specimens had not been considered. The presence of sliding in multi-layered graphene membranes increases the fracture mode ratio $G_I/G_{II}$, leading to a decrease in the adhesion toughness measurements. In the case of a silicon oxide substrate and pressure loading, the mode mixity jumps from 43% in the monolayer graphene membranes to over 71% in the two layer graphene membranes. This increase in the mode mixity has the effect of lowering the adhesion toughness $G_c$ from $0.424 \text{ J/m}^2$ to $0.362 \text{ J/m}^2$. As the number of graphene layers is increased further, the mode mixity increases slightly and the average adhesion energy toughness of multi-layered membranes is $G = 0.364 \text{ J/m}^2$. Using a linear failure criterion it has been identified that the critical mode I and mode II adhesion toughness’s are $G_{Ic} = 0.226 \text{ J/m}^2$ and $G_{IIc} = 0.683 \text{ J/m}^2$, respectively.

In the case of a silicon oxide substrate and point loading, the mode mixity jumps from 14% in the monolayer graphene membranes to above 38% in the multilayer graphene membranes, while the adhesion toughness $G_c$ falls from $0.550 \text{ J/m}^2$ to $0.438 \text{ J/m}^2$. It has been identified that the adhesion toughness $G_c$ in general loading conditions can be accurately determined using the critical mode I and mode II adhesion toughness $G_{Ic} = 0.226 \text{ J/m}^2$ and $G_{IIc} = 0.683 \text{ J/m}^2$ and a linear failure criterion.
Chapter 7: Conclusion

7.1. Conclusion

This thesis has presented three original mixed-mode partition theories based on 2D elasticity. The first of which is for an orthotropic laminated DCB with crack tip bending moments, axial forces and through-thickness shear forces. The second is for a bimaterial DCB with a mismatch in the Young’s modulus as well as a mismatch in the Poisson’s ratio, under tip bending moments and axial forces. The third is for the calculation of adhesion energies between multilayer graphene membranes and substrates.

Using the same orthogonal pure mode methodology as Wang and Harvey\(^{1–3,12,25}\), 2D-elasticity-based pure modes have been derived for an orthotropic DCB with tip bending moments and axial forces by introducing correction factors into beam-theory-based mechanical conditions. To determine the correction factors that are required in order to obtain the 2D-elasticity-based pure modes, previously determined pure modes based on Euler and Timoshenko beam theories\(^{1,2}\) are used as the upper and lower limits of the 2D elasticity pure modes. Once the correction factors have been identified, it is possible to calibrate the selection by comparing results with the current most accurate 2D elasticity partition theory\(^{21}\). The new 2D elasticity partition theory is then validated by comparison with other partition solutions and excellent agreement is obtained between the new theory and that from Suo and Hutchinson\(^{21}\), and FEM simulations. The new partition theory has the advantage over the work of Suo and Hutchinson\(^{21}\) as being completely analytical.

The mixed-mode partition theory for an orthotropic DCB has been extended to include crack tip through-thickness shear forces. Pure modes were obtained for crack tip through-thickness shear forces only, using the FEM and then lines are fitted to the results to obtain an analytical expression for the pure modes. It is identified that the pure mode I condition is the same when considering 2D elasticity and Timoshenko beam theory, the pure mode II condition is different. Due to the high mesh densities that are associated with very large and very small thickness ratios, the partition theory is only valid in the range \(-1.7 \leq \gamma_i \leq 1.7\) where \(\gamma_i = \log_{10}(1/\gamma)\). Results using Timoshenko beam theory show that shear forces only affect the mode I component of the ERR i.e. \(G_H = 0\), however, it has been shown that this isn’t the case for 2D elasticity. Finally, the
total ERR \( G \) is different when comparing results from Timoshenko beam theory to 2D elasticity theory as Timoshenko beam theory assumes a constant through-thickness shear correction factor, which is not that case for 2D elasticity.

Due to these findings, the Timoshenko beam partition theory is modified to account for 2D elasticity by introducing a thickness-ratio-dependent shear correction factor and a pure-mode-II ERR correction factor. Using the FEM to obtain the correction factors, it is interesting to note that they both follow a normal distribution around a symmetric DCB geometry and are independent of material properties. The theory has been validated using the FEM with the VCCT and it is identified that excellent agreement is obtained for all thickness ratios and loading conditions, particularly when the total ERR \( G \) is not close to zero.

The partition theory for an orthotropic laminated DCB with general loading has been extended and applied to the blister test for interface fracture toughness under a pressure and point load. From experimental results \(^{104}\) in the literature when comparing mono- and multi-layered graphene specimens under a pressure load, it was identified that there was a drop in the adhesion energy but it was unclear why. Using the present theory, it has been identified that the drop in adhesion energy is due to the mode mixity and sliding effect that is present in the multi-layered specimens, which is not considered in previous mechanical models. Using a linear failure criterion and the partition theory, the mode I and mode II fracture toughness’s were obtained. Then it was possible to use other experimental results \(^{108}\) for multi-layered graphene blisters under a point load to confirm the accuracy of the results and excellent agreement was obtained.

When considering a bimaterial DCB a new method to analytically obtain the complex SIFs and crack extension size-dependent mode I and II ERRs when there is a mismatch in the Young’s modulus and the beams have the same Poisson’s ratio is given. It has been identified that the mismatch in the Young’s modulus between the beams causes two distinct sets of orthogonal pure modes \((\theta_k, \beta_k)\) and \((\theta'_k, \beta'_k)\), and two sets of orthogonal approximate pure modes, \((\tilde{\theta}_k, \tilde{\beta}_k)\) and \((\tilde{\theta}'_k, \tilde{\beta}'_k)\), which are in terms of the SIFs, \(K_I\) and \(K_{II}\), and which are crack extension size-dependent.

A brittle interface causes the existence of two distinct sets of orthogonal pure modes, \((\theta_i, \beta_i)\) and \((\theta'_i, \beta'_i)\) (with \(i = 1, 2, 3\)), which are in terms of the crack tip loads. As the crack extension size decreases, the two sets of pure modes approach each other and
become coincident, however they do not converge and are crack extension size-dependent. When the crack extension size $\delta a = 0.05\,\text{mm}$, thickness ratio $\gamma = 1$ and Poisson’s ratio $\nu = 0.29$, the two distinct sets of orthogonal pure modes, $(\theta_i, \beta_i)$ and $(\theta_i', \beta_i')$ (with $i = 1, 2, 3$), approximately coincide with each other and are also approximately equal to the pure modes based on Timoshenko beam theory\(^1,\!^2\) for the entire modulus ratio range $1/100 \leq \eta \leq 100$. From which a shifting technique has been developed in order to obtain the 2D-elasticity-based pure modes using the Timoshenko beam partition theory\(^1,\!^2\). Once the pure modes are obtained, it is possible to partition the total ERR for the crack extension size $\delta a = 0.05\,\text{mm}$. Using the approximate SIF-based pure modes, it is possible to determine which solution of the SIFs is physically admissible, meaning the crack extension size-dependent ERRs, $G_I$ and $G_{II}$, can be analytically obtained for any crack extension size.

The theory has also been extended to account for a mismatch in the Poisson’s ratio as well as the Young’s modulus. By identifying that for a given geometry and loading conditions, the total ERR and the bimaterial mismatch coefficient are the two main factors affecting the ERR partitions; it is possible to derive equivalent material properties so that the total ERR and bimaterial mismatch coefficient are maintained. The equivalent material properties take the form of an equivalent Poisson’s ratio and an equivalent Young’s modulus. For the plane stress condition, as the total ERR is maintained regardless of the choice of the Poisson’s ratio, only an equivalent Poisson’s ratio is required. Results for the new theory are compared to 2D FEM simulations and excellent agreement is obtained for both plane stress and plane strain conditions under a number of loading conditions.

The work in this thesis has been published in the Journal Composite Structures\(^16–\!^20\).

7.2. **Further work**

Although the work in this thesis has provided a valuable contribution to the field of fracture mechanics, there still remain plenty of areas that will provide a novel contribution to the field. In this section a number of valuable extensions to the work in this thesis are given.
7.2.1. **Extension of bimaterial partition theory to include through-thickness shear forces**

The work based on a bimaterial DCB offers an analytical approach to obtain the complex SIFs and crack extension size-dependent ERR partitions for a combination of tip bending moments and axial forces. However, there remains no analytical method for the inclusion of crack tip through-thickness shear forces, therefore the theory should be extended to include crack tip through-thickness shear forces.

7.2.2. **Experimental validation of the bimaterial partition theory**

Although the work based on a bimaterial DCB in Chapter’s 4 and 5 has been validated using results obtained from 2D FEM solutions, it is also important to gain experimental validation of the theory. This would show a real life application and therefore reinforce the use of the partition theory for bimaterial DCBs.

7.2.3. **Application of the orthotropic partition theory to other experimental results**

Finally, the partition theory based on an orthotropic laminated DCB has shown excellent predictive capability when applied to experimental results obtained from mono- and multi-layered graphene blisters. Therefore further experimental validation would be extremely beneficial, for example in gas turbine engine thermal barrier coatings.
Appendices
Appendix A – Convergence study

A.1. Convergence study

This section presents the convergence study for the FEM meshes used in this thesis. The numerical simulations have been carried out using MSC/NASTRAN. Section A.1.1 contains the mesh which is applicable to Chapter’s 3 and 5. Section A.1.2 contains the mesh for Chapter 4.

As a DCB is being modelled, the boundary conditions were the same as that for a cantilever beam, therefore at the fixed support the translational and rotational degrees of freedom were restricted. The DCB geometry consisted of the uncracked length $L = 100 \text{ mm}$, the cracked length $a = 10 \text{ mm}$, the width $b = 10 \text{ mm}$ and the minimum beam thickness $h_{\text{min}} = 1 \text{ mm}$. The thickness of the upper and lower beams were therefore dependent on the thickness ratio $\gamma = h_2/h_1$, with $h_1 = h_{\text{min}}$ and $h_2 = \gamma h_{\text{min}}$ if $\gamma > 1$, and with $h_1 = h_{\text{min}}/\gamma$ and $h_2 = h_{\text{min}}$ if $\gamma < 1$.

The top and bottom layers in the DCB were modelled using quadrilateral plane-strain shell elements and isotropic material properties within each layer. The minimum Young’s modulus $E_{\text{min}} = 1000 \text{ N/mm}^2$. The Young’s modulus of the top and bottom layers therefore varied with $E_1 = E_{\text{min}}$ and $E_2 = \eta E_{\text{min}}$ respectively if $\eta > 1$, and with $E_1 = E_{\text{min}}/\eta$ and $E_2 = E_{\text{min}}$ if $\eta < 1$. When $\eta = 1$, $E_1 = E_2 = E_{\text{min}} = 1000 \text{ N/mm}^2$. The shear modulus was calculated as $\mu_i = E_i/[2(1 + \nu)]$ with $i = 1, 2$ and Poisson’s ratio $\nu = 0.29$.

Non uniform meshes were used and the procedure is now summarised. In the x-direction 2000 square elements of size $p \times p$ were centred on the crack tip and 100 square elements of the same size were centred on the crack tip in the y-direction. Beyond the region of uniform element size surrounding the crack tip, elements were allowed to grow at a constant rate of 1.1 in both the x- and y-directions up to maximum sizes of 1.0 and 0.1 respectively, after which they remained at these maximum sizes. Very small adjustments were made to the element size growth rate, or to the maximum element size, as appropriate, to satisfy the boundary geometry. The ERRs were calculated using the VCCT and the forces from the crack tip springs. Contact between the upper and lower surfaces of the crack was not considered.
A.1.1. Chapter’s 3 and 5 convergence study

To check convergence of the FEM mesh for Chapter’s 3 and 5 both the total ERR $G$ and mode partition $G_i/G$ will be considered for two loading ratios and two thickness ratios $\gamma$. The first loading ratio considered was $-10 \leq P_{2B}/P_{1B} \leq 10$ with $M_{1B} = M_{2B} = N_{1B} = N_{2B} = 0$. The second loading ratio was $-10 \leq P_{1B}/M_{1B} \leq 10$ with $M_{2B} = N_{1B} = N_{2B} = P_{2B} = 0$. The two thickness ratios were $\gamma = 10$ and $\gamma = 1$. These were selected as the mesh is $\gamma$-dependent, therefore both the maximum and minimum mesh densities have been checked for convergence.

A rigid interface between the upper and lower beams was modelled by ‘connecting’ the translational degrees of freedom of co-located interface nodes on the upper and lower beams using multi-point constraints; however, at the crack tip, instead of rigidly connecting the crack tip nodes, the interface was modelled with normal and shear point springs. The stiffness of both of the springs was selected using $k_s = E_{CT} A_{CT}/L_s$ where $A_{CT}$ is the element area at the crack tip calculated as $A_{CT} = bp$ where $b$ is the DCB width and $p$ the element pitch. The spring length was selected to be a unit value, however in reality the interface has zero thickness. Finally, $E_{CT} = 10^9 \text{ N/mm}^2$, which is the Young’s modulus of the interface at the crack tip. Therefore, the springs stiffness $k_s$ was sufficiently high in comparison to $E$ to simulate brittle interfacial cracking without introducing excessive numerical error.

In Figs. A.1–A.4 three element sizes at the crack tip have been considered. The first of which $p = 0.1$ gives a maximum mesh density of $1100 \times 110$ when $\gamma = 10$ and minimum mesh density of $1100 \times 20$ when $\gamma = 1$. The second element size has $p = 0.01$ which gives a maximum mesh density of $2128 \times 229$ when $\gamma = 10$ and minimum mesh density of $2128 \times 138$ when $\gamma = 1$. The third element size has $p = 0.001$ which gives a maximum mesh density of $2233 \times 285$ when $\gamma = 10$ and minimum mesh density of $2233 \times 196$ when $\gamma = 1$. From Figs. A.1–A.4 it can be seen that convergence has been achieved when the element size $p \leq 0.01$. Therefore a maximum element size of $p = 0.01 \times 0.01$ must be selected with the non-uniform mesh to achieve convergence.
Appendix A – Convergence study

Figure A.1: Convergence study for the total ERR $G$ and mode partition $G_i/G$ for an orthotropic DCB with $\gamma = 10$, $\eta = 1$ and $-10 \leq P_{2B}/P_{1B} \leq 10$.

Figure A.2: Convergence study for the total ERR $G$ and mode partition $G_i/G$ for an orthotropic DCB with $\gamma = 1$, $\eta = 1$ and $-10 \leq P_{2B}/P_{1B} \leq 10$. 
Figure A.3: Convergence study for the total ERR $G$ and mode partition $G_i/G$ for an orthotropic DCB with $\gamma = 10$, $\eta = 1$ and $-10 \leq P_{1B}/M_{1B} \text{[mm}^{-1}\text{]} \leq 10$.

Figure A.4: Convergence study for the total ERR $G$ and mode partition $G_i/G$ for an orthotropic DCB with $\gamma = 1$, $\eta = 1$ and $-10 \leq P_{1B}/M_{1B} \text{[mm}^{-1}\text{]} \leq 10$. 
A.1.2 Chapter 4 convergence study

For an interfacial crack between two dissimilar materials although the total ERR $G$ will converge to a well-defined value, the individual mode components $G_I$ and $G_{II}$ show non-convergence as the mesh density increases. Therefore in this section, only the total ERR $G$ can be checked for convergence.

The loading ratio that has been considered is $-20 \leq M_{2B}/M_{1B} \leq 20$ with $N_{1B} = N_{2B} = P_{1B} = P_{2B} = 0$. In Fig. A.5 $\gamma = 10$ and $\eta = 1/100$. In Fig. A.6 $\gamma = 1$ and $\eta = 100$. Again as the mesh density is dependent on the thickness ratio $\gamma$, the maximum and minimum values used in this thesis have been considered. Furthermore, in Chapter 4 there is a mismatch in the Young’s modulus of the upper and lower beams meaning that the minimum and maximum values are also considered in this section.

The interface between the top and bottom layers was modelled with normal and shear point springs with a stiffness of $k_s = 10^{11}$ N/mm, which was sufficiently high in comparison to $E_1$ and $E_2$ to simulate brittle interfacial cracking without introducing excessive numerical error.
Figure A.5: Convergence study for the total ERR $G$ for a bimaterial DCB with $\gamma = 10$, $\eta = 1/100$ and $-20 \leq M_{2B}/M_{1B} \leq 20$.

Figure A.6: Convergence study for the total ERR $G$ for a bimaterial DCB with $\gamma = 1$, $\eta = 100$ and $-20 \leq M_{2B}/M_{1B} \leq 20$. 

$G [\text{J/mm}^2]$ 

$M_{2B}/M_{1B}$
In Figs. A.5 and A.6 two element sizes at the crack tip have been considered. The first of which $p = 0.1$ gives a maximum mesh density of $1100 \times 110$ when $\gamma = 10$ and minimum mesh density of $1100 \times 20$ when $\gamma = 1$. The second element size has $p = 0.01$ which gives a maximum mesh density of $2128 \times 229$ when $\gamma = 10$ and minimum mesh density of $2128 \times 138$ when $\gamma = 1$. From Figs. A.5 and A.6 it can be seen that convergence has been achieved for both of the element sizes considered. In conclusion, a maximum element size of $p = 0.01 \times 0.01$ must be selected with the non-uniform mesh to achieve convergence.
Appendix B – Chapter 7 alternative calculations

B.1. Introduction

For the purpose of completeness and comparison, $\bar{\lambda}$ from Chapter 7 is calculated using the different definitions given, to see whether the total ERR $G$ and the mode mixity are affected.

B.2. Alternative calculations

B.2.1. Alternative calculation 1

Table B.1: Adhesion toughness of monolayer graphene membranes.

<table>
<thead>
<tr>
<th>$p$ (MPa)</th>
<th>$\delta$ (µm)</th>
<th>$R_y$ (µm)</th>
<th>$k$ ($pR_y$)</th>
<th>$G_y$ (J/m$^2$) ($p\delta$)</th>
<th>$G$ (J/m$^2$)</th>
<th>$\rho = G_y/G_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_s = 3.18$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.709</td>
<td>0.363</td>
<td>2.492</td>
<td>0</td>
<td>0.405</td>
<td>0.405</td>
<td>0.431</td>
</tr>
<tr>
<td>1.514</td>
<td>0.396</td>
<td>2.710</td>
<td>0</td>
<td>0.391</td>
<td>0.391</td>
<td>0.431</td>
</tr>
<tr>
<td>1.267</td>
<td>0.463</td>
<td>2.934</td>
<td>0</td>
<td>0.382</td>
<td>0.382</td>
<td>0.431</td>
</tr>
<tr>
<td>1.096</td>
<td>0.496</td>
<td>3.171</td>
<td>0</td>
<td>0.354</td>
<td>0.354</td>
<td>0.431</td>
</tr>
<tr>
<td>Group Average</td>
<td>0.383</td>
<td>0.383</td>
<td>0.431</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_s = 3.55$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.648</td>
<td>0.405</td>
<td>2.756</td>
<td>0</td>
<td>0.435</td>
<td>0.435</td>
<td>0.431</td>
</tr>
<tr>
<td>1.429</td>
<td>0.456</td>
<td>2.947</td>
<td>0</td>
<td>0.425</td>
<td>0.425</td>
<td>0.431</td>
</tr>
<tr>
<td>1.242</td>
<td>0.493</td>
<td>3.168</td>
<td>0</td>
<td>0.400</td>
<td>0.400</td>
<td>0.431</td>
</tr>
<tr>
<td>Group Average</td>
<td>0.420</td>
<td>0.420</td>
<td>0.431</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_s = 3.95$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.632</td>
<td>0.437</td>
<td>2.964</td>
<td>0</td>
<td>0.465</td>
<td>0.465</td>
<td>0.431</td>
</tr>
<tr>
<td>1.547</td>
<td>0.466</td>
<td>3.021</td>
<td>0</td>
<td>0.470</td>
<td>0.470</td>
<td>0.431</td>
</tr>
<tr>
<td>1.320</td>
<td>0.509</td>
<td>3.252</td>
<td>0</td>
<td>0.438</td>
<td>0.438</td>
<td>0.431</td>
</tr>
<tr>
<td>Group Average</td>
<td>0.458</td>
<td>0.458</td>
<td>0.431</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_s = 4.10$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.494</td>
<td>0.475</td>
<td>3.208</td>
<td>0</td>
<td>0.463</td>
<td>0.463</td>
<td>0.431</td>
</tr>
<tr>
<td>1.429</td>
<td>0.502</td>
<td>3.376</td>
<td>0</td>
<td>0.468</td>
<td>0.468</td>
<td>0.431</td>
</tr>
<tr>
<td>1.255</td>
<td>0.514</td>
<td>3.513</td>
<td>0</td>
<td>0.421</td>
<td>0.421</td>
<td>0.431</td>
</tr>
<tr>
<td>Group Average</td>
<td>0.450</td>
<td>0.450</td>
<td>0.431</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Average</td>
<td>0.424</td>
<td>0.424</td>
<td>0.431</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The Work$^{104}$</td>
<td>0.450</td>
<td>0.450</td>
<td>0.431</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table B.2: Adhesion toughness of two-layer graphene membranes.

<table>
<thead>
<tr>
<th>$p$ (MPa)</th>
<th>$\delta$ ((\mu)m)</th>
<th>$R$ ((\mu)m)</th>
<th>$k$ ((pR))</th>
<th>$G_c$ (J/m$^2$)</th>
<th>$\rho$ (J/m$^2$)</th>
<th>$\rho = G_c/G_{\text{GG}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_s = 3.25$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.684</td>
<td>0.288</td>
<td>2.401</td>
<td>0.215</td>
<td>0.316</td>
<td>0.380</td>
<td>0.721</td>
</tr>
<tr>
<td>1.471</td>
<td>0.319</td>
<td>2.573</td>
<td>0.210</td>
<td>0.306</td>
<td>0.366</td>
<td>0.714</td>
</tr>
<tr>
<td>1.284</td>
<td>0.345</td>
<td>2.738</td>
<td>0.205</td>
<td>0.289</td>
<td>0.344</td>
<td>0.706</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td>0.304</td>
<td>0.364</td>
<td>0.714</td>
</tr>
<tr>
<td>$p_s = 3.67$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.380</td>
<td>0.341</td>
<td>2.830</td>
<td>0.213</td>
<td>0.307</td>
<td>0.368</td>
<td>0.717</td>
</tr>
<tr>
<td>1.189</td>
<td>0.376</td>
<td>2.978</td>
<td>0.206</td>
<td>0.291</td>
<td>0.347</td>
<td>0.707</td>
</tr>
<tr>
<td>1.085</td>
<td>0.407</td>
<td>3.146</td>
<td>0.203</td>
<td>0.288</td>
<td>0.343</td>
<td>0.703</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td>0.295</td>
<td>0.353</td>
<td>0.709</td>
</tr>
<tr>
<td>$p_s = 4.35$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.076</td>
<td>0.456</td>
<td>3.322</td>
<td>0.206</td>
<td>0.320</td>
<td>0.381</td>
<td>0.708</td>
</tr>
<tr>
<td>0.901</td>
<td>0.542</td>
<td>3.467</td>
<td>0.197</td>
<td>0.318</td>
<td>0.377</td>
<td>0.694</td>
</tr>
<tr>
<td>0.756</td>
<td>0.583</td>
<td>3.679</td>
<td>0.190</td>
<td>0.287</td>
<td>0.338</td>
<td>0.683</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td>0.308</td>
<td>0.365</td>
<td>0.695</td>
</tr>
<tr>
<td>Total Average</td>
<td></td>
<td></td>
<td></td>
<td>0.303</td>
<td>0.361</td>
<td>0.706</td>
</tr>
<tr>
<td>The Work$^{104}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.431</td>
</tr>
</tbody>
</table>
Table B.3: Adhesion toughness of three-layer graphene membranes.

<table>
<thead>
<tr>
<th>$\rho$ (MPa)</th>
<th>$\delta$ (µm)</th>
<th>$R_s$ (µm)</th>
<th>$k \cdot (\rho R_g)$</th>
<th>$G_f$ (J/m$^2$)</th>
<th>$\rho$</th>
<th>$\rho = G_f / G_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.623</td>
<td>0.280</td>
<td>2.467</td>
<td>0.256</td>
<td>0.296</td>
<td>0.369</td>
<td>0.785</td>
</tr>
<tr>
<td>1.376</td>
<td>0.339</td>
<td>2.615</td>
<td>0.247</td>
<td>0.304</td>
<td>0.376</td>
<td>0.770</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.300</td>
<td>0.373</td>
</tr>
<tr>
<td>$\rho_s = 3.25$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.425</td>
<td>0.334</td>
<td>2.862</td>
<td>0.258</td>
<td>0.310</td>
<td>0.387</td>
<td>0.787</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.310</td>
<td>0.387</td>
</tr>
<tr>
<td>$\rho_s = 3.67$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.210</td>
<td>0.411</td>
<td>3.286</td>
<td>0.256</td>
<td>0.325</td>
<td>0.405</td>
<td>0.784</td>
</tr>
<tr>
<td>1.020</td>
<td>0.478</td>
<td>3.405</td>
<td>0.244</td>
<td>0.318</td>
<td>0.392</td>
<td>0.766</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.321</td>
<td>0.399</td>
</tr>
<tr>
<td>Total Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.311</td>
<td>0.386</td>
</tr>
<tr>
<td>The Work$^{104}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table B.4: Adhesion toughness of four-layer graphene membranes.

<table>
<thead>
<tr>
<th>$\rho$ (MPa)</th>
<th>$\delta$ (µm)</th>
<th>$R_s$ (µm)</th>
<th>$k \cdot (\rho R_g)$</th>
<th>$G_f$ (J/m$^2$)</th>
<th>$\rho$</th>
<th>$\rho = G_f / G_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.535</td>
<td>0.265</td>
<td>2.664</td>
<td>0.258</td>
<td>0.265</td>
<td>0.331</td>
<td>0.787</td>
</tr>
<tr>
<td>1.420</td>
<td>0.271</td>
<td>2.845</td>
<td>0.257</td>
<td>0.251</td>
<td>0.313</td>
<td>0.785</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.258</td>
<td>0.322</td>
</tr>
<tr>
<td>$\rho_s = 3.25$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.407</td>
<td>0.319</td>
<td>2.998</td>
<td>0.260</td>
<td>0.293</td>
<td>0.366</td>
<td>0.791</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.293</td>
<td>0.366</td>
</tr>
<tr>
<td>$\rho_s = 3.67$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.118</td>
<td>0.414</td>
<td>3.513</td>
<td>0.254</td>
<td>0.302</td>
<td>0.376</td>
<td>0.781</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.302</td>
<td>0.376</td>
</tr>
<tr>
<td>Total Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.278</td>
<td>0.347</td>
</tr>
<tr>
<td>The Work$^{104}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table B.5: Adhesion toughness of five-layer graphene membranes.

<table>
<thead>
<tr>
<th>The Work 104</th>
<th>Analytical Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p (\text{MPa}) )</td>
<td>( \delta (\mu \text{m}) )</td>
</tr>
<tr>
<td>( p_s = 3.25 \text{ MPa} )</td>
<td></td>
</tr>
<tr>
<td>1.700</td>
<td>0.244</td>
</tr>
<tr>
<td>1.621</td>
<td>0.252</td>
</tr>
<tr>
<td>1.417</td>
<td>0.305</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td>( p_s = 3.67 \text{ MPa} )</td>
<td></td>
</tr>
<tr>
<td>1.596</td>
<td>0.276</td>
</tr>
<tr>
<td>1.517</td>
<td>0.289</td>
</tr>
<tr>
<td>1.430</td>
<td>0.306</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td>( p_s = 4.35 \text{ MPa} )</td>
<td></td>
</tr>
<tr>
<td>1.297</td>
<td>0.376</td>
</tr>
<tr>
<td>1.181</td>
<td>0.384</td>
</tr>
<tr>
<td>1.056</td>
<td>0.436</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td>Total Average</td>
<td>0.288</td>
</tr>
<tr>
<td>The Work 104</td>
<td></td>
</tr>
</tbody>
</table>
### B.2.2. Alternative calculation 2

Table B.6: Adhesion toughness of monolayer graphene membranes.

<table>
<thead>
<tr>
<th>The Work$^{104}$</th>
<th>Analytical Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$ (MPa)</td>
<td>$\delta$ (μm)</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_s = 3.18$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.709</td>
<td>0.363</td>
</tr>
<tr>
<td>1.514</td>
<td>0.396</td>
</tr>
<tr>
<td>1.267</td>
<td>0.463</td>
</tr>
<tr>
<td>1.096</td>
<td>0.496</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_s = 3.55$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.648</td>
<td>0.405</td>
</tr>
<tr>
<td>1.429</td>
<td>0.456</td>
</tr>
<tr>
<td>1.242</td>
<td>0.493</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_s = 3.95$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.632</td>
<td>0.437</td>
</tr>
<tr>
<td>1.547</td>
<td>0.466</td>
</tr>
<tr>
<td>1.320</td>
<td>0.509</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_s = 4.01$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.494</td>
<td>0.475</td>
</tr>
<tr>
<td>1.429</td>
<td>0.502</td>
</tr>
<tr>
<td>1.255</td>
<td>0.514</td>
</tr>
<tr>
<td>Group Average</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Average</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Work$^{104}$ 0.45 0.45 0.431
Table B.7: Adhesion toughness of two-layer graphene membranes.

<table>
<thead>
<tr>
<th>$p$ (MPa)</th>
<th>$\delta$ (µm)</th>
<th>$R_s$ (µm)</th>
<th>$\lambda$ ($\delta/R_s$)</th>
<th>$G_j$ (J/m^2) ($p\delta$)</th>
<th>$G$ (J/m^2)</th>
<th>$\rho = G_j/G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.25 MPa</td>
<td>1.684 0.288</td>
<td>2.401</td>
<td>0.208</td>
<td>0.316</td>
<td>0.378</td>
<td>0.710</td>
</tr>
<tr>
<td></td>
<td>1.471 0.319</td>
<td>2.573</td>
<td>0.214</td>
<td>0.306</td>
<td>0.368</td>
<td>0.720</td>
</tr>
<tr>
<td></td>
<td>1.284 0.345</td>
<td>2.738</td>
<td>0.218</td>
<td>0.289</td>
<td>0.348</td>
<td>0.726</td>
</tr>
<tr>
<td></td>
<td>Group Average</td>
<td></td>
<td>0.304</td>
<td>0.365</td>
<td>0.718</td>
<td></td>
</tr>
<tr>
<td>3.67 MPa</td>
<td>1.380 0.341</td>
<td>2.830</td>
<td>0.208</td>
<td>0.307</td>
<td>0.366</td>
<td>0.711</td>
</tr>
<tr>
<td></td>
<td>1.189 0.376</td>
<td>2.978</td>
<td>0.218</td>
<td>0.291</td>
<td>0.351</td>
<td>0.726</td>
</tr>
<tr>
<td></td>
<td>1.085 0.407</td>
<td>3.146</td>
<td>0.224</td>
<td>0.288</td>
<td>0.349</td>
<td>0.735</td>
</tr>
<tr>
<td></td>
<td>Group Average</td>
<td></td>
<td>0.295</td>
<td>0.356</td>
<td>0.724</td>
<td></td>
</tr>
<tr>
<td>4.35 MPa</td>
<td>1.076 0.456</td>
<td>3.322</td>
<td>0.237</td>
<td>0.320</td>
<td>0.392</td>
<td>0.755</td>
</tr>
<tr>
<td></td>
<td>0.901 0.542</td>
<td>3.467</td>
<td>0.270</td>
<td>0.318</td>
<td>0.402</td>
<td>0.807</td>
</tr>
<tr>
<td></td>
<td>0.756 0.583</td>
<td>3.679</td>
<td>0.274</td>
<td>0.287</td>
<td>0.364</td>
<td>0.813</td>
</tr>
<tr>
<td></td>
<td>Group Average</td>
<td></td>
<td>0.308</td>
<td>0.386</td>
<td>0.792</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Total Average</td>
<td></td>
<td>0.303</td>
<td>0.369</td>
<td>0.745</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The Work</td>
<td></td>
<td></td>
<td></td>
<td>0.431</td>
<td></td>
</tr>
</tbody>
</table>
Table B.8: Adhesion toughness of three-layer graphene membranes.

<table>
<thead>
<tr>
<th>The Work\textsuperscript{104}</th>
<th>Analytical Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$ (MPa)</td>
<td>$\delta$ (µm)</td>
</tr>
<tr>
<td>$p_s = 3.25$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.623</td>
<td>0.280</td>
</tr>
<tr>
<td>1.376</td>
<td>0.339</td>
</tr>
<tr>
<td>Group Average</td>
<td>&amp;</td>
</tr>
<tr>
<td>$p_s = 3.67$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.425</td>
<td>0.334</td>
</tr>
<tr>
<td>Group Average</td>
<td>&amp;</td>
</tr>
<tr>
<td>$p_s = 4.35$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.210</td>
<td>0.411</td>
</tr>
<tr>
<td>1.020</td>
<td>0.478</td>
</tr>
<tr>
<td>Group Average</td>
<td>&amp;</td>
</tr>
<tr>
<td>Total Average</td>
<td>&amp;</td>
</tr>
<tr>
<td>The Work\textsuperscript{104}</td>
<td>&amp;</td>
</tr>
</tbody>
</table>

Table B.9: Adhesion toughness of four-layer graphene membranes.

<table>
<thead>
<tr>
<th>The Work\textsuperscript{104}</th>
<th>Analytical Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$ (MPa)</td>
<td>$\delta$ (µm)</td>
</tr>
<tr>
<td>$p_s = 3.25$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.535</td>
<td>0.265</td>
</tr>
<tr>
<td>1.420</td>
<td>0.271</td>
</tr>
<tr>
<td>Group Average</td>
<td>&amp;</td>
</tr>
<tr>
<td>$p_s = 3.67$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.407</td>
<td>0.319</td>
</tr>
<tr>
<td>Group Average</td>
<td>&amp;</td>
</tr>
<tr>
<td>$p_s = 4.35$ MPa</td>
<td></td>
</tr>
<tr>
<td>1.118</td>
<td>0.414</td>
</tr>
<tr>
<td>Group Average</td>
<td>&amp;</td>
</tr>
<tr>
<td>Total Average</td>
<td>&amp;</td>
</tr>
<tr>
<td>The Work\textsuperscript{104}</td>
<td>&amp;</td>
</tr>
</tbody>
</table>
Table B.10: Adhesion toughness of five-layer graphene membranes.

<table>
<thead>
<tr>
<th>$p$ (MPa)</th>
<th>$\delta$ (µm)</th>
<th>$R_s$ (µm)</th>
<th>$\lambda \left( \delta/R_s \right)$</th>
<th>$G_c$ (J/m²)</th>
<th>$G$ (J/m²)</th>
<th>$\rho = G_c/G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_s = 3.25$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.700</td>
<td>0.244</td>
<td>2.459</td>
<td>0.267</td>
<td>0.271</td>
<td>0.341</td>
<td>0.802</td>
</tr>
<tr>
<td>1.621</td>
<td>0.252</td>
<td>2.587</td>
<td>0.262</td>
<td>0.267</td>
<td>0.335</td>
<td>0.794</td>
</tr>
<tr>
<td>1.417</td>
<td>0.305</td>
<td>2.686</td>
<td>0.305</td>
<td>0.282</td>
<td>0.367</td>
<td>0.864</td>
</tr>
<tr>
<td><strong>Group Average</strong></td>
<td></td>
<td></td>
<td></td>
<td>0.273</td>
<td>0.347</td>
<td>0.820</td>
</tr>
<tr>
<td>$p_s = 3.67$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.596</td>
<td>0.276</td>
<td>2.861</td>
<td>0.259</td>
<td>0.287</td>
<td>0.358</td>
<td>0.789</td>
</tr>
<tr>
<td>1.517</td>
<td>0.289</td>
<td>2.961</td>
<td>0.262</td>
<td>0.286</td>
<td>0.358</td>
<td>0.794</td>
</tr>
<tr>
<td>1.430</td>
<td>0.306</td>
<td>3.017</td>
<td>0.273</td>
<td>0.285</td>
<td>0.361</td>
<td>0.811</td>
</tr>
<tr>
<td><strong>Group Average</strong></td>
<td></td>
<td></td>
<td></td>
<td>0.286</td>
<td>0.359</td>
<td>0.798</td>
</tr>
<tr>
<td>$p_s = 4.35$ MPa</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.297</td>
<td>0.376</td>
<td>3.276</td>
<td>0.308</td>
<td>0.318</td>
<td>0.415</td>
<td>0.889</td>
</tr>
<tr>
<td>1.181</td>
<td>0.384</td>
<td>3.372</td>
<td>0.308</td>
<td>0.293</td>
<td>0.385</td>
<td>0.886</td>
</tr>
<tr>
<td>1.056</td>
<td>0.436</td>
<td>3.483</td>
<td>0.336</td>
<td>0.300</td>
<td>0.402</td>
<td>0.916</td>
</tr>
<tr>
<td><strong>Group Average</strong></td>
<td></td>
<td></td>
<td></td>
<td>0.305</td>
<td>0.401</td>
<td>0.884</td>
</tr>
<tr>
<td><strong>Total Average</strong></td>
<td></td>
<td></td>
<td></td>
<td>0.288</td>
<td>0.369</td>
<td>0.834</td>
</tr>
</tbody>
</table>

The Work$^{104}$ | 0.431
References


27. Inglis, C. E. Stresses in a plate due to the presence of cracks and sharp corners.


44. Sun, C. T. & Qian, W. The use of finite extension strain energy release rate in


References


References


109. Cao, Z. *et al.* A blister test for interfacial adhesion of large-scale transferred


