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NORMAL MODES OF THE ELEPHANT BELL

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ABSTRACT

The normal modes of a 16-tine elephant bell have been investigated using finite-element modeling, group representation methods and electronic speckle pattern interferometry. The experimental results are in good agreement with the model for about the first ten frequencies, and both are consistent with the predictions of group theory. It is found that the vibration spectrum can be understood by regarding the tines as a set of identical oscillators coupled in series in a closed loop with the hemispherical crown holding them in place, providing the coupling and, perhaps, acting as a sounding board.

1. INTRODUCTION

Beautiful Bangaloreware elephant bells were frequently on sale as souvenirs at exhibitions in the UK in the middle decades of the last century and were often seen on display in the nations' living rooms. Today they have become collectors' items but relatively crude brass bells of similar general type are still available commercially, and it is with one of these that this article is primarily concerned.

The "modern" elephant bell, as shown in Figure 1, consists of a hemispherical shell from whose rim hang roughly identical and equally spaced tines which have a slight inward curvature. There are between 10 and 20 of these, the number increasing with the overall size of the bell and always seeming to be even. There is a caste-in handle at the top of the hemisphere and the bell is rung by means of a clapper consisting of a metal ball suspended from a wire attached to the underside of the hemisphere. This strikes the tines at approximately the point where the hemispherical cap meets the tines. With the Bangaloreware bells the "hemisphere" is rather flattened, the tines are more curved and always seem to be odd in number.

The only previous study of an elephant bell to our knowledge was published in 1944 by Brailsford [1] who looked at a Bangaloreware bell with 15 tines. Looking at the vibration spectrum he found a number of very weak peaks below about 200Hz which he considered to be due to the tines vibrating as individual cantilevers. Starting at 900 Hz he found a sequence of much more prominent peaks, which he seems to have regarded as being due to the bell as a whole.

Figure 1: *The modern elephant bell.*

2. THE FINITE-ELEMENT MODEL

An elephant bell is harder to model than a conventional one [2, 3] because it does not have complete axial symmetry. It does, however, have a high level of symmetry which can be exploited. We take a "unit cell" to be a vertical segment contained between two planes containing the symmetry axis. These are chosen so that one touches the left hand side of a typical tine at its widest point and the other touches its right-hand neighbour at the corresponding point. The unit cell thus contains one tine plus one gap and that segment of the hemisphere joining them to its pole. If we make a finite-element model of this unit cell then one for the entire bell can be made by copying it (r-1) times while rotating about the symmetry axis through an angle of 360/r where r is the number of tines.

To produce the unit cell careful measurements were made of the overall dimensions and profile of a 16-tine bell. The tines and gaps were all measured and averages taken. The unit cell was then modeled by means of the LUSAS package using thin-shell elements chosen to preserve the overall shape of the outside of the bell. The thickness of the tine was taken as a constant 3mm and that of the slice of hemisphere as 4.5 mm.

The bell was about 8 cm tall and 8 cm in diameter at its widest point. The effect of the handle was modeled by constraining the bell to be fixed at the handle's edge. This had the added bonus of making sure that rigid body modes were excluded from the calculation. Typical values for brass of 8500 kgm^{-3} for density, 104 GPa for Young's modulus and 0.37 for Poisson's ratio were input to the model.

3. SYMMETRY CONSIDERATIONS

3.1 Perturbed bell approach

If the elephant bell did not have inter-tine gaps it would be just one more convex bell with axial symmetry group $C_{\equiv V}$ and would be subject to the usual consequences.[4] The nodal patterns would consist of m equally spaced "diameters" plus n circles parallel to the rim. Those with m > 0 would be in degenerate pairs with modal functions varying like sin(mφ) and cos(mφ), where φ is the polar angle, while those with m = 0 would be singlets. The number pairs (m, n) could be used to specify the modes.

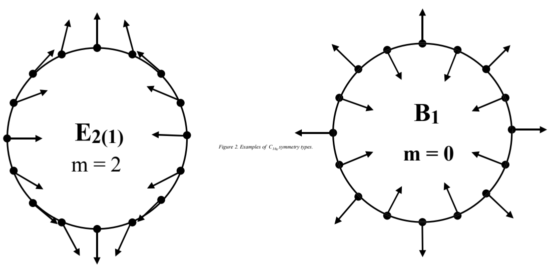

Figure 2. Examples of C_{16v} symmetry types.

The r inter-tine gaps constitute a large perturbation with symmetry group C_{rV} to be applied to this standard convex bell. Because some, but not all, of the original symmetries are now removed, some but not necessarily all of the doublets will become split. One of the present authors (RP) [5] has shown that, in these circumstances, the majority of doublets actually remain as degenerate pairs, although their common frequency may change. Only those with m/r = half integer, or non-zero integer, will split. Thus, with 16-tines, only doublets with m = 8,16,… will split.

3.2 Coupled cantilever approach

Alternatively one could regard the elephant bell as a collection of r identical cantilevers coupled together in a closed loop by the hemisphere. The collective modes of these cantilevers must be classifiable as symmetry types of the group C_{rV}. It is possible to construct the appearance of each of these types, as seen in any plane normal to the bell's symmetry axis, from symmetry arguments alone. No information about the coupling forces is required. Details of the method have been given by two of the present authors [6]. With 16 tines there are 4 singlet types and 14 pairs of doublets. Details of some of these are given in Figure 2.

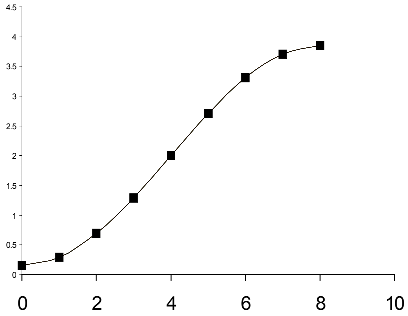

Figure 3. Frequency v m for vibrating polygon

The analogous problem of r identical masses at the vertices of a polygon, joined together in pairs by identical springs along the sides to form a closed loop, has previously be solved in detail by one of us (RP) [7]. In this case, because the details of the coupling force were known, it was possible, using group theoretical arguments, to obtain the actual frequencies for the normal modes. When these are plotted in mode order they form a very distinctive "saturation" curve as shown in Figure 3 for the case of 16 sides. This proves to be very significant in understanding the elephant bell, as we shall see.

4. FINITE-ELEMENT MODEL RESULTS

LUSAS was used to calculate the frequencies and display the modal forms for all the modes it could find up to about 6 kHz. The results up to 4 kHz are summarised in Table 1. In nearly all cases it was easy to establish an m value by looking at the mode's behaviour on the hemisphere, although this became harder as m increased due to evanescence setting in lower down the bell [8]. For $0 < m < 8$ the modes were all in degenerate pairs, as expected, with frequency differences between pair members never being more than ¼ Hz. Being so close we list only the higher frequency of each pair in Table 1. Modes with m = 0 were all singlets, again as expected. Those with m = 8 were also singlets, not what one expects in a normal bell but which was anticipated in section 3.1 due to the 16-tine perturbation. There was no prediction of Brailsford's low-frequency single-tine cantilever modes. Nor were any modes predicted with m > 8.

In Figure 4 we plot frequency verses m for all predicted modes with m ∩ 0 up to 4 kHz. The curves are not even remotely similar to those for a conventional bell. Instead of the frequency rising steadily and indefinitely with m, the curves reach limiting values at m = 8. This behaviour is exactly what was described in section 3.2 as being due to a closed loop of identical coupled oscillators. This being the case it should be possible to identify the C_{16V} symmetry type for each mode by comparing the LUSAS solutions with the group theoretical predictions mentioned in section 3.2. It was quite easy to do this for the first dozen or so modes and the results are incorporated into Table 1. As frequencies got higher and modes more complicated the identification became increasingly difficult. Likewise with identifying values for n. For modes on the lower curve in Figure

4 it was very easy to see that they had n = 0, as one would expect. With those on the upper

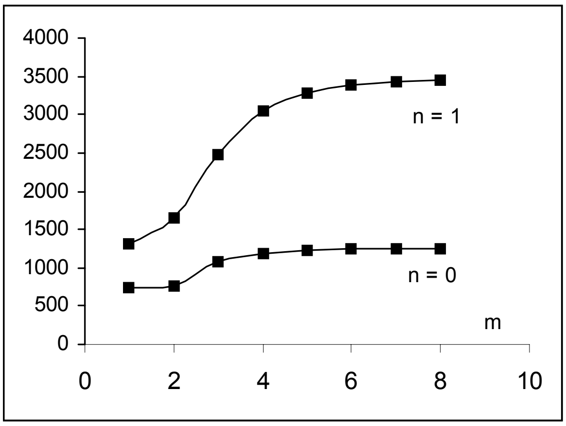

Figure 4. Frequency (Hz) v m for LUSAS predictions.

curve many appeared to have one nodal circle at around the mid points of the tines. However, because of twisting and/or transverse motion of the tines, it was often difficult to decide where the circle might actually be. Indeed it sometimes appeared to move up and down from tine to tine. In the case of singlet modes it was always very easy to identify both their symmetry types and the location and numbers of their circles

From those LUSAS results so far analysed it is clear that the n = 0 modes on the lower saturation curve are all due to the tines going into fundamental cantilever modes, but in directions varying from tine to tine in a way determined by their symmetry types – essentially by their m values. The higher modes seem to be due to the tines going into cantilever modes of higher orders plus, in some cases, twisting about their individual axes. Further study using the LUSAS animation facility should produce firmer identifications.

5. EXPERIMENTS

In order to compare the predictions of the finite element analysis with the modes of an actual elephant bell, a 16 tine modern bell similar to that shown in Figure 1 was studied using electronic speckle pattern interferometry (ESPI) [9]. The ESPI system images out of plane vibrations by digitally subtracting a speckle interferogram of an object illuminated by coherent radiation before the object begins to vibrate, from one imaged subsequent to its movement.

The system used to image the vibrations of elephant bells was constructed from discrete components on a vibration-isolated optical table that was inside an anechoic chamber. The laser used to illuminate the bell was a diode-pumped, frequency-doubled, Nd:YVO_4 laser with an output of 532nm. The laser was mounted on a vibration isolated optical table outside of the anechoic chamber in order to minimize the ambient noise. The light entered the chamber through a small hole in the wall. All of the data acquisition and analysis was performed by a computer located outside of the chamber.

The vibrations of the bell were driven by a piezoelectric disk that was mounted to either the hemispheric cap or the tines using putty. The piezoelectric driver was connected to a high-quality function generator that produced a sine wave with frequency accuracy and precision exceeding 0.1 Hz. The frequencies of the normal modes of vibration were determined by striking the bell and performing a spectral analysis of the resulting sound. Additionally, the frequency of the function generator was scanned while observing the ESPI image in real time to ensure that no normal modes were neglected.

A typical electronic speckle pattern interferogram is shown in Figure 5. In Figure 5, the bell is vibrating in the (3,0) mode and is viewed both from the side and the top. The dark areas indicate nodes while the light areas indicate antinodes.

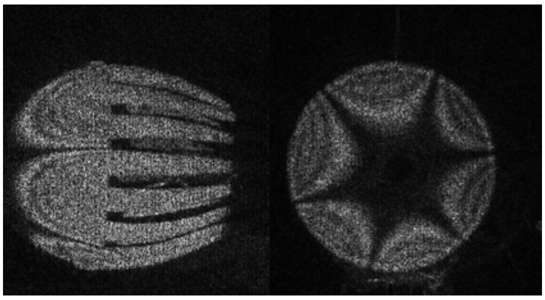

Figure 5: Typical electronic speckle pattern interferogram of a 16-tine modern elephant bell. The view is from the side and top, respectively. The mode is (3,0).

6. COMPARISON OF RESULTS

The interpretation of fringe patterns in interferograms of vibrating three dimensional objects is famously difficult. Although a large number of resonant frequencies of the 16-tine bell were detected, it has so far proved possible to fully identify only a few of them. In those cases where m was identified the degeneracy was always as predicted by group theory. The doublets were all slightly split so, when it is required, we quote the higher value. The lowest frequency modes to be positively identified were a (2, 0) pair at 753 Hz and a (3, 0) pair at 1071 Hz. The LUSAS results were therefore scaled, as in Table 2, to bring its lowest (2, 0) pair into alignment with experiment. This was legitimate because of the use of standard values for density and Young's modulus in the model calculations. If the two lists of results are now matched in sequence, then provided one experimental point is missed out and another is accepted as not having been excited, the agreement of frequencies is excellent up to about 1300 Hz. Above that it is relatively poor and gets progressively worse with further increase in frequencies.

7. CONCLUSIONS

The agreement between the finite-element results and the ESPI frequency data for the first set of modes is impressive and suggests that the model is basically sound. The modes that interferometry failed to detect are probably only missing because a more subtle excitation is required. Those modes which it finds that are not predicted may well be due to non-linear effects. Clearly, a more sophisticated means of identifying the nodal patterns and the symmetry type via interferometry would be a great help in matching the modes up with those predicted by LUSAS. There is little doubt that all of the predictions of group theory will be borne out, always subject to allowances for slight splitting due to small symmetry breaking because of asymmetry in the bell. The failure of the model to give good frequency predictions at higher frequencies is probably because the modes concerned involve the tines twisting about their own axes. The thin shell elements used in the model are probably inadequate to deal with this. In order to arrive at complete agreement, it is probably necessary to develop a more sophisticated model based on true three dimensional elements.

Frequency (Hz)	Degeneracy (S or D)	m	n	Symmetry type
748	D	1	0	E1(1)
765	D	2	0	E2(1)
1087	D	3	0	E3(1)
1181	D	4	0	E4(1)
1197	S	0	1	A1
1221	D	5	0	E5(1)
1241	D	6	0	E6(1)
1251	D	7	0	E7(1)
1254	S	8	0	B1
1316	D	1	1	E1(1)
1661	D	2	1	E2(1)
2467	D	3	1	E3(1)
2901	S	0	1	A2
3044	D	4	1?	E4(1)
3289	D	5	1?	E5(1)
3390	D	6	1?	E6(1)
3432	D	7	1?	E7(1)
3432	S	8	0	B2
3444	S	0	1	A1
3559	D	1	1	E1(1)

Table 1: LUSAS results for the 16-tine bell up to 4000Hz

m	n	Scaled LUSAS frequency (Hz)	Experimental frequency (Hz)
1	0	737	*
2	0	753	753
*	*	*	1004
3	0	1070	1071
4	0	1163	*
0	1	1179	1182
5	0	1202	1206
6	0	1222	1222
7	0	1232	1233
8	0	1235	1242
1	1	1296	1299
*	*	*	1353
*	*	*	1406
2	1	1635	1593
3	1	2429	2050

Table 2: Comparison of scaled LUSAS predictions with interferometric results. An * marks missing data.

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