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RAYLEIGH'S BELL MODEL REVISITED

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ABSTRACT

It is now well over a century since Lord Rayleigh published his model for western-style bells. He used a hyperboloid of revolution plus a flat circular plate for the crown. By limiting himself to inextensional modes of a very restricted type, and exploiting the hyperbola's parametric form, he produced an equation whose roots give the locations of nodal circles. Remarkably this equation involves neither the wall thickness nor physical properties of the bell material and this approach remains the only available analytical way of making such predictions. Although he gave adequate accounts of the derivation and method of solution of his equation, Rayleigh did not present much in the way of comparison of its predictions with experiment. Rather he focussed on using it to explain the fact that the Hum note never has any nodal circles. In the present paper we consider how well profiles of some modern church and handbells can be fitted by hyperbolae. We compare the model's predictions for these bells with data for a range of inextensional modes and report a new, surprisingly accurate, approximate analytical solution of Rayleigh's equation.

1. INTRODUCTION

Due to its axial symmetry, it is convenient to discuss a bell using cylindrical polar co-ordinates with z-axis defined by the axis of symmetry. Thus a typical point (r, $\hbar$, z) on the bell undergoes displacements (u, v, w) in radial, transverse and axial directions. Rayleigh [1] pointed out that normal modes of bells cannot have any nodes in the sense of their being points of zero motion. However, if one considers nodes in the more limited sense of being points of zero amplitude in any one cylindrical polar direction, e.g. radial, then nodal patterns do arise consisting of m equally spaced "diameters" and n circles parallel to the rim. This is because axial symmetry requires the normal modes to occur in degenerate pairs, for non-zero m, whose modal functions, in any one cylindrical polar direction, vary like sin(m$\hbar$) and cos(m$\hbar$) [2].

Rayleigh's professional interest in church bells seems to have gone back to experiments he conducted on the specimens in his local church tower in 1879. His subsequent application of his general theory for thin curved plates and shells to the strictly inextensional modes of concave bells is well known [1]. There exists only one of these modes for each value on m. This application is very unusual in that, being based purely on geometry and three inextensibility conditions, it does not predict frequencies but only the locations of nodal circles. Rayleigh's use of this to explain the fact that the Hum (m=2) never has any nodal circles while the Tierce (m=3) may, or may not, do so is very convincing and remains the best available clue to what is really happening with these modes. However he never explored his model's predictions for higher m modes and so was not drawn to consider whether they could have more than one nodal circle. The predictions for these higher modes are interesting but prove to be at odds with both experimental and finite-element studies of both church bells and handbells.

2. RAYLEIGH'S MODEL

Rayleigh assumed that the profile of a (thin) bell, as seen in any plane containing the symmetry axis, could be approximated by part of a hyperbola with its pole at the bell's shoulder. For convenience we shall use the upper half of the right hand branch of the hyperbola and regard the bell as being stood on its crown, as shown in Figure 1. The complete bell wall is thus the upper half of the hyperboloid of revolution produced by rotating the hyperbola about the z-axis. The crown Rayleigh considered to be a flat rigid circular plate causing the pole of the hyperbola to be rigidly fixed. Surprisingly no boundary conditions need to be imposed on the rim of the bell in order to solve the equations. Indeed the rim's location is only required at all when one wants to compare the predictions of the model with experiment.

3. THE HYPERBOLA

Choosing a plane containing the symmetry axis (i.e. fixed $\hbar$), and retaining cylindrical polar variables, the hyperbola's equation is

$$\frac{r^2}{a^2} - \frac{z^2}{b^2} = 1 \qquad (1)$$

where a and b have their usual geometrical meanings and are related to the eccentricity e by $e^2 = 1 + (b/a)^2$ so that $e > 1$. In his model Rayleigh chose to work with the parametric form for the hyperbola:

$$z = b\tan\chi \quad , \; r = a\sec\chi \qquad (2)$$

Geometrical interpretations of a, b and χ are shown in Figure 1. Note the role of the point (b, 0) in defining the angle χ. This is not usually discussed in texts on conic geometry. Since the solutions of Rayleigh's model come out in terms of χ it is important to note its interpretation and range of $0 \le \chi \le \pi / 2$.

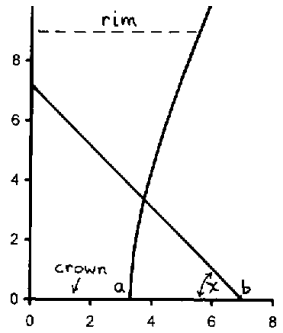

Figure 1: Meaning of the parametric angle χ

4. SUMMARY OF RAYLEIGH'S THEORY

By considering the change in length of an arbitrary element traced on the surface and imposing axial symmetry, Rayleigh derives three conditions for inextensibility of the element. These are:

$$\frac{\partial w}{\partial z} + \frac{dr}{dz}\frac{\partial u}{\partial z} = 0 \qquad (3a)$$

$$\frac{\partial v}{\partial \theta} + u = 0 \qquad (3b)$$

$$\frac{\partial w}{\partial \theta} + r\frac{\partial v}{\partial z} + \frac{dr}{dz}\frac{\partial u}{\partial \theta} = 0 \qquad (3c)$$

To achieve strict inextensibility all three of these conditions must be satisfied. However, if the element selected lies in the plane of fixed z, then only the second condition applies. This equation is well known and is sometimes regarded as "the" inextensibility condition [3]. This is because, if one considers the bell's cross-section in a plane of fixed z, then this is the condition a neutral circle, whose total length remains unchanged throughout the cycle, must satisfy. It requires that if $u = A(z)\sin(m\hbar)$ then $mv = A(z)\cos(m\hbar)$ so the radial and transverse components are locked together. It is well established that all the acoustically important bell modes obey this condition to a good degree of approximation.

Starting from equations (1) and (3), using the known angular form for $v(z,\hbar)$, converting to parametric forms of the co-ordinates and imposing the boundary condition at the pole, Rayleigh shows that to get zero motion normal to the bell's surface requires

$$\sin(2\chi) + 2m\tan(m\chi)\left(e^2 - \cos^2\chi\right) = 0 \qquad (4)$$

Note that the root $\chi = 0$ is built into this equation for all values of m due to the boundary condition at the crown. To locate the nodal circles for the one strictly inextensional mode for each value of m requires a solution of the equation for that m value. It is easy to see that no further solutions occur for m = 0 or 1, simply by remembering that $0 \le \chi \le \pi / 2$. In that range $\sin(2\chi) > 0$ so equation 4 can only have solutions if $\tan(m\chi) < 0$, which can never be the case for m = 0 or 1.

5. ASYMPTOTIC SOLUTION

As the eccentricity of the hyperbola increases so the model bell approaches a right circular cylinder. As $e \to \infty$ and/or $m \to \infty$ the only possibility of solutions of equation 4 is for

$$\tan(m\chi) \to 0 \qquad (5)$$

The limit needs to be from the negative side because $\sin(2\chi) \ge 0$ and $(e^2 - \cos^2\chi) \ge 0$. Thus

$$\chi \approx 0, \frac{\pi}{m}, \frac{2\pi}{m}, ... \qquad (6)$$

so, depending on m, there could be any number of nodal circles up to a maximum of m/2 because $\chi < \pi / 2$. If this asymptotic solution is a reasonably accurate approximation to the exact one, which we show to be the case in the next section, then a nodal circle for m=2 would be ruled out because its location at $\chi < \pi/2$ puts it at the far end of an infinitely long bell, see Table 1.

Maximum possible number of nodal circles	m values
0	0, 1, 2
1	3, 4
2	5, 6
3	7, 8

Table 1. *Possible m values*

6. EXACT SOLUTIONS

It has already been pointed out that $\chi = 0$ is always a solution of equation 4 and that no other exists for m = 0 or 1. For m = 2 the equation can be rewritten as

$$\sin(2\chi)\left[1 + \frac{4}{\cos(2\chi)}\left(e^2 - \cos^2\chi\right)\right] = 0 \qquad (7)$$

Thus either $$\sin(2\chi) = 0 \tag{8}$$

or $$4e^2 - 1 - 2\cos^2\chi = 0 \tag{9}$$

The latter condition can never be satisfied because e >1 so the only solutions are $\chi = 0$, as expected, and $\chi = \pi/2$ which is of no practical significance.

Putting m = 3 into equation 4, expanding the multiple angle terms, cancelling $\sin\chi$ throughout, rewriting the remaining terms in powers of $\cos\chi$ and collecting them up yields a quadratic equation in $\cos^2\chi$:

$$8\cos^4\chi - 12e^2\cos^2\chi + 3e^2 = 0 \tag{10}$$

whose roots can be written

$$\cos^2\chi = \frac{3}{4}e^2\left[1 \pm \sqrt{1 - \frac{2}{3e^2}}\right] \tag{11}$$

where the upper root is unphysical, leading to $\cos^2\chi > 1$ as e > 1. The values of χ for the lower root as e varies are given in Table 2 from which it will be seen that it approaches the asymptotic value of 60° very rapidly as e increases. When e = 2, a typical value for church and handbells, the exact value is already less than 1° away. The reason for this is easier to see if one uses the Binomial theorem to expand the square root in equation 11 and then collects terms to give

$$\cos^2\chi = \frac{1}{4} + \frac{1}{24e^2} + \frac{1}{72e^4} + O\left(\frac{1}{e^6}\right) \tag{12}$$

where the leading term is just the asymptotic value.

If one proceeds in a similar fashion for m = 4 one obtains a different quadratic in $\cos^2\chi$ whose roots are given by

$$\cos^2\chi = \frac{1}{6}\left(4e^2 + 1\right) \pm \frac{1}{12}\sqrt{64e^4 - 64e^2 + 10} \tag{13}$$

Again the upper root is unphysical and values of χ for the lower root are listed in Table 2. As expected the asymptotic value is reached even more quickly and the e = 2 value is now to within half a degree of it.

Because the exact roots of equation 4 are expected to get ever closer to the asymptotic values as m increases, and it is already so good for m = 4 there may be little point in bothering with it. However we decided to check this for the next two values of m because they are expected to give two physical roots. Proceeding as before one now obtains cubic equations in $\cos^2\chi$ which can be solved exactly using Cardin's formula. In each case the three roots were all real but only two were physical. Their algebraic forms not being very enlightening we limit ourselves to including their values in Table 2.

m→ e↓	3	4	5		6	
1.1	56.83	42.35	33.99	71.38	28.49	59.26
1.5	58.57	43.97	35.32	71.68	29.54	59.65
2.0	59.25	44.49	35.68	71.83	29.79	59.82
5.0	59.89	44.93	35.96	71.97	29.97	59.97
limit	60.00	45.00	36.00	72.00	30.00	60.00

Table 2. *Exact solutions in degrees to Rayleigh's equation for a range of values of m and of eccentricity.*

7. HYPERBOLIC FITS TO BELL PROFILES

It is well known that both inner and outer profiles of most modern bells are based on either elliptical arcs, circular arcs, or both [3]. Since such curves can never be fitted by a single hyperbola it is tempting to dismiss Rayleigh's model on that basis [4]. However it is not these profiles which are important here but rather that of the neutral bell, which is better approximated by their average. We have therefore tried fitting hyperbolae to the average profiles of two very different modern bells whose details we had previously measured with some accuracy [5,6]. These were a Malmark C_5 handbell and a Taylor D_5 church bell. The parameters to be fitted were the usual hyperbola parameters a and b plus the origin of co-ordinates for the hyperbola as seen from the co-ordinate system for the empirical measurements. The latter were expected to be small and it proved possible to set the x co-ordinate of the origin to zero. The fitting routine required us to input analytical forms for both the fitting curve and its partial derivatives with respect to the parameters. The results are shown in Figures 2 and 3 from which it can be seen that, in the case of the handbell , the fit is quite good, apart from the region close to the rim. The optimum fit eccentricity was 2.33. When the two points closest to the rim were excluded from the fit this increased to 2.50. In the case of the church bell the fit was worse and had an eccentricity of 1.96. When the three points nearest to the rim were removed the fit improved markedly and the eccentricity went up to 2.19.

8. CIRCLE LOCATIONS

8.1 The handbell

In Figure 2 we show the hyperbola of best fit to the handbell with lines of fixed χ drawn corresponding to asymptotic solutions of Rayleigh's model for various values of m. From the points where these cut the z-axis one can see by inspection where nodal circles are predicted. Since the 60 degree line cuts

the axis beyond the bell's rim there should be no nodal circle for m = 3. There should however be one for m = 4 about a quarter of the way up the bell. This is in good agreement with experiment. However, while Rayleigh's model predicts that this circle should get ever closer to the crown as m increases, experiment shows that it reaches a limiting point about half way up [5, 6]. When m = 8 is reached Rayleigh is predicting two nodal circles at χ = 22.5^{o} and 45° degrees. There is no evidence for a second circle arising in practice.

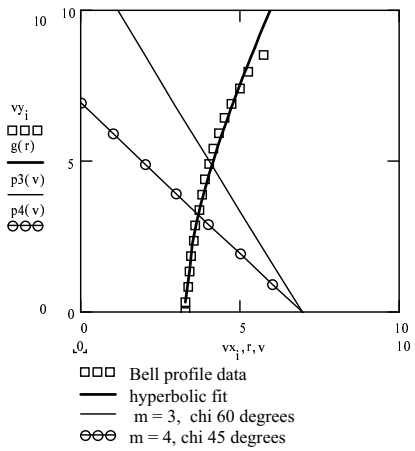

Figure 2: Best fit and theory predictions for a handbell

8.2 The church bell

In Figure 3 we show the corresponding diagram for the church bell. Again no circles are predicted for m = 2 or 3 but one is for higher m values until a second circle is also predicted at m = 8. This is in contrast to experiment where one finds that m = 2 is the only mode with no circles. There is always a circle for m = 3 and for higher m values. As with the handbell, experiment shows that the one circle moves to a limiting position near the waist of the bell as m increases. It never approaches the shoulder.

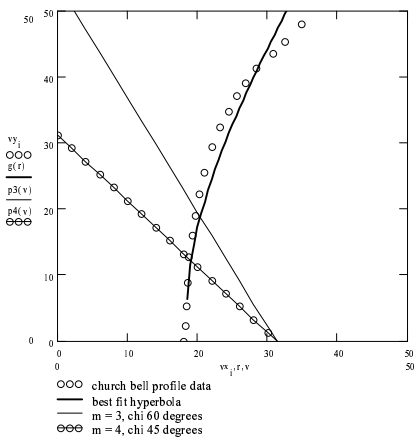

Figure 3: Best fit and theory predictions for a church bell.

9. DISCUSSION

Handbells are certainly thin axisymmetric shells and they do not deviate badly from a hyperbolic shape. The correct prediction of zero or one circle for m values up to 7 is impressive but the failure to predict the correct location of this circle, as m increases, is a problem as is the prediction of extra circles at higher m values.. It appears this must be due to the incorrect treatment of the crown. Bells' modal functions do not just disappear at the shoulder but fall to zero at the center of the crown in evanescent fashion [5,7]. An incorrect boundary condition thus seems likely to be to blame but there is no obvious way of correcting it. Exactly the same is true of the church bell but there the situation is worse because the hyperbolic fit is worse and the thickness of the wall varies much more. The failure to predict a circle for the case of m = 3 is a real problem.

10. CONCLUSIONS

Rayleigh's model fails when one looks at its predictions in detail but it remains of value for two main reasons. Firstly it does give a qualitative explanation of why all concave bells have (2,0) modes and why, as m becomes larger, a point is reached where the otherwise expected (m,0) mode is replaced by a second (m,1). Secondly the fact that it predicts only one mode for each m emphasises the point that the first mode for each m is strictly inextensional and so differs in a basic way from all the other "inextensional" modes. In addition to this the model remains a remarkable example of Rayleigh's sheer inventive genius. After more than a century a better analytical approach to the bell has yet to be produced.

11. REFERENCES

[1] Strutt J. W. (Lord Rayleigh), "On Bells", Philos. Mag.., Vol.29, 1890, pp. 1-17.

[2] Perrin R. and Charnley T., "Group Theory and the Bell", J. Sound Vib., Vol. 31, 1973, pp. 411-418.

[3] Rossing T. D. and Perrin R., "Vibrations of Bells", App. Acoust., Vol 20, 1987, pp. 41-70.

[4] Perrin R., Charnley T., Samson J. and Gottlieb H. P. W., "The Campanoid: an Equation for the Church Bell Profile", J. Sound Vib., Vol. 151, 1991, pp. 163-167.

[5] Perrin R., Charnley T. and dePont J., "Normal Modes of the Modern English Church Bell", J. Sound Vib., Vol. 90, 1983, pp. 29-49.

[6] Rossing T. D.,, Perrin R., Sathoff H.J. and Peterson R.W., "Vibrational Modes of a Tuned Handbell", J. Acoust. Soc. Am., Vol. 76, 1984, pp. 1263-1267.

[7] Perrin R. and Gottlieb H. P. W., "Evanescence and Bessel Functions in the Vibrating Circular Membrane", Eur. J. Phys., Vol. 15, pp. 293-299.