Multiple scattering by random configurations of circular cylinders: second-order corrections for the effective wavenumber

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Additional Information:


Metadata Record: https://dspace.lboro.ac.uk/2134/246

Please cite the published version.
Multiple scattering by random configurations of circular cylinders: second-order corrections for the effective wavenumber

C. M. Linton

*Department of Mathematical Sciences*
*Loughborough University, Leicestershire LE11 3TU, United Kingdom*

and

P. A. Martin

*Department of Mathematical and Computer Sciences*
*Colorado School of Mines, Golden, Colorado 80401-1887*

3rd September, 2004

Running title: Scattering by random arrangements of cylinders

Abstract

A formula for the effective wavenumber in a dilute random array of identical scatterers in two dimensions is derived, based on Lax’s quasi-crystalline approximation. This formula replaces a widely-used expression due to Twersky, which is shown to be based on an inappropriate choice of pair-correlation function.

PACS number: 43.20.Fn, 43.20.Bi, 43.20.Hq, 43.28.En
I INTRODUCTION

Multiple scattering by random arrangements of scatterers is a topic with an extensive literature. The modern era dates from the work of Foldy (1945), Lax (1951, 1952), Waterman and Truell (1961), and Twersky (1962). Major applications include wave propagation through suspensions (see, for example (McClements et al., 1990), (Povey, 1997), (Dukhin and Goetz, 2001), and (Spelt et al., 2001)) and through elastic composites (see (Mal and Knopoff, 1967), (Kim et al., 1995), and (Kanaun, 2000)). In this paper, we are mainly interested in two-dimensional problems, motivated by the calculation of sound propagation through forests (Embleton, 1966; Price et al., 1988). An important paper on acoustic scattering by arrays of circular cylinders is that of Bose and Mal (1973); see Sec. IV below. For subsequent work, see (Varadan et al., 1978), (Yang and Mal, 1994), (Bose, 1996), (Kanaun and Levin, 2003), and (Kim, 2003). For analogous plane-strain elastodynamics, see (Varadan et al., 1986), (Yang and Mal, 1994), (Bussink et al., 1995) and (Verbis et al., 2001).

A typical problem is the following. The region \( x < 0 \) is filled with a homogeneous compressible fluid of density \( \rho \) and sound-speed \( c \). The region \( x > 0 \) contains the same fluid and many scatterers; to fix ideas, we suppose that the scatterers are identical circles (parallel circular cylinders). Then, a time-harmonic plane wave with wavenumber \( k = \omega/c \) (\( \omega \) is the angular frequency) is incident on the scatterers: what is the reflected wave field? This field may be computed exactly for any given configuration (ensemble) of \( N \) circles, but the cost increases as \( N \) increases. If the computation can be done, it may be repeated for other configurations, and then the average reflected field could be computed (this is the Monte Carlo approach). Instead of doing this, one can try to do some ensemble averaging in order to calculate the average (coherent) field. One result of this is a formula for the effective wavenumber \( K \). This can then be used to replace the “random medium” occupying \( x > 0 \) by a homogeneous effective medium.

Foldy (1945) began by considering isotropic point scatterers; this is an appropriate model for small sound-soft scatterers. He obtained the formula

\[
K^2 = k^2 - 4\pi g n_0, \tag{1}
\]

where \( n_0 \) is the number of circles per unit area and \( g \) is the scattering coefficient for an individual scatterer. (In fact, Foldy considered scattering in three dimensions; the two-dimensional formula, Eq. (1), can be found as Eq. (3.20) in (Twersky, 1962) and Eq. (26) in (Aristegui and Angel, 2002), for example.) The formula (1) assumes that the scatterers are independent and that \( n_0 \) is small. We are interested in calculating the correction to Eq. (1) (a term proportional to \( n_0^2 \)), and this will require saying more about the distribution of the scatterers; specifically, we shall use pair correlations. Thus, our goal is a formula of the form

\[
K^2 = k^2 + \delta_1 n_0 + \delta_2 n_0^2, \tag{2}
\]

with computable expressions for \( \delta_1 \) and \( \delta_2 \). Moreover, we do not only want to restrict our formula to sound-soft scatterers.

There is some controversy over the proper value for \( \delta_2 \). In order to state one such formula, we introduce the far-field pattern \( f \). For scattering by one circular cylinder, we have

\[
u_{\text{in}} = e^{ikr \cos(\theta - \theta_{\text{in}})}
\]
for the incident plane wave and then the scattered waves satisfy

\[ u_{sc} \sim \sqrt{2/(\pi kr)} f(\theta - \theta_{in}) \exp (ikr - i\pi/4) \quad \text{as} \ r \to \infty, \]

(3)

where \((r, \theta)\) are plane polar coordinates. Then, Twersky (1962) has given the following formula:

\[ K^2 = k^2 - 4i n_0 f(0) + (2n_0 / k)^2 \sec^2 \theta_{in} \left\{ [f(\pi - 2\theta_{in})]^2 - [f(0)]^2 \right\}. \]

(4)

This formula involves \(\theta_{in}\), so that it gives a different effective wavenumber for different incident fields.

The three-dimensional version of Eq. (4) is older. For a random collection of identical spheres, it is

\[ K^2 = k^2 - 4\pi i (n_0 / k) f(0) + \delta_2 n_0^2 \]

with

\[ \delta_2 = (4\pi^2 / k^4) \sec^2 \theta_{in} \left\{ [f(\pi - 2\theta_{in})]^2 - [f(0)]^2 \right\}, \]

(5)

where the far-field pattern is now defined by \(u_{sc} \sim (ikr)^{-1} e^{ikr} f(\vartheta)\), and \(r\) and \(\vartheta\) are spherical polar coordinates. The same formula but with \(\theta_{in} = 0\) (normal incidence) was given by Waterman and Truell (1961). However, it was shown by Lloyd and Berry (1967) that Eq. (5) is incorrect; they obtained

\[ \delta_2 = \frac{4\pi^2}{k^4} \left\{ -[f(\pi)]^2 + [f(0)]^2 + \int_0^\pi \frac{1}{\sin(\vartheta/2)} \frac{d}{d\vartheta} [f(\vartheta)]^2 d\vartheta \right\} \]

(6)

(with no dependence on \(\theta_{in}\)). Lloyd and Berry (1967) used methods (and language) coming from nuclear physics. Thus, in their approach, which they “call the ‘resummation method’, a point source of waves is considered to be situated in an infinite medium. The scattering series is then written out completely, giving what Lax has called the ‘expanded’ representation. In this expanded representation the ensemble average may be taken exactly [but then] the coherent wave does not exist; the series must be resummed in order to obtain any result at all.” One purpose of the present paper is to demonstrate that a proper analysis of the semi-infinite two-dimensional model problem (with arbitrary angle of incidence) leads to a formula that is reminiscent of the (three-dimensional) Lloyd–Berry formula; specifically, instead of Eq. (4), we obtain

\[ K^2 = k^2 - 4i n_0 f(0) + \frac{8n_0^2}{\pi k^2} \int_0^\pi \cot(\theta/2) \frac{d}{d\theta} [f(\theta)]^2 d\theta. \]

(7)

Our analysis does not involve “resumming” series or divergent integrals. It builds on a conventional approach, in the spirit of the papers by Fikioris and Waterman (1964) and by Bose and Mal (1973).

In outline, the paper is as follows. Some elementary probability theory is recalled in Sec. II. In particular, the pair-correlation function is introduced; this leads to the notion of “hole correction”—individual cylinders must not be allowed to overlap during the averaging process. In Sec. III, we derive the integral equations of Foldy (isotropic scatterers, no hole correction) and of Lax (isotropic scatterers, hole correction included). Foldy’s integral equation can be solved exactly whereas we have been unable to solve Lax’s integral equation.
Nevertheless, we have developed a rigorous method for extracting an expression for $K$ from these integral equations without actually solving the integral equations themselves. Then, we use the same method in Sec. IV but without the restriction to isotropic scatterers. We start by following Bose and Mal (1973), and use an exact (deterministic) theory for scattering by $N$ circles followed by ensemble averaging. We give a clear derivation of a certain homogeneous infinite linear system of algebraic equations, obtained previously by Bose and Mal (1973) for the case of normal incidence; the system does not depend on $\theta_{in}$ and the existence of a non-trivial solution determines $K$. We solve the system for small $n_0$, and obtain Eq. (7). We also show that Eq. (4) is obtained if the hole correction is not done correctly. Concluding remarks are given in Sec. V.

II SOME PROBABILITY THEORY

In this section, we give a very brief summary of the probability theory needed. For more information, see (Foldy, 1945), (Lax, 1951), (Aristegui and Angel, 2002) or chapter 14 of (Ishimaru, 1978).

Suppose we have $N$ scatterers located at the points $r_1, r_2, \ldots, r_N$; denote the configuration of points by $\Lambda_N = \{r_1, r_2, \ldots, r_N\}$. Then, the ensemble (or configurational) average of any quantity $F(r|\Lambda_N)$ is defined by

$$\langle F(r) \rangle = \int \cdots \int p(r_1, r_2, \ldots, r_N) F(r|\Lambda_N) \, dV_1 \cdots dV_N$$

where the the integration is over $N$ copies of the volume $B_N$ containing $N$ scatterers. Here, $p(r_1, \ldots, r_N) \, dV_1 \, dV_2 \cdots dV_N$ is the probability of finding the scatterers in a configuration in which the first scatterer is in the volume element $dV_1$ about $r_1$, the second scatterer is in the volume element $dV_2$ about $r_2$, and so on, up to $r_N$. The joint probability distribution $p(r_1, \ldots, r_N)$ is normalised so that $\langle 1 \rangle = 1$. Similarly, the average of $F(r|\Lambda_N)$ over all configurations for which the first scatterer is fixed at $r_1$ is given by

$$\langle F(r) \rangle_1 = \int \cdots \int p(r_2, \ldots, r_N|r_1) F(r|\Lambda_N) \, dV_2 \cdots dV_N,$$

where the conditional probability $p(r_2, \ldots, r_N|r_1)$ is defined by

$$p(r_1, r_2, \ldots, r_N) = p(r_1) p(r_2, \ldots, r_N|r_1).$$

If two scatterers are fixed, say the first and the second, we can define

$$\langle F(r) \rangle_{12} = \int \cdots \int p(r_3, \ldots, r_N|r_1, r_2) F(r|\Lambda_N) \, dV_3 \cdots dV_N,$$

where

$$p(r_2, \ldots, r_N|r_1) = p(r_2|r_1) p(r_3, \ldots, r_N|r_1, r_2).$$

Higher-order probabilities can be defined, but we shall not need them.
Now, as each of the \( N \) scatterers is equally likely to occupy \( dV_1 \), the density of scatterers at \( r_1 \) is \( N p(r_1) = n_0 \), the (constant) number of scatterers per unit volume. Thus

\[
p(r) = n_0 / N = |B_N|^{-1},
\]

(11)

where \( |B_N| \) is the volume of \( B_N \). Also, as \( p(r_1, r_2) = p(r_1) p(r_2|r_1) \), we obtain

\[
\int \int p(r_2|r_1) \, dV_1 \, dV_2 = \frac{N}{n_0} = |B_N|.
\]

(12)

We have to specify \( p(r_2|r_1) \), consistent with Eq. (12). Also, we want to ensure that scatterers do not overlap. For circular cylinders of radius \( a \), a simple choice is \( p(r_2|r_1) = p_0 H(R_{12} - b) \) with \( b \geq 2a \), where \( H(x) \) is the Heaviside unit function, \( R_{12} = |r_1 - r_2| \) and \( p_0 \) is a constant determined by Eq. (12). Thus,

\[
p_0 = \left\{ |B_N| - \pi b^2 \right\}^{-1} \simeq n_0 / N,
\]

(13)

assuming that \( b^2 n_0 / N \ll 1 \). (The equality in Eq. (13) assumes that the “hole” at \( r_1 \) of radius \( b \) does not cut the boundary of \( B_N \). Evidently, taking this possibility into account would not change the approximation \( p_0 \simeq n_0 / N \).) Hence, the simplest sensible choice for the pair-correlation function is

\[
p(r_2|r_1) = \begin{cases} 0, & R_{12} < b, \\ n_0 / N, & R_{12} > b. \end{cases}
\]

(14)

This simple choice will be used for most of our analysis. More generally, we could use

\[
p(r_2|r_1) = \begin{cases} 0, & R_{12} < b, \\ (n_0 / N)[1 + \chi(R_{12}; n_0)], & R_{12} > b, \end{cases}
\]

(15)

where the function \( \chi \) is to be chosen, subject to some constraints. The effect of using Eq. (15) instead of Eq. (14) is calculated in Sec. IV.D. One could also consider functions \( \chi \) that depend on \( r_1 - r_2 \) (instead of just \( |r_1 - r_2| \)); such possibilities are discussed in (Twersky, 1978) and (Siqueira et al., 1995).

### III FOLDY–LAX THEORY: ISOTROPIC SCATTERERS

Foldy’s theory begins with a simplified deterministic model for scattering by \( N \) identical scatterers, each of which is supposed to scatter isotropically. Thus, the total field is assumed to be given by the incident field plus a point source at each scattering centre, \( r_j \):

\[
u(r|A_N) = u_{ii}(r) + g \sum_{j=1}^{N} u_{ex}(r_j; r_j|A_N) H_0(k|r - r_j|).
\]

(16)
Here, $H_n(w) \equiv H_n^{(1)}(w)$ is a Hankel function, $g$ is the (assumed known) scattering coefficient, and the exciting field $u_{ex}$ is given by

$$u_{ex}(\mathbf{r}; r_n|A_N) = u_{in}(\mathbf{r}) + g \sum_{j=1, j \neq n}^{N} u_{ex}(\mathbf{r}_j; r_j|A_N) H_0(|k|\mathbf{r} - \mathbf{r}_j|).$$

(17)

The $N$ numbers $u_{ex}(\mathbf{r}_j; r_j|A_N)$ ($j = 1, 2, \ldots, N$) required in Eq. (16) are to be determined by solving the linear system obtained by evaluating Eq. (17) at $\mathbf{r} = r_n$; direct numerical solutions of this system have been given by Fikioris (1966) and by Groenenboom and Snieder (1995).

Let us try to compute the ensemble average of $u$, using Eqs. (16) and (8). The result is

$$\langle u(\mathbf{r}) \rangle = u_{in}(\mathbf{r}) + g n_o \int_{B_N} \langle u_{ex}(\mathbf{r}_1) \rangle_1 H_0(|k|\mathbf{r} - \mathbf{r}_1|) \, dV_1,$$

(18)

where we have used Eqs. (9) and (11), and the indistinguishability of the scatterers. For $\langle u_{ex}(\mathbf{r}_1) \rangle_1$, we obtain

$$\langle u_{ex}(\mathbf{r}) \rangle_1 = u_{in}(\mathbf{r}) + g(N - 1) \int_{B_N} p(\mathbf{r}_2|\mathbf{r}_1) \langle u_{ex}(\mathbf{r}_2) \rangle_2 \, dV_2,$$

(19)

where we have used Eqs. (10) and (17). Equations (18) and (19) are the first two in a hierarchy, involving more and more complicated information on the statistics of the scatterer distribution. In practice, the hierarchy is broken using an additional assumption. At the lowest level, we have Foldy’s assumption,

$$\langle u_{ex}(\mathbf{r}) \rangle_1 \simeq \langle u(\mathbf{r}) \rangle,$$

(20)

at least in the neighbourhood of $\mathbf{r}_1$. When this is used in Eq. (18), we obtain

$$\langle u(\mathbf{r}) \rangle = u_{in}(\mathbf{r}) + g n_o \int_{B_N} \langle u(\mathbf{r}_1) \rangle \, H_0(|k|\mathbf{r} - \mathbf{r}_1|) \, dV_1, \quad \mathbf{r} \in B_N.$$

(21)

We call this Foldy’s integral equation for $\langle u(\mathbf{r}) \rangle$. The integral on the right-hand side is an acoustic volume potential. Hence, an application of $(\nabla^2 + k^2)$ to Eq. (21) eliminates the incident field and shows that $(\nabla^2 + K^2) \langle u \rangle = 0$ in $B_N$, where $K^2$ is given by Foldy’s formula, Eq. (1).

At the next level, we have the Lax (1952) quasi-crystalline assumption (QCA),

$$\langle u_{ex}(\mathbf{r}) \rangle_2 \simeq \langle u_{ex}(\mathbf{r}) \rangle_2.$$

(22)

When this is used in Eq. (19) evaluated at $\mathbf{r} = r_1$, we obtain

$$v(\mathbf{r}) = u_{in}(\mathbf{r}) + g(N - 1) \int_{B_N} p(\mathbf{r}_1|\mathbf{r}) v(\mathbf{r}_1) \, H_0(|k|\mathbf{r} - \mathbf{r}_1|) \, dV_1, \quad \mathbf{r} \in B_N,$$

(23)

where $v(\mathbf{r}) = \langle u_{ex}(\mathbf{r}) \rangle_1$. We call this Lax’s integral equation.

In what follows, we let $N \to \infty$ so that $B_N \to B_\infty$, a semi-infinite region, $x > 0$. 
A Foldy’s integral equation: exact treatment

Consider a plane wave at oblique incidence, so that

\[ u_{in} = e^{i(\alpha x + \beta y)} \quad \text{with} \quad \alpha = k \cos \theta_{in} \quad \text{and} \quad \beta = k \sin \theta_{in}. \]  

(24)

For a semi-infinite domain \( B_\infty \), Foldy’s integral equation, Eq. (21), becomes

\[ \langle u(x, y) \rangle = e^{i(\alpha x + \beta y)} + g n_0 \int_0^\infty \int_{-\infty}^\infty \langle u(x_1, y + Y) \rangle H_0(k \rho_1) \, dY \, dx_1, \quad x > 0, \quad -\infty < y < \infty, \]

where \( \rho_1 = \sqrt{(x - x_1)^2 + Y^2} \). This equation can be solved exactly. Thus, writing

\[ \langle u(x, y) \rangle = U(x) e^{i\mu y}, \quad x > 0, \quad -\infty < y < \infty, \]

(25)

we obtain

\[ U(x) = e^{i\alpha x} e^{i(\beta - \mu)y} + g n_0 \int_0^\infty \int_{-\infty}^\infty U(x_1) H_0(k \rho_1) e^{i\mu Y} \, dY \, dx_1, \quad x > 0, \quad -\infty < y < \infty. \]

(26)

Hence, for a solution in the form Eq. (25), we must have

\[ \mu = \beta = k \sin \theta_{in}. \]

Now,

\[ \int_{-\infty}^{\infty} H_0(\kappa \rho_1) e^{i\beta y} \, dY = \frac{2}{\alpha} e^{i\alpha |x-x_1|}, \]

(27)

where \( \alpha = \sqrt{k^2 - \beta^2} = k \cos \theta_{in} \). Thus, we see that \( U \) solves

\[ U(x) = e^{i\alpha x} + \frac{2 gn_0}{\alpha} \int_0^\infty U(x_1) e^{i\alpha |x-x_1|} \, dx_1, \quad x > 0. \]

(28)

Now, put \( U(x) = U_0 e^{i\lambda x} \), so that Eq. (25) gives

\[ \langle u(x, y) \rangle = U_0 e^{i(\lambda x + \beta y)}, \quad x > 0, \quad -\infty < y < \infty, \]

(29)

and Eq. (28) gives

\[ U_0 e^{i\lambda x} - e^{i\alpha x} = \frac{2 gn_0 U_0}{i\alpha} \left( \frac{2 \alpha e^{i\lambda x}}{\lambda^2 - \alpha^2} - \frac{e^{i\alpha x}}{\lambda - \alpha} \right), \]

where we have assumed that \( \text{Im} \lambda > 0 \). If we compare the coefficients of \( e^{i\lambda x} \), we see that \( U_0 \) cancels, leaving

\[ \lambda^2 - \alpha^2 = -4i gn_0, \]

(30)

which determines \( \lambda \). Then, the coefficients of \( e^{i\alpha x} \) give \( U_0 = 2\alpha/(\lambda + \alpha) \). A similar method can be used to find \( \langle u \rangle \) when \( B_\infty \) is a slab of finite thickness, \( 0 < x < h \), say; see Aristegui and Angel (2002).

From Eq. (29), it is natural to write

\[ \lambda = K \cos \varphi \quad \text{and} \quad \beta = K \sin \varphi = k \sin \theta_{in}. \]

(31)

These define the effective wavenumber \( K \); the last equality is recognised as Snell’s law, even though \( K \) and \( \varphi \) are complex, with \( \text{Im} \, K > 0 \). Hence, we see that

\[ \lambda^2 - \alpha^2 = K^2 - k^2, \]

(32)

whence Eq. (30) reduces to Foldy’s formula, Eq. (1).
B  Foldy’s integral equation: approximate treatment

We have seen that Foldy’s integral equation can be solved exactly, and that the solution process has two parts: first find \( \lambda \) (and hence the effective wavenumber) and then find \( U_0 \).

In fact, \( \lambda \) can be found without finding the complete solution; the reason for pursuing this is that we cannot usually find exact solutions. Thus, consider Eq. (28), and suppose that

\[
U_0 e^{i\lambda x} = \frac{2i\alpha n_0}{\alpha} \int_0^\ell U(t) e^{-iat} \, dt + \frac{2\alpha n_0}{\alpha} \int_\ell^\infty U(t) e^{ia|x-t|} \, dt
\]

where \( U_0, \alpha \) and \( \ell \) are unknown. To proceed, we need say nothing about the solution \( U \) in the “boundary layer” \( 0 < x < \ell \). Now, evaluate the integral equation for \( x > \ell \); we find that

\[
U_0 e^{i\lambda x} - e^{i\alpha x} = \frac{2i\alpha n_0}{\alpha} \int_0^\ell U(t) e^{-iat} \, dt + \frac{2i\alpha n_0 U_0}{\alpha(\alpha - \alpha) \ell}.
\]

Then, setting \( U_0 = A \) gives Eq. (30) again, without knowing the solution \( U \) everywhere. This basic method will be used again below.

C  Lax’s integral equation

Using the approximation \( p(r_1|r) = (n_0/N)H(R_1 - b) \) in Lax’s integral equation, Eq. (23) gives

\[
v(r) = u_{in}(r) + gn_0 \frac{N-1}{N} \int_{B_N^b(r)} v(r_1) H_0(kR_1) \, dr_1, \quad r \in B_N,
\]

where \( B_N^b(r) = \{ r_1 \in B_N : R_1 = |r - r_1| > b \} \), which is \( B_N \) with a (possibly incomplete) disc excluded.

Let \( N \to \infty \) and take an incident plane wave, Eq. (24), giving

\[
v(x, y) = e^{i(\alpha x + \beta y)} + gn_0 \int_{x_1 > 0, \, \rho_1 > b} v(x_1, y + Y) H_0(k\rho_1) \, dY \, dx_1, \quad x > 0, \quad -\infty < y < \infty.
\]

As in Sec. III.A, we write

\[
v(x, y) = V(x) e^{i\beta y}, \quad x > 0, \quad -\infty < y < \infty,
\]

giving

\[
V(x) = e^{i\alpha x} + gn_0 \int_{x_1 > 0, \, \rho_1 > b} V(x_1) H_0(k\rho_1) e^{i\beta Y} \, dY \, dx_1, \quad x > 0, \quad -\infty < y < \infty.
\]

Then, using Eq. (27), we see that \( V \) solves

\[
V(x) = e^{i\alpha x} + gn_0 \int_0^\infty V(x_1) L(x - x_1) \, dx_1, \quad x > 0,
\]

(36)
where the kernel, \( L(x - x_1) \), is given by

\[
L(X) = \frac{2}{\alpha} e^{i\alpha |X|} - 2 \int_0^c(X) H_0(k\sqrt{X^2 + Y^2}) e^{i\beta Y} \, dY
\]

(37)

with \( c(X) = \sqrt{b^2 - X^2} H(b - |X|) \); here, we have written the integral over \( Y \) in Eq. (35) as an integral over all \( Y \) minus an integral through the disc, if necessary.

We have been unable to solve Eq. (36) exactly (even though it is an integral equation of Wiener–Hopf type). However, the approximate method described in Sec. III.B can be used. Thus, let us suppose that

\[
V(x) = V_0 e^{i\lambda x} \quad \text{for} \quad x > \ell,
\]

(38)

where \( V_0, \lambda \) and \( \ell \) are unknown. Then, consider Eq. (36) for \( x > \ell + b \), so that the interval \( |x - x_1| < b \) is entirely within the range \( x_1 > \ell \). Making use of Eq. (37), Eq. (36) gives

\[
\frac{V_0 e^{i\lambda x} - e^{i\alpha x}}{g_0} = \frac{2}{\alpha} e^{i\alpha x} \int_0^\ell V(t) e^{-i\alpha t} \, dt + \frac{2}{\alpha} \int_\ell^\infty V(t) e^{i\alpha |x-t|} \, dt
\]

\[
- 2 \int_{x-b}^{x+b} V(t) \int_0^{c(x-t)} H_0(k\sqrt{(x-t)^2 + Y^2}) \, dY \, dt
\]

for \( x > \ell + b \). Equation (38) can be used in the second and third integrals. The second integral is elementary, and has the value

\[
\frac{2iV_0}{\alpha(\lambda - \alpha)} e^{i(\lambda - \alpha)\ell} e^{i\alpha x} - \frac{4iV_0}{\lambda^2 - \alpha^2} e^{i\lambda x}.
\]

The third integral becomes

\[
-2V_0 \int_{-b}^b e^{i\lambda(x+\xi)} \int_0^{b^2-\xi^2} H_0(k\sqrt{\xi^2 + Y^2}) e^{i\beta Y} \, dy \, d\xi
\]

\[
= -V_0 e^{i\lambda x} \int_0^{2\pi} \int_0^b e^{iK r \cos(\theta - \phi)} H_0(kr) \, r \, d\theta \, dr
\]

\[
= -2\pi V_0 e^{i\lambda x} \int_0^b J_0(kr) H_0(kr) \, r \, dr
\]

\[
= V_0 e^{i\lambda x} \left\{ \frac{4i}{K^2 - k^2} - \frac{2\pi N_0(Kb)}{K^2 - k^2} \right\},
\]

where

\[
N_0(Kb) = Kb H_0(\ell b) J_1(Kb) - \ell b H_1(\ell b) J_0(Kb).
\]

(40)

Using these results in Eq. (39), noting Eq. (32), we obtain

\[
V_0 e^{i\lambda x} - e^{i\alpha x} = A e^{i\lambda x} + B e^{i\alpha x} \quad \text{for} \quad x > \ell + b,
\]

where

\[
A = \frac{2\pi g_0 V_0}{k^2 - K^2} N_0(Kb),
\]

\[
B = \frac{2g_0}{\alpha} \int_0^\ell V(t) e^{-i\alpha t} \, dt + \frac{2i g_0 V_0}{\alpha(\lambda - \alpha)} e^{i(\lambda - \alpha)\ell}.
\]
For a solution, we must have \( A = V_0 \), whence

\[
K^2 = k^2 - 2\pi gn_0 N_0(Kb),
\]

(41)

which is a nonlinear equation for \( K \). Notice that this equation does not depend on the angle of incidence, \( \theta_{in} \).

We have \( N_0(Kb) \rightarrow 2i/\pi \) as \( b \rightarrow 0 \) so that, in this limit, we recover Foldy's formula for the effective wavenumber, Eq. (1).

Let us solve Eq. (41) for small \( n_0 \). (Alternatively, we could use the dimensionless area fraction \( \pi a^2 n_0 \).) Begin by writing

\[
K^2 = k^2 + \delta_1 n_0 + \delta_2 n_0^2 + \cdots,
\]

(42)

where \( \delta_1 \) and \( \delta_2 \) are to be found; for \( \delta_1 \), we expect to obtain the result given by Eq. (1). It follows from Eq. (42) that

\[
K = k + \frac{1}{2} \delta_1 n_0/k + O(n_0^2)
\]

and then

\[
N_0(Kb) = N_0(kb) + (Kb - kb)N_0'(kb) + \cdots = 2i/\pi + \frac{1}{2} b^2 \delta_1 d_0(kb) n_0 + O(n_0^2),
\]

where \( d_0(x) = J_0(x) H_0(x) + J_1(x) H_1(x) \). When this approximation for \( N_0(Kb) \) is used in Eq. (41), we obtain

\[
K^2 = k^2 - 4i gn_0 - \pi b^2 g \delta_1 d_0(kb) n_0^2.
\]

Comparison of this formula with Eq. (42) gives \( \delta_1 = -4i g \) (as expected) and \( \delta_2 = 4\pi i (gb)^2 d_0(kb) \), so that we obtain the approximation

\[
K^2 = k^2 - 4i gn_0 + 4\pi i (gb n_0)^2 d_0(kb).
\]

(43)

Note that the second-order term in Eq. (43) vanishes in the limit \( kb \rightarrow 0 \).

### IV FINITE-SIZE EFFECTS

The theory described above relies on the assumption of isotropy. Here, we use a more complete theory. We start with an exact theory (due to Záviška) for acoustic scattering by \( N \) identical circular cylinders of radius \( a \); for details and references, see p. 173 of (Linton and McIver, 2001). The cylinders can be soft, hard or penetrable. Then (in Sec. IV.B), we form averaged equations, and we invoke the QCA. This leads to an infinite homogeneous system of linear algebraic equations from which the effective wavenumber, \( K \), is to be determined; the equations are independent of the angle of incidence. An approximate solution for \( K \) is found in Sec. IV.C, correct to \( O(n_0^2) \). In Sec. IV.D, it is shown that this approximation does not depend on the choice of the function \( \chi(r; n_0) \), appearing in the pair-correlation function, Eq. (15). In Sec. IV.E, it is shown how Twersky’s formula for \( K \) can be derived, using an unreasonable choice for the pair-correlation function.
A finite array of identical circular cylinders: exact theory

We use polar coordinates \((r, \theta)\) centred at the origin and \((r_j, \theta_j)\), centred at \(r_j = (x_j, y_j)\), the centre of the \(j^{th}\) cylinder. The various parameters relating to the relative positions of the cylinders are shown in Figure 1.

Exterior to the cylinders the pressure field is \(u\), where

\[
\nabla^2 u + k^2 u = 0. \quad (44)
\]

In the interior of cylinder \(j\), the field is \(u_j\), where

\[
\nabla^2 u_j + \kappa^2 u_j = 0. \quad (45)
\]

A plane wave, given by Eq. (24), is incident on the cylinders. A phase factor for each cylinder, \(I_j\), is defined by

\[
I_j = e^{i(\alpha x_j + \beta y_j)} \quad (46)
\]

and then we can write

\[
u_{in} = I_j e^{ikr_j \cos(\theta_j - \theta_{in})} = I_j \sum_{n=-\infty}^{\infty} e^{in(\pi/2 - \theta_j + \theta_{in})} J_n(\kappa r_j). \quad (47)
\]

We seek a solution to Eqs. (44) and (45) in the form

\[
u = u_{in} + \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} A_n^j Z_n H_n(\kappa r_j) e^{in\theta_j}, \quad (48)
\]

\[
u_j = \sum_{n=-\infty}^{\infty} B_n^j J_n(\kappa r_j) e^{in\theta_j}, \quad (49)
\]
for some set of unknown complex coefficients $A^s_j$ and $B^s_n$. The factor
\[ Z_n = \frac{qJ_n'(ka)J_n(\kappa a) - J_n(ka)J_n'(\kappa a)}{qH_n'(ka)J_n(\kappa a) - H_n(ka)J_n'(\kappa a)} = Z_{-n} \]
has been introduced for later convenience. Here $\kappa = \omega / \bar{c}$ and $q = \tilde{\rho} \bar{c} / (\rho c)$, where $\tilde{\rho}$ and $\bar{c}$ are the density and sound speed, respectively, inside the cylinders. Note that we recover the sound-soft results in the limit $q \to 0$, whereas the limit $q \to \infty$ gives the sound-hard results. The boundary conditions on the cylinders are
\[ u = u_s, \quad \frac{1}{\rho} \frac{\partial u}{\partial r_s} = -\frac{1}{\rho} \frac{\partial u_s}{\partial r_s} \quad \text{on} \quad r_s = a, \quad s = 1, \ldots, N. \]

Using Graf’s addition theorem for Bessel functions, it can be shown that provided $r_s < R_{js}$ for all $j$, we can write the field exterior to cylinder $s$ as
\[ u(r, \theta) = \sum_{n=-\infty}^{\infty} \left( I_s J_n(kr_s)e^{i(n/2-\theta_s+\theta_m)} + A^s_n Z_n H_n(kr_s)e^{in\theta_s} \right) \]
\[ + \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} A^j_n Z_n \sum_{m=-\infty}^{\infty} J_m(kr_s)H_{n-m}(kR_{js})e^{im\theta_s}e^{i(n-m)\alpha_{js}}. \] (52)

The geometrical restriction implies that this expression is only valid if the point $(r_s, \theta_s)$ is closer to the centre of cylinder $s$ than the centres of any of the other cylinders. This is certainly true on the surface of cylinder $s$ and so Eq. (52) can be used to apply the body boundary conditions which leads, after using the orthogonality of the functions $\exp(im\theta_s)$, $m \in \mathbb{Z}$, and eliminating the coefficients $B^s_j$, to the system of equations
\[ A^s_m + \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} A^j_n Z_n e^{i(n-m)\alpha_{js}} H_{n-m}(kR_{js}) = -I_s e^{im(\pi/2-\theta_m)}, \quad s = 1, 2, \ldots, N, \]
\[ m \in \mathbb{Z}. \] (53)

Note that the quantities $q$, $\kappa$ and $a$ only enter the equations through the terms $Z_n$.

For a single cylinder the solution is immediate: $A^s_1 = -i^m I_1 \exp(-im\theta_m)$ and then the far-field pattern, defined by Eq. (3), is given by
\[ f(\theta) = -\sum_{n=-\infty}^{\infty} Z_n e^{in\theta}. \] (54)

**B Arrays of circular cylinders: averaged equations**

The above analysis applies to a specific configuration of scatterers. Now we follow Bose and Mal (1973) and take ensemble averages. Specifically, setting $s = 1$ in Eq. (53) and then taking the conditional average, using Eq. (14), we get
\[ \langle A^1_m \rangle_1 + n_0 \frac{N-1}{N} \sum_{n=-\infty}^{\infty} Z_n \int_{B \cap R_{12} > b} H_{n-m}(kR_{21}) e^{i(n-m)\alpha_{21}} \langle A^2_n \rangle_2 \, dV_2 \]
\[ = -I_1 e^{im(\pi/2-\theta_m)}, \quad m \in \mathbb{Z}. \] (55)
Now we let \( N \to \infty \) so that \( B_N \) becomes the half-space \( x > 0 \), and invoke Lax’s QCA, Eq. (22). This implies that

\[
\langle A^m_1 \rangle_{12} = \langle A^m_2 \rangle.
\]

(56)

We seek a solution to Eq. (55) in the form

\[
\langle A^m_s \rangle = i^m e^{i\beta y_s} \Phi_m(x_s)
\]

(57)

so that

\[
\Phi_m(x_1) + n_0 \sum_{n=-\infty}^{\infty} Z_n(-i)^{n-m} \int_{x_2 > 0, R_{12} > b} \psi_{n-m}(x_{21}, y_{21}) e^{i\beta y_{21}} \Phi_n(x_2) \, dx_2 \, dy_2 = -e^{-im\theta_{in}} e^{i\alpha x_1}, \quad m \in \mathbb{Z}, \quad (58)
\]

where we have written \( x_{21} = x_2 - x_1 \) and \( y_{21} = y_2 - y_1 \), used \( \alpha_{21} = \alpha_{12} - \pi \), and defined

\[
\psi_n(X, Y) = H_n(kR) e^{i\alpha Y}
\]

with \( X = R \cos \Theta \) and \( Y = R \sin \Theta \).

Proceeding as before, suppose that for sufficiently large \( x \) (say \( x > \ell \)) we can write

\[
\Phi_m(x) = F_m e^{-i\mu \varphi} e^{i\lambda x},
\]

(59)

where \( \lambda \) and \( \varphi \) are defined by Eq. (31). We assume that \( \text{Im} \lambda > 0 \) so that \( \Phi_m \to 0 \) as \( x \to \infty \). Then if \( x_1 > \ell + b \), Eq. (58) becomes

\[
F_m e^{-i\mu \varphi} e^{i\lambda x_1} + n_0 \sum_{n=-\infty}^{\infty} Z_n(-i)^{n-m} \left\{ \int_0^\ell \Phi_n(x_2) L_{n-m}(x_{21}) \, dx_2 + F_n e^{-i\mu \varphi} e^{i\lambda x_1} M_{n-m} \right\} \quad (60)
\]

where

\[
L_n(X) = \int_{-\infty}^{\infty} \psi_n(X, Y) e^{i\beta Y} \, dY,
\]

(61)

\[
M_n = \int_{x_2 > \ell, R_{12} > b} \psi_n(x_{21}, y_{21}) \Psi(x_{21}, y_{21}) \, dx_2 \, dy_2,
\]

(62)

\[
\Psi(X, Y) = e^{i(\lambda X + \beta Y)} = e^{ikR \cos(\Theta - \varphi)},
\]

(63)

and we have used Eq. (31). Next, we shall evaluate \( L_n \) and \( M_n \); note that we have \( x_2 < \ell < x_1 \) in Eq. (60) so that \( x_{21} < 0 \).

Consider the integral \( L_n(X) \) for \( X < 0 \). From Eq. (27), we have

\[
L_0(X) = (2/\alpha)e^{-i\alpha X} \quad \text{and} \quad L'_0 = -i\alpha L_0.
\]

(64)

For \( L_n \) with \( n > 0 \), we use the fact that

\[
L_n(X) = \int_{-\infty}^{\infty} \frac{1}{k} e^{i\beta Y} \left( \frac{\partial}{\partial X} + i \frac{\partial}{\partial Y} \right) \psi_{n-1}(X, Y) \, dY.
\]

(65)
Then, as the $\partial/\partial X$ can be taken outside the integral and the $\partial/\partial Y$ can be removed using an integration by parts, we have

$$L_n = -\frac{1}{k} L'_n - \frac{\beta}{k} L_{n-1},$$

which expresses $L_n$ in terms of $L_{n-1}$ and $L'_n$. It follows from Eq. (64) that

$$L_n = \frac{2(i\alpha - \beta)^n}{\alpha k^n} e^{-i\alpha x} = \frac{2i^n}{\alpha} e^{in\theta_n} e^{-i\alpha x}.$$  

This formula also holds for $n < 0$. Hence, for $x_1 > x_2$,

$$L_n(x_2 - x_1) = (2/\alpha)i^n e^{in\theta_n} e^{i\alpha(x_1-x_2)}.$$  

The double integral $M_n$ can be evaluated using Green’s theorem as follows. We have

$$\psi_n \nabla^2 \Psi - \Psi \nabla^2 \psi_n = (k^2 - K^2)\psi_n \Psi.$$  

It follows that

$$M_m = \frac{1}{k^2 - K^2} \int_{\partial B} \left[ \psi_m \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \psi_m}{\partial n} \right] ds_2,$$

where $\partial B$ consists of two parts, the line $x_2 = \ell$ and the circle $R_{12} = b$. Now, on $x_2 = \ell$, $\partial/\partial n = -\partial/\partial x_2$ and so we have

$$- \int_{x_2=\ell} \left[ \psi_n \frac{\partial \Psi}{\partial x_2} - \Psi \frac{\partial \psi_n}{\partial x_2} \right] dy_2 = e^{i\lambda(x_1-x_2)} \int_{-\infty}^{\infty} e^{iby_2} \left[ -i\lambda \psi_n + \cos \alpha_1 \frac{\partial \psi_n}{\partial R_1} - \frac{\sin \alpha_1}{R_1} \frac{\partial \psi_n}{\partial \alpha_1} \right] dy_2$$

$$= e^{i\lambda(x_1-x_2)} \int_{-\infty}^{\infty} e^{iby_2} \left[ -i\lambda \psi_n + \frac{k}{2} (\psi_n - \psi_{n+1}) \right] dy_2$$

$$= \frac{2}{\alpha} e^{i(\alpha-\lambda)(x_1-\ell)} i^n e^{in\theta_n} (\lambda + \alpha),$$

using $2H'_n(x) = H_{n-1}(x) - H_{n+1}(x)$, $(2n/x)H_n(x) = H_{n-1}(x) + H_{n+1}(x)$, and Eq. (68) thrice.

The contribution from the circle $R_{12} = b$ is

$$- \int_0^{2\pi} \left[ \psi_n \frac{\partial}{\partial R} \left( e^{iKR \cos (\Theta - \varphi)} - e^{iKR \cos (\Theta + \varphi)} \frac{\partial \psi_n}{\partial R} \right) \right]_{R=b} b d\Theta$$

$$= -b \int_0^{2\pi} e^{iKb \cos (\Theta - \varphi)} e^{i\Theta} [iKH_n(kb) \cos (\Theta - \varphi) - kH'_n(kb)] d\Theta$$

$$= -b e^{i\varphi} \int_0^{2\pi} e^{i\Theta} \sum_{q=-\infty}^{\infty} i^n J_q(kb) e^{-i\Theta} \left[ \frac{iK}{2} H_n(kb)(e^{i\Theta} + e^{-i\Theta}) - kH'_n(kb) \right] d\Theta$$

$$= -2\pi b i^n e^{i\varphi} \left[ KH_n(kb) J'_n(kb) - kH'_n(kb) J_n(kb) \right].$$

Thus, the system (60) can be written as

$$A_m e^{-im\theta_n} e^{i\lambda x} + B e^{-im\theta_n} e^{i\lambda x} = e^{-im\theta_n} e^{i\lambda x}, \quad x > \ell + b, \quad m \in \mathbb{Z},$$

$$Linton & Martin, JASA \ 14$$
where

\[ A_m = F_m + \frac{2n_0 \pi}{k^2 - K^2} \sum_{n=-\infty}^{\infty} F_n Z_n N_{n-m}(Kb), \]

\[ B = \frac{2n_0}{\alpha} \sum_{n=-\infty}^{\infty} Z_n e^{i\theta_{in}} \left\{ \int_0^\ell \Phi_n(t) e^{-i\alpha t} dt + \frac{iF_n e^{-i\phi}}{\lambda - \alpha} e^{i(\lambda - \alpha)\ell} \right\}, \]

and

\[ N_n(Kb) = kbH_n'(kb) J_n(Kb) - KbH_n(kb) J_n'(Kb). \]  \hspace{1cm} (72)

Notice that \( N_0 \), defined by Eq. (40), appeared in Sec. III.C during our analysis of Lax’s integral equation.

From Eq. (71), we immediately obtain \( B = -1 \) and \( A_m = 0 \) for all \( m \); the second of these, namely

\[ F_m + \frac{2n_0 \pi}{k^2 - K^2} \sum_{n=-\infty}^{\infty} F_n Z_n N_{n-m}(Kb) = 0, \quad m \in \mathbb{Z}, \]  \hspace{1cm} (73)

is of most interest to us. It is an infinite homogeneous system of linear algebraic equations for \( F_m, m \in \mathbb{Z} \). The existence of a non-trivial solution to Eq. (73) determines \( K \). Notice that Eq. (73) does not depend on \( \theta_{in} \), so that the effective wavenumber cannot depend on \( \theta_{in} \).

Equation (73) is the same as Eq. (33) in (Bose and Mal, 1973) (with the choice Eq. (14)); these authors began by considering normal incidence, \( \theta_{in} = 0 \). However, the derivation of Eq. (73) given here has some advantages over that given by Bose and Mal (1973). First, we do not invoke “the so-called ‘extinction theorem’” of Lax; this is described in Sec. VI of (Lax, 1952). Roughly speaking, this “theorem” asserts that one may simply delete the incident field when calculating the effective wavenumber, in the limit \( N \to \infty \). Along with this come some divergent integrals; for example, the integrals in the unnumbered equation between eqns. (32) and (33) of (Bose and Mal, 1973) are divergent, because \( e^{iKx} \) is exponentially large as \( x \to -\infty \). In fact, we can say that our analysis proves Lax’s theorem in our particular case.

Second, when dealing with a half-space containing scatterers, we know from the work of Lloyd and Berry (1967) that the boundary of the half-space can cause difficulties. Here, we give a proper treatment of this boundary. In particular, we do not assume that all fields are proportional to \( e^{i\lambda x} \) everywhere inside the half-space, \( x > 0 \), but only in \( x > \ell \), away from the boundary: the width of the boundary layer, \( \ell \), is not specified, and need not be specified if one only wants to calculate \( K \).

A more recent analysis was given by Siqueira and Sarabandi (1996). They allow non-circular and non-identical cylinders (using a \( T \)-matrix formulation) but they do assume that the effective field is proportional to \( e^{i\lambda x} \) for all \( x > 0 \).

## C Approximate determination of \( K \) for small \( n_0 \)

The only approximation made in the derivation of Eq. (73) is the QCA, which is expected to be valid for small values of the scatterer concentration \((n_0a^2 \ll 1)\). We now assume (as in Sec. III.C) that \( n_0/k^2 \) is also small and write \( K^2 = k^2 + \delta_1 n_0 + \delta_2 n_0^2 + \ldots \). We then have

\[ N'_n(Kb) = 2i/\pi + \frac{1}{2} b^2 \delta_1 d_n(kb) n_0 + O(n_0^2), \]  \hspace{1cm} (74)
where

\[ d_n(x) = J'_n(x)H'_n(x) + [1 - (n/x)^2]J_n(x)H_n(x) \]  

(75)

and so

\[ \frac{N_n(Kb)}{k^2 - K^2} = -\frac{2i}{\pi \delta_1 n_0} - \frac{b^2 d_n(kb)}{2} + \frac{2i\delta_2}{\pi \delta_1^2} + O(n_0). \]  

(76)

If Eq. (76) is substituted in Eq. (73) and \( O(n_0^2) \) terms neglected we get

\[ F_m = \frac{4i}{\delta_1} \sum_{n=-\infty}^{\infty} Z_n F_n + n_0 \sum_{n=-\infty}^{\infty} Z_n F_n \left( \pi b^2 d_{n-m}(kb) - \frac{4i\delta_2}{\delta_1^2} \right), \quad m \in \mathbb{Z}. \]  

(77)

At leading order this gives

\[ F_m = \frac{4i}{\delta_1} \sum_{n=-\infty}^{\infty} Z_n F_n, \quad m \in \mathbb{Z}, \]  

(78)

which implies that all the \( F_m \) are equal. If we write \( F_m = F \), Eq. (78) becomes

\[ \delta_1 = 4i \sum_{s=-\infty}^{\infty} Z_s = -4if(0), \]  

(79)

where \( f \) is the far-field pattern, given by Eq. (54).

Returning to Eq. (77), we now put \( F_m = F + n_0 q_m \), and then the \( O(n_0) \) terms give

\[ q_m = -\frac{1}{f(0)} \sum_{n=-\infty}^{\infty} Z_n q_n + \pi b^2 F \sum_{n=-\infty}^{\infty} Z_n d_{n-m}(kb) - \frac{iF\delta_2}{4f(0)}, \quad m \in \mathbb{Z}. \]  

(80)

It follows that \( q_m - \pi b^2 F \sum_{n=-\infty}^{\infty} Z_n d_{n-m} \) must be independent of \( m \), call it \( Q \):

\[ Q = -\frac{1}{f(0)} \sum_{n=-\infty}^{\infty} Z_n q_n - \frac{iF\delta_2}{4f(0)} \]
\[ = -\frac{1}{f(0)} \sum_{n=-\infty}^{\infty} Z_n \left( Q + F \pi b^2 \sum_{s=-\infty}^{\infty} Z_s d_{s-n}(kb) \right) - \frac{iF\delta_2}{4f(0)}. \]  

(81)

Hence

\[ \delta_2 = 4\pi ib^2 \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Z_s d_{s-n}(kb) \]  

(82)

and so we obtain the approximation

\[ K^2 = k^2 - 4in_0 f(0) + 4\pi ib^2 n_0^2 \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Z_s d_{s-n}(kb) + \ldots. \]  

(83)

For isotropic point scatterers, we have \( |Z_0| \gg |Z_n| \) for all \( n \neq 0 \) and \( g = -Z_0 \), so that Eq. (83) reduces to Eq. (43) in this limit.
So far we have not made any assumptions about the size of $ka$ or $kb$ (though clearly $kb \geq 2ka$). Now we will assume that $kb$ is small. In the limit $x \to 0$, we have

$$x^2 d_n(x) \sim 2i |n|/\pi.$$  

(84)

Hence as $kb \to 0$,

$$\delta_2 \sim -\frac{8}{k^2} \sum_{n=\infty}^{\infty} \sum_{s=\infty}^{\infty} |s - n| Z_n Z_s.$$  

(85)

Now

$$[f(\theta)]^2 = \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Z_s e^{i(n+s)\theta} = \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Z_s \cos(n-s)\theta$$  

(86)

since $Z_n = Z_{-n}$. Thus

$$\frac{d}{d\theta} [f(\theta)]^2 = -\sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (n-s)Z_n Z_s \sin(n-s)\theta.$$  

(87)

Also

$$\int_0^\pi \cot \frac{1}{2} \theta \sin m\theta \, d\theta = \pi \text{sgn}(m),$$  

(88)

see Eq. 3.612(7) in (Gradshteyn and Ryzhik, 2000). Thus, setting $kb = 0$ gives

$$K^2 = k^2 - 4i n_0 f(0) + \frac{8n_0^2}{\pi k^2} \int_0^\pi \cot \frac{1}{2} \theta \frac{d}{d\theta} [f(\theta)]^2 \, d\theta.$$  

(89)

The integral appearing here is convergent because $f'(0) = 0$.

**D Effect of pair-correlation function choice**

Here, we consider the effect of using a more complicated pair-correlation function, defined by Eq. (15) in terms of the function $\chi(r; n_0)$. This function must decay rapidly to zero as $r \to \infty$ and, in addition, $\chi(r; n_0) \to 0$ as $n_0 \to 0$ for any fixed $r$. For example, Bose and Mal (1973) suggest using $\chi(r; n_0) = e^{-r/L(n_0)}$, where the correlation length $L(n_0) \to 0$ as $n_0 \to 0$. Other authors have supposed that $\chi(r; n_0) = 0$ for $r > b' > b$, where the radius $b'$ may be taken as $2b$; see, for example, p. 1072 of (Bose, 1996) or Eq. (27) in (Twersky, 1978).

Proceeding as in Sec. IV.B, we obtain Eq. (58) with an additional factor of $[1 + \chi(R_{12}; n_0)]$ in the integrand. Evaluating this equation for $x_1 > \ell + b'$, assuming that $\chi(r; n_0) = 0$ for $r > b'$, we obtain Eq. (60) with $M_{n-m}$ replaced by $M'_{n-m}$, where

$$M'_n = M_n + \int_{b<R_{12}<b'} \psi_n(x_{21}, y_{21}) \Psi(x_{21}, y_{21}) \chi(R_{12}; n_0) \, dx_{21} \, dy_{21}$$  

$$= M_n + 2\pi i^n e^{in\varphi} W_n,$$

$$W_n = \int_b^{b'} H_n(kR) J_n(KR) \chi(R; n_0) \, R \, dR,$$
and $M_n$ is defined by Eq. (62). Hence, we obtain a modified form of Eq. (73), namely

$$ F_m + 2n_0\pi\sum_{n=-\infty}^{\infty} F_n Z_n \left\{ \frac{N_{n-m}(Kb)}{k^2 - K^2} + W_{n-m} \right\} = 0, \quad m \in \mathbb{Z}, \quad (90) $$

from which $K$ is to be determined. This homogeneous system for $F_n$ is Eq. (33) in (Bose and Mal, 1973) and it is a special case of Eq. (24) in (Siqueira and Sarabandi, 1996). Moreover, the fact that $W_n = o(1)$ as $n_0 \to 0$ means that the approximations for $K$ obtained in Sec. IV.C, namely Eqs. (83) and (89), are unchanged by the presence of $\chi$.

### E Reproducing Twersky’s formula

It is implicit in the work of Twersky (1962) (and others) that the complications arising when a scatterer centre is closer to the boundary $x = 0$ than its radius are ignored. It was pointed out by Lloyd and Berry (1967) that, since all scatterers are treated equally, ignoring the boundary-layer effects is equivalent to using a pair-correlation function with the following property: if one scatterer is at $(x_1, y_1)$, then no other scatterer (with centre $(x_2, y_2)$) can occupy the infinite strip $x_1 - a < x_2 < x_1 + a$. Thus, instead of Eq. (14), the choice

$$ N_p(r_2|r_1) = \begin{cases} 0, & |x_{21}| < a, \\ n_0, & |x_{21}| > a, \end{cases} \quad (91) $$

was made. We shall show that use of Eq. (91) leads to Twersky’s formula, Eq. (4).

Setting $s = 1$ and taking the conditional average of Eq. (53) in the usual way, and looking for a solution in the form of Eq. (57) now leads to

$$ \Phi_m(x_1) + n_0 \sum_{n=-\infty}^{\infty} Z_n(-i)^{n-m} \left( \int_0^{x_1-a} + \int_{x_1+a}^{\infty} \right) L_{n-m}(x_{21}) \Phi_n(x_2) \, dx_2 $$

$$ = -e^{-im\theta_n} e^{i\lambda x_1}, \quad m \in \mathbb{Z}, \quad (92) $$

where $L_n(X)$ is defined by Eq. (61).

Suppose that for $x > \ell$ we can write (cf. Eq. (59))

$$ \Phi_m(x) = F_m e^{-im\theta_n} e^{i\lambda x}, \quad (93) $$

where $\text{Im} \lambda > 0$. Then if $x_1 > \ell + a$, Eq. (92) becomes

$$ F_m e^{i\lambda x_1} + n_0 \sum_{n=-\infty}^{\infty} Z_n(-i)^{n-m} e^{im\theta_n} \int_{0}^{\ell} \Phi_n(x_2) L_{n-m}(x_{21}) \, dx_2 $$

$$ + n_0 e^{i\lambda x_1} \sum_{n=-\infty}^{\infty} F_n Z_n e^{-i(n-m)(\pi/2+\theta_n)} \left( \int_{\ell}^{x_1-a} + \int_{x_1+a}^{\infty} \right) L_{n-m}(x_{21}) e^{i\lambda x_{21}} \, dx_2 $$

$$ = -e^{i\lambda x_1}, \quad m \in \mathbb{Z}. \quad (94) $$
We have already evaluated \( L_n(x) \) for \( x < 0 \), see Eq. (68). Now, we also need its value for \( x \geq 0 \); we have

\[
\alpha L_n(x) = \begin{cases} 
2(-i)^n e^{-in\theta_in} e^{i\alpha x} & x \geq 0 \\
2i^n e^{in\theta_in} e^{-i\alpha x} & x < 0.
\end{cases}
\]  

(95)

Using these in Eq. (94) gives

\[
\widetilde{\mathcal{A}}_m e^{i\lambda x} + \mathcal{B} e^{i\alpha x} = -e^{i\alpha x}, \quad x > \ell + a, \quad m \in \mathbb{Z},
\]

where

\[
\begin{align*}
\widetilde{\mathcal{A}}_m &= F_m - \frac{2in_0}{\alpha} \sum_{n=-\infty}^{\infty} F_n Z_n \left\{ \frac{e^{-i(\lambda-\alpha)a}}{\lambda - \alpha} - \frac{e^{i(\lambda+\alpha)a}}{\lambda + \alpha} e^{i(n-m)\theta_T} \right\}, \\
\mathcal{B} &= \frac{2n_0}{\alpha} \sum_{n=-\infty}^{\infty} Z_n \left\{ e^{in\theta_in} \int_0^\ell \Phi_n(t) e^{-i\alpha t} dt + \frac{iF_n e^{i(\lambda-\alpha)t}}{\lambda - \alpha} \right\}
\end{align*}
\]

and \( \theta_T = \pi - 2\theta_in \). Thus, \( \lambda \) is to be found from \( \widetilde{\mathcal{A}}_m = 0 \) for all \( m \in \mathbb{Z} \).

As before, we write \( K^2 - k^2 = \lambda^2 - \alpha^2 = \delta_1 n_0 + \delta_2 n_0^2 + \ldots \). Hence,

\[
\frac{e^{-i(\lambda-\alpha)a}}{\lambda - \alpha} - \frac{e^{i(\lambda+\alpha)a}}{\lambda + \alpha} e^{i(n-m)\theta_T} = \frac{2\alpha}{\delta_1 n_0} + \frac{1}{2\alpha} \left\{ 1 - 2i\alpha a - e^{2i\alpha a} e^{i(n-m)\theta_T} \right\} - \frac{2\alpha \delta_2}{\delta_1^2} + O(n_0).
\]

Substituting in \( \widetilde{\mathcal{A}}_m = 0 \) and neglecting terms that are \( O(n_0^2) \), we obtain

\[
F_m - \sum_{n=-\infty}^{\infty} F_n Z_n \left\{ \frac{4i}{\delta_1} + \frac{in_0}{\alpha^2} \left\{ 1 - 2i\alpha a - e^{2i\alpha a} e^{i(n-m)\theta_T} \right\} - \frac{4in_0 \delta_2}{\delta_1^2} \right\} = 0
\]

for \( m \in \mathbb{Z} \). Proceeding as in Sec. IV.C, we obtain Eq. (79), as before. Then, the \( O(n_0) \) terms give

\[
\delta_2 = \frac{\delta_1^2}{4\alpha^2} \left\{ 1 - 2i\alpha a + \frac{4}{\alpha^2} e^{2i\alpha a} \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Z_s e^{i(s-n)\theta_T} \right\}
\]

\[
= \frac{4}{\alpha^2} \left\{ e^{2i\alpha a} [f(\theta_T)]^2 - (1 - 2i\alpha a) [f(0)]^2 \right\}.
\]

Hence, if we let \( \alpha a \to 0 \) in this formula, we recover Twersky’s (erroneous) formula, Eq. (4).

V CONCLUDING REMARKS

We have derived a two-dimensional version of the three-dimensional Lloyd–Berry formula for the effective wavenumber in a dilute random configuration of scatterers, using methods that differ from those used by Lloyd and Berry (1967). Much remains to be done in order to validate the new formula. Specifically, it should be possible to compare its predictions with those obtained from full numerical simulations (using Monte Carlo methods) and from experiments. Some comparisons between Monte Carlo results and solutions of the infinite
system (90), for various choices of $\chi$, have been reported by Siqueira and Sarabandi (1996). They used lossy cylinders, and found good agreement for low area fractions, with little dependence on $\chi$.

Price et al. (1988) have compared the predictions of Twersky’s formula, Eq. (4), with experimental results obtained from sound propagation through three forests; they found “poor” agreement, but perhaps this could be attributed to errors in the formula and the crude approximation of an actual forest by a random array of sound-hard circular cylinders. For a recent review of the quantification of attenuation effects due to trees, see (Attenborough, 2002). Several other experimental studies, in the context of fiber-reinforced materials, are cited in the paper by Verbist et al. (2001).

In three dimensions, there is an extensive literature on comparisons between experiments, direct numerical simulations, and various theories, including the Lloyd–Berry formula; see, for example, Sec. 4.3.12 of (Povey, 1997), (Hipp et al., 1999), and references therein. For low volume fractions and properly modelled spheres, the agreement is generally good: according to Povey (1997, p. 133), there is “a sizeable body of evidence in support of the acoustic multiple scattering theory.”

Acknowledgment

This work was supported by EPSRC grant GR/S35585/01.


