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VARIABILITY OF “GEOSTROPHIC MASS” IN THE PRESENCE OF A LONG BOUNDARY AND RELATED KELVIN WAVE

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Abstract

We analyze an evolution of a localized flow in a half-plane bounded by a rigid wall when the total mass is not conserved within the equivalent-barotropic quasigeostrophic (QG) approximation. A simple formula expressing the total geostrophic mass in terms of the QG potential vorticity is derived and used to estimate the range of the geostrophic mass variability. Behaviour of the total mass is analysed for the system of two point vortices interacting with the wall. Distributed localized perturbations are examined by means of numerical experiments using the QG model. Two types of time variability of the total geostrophic mass are revealed: oscillating (the mass oscillates near some mean value) and limiting one (the mass tends to some constant value with increasing times).

In the framework of a rotating shallow water model, the QG model is known to describe the slow evolution of the geostrophic vorticity assuming the Rossby number to be small. Consideration of the next-order dynamics shows that conservation of the total mass and circulation is provided by a compensating jet taking away the surplus, or shortage of mass from the localized geostrophic disturbance. The along-wall jet expands with the fast speed of Kelvin waves to the right of the initial perturbation. The slow time-dependent amplitude determines the jet sign and intensity at each moment. Dynamics of the compensating jet are discussed for both oscillating and limiting regimes revealed by the QG analysis.

The role of Kelvin waves in establishing the usual Phillips condition for conservation of circulation of the along-wall QG velocity is discussed. In the case of periodic motion or motion in a finite domain, the considered approximation of infinitely long boundary can be used if 1) the typical basin scale greatly exceeds typical size of the localized perturbation and the Rossby scale; 2) the time does not exceed the typical time which is required for the Kelvin wave to travel the typical basin scale.
1. Introduction

This work is motivated by the fact (not well-known) that for a localized flow near infinitely long boundary (e.g., in a channel or half-plane), the standard derivation of quasigeostrophic (QG) equations is incomplete. In particular, it cannot provide: 1) conservation of geostrophic mass; 2) locality of all successive approximations. The QG approximation is based on taking into account only slow rate of evolution for fast rotating fluid (small Rossby number). In this work we elucidate that is not enough for domains with long boundaries (much longer than the flow scale that is typical for baroclinic variability in the ocean) and look closely at the interaction of slow and fast modes through analytical and numerical modelling.

First, we consider the standard derivation of QG equations on the semi-infinite $f$-plane from full rotating shallow water (RSW) system which is often used to reproduce the most important vertical mode of large-scale planetary flows, when their vertical scale is much less than the horizontal. If we assume that the motion is slow and the Rossby number $\varepsilon$ is small, the non-dimensional RSW equations take the form

\[ \frac{\partial U}{\partial T} + \varepsilon \left( U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right) - V = -\frac{\partial P}{\partial x}, \]  

\[ \frac{\partial V}{\partial T} + \varepsilon \left( U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} \right) + U = -\frac{\partial P}{\partial y}, \]

\[ \frac{\partial \Pi}{\partial T} + \varepsilon \left( \frac{\partial (UT)}{\partial x} + \frac{\partial (VT)}{\partial y} \right) = 0, \quad \Pi = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} - P. \]

Here $\mathbf{V} = (U(x,y,T), V(x,y,T))$ is the two-dimensional velocity field, $P$ is the geopotential perturbation equal to the dimensionless elevation as $H = H_0(1 + \varepsilon P(x,y,T))$ is the layer depth, the parameter $\varepsilon = U_0 / f R_d$ is the Rossby number, and $U_0$ is the horizontal velocity scale. The Rossby scale $R_d = \sqrt{gH_0} / f$ and the typical geostrophic time $(\varepsilon f)^{-1}$ are chosen as the spatial and time scales, respectively; $g$ and $f$ are the gravity acceleration and Coriolis parameter, respectively.

The velocity obeys the no flux boundary condition at the rigid wall $y = 0$

\[ V = 0 \text{ at } y = 0. \]
As usually, the QG solution is sought in the form of the following asymptotic expansions (see, e.g. Pedlosky 1987):

\[
(U, V, P) = (U_0, V_0, P_0)(x, y, T) + \varepsilon(U_1, V_1, P_1)(x, y, T) + \ldots
\]

(1.3)

In the lowest order we have from (1.1), (1.2):

\[
V_0 = \frac{\partial \Psi}{\partial x}, \quad U_0 = -\frac{\partial \Psi}{\partial y}, \quad \Psi = P_0,
\]

(1.4a,b,c)

\[
\frac{\partial (\nabla^2 \Psi - \Psi)}{\partial T} + J(\Psi, \nabla^2 \Psi) = 0, \quad \frac{\partial \Psi}{\partial x} = 0 \text{ at } y = 0.
\]

(1.4d,e)

Here \( J(a, b) = \partial(a, b) / \partial(x, y) \) is the Jacobian operator. Let the initial field

\[
\Psi|_{t=0} = \Psi_I(x, y)
\]

(1.5)

be localized, i.e.

\[
\Psi_I \rightarrow 0, \quad y \rightarrow +\infty, \quad x \rightarrow \pm \infty.
\]

(1.6)

Here and below the subscript “\( I \)” denotes initial fields.

Obviously, for conditions (1.4e), (1.6) to be consistent, it is necessary that

\[
\Psi_I = 0 \text{ at } y = 0.
\]

(1.7)

We assume that the decay (1.6) is sufficiently rapid so that the initial energy and enstrophy are finite.

We are looking for the solution conserving energy and enstrophy (note that these quantities are conserved under condition (1.4e)). It means that conditions of locality (1.6) are valid at any time, i.e

\[
\Psi \rightarrow 0, \quad y \rightarrow +\infty, \quad x \rightarrow \pm \infty, \quad (1.8)
\]

since in the opposite case the energy and enstrophy become infinite. Therefore, by virtue of (1.4e) and (1.8)

\[
\Psi = 0 \text{ at } y = 0.
\]

(1.9)

Thus, in the localized case the boundary streamfunction is always identical zero.
It readily follows from (1.4d) that the total geostrophic mass \( \overline{M} = \int_{y>0} \Psi dx dy \) is conserved only if
\[
\int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0} dx = 0.
\]
(1.10)

But the problem (1.4d), (1.5), (1.9) is well-posed and, therefore, condition (1.10), generally, is not satisfied (see Section 2 below and Reznik & Grimshaw 2002; hereafter referred to as RG).

The first-order velocity normal to the wall has the form
\[
V_1 = \frac{\partial P_1}{\partial x} + \frac{\partial U_0}{\partial T} + U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial y},
\]
(1.11)

therefore, the first-order no-flux condition is written as
\[
\frac{\partial P_1}{\partial x} = -\left( \frac{\partial U_0}{\partial T} + U_0 \frac{\partial U_0}{\partial x} \right) \text{ at } y = 0.
\]
(1.12)

For the asymptotic procedure to be self-consistent one has to require that \( P_1 \) is also localized. By virtue of (1.4b), (1.12) this is possible only under the condition (1.10) which, generally, is not satisfied, as mentioned above. Thus, the “slow” solution (1.3) of problem (1.1), (1.2), (1.5), (1.6) does not conserve the total mass and is not localized.

The locality of solution and the mass conservation can be satisfied only in the framework of the full RSW model taking into account the fast Kelvin waves as demonstrated in RG. It was shown that the total (geostrophic + ageostrophic) mass is conserved due to the compensating along-shore jet “emanating” from the localized QG disturbance. This jet, formed by the first-order Kelvin wave plus the slow correction \( P_1 \), provides sink or source of the QG mass. It is worthwhile to note, that the composite (geostrophic + ageostrophic) first-order solution is localized, although the correction \( P_1 \) and the first-order Kelvin wave are non-localized separately.

One can see, that the locality condition (1.8) is crucial in the above consideration, since in the case of non-localized (periodic or non-periodic along the \( y \)-axis) motion the boundary streamfunction \( \Psi \big|_{y=0} = c(T) \) is, generally, non-zero being determined.
from the no-flux condition for the ageostrophic first-order velocity field (cf. Phillips, 1954, McWilliams, 1977). We emphasize also that the above consideration does not mean that the QG approximation is “deficient” or breaks down in the localized case: the equations (1.4), (1.5), (1.9) correctly describe the slow component of motion as demonstrated in RG. But, contrary to non-localized motion, in this case the lowest-order QG dynamics should be supplement with the fast Kelvin wave to provide the solution locality and mass conservation.

Note that Dorofeyev & Larichev (1992) met an analogous problem when considering the reflection of linear Rossby waves from the meridional boundary in the framework of the RSW model on the $\beta$-plane. They revealed that the total mass of Rossby waves is not conserved, and the surplus or shortage of mass is taken away by fast Kelvin waves. Also, Helrich and Pedlosky (1995) examined the QG motion in periodic and unbounded channels and indicated that in the localized case non-conservation of slow circulation results in radiation of Kelvin waves “emanating from the local region” of slow motion.

Key role of the Kelvin waves in the process of geostrophic adjustment near boundaries was investigated in a number of studies after the pioneering work by Gill (1976) (e.g. Hermann et al. 1989; Tomasson & Melville 1992; Helfrich et al. 1999; RG). Interaction of Kelvin waves with other modes of the system was examined, for example, by Miles 1972; Grimshaw and Allen 1988; McCalpin 1995. In the presence of slowly varying environmental parameters, the Kelvin wave can be described by an approximate solution where longshore geostrophy is typically retained (e.g., Fedorov & Melville 1996; Helfrich et al. 1999).

The aim of this paper is to examine the variability of the total mass for the slow geostrophic flow in the localized case, and the related Kelvin wave responsible for the mass conservation. The paper is organized as follows. In Section 2 we derive the general formula expressing the mass of slow QG component in terms of the QG potential vorticity (PV). Using this formula we obtain limits of the QG mass variability and show that the mass variations are determined by the displacements of fluid elements with non-zero PV perpendicular to the boundary. The simplest system possessing this property consists of two point vortex interacting with each other and with the wall; dynamics of this system is analyzed in Section 3. Distributed localized perturbations are examined by means of numerical experiments using the QG model;
the experiments are described in Section 4. Two types of time variability of the QG mass are revealed: oscillating (the mass oscillates near some mean value) and limiting one (the mass tends to some constant value with increasing times). The first-order correction to the QG solution describing the compensating jet is derived in Section 5 by asymptotic analysis of the full RSW model for small Rossby number. Dynamics of the compensating jet for various regimes of QG mass variability are examined. Comparison between the localized and periodic solutions is discussed in section 6 where limits of validity of the considered model for localized disturbance interacting with infinitely long boundary are obtained. In Section 7, our results are summarized.

2. Some general results for QG model

2.1. Formulae for the geostrophic mass in the presence of a boundary

We start with the QG potential vorticity equation (1.4d) with initial and boundary conditions (1.5), (1.6), (1.7). It was demonstrated in RG (Subsection 4.3) that the energy and enstrophy of the localized slow motion are conserved but the total geostrophic mass

$$\bar{M} = \int \Psi dx dy \quad (2.1)$$

is not conserved. To show this we integrate (1.4d) over the half-plane $y > 0$ to obtain the equation

$$\frac{\partial \bar{M}}{\partial T} = F = -\int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0} dx , \quad (2.2)$$

and take into account that the lowest-order along wall velocity circulation

$$\Gamma = -\int_{-\infty}^{\infty} \frac{\partial \Psi}{\partial y} \bigg|_{y=0} dx \quad (2.3)$$

is not a conserved quantity.

Here we express $\Psi$ and the total geostrophic mass (2.1) in terms of the potential vorticity $Q = \nabla^2 \Psi - \Psi$. To find $\bar{M}$ we integrate this equation over $x$:
\[
\frac{d^2 H}{dy^2} - H = R = \int_{-\infty}^{\infty} Q dx, \quad H = \int_{-\infty}^{\Psi} \Psi dx .
\]  

(2.4a,b)

Using (1.9) we have from (2.4a):

\[
H = -\frac{1}{2} \left( e^y \int_{-y}^{\infty} Re^{-y} dy + e^{-y} \int_{0}^{y} Re^{y} dy \right) + \frac{1}{2} e^{-y} \int_{0}^{\infty} Re^{-y} dy .
\]  

(2.4c)

Integration of (2.4a) over \(y\) gives:

\[
\bar{M} = -\int_{y>0} Q dx dy - \frac{dH}{dy} \bigg|_{y=0} .
\]  

(2.5)

Substituting (2.4c) into (2.5) we come to the following formula for the total geostrophic mass:

\[
\bar{M} = -\int_{y>0} Q(x,y,t)(1-e^{-y})dy
\]  

(2.6)

It is useful to represent integral in (2.6) in terms of the Lagrangian coordinates \(x_0, y_0\):

\[
\bar{M} = -\int_{y_0>0} Q_0(x_0, y_0)(1-e^{-y(x_0,y_0,t)})dx_0 dy_0 .
\]  

(2.7)

Here \(Q_0(x_0, y_0)\) is the initial potential vorticity field and the Lagrangian coordinates \(x_0, y_0\) are related to the Eulerian coordinates \(x, y\) in the following way:

\[
x = x(x_0, y_0, t), \quad y = y(x_0, y_0, t), \quad \mathbf{r} \bigg|_{t=0} = (x_0, y_0),
\]  

(2.8)

\[
\frac{\partial x}{\partial \mathbf{r}} \bigg|_{x_0, y_0} = u(x, y, t) = -\frac{\partial \Psi}{\partial y}, \quad \frac{\partial y}{\partial \mathbf{r}} \bigg|_{x_0, y_0} = v(x, y, t) = \frac{\partial \Psi}{\partial x},
\]  

(2.9a,b)

where \(u, v\) are the geostrophic velocity components.
Representations (2.6), (2.7) allow us to make two simple but important conclusions. First, the mass of perturbation steadily translated along the boundary $y = 0$ does not change in time as readily seen from (2.6). Second, it follows from (2.7) that the total mass changes are determined by the displacements of fluid particles perpendicular to the boundary $y = 0$; displacements along the boundary do not influence on the total mass $M$. Note also that only the displacements of fluid particles with non-zero PV make a contribution to the integral (2.7).

Multiplying (1.4d) by $y$ and integrating the resulting equation over the half-plane $y > 0$ we obtain a useful relationship

$$\int_{y>0} yQ(x,y,t)dx dy = const = C_0,$$  
(2.10)

which means that the “PV centroid” remains on a fixed distance from the wall. In the Lagrangian variables (2.10) takes the form

$$\int_{y_0>0} y(x_0,y_0,t)Q_0(x_0,y_0)dx_0 dy_0 = const = C_0.$$  
(2.11)

2.2. Limits of the geostrophic mass variability

It readily follows from (2.6), (2.7) that the absolute value of the geotrophic mass $|M|$ is bounded from above by the integral of the PV modulus:

$$|M| \leq \int_{y>0} |Q(x,y,t)|dx dy = \int_{y_0>0} |Q_0(x_0,y_0)|dx_0 dy_0.$$  
(2.12)

It is possible, however, to obtain a subtler estimate for $|M|$ in the case of sign-defined PV. In the integrand in (2.7) the initial PV $Q_0$ is given by the initial conditions and the total mass changes only due to the time-dependent displacements $y$ of the fluid particles perpendicular to the wall. In its turn, the displacements obey the restriction (2.11). What is the range of the total mass variability generated by the displacements $y$ satisfying (2.11) given initial PV distribution $Q_0$? We emphasize that we mean all possible displacements, not only those obeying equations (2.8), (2.9) determining the Lagrangian coordinates.

To answer this question we investigate extrema of the functional
\[ \bar{M}_a = - \int_{y_0 > 0} Q_0(x_0, y_0)[1 - e^{-\bar{y}(x_0, y_0)}]dx_0dy_0 \quad (2.13a) \]

under the condition

\[ \int_{y_0 > 0} \bar{y}(x_0, y_0)Q_0(x_0, y_0)dx_0dy_0 = \text{const} = C_0, \quad (2.13b) \]

assuming \( Q_0(x_0, y_0) \) to be given and varying \( \bar{y}(x_0, y_0) \). Simple analysis using the Lagrange multipliers method (see Appendix A) gives that the extremal of the problem (2.13) is given by the equation

\[ \bar{y} = \bar{y}_e = \frac{\int_{y_0 > 0} y_0Q_0(x_0, y_0)dx_0dy_0}{\int_{y_0 > 0} Q_0(x_0, y_0)dx_0dy_0} \quad (2.14) \]

where \( \bar{y}_e \) is a constant distance of the PV centroid from the wall.

Calculation of the second variation on the extremal (2.14) shows that the second variation is sign-defined only if the PV \( Q_0 \) does not change its sign i.e., the sufficient conditions for existence of maximum or minimum of the functional \( \bar{M}_a \) are satisfied only in the case of PV of one sign. Also, the coordinate \( \bar{y}_e \) is positive for \( Q_0 \) of a fixed sign. Thus, for PV of one sign the absolute value of functional (2.13a) is maximal on extremal (2.14), i.e. we have the following inequality:

\[ |\bar{M}| \leq |\bar{M}_a|_{\text{max}} = (1 - e^{-\bar{y}_e}) \int_{y_0 > 0} Q_0(x_0, y_0)dx_0dy_0. \quad (2.15) \]

Obviously, (2.15) gives the better estimation of the upper bound of \( |\bar{M}| \) than equation (2.12). Physically (2.15) means that for a given sign-defined initial PV distribution the modulus of the total mass \( |\bar{M}| \) is maximal if the disturbance is elongated parallel to the boundary at the level \( y = \bar{y}_e \), i.e., if all fluid particles are at the equal distance \( \bar{y}_e \) from the wall.

Restriction to \( |\bar{M}| \) from below follows from the energy conservation:
\[
\int_{y>0} \Psi Q \, dx \, dy = -2E = \text{const}.
\]  
(2.16)

One can readily obtain from (2.16) that

\[
\int_{y>0} |\Psi| Q \, dx \, dy \geq \frac{2E}{|Q|_\max}.
\]  
(2.17)

If \(Q_0\) is of one sign then the streamfunction \(\Psi\) is also of one sign (opposite to the sign of \(Q_0\)) by virtue of the equation \(Q = \nabla^2 \Psi - \Psi\) and PV conservation. Therefore for this case we have

\[
|\bar{M}| = \int_{y>0} |\Psi| Q \, dx \, dy \geq \frac{2E}{|Q_0|_\max}.
\]  
(2.18)

Thus the total geostrophic mass \(\bar{M}\) is bounded in absolute value by the parameter \(|\bar{M}_a|_\max\) (see (2.15)) from above and by the value \(2E / |Q_0|_\max\) from below. We note that both these bounds do not depend on time and are determined in terms of the initial conditions.

Restrictions (2.15), (2.18) give a rough estimate of the variability of the total geostrophic mass given a sign-defined initial PV. Of course, these estimates do not take into account details of dynamics describing by (1.4d), therefore maximum (2.15) and minimum (2.18) are hardly achieved in the general case. However, (2.15) can be achieved in the simplest case of two point vortices (see below Section 3). Also, (2.15) turns out to be useful for interpreting the numerical experiments in Section 4.

3. Point vortex systems

It was shown in Subsection 2.1 that the total mass of localized QG perturbation changes in time if i) the perturbation is not steadily translated along the wall, and ii) the fluid particles with non-zero PV displace perpendicular to the boundary \(y = 0\). Obviously, the simplest system possessing these properties consists of two point vortex interacting with each other and with the wall; dynamics of this system is analyzed below.
3.1. General equations

The localized solution to the equation \( \nabla^2 \Psi - \Psi = Q \) obeying the no-flux condition (1.9) is written as follows:

\[
\Psi(x, y, t) = -\frac{1}{2\pi} \int_{y > 0} Q(x', y', t) \left[ K_0(r_-) - K_0(r_+) \right] dx' dy',
\]

(3.1a)

where

\[
r_{\pm} = \sqrt{(x - x')^2 + (y \pm y')^2}
\]

(3.1b)

and \( K_n(z) \), \( n = 0, 1, \ldots \) here and below denotes the \( n \)-order modified Bessel function. Inside the integrand in (3.1a) the terms proportional to \( K_0(r_-) \) and \( K_0(r_+) \) correspond to the contribution made by the “real” and “image” point vortices, respectively.

We now consider the system of point vortices interacting with each other and with their images, and moving along some trajectories \( \mathbf{r} = \mathbf{r}_k(t) \). Here \( \mathbf{r} = (x, y) \) and \( k = 1, 2, \ldots N \) denote radius-vector and number of the vortex, respectively. The streamfuction \( \Psi \) obeys the equations:

\[
\nabla^2 \Psi - \Psi = \sum_{k=1}^{N} A_k \delta(x - x_k(t)) \delta(y - y_k(t)), \quad \left. \Psi \right|_{y=0} = 0
\]

(3.2a, b)

where \( A_k \) is a constant amplitude of \( k \)-th point vortex and \( \delta(z) \) is the Dirac delta-function. The solution to (3.2) is written in the form

\[
\Psi = \frac{1}{2\pi} \sum_{k} A_k \left[ K_0(|\mathbf{r} - \mathbf{r}^{(k)}_-|) - K_0(|\mathbf{r} - \mathbf{r}^{(k)}_+|) \right], \quad \mathbf{r}^{(k)}_{\pm} = (x_k, y_k). \]

(3.3a, b)

In (3.3) the radius-vectors \( \mathbf{r}^{(k)}_+ \) and \( \mathbf{r}^{(k)}_- \) correspond to the “real” and “image” vortices, respectively.

The motion of \( n \)-th vortex is due to its interaction with its own image and with other vortices and their images. Equations describing this motion are obtained in a usual way and can be written in the form:

\[
\dot{x}_n = \frac{A_n}{2\pi} K_1(2x_n) + \frac{1}{2\pi} \sum_{k \neq n} A_k \left[ \frac{y^{(nk)}_+ K_1(r^{(nk)}_+)}{r^{(nk)}_+} - \frac{y^{(nk)}_- K_1(r^{(nk)}_-)}{r^{(nk)}_-} \right],
\]

(3.4a)
\[
\dot{y}_n = \frac{1}{2\pi} \sum_{k \neq n} A_k x_{nk} \left[ \frac{K_1(r_{+}^{(nk)})}{r_{+}^{(nk)}} - \frac{K_1(r_{-}^{(nk)})}{r_{-}^{(nk)}} \right] \tag{3.4b}
\]

where
\[
r_{\pm}^{(nk)} = \sqrt{(x_n - x_k)^2 + (y_n \pm y_k)^2}, \quad x_{nk} = x_n - x_k, \quad y_{\pm}^{(nk)} = y_n \pm y_k. \tag{3.4c}
\]

The first term in the r.h.s. of (3.4a) describes an effect on the vortex of its own image; obviously, this effect initiated displacement of the vortex along the boundary.

Multiplying (3.4b) by \(A_n\) and making summation over \(n\) from 1 to \(N\) we come to an analogue of integral (2.11) for the system of point vortices:
\[
\sum_{n} A_n y_n = \text{const}. \tag{3.5}
\]

Integrating \(\Psi\) in (3.3) over the plane \(y > 0\) we obtain the total mass of the system of \(N\) point vortices (c.f. analogous formula (2.6) for the distributed fields):
\[
\overline{M}_N = -\sum_{k} A_k (1 - e^{-y_k}). \tag{3.6}
\]

One can derive an estimate analogous to (2.15) using the Lagrange multipliers method to analyze the extrema of the function
\[
\overline{M}_N^{(a)} = \overline{M}_N^{(a)} (y_1, y_2, \ldots, y_N) = -\sum_{k} A_k (1 - e^{-y_k}) \tag{3.7}
\]
under condition (3.5) given the amplitudes \(A_k\). The analysis shows that if \(A_k\) are of the same sign then the modulus \(\left|\overline{M}_N^{(a)}\right|\) has maximum equal to
\[
\left|\overline{M}_N^{(a)}\right|_{\text{max}} = (1 - e^{-y_c}) \sum_{k} |A_k| \tag{3.8}
\]
at the point

\(^{1}\) We use the formula \(\int_{-\infty}^{\infty} K_0(\sqrt{x^2 + y^2})dx = \pi e^{-1}\) when deriving (3.6) (Gradshtein, Ryzhik, 1980).
\[ y_1 = y_2 = \ldots = y_N = y_c = \frac{\sum A_k y_k}{\sum A_k}. \]  
(3.9)

Respectively, the estimate (2.15) in the case of a system of \( N \) point vortices is rewritten as

\[ |\overline{M}_N| \leq \left| \overline{M}_N^{(a)} \right|_{\text{max}} = (1 - e^{-y_c}) \sum_k |A_k|. \]  
(3.10)

Using (3.4b) one can derive the useful formula for the mass time derivative:

\[ \dot{\overline{M}}_N = -\frac{1}{4\pi} \sum_{k \neq n} A_k A_n x_{nk} \left[ \frac{K_1(r_{-nk})}{r_{-nk}} - \frac{K_1(r_{+nk})}{r_{+nk}} \right] (e^{-y_n} - e^{-y_k}). \]  
(3.11)

In accordance to (3.4) the separate point vortex moves along the boundary with a constant speed; the corresponding total mass \( \overline{M}_1 \) does not change in time. Detailed analysis of the two point vortex is given in next Subsection.

3.2. System of two point vortices

Dynamics of two point vortices near the wall is described by the system of four vortices in unbounded fluid with the conditions \( A_1 = -A_4, A_2 = -A_3 \) which is known to be integrable (e.g. Borisov et al., 2003). In the case \( N = 2 \) the system (3.4) takes the form:

\[ \dot{x}_1 = A_1 K_1(2y_1) + A_2 \frac{\bar{y}_{12} K_1(\bar{\eta}_{12})}{\bar{\eta}_{12}} - \frac{y_{12} K_1(\eta_{12})}{\eta_{12}}, \]  
(3.12a)

\[ \dot{y}_1 = A_2 x_{12} \left[ \frac{K_1(\eta_{12})}{\eta_{12}} - \frac{K_1(\bar{\eta}_{12})}{\bar{\eta}_{12}} \right], \]  
(3.12b)

\[ \dot{x}_2 = A_2 K_1(2y_2) + A_1 \frac{\bar{y}_{12} K_1(\bar{\eta}_{12})}{\bar{\eta}_{12}} + \frac{y_{12} K_1(\eta_{12})}{\eta_{12}}, \]  
(3.12c)

\[ \dot{y}_2 = A_1 x_{12} \left[ \frac{K_1(\bar{\eta}_{12})}{\bar{\eta}_{12}} - \frac{K_1(\eta_{12})}{\eta_{12}} \right], \]  
(3.12d)

where
\[ \eta_2 = \sqrt{x_{12}^2 + y_{12}^2}, \quad \bar{\eta}_2 = \sqrt{x_{12}^2 + (y_1 + y_2)^2} . \]  
\[ \text{(3.13a,b)} \]

Relationship (3.5) can be written as follows:

\[ A_1y_1 + A_2y_2 = (A_1 + A_2)y_c ; \]  
\[ \text{(3.14)} \]

where \( y_c \) is a constant distance of the vortex pair centroid from the wall.

Mass \( \bar{M}_2 \) and its time derivative \( \dot{\bar{M}}_2 \) are equal to

\[ \bar{M}_2 = -A_1(1 - e^{-y_1}) - A_2(1 - e^{-y_2}) , \]  
\[ \text{(3.15a)} \]

\[ \dot{\bar{M}}_2 = \frac{A_1A_2}{4\pi}x_{12} \left[ \frac{K_1(\eta_2)}{\eta_2} - \frac{K_1(\bar{\eta}_2)}{\bar{\eta}_2} \right] (e^{-y_1} - e^{-y_2}) . \]  
\[ \text{(3.15b)} \]

The set of equation (3.12) is readily reduced to two equations for the distances between the vortices \( x_{12} = x_1 - x_2, \ y_{12} = y_1 - y_2 \):

\[ \dot{x}_{12} = \bar{A}_1K_1(2y_1) - \bar{A}_2K_1(2y_2) - (\bar{A}_1 + \bar{A}_2) \frac{y_{12}K_1(\eta_2)}{\eta_2} + (\bar{A}_2 - \bar{A}_1) \frac{\bar{y}_{12}K_1(\bar{\eta}_2)}{\bar{\eta}_2} , \]  
\[ \text{(3.16a)} \]

\[ \dot{y}_{12} = (\bar{A}_1 + \bar{A}_2)x_{12} \left[ \frac{K_1(\eta_2)}{\eta_2} - \frac{K_1(\bar{\eta}_2)}{\bar{\eta}_2} \right] \]  
\[ \text{(3.16b)} \]

where \( \bar{A}_k = A_k / 2\pi, \ k = 1,2 \). The coordinates \( y_1, y_2 \) in (3.16) are expressed in terms of the constant \( y_c \) and variable \( y_{12} \) using equation (3.14):

\[ y_1 = y_c + \frac{A_2}{A_1 + A_2}y_{12}, \quad y_2 = y_c - \frac{A_1}{A_1 + A_2}y_{12} . \]  
\[ \text{(3.17a,b)} \]

Equations (3.16), (3.17) allow to calculate the variables \( x_{12}, y_{12}, y_1, y_2 \) depending on the time \( t \) and, therefore, the mass \( \bar{M}_2 \). To obtain the phase portraits (the relation between \( x_{12} \) and \( y_{12} \)) of system (3.16), (3.17) we use the energy conservation integral, which is written here as follows

\[ 2A_1A_2[K_0(\eta_2) - K_0(\bar{\eta}_2)] - \bar{A}_1^2K_0(2y_1) - \bar{A}_2^2K_0(2y_2) = \text{const} . \]  
\[ \text{(3.18)} \]

We carried on a set of numerical experiment with equations (3.16), (3.17) for various amplitudes \( A_1, A_2 \) of the vortices and initial distances \( x_{12}, y_{12} \) between
them. Typical behavior of the vortex pair near the rectilinear wall is presented in Fig. 1-3. It is seen from the phase portraits (upper panels) that if the initial distance between the vortices is sufficiently small then the vortices move periodically rotating around some moving centre.

Correspondingly, the total mass $M_2$ and its time derivative $\dot{M}_2$ are periodical functions of time (middle and lower panels). In the case when the amplitudes $A_1, A_2$ are positive (Figs. 1,2) the minimal values of $M_2$ (see the middle panels) are equal to $-\left|M_2^{(a)}\right|$. To determine the maximum values of $M_2$ we use the fact that in accordance with (3.15b) (see also the lower panels) the time derivative $\dot{M}_2$ vanishes at two points: at the point $y_1 = y_2$ which corresponds to the minima of mass, and at the point $x_1 = x_2$ corresponding to the maximums of mass $M_2$. Thus we come to the simple rule: absolute value of the mass of the two-point vortex system is maximal when the system is elongated along the wall and minimal for the system elongated perpendicular the wall. Analogous effect takes place for the distributed perturbations (see the next Section).

If the initial distance is sufficiently large then the motion of the pair is non-periodic, the distance between the vortices increasing monotonically from some time on. Correspondingly, the interaction between the vortices weakens and as time passes the vortices move practically rectilinearly and uniformly along the boundary. The total mass $M_2$ and derivative $\dot{M}_2$ tend to some constant value and to zero, respectively, with increasing time.

Shapes of the phase trajectories depend substantially on the vortex signs. If the vortices are of the same sign then the periodic trajectories are lens-like, longitude size of the lens (along x-axis) increasing and tending to infinity with tending of its transversal size (along y-axis) to some limiting value $\Delta$. For example if $A_1 = A_2 = 1$ the parameter $\Delta = 2.4516$. The “limit” infinitely long lens is a separatrix separating periodic and non-periodic regimes.

In the case of the vortices of opposite signs the shape of the phase trajectories is more “usual” and the separatrix is a loop with two infinite “tendrils” (see Fig. 3). We note that this phase portrait is valid only if the vortex amplitudes are different i.e. $|A_1| \neq |A_2|$. If $A_1 = -A_2$ then the periodic regimes are impossible.
Thus the total mass of two point vortices interacting near the wall either oscillates periodically in time in the vicinity of some mean value (small distance between the vortices) or tends to some constant (large distance between the vortices). In the case of three point vortices (not shown) the total mass behaves analogously but the oscillating regime, generally, is not periodic one.

What is the evolution of mass of the system of distributed vortices? Let the vortices be monopoles and the distance between them greatly exceeds their typical sizes. One can assume that in this case the system behavior would be qualitatively the same as the behavior of the system of point vortices. But if the vortex size is comparable to the distance between them, then changes in the vortex structure can affect the total mass of the system. To study this effect we consider numerical experiments with various localized initial states.

4. Numerical experiments with distributed vortices

In the numerical model for equation (1.4d), Arakawa scheme was used to represent the Jacobian; the time derivative was approximated either by central differences or by Adams-Bashforth formula. Biharmonic viscosity provided a sink for the small-scale vorticity. The calculations were made in a channel bounded by two solid walls at \( y = 0, L \), the motion along the channel (x-axis) assuming to be periodic. The no-flux condition \( \Psi = 0 \) at \( y = 0, L \) and the no-slip condition \( Q = 0 \) at \( y = 0, L \) were used at the channel boundaries. Initial perturbation was located near the south boundary \( y = 0 \); the channel width \( L \) and period along the x-axis were sufficiently large to neglect the interaction of the perturbation with the north wall \( y = L \) and adjacent boxes. Numerical experiments with various initial perturbations showed that the total mass is rather robust characteristic depending weakly on space-time resolution and the domain size.

Initial streamfunction \( \Psi_0 \) in the experiments was calculated given initial potential vorticity \( Q_0 \) by solving the equation

\[
\nabla^2 \Psi_0 - \Psi_0 = Q_0, \tag{4.1}
\]

where \( Q_0 \) is conveniently determined by the finite-area function
\[ F(z) = \begin{cases} \frac{A}{z} \left( \frac{J_1(kz)}{J_1(ka)} - \frac{z}{a} \right), & z < a \\ 0, & z \geq a \end{cases} \] (4.2)

Here \( A, k, \) and \( a \) are some constants; \( J_1(z) \) denotes the first-order Bessel function, and the variable \( z \) is defined by

\[ z = \sqrt{\frac{x^2}{s^2} + (y - d)^2}, \] (4.3)

where \( s \) determines the perturbation size along the \( x \)-axis; \( d \) is the distance of the center of the perturbation from the boundary \( y = 0 \). For \( s < 1 \) the perturbation is elongated along the \( y \)-axis, for \( s > 1 \) it is elongated along the \( x \)-axis. In numerical experiments represented in Figs. 4-7 the following parameter values were adopted: \( A = 2, k = 5.136, a = 1, d = 1 \).

For a vortex dipole the initial PV was prescribed in the form

\[ Q_0 = \frac{y - d}{z} F(z), \quad s = 1. \] (4.4)

The dipole evolution is qualitatively similar to the evolution of pair of the point vortices of opposite sign (Fig.4). Interaction with the boundary results in the destruction of the dipole, the vortices gradually move apart each other, and the pair eventually splits into two monopoles moving in opposite directions along the boundary. Respectively, the total mass increases and tends to a new constant value with increasing time. The change of the total mass exceeds 50% of its initial value in this case, i.e., it is quite significant.

Evolution of the pair of vortices of the same sign was examined for the initial state

\[ Q_0 = \frac{|y - d|}{z} F(z), \quad s = 1. \] (4.5)

The vortices rotate around some common center, elongate and deform as it is seen from the evolution of PV field (Fig. 5). The total mass variability in this case is substantially less than for the dipole. Main change of the mass takes place at the initial adaptation period, then the pair transforms into an approximately axisymmetric vortex and the mass oscillates with small amplitude near its mean value. The mass is maximal at \( T = 1 \) (minimal at \( T = 0 \)) when the perturbation is elongated along (perpendicular) the wall as it was found for the point vortex pair in previous Section.
Thus, in both these cases the vortex system evolution results in one or two monopole vortices interacting with the boundary. We carried out special numerical experiments to study dynamics of the monopoles near the wall. The calculations were made for the initially circular vortex with \( Q_0 = F(z), \ s = 1 \) (Fig. 6) and for the vortex initially elongated along \( y \)-axis with \( Q_0 = F(z), \ s = 0.5 \) (Fig. 7). It is seen that in both these cases the vortex mass oscillates in time similarly to the total mass of two same sign point vortices separated by a small distance. Variability of mass of the elongated vortex is somewhat larger than of the circular one, which is obviously related with larger displacements of fluid particles along the \( y \)-axis when rotating the elongated vortex around its centre. Again, the total mass is minimal (maximal) when the perturbation is elongated perpendicular (along) the wall.

It should be noted that the amplitude of mass oscillations for the monopoles, being rather small, remains nearly the same practically after initial adaptation and this fact does not depend practically on the time and spatial resolution of our numerical scheme with small viscosity. By virtue of symmetry of the problem (1.4d), (1.5), (1.9) the monopole evolution is equivalent to the evolution of the dipole composed of the vortex and its “image”, on the unbounded plane. The described above non-decaying oscillations of the vortices in the dipole indicate that their inviscid evolution may not tend to a stationary limit.

5. First-order correction

Analytical calculations and numerical simulations in preceding Sections show that the mass of localized QG disturbance varies in time. At the same time, the total (geostrophic + ageostrophic) mass in full RSW model (which the QG equations follow for the small Rossby numbers from) should be conserved. To provide the mass conservation we have to take into account the fast ageostrophic waves, i.e. to refuse from the assumption that the motion is slow. The non-dimensional equations of the RSW model take the following form (c.f. (1.1)):

\[
\frac{\partial U}{\partial t} + \varepsilon \left( U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right) - V = -\frac{\partial P}{\partial x}, \quad (5.1a)
\]

\[
\frac{\partial V}{\partial t} + \varepsilon \left( U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} \right) + U = -\frac{\partial P}{\partial y}, \quad (5.1b)
\]
\[
\frac{\partial \Pi}{\partial t} + \epsilon \left( \frac{\partial (UTI)}{\partial x} + \frac{\partial (VTI)}{\partial y} \right) = 0.
\]

(5.1c)

The solution obeys the same no flux condition (1.2).

Here we consider localized and geostrophically balanced initial fields, i.e.

\[
P = \Psi_j(x, y), \quad U = U_j(x, y) = -\frac{\partial \Psi_j}{\partial y}, \quad V = V_j(x, y) = \frac{\partial \Psi_j}{\partial x} \quad \text{at} \quad t = 0.
\]

(5.2)

Following RG we represent the solution in the form of the asymptotic expansions:

\[
U = U_0(x, y, T, T_2, \ldots) + \epsilon u(x, y, t, T, T_2, \ldots) + \ldots \quad (5.3a)
\]

\[
V = V_0(x, y, T, T_2, \ldots) + \epsilon v(x, y, t, T, T_2, \ldots) + \ldots \quad (5.3b)
\]

\[
P = \Psi(x, y, t, T, T_2, \ldots) + \epsilon \nu(x, y, t, T, T_2, \ldots) + \ldots \quad (5.3c)
\]

Here \(T = \epsilon t, \quad T_n = \epsilon^n t, \quad n = 2, 3, \ldots\) are the slow times introduced to prevent the solution from a secular growth in time.

Due to geostrophically balanced initial conditions (5.2) in the lowest order the solution does not depend on the fast time \(t\) and obeys the geostrophic relationships

\[
U_0 = -\frac{\partial \Psi}{\partial y}, \quad V_0 = \frac{\partial \Psi}{\partial x}; \quad \Psi = \Psi(x, y, T, \ldots).
\]

(5.4)

To describe the first-order solution, we obtain from (5.1)-(5.3) the equations

\[
\frac{\partial u}{\partial t} - v + \frac{\partial h}{\partial x} = R_u, \quad \frac{\partial v}{\partial t} + u + \frac{\partial h}{\partial y} = R_v, \quad \frac{\partial (\zeta - h)}{\partial t} = R_\zeta,
\]

(5.5a,b,c)

\[
\nu|_{t=0} = 0, \quad (u, v, h)|_{t=0} = 0,
\]

(5.5d,e)

\[
R_u = \left( \frac{\partial}{\partial T} - \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} \right) \frac{\partial \Psi}{\partial y}, \quad R_v = \left( \frac{\partial}{\partial T} - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} \right) \frac{\partial \Psi}{\partial x},
\]

(5.6a,b)

\[
R_\zeta = -\frac{\partial (\nabla^2 \Psi - \Psi)}{\partial T} - J(\Psi, \nabla^2 \Psi).
\]

(5.6c)

Here \(\zeta - h = \partial_x v - \partial_y u - h\) is the first-order potential vorticity.

The streamfunction \(\Psi\) does not depend on the fast time \(t\), therefore, by virtue of (5.5c) for \(\zeta - h\) to be bounded as \(t \to \infty\), the function \(R_\zeta\) must be identically zero,

\[
R_\zeta = 0,
\]

(5.7)
whence we obtain that $\Psi$ satisfies equation (1.4d).

Taking into account (5.7), we obtain the first-order mass conservation equation from (5.5a,b,c) in the form

$$\frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial R_y}{\partial x} - \frac{\partial R_u}{\partial y}, \quad (5.8)$$

which shows that changes of the first-order total mass $M_1 = \int h dx dy$ in the localized case are expressed as follows

$$\frac{\partial M_1}{\partial t} = \int_{-\infty}^{\infty} R_u(x,0,t) dx = \int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0} dx. \quad (5.9)$$

We see that these changes compensate exactly the changes of the lowest-order geostrophic mass given by (2.2). Therefore, instead of splitting the solution to (5.5), (5.6) into the fast and slow components used in RG, here we represent it as a sum

$$(u, v, h) = (u_c,v_c,h_c) + (u_j,v_j,h_j), \quad (5.10)$$

where the components conserving the corresponding total mass fields $u_c,v_c,h_c$ obey the equations

$$\frac{\partial u_c}{\partial t} - v_c + \frac{\partial h_c}{\partial x} = R'_u = R_u - e^{-y} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0}, \quad \frac{\partial v_c}{\partial t} + u_c + \frac{\partial h_c}{\partial y} = R_v, \quad \frac{\partial (\zeta_c - h_c)}{\partial t} = 0, \quad (5.11a,b,c)$$

$$v_c \bigg|_{y=0} = 0, \quad (u_c,v_c,h_c)_{t=0} = 0, \quad (5.11d,e)$$

while the residual fields $u_j,v_j,h_j$ responsible for the mass changes (5.9) satisfy the system

$$\frac{\partial u_j}{\partial t} - v_j + \frac{\partial h_j}{\partial x} = e^{-y} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0}, \quad \frac{\partial v_j}{\partial t} + u_j + \frac{\partial h_j}{\partial y} = 0, \quad \frac{\partial (\zeta_j - h_j)}{\partial t} = 0, \quad (5.12a,b,c)$$

$$v_j \bigg|_{y=0} = 0, \quad (u_j,v_j,h_j)_{t=0} = 0. \quad (5.12d,e)$$
Analysis of system (5.11) is performed exactly in the same way as in Section 5 of RG and reveals that the fields $u_c, v_c, h_c$ are localized; therefore, they are unimportant if we are interested in the leading-order solution.

The solution to system (5.12) is sought in the form

$$ (u_j, v_j, h_j) = (\phi_j, 0, \phi_j) e^{-y}, $$

which satisfies no-flux boundary condition (5.12d). Evolution of $\phi_j$ is described by the equation

$$ \frac{\partial \phi_j}{\partial t} + \frac{\partial \phi_j}{\partial x} = \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0}. $$

The solution for $\phi_j$ satisfying zero initial condition is written as

$$ \phi_j(x, t, T) = \int_{x-t}^{x} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0} dy'. $$

It is seen from (5.14) that the solution (5.13), (5.15) can be interpreted as a forced Kelvin wave generated by the along wall slow velocity varying in time. This wave looks like an injected jet propagating along the wall to the right of the localized lowest-order disturbance as shown in Figs. 8, 9. The time derivative of the total mass of the jet is equal to

$$ \frac{\partial}{\partial t} \int_{y>0} e^{-y} \phi_j dx dy = \int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0} dx = -\frac{\partial \overline{M}}{\partial T}. $$

Comparing (2.2) to (5.16) we conclude that this compensating jet takes away the surplus or shortage (depending on the sign of $\frac{\partial \overline{M}}{\partial T}$) of the mass from the localized lowest-order disturbance; if the geostrophic mass $\overline{M}$ decreases (increases) then the jet velocity $u_j$ is positive (negative) for sufficiently large $x$ and $t \gg x$.

We now consider the dynamics of compensating jet (5.13) in details representing the amplitude $\phi_j$ in the form

$$ \phi_j = \int_{x-t}^{x} a(x, T) dx + \left[ \int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0} \right]_{y=0}^{x} \phi(x) dx, $$

where
\[ a(x, T) = \left[ \frac{\partial^2 \Psi}{\partial y \partial T} - \varphi(x) \int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial T} \right]_{y=0} \]  

(5.18)

and \( \varphi(x) \) is an arbitrary localized function satisfying the condition

\[ \int_{-\infty}^{\infty} \varphi(x) dx = 1. \]  

(5.19)

The function \( a(x, T) \) is also localized in \( x \) and

\[ \int_{-\infty}^{\infty} a(x, T) dx = 0 \]  

(5.20)

by virtue of (5.19). One can readily show that the first term in the r.h.s. of (5.17) does not make any contribution to the integral in the l.h.s. of (5.16) and the mass conservation is provided by the second term in the r.h.s. of (5.17),

\[ \phi_{j}^{(c)} = \left[ \int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial T} \right] \int_{y=0}^{x} \varphi(x) dx = -\frac{\partial M}{\partial T} \int_{x-t}^{x} \varphi(x) dx. \]  

(5.21)

For example, we can choose the function \( \varphi(x) \) as

\[ \varphi(x) = \frac{c}{\sqrt{\pi}} e^{-c^2 x^2} \]  

(5.22)

where \( c \) is a constant. Note, that the arbitrariness of \( \varphi(x) \) is related to the fact that representation of function as a sum of localized and non-localized parts is not uniquely defined.

Integral \( \int_{x-t}^{x} \varphi(x) dx \) represents the along wall jet expanding with a constant speed of Kelvin wave to the right of the initial perturbation. The slow time-dependent amplitude \( -\partial M / \partial T \) - time rate of the total mass with a minus – determines the jet sign and intensity at each moment. In the oscillating regime the jet sign and intensity also slowly (in comparison with the jet propagation) oscillate as it is shown in Fig. 8. The limiting regime is represented in Fig. 9. In this case the jet sign does not change and the jet amplitude slowly tends to zero.
6. Comparison of localized and periodically-localized solutions

We have considered situation when the localized disturbance interacts with infinitely long boundary and shown that the QG component of the motion obeys zero boundary condition for QG streamfunction. In the case of periodic motion or motion in a finite domain the boundary conditions for QG component are more complicated: the boundary streamfunction is an unknown function of the slow time $T$ and is determined using the condition of conservation of circulation of the along-wall QG velocity (see e.g. Phillips, 1954, McWilliams, 1977 and below). One can think, however, that for some conditions the zero boundary conditions for the QG streamfunction can be used approximately, if the motion is sufficiently localized and the domain is sufficiently large. The using of zero boundary conditions for QG component instead of more complicated ones will be referred to as approximation of infinite domain. An important question is what are the conditions for the approximation of infinite domain to be applied? To answer this question we ought to investigate evolution of a localized perturbation in a bounded domain having a size greatly exceeding typical size of the perturbation. However, the problem of geostrophic adjustment in a closed basin is not yet solved even for the linear case. Instead, we consider problem (5.1) with periodic initial conditions of the form:

$$\Psi_f^{(P)} = \sum_{n=-\infty}^{\infty} \Psi_f^{(L)}(x + nL_x, y),$$

(6.1)

where $\Psi_f^{(L)}(x, y)$ is a localized function sufficiently rapidly decaying at $x \to \pm \infty, \ y \to \infty$; $L_x$ is the period. Superscripts $(P), (L)$ here and below denote the periodic and localized solutions, respectively. For convenience initial condition (6.1) and the corresponding solution $\Psi_f^{(P)}(x, y, t)$ will be referred to as periodically-localized.

The main aim here is to compare the solutions of localized problem for initial streamfunction $\Psi_f^{(L)}(x, y)$ with periodically-localized problem for initial streamfunction (6.1). Such a comparison, being interesting per se, allows us to understand better evolution of localized disturbances in closed domains.
Solution to the RSW system (5.1) for periodic initial conditions is represented in the same way as in the preceding Section; equations (5.1) to (5.15) (excluding (5.9)) remain valid in the periodic case. The main difference is in boundary conditions for QG equation (1.4d). For periodic QG motion condition (1.9) should be changed by the more “general” condition:

\[ \Psi^{(P)} = \Psi^{(B)}(T) \text{ at } y = 0, \]  

(6.2a)

where \( \Psi^{(B)}(T) \) is an unknown function of the slow time. Because the amplitude (5.15) of the forced Kelvin wave as \( t \to \infty \) must remain finite, another boundary condition is

\[ \int_{0}^{L_x} \frac{\partial^2 \Psi^{(P)}}{\partial y \partial T} \bigg|_{y=0} \, dx = 0; \]  

(6.2b)

(it is easy to check that if (6.2b) is not satisfied then \( \phi_j \) grows linearly in time). Note, that traditionally condition (6.2b) is obtained from conservation of the QG velocity circulation or conservation of QG mass (e.g. Phillips, 1954; McWilliams, 1977; Kamenkovich & Reznik, 1978). Simple analysis shows that solution to system (5.11) is always periodic, bounded as \( t \to \infty \), and does not impose additional restrictions on the lowest-order fields.

Thus, in the periodic case, the lowest-order QG component obeys equation (1.4d) and usual boundary conditions (6.2a,b) - contrary to the localized case, the total QG mass in each periodic box is conserved.

Approximate localized and periodically-localized solutions can be written in the same form; for example, for the geopotential we have

\[ p^{(L)} \simeq \Psi^{(L)} + \epsilon e^{-y} \int_{x-t}^{x} \frac{\partial^2 \Psi^{(L)}}{\partial y \partial T} \bigg|_{y=0} \, dx, \]  

(6.3)

\[ p^{(P)} \simeq \Psi^{(P)} + \epsilon e^{-y} \int_{x-t}^{x} \frac{\partial^2 \Psi^{(P)}}{\partial y \partial T} \bigg|_{y=0} \, dx. \]  

(6.4)

We neglect parts of the first-order solution which conserve mass and are insignificant physically when writing (6.3), (6.4).
We now represent the QG solution of periodic problem (1.4d), (6.2a,b) in the form
\[ \Psi^{(P)} = \Psi_B(T) e^{-\gamma} + \tilde{\Psi}^{(P)}. \] (6.5)

The function \( \tilde{\Psi}^{(P)} \) obeys the equations:
\[
\frac{\partial (\nabla^2 \tilde{\Psi}^{(P)} - \tilde{\Psi}^{(P)})}{\partial T} + J(\tilde{\Psi}^{(P)}, \nabla^2 \tilde{\Psi}^{(P)}) + \Psi_B e^{-\gamma} \frac{\partial (\nabla^2 \tilde{\Psi}^{(P)} - \tilde{\Psi}^{(P)})}{\partial x} = 0, \quad (6.6a)
\]

\[ \tilde{\Psi}^{(P)} \bigg|_{y=0} = 0, \quad \tilde{\Psi}^{(P)} \bigg|_{T=0} = \Psi_I^{(P)}. \] (6.6b,c)

By virtue of (6.2b) the function \( \Psi_B \) is related to \( \tilde{\Psi}^{(P)} \) in the following way:
\[
\int L_y \frac{\partial^2 \tilde{\Psi}^{(P)}}{\partial y \partial T} \bigg|_{y=0} \, dx. \quad (6.7)
\]

We now construct an approximate solution to the problem (6.6), (6.7) for \( L_x \gg 1 \). At the initial moment \( \Psi_B = 0, \frac{\partial \Psi_B}{\partial T} = O \left( \frac{1}{L_x} \right) \) (see Appendix B), therefore the solution is sought in the form of asymptotic expansions in the parameter
\[ \delta = \frac{1}{L_x} \ll 1: \]
\[ \tilde{\Psi}^{(P)} = \tilde{\Psi}_0^{(P)} + \delta \tilde{\Psi}_1^{(P)} + \ldots, \quad \Psi_b = \delta \Psi_{B0} + \delta^2 \Psi_{B1} + \ldots \] (6.8a,b)

In the lowest order we have:
\[
\frac{\partial (\nabla^2 \tilde{\Psi}_0^{(P)} - \tilde{\Psi}_0^{(P)})}{\partial T} + J(\tilde{\Psi}_0^{(P)}, \nabla^2 \tilde{\Psi}_0^{(P)}) = 0, \quad \tilde{\Psi}_0^{(P)} \bigg|_{y=0} = 0, \quad \tilde{\Psi}_0^{(P)} \bigg|_{T=0} = \Psi_I^{(P)}. \] (6.9a,b,c)

\[ \frac{\partial \Psi_{B0}}{\partial T} = \int L_y \frac{\partial^2 \tilde{\Psi}_0^{(P)}}{\partial y \partial T} \bigg|_{y=0} \, dx. \] (6.9d)

If the initial function \( \Psi_I^{(L)} \) is strongly localized, for example, behaves as \( O(e^{-r}) \) for \( r \to \infty \) then the localized solution \( \Psi^{(L)} \) also behaves as \( O(e^{-r}) \) for \( r \to \infty \) (see
Appendix B). Therefore the approximate solution \( \tilde{\Psi}_0^{(P)} \) satisfying (6.9a) within to exponentially small terms \( O(e^{-1/\delta}) \) can be written in the form:

\[
\tilde{\Psi}_0^{(P)} = \sum_{n=-\infty}^{\infty} \Psi^{(L)}(x + nL_x, y, T).
\]

(6.10)

Substituting (6.10) into (6.9d) we obtain:

\[
\frac{\partial \Psi_{B0}}{\partial T} = \int_{-\infty}^{\infty} \left. \frac{\partial^2 \Psi^{(L)}}{\partial y \partial T} \right|_{y=0} \, dx.
\]

(6.11)

Using (6.8) solution (6.4) within to small values can be written as follows:

\[
P^{(P)} \approx \tilde{\Psi}_0^{(P)} + e e^{-y} \int_{-t}^{t} \left[ \frac{\partial^2 \tilde{\Psi}_0^{(P)}}{\partial y \partial T} \right]_{y=0} \, dx + \delta e^{-y} \left[ \Psi_{B0} - \epsilon t \frac{\partial \Psi_{B0}}{\partial T} \right].
\]

(6.12)

We now consider periodic solution (6.12) in the domain \(|x| \leq L_x / 2\), i.e. in the central box for \(n = 0\) (see Fig. 10); because of periodicity the analogous results will be valid for any box. Obviously, due to the rapid decay of \( \Psi^{(L)} \) at infinity we have

\[
\tilde{\Psi}_0^{(P)} = \Psi^{(L)} + O(e^{-1/\delta}), \quad |x| \leq L_x / 2,
\]

(6.13)

i.e. in this box the first term in (6.12) is close to the first term in localized solution (6.3).

Using (6.10) the integral in second term in (6.12) can be written as

\[
J^{(P)} = \int_{-t}^{t} \left[ \frac{\partial^2 \tilde{\Psi}_0^{(P)}}{\partial y \partial T} \right]_{y=0} \, dx + \sum_{n=1}^{\infty} \int_{-t+nl_x}^{t+nl_x} \left. \frac{\partial^2 \Psi^{(L)}}{\partial y \partial T} \right|_{y=0} \, dx.
\]

(6.14)

The integrals with \(n \neq 0\) in (6.14) can be neglected for \(t << L_x\), \(|x| \leq L_x / 2\), i.e.

\[
J^{(P)} \approx \int_{-t}^{t} \left[ \frac{\partial^2 \Psi^{(L)}}{\partial y \partial T} \right]_{y=0} \, dx, \quad t << L_x, \quad |x| \leq L_x / 2.
\]

(6.15)

However, for larger times \(t = O(L_x)\) one should take into account the integral with \(n = 1\):
The formulae (6.15), (6.16) are simply interpreted. The second term in (6.12) is the Kelvin wave generated by the periodically-localized structure \( \tilde{\Psi}_0^{(P)} \). In each periodic box this wave coincides practically with corresponding compensating jet radiated by the corresponding localized QG motion in the middle of the box up to the time \( t << t_0 = O(L_x) \). Here \( t_0 \) is the time for the compensating jet produced in the box \( n \) to reach the localized motion domain in the box \( n - 1 \) located to the right from the box \( n \) as it is shown in Fig. 10.

We now consider the last term in (6.12) related to the along-shore current \( \delta \Psi_{B0} e^{-\gamma t} \) providing QG mass conservation. If the period \( L_x \leq 1/\varepsilon \) then on times \( t << L_x \) the last term in (6.12) is small,

\[
\Psi_{B0} - \varepsilon L \frac{\partial \Psi_{B0}}{\partial t} = O(\delta \varepsilon^2 L^2) << \varepsilon O\left(\frac{t^2}{L_x^2}\right) << \varepsilon ,
\]

and can be neglected. In the case \( L_x >> 1/\varepsilon \), i.e. \( \delta << \varepsilon \), this term also can be neglected at \( t << L_x \). Thus, for any period \( L_x >> 1 \) the periodic solution \( \Psi^{(P)} \) in each periodic box is closed to the corresponding localized solution \( \Psi^{(L)} \) at times \( t << L_x \).

Thus, our comparison shows that for a fixed period \( L_x \) larger than the typical scale \( L_{loc} \) of the localized field \( \Psi_i^{(L)} \) the interaction between the adjacent boxes can be practically neglected up to time \( T_r = O(L_x / c_K) \) where \( c_K \) is a Kelvin wave phase speed (equal to 1 in our scaling). This is related to fact that the motion in a given box \( m \) does not depend on the adjacent boxes until the lowest-order Kelvin wave and the compensating jet radiating by the localized perturbation \( \Psi_i^{(L)}(x + mL_x, y) \) reach the “next” perturbation \( \Psi_i^{(L)}(x + (m - 1)L_x, y) \). At times \( t << T_r \) the periodic boxes can be approximately considered as isolated ones and the zero boundary condition (1.9) can be used to describe the geostrophic component in...
each of these boxes. For times $t \geq T_r$ the boxes influence upon each other and one should take into account the along-wall uniform current $\Psi_B(T)e^{-y}$ evolving in the slow time $T$ and providing the slow mass conservation in the periodic case. Note that the magnitude of this current is $O(1/L_x)$ and tends to zero with increasing $L_x$.

The above periodic case is in some sense analogous to the case of localized perturbation in a closed basin. Our analysis allows us to expect that the geostrophic adjustment in the closed basin develops in the same way as in the periodic case (6.1) i.e., the adjustment is accompanied by the radiation of the lowest-order Kelvin wave and the compensating jet propagating around the basin to the right of the localized perturbation, the signal being exponentially small ahead of the wave and jet. For the wave and jet to reach the location of the initial perturbation from the “other” side the time $T_r = O(L_b/c_K)$ is required where $L_b$ is a typical basin size. Obviously, the time $T_r$ increases with increasing the basin scale $L_b$. On times $t \leq T_r$ zero boundary condition like (1.9) can be used for the geostrophic component, for times $t > T_r$ - conditions like (6.2) should be posed. For “small” basins with $L_b \leq R_d/\epsilon$ the time $T_r \leq 1/f\epsilon$, and the conditions like (6.2) are required to describe the slow geostrophic evolution. For larger basins with $L_b > R_d/\epsilon$ (for example, in the ocean) the condition like (1.9) can be used up to times of the order of $T_r > 1/f\epsilon$ for the localized QG perturbations.

Thus, the approximation of infinite domain can be used if 1) the typical basin scale greatly exceeds typical size of the localized perturbation and the Rossby scale; 2) the time does not exceed the typical time $T_r$ which is required for the Kelvin wave to travel the typical basin scale.

Finally, we consider dynamics of the along wall current $\Psi_B(T)e^{-y}$. For the periodic motion the total mass of QG component is conserved in each periodic box therefore by virtue of (6.5) the along wall current $\Psi_B(T)e^{-y}$ can be considered as a flow compensating variation of mass of the component $\overline{\Psi}^{(P)}$. One can show that this current is formed by the periodic forced Kelvin wave generated by the component $\overline{\Psi}^{(P)}$. This is especially clearly seen on times $t << 1/\epsilon$ when solution (6.12) takes the form
\[ p^{(P)} \approx \tilde{\Psi}_0^{(P)} \bigg|_{T=0} + \varepsilon t \frac{\partial \tilde{\Psi}_0^{(P)}}{\partial T} \bigg|_{T=0} + \varepsilon e^{-y} \int_{x-t}^{x} \frac{\partial^2 \tilde{\Psi}_0^{(P)}}{\partial y \partial T} \bigg|_{y=0, T=0} \, dx = \quad (6.18) \]

\[ \tilde{\Psi}_0^{(P)} \bigg|_{T=0} + \varepsilon t \frac{\partial \tilde{\Psi}_0^{(P)}}{\partial T} \bigg|_{T=0} + \varepsilon e^{-y} \left( \int_{x-t}^{x} \frac{\partial^2 \tilde{\Psi}_0^{(P)}}{\partial y \partial T} \bigg|_{y=0, T=0} \right) + \varepsilon e^{-y} s \]

where \( s \) is a function periodic in \( x \) and bounded in \( t \). The angle brackets denote the averaging in \( x \)

\[ \langle a \rangle = \frac{1}{L_x} \int_{0}^{L_x} adx \, . \quad (6.19) \]

It is seen from (6.18) that the propagation of the forced periodic Kelvin wave gives rise to along wall current linearly growing in time \( t \) (the last term in r.h.s. of (6.18)). This current within to small values is equal to the streamfunction \( \Psi_B(T) e^{-y} \) at times \( t \ll 1/\varepsilon \) and the boundary value \( \Psi_B(T) \) coincides with the linearly growing mass of the forced Kelvin wave.

On larger times analogous conclusion can be made using the equation

\[ \frac{\partial \Psi_B}{\partial T} = \frac{\partial}{\partial t} \left( \int_{x-t}^{x} \frac{\partial^2 \tilde{\Psi}^{(P)}}{\partial y \partial T} \bigg|_{y=0} \, dx \right) \, , \quad (6.20) \]

which readily follows from (6.7). By virtue of (6.20) the change of mass of the QG along wall current is exactly equal to the change of mass of the forced Kelvin wave.

7. Summary and conclusion

We analysed an evolution of a localized flow near a straight infinite boundary when the total mass is not conserved within the equivalent-barotropic quasigeostrophic (QG) approximation. A simple formula expressing the total geostrophic mass in terms of the QG potential vorticity is derived and used to estimate the range of the geostrophic mass variability. Behaviour of the total mass is analysed for the system of two point vortices interacting with the wall. Evolution of distributed localised perturbations is examined by means of numerical experiments using the QG model. Two types of time variability of the total geostrophic mass are revealed: oscillating
(the mass oscillates near some mean value) and limiting one (the mass tends to some constant value with increasing times).

The equation related the geostrophic mass to the QG potential vorticity shows that the total geostrophic mass changes are related to the displacements of fluid particles perpendicular to the boundary \( y = 0 \); displacements along the boundary do not affect the total mass. This equation written in the Lagrangian variables is used to estimate the range of the geostrophic mass variability. The bounds of the mass are expressed in terms of initial geostrophic elevation field. If the QG potential vorticity has one sign then the absolute value of total geostrophic mass is maximal when all fluid particles are located along the straight line \( y = \bar{y}_e \) where \( \bar{y}_e \) is the distance of the quasigeostrophic PV centroid from the wall (a conserved quantity).

Those formulas for total geostrophic mass and corresponding estimates were generalized to the case of a system of point vortices interacting with the wall and each other. The system of two point vortices was examined in detail. It was shown that if the initial distance between the vortices is sufficiently small then the vortices move periodically rotating around some moving centre. Correspondingly, the total mass \( M_2 \) and its time derivative \( \dot{M}_2 \) are periodical functions of time, absolute value of the mass being maximal when the system is elongated along the wall and minimal for the system elongated perpendicular the wall. If the initial distance is sufficiently large then the motion of the pair is non-periodic, the distance between the vortices increases monotonically starting from some time. Correspondingly, the interaction between the vortices weakens and as time passes the vortices move practically rectilinearly and uniformly along the boundary. With increasing time, the total mass \( M_2 \) tends to a constant value.

Distributed localised perturbations were examined by means of numerical using the QG model. Evolution of the distributed dipole is qualitatively similar to the evolution of pair of the point vortices opposite in sign. Interaction with the boundary results in the destruction of the dipole, the vortices gradually move apart each other: eventually the couple splits into two monopoles moving in opposite directions along the boundary. With increasing time, the total mass increases and tends to a new constant value. The change of the mass can exceed 50% of the initial value in this case i.e., it could be rather significant.
Evolution of monopolar vortices is qualitatively similar to the behaviour of the pair of vortices of the same sign. The total mass variability in this case is substantially less than for the dipole. Main change of the mass takes place at an initial adaptation period, and later the mass oscillates with small amplitude near some mean value. The mass is minimal (maximal) when the perturbation is elongated perpendicular (along) the wall as it is for the point vortex pair.

Considering the next-order dynamics, we found that the conservation of the total mass and circulation is provided by a compensating jet taking away the surplus, or shortage of mass from the localized lowest-order slow disturbance. A simple representation for the compensating jet was obtained for both oscillating and limiting regimes revealed by the QG analysis. It is seen from this representation that the along-wall jet expands with the fast speed of Kelvin wave to the right of the initial perturbation. The slow time-dependent amplitude \(-\frac{\partial \bar{M}}{\partial T}\) - time rate of the total mass with a minus – determines the jet sign and intensity at each moment. In the oscillating regime the jet sign and intensity also slowly (in comparison with the jet propagation) oscillate. If the mass \(\bar{M}\) tends to a constant value with increasing time then the jet sign does not change and the jet amplitude slowly tends to zero.

Limits of validity of the considered model when localized disturbance interacts with infinitely long boundary are discussed. The approximation of infinite domain can be used if 1) the typical basin scale greatly exceeds typical size of the localized perturbation and the Rossby scale; 2) the time does not exceed the typical time which is required for the Kelvin wave to travel the typical basin scale. These conditions are typical for synoptic eddies of a hundred km spatial scale and 10 days temporal scale in the ocean with a few thousands km basin scale and baroclinic Kelvin wave speed 2 m/s.

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References


Appendix A

In accordance with the Lagrange multipliers method the following functional is introduced:

$$ N = - \int_{y_0 > 0} Q_0(x_0, y_0)[1 - \lambda y - e^{-\eta y}] dx_0 dy_0 $$  \hspace{1cm} (A.1)

where $\lambda$ is a Lagrange multiplier. The corresponding Euler equation for the extremal $y = y_e(x_0, y_0)$ is written in the form:

$$ \lambda - e^{-y_e} = 0, $$  \hspace{1cm} (A.2)

whence it follows that $y_e$ is a constant related to $\lambda$ by the equation

$$ y_e = \ln \lambda. $$  \hspace{1cm} (A.3)

The constants $y_e$ and $\lambda$ are determined from equation (2.13b),

$$ y_e = \frac{\int_{y_0 > 0} y_0 Q_0(x_0, y_0) dx_0 dy_0}{\int_{y_0 > 0} Q_0(x_0, y_0) dx_0 dy_0}, $$  \hspace{1cm} (A.4)

and (A.2).

The second variation of the functional $N$ on the extremal $y = y_e$ is equal to

$$ \delta^2 N = - \frac{\alpha^2}{2} e^{-y_e} \int_{y_0 > 0} \eta^2 Q_0(x_0, y_0) dx_0 dy_0. $$  \hspace{1cm} (A.5)

Here $\alpha \eta$ is a perturbation to $y_e$ such that $\eta$ is an arbitrary bounded function and $\alpha \to 0$. Obviously, $\delta^2 N$ is sign-defined only if the PV $Q_0$ does not change its sign i.e., the sufficient conditions for existence of maximum or minimum of the functional $M_0$ are satisfied only in the case of PV of fixed sign. Also, the coordinate $y_e$ is
positive for $Q_0$ of a fixed sign. Thus, for $Q_0$ of a fixed sign the absolute value of functional (2.13a) is maximal on the extremal $\bar{y} = \bar{y}_e$.

**Appendix B**

**Behavior of $\Psi^{(L)}$ at infinity**

Let us write equations (1.4d), (1.5), and (1.9) for $\Psi^{(L)}$ in the form:

$$\nabla^2 \frac{\partial \Psi^{(L)}}{\partial T} - \frac{\partial \Psi^{(L)}}{\partial T} = -J(\Psi^{(L)}, \nabla^2 \Psi^{(L)}), \quad \Psi^{(L)} \bigg|_{T=0} = \Psi_I^{(L)}(x,y) \quad (B.1)$$

$\Psi^{(L)} = 0$ at $y = 0$. \hfill (B.1c)

Let at some moment $T = T_0$ the field $\Psi^{(L)}$ is strongly localized, so that

$$\Psi^{(L)} = O(e^{-r}), \quad r = \sqrt{x^2 + y^2} \to \infty. \quad (B.2)$$

By virtue of (B.1a,c) the same estimate is valid for the time derivative $\frac{\partial \Psi^{(L)}}{\partial T}$:

$$\frac{\partial \Psi^{(L)}}{\partial T} = O(e^{-r}), \quad r \to \infty. \quad (B.3)$$

It follows from (B.2), (B.3) that the estimate (B.2) is valid at the next moment $T = T_0 + dT$, i.e. during all time. Thus if the initial field $\Psi_I^{(L)}$ is $O(e^{-r})$ then this estimate remains unchanged in time.

**Behavior of $\frac{\partial \Psi_B}{\partial T}$ at $T = 0$**

Let us write (1.4d) for $\Psi^{(P)}$ in the form analogous (B.1a):

$$\nabla^2 \frac{\partial \Psi^{(P)}}{\partial T} - \frac{\partial \Psi^{(P)}}{\partial T} = -J(\Psi^{(P)}, \nabla^2 \Psi^{(P)}), \quad (B.4)$$

and integrate (B.4) in $x$ from 0 to $L_x$. As a result we obtain:
\[
s_{yy} - s = - \int_{0}^{L_x} J(\Psi^{(P)}, \nabla^2 \Psi^{(P)}) \, dx, \quad s = \int_{0}^{L_x} \frac{\partial \Psi^{(P)}}{\partial T} \, dx. \quad (B.5a,b)
\]

We now multiply (B.5a) by \( e^{-y} \) and integrate the resulting equation in \( y \) from 0 to \( \infty \); using boundary conditions (6.2) we have:

\[
\frac{\partial \Psi_B}{\partial T} = \frac{1}{L_x} \int_{0}^{\infty} e^{-y} \, dy \int_{0}^{L_x} \frac{\partial \Psi^{(P)}}{\partial x} \frac{\partial^2 \Psi^{(P)}}{\partial y^2} \, dx. \quad (B.6)
\]

Correspondingly, at the initial moment:

\[
\frac{\partial \Psi_B}{\partial T} \bigg|_{T=0} = \frac{1}{L_x} \int_{0}^{\infty} e^{-y} \, dy \int_{0}^{L_x} \frac{\partial \Psi^{(P)}}{\partial x} \frac{\partial^2 \Psi^{(P)}}{\partial y^2} \, dx. \quad (B.7)
\]

By virtue of initial condition (6.1) we have to within exponentially small values:

\[
\frac{\partial \Psi_L^{(P)}}{\partial x} \frac{\partial^2 \Psi_L^{(P)}}{\partial y^2} = \sum_{-\infty}^{\infty} g(x + nL_x, y), \quad g(x, y) = \frac{\partial \Psi_L^{(L)}}{\partial x} \frac{\partial^2 \Psi_L^{(L)}}{\partial y^2}. \quad (B.8a,b)
\]

It readily follows from (B.7), (B.8) that

\[
\frac{\partial \Psi_B}{\partial T} \bigg|_{T=0} = \frac{1}{L_x} \int_{-\infty}^{\infty} e^{-y} \, dy \int_{-\infty}^{\infty} \frac{\partial \Psi_L^{(L)}}{\partial x} \frac{\partial^2 \Psi_L^{(L)}}{\partial y^2} \, dx = O\left(\frac{1}{L_x}\right).
\]
Fig. 1. Behavior of the pair of point vortices of the same magnitude and sign interacting with the wall. Upper panel represents the phase plot of the system, lower panels – time evolution of the total mass of the pair. The solid (dashed) curves correspond to a small (large) distances between the vortices, and, respectively, to a closed (open) phase trajectories.
Fig. 2. The same as in Fig. 1 but for the vortices having different amplitudes; $A_1 = 0.5, \ A_2 = 1$
Fig. 3. The same as in Fig. 1 but for the point vortices of different signs
Fig. 4. Evolution of the distributed dipole (5.4). The left (right) panels display the streamfunction (PV) contours for different time $T$. The normalized deviation of mass $(M - M_I)/M_I$ from its initial value $M_I$ is shown in the lower panel.
Fig. 5. The same as in Fig. 4 but for the distributed vortex pair (5.5)
Fig. 6. The same as in Fig. 6 but for the initially circular vortex; $Q_0 = F(z), s = 1$
Fig. 7. The same as in Fig. 4 but for the vortex elongated along the $y$-axis; $Q_0 = F(z), s = 0.5$
Fig. 8 Schematic representation of time evolution of the compensating jet in the oscillating regime. Plots of the amplitude $\phi_j^{(c)}$ in (5.21) are shown for different times for the model case

$$\int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial T} \bigg|_{y=0} d\psi = (\sqrt{\pi} / 2) \sin(0.5t).$$

Function $\varphi(x) = (2 / \sqrt{\pi}) \exp(-4x^2)$. Figures near the curves denote the scaled time $t / \pi$.

It is seen that the jet length increases monotonic with increasing time but its intensity oscillates near zero changing its sign. Arrows show the direction of the jet velocity.
Fig. 9. Schematic representation of time evolution of the compensating jet in the “limiting” regime. Plots of the amplitude $\phi_j^{(c)}$ in (5.21) are shown for different times for the model case

$$\int_{-\infty}^{\infty} \frac{\partial^2 \Psi}{\partial y \partial t} \bigg|_{y=0} dx = (\sqrt{\pi} / 2) \exp(-0.5t).$$

Function $\varphi(x) = (2 / \sqrt{\pi}) \exp(-4x^2).$

Figures near the curves denote the time $t$. It is seen that the jet length increases monotonic with increasing time but its intensity gradually tends to zero. Arrow shows the direction of the jet velocity.
Fig. 10. Schematic representation of evolution of periodically-localized solution. Large peaks correspond to domains of localized motion, upper figures $n$ - to the number of boxes. Propagation of the forced periodic Kelvin wave is shown at the time $t < T_r$ (solid line) when the approximation of infinite domain can be still used, and at the time $t \geq T_r$ (dashed line) when condition (6.2) should be posed.