A novel method for the partition of mixed-mode fractures in 2D elastic laminated composite beams

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A Novel Method for the Partition of Mixed-Mode Fractures in 2D Elastic Laminated Composite Beams

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Cracks tend to propagate along the interfaces in laminated materials because they represent a plane of weakness.

They do not kink in order to propagate in pure mode I opening, as they would tend to in an isotropic material.

Interfacial cracks therefore propagate as mixed-mode cracks, with a combination of mode I opening, mode II shearing, and mode III tearing.

Fracture toughness is not an intrinsic material property but depends on the fracture mode partition, i.e. it is load-dependent.

Therefore, to predict fracture toughness, it is essential to know the partition of a mixed fracture mode.

...a particularly complex problem with many confusing aspects.
Introduction

- **Aim 1**: To develop and numerically validate a *novel* mixed-mode partition methodology for interfacial brittle fracture in UD laminated composite beams.
  - It should have a stronger capability for solving more complex mixed-mode partition problems.

- **Aim 2**: To extend this methodology to interfacial brittle fracture in bimaterial beams, and to validate it numerically.

- **Motivation**: Limitations in the conventional approach when dealing with more complex problems, e.g.
  - Bimaterial case: partition relies on extensively tabulated numerical results over a finite range of geometries and material configurations\(^1\)

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Contents

- **Part 1: Background to mixed-mode fracture partitions**

- **Part 2: Laminated unidirectional (UD) composite beams**
  - Development of 2D-elasticity-based partition theory
  - Comparisons with exact 2D-elasticity-based partition theory

- **Part 3: Bimaterial beams**
  - Extension of methodology to bimaterial beams
  - Finite element method (FEM) calculation of fracture mode partitions
  - Comparisons of analytical and FEM results

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Mixed-mode partition theories

- Double cantilever beam (DCB)
- Fundamental case for in-depth study
- Two objectives:
  - Pure mode relationships
  - Mixed-mode partition
The total energy release rate is (UD beams only)

\[
G = \frac{1}{2bE_L} \left[ \frac{M_{1B}^2}{l_1} + \frac{M_{2B}^2}{l_2} - \frac{1}{l} \left( M_{1B} + M_{2B} - \frac{h_2 N_{1Be}}{2} \right)^2 \right. \\
\left. + \left( \frac{1}{A_1} - \frac{1}{A} \right) N_{1Be}^2 \right]
\]

\[
= \{ M_{1B} \quad M_{2B} \quad N_{1Be} \} [C] \{ M_{1B} \quad M_{2B} \quad N_{1Be} \}^T
\]

where

\[
N_{1Be} = N_{1B} - \frac{N_{2B}}{\gamma} \quad \text{and} \quad \gamma = \frac{h_2}{h_1}
\]

ERR is in **quadratic form** in terms of \( M_{1B}, M_{2B} \) and \( N_{1Be} \) and positive definite.
Orthogonal pure fracture modes

First set of orthogonal pure fracture modes

- Pure mode I $\theta_i$: zero relative shearing displacement just behind crack tip.
- Pure mode II $\beta_i$: zero crack tip opening force
Orthogonal pure fracture modes

- **First set of orthogonal pure fracture modes**
  - Pure mode I $\theta_i$: zero relative shearing displacement just behind crack tip.
  - Pure mode II $\beta_i$: zero crack tip opening force

- **Second set of pure fracture modes**
  - Pure mode I $\theta'_i$: zero crack tip shearing force
  - Pure mode II $\beta'_i$: zero relative opening displacement just behind crack tip
First set of orthogonal pure fracture modes

- Pure mode I $\theta_i$: zero relative shearing displacement just behind crack tip.
- Pure mode II $\beta_i$: zero crack tip opening force

Second set of pure fracture modes

- Pure mode I $\theta'_i$: zero crack tip shearing force
- Pure mode II $\beta'_i$: zero relative opening displacement just behind crack tip

The general expressions for the ERR partitions are

\[
G_I = c_I \left( M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1Be}}{\beta_2} \right) \left( M_{1B} - \frac{M_{2B}}{\beta'_1} - \frac{N_{1Be}}{\beta'_2} \right)
\]

\[
G_{II} = c_{II} \left( M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1Be}}{\theta_2} \right) \left( M_{1B} - \frac{M_{2B}}{\theta'_1} - \frac{N_{1Be}}{\theta'_2} \right)
\]
Orthogonal pure fracture modes

\[ G_I = c_I \left( \frac{M_{1B}}{\beta_1} - \frac{N_{1Be}}{\beta_2} \right) \left( \frac{M_{1B}}{\beta'_1} - \frac{N_{1Be}}{\beta'_2} \right) \]

\[ G_{II} = c_{II} \left( \frac{M_{1B}}{\theta_1} - \frac{N_{1Be}}{\theta_2} \right) \left( \frac{M_{1B}}{\theta'_1} - \frac{N_{1Be}}{\theta'_2} \right) \]

Examples:
- \( M_{2B} = \theta_1 M_{1B} \) and \( N_{1Be} = 0 \) \( \implies \) pure mode I (zero relative shearing displacement just behind the crack tip)
- Same for \( N_{1Be} = \theta_2 M_{1B} \) and \( M_{2B} = 0 \).
Orthogonal pure fracture modes

\[ G_I = c_I \left( M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1Be}}{\beta_2} \right) \left( M_{1B} - \frac{M_{2B}}{\beta'_1} - \frac{N_{1Be}}{\beta'_2} \right) \]

\[ G_{II} = c_{II} \left( M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1Be}}{\theta_2} \right) \left( M_{1B} - \frac{M_{2B}}{\theta'_1} - \frac{N_{1Be}}{\theta'_2} \right) \]

- Examples:
  - \( M_{2B} = \theta_1 M_{1B} \) and \( N_{1Be} = 0 \) \( \implies \) pure mode I (zero relative shearing displacement just behind the crack tip)
  - Same for \( N_{1Be} = \theta_2 M_{1B} \) and \( M_{2B} = 0 \).
  - \( M_{2B} = \beta'_1 M_{1B} \) and \( N_{1Be} = 0 \) \( \implies \) pure mode II (zero relative opening displacement just behind the crack tip)
  - Same for \( N_{1Be} = \beta'_2 M_{1B} \) and \( M_{2B} = 0 \).
Orthogonal pure fracture modes

Partitions $G_{I}/G$ and $G_{II}/G$ with $M_{1B} = 1$
A set’s mode I and II conditions are orthogonal.

Consider bending moments only and the first set:

- \( M_2B/M_1B = \theta_1 \) gives pure mode I.
- \( M_2B/M_1B = \beta_1 \) gives the orthogonal pure mode II.

ERR \( G \) is given by

\[
G = \{ M_{1B} \ M_{2B} \ N_{1Be} \} \ [C] \ \{ M_{1B} \ M_{2B} \ N_{1Be} \}^T
\]

Therefore orthogonality here means that

\[
\{ 1 \ \theta_1 \ 0 \} \ [C] \ \{ 1 \ \beta_1 \ 0 \}^T = 0
\]
Orthogonal pure fracture modes

- **Euler beam theory** with rigid interface
  - Two sets of pure modes are unique and do not coincide, i.e.
    \[ \theta_{iE} \neq \theta'_{iE}, \quad \beta_{iE} \neq \beta'_{iE}. \]

  \[
  \theta_{1E} = -\gamma^2, \quad \theta'_{1E} = -1, \quad \beta_{1E} = \gamma^2 \frac{(3 + \gamma)}{(1 + 3\gamma)}, \quad \beta'_{1E} = \gamma^3
  \]

Orthogonal pure fracture modes

- **Euler beam theory** with rigid interface
  - Two sets of pure modes are unique and do not coincide, i.e.
    \[ \theta_i^E \neq \theta_i'^E, \ \beta_i^E \neq \beta_i'^E. \]
    \[ \theta_1^E = -\gamma^2, \ \theta_1'^E = -1, \ \beta_1^E = \gamma^2 (3 + \gamma) / (1 + 3\gamma), \ \beta_1'^E = \gamma^3 \]

- **Timoshenko beam theory** with rigid interface
  - Two sets of pure modes coincide on the Euler theory’s first set, i.e. \( \theta'_i \rightarrow \theta_i, \ \beta'_i \rightarrow \beta_i. \)
    \[ \theta_1^T = \theta_1'^T = \theta_1^E = -\gamma^2, \ \beta_1^T = \beta_1'^T = \beta_1^E = \gamma^2 (3 + \gamma) / (1 + 3\gamma) \]

---

\(^1\text{Suo, Hutchinson. Int J Fract Mech 1990;43:1–18.}\)
Orthogonal pure fracture modes

- **Euler beam theory** with rigid interface
  - Two sets of pure modes are unique and **do not coincide**, i.e. \( \theta_{iE} \neq \theta'_{iE}, \beta_{iE} \neq \beta'_{iE}. \)

\[
\begin{align*}
\theta_{1E} &= -\gamma^2, & \theta'_{1E} &= -1, & \beta_{1E} &= \gamma^2 (3 + \gamma) / (1 + 3\gamma), & \beta'_{1E} &= \gamma^3
\end{align*}
\]

- **Timoshenko beam theory** with rigid interface
  - Two sets of pure modes **coincide on the Euler theory’s first set**, i.e. \( \theta'_i \rightarrow \theta_i, \beta'_i \rightarrow \beta_i. \)

\[
\begin{align*}
\theta_{1T} &= \theta'_{1T} = \theta_{1E} = -\gamma^2, & \beta_{1T} &= \beta'_{1T} = \beta_{1E} = \gamma^2 (3 + \gamma) / (1 + 3\gamma)
\end{align*}
\]

- **2D elasticity theory** with rigid interface\(^1\)
  - Two sets of pure modes **coincide**, i.e. \( \theta_{i-2D} = \theta'_{i-2D}, \beta_{i-2D} = \beta'_{i-2D}. \) **Where do they coincide?**

---

Comparison of partition theories

\[ M_{2B} / M_{1B} \] (with \( \gamma = 2, M_{1B} = 1 \) and \( N_{1Be} = 0 \))

- Euler beam partition theory\(^2\) \( G_i / G \)
- Euler beam partition theory\(^2\) \( G_{II} / G \)
- Timo. beam partition theory\(^2\) \( G_i / G \)
- Timo. beam partition theory\(^2\) \( G_{II} / G \)
- Suo-Hutchinson theory\(^1\) \( G_i / G \)
- Suo-Hutchinson theory\(^1\) \( G_{II} / G \)

\( \theta_{1-2D} = ? \)
\( \beta_{1-2D} = ? \)

---

Objective: Determine $\theta_{1-2D}$ as a function of $\gamma$.

Then use orthogonality condition to determine the other pure modes.

For $\beta_{1-2D}$, solve...
\[
\begin{bmatrix} 1 & \theta_{1-2D} & 0 \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} 1 & \beta_{1-2D} & 0 \end{bmatrix}^T = 0
\]

For $\beta_{2-2D}$, solve...
\[
\begin{bmatrix} 1 & \theta_{1-2D} & 0 \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta_{2-2D} \end{bmatrix}^T = 0
\]

For $\theta_{2-2D}$, solve...
\[
\begin{bmatrix} 1 & \beta_{1-2D} & 0 \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} 1 & 0 & \theta_{2-2D} \end{bmatrix}^T = 0
\]

Then partition the ERR as

\[
G_{I-2D} = c_I \left( M_{1B} - \frac{M_{2B}}{\beta_{1-2D}} - \frac{N_{1Be}}{\beta_{2-2D}} \right)^2
\]

\[
G_{II-2D} = c_{II} \left( M_{1B} - \frac{M_{2B}}{\theta_{1-2D}} - \frac{N_{1Be}}{\theta_{2-2D}} \right)^2
\]
Contents

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Calculating $\theta_{1-2D}$

- The beam-theory-based (Euler and Timoshenko) **pure mode II condition** is
  
  \[ \frac{M_{1B}}{l_1} - \frac{M_{1B} + M_{2B}}{l} = \gamma^2 \left( \frac{M_{2B}}{l_2} - \frac{M_{1B} + M_{2B}}{l} \right) \]

- LHS gives difference between the curvature at the crack tip and the curvature at point A on beam 1.
- RHS gives difference between the curvature at the crack tip and the curvature at point A on beam 2.
Obviously, this condition does not represent the pure mode II condition under 2D elasticity. **A correction is needed.**

It is expected that these curvature differences will be different in 2D elasticity theory.

Hence, a correction factor $c_\beta (\gamma)$ is introduced, as follows:

$$\frac{M_{1B}}{l_1} - c_\beta \left( \frac{M_{1B} + M_{2B}}{l} \right) = \gamma^2 \left[ \frac{M_{2B}}{l_2} - c_\beta \left( \frac{M_{1B} + M_{2B}}{l} \right) \right]$$

This gives the $M_{2B}/M_{1B}$ ratio, denoted by $\beta_1$, as

$$\beta_{1-2D} = \gamma \left[ \frac{(1 + \gamma)^2 + c_\beta (\gamma - 1)}{(1 + \gamma)^2 - c_\beta \gamma (\gamma - 1)} \right]$$
Calculating $\theta_{1-2D}$

- The beam-theory-based (Euler and Timoshenko) **pure mode I condition** is
  \[ M_{1B}/l_1 = -\gamma (M_{2B}/l_2) \]
  (zero relative shearing displacement at the crack tip).

- Similarly, a correction factor $c_\theta (\gamma)$ is introduced:
  \[
  \frac{M_{1B}}{l_1} + c_\theta \left( \frac{M_{1B} + M_{2B}}{l} \right) = -\gamma \left[ \frac{M_{2B}}{l_2} + c_\theta \left( \frac{M_{1B} + M_{2B}}{l} \right) \right]
  \]

- This gives the $M_{2B}/M_{1B}$ ratio, denoted by $\theta_1$, as
  \[
  \theta_{1-2D} = -\gamma^2 \left[ \frac{(1 + \gamma)^2 + c_\theta}{(1 + \gamma)^2 + c_\theta \gamma^2} \right]
  \]
Calculating $\theta_{1-2D}$

Since $\theta_{1-2D}$ and $\beta_{1-2D}$ are orthogonal, the relationship between the correction factors, $c_\theta(\gamma)$ and $c_\beta(\gamma)$, is determined.

$$c_\theta(\gamma) = \frac{(1 - c_\beta)(1 + \gamma)^3}{(1 + \gamma^3)c_\beta - (1 + \gamma)^3} \quad \text{and} \quad c_\beta(\gamma) = \frac{(1 + c_\theta)(1 + \gamma)^3}{(1 + \gamma^3)c_\theta + (1 + \gamma)^3}$$

**How to determine these correction factors?**

- They represent the contributions of the intact beam’s curvature to beam 1 and 2’s curvature at the crack tip.
- For a symmetric beam ($\gamma = 1$), it is reasonable to expect that the corrected curvatures are the average of the intact beam’s and two separated beams’ curvatures, i.e. $c_\theta(1) = 1$. This then gives $c_\beta(1) = 1.6$. 
Now consider the limit where $\gamma \to \infty$ or $\gamma \to 0$.

$c_\beta (\gamma)$ tends to a constant, $\overline{c}_\beta = 1$, in both limits.

$c_\theta (\gamma)$ also tends a constant, $\overline{c}_\theta$, in both limits, however the equation for $c_\theta (\gamma)$ gives $0/0$ so cannot help here.

Instead consider gradients of $c_\beta (\gamma)$, as follows:

$$\lim_{\gamma \to \infty} \left\{ \frac{dc_\beta (\gamma)}{d\gamma} \right\} = 0 \quad \text{and} \quad \lim_{\gamma \to 0} \left\{ \frac{dc_\beta (\gamma)}{d\gamma} \right\} = \frac{3\overline{c}_\theta}{1 + \overline{c}_\theta}$$
Calculating $\theta_{1-2D}$

From the figure, the following approximate assumption can be made:

$$\lim_{\gamma \to 0} \left\{ \frac{dc_\beta(\gamma)}{d\gamma} \right\} \approx \frac{c_\beta(1)}{1} \quad \Rightarrow \quad \frac{3c_\theta}{1 + c_\theta} \approx 1.6$$

This gives $\overline{c}_\theta \approx \frac{8}{7}$. Later we will see that $\overline{c}_\theta \approx \frac{6}{5}$ is even more accurate.
Calculating $\theta_{1-2D}$

- $c_\theta (1) = 1$  
- $c_\theta (\infty) \approx 6/5$  
- $c_\theta (0) \approx 6/5$

Variation of $c_\theta (\gamma)$ can be expressed as

$$c_\theta (\gamma) = \overline{c_\theta \left[2-\hat{c}_\beta^{1/2}(\gamma)\right]}$$

$$\hat{c}_\beta (\gamma) = (1 + \gamma)^3 / (1 + \gamma^3)$$

where $\hat{c}_\beta (\gamma)$ is the $c_\beta (\gamma)$ correction factor which recovers the Timoshenko-based pure modes.
Calculating $\theta_{1-2D}$

- $c_\theta (1) = 1$  
- $c_\theta (\infty) \approx 6/5$  
- $c_\theta (0) \approx 6/5$

Variation of $c_\theta (\gamma)$ can be expressed as

$$c_\theta (\gamma) = \frac{1}{\hat{c}_\beta (\gamma)^{1/2}}$$

$$\hat{c}_\beta (\gamma) = (1 + \gamma)^3 / (1 + \gamma^3)$$

where $\hat{c}_\beta (\gamma)$ is the $c_\beta (\gamma)$ correction factor which recovers the Timoshenko-based pure modes.

**As explained:** Pure modes are given by:

$$\theta_{1-2D} = -\gamma^2 \left[ (1 + \gamma)^2 + c_\theta \right] / \left[ (1 + \gamma)^2 + c_\theta \gamma^2 \right]$$

and the orthogonality conditions.
Calculating $\theta_{1-2D}$

- $c_\theta (1) = 1$  
- $c_\theta (\infty) \approx 6/5$  
- $c_\theta (0) \approx 6/5$

- Variation of $c_\theta (\gamma)$ can be expressed as

$$c_\theta (\gamma) = \frac{2 - \hat{c}_\beta^{1/2} (\gamma)}{c_\theta}$$

$$\hat{c}_\beta (\gamma) = (1 + \gamma)^3 / (1 + \gamma^3)$$

where $\hat{c}_\beta (\gamma)$ is the $c_\beta (\gamma)$ correction factor which recovers the Timoshenko-based pure modes.

- **As explained:** Pure modes are given by:

$$\theta_{1-2D} = -\gamma^2 \left[ (1 + \gamma)^2 + c_\theta \right] / \left[ (1 + \gamma)^2 + c_\theta \gamma^2 \right]$$

and the orthogonality conditions.

- And the ERR partition is given by

$$G_{I-2D} = c_I \left( M_{1B} - \frac{M_{2B}}{\beta_{1-2D}} - \frac{N_{1Be}}{\beta_{2-2D}} \right)^2$$

$$G_{II-2D} = c_{II} \left( M_{1B} - \frac{M_{2B}}{\theta_{1-2D}} - \frac{N_{1Be}}{\theta_{2-2D}} \right)^2$$

Performance of novel methodology

\[ M_{1B} = 1, \; M_{2B} = 0 \; \text{and} \; N_{1Be} = 0 \]

\[ \log_{10}(1/\gamma) \]

Partition \( G_i/G \)

- Suo and Hutchinson’s theory\(^1\)
- Present theory with \( \bar{c}_\theta = 8/7 \)
- Present theory with \( \bar{c}_\theta = 6/5 \)
- Luo and Tong’s theory\(^3\)

Performance of novel methodology

Present theory with $\bar{c}_\theta = 8/7$

Present theory with $\bar{c}_\theta = 6/5$

Luo and Tong’s theory

$L_1 B$ and $L_2 B$ only with $N_{1Be} = 0$

Magnitude of $G_1/G$ error relative to Suo and Hutchinson’s theory


Performance of novel methodology

Present theory with $\bar{c}_\theta = 8/7$

Luo and Tong’s theory$^3$

$M_{1B}$ and $N_{1B}$ only
with $M_{2B} = N_{2B} = 0$

Magnitude of $G_I/G$ error relative
to Suo and Hutchinson’s theory$^1$

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Part 1: Background to mixed-mode fracture partitions

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Extension to bimaterial beams

Material ratio

\[ \eta = \frac{E_2}{E_1} \]

as well as

\[ \gamma = \frac{h_2}{h_1}. \]

Harvey and Wang\textsuperscript{4} have defined all the composite-beam-theory-based conditions. Repeat methodology.

3 components to the \( c_\theta (\gamma, \eta) \) correction factor:

Extension to bimaterial beams

Material ratio
\[ \eta = \frac{E_2}{E_1} \] as well as
\[ \gamma = \frac{h_2}{h_1}. \]

Harvey and Wang have defined all the composite-beam-theory-based conditions. Repeat methodology.

3 components to the \( c_\theta (\gamma, \eta) \) correction factor:

- \( \eta = 1 \) with variable \( \gamma \), \( c_\theta (\gamma, 1) = \bar{c}_{\theta \gamma} \left[ 2 - \hat{c}^{1/2}_\beta (\gamma, 1) \right] \) with \( \bar{c}_{\theta \gamma} \approx \frac{6}{5} \)

---

Extension to bimaterial beams

Material ratio
\[ \eta = \frac{E_2}{E_1} \]
as well as
\[ \gamma = \frac{h_2}{h_1}. \]

Harvey and Wang\textsuperscript{4} have defined all the composite-beam-theory-based conditions. **Repeat methodology.**

3 components to the \( c_\theta (\gamma, \eta) \) correction factor:

- \( \eta = 1 \) with variable \( \gamma \),
  \[ c_\theta (\gamma, 1) = \overline{c}_{\theta \gamma} \left[ 2 - \hat{c}_\beta^{1/2}(\gamma, 1) \right] \]
  with \( \overline{c}_{\theta \gamma} \approx \frac{6}{5} \)

- \( \gamma = 1 \) with variable \( \eta \),
  \[ c_\theta (1, \eta) = \overline{c}_{\theta \eta} \left[ 2 - \hat{c}_\beta^{1/2}(1, \eta) \right] \]
  with \( \overline{c}_{\theta \eta} \approx \frac{2}{13} \)

---

Extension to bimaterial beams

Material ratio
\[ \eta = \frac{E_2}{E_1} \]
as well as
\[ \gamma = \frac{h_2}{h_1}. \]

- Harvey and Wang\(^4\) have defined all the composite-beam-theory-based conditions. Repeat methodology.

3 components to the \(c_\theta(\gamma, \eta)\) correction factor:

- \(\eta = 1\) with variable \(\gamma\), \(c_\theta(\gamma, 1) = \tilde{c}_{\theta\gamma} \left[ 2 - \hat{c}_{\beta}^{1/2}(\gamma, 1) \right]\) with \(\tilde{c}_{\theta\gamma} \approx \frac{6}{5}\)

- \(\gamma = 1\) with variable \(\eta\), \(c_\theta(1, \eta) = \tilde{c}_{\theta\eta} \left[ 2 - \hat{c}_{\beta}^{1/2}(1, \eta) \right]\) with \(\tilde{c}_{\theta\eta} \approx \frac{2}{13}\)

- Length scale term, \(f(\gamma, \eta, L)\)

---

Extension to bimaterial beams

Material ratio

\[ \eta = \frac{E_2}{E_1} \]
as well as
\[ \gamma = \frac{h_2}{h_1}. \]

Harvey and Wang\(^4\) have defined all the composite-beam-theory-based conditions. **Repeat methodology.**

3 components to the \( c_\theta (\gamma, \eta) \) correction factor:

- \( \eta = 1 \) with variable \( \gamma \), \( c_\theta (\gamma, 1) = \bar{c}_{\theta\gamma} \left[ 2 - \hat{c}_{\beta}^{1/2}(\gamma, 1) \right] \) with \( \bar{c}_{\theta\gamma} \approx \frac{6}{5} \)

- \( \gamma = 1 \) with variable \( \eta \), \( c_\theta (1, \eta) = \bar{c}_{\theta\eta} \left[ 2 - \hat{c}_{\beta}^{1/2}(1, \eta) \right] \) with \( \bar{c}_{\theta\eta} \approx \frac{2}{13} \)

- Length scale term, \( f (\gamma, \eta, L) \)

\[ c_\theta (\gamma, \eta) = c_\theta (\gamma, 1) \times c_\theta (1, \eta) \times f (\gamma, \eta, L) \]

- Very fine mesh at crack tip: $\delta a \to 0$, $r = 1.1$
- High-stiffness normal and shear interface springs
- Virtual crack closure technique (VCCT)

$$G_I = Z_d \left( w_{c1} - w_{c2} \right) / (2b \delta a) \quad G_{II} = X_d \left( u_{c1} - u_{c2} \right) / (2b \delta a)$$
Length scale dependence

Global (Euler) partition

Local partition

\( h_1 = 1 \quad \overline{E}_1 = 10^3 \)

\( h_2 = 2 \quad \overline{E}_2 = 10^3 \)

\( M_1 = 1 \quad \gamma = 2 \quad \eta = 1 \)

\( \log_{10}(\text{length scale, } L) \)

Partition \( G_I / G \)
Length scale dependence

\[ h_2 = 10 \]
\[ E_2 = 10^4 \]

\[ h_1 = 1 \]
\[ E_1 = 10^3 \]

\[ M_1 = 1 \]
\[ \gamma = 10 \]
\[ \eta = 10 \]

\[ \log_{10}(\delta a) \]

Global (Euler) partition

\[ \frac{G_I}{G} \]
Pure mode convergence

**Equations:**

- \( h_2 = 10 \)
- \( E_2 = 10^4 \)
- \( M_2 = \text{variable} \)
- \( M_1 = 1 \)
- \( 
\gamma = 10 \\
\eta = 10 
\)

**Graph:**

- First pure mode, \( G_I/G = 1 \)
- Second pure mode, \( G_I/G = 1 \)

**Legend:**

- Green line: First pure mode, \( G_I/G = 1 \)
- Red line: Second pure mode, \( G_I/G = 1 \)

**Axes:**

- Y-axis: Bending moment ratio \( M_{2B}/M_{1B} \)
- X-axis: \( \log_{10}(\delta a) \)

**Graph Annotation:**

- Pure modes coincide here.
Conclusions

- Methodology developed for partition of mixed-mode rigid interface fractures in laminated UD DCBs
  - Takes 2D elasticity into consideration in a novel way
  - Agrees very well with Suo and Hutchinson’s partition theory\(^1\)
  - More easily extendable to more complex problems

- Extension to fractures on bimaterial interfaces is in progress
  - Local partitions of ERR do not exist
  - ERR partitions are length-scale dependent
  - Euler theory gives correct ERR partition for large damage region
  - Analytical method for interfacial fracture in bimaterials is expected to work well

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