Finite orbits of the action of the pure braid group on the character variety of the Riemann sphere with five boundary components

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Finite Orbits of the Action of the Pure Braid Group on the Character Variety of the Riemann Sphere with 5 Boundary Components

by

Pierpaolo Calligaris

A Doctoral Thesis
Submitted in partial fulfilment of the requirements for the award of
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Abstract

In this thesis, we classify finite orbits of the action of the pure braid group over a certain large open subset of the $\text{SL}_2(\mathbb{C})$ character variety of the Riemann sphere with five boundary components, i.e. $\Sigma_5$. This problem arises in the context of classifying algebraic solutions of the Garnier system $\mathcal{G}_2$, that is the two variable analogue of the famous sixth Painlevé equation $\text{PVI}$. The structure of the analytic continuation of these solutions is described in terms of the action of the pure braid group on the fundamental group of $\Sigma_5$.

To deal with this problem, we introduce a system of co-adjoint coordinates on a big open subset of the $\text{SL}_2(\mathbb{C})$ character variety of $\Sigma_5$. Our classification method is based on the definition of four restrictions of the action of the pure braid group such that they act on some of the co-adjoint coordinates of $\Sigma_5$ as the pure braid group acts on the co-adjoint coordinates of the character variety of the Riemann sphere with four boundary components, i.e. $\Sigma_4$, for which the classification of all finite orbits is known. In order to avoid redundant elements in our final list, a group of symmetries $G$ of the large open subset is introduced and the final classification is achieved modulo the action of $G$. We present a final list of 54 finite orbits.
Acknowledgements

I am particularly grateful to my supervisor Prof. Marta Mazzocco, for her continuous guidance, technical and moral support during this project.

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# Contents

**Introduction** vii

1. **Action of the braid group $B_4$ on $M_{G_2}$ and restrictions** 1
   1.1 Co-adjoint coordinates on $M_{G_2}$ ......................... 2
   1.2 Braid group action on $M_{G_2}$ ............................... 19
   1.3 The problem in terms of co-adjoint coordinates ............. 30
   1.4 Restrictions .................................................. 30

2. **Garnier system $G_2$ and symmetry group G** 39
   2.1 Garnier systems $G_n$ .......................................... 39
   2.2 Fuchsian system and its monodromy data ..................... 41
   2.3 Isomonodromic deformations .................................. 46
   2.4 Braid group $B_n$ and analytic continuation .................. 50
   2.5 Symmetry group G .............................................. 54
      2.5.1 Birational transformations of $G_2$ ....................... 54
      2.5.2 Symmetries of the monodromy matrices ................. 61

3. **Input set of the matching procedure** 67
   3.1 PVI: Braid group and analytic continuation .................. 69
   3.2 PVI: symmetry group $G_{PVI}$ ............................... 74
      3.2.1 Okamoto transformations of PVI .......................... 74
### 3.2.2 Symmetries of monodromy matrices

85

### 3.3 PVI: Classification of finite orbits

91

### 3.3.1 Expansion algorithm for Table 3.4

96

### 4 Matching

101

#### 4.1 Outline of the procedure

102

#### 4.2 Matching with three of the PVI 45 exceptional algebraic solutions

108

#### 4.3 Matching with Okamoto solutions

110

#### 4.4 List of finite orbits

119

#### 4.5 Proofs of Theorems 62-63 and Lemmata 66-70

129

### 5 Outlook

137
Introduction

The topic of this thesis is the classification of finite orbits of a certain action of the pure braid group on the $\text{SL}_2(\mathbb{C})$ character variety of $\Sigma_5$, i.e. the Riemann sphere with five boundary components:

$$\mathcal{M}_{G_2} := \text{Hom}(\pi_1(\Sigma_5), \text{SL}_2(\mathbb{C}))/\text{SL}_2(\mathbb{C}).$$

After fixing a basis of oriented loops $\gamma_1, \ldots, \gamma_4, \gamma_\infty$ for $\pi_1(\Sigma_5)$ such that $\gamma_\infty^{-1} = \gamma_1 \cdots \gamma_4$, as in Figure 1, an equivalence class of an homomorphism in the character variety $\mathcal{M}_{G_2}$ can be determined by the five matrices $M_1, \ldots, M_4, M_\infty \in \text{SL}_2(\mathbb{C})$, that are images of $\gamma_1, \ldots, \gamma_4, \gamma_\infty$. These matrices must satisfy the relation:

$$M_\infty M_4 M_3 M_2 M_1 = 1,$$  \hspace{1cm} (1)

up to global conjugation. Assuming $M_\infty$ diagonalizable, then by (1) and global conjugation, $M_\infty$ can be brought to diagonal form:

$$M_\infty = \begin{pmatrix} e^{\pi i \theta_\infty} & 0 \\ 0 & e^{-\pi i \theta_\infty} \end{pmatrix}, \theta_\infty \in \mathbb{C}. $$

vii
As a consequence the character variety $\mathcal{M}_{G_2}$ is identified with the quotient space $\widehat{\mathcal{M}}_{G_2}$, defined as:

$$\widehat{\mathcal{M}}_{G_2} := \{(M_1, \ldots, M_4) \in \text{SL}_2(\mathbb{C}) | M_\infty M_4 M_3 M_2 M_1 = 1,$$

$$M_\infty = \text{diag}(e^{\pm i\theta_\infty}) \} / \sim, \quad (2)$$

where $\sim$ is equivalence up to simultaneous conjugation of $M_1, \ldots, M_4$ by a diagonal matrix in $\text{SL}_2(\mathbb{C})$. The action:

$$B_4 \times \widehat{\mathcal{M}}_{G_2} \rightarrow \widehat{\mathcal{M}}_{G_2}, \quad (3)$$

of the braid group $B_4$ on an element in $\widehat{\mathcal{M}}_{G_2}$ is defined in terms of the following generators:

$$\sigma_1 : (M_1, M_2, M_3, M_4) \rightarrow (M_2, M_2 M_1 M_2^{-1}, M_3, M_4),$$

$$\sigma_2 : (M_1, M_2, M_3, M_4) \rightarrow (M_1, M_3, M_3 M_2 M_3^{-1}, M_4), \quad (4)$$

$$\sigma_3 : (M_1, M_2, M_3, M_4) \rightarrow (M_1, M_2, M_4, M_4 M_3 M_4^{-1}).$$
so that $M_\Sigma$ is preserved and the generators $\sigma_i$ satisfy the following braid relations:

$$\sigma_1 \sigma_3 = \sigma_3 \sigma_1, \quad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \quad \sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3.$$ 

In this thesis, we classify finite orbits of this action (actually of the action of the pure braid group $P_4 \subset B_4$) on the space $\mathcal{M}_{g_2}$.

In Chapter 2, we show that this problem arises in the context of classifying algebraic solutions of the $2 \times 2$ Schlesinger equations in four variables:

$$\frac{\partial}{\partial u_j} A_i = \left[ A_i, A_j \right] \frac{u_i - u_j}{u_i - u_j}, \quad \frac{\partial}{\partial u_i} A_i = -\sum_{j \neq i} \left[ A_i, A_j \right] \frac{u_i}{u_i - u_j}, \quad i \neq j, \quad i, j = 1, \ldots, 4. \quad (5)$$

These equations are the isomonodromic deformations equations of the following Fuchsian system with five singularities $u_1, \ldots, u_4, \infty \in \mathbb{C}$:

$$\frac{d\Psi}{dz} = \left( \frac{A_1}{z - u_1} + \frac{A_2}{z - u_2} + \frac{A_3}{z - u_3} + \frac{A_4}{z - u_4} \right) \Psi, \quad z \in \mathbb{C} \setminus \{u_1, \ldots, u_4\}, \quad (6)$$

where the residue matrices $A_i$, for $i = 1, \ldots, 4$, are traceless and the residue at infinity, i.e. $A_\infty$, defined by:

$$A_\infty := -\left( A_1 + A_2 + A_3 + A_4 \right),$$

is assumed to be diagonal:

$$A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \quad \theta_\infty \in \mathbb{C}.$$ 

Solutions $A_i(u)$, where $u = (u_1, \ldots, u_4)$, of (5) locally are (up to Bäcklund
transformations) in one to one correspondence with points on $\mathcal{M}_{G_2}$. The analytic continuation of the solution $A_i(u)$ along a loop on the universal cover of the configuration space of four points, i.e. $\mathbb{C}^4 \backslash \{\text{diags}\}$, corresponds to another point on $\mathcal{M}_{G_2}$ that is given by the action (4) of the braid group on $(M_1, \ldots, M_4)$, as introduced by Dubrovin-Mazzocco in [DM00] for the Schlesinger equations in three variables. Then, by the generalization due to Cousin [Cou16] of the results of [DM00] and Iwasaki in [Iwa03], algebraic solutions of (5) must correspond to finite orbits of the action (4).

System (5) is equivalent to the Garnier system $\mathcal{G}_2$:

$$\begin{align*}
\frac{\partial \nu_i}{\partial u_i} = \frac{\partial K_i}{\partial \rho_j}, & \quad i, j = 1, 2, \\
\frac{\partial \rho_j}{\partial u_i} = -\frac{\partial K_i}{\partial \nu_j}, & \quad i, j = 1, 2,
\end{align*}$$

(7)
defined in Chapter 2, that is the two variables analogue of the famous Sixth Painlevé equation, PVI: to be more precise, the Garnier system $\mathcal{G}_2$ is the reduction of the Schlesinger equations (5) to Darboux coordinates on the symplectic leaves. Therefore finite orbits will correspond to algebraic solutions of the Garnier system $\mathcal{G}_2$, see [Cou16]. The simplest example of algebraic solution of $\mathcal{G}_2$ is given by Tsuda in [Tsu06]:

$$(\tilde{\nu}_i, \tilde{\rho}_i) = \left( \frac{\theta_i \sqrt{u_i}}{\theta_x}, \frac{\theta_x}{2 \sqrt{u_i}} \right), \quad i = 1, 2,$$

that is algebraic for $\theta_3 = \theta_4 = \frac{1}{2}$ and it satisfies (7) after a suitable change of variables $(\nu_i, \rho_i, u_i) \rightarrow (\tilde{\nu}_i, \tilde{\rho}_i, \tilde{u}_i)$. In our classification we are going to exclude both cases either when the monodromy group $\langle M_1, M_2, M_3, M_4 \rangle$ is reducible or there exists an index $i = 1, \ldots, 4, \infty$ such that $M_i = \pm 1$. Indeed if the monodromy group is reducible the associated solution of $\mathcal{G}_2$ can be reduced to classical solutions in terms of Lauricella hypergeometric
functions as proved by Mazzocco in [Maz01a]. Moreover, in case $M_i = \pm 1$ for some index $i$, again following [Maz01a], the solution of $\mathcal{G}_2$ can be reduced to a solution of PVI. This leads us to define the following big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$:

$$\mathcal{U} = \{(M_1, \ldots, M_4) \in \widehat{\mathcal{M}}_{\mathcal{G}_2}|\langle M_1, \ldots, M_4 \rangle \text{ irreducible}, \quad M_i \neq \pm 1, \forall i = 1, \ldots, 4, \}|/\sim.$$ 

To explain our classification result, we identify the open subset $\mathcal{U}$ with an affine algebraic variety:

**Lemma 1.** Let the functions $p_i, p_{ij}, p_{ijk}$ be defined as:

\begin{align}
    p_i &= \text{Tr} M_i, \quad i = 1, \ldots, 4, \\
    p_{ij} &= \text{Tr} M_i M_j, \quad i, j = 1, \ldots, 4, \quad i > j, \\
    p_{ijk} &= \text{Tr} M_i M_j M_k, \quad i, j, k = 1, \ldots, 4, \quad i > j > k, \\
    p_\infty &= \text{Tr} M_4 M_3 M_2 M_1,
\end{align}

then for every choice of $p_1, \ldots, p_4, p_\infty$, the open subset $\mathcal{U}$ is a four dimensional affine algebraic variety isomorphic to:

$$\mathbb{C}[p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}]/I,$$

where the ideal $I$ is the ideal generated by the polynomials $f_1, \ldots, f_{15}$ defined in (1.53)-(1.67).

Therefore, we think of $p_i, p_{ij}, p_{ijk}$ as an overdetermined system of coordinates on a big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{\mathcal{G}_2}$ and we express the action (3) in terms of $p_i, p_{ij}, p_{ijk}$:
Lemma 2. The transformations $\sigma_i : \hat{M}_{G_2} \to \hat{M}_{G_2}$ act on the coordinates $p_i, p_{ij}, p_{ijk}$ in the open subset $U$ as follows:

$$\sigma_1 : (p_1, p_2, p_3, p_4, p_{12}, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto$$

$$(p_2, p_1, p_3, p_4, p_{12}, p_{21}, p_{31}, p_{32}, p_{41}, p_{42} - p_{31} - p_{21} + p_{21}^2, p_{21}, p_{42},)$$

$$p_1p_4 - p_{41} - p_{21}p_{42} + p_2p_{421}, p_{43}, p_{321}, p_{143} - p_{431} - p_{21}p_{432} + p_2p_{321},$$

$$p_432, p_{421}),$$

$$\sigma_2 : (p_1, p_2, p_3, p_4, p_{12}, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto$$

$$(p_1, p_2, p_4, p_{12}, p_{31}, p_{142} - p_{21} - p_{31}p_{32} + p_3p_{321}, p_{32}, p_{41}, p_{43},)$$

$$p_2p_4 - p_{42} - p_{32}p_{43}, p_{321}, p_{432}, p_{321}, p_{241} - p_{421} - p_{321}p_{431} + p_3p_{431},$$

$$p_{431}),$$

$$\sigma_3 : (p_1, p_2, p_3, p_4, p_{12}, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \mapsto$$

$$(p_1, p_2, p_4, p_{12}, p_{31}, p_{42} - p_{31} - p_{41}p_{43} + p_4p_{431},)$$

$$p_2p_3 - p_{32} - p_{42}p_{43}, p_{432}, p_{43}, p_{421}, p_{432}, p_{431},$$

$$p_{321} - p_{421}p_{43} + p_4p_{431}),$$

and they define an action of the braid group $B_4$.

Hence, our problem can be reformulated as: find all $p_i, p_{ij}, p_{ijk}$ in the big open subset $U$ such that:

- they satisfy the constraints given by $I$ in Lemma 1,

- their orbit under the action of the pure braid group $P_4$ is finite.

Our approach is based on the observation that given $p_i, p_{ij}, p_{ijk}$ such that they generate a finite orbit under the action of the pure braid group $P_4$, then for any subgroup $H \subset P_4$ the restriction of the action to $H$ produces a finite orbit as well. Such restriction only acts on some of the $p_i, p_{ij}, p_{ijk}$ and
leaves others invariant. We select subgroups $H \subset P_4$ acting on the set (9) so that the restricted action is isomorphic to the action of the pure braid group $P_3$ on the $SL_2(\mathbb{C})$ character variety of $\Sigma_4$, i.e. the Riemann sphere with four boundary components, for which all finite orbits are classified in Lisovyy and Tykhyy’s work [LT14].

Furthermore, we show that there exist four restrictions $H_1, \ldots, H_4$ isomorphic to $P_3$. Each one of these restrictions allows us to identify some of the $p_i,p_{ij},p_{ijk}$ with coordinates on the $SL_2(\mathbb{C})$ character variety of $\Sigma_4$, as in Table 1: each line shows which $p_i,p_{ij},p_{ijk}$ can be found by imposing that the restriction gives a finite orbit of $P_3$. We recall the list of all finite orbits of the action of $P_3$ on the $SL_2(\mathbb{C})$ character variety of $\Sigma_4$ in Chapter 3.

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Table 1: Action on $p_i,p_{ij},p_{ijk}$ defined in (10) of subgroups of $P_4$ isomorphic to $P_3$.

In order to avoid redundant solutions to this classification problem, such as for example equivalent solutions obtained by simple cyclic relabelling of indices in (8), in Chapter 2, we introduce the symmetry group $G$ of the big open subset $\mathcal{U}$ and factorize our classification modulo the action of $G$. The symmetry group $G$ can be calculated using known results about Bäcklund transformations of Schlesinger equations (5) and permutations and sign flips on the monodromy matrices.
In Chapter 4, we present a list of 54 finite orbits of action (10) obtained up to the action of the group of symmetries $G$. Due to the identification of each action of the restriction $H_i$ (determined by the rows in Table 1) with the finite action of $P_3$ over the $SL_2(\mathbb{C})$ character variety of $\Sigma_4$, we can associate to each restriction an algebraic solution of PVI (see [DM00, Iwa03, Cou16, LT14]). Then in our list each orbit’s member has the following properties:

- no more than one restriction (determined by the rows of Table 1) is associated to algebraic solutions of PVI obtained by the pull-back of the hypergeometric equation, see Doran [Dor01] and Andreev-Kitaev [AK02],

- no more than one restriction corresponds to the so-called Picard solutions of PVI, see the work of Picard [Pic89] and Mazzocco [Maz01b].

Moreover, we do not allow any orbit’s member such that:

- one restriction is associated to algebraic solutions of PVI obtained by the pull-back of the hypergeometric equation and another restriction is associated to the so-called Picard solutions of PVI.

Accordingly, our solutions do not include the before mentioned solution obtained by Tsuda in [Tsu06] as a fixed point of a certain birational symmetry of $G_2$, nor the solutions found by Diarra in [Dia13], who presents all finite orbits that can be obtained using the method of pull-back introduced in [Dor01] and [AK02], nor the one found by Girand in [Gir16a, Gir16b], who presents two-parameter families of algebraic solutions of $G_2$ obtained restricting a logarithmic flat connection defined on the complement of a quintic curve on $\mathbb{P}^2$ on generic lines of the projective plane, these solutions have at least two restrictions obtained by pull-back of the hypergeometric
equation.

From the monodromy data $M_1, \ldots, M_4$, it is possible to recover the explicit formulation of the associated solution of $G_2$ using the method developed by Lisovyy and Gavrylenko in [GL16] of Fredholm determinant representation for isomonodromic tau functions of Fuchsian systems of the form (6).

The shortest finite orbit classified has length 36, for this reason the associated algebraic solution of $G_2$ has eventually 36 branches and we doubt that the expression of this solution can have a nice and compact form.
Chapter 1

Action of the braid group $B_4$
on $\mathcal{M}_{\mathcal{G}_2}$ and restrictions

In this Chapter we are going to describe in details the action:

$$P_4 \times \mathcal{M}_{\mathcal{G}_2} \longrightarrow \mathcal{M}_{\mathcal{G}_2}, \quad (1.1)$$

of the pure braid group $P_4$ on $\mathcal{M}_{\mathcal{G}_2}$, i.e. the SL$_2(\mathbb{C})$ character variety of the Riemann sphere $\Sigma_5$ with five boundary components. In Theorem 3, Lemma 4 and Proposition 5, we will show that there exists a system of co-adjoint coordinates $p_i, p_{ij}, p_{ijk}$ on a big open subset $\mathcal{U}$ of $\mathcal{M}_{\mathcal{G}_2}$. Furthermore, in Theorem 6, the big open subset $\mathcal{U}$ is identified with an affine algebraic variety that is the zero locus of a particular family $\mathcal{F}$ of polynomials.

In Section 1.2, the action (1.1) on $p_i, p_{ij}, p_{ijk}$ is presented explicitly in Lemma 10. In Section 1.3, the problem is reformulated as the classification of finite orbits of the $P_4$ action over the $p_i, p_{ij}, p_{ijk}$ such that they are in the zero locus of $\mathcal{F}$.

Moreover, in Section 1.4, we discuss the methodology used to achieve
this classification problem: indeed, if \( p_i, p_{ij}, p_{ijk} \) are known such that they
generates a finite \( P_4 \) orbit, then for any subgroup \( H \subset P_4 \), the action of \( H \)
over \( p_i, p_{ij}, p_{ijk} \) still generate a finite orbit. In Theorem 12, we identify four
subgroups \( H_1, \ldots, H_4 \) acting as the pure braid group \( P_3 \) over the \( SL_2(\mathbb{C}) \)
character variety of the Riemann sphere \( \Sigma_4 \) with four boundary components, that we will denote \( \mathcal{M}_{PV1} \): so that we can use the classification result
obtained by Lisovyy and Tykhyy in [LT14].

1.1 Co-adjoint coordinates on \( \mathcal{M}_{G_2} \)

We identify the character variety \( \mathcal{M}_{G_2} \) with the quotient space:

\[
\mathcal{M}_{G_2} = \{(M_1, M_2, M_3, M_4) \mid M_i \in SL_2(\mathbb{C}), \ M_\infty M_4 M_3 M_2 M_1 = 1\} / \sim,
\]

where \( \sim \) is equivalence under global diagonal conjugation. Without loss of
generality, the matrix \( M_\infty \) can be brought to diagonal form:

\[
M_\infty = \begin{pmatrix}
e^{i\theta_\infty} & 0 \\
0 & e^{-i\theta_\infty}
\end{pmatrix}, \ \theta_\infty \in \mathbb{C}, \quad (1.2)
\]

then, since the trace of \( M_\infty \) is a given parameter, \( \mathcal{M}_{G_2} \) is an eight dimen-
sional affine algebraic variety: indeed each \( M_i \) is an element of \( SL_2(\mathbb{C}) \), up
to global diagonal conjugation, and \( M_1, \ldots, M_4 \) satisfy the cyclic relation
\( M_\infty M_4 M_3 M_2 M_1 = 1 \) and (1.2).

It is possible to endow the space of functions on \( \mathcal{M}_{G_2} \) with a system of
co-adjoint coordinates, this is a generalization of a result proved by Iwasaki
for the Sixth Painlevé equation [Iwa03]:

**Theorem 3.** Let \( (M_1, \ldots, M_4) \in \mathcal{M}_{G_2} \) and define the following complex
Action of the braid group \( B_4 \) on \( \mathcal{M}_{G_2} \) and restrictions

\[ (p_1, p_2, p_3, p_4, p_x, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \in \mathbb{C}^{15}, \quad (1.3) \]

to be:

\[
\begin{align*}
p_i &= \text{Tr} M_i, \quad i = 1, \ldots, 4, \\
p_{ij} &= \text{Tr} M_i M_j, \quad i, j = 1, \ldots, 4, \quad i > j, \\
p_{ijk} &= \text{Tr} M_i M_j M_k, \quad i, j, k = 1, \ldots, 4, \quad i > j > k, \\
p_x &= p_{4321} = \text{Tr} M_4 M_3 M_2 M_1. \quad (1.4)
\end{align*}
\]

Let \( g(x, y, z) := x^2 + y^2 + z^2 - xyz - 4 \), then in the open subset:

\[ \mathcal{U}^{(0)}_{jk} := \mathcal{M}_{G_2} \cap \{ (p_{jk}^2 - 4) g(p_{jk}, p_{jc}, p_{jkd}) \neq 0 \}, \quad (1.5) \]

the matrices \( M_1, \ldots, M_4 \) can be parametrized as follows:

\[
PM_\ell P^{-1} = \begin{pmatrix}
p_{jkl} - p_{k} \lambda_{j}^{\ell}
\end{pmatrix}
\end{pmatrix}
\end{equation}

\[
PM_{k} P^{-1} = \begin{pmatrix}
-\frac{p_{k} - p_{j} \lambda_{j}^{k}}{r_{jk}}
-\frac{y_{kl} - y_{jk} \lambda_{jk}^{k}}{r_{jk}}
\end{pmatrix},
\]

\[
PM_{j} P^{-1} = \begin{pmatrix}
-\frac{p_{j} - p_{k} \lambda_{k}^{j}}{r_{jk}}
-\frac{y_{il} - y_{jk} \lambda_{jk}^{l}}{r_{jk}}
\end{pmatrix},
\]

\[
PM_{i} P^{-1} = \begin{pmatrix}
tp_{ij} - p_{i} \lambda_{jk}^{i}
\end{pmatrix}.
\]
alternatively on the open subset:

\[ \mathcal{U}_{jk}^{(1)} := \widetilde{\mathcal{M}}_{G_2} \cap \{ (p_{jk}^2 - 4)g(p_j, p_k, p_{ijk}) \neq 0 \}, \quad (1.10) \]

the matrices \( M_1, \ldots, M_4 \) can be parametrized as follows:

\[ PM_1^{-1} = \begin{pmatrix} \frac{p_{jk} - p_k \lambda_j^+}{r_{jk}} & \frac{y_{ik} - y_{ij} \lambda_j^+}{g(p_{jk}, p_j, p_k)} \\ \frac{y_{ik} - y_{ij} \lambda_j^+}{g(p_{jk}, p_j, p_k)} & \frac{p_{jk} - p_k \lambda_j^+}{r_{jk}} \end{pmatrix}, \quad (1.11) \]

\[ PM_2^{-1} = \begin{pmatrix} \frac{p_j - p_k \lambda_j^+}{r_{jk}} & \frac{g(p_{jk}, p_j, p_k)}{p_{jk} - p_k \lambda_j^+} \\ \frac{p_j - p_k \lambda_j^+}{r_{jk}} & \frac{-\lambda_j^-}{r_{jk}} \end{pmatrix}, \quad (1.12) \]

\[ PM_3^{-1} = \begin{pmatrix} \frac{p_{jk} - p_i \lambda_{jk}^-}{r_{jk}} & \frac{y_{ik} - y_{ij} \lambda_{jk}^-}{g(p_{jk}, p_j, p_k)} \\ \frac{y_{ik} - y_{ij} \lambda_{jk}^-}{g(p_{jk}, p_j, p_k)} & \frac{p_{jk} - p_i \lambda_{jk}^-}{r_{jk}} \end{pmatrix}, \quad (1.13) \]

\[ PM_4^{-1} = \begin{pmatrix} \frac{p_{jk} - p_i \lambda_{jk}^-}{r_{jk}} & \frac{y_{ik} - y_{ij} \lambda_{jk}^-}{g(p_{jk}, p_j, p_k)} \\ \frac{y_{ik} - y_{ij} \lambda_{jk}^-}{g(p_{jk}, p_j, p_k)} & \frac{p_{jk} - p_i \lambda_{jk}^-}{r_{jk}} \end{pmatrix}, \quad (1.14) \]

and on the open subset:

\[ \mathcal{U}_{jk}^{(2)} := \widetilde{\mathcal{M}}_{G_2} \cap \{ (p_{jk}^2 - 4)g(p_j, p_i, p_{ijk}) \neq 0 \}, \quad (1.15) \]

the matrices \( M_1, \ldots, M_4 \) can be parametrized as follows:

\[ PM_1^{-1} = \begin{pmatrix} \frac{p_{jk} - p_k \lambda_j^+}{r_{jk}} & \frac{y_{ik} + y_{ijkl} \lambda_j^+}{g(p_{jk}, p_j, p_k)} \\ \frac{y_{ik} + y_{ijkl} \lambda_j^+}{g(p_{jk}, p_j, p_k)} & \frac{p_{jk} - p_k \lambda_j^+}{r_{jk}} \end{pmatrix}, \quad (1.16) \]

\[ PM_2^{-1} = \begin{pmatrix} \frac{p_j - p_k \lambda_j^+}{r_{jk}} & \frac{y_{ik} - y_{ij} \lambda_j^+}{g(p_{jk}, p_j, p_k)} \\ \frac{y_{ik} - y_{ij} \lambda_j^+}{g(p_{jk}, p_j, p_k)} & \frac{p_j - p_k \lambda_j^+}{r_{jk}} \end{pmatrix}, \quad (1.17) \]
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

\[ PM_j P^{-1} = \begin{pmatrix} -\frac{p_{jk} - p_j \lambda^+_{jk}}{r_{jk}} & -\frac{y_{ij} - y_{jk} \lambda^+_{jk}}{r_{jk}} \\ \frac{y_{ij} - y_{jk} \lambda^+_{jk}}{g(p_{jk}, p_j, p_{ijk})} & -\frac{p_{jk} - p_j \lambda^+_{jk}}{r_{jk}} \end{pmatrix}, \tag{1.18} \]

\[ PM_i P^{-1} = \begin{pmatrix} \frac{p_{ijk} - p_i \lambda^+_{jk}}{r_{jk}} & g(p_{ijk}, p_j, p_{ijk}) \\ 1 & -\frac{p_{ijk} - p_i \lambda^+_{jk}}{r_{jk}} \end{pmatrix}, \tag{1.19} \]

where $P \in \text{GL}_2(\mathbb{C})$ and:

\[ \lambda^+_{jk} := \frac{p_{jk} + r_{jk}}{2}, \quad \lambda^-_{jk} = \frac{1}{\lambda^+_{jk}}; \tag{1.20} \]

\[ r_{jk} := \sqrt{p_{jk}^2 - 4}; \tag{1.21} \]

\[ y_{kl} := 2p_{k \ell} + p_{jk} p_{j \ell} - p_j p_{jk \ell} - p_k p_{j \ell}, \tag{1.22} \]

\[ y_{ji} := 2p_{ji} + p_{jk} p_{j k \ell} - p_j p_{jk \ell} - p_k p_{j \ell}, \tag{1.23} \]

\[ y_{ik} := 2p_{ik} + p_{ij} p_{j k} - p_j p_{ijk} - p_i p_k, \tag{1.24} \]

\[ y_{ij} := 2p_{ij} + p_{ik} p_{j k} - p_k p_{ijk} - p_i p_j, \tag{1.25} \]

\[ y_{il} := 2p_{i l} + p_{ij} p_{j k \ell} - p_j p_{ijk \ell} - p_i p_{j \ell}, \tag{1.26} \]

\[ y_{ijkl} := 2p_{ijkl} - p_i p_{jk} - p_j p_{ik} - p_k p_{ij} + p_{ij} p_{jk} p_{i \ell} + p_{ij} p_{jk} p_{i \ell}. \tag{1.27} \]

**Proof.** Consider $(M_1, \ldots, M_4) \in \hat{\mathcal{M}}_{G_2}$. We are going to prove that there exists a parametrization of $M_1, \ldots, M_4$ in terms of the invariants $p_i, p_{ij}, p_{ijk}$ in the open subset $\mathcal{U}^{(0)}_{jk} = \hat{\mathcal{M}}_{G_2} \cap \{(p_{jk}^2 - 4)g(p_{jk}, p_j, p_{jk}) \neq 0\}$. For the parametrizations on the open subsets $\mathcal{U}^{(1)}_{jk}$ and $\mathcal{U}^{(2)}_{jk}$ a similar proof applies.

Under the generic hypothesis that there exist two indices $j$ and $k$ such that $p_{jk} \neq \pm 2$, the product $M_j M_k$ has two distinct eigenvalues $\lambda^\pm_{jk}$, namely:

\[ \lambda^+_{jk} = \frac{p_{jk} + r_{jk}}{2}, \quad \lambda^-_{jk} = \frac{1}{\lambda^+_{jk}}, \quad r_{jk} = \sqrt{p_{jk}^2 - 4}, \tag{1.28} \]

where the positive branch of the square root is chosen. Consequently, there
exists a matrix $P \in \text{GL}_2(\mathbb{C})$ such that the product matrix $M_jM_k$ can be brought into diagonal form:

$$\Lambda_{jk} := P(M_jM_k)P^{-1} = \text{diag}\{\lambda_{jk}^+, \lambda_{jk}^-\}, \quad (1.29)$$

and we conjugate by $P$ the matrices $M_\ell, M_k, M_j, M_i$ as follows:

$$P(M_\ell, M_k, M_j, M_i)P^{-1} = (U, V, W, T). \quad (1.30)$$

Since, $W = \Lambda_{jk}^{-1}$, we proceed with the parametrization of the matrices $U, V, T$. First, we parametrize the diagonal elements of $U, V, T$. Indeed, solving the equations $\text{Tr } U = p_\ell$ and $\text{Tr } \Lambda_{jk}U = p_{jkl}$, we get the diagonal elements of $U$:

$$\begin{cases}
    u_{11} = \frac{p_{jkl} - p_k\lambda_{jk}^+}{r_{jk}}, \\
    u_{22} = -\frac{p_{jkl} - p_k\lambda_{jk}^-}{r_{jk}}.
\end{cases} \quad (1.31)$$

Next, solving the equations $\text{Tr } V = p_k$ and $\text{Tr } \Lambda_{jk}V^{-1} = p_j$, we obtain the diagonal elements of $V$:

$$\begin{cases}
    v_{11} = \frac{p_j - p_k\lambda_{jk}^+}{r_{jk}}, \\
    v_{22} = \frac{p_j - p_k\lambda_{jk}^-}{r_{jk}}.
\end{cases} \quad (1.32)$$

Finally, equations $\text{Tr } T = p_i$ and $\text{Tr } TWV = \text{Tr } TA_{jk} = p_{ijk}$, determine the diagonal elements of $T$:

$$\begin{cases}
    t_{11} = \frac{p_{ijk} - p_i\lambda_{jk}^+}{r_{jk}}, \\
    t_{22} = -\frac{p_{ijk} - p_i\lambda_{jk}^-}{r_{jk}}.
\end{cases} \quad (1.33)$$

At this point, we calculate the off-diagonal elements of $U, V, T$ respec-
tively. Consider the matrix $U$. Once calculated the diagonal elements of $U$, since $\det U = 1$, then the following identity holds:

$$u_{12}u_{21} = -\frac{g(p_{jk}, p_{\ell}, p_{jk\ell})}{r_{jk}^2},$$  \hspace{1cm} (1.34)

and we suppose $g(p_{jk}, p_{\ell}, p_{jk\ell}) \neq 0$. This leads us to define the open subset $U^{(0)}_{jk}$, as follows:

$$U^{(0)}_{jk} := \mathcal{M}_{\mathbb{C}^2} \cap \{(p_{jk}^2 - 4)g(p_{jk}, p_{\ell}, p_{jk\ell}) \neq 0\}. \hspace{1cm} (1.35)$$

Moreover, note that, since $P$ is unique up to left multiplication by a diagonal matrix $D \in \text{GL}_2(\mathbb{C})$, we are allowed to fix $u_{21} = 1$. Then equation (1.34) gives us the element $u_{12}$. Next, consider the matrix $V$. The system of equations $\text{Tr} \ V \ U = p_{kl}$ and $\text{Tr} \ A_{jk} V^{-1} U = p_{jk\ell}$ gives us a parametrization for the off-diagonal elements of $V$:

$$\begin{cases}
\begin{align*}
v_{12} &= -\frac{y_{ik} - y_{ij} \lambda_{jk}}{r_{jk}^2}, \\
v_{21} &= \frac{y_{ik} - y_{ij} \lambda_{jk}^{+}}{g(p_{jk}, p_{\ell}, p_{jk\ell})},
\end{align*}
\end{cases} \hspace{1cm} (1.36)$$

where $y_{ik}$ and $y_{ij}$ are defined in (1.24) and (1.25) respectively. Finally we calculate the off-diagonal elements of the matrix $T$. Consider the system of equations $\text{Tr} \ T \ U = p_{id}$ and $\text{Tr} \ T \ W \ V \ U = \text{Tr} \ T A_{jk} U = p_{ijkl}$, then we have the following parametrization for $t_{12}$ and $t_{21}$:

$$\begin{cases}
\begin{align*}
t_{12} &= -\frac{y_{id} + y_{ijkl} \lambda_{jk}}{r_{jk}^2}, \\
t_{21} &= \frac{y_{id} + y_{ijkl} \lambda_{jk}^{+}}{g(p_{jk}, p_{\ell}, p_{ijkl})},
\end{align*}
\end{cases} \hspace{1cm} (1.37)$$

where $y_{id}$ and $y_{ijkl}$ are defined in (1.26) and (1.27) respectively. This con-
Theorem 3 shows that $p$, defined in (1.3), parametrizes the following open subset of $\mathcal{M}_G$:

$$\bigcup_{j > k} U_{jk}^{(0)} \cup U_{jk}^{(1)} \cup U_{jk}^{(2)}$$  \hspace{1cm} (1.38)

We now show that, when the monodromy group is not reducible, and none of the monodromy matrices $M_1, \ldots, M_4$ is a multiple of the identity, it is possible to parametrize the monodromy matrices in terms of $p$, defined in (1.3) and (1.4), also outside of the open subset (1.38).

**Lemma 4.** Let $(M_1, \ldots, M_4) \in \mathcal{M}_G$ and define the complex quantities (1.3) as in (1.4). Assume that: none of the matrices $M_1, \ldots, M_4$ is a multiple of the identity, the monodromy group is not reducible, and $p_{jk} \neq \pm 2$ for at least one choice of $j \neq k, j, k = 1, \ldots, 4$. Moreover, assume that

$$g(p_{jk}, p_\ell, p_{jk\ell}) = g(p_j, p_k, p_{jk}) = g(p_{jk}, p_\ell, p_{jk\ell}) = 0, \hspace{1cm} (1.39)$$

then there exists at least an index $\ell$ for which $p_{jk} \neq \lambda_\ell \lambda_k + \frac{1}{\lambda_\ell \lambda_k}$ and a global conjugation $P \in \mathrm{GL}_2(\mathbb{C})$ such that:

$$PM_kP^{-1} = \begin{pmatrix} \lambda_k & 1 \\ 0 & \frac{1}{\lambda_k} \end{pmatrix}, \hspace{1cm} (1.40)$$

$$PM_jP^{-1} = \begin{pmatrix} \lambda_j & -\lambda_j \lambda_k \\ 0 & \frac{1}{\lambda_j} \end{pmatrix}, \hspace{1cm} (1.41)$$

$$PM_{\ell}P^{-1} = \begin{pmatrix} \lambda_\ell & 0 \\ p_{jk} - \lambda_\ell \lambda_k - \frac{1}{\lambda_\ell \lambda_k} & \frac{1}{\lambda_\ell} \end{pmatrix}, \hspace{1cm} (1.42)$$
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

$$PM_iP^{-1} = \begin{cases} 
\begin{pmatrix} 
\lambda_i & 0 \\
p_{ik} - \lambda_i \lambda_k - \frac{1}{\lambda_i \lambda_k} & \frac{1}{\lambda_i} 
\end{pmatrix}, & \text{for } p_{il} = \lambda_i \lambda_\ell + \frac{1}{\lambda_i \lambda_\ell}, \\
\begin{pmatrix} 
\lambda_i & \frac{p_{il} - \lambda_i \lambda_\ell - \frac{1}{\lambda_i \lambda_\ell}}{p_{lk} - \lambda_\ell \lambda_k - \frac{1}{\lambda_\ell \lambda_k}} \\
0 & \frac{1}{\lambda_i} 
\end{pmatrix}, & \text{for } p_{il} \neq \lambda_i \lambda_\ell + \frac{1}{\lambda_i \lambda_\ell}, 
\end{cases}$$

(1.43)

where $\lambda_s + \frac{1}{\lambda_s} = p_s$, $\forall s = 1, \ldots, 4$.

**Proof.** Proceeding as before, we bring the product matrix $M_jM_k$ into the diagonal form (1.29). Condition (1.39) implies that the following equations must be satisfied:

$$(M_1)_{12}(M_1)_{21} = (M_2)_{12}(M_2)_{21} = (M_3)_{12}(M_3)_{21} = (M_4)_{12}(M_4)_{21} = 0.$$

By global conjugation by a permutation matrix, we can assume that $(M_k)_{12} \neq 0$ and then by global diagonal conjugation we can put $M_k$ in the form (1.40). Then, since $M_j = \Lambda_{jk}M_k^{-1}$ we immediately obtain (1.41).

Now, since the monodromy group must be irreducible, one of the two remaining matrices, call it $M_\ell$, must have non zero 21 entry. Then since $\text{Tr}(M_\ell M_k) = p_{ik}$, we obtain $(M_\ell)_{21} = p_{ik} - \lambda_\ell \lambda_k - \frac{1}{\lambda_\ell \lambda_k} \neq 0$, and therefore:

$$M_\ell = \begin{pmatrix} 
\lambda_\ell & 0 \\
p_{ik} - \lambda_\ell \lambda_k - \frac{1}{\lambda_\ell \lambda_k} & \frac{1}{\lambda_\ell} 
\end{pmatrix}.$$ 

Now, if the last matrix is also lower triangular, by imposing $\text{Tr} M_\ell M_k = p_{ik}$, we obtain the first formula in (1.43), and it is immediate to check that $p_{il} = \lambda_i \lambda_\ell + \frac{1}{\lambda_i \lambda_\ell}$. Otherwise, if $M_i$ is upper triangular, by imposing $\text{Tr} M_i M_\ell = p_{il}$, we obtain the second formula (1.43), and it is immediate to
check that \( p_{il} \neq \lambda_i \lambda_l + \frac{1}{\lambda_i \lambda_l} \). \( \square \)

**Proposition 5.** Let \((M_1, \ldots, M_4) \in \hat{M}_{G_2}\) and define the complex quantities (1.3) as in (1.4). Assume that: none of the matrices \( M_1, \ldots, M_4 \) is a multiple of the identity, the monodromy group is not reducible, and \( p_{jk} = 2 \epsilon_{jk} \) for all \( j, k = 1, \ldots, 4 \), where \( \epsilon_{jk} = \pm 1 \). Then, if at least one matrix \( M_i \) is diagonalizable, there exists a choice of the ordering of the indices \( i, j, k, \ell \in \{1, 2, 3, 4\} \) and a global conjugation \( P \in \text{GL}_2(\mathbb{C}) \) such that the following parametrization holds true:

\[
PM_i P^{-1} = \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix}, \quad \lambda_i \neq \pm 1, \quad \lambda_i + \frac{1}{\lambda_i} = p_i, \tag{1.44}
\]

\[
PM_k P^{-1} = \begin{pmatrix} \frac{p_k - 2 \epsilon_{ki} \lambda_i}{\lambda_i - 1} & \frac{(p_k \lambda_i - \epsilon_{ki} (\lambda_i^2 + 1))^2}{(\lambda_i^2 - 1)^2} \\ 1 & \frac{\lambda_i (p_k \lambda_i - 2 \epsilon_{ki})}{\lambda_i - 1} \end{pmatrix}, \tag{1.45}
\]

\[
PM_j P^{-1} = \begin{pmatrix} \frac{p_j - 2 \epsilon_{kj} \lambda_i}{\lambda_i - 1} & \frac{(p_j \lambda_i - \epsilon_{kj} (\lambda_i^2 + 1))^2}{(\lambda_i^2 - 1)^2} \\ \frac{(\lambda_i^2 - 1)(2 \epsilon_{kj} - p_k \lambda_i \lambda_i + (p_k \lambda_i - 2 \epsilon_{ki}) (p_j \lambda_i - 2 \epsilon_{kj})}{(p_k \lambda_i - \epsilon_{ki} (\lambda_i^2 + 1))^2} & \frac{\lambda_i (p_j \lambda_i - 2 \epsilon_{kj})}{\lambda_i - 1} \end{pmatrix}, \tag{1.46}
\]

\[
PM_\ell P^{-1} = \begin{pmatrix} \frac{p_\ell - 2 \epsilon_{i\ell} \lambda_i}{\lambda_i - 1} & \frac{(p_\ell \lambda_i - \epsilon_{i\ell} (\lambda_i^2 + 1))^2}{(\lambda_i^2 - 1)^2} \\ \frac{(\lambda_i^2 - 1)(2 \epsilon_{i\ell} - p_k \lambda_i \lambda_i + (p_k \lambda_i - 2 \epsilon_{ki}) (p_\ell \lambda_i - 2 \epsilon_{i\ell})}{(p_k \lambda_i - \epsilon_{ki} (\lambda_i^2 + 1))^2} & \frac{\lambda_i (p_\ell \lambda_i - 2 \epsilon_{i\ell})}{\lambda_i - 1} \end{pmatrix}. \tag{1.47}
\]

If none of the monodromy matrices is diagonalizable, then there exists a
choice of the ordering of the indices \(i, j, k, \ell \in \{1, 2, 3, 4\}\) and a global conjugation \(P \in \text{GL}_2(\mathbb{C})\) such that the following parametrization holds true:

\[
PM_i P^{-1} = \begin{pmatrix} \epsilon_i & 1 \\ 0 & \epsilon_i \end{pmatrix}, \quad PM_j P^{-1} = \begin{pmatrix} \epsilon_j & 0 \\ 4\epsilon_{ij} & \epsilon_j \end{pmatrix}, \quad (1.48)
\]

\[
PM_k P^{-1} = \begin{pmatrix} 
\frac{p_{ijkl} - 2\epsilon_{ik}\epsilon_{j} - 2\epsilon_{jk}\epsilon_{i} + 2\epsilon_{ij}\epsilon_{k}}{4\epsilon_{ij}} & \frac{\epsilon_{jk} - \epsilon_{j}\epsilon_{k}}{2\epsilon_{ij}} \\
2(\epsilon_{ik} - \epsilon_{i}\epsilon_{k}) & \frac{2\epsilon_{ik}\epsilon_{j} + 2\epsilon_{ik}\epsilon_{i} + 8\epsilon_{ij}\epsilon_{k} - 2\epsilon_{ij}\epsilon_{k} - p_{ijkl}}{4\epsilon_{ij}}
\end{pmatrix}, \quad (1.49)
\]

\[
PM_\ell P^{-1} = \begin{pmatrix} 
\frac{p_{ijkl} - 2\epsilon_{il}\epsilon_{j} - 2\epsilon_{jl}\epsilon_{i} + 2\epsilon_{ij}\epsilon_{l}}{4\epsilon_{ij}} & \frac{\epsilon_{lj} - \epsilon_{l}\epsilon_{j}}{2\epsilon_{ij}} \\
2(\epsilon_{il} - \epsilon_{i}\epsilon_{l}) & \frac{2\epsilon_{il}\epsilon_{j} + 2\epsilon_{il}\epsilon_{i} + 8\epsilon_{ij}\epsilon_{l} - 2\epsilon_{ij}\epsilon_{l} - p_{ijkl}}{4\epsilon_{ij}}
\end{pmatrix}. \quad (1.50)
\]

**Proof.** First, let us assume that at least one matrix \(M_i\) is diagonal and work in the basis in which \(M_i\) assumes the form (1.44) with \(\lambda_i \neq \pm 1\).

Let \(j \neq i\), then we have a set of linear equations in the diagonal elements of \(M_j\):

\[
\text{Tr}(M_i M_j) = 2\epsilon_{ji}, \quad \text{Tr} M_j = p_j, \quad \epsilon_{ji} = \pm 1,
\]

that it is solved by

\[
(M_j)_{11} = -\frac{p_j - 2\epsilon_{ji} \lambda_i}{\lambda_i^2 - 1}, \quad (M_j)_{22} = \frac{\lambda_i(p_j \lambda_i - 2\epsilon_{ji})}{\lambda_i^2 - 1}, \quad (1.51)
\]

for \(j = 1, \ldots, 4, j \neq i\).

Since the monodromy group is not reducible, there is at least one matrix \(M_k, k \neq i\) such that in the chosen basis, \((M_k)_{21} \neq 0\), then we use the freedom of global diagonal conjugation to set \((M_k)_{21} = 1\). Since \(\det(M_k) = 1\) we obtain the formula (1.45). Observe that:

\[
\frac{(p_k \lambda_i - \epsilon_k \lambda_i^2 + 1))^2}{(\lambda_i^2 - 1)^2} \neq 0
\]
otherwise \( p_k = \epsilon_{ki} p_i \) and by using \( \text{Tr} M_i M_k = 2 \epsilon_{ki} \) we would find \( \lambda_i = \pm 1 \).

We now deal with the other two matrices. We only need to find the off-diagonal elements of these matrices. To this aim we use the following equations for \( s = j, \ell \):

\[
\text{Tr}(M_s M_k) = 2 \epsilon_{sk}, \quad \text{Tr}(M_i M_k M_s) = p_{iks},
\]

which, combined with (1.51) lead to (1.46) and (1.45). This concludes the proof of the first case.

To prove the second case, assume none of the matrices \( M_1, \ldots, M_4 \) are diagonalizable, then \( \text{eigen}(M_i) = \{ \epsilon_i, \epsilon_i \} \), \( \forall i = 1, \ldots, 4 \), where \( \epsilon_i = \pm 1 \). Let us choose a global conjugation such that one of the matrices \( M_i \) is in upper triangular form as in (1.48).

Now, since the monodromy group is not reducible, there exists at least one \( j \) such that \( (M_j)_{21} \neq 0 \). From \( \text{Tr} M_i M_j = 2 \epsilon_{ij} \) we have \( 2 \epsilon_i \epsilon_j + (M_j)_{21} = 2 \epsilon_{ij} \), so that \( (M_j)_{21} \neq 0 \) implies \( \epsilon_i \epsilon_j = -\epsilon_{ij} \). We perform a conjugation by a unipotent upper triangular matrix to impose \( (M_j)_{12} = 0 \), so that the second equation in (1.48).

For all other matrices we use \( \text{Tr} M_i M_s = 2 \epsilon_{is} \) and \( \text{Tr} M_j M_s = 2 \epsilon_{js} \), \( s = k, \ell \) to find:

\[
(M_s)_{21} = 2(\epsilon_{is} - \epsilon_i \epsilon_s), \quad (M_s)_{12} = \frac{\epsilon_{js} - \epsilon_j \epsilon_s}{2 \epsilon_{ij}},
\]

From \( \text{Tr} M_s = 2 \epsilon_s \) and \( \text{Tr}(M_i M_j M_s) = p_{ij s} \) we find the final formula (1.49) for \( s = k \) and (1.50) for \( s = \ell \) respectively.

In the following Theorem we show that \( \mathcal{M}_{G_2} \) can be identified with an affine algebraic variety that is the zero locus of a family \( \mathcal{F} \) of 15 polynomials.
in the ring:

\[ \mathbb{C}[p_1, p_2, p_3, p_4, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}]. \quad (1.52) \]

**Theorem 6.** Consider \( m := (M_1, \ldots, M_4) \in \tilde{\mathcal{M}}_{G_2} \).

(i) The co-adjoint coordinates of \( m \) defined in (1.3) and (1.4) belong to the zero locus of the following 15 polynomials in the ring (1.52):

\[
f_1(p) := p_{32}p_{31}p_{21} + p_3^2 + p_3^2 + p_{21} - (p_1p_{321} + p_2p_{321} - (p_2p_{321} + p_1p_3)p_{31} - (p_3p_{321} + p_1p_2)p_{21} + p_2^2 + p_2^2 + p_1^2 + p_{321}^2 + p_3p_2p_1p_{321} - 4, \]

\[
(1.53)
\]

\[
f_2(p) := p_{42}p_{41}p_{21} + p_4^2 + p_4^2 + p_{21} - (p_1p_{421} + p_2p_{421} - (p_2p_{421} + p_1p_4)p_{41} - (p_4p_{421} + p_1p_2)p_{21} + p_4^2 + p_4^2 + p_1^2 + p_{421}^2 + p_4p_2p_1p_{421} - 4, \]

\[
(1.54)
\]

\[
f_3(p) := p_{43}p_{41}p_{31} + p_4^2 + p_4^2 + p_{31}^2 - (p_1p_{431} + p_3p_4)p_{43} - (p_3p_{431} + p_1p_4)p_{41} - (p_4p_{431} + p_1p_3)p_{31} + p_4^2 + p_3^2 + p_1^2 + p_{431}^2 + p_4p_3p_1p_{431} - 4, \]

\[
(1.55)
\]

\[
f_4(p) := p_{43}p_{42}p_{32} + p_4^2 + p_4^2 + p_{32}^2 - (p_2p_{432} + p_4p_3)p_{43} - (p_3p_{432} + p_2p_4)p_{42} - \]

\[
(1.56)
\]
\[(p_4 p_{432} + p_2 p_3)p_{32} + p_1^2 + p_3^2 + p_2^2 + p_{432}^2 + p_4 p_3 p_2 p_{432} - 4,\]

(1.56)

\[f_5(p) := -2p_\infty + p_1 p_2 p_3 p_4 + p_1 p_{432} + p_2 p_{431} + p_3 p_4 + p_3 p_{41} + p_3 p_{42} -\]

\[p_{21} p_{43} + p_{32} p_{41} - p_1 p_2 p_{43} - p_1 p_4 p_{32} - p_2 p_{3} p_{41} - p_3 p_{4} p_{21} -\]

\[p_4 p_{31},\]

(1.57)

\[f_6(p) := p_2 p_3 p_4 - p_3 p_4 - p_{21} p_3 p_{41} + p_{32} p_{41} - p_3 p_{42} + p_1 p_3 p_{42} -\]

\[p_{31} p_{42} - p_2 p_{43} + p_2 p_{431} + 2p_{432} - p_1 p_\infty,\]

(1.58)

\[f_7(p) := -p_1 p_4 + 2p_{41} + p_{21} p_{42} - p_2 p_{421} + p_{31} p_{43} + p_{21} p_3 p_{43} -\]

\[p_2 p_{321} p_{43} - p_3 p_{431} - p_{21} p_3 p_{432} + p_{321} p_{432} + p_2 p_{5} p_\infty - p_3 p_{5} p_\infty,\]

(1.59)

\[f_8(p) := -p_1 p_2 p_3 + p_{21} p_3 + p_2 p_{31} + p_1 p_{32} - 2p_{321} + p_2 p_{41} p_{43} -\]

\[p_{41} p_{43} - p_2 p_4 p_{431} + p_4 p_{431} - p_4 p_{432} + p_1 p_\infty,\]

(1.60)

\[f_9(p) := -p_1 p_2 + 2p_{21} + p_{31} p_{32} - p_3 p_{321} + p_{41} p_{42} - p_4 p_{421} +\]

\[p_{32} p_{41} p_{43} - p_3 p_{41} p_{431} - p_3 p_{41} p_{432} + p_{431} p_{432} + p_3 p_4 p_\infty -\]

\[p_4 p_\infty,\]

(1.61)

\[f_{10}(p) := -p_1 p_2 p_4 + p_{21} p_4 + p_2 p_{41} + p_1 p_{42} - 2p_{421} + p_1 p_{32} p_{43} -\]

\[p_{32} p_{43} - p_3 p_{431} - p_1 p_3 p_{432} + p_{31} p_{432} + p_3 p_\infty,\]

(1.62)
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

\( f_{11}(p) := p_1 p_3 p_4 - p_3 p_4 - p_{21} p_{32} p_4 + p_2 p_{32} p_4 - p_3 p_{41} - p_3 p_{42} + p_{32} p_{421} - p_1 p_{43} + 2 p_{431} + p_{21} p_{432} - p_2 p_x, \) \tag{1.63}

\( f_{12}(p) := -p_2 p_4 + p_{21} p_{41} + 2 p_{42} - p_1 p_{421} + p_3 p_{43} - p_{32} p_{431} - p_3 p_{432} + p_{31} p_x, \) \tag{1.64}

\( f_{13}(p) := p_1 p_3 - 2 p_{31} - p_{21} p_{32} + p_2 p_{321} - p_{41} p_{43} + p_4 p_{431} + p_{421} p_{432} - p_{42} p_x, \) \tag{1.65}

\( f_{14}(p) := p_2 p_3 - p_{21} p_{31} - 2 p_{32} + p_1 p_{321} - p_{21} p_{41} p_{43} - p_{42} p_{43} + p_{1} p_{421} p_{43} + p_{21} p_{432} - p_{41} p_{431} + p_{42} p_x + p_{41} p_x, \) \tag{1.66}

\( f_{15}(p) := -p_3 p_4 + p_{31} p_{41} + p_{21} p_{32} p_{41} - p_2 p_{32} p_{41} + p_{32} p_{42} - p_{1} p_{32} p_{421} + p_{321} p_{421} + 2 p_{43} - p_{1} p_{431} - p_2 p_{432} + p_1 p_2 p_x - p_{21} p_x. \) \tag{1.67}

(ii) For every given generic \( p_1, p_2, p_3, p_4, p_x \), the affine algebraic variety:

\[ \widehat{\mathcal{M}}_{G_2} = \mathbb{C}[(p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421})]/I, \] \tag{1.68}

where \( I = \langle f_1, \ldots, f_{15} \rangle \), is four dimensional.

Proof. We proceed proving point by point the statement of the Theorem.

(i) We give a detailed proof for the polynomial (1.57), while all the others polynomials are calculated in a similar way and hence we omit their
proof. Before proceeding, it is useful to remind the so-called skein relation:

\[ \text{Tr } AB + \text{Tr } A^{-1}B = \text{Tr } A \text{Tr } B, \ \forall A, B \in \text{SL}_2(\mathbb{C}), \]  
(1.69)

and the following well-known properties of matrices in \( \text{SL}_2(\mathbb{C}) \):

\[ \text{Tr } A^{-1} = \text{Tr } A, \ \forall A \in \text{SL}_2(\mathbb{C}), \]  
(1.70)

\[ \text{Tr } AB = \text{Tr } BA, \ \forall A, B \in \text{SL}_2(\mathbb{C}), \]  
(1.71)

\[ \text{Tr } ABC = \text{Tr } CAB = \text{Tr } BCA, \ \forall A, B, C \in \text{SL}_2(\mathbb{C}). \]  
(1.72)

We can now start the proof of (1.57). Firstly rewrite (1) as:

\[ M_4 M_3 M_1 = (M_1^{-1} M_2 M_1 M_\infty)^{-1}. \]  
(1.73)

Then apply the trace operator:

\[ \text{Tr } M_4 M_3 M_1 = \text{Tr } M_1^{-1} M_2 M_1 M_\infty, \]  
(1.74)

and expand the right hand side of (1.74) using rules (1.69) and (1.72):

\[
\begin{align*}
\text{Tr } M_4^{-1} M_2 M_1 M_\infty &= \text{Tr } M_4^{-1} M_2 M_1 \text{Tr } M_\infty - \text{Tr } M_4 M_3 M_2 M_1 = \\
\text{Tr } M_2 M_1 M_\infty - \text{Tr } M_2 M_1 \text{Tr } M_4 M_3 M_2 + \text{Tr } M_4 M_3 M_2 M_1^{-1} M_2^{-1} = \\
\text{Tr } M_2 M_1 M_\infty - \text{Tr } M_2 M_4 M_3 M_2 + \text{Tr } M_4 M_2^{-1} \text{Tr } M_3 M_2 M_1^{-1} - \\
\text{Tr } M_4^{-1} M_2 M_3 M_1^{-1} &= \text{Tr } M_2 M_1 M_\infty - \text{Tr } M_2 M_1 \text{Tr } M_4 M_3 M_2 + \\
(\text{Tr } M_4 \text{Tr } M_2 - \text{Tr } M_4 M_2)(\text{Tr } M_3 M_2 \text{Tr } M_1 - \text{Tr } M_3 M_2 M_1) - \\
\text{Tr } M_4^{-1} M_2 M_3 \text{Tr } M_2 M_1^{-1} + \text{Tr } M_2 M_3 M_1 M_2^{-1} M_4^{-1} =
\end{align*}
\]
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

\[
\begin{align*}
\text{Tr} M_2 \text{Tr} M \text{-} \text{Tr} M_2 M_1 \text{Tr} M_3 M_2 \text{+}
(\text{Tr} M_4 \text{Tr} M_2 - \text{Tr} M_4 M_2)(\text{Tr} M_3 M_2 \text{Tr} M_1 - \text{Tr} M_3 M_2 M_1) - \\
(\text{Tr} M_4 \text{Tr} M_3 M_2 - \text{Tr} M_4 M_2 M_3)(\text{Tr} M_2 \text{Tr} M_1 - \text{Tr} M_2 M_1) + \\
\text{Tr} M_3 M_1 M_2 \text{Tr} M_4 M_2 - \text{Tr} M_2 M_3 M_1 M_4 M_2 = \\
\text{Tr} M_2 \text{Tr} M - \text{Tr} M_2 M_1 \text{Tr} M_4 M_3 M_2 \text{+}
\text{Tr} M_3 M_1 M_2 \text{Tr} M_4 M_2 M_3 M_1 + \text{Tr} M_4 M_3 M_1.
\end{align*}
\]

(1.75)

The traces $\text{Tr} M_3 M_1 M_2$, $\text{Tr} M_4 M_2 M_3$ and $\text{Tr} M_4 M_2 M_3 M_1$, satisfy the following relations:

\[
\begin{align*}
\text{Tr} M_3 M_1 M_2 &= \text{Tr} M_3 \text{Tr} M_2 M_1 + \text{Tr} M_2 \text{Tr} M_3 M_1 + \text{Tr} M_1 \text{Tr} M_3 M_2 - \\
\text{Tr} M_3 \text{Tr} M_2 \text{Tr} M_1 - \text{Tr} M_3 M_2 M_1, \quad (1.76) \\
\text{Tr} M_4 M_2 M_3 &= \text{Tr} M_4 \text{Tr} M_3 M_2 + \text{Tr} M_3 \text{Tr} M_4 M_2 + \text{Tr} M_2 \text{Tr} M_4 M_3 - \\
\text{Tr} M_4 \text{Tr} M_3 \text{Tr} M_2 - \text{Tr} M_4 M_3 M_2, \quad (1.77) \\
\text{Tr} M_4 M_2 M_3 M_1 &= \text{Tr} M_4 M_2 M_1 \text{Tr} M_3 - \text{Tr} M_3 M_2^{-1} M_4^{-1} M_1^{-1} = \\
\text{Tr} M_4 M_2 M_1 \text{Tr} M_3 - \text{Tr} M_3 M_2^{-1} \text{Tr} M_4 M_1 + \text{Tr} M_2 M_3^{-1} M_4^{-1} M_1^{-1} = \\
\text{Tr} M_4 M_2 M_1 \text{Tr} M_3 - (\text{Tr} M_3 \text{Tr} M_2 - \text{Tr} M_3 M_2) \text{Tr} M_4 M_1 + \\
\text{Tr} M_2 \text{Tr} M_4 M_3 M_1 - \text{Tr} M_4 M_3 M_2 M_1. \quad (1.78)
\end{align*}
\]

Substitute back in (1.75) the equations (1.76)-(1.78) and apply the definitions given in (1.4), in order to get the following relation:

\[
p_2 (-p_1 p_2 p_3 p_4 + p_2 p_3 p_4 + p_1 p_3 p_4 - p_3 p_1 p_4 + p_2 p_3 p_4 - p_3 p_4) +
\]
\( p_{31}p_{42} - p_{31}p_{421} + p_{1}p_{2}p_{43} - p_{21}p_{43} - p_{2}p_{431} - p_{1}p_{432} + 2p_x \) = 0. \tag{1.79}

Since \( p \in \hat{\mathcal{M}}_{G_2} \) is arbitrary, then (1.79) must be true independently from the value of \( p_2 \), then \( f_{5}(p) = 0 \).

(ii) For given \( p_1, p_2, p_3, p_4, p_x \), we used Macaulay2 [GS], a software for algebraic geometry, in order to compute the dimension of the algebraic variety:

\[ \hat{\mathcal{M}}_{G_2} = \mathbb{C}[(p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421})]/I. \tag{1.80} \]

The result is that (1.80) has dimension four.

This concludes the proof.

\[ \Box \]

**Corollary 7.** The quantities \((p_{21}, \ldots, p_{43}, p_{321}, \ldots, p_{421})\) give a set of overdetermined coordinates on the open subset \( \mathcal{U} \subset \hat{\mathcal{M}}_{G_2} \), where:

\[ \mathcal{U} = \{(M_1, \ldots, M_4) \in \hat{\mathcal{M}}_{G_2}|\langle M_1, \ldots, M_4 \rangle \text{ irreducible}, \]
\[ M_i \neq \pm 1, \forall i = 1, \ldots, 4, \} / \sim, \tag{1.81} \]

**Proof.** Thanks to Theorem 3, Lemma 4 and Proposition 5 the quantities \( p_i, p_{ij}, p_{ijk} \) parameterize the monodromy matrices up to global conjugation. Thanks to Theorem 6 for every fixed choice of \( p_1, p_2, p_3, p_4, p_x \) only 4 among the quantities \( p_{ij}, p_{ijk} \) for \( i, j, k = 1, \ldots, 4, i > j > k \), are independent. This concludes the proof. \[ \Box \]
1.2 Braid group action on $\mathcal{M}_{g_2}$

The braid group $B_n$, $n \in \mathbb{N}$, was firstly introduced by Artin in [Art25]. $B_n$ is defined as the infinite group that can be generated by $n - 1$ elementary braids $\sigma_i$, for $i = 1, \ldots, n - 1$, and each $\sigma_i$ is a collection of $n$-strands such that the $i$-th strand pass over the $(i + 1)$-th strand.

**Definition 8.** The so-called Artin’s presentation of $B_n$ is given by:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ 1 \leq i \leq n-2, \sigma_i \sigma_j = \sigma_j \sigma_i,\ |i-j| > 1 \rangle. \quad (1.82)$$

There exists a natural surjective group homomorphism between $B_n$ and the symmetric group $S_n$. The kernel of this homomorphism is denoted $P_n$ and is called the pure braid group. A complete set of generators for $P_n$ is given by formulae:

$$\beta_{ij} = \sigma_{i-1}^{-1} \sigma_{i-2}^{-1} \cdots \sigma_{j+1}^{-1} \sigma_j \sigma_{j+1} \cdots \sigma_{i-1}^{-1} \sigma_i,\ 1 \leq j < i \leq n, \quad (1.83)$$

and relations:

$$\beta_{rs}^{-1} \beta_{ij} \beta_{rs} = \begin{cases} 
\beta_{ij}, & \text{if } j < s < r < i, \\
\beta_{rj}^{-1} \beta_{ij} \beta_{rj}, & \text{if } s < r < j < i, \\
\beta_{j}^{-1} \beta_{sj}^{-1} \beta_{ij} \beta_{sj} \beta_{rj}, & \text{if } j = s < r < i, \\
\beta_{rj}^{-1} \beta_{sj}^{-1} \beta_{ij} \beta_{sj} \beta_{rj}^{-1} \beta_{sj} \beta_{rj}, & \text{if } s < j < r < i. 
\end{cases} \quad (1.84)$$

We show now that formulae given in (4) express the action of the braid
Lemma 9. Formulae (4) define an action of the braid group $B_4$ over $\widehat{M}_{G_2}$.

Proof. Firstly, we prove that the $\sigma_i$ for $i = 1, 2, 3$, see (4), define an action over $\widehat{M}_{G_2}$, i.e. $\sigma_i(\widehat{M}_{G_2}) = \widehat{M}_{G_2}$. In order to do this, it is sufficient to prove that the cyclic relation (1) is preserved by the action. If we consider $(M'_1, \ldots, M'_4) = \sigma_i(M_1, \ldots, M_4)$, then, for every $i = 1, 2, 3$, the $M'_i$ satisfy the cyclic relation and consequently the action is well defined.

Next we prove that the $\sigma_i$ are generators of the braid group $B_4$. Suppose $m = (M_1, \ldots, M_4) \in \widehat{M}_{G_2}$, it is straightforward calculation to check that the $\sigma_i$ satisfy the so-called “braid relations”:

$$\begin{align*}
\sigma_1\sigma_3(m) &= \sigma_3\sigma_1(m), \\
\sigma_1\sigma_2\sigma_1(m) &= \sigma_2\sigma_1\sigma_2(m), \\
\sigma_2\sigma_3\sigma_2(m) &= \sigma_3\sigma_2\sigma_3(m).
\end{align*}$$

Then the $\sigma_i$ generate the full braid group $B_4$. $\square$

The action of $\sigma_i$ in (4) can be expressed in terms of co-adjoint coordinates (1.4) on $U \subset \widehat{M}_{G_2}$. This is given by:

$$\begin{align*}
\sigma_1 : (p_1, p_2, p_3, p_4, p_x, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{341}, p_{421}) &\mapsto \\
(p_2, p_1, p_3, p_4, p_x, p_{21}, p_{32}, p_1p_3 - p_{31} - p_{21}p_{32} + p_2p_{321}, p_{42}, \\
p_1p_4 - p_{41} - p_{21}p_{42} + p_2p_{421}, p_{43}, p_{321}, p_1p_{43} - p_{431} - p_{21}p_{432} + p_2p_x, \\
p_{432}, p_{421}), \\
\sigma_2 : (p_1, p_2, p_3, p_4, p_x, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{341}, p_{421}) &\mapsto \\
(p_1, p_3, p_2, p_4, p_x, p_{31}, p_1p_2 - p_{21} - p_{31}p_{32} + p_{32}p_{321}, p_{32}, p_{41}, p_{43}.
\end{align*}$$
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

$$p_2p_4 - p_42 - p_32p_43 + p_3p_432, p_321, p_432, p_2p_41 - p_421 - p_32p_431 + p_3p_\infty, p_431),$$

$$\sigma_3 : (p_1, p_2, p_3, p_4, p_\infty, p_21, p_31, p_32, p_41, p_42, p_43, p_321, p_432, p_431, p_421) \mapsto$$

$$(p_1, p_2, p_4, p_3, p_\infty, p_21, p_41, p_42, p_1p_3 - p_31 - p_41p_43 + p_4p_431,$$

$$p_2p_3 - p_32 - p_42p_43 + p_4p_432, p_43, p_421, p_432, p_431, p_321, p_432, p_431, p_421,$$

$$p_21p_3 - p_321 - p_421p_43 + p_4p_\infty), \quad (1.86)$$

With the following Lemma we prove that the braids, defined in (1.86), still define the action of $B_4$ over the co-adjoint coordinates defined in the open subset $\mathcal{U} \subset \widetilde{\mathcal{M}}_{G_2}$:

**Lemma 10.** Formulae (1.86) define the action of the braid group $B_4$ over the open subset $\mathcal{U} \subset \widetilde{\mathcal{M}}_{G_2}$.

**Proof.** As in the previous proof, we begin proving that action (1.86) is well defined, namely consider the ideal $I = \langle \mathcal{F} \rangle = \{f_1, \ldots, f_{15}\}$, where $f_i$ for $i = 1, \ldots, 15$ are given in (1.53)-(1.67), then $I$ is invariant under the $B_4$ action, i.e. for every $i = 1, 2, 3$:

$$\sigma_i(I) = I.$$

Consider $p \in \mathcal{U} \subset \widetilde{\mathcal{M}}_{G_2}, \sigma_i$ and $f_j \in \mathcal{F}$. We proceed computing $f_j(\sigma_i(p))$ for $i = 1, \ldots, 4$ and $j = 1, \ldots, 15$. For $\sigma_1$, we obtain:

$$f_1(\sigma_1(p)) = f_1(p),$$

$$f_2(\sigma_1(p)) = f_2(p),$$

$$f_3(\sigma_1(p)) = f_4(p),$$

$$f_4(\sigma_1(p)) = f_3(p) + (p_{21}p_{42} - p_{2p_{421}})f_6(p) + (p_{21}p_{431} - p_{2p_{\infty}})f_{11}(p)+$$
\[(p_2p_{321} - p_{21}p_{32})f_{13}(p),\]
\[f_5(\sigma_1(p)) = f_1(p) - p_2f_7(p),\]
\[f_6(\sigma_1(p)) = f_8(p) - p_{21}f_2(p),\]
\[f_7(\sigma_1(p)) = f_3(p) + p_2f_0(p),\]
\[f_8(\sigma_1(p)) = f_4(p) - p_{42}f_2(p) - p_{432}f_7(p) + p_{32}f_9(p),\]
\[f_9(\sigma_1(p)) = f_5(p) - p_2f_2(p),\]
\[f_{10}(\sigma_1(p)) = -f_7(p),\]
\[f_{11}(\sigma_1(p)) = f_6(p) - p_{21}f_7(p),\]
\[f_{12}(\sigma_1(p)) = -f_2(p),\]
\[f_{13}(\sigma_1(p)) = -p_{21}f_6(p) + f_{10}(p),\]
\[f_{14}(\sigma_1(p)) = -f_9(p),\]
\[f_{15}(\sigma_1(p)) = f_{11}(p) + p_1f_7(p).

While, for the inverse \(\sigma_1^{-1}\), we obtain:

\[f_1(\sigma_1^{-1}(p)) = f_1(p),\]
\[f_2(\sigma_1^{-1}(p)) = f_2(p),\]
\[f_3(\sigma_1^{-1}(p)) = f_4(p) + (p_{21}p_{321} - p_{1}p_{x})f_6(p) + (p_{21}p_{41} - p_{1}p_{421})f_{12}(p) + (p_{1}p_{321} - p_{21}p_{31})f_{14}(p),\]
\[f_4(\sigma_1^{-1}(p)) = f_3(p),\]
\[f_5(\sigma_1^{-1}(p)) = f_5(p) - p_1f_6(p),\]
\[f_6(\sigma_1^{-1}(p)) = f_{11}(p) - p_{21}f_6(p),\]
\[f_7(\sigma_1^{-1}(p)) = -f_{12}(p),\]
\[f_8(\sigma_1^{-1}(p)) = f_8(p) + p_1f_{14}(p),\]
\[f_9(\sigma_1^{-1}(p)) = f_9(p) - p_{431}f_6(p) - p_{41}f_{12}(p) + p_{31}f_{14}(p).\]
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

\[
f_{10}(\sigma_1^{-1}(p)) = f_{10}(p) - p_1 f_{12}(p),
\]
\[
f_{11}(\sigma_1^{-1}(p)) = -f_6(p),
\]
\[
f_{12}(\sigma_1^{-1}(p)) = f_7(p) - p_{21} f_{12}(p),
\]
\[
f_{13}(\sigma_1^{-1}(p)) = -f_{14}(p),
\]
\[
f_{14}(\sigma_1^{-1}(p)) = -p_{21} f_{14}(p) + f_{13}(p),
\]
\[
f_{15}(\sigma_1^{-1}(p)) = f_{15}(p) + p_2 f_6(p).
\]

For $\sigma_2$, we obtain:

\[
f_{1}(\sigma_2(p)) = f_1(p),
\]
\[
f_{2}(\sigma_2(p)) = f_3(p),
\]
\[
f_{3}(\sigma_2(p)) = f_2(p) + (p_{32} p_{31} - p_{32} p_{321}) f_8(p) + (p_3 p_{20} - p_{32} p_{431}) f_9(p) +
(p_{32} p_{43} - p_{32} p_{432}) f_{12}(p),
\]
\[
f_{4}(\sigma_2(p)) = f_4(p),
\]
\[
f_{5}(\sigma_2(p)) = f_1(p) + p_3 f_5(p),
\]
\[
f_{6}(\sigma_2(p)) = f_2(p) - p_2 f_5(p),
\]
\[
f_{7}(\sigma_2(p)) = f_3(p) - p_3 f_4(p),
\]
\[
f_{8}(\sigma_2(p)) = -p_{32} f_4(p) - f_9(p),
\]
\[
f_{9}(\sigma_2(p)) = -p_{32} f_5(p) - f_7(p),
\]
\[
f_{10}(\sigma_2(p)) = f_6(p) + p_3 f_8(p),
\]
\[
f_{11}(\sigma_2(p)) = f_5(p),
\]
\[
f_{12}(\sigma_2(p)) = -p_{32} f_8(p) + f_{11}(p),
\]
\[
f_{13}(\sigma_2(p)) = f_4(p),
\]
\[
f_{14}(\sigma_2(p)) = f_{10}(p) + p_{31} f_4(p) + p_{43} f_8(p) - p_{431} f_5(p),
\]
\[
f_{15}(\sigma_2(p)) = -f_8(p).
\]
While, for the inverse $\sigma_2^{-1}$, we obtain:

$$f_1(\sigma_2^{-1}(p)) = f_1(p),$$
$$f_2(\sigma_2^{-1}(p)) = f_3(p) + (p_3p_4p_2 - p_2p_4) f_{11}(p) + (p_2p_3 - p_{21}p_{32}) f_{13}(p) + (p_3p_4 - p_2p_{43}) f_{15}(p),$$
$$f_3(\sigma_2^{-1}(p)) = f_2(p),$$
$$f_4(\sigma_2^{-1}(p)) = f_4(p),$$
$$f_5(\sigma_2^{-1}(p)) = f_5(p) - p_2 f_{11}(p),$$
$$f_6(\sigma_2^{-1}(p)) = f_6(p) + p_2 f_{15}(p),$$
$$f_7(\sigma_2^{-1}(p)) = f_7(p) + p_3 f_{11}(p),$$
$$f_8(\sigma_2^{-1}(p)) = f_8(p) + p_2 f_{13}(p),$$
$$f_9(\sigma_2^{-1}(p)) = f_{13}(p),$$
$$f_{10}(\sigma_2^{-1}(p)) = f_{11}(p),$$
$$f_{11}(\sigma_2^{-1}(p)) = -f_{10}(p) - p_{32} f_{11}(p),$$
$$f_{12}(\sigma_2^{-1}(p)) = -f_{15}(p),$$
$$f_{13}(\sigma_2^{-1}(p)) = -f_9(p) - p_{32} f_{13}(p),$$
$$f_{14}(\sigma_2^{-1}(p)) = f_{14}(p) + p_{421} f_{11}(p) - p_{21} f_{13}(p) + p_{42} f_{15}(p),$$
$$f_{15}(\sigma_2^{-1}(p)) = f_{12}(p) - p_{32} f_{15}(p).$$

Finally for $\sigma_3$:

$$f_1(\sigma_3(p)) = f_2(p),$$
$$f_2(\sigma_3(p)) = f_1(p) + (p_4p_2 - p_{421}p_{43}) f_{17}(p) + (p_{4p431} - p_{43}p_{41}) f_{13}(p) + (p_4p_{432} - p_{42}p_{43}) f_{14}(p),$$
$$f_3(\sigma_3(p)) = f_3(p),$$
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

$$f_4(\sigma_3(p)) = f_4(p),$$
$$f_5(\sigma_3(p)) = f_1(p) + p_4f_3(p),$$
$$f_6(\sigma_3(p)) = f_0(p),$$
$$f_7(\sigma_3(p)) = f_5(p) - p_{43}f_3(p),$$
$$f_8(\sigma_3(p)) = f_4(p) - p_3f_3(p),$$
$$f_9(\sigma_3(p)) = -f_3(p),$$
$$f_{10}(\sigma_3(p)) = f_6(p) - p_4f_{10}(p),$$
$$f_{11}(\sigma_3(p)) = f_7(p) - p_4f_9(p),$$
$$f_{12}(\sigma_3(p)) = f_{10}(p),$$
$$f_{13}(\sigma_3(p)) = -f_2(p) - p_{43}f_9(p),$$
$$f_{14}(\sigma_3(p)) = -p_{43}f_{10}(p) - f_8(p),$$
$$f_{15}(\sigma_3(p)) = f_{11}(p) + p_{42}f_3(p) + p_{41}f_9(p) + p_{42}f_{10}(p).$$

While, for the inverse $\sigma_3^{-1}$, we obtain:

$$f_1(\sigma_3^{-1}(p)) = f_2(p) + (p_{43}p_{31} - p_{3}p_{431})f_7(p) + (p_{3}p_{43} - p_{43}p_{321})f_{10}(p) + (p_{43}p_{32} - p_{3}p_{432})f_{12}(p),$$
$$f_2(\sigma_3^{-1}(p)) = f_1(p),$$
$$f_3(\sigma_3^{-1}(p)) = f_3(p),$$
$$f_4(\sigma_3^{-1}(p)) = f_4(p),$$
$$f_5(\sigma_3^{-1}(p)) = f_5(p) + p_4f_{10}(p),$$
$$f_6(\sigma_3^{-1}(p)) = f_6(p) + p_3f_{12}(p),$$
$$f_7(\sigma_3^{-1}(p)) = -p_{43}f_7(p) - f_{13}(p),$$
$$f_8(\sigma_3^{-1}(p)) = -f_{10}(p),$$
$$f_9(\sigma_3^{-1}(p)) = f_9(p) + p_{4}f_{10}(p).$$
Then we conclude that the action (1.86) is well defined over the co-adjoint coordinates defined in $\mathcal{U}$.

In order to prove that $\sigma_i$ for $i = 1, 2, 3$, defined in (1.86), are generators of the braid group $B_4$, the “braid relations” (1.85) must be satisfied. Consider $p \in \mathcal{U} \subseteq \mathcal{M}_G$, as defined in (1.3), then relation:

$$\sigma_1\sigma_3(p) = \sigma_3\sigma_1(p),$$

holds true, and the following relations:

$$\sigma_1\sigma_2\sigma_1(p) = \sigma_2\sigma_1\sigma_2(p),$$
$$\sigma_2\sigma_3\sigma_2(p) = \sigma_3\sigma_2\sigma_3(p),$$

follow from the fact that polynomials (1.63),(1.64),(1.65) and (1.66) are zero for every $p \in \mathcal{U}$.

In order to be consistent, we give explicitly the action of the pure braid group $P_4$, namely we are going to define the generators of the subgroup $P_4 \subset B_4$. By formulae (1.83), the group $P_4$ has generators:

$$\beta_{21} = \sigma_1^2,$$
\[ \beta_{31} = \sigma_2^{-1}\sigma_1^2\sigma_2, \]
\[ \beta_{32} = \sigma_2^2, \]
\[ \beta_{41} = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3, \]
\[ \beta_{42} = \sigma_3^{-1}\sigma_2^2\sigma_3, \]
\[ \beta_{43} = \sigma_3^2. \]

In the last part of this Section, we state a Lemma that is a necessary condition for variables:

\[(p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}),\]

in order to generate a finite orbit under the action of the group \( P_4 \):

**Lemma 11.** Suppose \( p \in \mathcal{U} \subset \mathcal{M}_{G_2} \) is such that it generates a finite \( P_4 \)'s orbit, then only two possibilities arise:

(i) Or \( p \) satisfies:

\[ p_{ij} = 2 \cos \pi r_{ij}, \quad r_{ij} \in \mathbb{Q}, \quad 0 \leq r_{ij} \leq 1, \quad i, j = 1, 2, 3, 4, \quad i > j. \]  \[(1.88)\]

(ii) Or there exists a pure braid \( \beta_{ij} \), for some choice of indices \( i, j, k, \ell = 1, 2, 3, 4 \) such that \( \beta_{ij}(p) = p \). Then \( p_{ij} \) is a complex parameter satisfying:

\[ p_{ki} = \frac{p_{ij}(p_j p_k + p_i p_{ijk}) - 2(p_i p_i + p_{ij} p_{j\ell})}{p_{ij}^2 - 4}, \]  \[(1.89)\]
\[ p_{ij} = \frac{p_{ij}(p_i p_i + p_{ij} p_{j\ell}) - 2(p_i p_j + p_{ij} p_{j\ell})}{p_{ij}^2 - 4}, \]  \[(1.90)\]
\[ p_{ki} = \frac{p_{ij}(p_j p_k + p_i p_{ijk}) - 2(p_k p_i + p_j p_{ijk})}{p_{ij}^2 - 4}, \]  \[(1.91)\]
\[ p_{kj} = \frac{p_{ij}(p_ip_k + p_jp_{ij}) - 2(p_kp_j + p_i p_{ijk})}{p_{ij}^2 - 4}. \] (1.92)

**Proof.** We are going to prove the statement for the generator \( \beta_{21} \), then in a similar way the statement can be proven for all other five generators (1.87).

The braid \( \beta_{21} \) fixes quantities \( p_1, p_2, p_3, p_4, p_{321} \) and \( p_{421} \) and the resulting action on \( (p_1, p_2, p_3, p_4, p_{321}, p_{421}) \) is:

\[
\begin{align*}
\beta_{21}(p_1) &= p_1, \\
\beta_{21}(p_2) &= p_4 - 2p_3 - p_2p_4 + p_2p_{321}, \\
\beta_{21}(p_3) &= p_2p_3 - 2p_3 + p_1p_3 - p_1p_2p_3 + p_1p_{321}, \\
\beta_{21}(p_4) &= p_1p_4 - 2p_1 - p_2p_2 + p_2p_{321}, \\
\beta_{21}(p_{321}) &= p_2p_{321} - 2p_{321} + p_4p_{421} - p_1p_{321} + p_1p_{421} + p_1p_{421}, \\
\beta_{21}(p_{421}) &= p_4p_{421} - 2p_{421} + p_4p_{421} + p_4p_{421} + p_4p_{421} + p_4p_{421}.
\end{align*}
\]

Next we show that the pure braid \( \beta_{21} \) acts as linear transformation on variables \( (p_{31}, p_{32}, p_{41}, p_{42}) \). The cubic relations \( f_1(p) = 0 \) and \( f_2(p) = 0 \) are invariant during the action of \( \beta_{21} \), moreover they are two conics in the variables \( (p_{31}, p_{32}) \) and \( (p_{41}, p_{42}) \) respectively:

\[
\begin{align*}
(p_1^2 + p_2^2 + p_3^2 + p_4^2 \cdot p_{321})p_{321} - (p_3p_2 + p_1p_{321})p_{321} - (p_4p_1 + p_2p_{321})p_{31} - \\
((p_2p_1 + p_3p_{321})p_{321} - (p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_1p_2p_3p_{321}) + 4) &= 0, \\
(1.93) \\
(p_1^2 + p_2^2 + p_3^2 + p_4^2 \cdot p_{421})p_{421} - (p_4p_2 + p_1p_{421})p_{421} - (p_4p_1 + p_2p_{421})p_{41} - \\
((p_2p_1 + p_4p_{421})p_{421} - (p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_1p_2p_4p_{421}) + 4) &= 0. \\
(1.94)
\end{align*}
\]

If \( p_{21} = \pm 2 \) then \( r_{21} = 0 \) or \( r_{21} = 1 \) and the statement follows. Then,
hereafter, we suppose $p_{21} \neq \pm 2$:

(i) The linear action of $\beta_{21}$ on $(p_{31}, p_{32}, p_{41}, p_{42})$ describes simultaneously a rotation $R$ of $(p_{31}, p_{32})$ and $(p_{41}, p_{42})$ on the conics (1.93) and (1.94) respectively. Suppose angle of the rotation $R$ is $\theta$ such that $p_{21} = 2 \cos \theta$ and if $\theta$ is a rational multiple of $\pi$ then:

$$\exists n \in \mathbb{N} \text{ s.t. } R^n = Id. \tag{1.95}$$

As a consequence the action of $\beta_{21}$ produces a finite orbit in $(p_{31}, p_{32})$ and $(p_{41}, p_{42})$ if and only if $q_{21} = 2 \cos \theta$ where $\theta$ is a rational multiple of $\pi$.

(ii) Suppose $p$ to be a fixed point of the braid $\beta_{21}$, i.e. $\beta_{21}(p) = p$, then:

$$p_{32} = \frac{p_{21}(p_1p_3 + p_2p_{32}) - 2(p_3p_2 + p_1p_{32})}{p_{21}^2 - 4}, \tag{1.96}$$

$$p_{31} = \frac{p_{21}(p_2p_3 + p_1p_{32}) - 2(p_3p_1 + p_2p_{32})}{p_{21}^2 - 4}, \tag{1.97}$$

$$p_{42} = \frac{p_{21}(p_1p_4 + p_2p_{42}) - 2(p_4p_2 + p_1p_{42})}{p_{21}^2 - 4}, \tag{1.98}$$

$$p_{41} = \frac{p_{21}(p_2p_4 + p_1p_{42}) - 2(p_4p_1 + p_2p_{42})}{p_{21}^2 - 4}. \tag{1.99}$$

Then $p_{21}$ plays the role of a complex parameter.

This concludes the proof. \qed
1.3 The problem in terms of co-adjoint co-ordinates

The aim of this thesis is to classify all finite orbits:

\[ \mathcal{O}_{P_4}(p) = \{ \beta(p) | \beta \in P_4 \} , \]

where \( p \in \mathcal{U} \subset \hat{\mathcal{M}}_{g_2} \) is the following 15-tuple of complex quantities:

\[ p = (p_1, p_2, p_3, p_4, p_{x}, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{331}, p_{432}, p_{431}, p_{421}) \in \mathbb{C}^{15}, \]

defined in (1.4), and \( P_4 \) is the pure braid group defined in (1.87).

Our classification of finite orbits will be presented modulo symmetries \( \Phi \), where \( \Phi \) is an invertible map such that:

\[ \Phi : \hat{\mathcal{M}}_{g_2} \rightarrow \hat{\mathcal{M}}_{g_2}, \]

and given an element \( p \in \mathcal{U} \subset \hat{\mathcal{M}}_{g_2} \) and its orbit \( \mathcal{O}_{P_4}(p) \), the following is true:

\[ |\mathcal{O}_{P_4}(\Phi(p))| = |\mathcal{O}_{P_4}(p)|. \]

1.4 Restrictions

Our approach is based on the observation that given \( p_i, p_{ij}, p_{ijk} \), defined in (1.4), on the big open subset \( \mathcal{U} \subset \hat{\mathcal{M}}_{g_2} \), defined in (1.38) such that they generate a finite orbit under the action of the pure braid group \( P_4 \), then for any subgroup \( H \subset P_4 \) the action of \( H \) over \( p_i, p_{ij}, p_{ijk} \) produces again a finite orbit. Such restriction \( H \) only acts on some of the \( p_i, p_{ij}, p_{ijk} \) and it
leaves others invariant. We select subgroups $H \subset P_4$ acting on $p_i,p_{ij},p_{ijk}$ in the big open subset $\mathcal{U} \subset \widetilde{\mathcal{M}}_{G_2}$, such that the restricted action is isomorphic to the action of the pure braid group $P_3$ on the $SL_2(\mathbb{C})$ character variety of the Riemann sphere with four boundary components, i.e. $\mathcal{M}_{\text{PV1}}$: indeed in this case all finite orbits of the action of $P_3$ over the quotient space:

$$\widetilde{\mathcal{M}}_{\text{PV1}} := \{ (N_1, N_2, N_3) \in SL_2(\mathbb{C}) \mid N_{∞}N_3N_2N_1 = 1, \quad N_{∞} = \text{diag}(e^{±i\pi\theta_{∞}}), \quad \theta_{∞} \in \mathbb{C} \} / \sim,$$

(1.100)

where $\sim$ is the usual equivalence relation up to global diagonal conjugation, are classified in Lisovyy and Tykhyy’s work [LT14]. This will be discussed in details in Chapter 3.

There exist four well defined restrictions $H_1,\ldots,H_4$ isomorphic to $P_3$ and each of these restrictions allows us to identify some of the $p_i,p_{ij},p_{ijk}$ with coordinates on $\mathcal{M}_{\text{PV1}}$. The four subgroups $H_i$ are defined in the following Theorem:

**Theorem 12.** There exist four subgroups $H_i \subset P_4$ with $i = 1,\ldots,4$, such that they are generated by:

- $H_1 = \langle \beta_{32}, \beta_{43}, \beta_{42} \rangle$,
- $H_2 = \langle \beta_{43}, \beta_{31}, \beta_{41} \rangle$,
- $H_3 = \langle \beta_{21}, \beta_{42}, \beta_{41} \rangle$,
- $H_4 = \langle \beta_{21}, \beta_{32}, \beta_{31} \rangle$,

where generators $\beta_{jk}$, with $j,k = 1,\ldots,4$ and $j > k$, are defined in (1.87). The subgroups $H_i$ satisfy:
(i) $H_i$ is isomorphic to the pure braid group $P_3$ for $i = 1, \ldots, 4$.

(ii) Consider $(M_1, M_2, M_3, M_4) \in \mathcal{M}_{G_2}$ as an ordered 4-tuple of matrices, then each $H_i$, for $i = 1, \ldots, 4$, acts as pure braid group $P_3$ on $\hat{\mathcal{M}}_{PV}$ leaving matrix $M_i$ out of action, where $N_1, N_2, N_3$ are given by:

\begin{align*}
H_1 : \tilde{N}_1 &= M_2, \quad \tilde{N}_2 = M_3, \quad \tilde{N}_3 = M_4, \quad \tilde{N}_x = (M_4M_3M_2)^{-1}, \\
H_2 : \tilde{N}_1 &= M_1, \quad \tilde{N}_2 = M_3, \quad \tilde{N}_3 = M_4, \quad \tilde{N}_x = (M_4M_3M_1)^{-1}, \\
H_3 : \tilde{N}_1 &= M_1, \quad \tilde{N}_2 = M_2, \quad \tilde{N}_3 = M_4, \quad \tilde{N}_x = (M_4M_2M_1)^{-1}, \\
H_4 : \tilde{N}_1 &= M_1, \quad \tilde{N}_2 = M_2, \quad \tilde{N}_3 = M_3, \quad \tilde{N}_x = (M_3M_2M_1)^{-1}. \quad (1.104)
\end{align*}

Proof. We prove explicitly the statements (i) and (ii) for the subgroup:

$$H_1 = \langle \beta_{32}, \beta_{43}, \beta_{42} \rangle \subset P_4,$$

then for the other subgroups a similar proof applies.

(i) We are going to show that generators of $H_1$ satisfy the presentation of the pure braid group $P_3$ given in formulae (1.83)-(1.84), for $n = 3$. In other words generators $\beta_{32}, \beta_{42}, \beta_{43}$ must satisfy:

\begin{align*}
\beta_{32} \beta_{43} \beta_{32}^{-1} &= \beta_{42}^{-1} \beta_{43} \beta_{42}, \\
\beta_{32} \beta_{42} \beta_{32}^{-1} &= \beta_{42}^{-1} \beta_{43}^{-1} \beta_{42} \beta_{43} \beta_{42}. \quad (1.105)
\end{align*}

Relations (1.105) are true and they can be checked by direct computations. This implies the isomorphism between $H_1$ and $P_3$.

(ii) We prove that $\hat{n} = (\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$ is in $\hat{\mathcal{M}}_{PV}$. Suppose $m = (M_1, M_2, M_3, M_4) \in \hat{\mathcal{M}}_{G_2}$ and consider the identities (1.101): by definition of
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

$\widehat{\mathcal{M}}_{G_2}$, matrices $\widehat{N}_i$ for $i = 1, 2, 3, \infty$ are in $SL_2(\mathbb{C})$, moreover, (1.101) transforms the cyclic relation satisfied by $m$:

$$M_x M_4 M_3 M_2 M_1 = 1 \iff M_1 M_x M_4 M_3 M_2 = 1,$$

to the following cyclic relation:

$$\widehat{N}_x \widehat{N}_3 \widehat{N}_2 \widehat{N}_1 = 1,$$

This implies $\widehat{n} \in \widehat{\mathcal{M}}_{PV/I}$. Now we show that the subgroup $H_1$ acts as pure braid group $P_3$ on $\widehat{\mathcal{M}}_{PV/I}$ leaving matrix $M_i$ out of action. Since the generators of $H_1$ are defined in terms of generators $\sigma_2$ and $\sigma_3$ of the full braid group $B_4$, see definition (1.87), then, it is enough to prove that $\sigma_2$ and $\sigma_3$, by (1.101), act as generators of the full braid group $B_3$. Consider (1.101), then the following relations hold:

$$\sigma_2(m) = (M_1, M_3, M_3 M_2 M_3^{-1}, M_4) \simeq (\widehat{N}_2, \widehat{N}_2 \widehat{N}_1 \widehat{N}_2^{-1}, \widehat{N}_3) = \sigma_1^{(PV/I)}(\widehat{n}),$$

$$\sigma_3(m) = (M_1, M_2, M_4, M_4 M_3 M_4^{-1}) \simeq (\widehat{N}_1, \widehat{N}_3, \widehat{N}_3 \widehat{N}_2 \widehat{N}_3^{-1}) = \sigma_2^{(PV/I)}(\widehat{n}).$$

(1.106)

Furthermore, the generators $\sigma_2$ and $\sigma_3$ satisfy the “braid relations” (1.82), and then they generate the braid group $B_3$. Moreover in (1.106) the matrix $M_1$ remains out of the action as expected.

This completes the proof. \qed

To avoid extra complications due to the freedom of global conjugation in (1.101),(1.102),(1.103),(1.104), we consider the action of the subgroups $H_i$ for $i = 1, \ldots, 4$ in terms of co-adjoint coordinates. In order to do this,
we define:

\[ \hat{\mathcal{q}} := (\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_x, \hat{q}_{21}, \hat{q}_{31}, \hat{q}_{32}) \in \mathbb{C}^7, \]
\[ \bar{\mathcal{q}} := (\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_x, \bar{q}_{21}, \bar{q}_{31}, \bar{q}_{32}) \in \mathbb{C}^7, \]
\[ \check{\mathcal{q}} := (\check{q}_1, \check{q}_2, \check{q}_3, \check{q}_x, \check{q}_{21}, \check{q}_{31}, \check{q}_{32}) \in \mathbb{C}^7, \]
\[ \tilde{\mathcal{q}} := (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_x, \tilde{q}_{21}, \tilde{q}_{31}, \tilde{q}_{32}) \in \mathbb{C}^7, \] (1.107)

where \( \hat{q}_i = \text{Tr} \hat{N}_i \) for \( i = 1, 2, 3, \infty \), \( \hat{q}_{jk} = \text{Tr} \hat{N}_j \hat{N}_k \) for \( j > k \), \( j, k = 1, 2, 3 \), etc. As it will be reminded in Chapter 3, the \( \hat{q}, \bar{q}, \check{q} \) are respectively co-adjoint coordinates on the \( \text{SL}_2(\mathbb{C}) \) character variety of the Riemann sphere with four boundary components. Then identifications (1.101)-(1.104) imply:

\[ \hat{\mathcal{q}}_1 = p_2, \hat{\mathcal{q}}_2 = p_3, \hat{\mathcal{q}}_3 = p_4, \hat{\mathcal{q}}_x = p_{432}, \hat{\mathcal{q}}_{21} = p_{32}, \hat{\mathcal{q}}_{31} = p_{42}, \hat{\mathcal{q}}_{32} = p_{43}, \] (1.108)
\[ \bar{\mathcal{q}}_1 = p_1, \bar{\mathcal{q}}_2 = p_3, \bar{\mathcal{q}}_3 = p_4, \bar{\mathcal{q}}_x = p_{431}, \bar{\mathcal{q}}_{21} = p_{31}, \bar{\mathcal{q}}_{31} = p_{41}, \bar{\mathcal{q}}_{32} = p_{43}, \] (1.109)
\[ \check{\mathcal{q}}_1 = p_1, \check{\mathcal{q}}_2 = p_2, \check{\mathcal{q}}_3 = p_4, \check{\mathcal{q}}_x = p_{421}, \check{\mathcal{q}}_{21} = p_{21}, \check{\mathcal{q}}_{31} = p_{41}, \check{\mathcal{q}}_{32} = p_{42}, \] (1.110)
\[ \tilde{\mathcal{q}}_1 = p_1, \tilde{\mathcal{q}}_2 = p_2, \tilde{\mathcal{q}}_3 = p_3, \tilde{\mathcal{q}}_x = p_{321}, \tilde{\mathcal{q}}_{21} = p_{21}, \tilde{\mathcal{q}}_{31} = p_{31}, \tilde{\mathcal{q}}_{32} = p_{32}, \] (1.111)

where \( p_i, p_{ij}, p_{ijk} \) are defined in (1.4). We summarize identities (1.108)-(1.111) in Table 1.1.

In terms of analytic continuation, this means that we are extending our solution on the Riemann sphere with five boundary components in such a way that this continuation doesn’t go around one of the singularities \( \{0, 1, u_1, u_2, \infty\} \). Moreover, the suborbit of a finite orbit \( \mathcal{O}_{P_i} \), generated
Action of the braid group $B_4$ on $\mathcal{M}_{G_2}$ and restrictions

Table 1.1: Matching using traces: elements on the same column must be equal.

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$\tilde{q}_1$</th>
<th>$\tilde{q}_2$</th>
<th>$\tilde{q}_3$</th>
<th>$\tilde{q}_{21}$</th>
<th>$\tilde{q}_{31}$</th>
<th>$\tilde{q}_{32}$</th>
<th>$\tilde{q}_{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_2$</td>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
<td>$\tilde{q}_3$</td>
<td>$\tilde{q}_{21}$</td>
<td>$\tilde{q}_{31}$</td>
<td>$\tilde{q}_{32}$</td>
<td>$\tilde{q}_{x}$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
<td>$\tilde{q}_3$</td>
<td>$\tilde{q}_{21}$</td>
<td>$\tilde{q}_{31}$</td>
<td>$\tilde{q}_{32}$</td>
<td>$\tilde{q}_{x}$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
<td>$\tilde{q}_3$</td>
<td>$\tilde{q}_{21}$</td>
<td>$\tilde{q}_{31}$</td>
<td>$\tilde{q}_{32}$</td>
<td>$\tilde{q}_{x}$</td>
</tr>
</tbody>
</table>

fixing a conjugacy class of a monodromy matrix $M_i$ must be a finite suborbit describing the analytic continuation of an algebraic solution of PVI: this provides the basis of our method.

Furthermore, by relations (1.101)-(1.104) and (1.108)-(1.111), the following four projections:

$$\tilde{\pi}, \hat{\pi}, \bar{\pi}, \bar{\pi} : \tilde{\mathcal{M}}_{G_2} \rightarrow \tilde{\mathcal{M}}_{PVI},$$

(1.112)

can be defined in such a way that, if we know a 4-tuple $m$ of monodromy matrices in $\tilde{\mathcal{M}}_{G_2}$, then we can project it to four 3-tuples $\tilde{n}, \bar{n}, \tilde{n}, \bar{n} \in \tilde{\mathcal{M}}_{PVI}$ as follows:

$$\tilde{\pi}(m) := (M_1, M_2, M_3) = \tilde{n},$$
$$\hat{\pi}(m) := (M_2, M_3, M_4) = \hat{n},$$
$$\bar{\pi}(m) := (M_1, M_3, M_4) = \bar{n},$$
$$\bar{\pi}(m) := (M_1, M_2, M_4) = \bar{n}.$$

(1.113)

Equivalently, we can define each projection in terms of co-adjoint coordi-
nates $p, p_{ij}, p_{ijk}$ on $\mathcal{U} \subset \mathcal{M}_{G_2}$ and co-adjoint coordinates $\tilde{q}, \tilde{q}, \tilde{q}, \tilde{q}$ on $\tilde{\mathcal{M}}_{PV/1}$:

\[
\begin{align*}
\tilde{\pi}(p) & := (p_1, p_2, p_3, p_{321}, p_{21}, p_{31}, p_{32}) = \tilde{q}, \\
\hat{\pi}(p) & := (p_2, p_3, p_4, p_{432}, p_{32}, p_{42}, p_{43}) = \hat{q}, \\
\tilde{\pi}(p) & := (p_1, p_2, p_4, p_{421}, p_{41}, p_{31}) = \tilde{q}, \\
\hat{\pi}(p) & := (p_1, p_3, p_4, p_{331}, p_{31}, p_{41}, p_{43}) = \hat{q}.
\end{align*}
\]

(1.114)

In second instance, given four triples $\tilde{n}, \tilde{n}, \tilde{n}, \tilde{n} \in \tilde{\mathcal{M}}_{PV/1}$ or the associated co-adjoint coordinates $\tilde{q}, \tilde{q}, \tilde{q}, \tilde{q}$, we can lift them to the respective 4-tuple of matrices $m \in \mathcal{M}_{G_2}$ or to co-adjoint coordinates $p$. In general we can not do this for every choice of $\tilde{q}, \tilde{q}, \tilde{q}, \tilde{q}$, indeed these four points must satisfy the following relations:

\[
\begin{align*}
\tilde{q}_1 &= \tilde{q}_1 = \tilde{q}_1, \\
\tilde{q}_2 &= \tilde{q}_1 = \tilde{q}_2, \\
\tilde{q}_3 &= \tilde{q}_2 = \tilde{q}_2, \\
\tilde{q}_3 &= \tilde{q}_3 = \tilde{q}_3, \\
\tilde{q}_{21} &= \tilde{q}_{21}, \\
\tilde{q}_{31} &= \tilde{q}_{21}, \\
\tilde{q}_{32} &= \tilde{q}_{21}, \\
\tilde{q}_{31} &= \tilde{q}_{31}, \\
\tilde{q}_{31} &= \tilde{q}_{32}, \\
\tilde{q}_{32} &= \tilde{q}_{32},
\end{align*}
\]

(1.115)

derived from (1.108)-(1.111). We give here only the relations between co-adjoint coordinates $\hat{q}, \hat{q}, \hat{q}, \hat{q}$, so that we can forget about the global conjugation.
Definition 13. We call *matching on four points* the procedure of applying the identities (1.115) to four points $\hat{q}, \tilde{q}, \hat{q}, \tilde{q}$ in $\hat{\mathcal{M}}_{PVI}$.

Note that in Table 1.1 the column relative to $p_x$ is empty, then $p_x$ is not determined by the *matching* procedure but it can be recovered using relation (1.62). Indeed, by (1.108)-(1.111), we can rewrite (1.62) as follows:

$$p_x = \frac{1}{2} (\hat{q}_1 \hat{q}_2 \hat{q}_3 \hat{q}_1 + \hat{q}_1 \hat{q}_3 \hat{q}_1 + \hat{q}_2 \hat{q}_3 \hat{q}_1 + \hat{q}_2 \hat{q}_1 \hat{q}_3 + \hat{q}_2 \hat{q}_1 \hat{q}_3 + \hat{q}_1 \hat{q}_2 \hat{q}_3 + \hat{q}_1 \hat{q}_2 \hat{q}_3)$$

(1.116)

where the r.h.s of equations (1.116) depends only on the known coordinates of $\hat{q}, \tilde{q}, \hat{q}, \tilde{q}$.

Clearly, before to proceed describing in depth the matching procedure, we need to define the space of all possible $\hat{q}, \tilde{q}, \hat{q}, \tilde{q}$; therefore in Chapter 3 we will review the main results about PVI.
Chapter 2

Garnier system $\mathcal{G}_2$ and symmetry group $G$

In this Chapter we recall some known facts about the Garnier system $\mathcal{G}_2$ and we use its birational canonical transformations to define the group $G$ of symmetries acting on the character variety of the Riemann sphere with five boundary components $\mathcal{M}_{\mathcal{G}_2}$.

2.1 Garnier systems $\mathcal{G}_n$

The Garnier system $\mathcal{G}_n$ is a completely integrable Hamiltonian system [Gar12, Gar26, Oka81] in $n$ variables $u_1, \ldots, u_n \in \mathbb{C}$ with $u_i \neq u_j$ when $i \neq j$:

\[
\begin{align*}
\frac{\partial v_i}{\partial u_i} &= \frac{\partial K_1}{\partial \rho_j}, & i, j = 1, \ldots, n, \\
\frac{\partial \rho_j}{\partial u_i} &= -\frac{\partial K_i}{\partial \rho_j}, & i, j = 1, \ldots, n.
\end{align*}
\]

(2.1)
The Hamiltonians $K_i$ are defined as:

$$K_i = -\frac{\Lambda(u_i)}{T'(u_i)} \left[ \sum_{k=1}^{n} \frac{T(\nu_k)}{(\nu_k - u_i)\Lambda'(\nu_k)} \left( \rho_k^2 - \sum_{m=1}^{n+2} \frac{\theta_m - \delta_{im}}{\nu_k - u_m} \rho_k + \frac{\kappa}{\nu_k(\nu_k - 1)} \right) \right],$$

(2.2)

where $\theta_1, \ldots, \theta_{n+2}, \theta_\infty$ are constant parameters and:

$$\Lambda(u) := \Pi_{k=1}^{n}(u - \nu_k), \quad T(u) := \Pi_{k=1}^{n+2}(u - u_k),$$

(2.3)

$$\kappa = \frac{1}{4} \left\{ \left( \sum_{m=1}^{n+2} \theta_m - 1 \right)^2 - (\theta_\infty + 1)^2 \right\}.$$  

(2.4)

Without loss of generality we fix $u_{n+1} = 0$ and $u_{n+2} = 1$.

When $n = 1$, there is only one complex variable $u = u_1$ and the Hamiltonian $K = K_1$ reads:

$$K = \frac{1}{u(u - 1)} \left[ \nu(\nu - 1)(\nu - u)\rho^2 - \{\theta_2(\nu - 1)(\nu - u) + \theta_3\nu(\nu - u) + (\theta_1 - 1)\nu(\nu - 1)\} \rho + \kappa \nu \right],$$

(2.5)

where $\kappa = \frac{1}{4} [(\theta_1 + \theta_2 + \theta_3 - 1)^2 - (\theta_\infty + 1)^2]$. The system $G_1$ becomes a system of two first order equations or equivalently a scalar second order ODE that is the famous Painlevé Sixth equation PVI:

$$\nu_{uu} = \frac{1}{2} \left( \frac{1}{\nu} + \frac{1}{\nu - 1} + \frac{1}{\nu - u} \right) \nu_u^2 - \left( \frac{1}{u} + \frac{1}{u - 1} + \frac{1}{\nu - u} \right)\nu_u + \nu(\nu - 1)(\nu - u) \left[ \alpha + \beta \frac{u}{\nu^2} + \gamma \frac{u - 1}{(\nu - 1)^2} + \delta \frac{u(u - 1)}{(\nu - u)^2} \right],$$

(2.6)

with parameters:

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_1^2}{2}, \quad \gamma = \frac{\theta_2^2}{2}, \quad \delta = \frac{1 - \theta_3^2}{2}.$$  

(2.7)
Hence, the Garnier systems $\mathcal{G}_n$ can be thought as a multivariable generalization of the PVI equation. In this thesis we focus on the first such generalization: $\mathcal{G}_2$.

### 2.2 Fuchsian system and its monodromy data

As mentioned in the Introduction, the Garnier system $\mathcal{G}_n$ describes isomonodromic deformations of the following Fuchsian system of linear differential equations for a $2 \times 2$ matrix valued function $\Psi(z)$ defined over $\mathbb{C}$ (see [Gar12, Gar26, Oka81, Iwa91]):

$$\frac{d\Psi}{dz} = A(z)\Psi, \quad (2.8)$$

where $A(z)$ is the following matrix function:

$$A(z) = \sum_{i=1}^{n+2} \frac{A_i}{z - u_i}, \ A_i \in \mathfrak{sl}_2(\mathbb{C}).$$

The singularity at $\infty$ is also Fuchsian so that the residue at $\infty$ is defined by:

$$A_{\infty} := -\sum_{i=1}^{n+2} A_i.$$

Without loss of generality, for $i = 1, \ldots, n+2, \infty$, we can choose the matrices $A_i$ traceless and set the eigenvalues of $A_i$ to be $\pm \frac{\theta_i}{2}$ with $\theta_i \in \mathbb{C}$. In addition, by a proper gauge transformation, it is possible to assume the matrix $A_{\infty}$ to be diagonal:

$$A_{\infty} = \frac{1}{2} \begin{pmatrix} \theta_{\infty} & 0 \\ 0 & -\theta_{\infty} \end{pmatrix}, \ \theta_{\infty} \in \mathbb{C}^*. \quad (2.9)$$
In order to define the monodromy data of system (2.8), according to the classical results of Wasow [Was65] and Sibuya [Sib90], we choose to fix a fundamental matrix solution $\Psi_\infty$ of (2.8) near $\infty$. The behaviour of a fundamental solution of (2.8), near the regular singular points, is described in the following Theorem, see [Dub96]:

**Theorem 14.** Suppose that $\infty \in \bar{\mathbb{C}}$ is a simple pole of (2.8), and the residue matrix $A_\infty$ of the coefficient $A(z)$ near $\infty$ is in diagonal form (2.9), then system (2.8) has a fundamental solution $\Psi_\infty$ such that:

$$\Psi_\infty(z) = \left(1 + O\left(\frac{1}{z}\right)\right) z^{-J_\infty} z^{-R_\infty}, \ z \to \infty, \quad (2.10)$$

where $J_\infty = A_\infty$ and:

$$(R_\infty)_{12} \neq 0 \text{ if } \theta_\infty \in \mathbb{N}, \quad (R_\infty)_{11} = (R_\infty)_{22} = (R_\infty)_{21} = 0,$$

$$(R_\infty)_{21} \neq 0 \text{ if } -\theta_\infty \in \mathbb{N}, \quad (R_\infty)_{11} = (R_\infty)_{22} = (R_\infty)_{12} = 0.$$

A branch of the logarithm in the function $\Psi_\infty$ must be chosen.

Near a singular point $u_i$, system (2.8) has a local fundamental solution $\Psi_i$ such that:

$$\Psi_i(z) = G_i \left(1 + O\left(z - u_i\right)\right) (z - u_i)^{J_i} (z - u_i)^{R_i}, \ z \to u_i, \quad (2.11)$$

where $G_i$ is an invertible matrix and $J_i = G_i^{-1} A_i G_i$ is the Jordan normal form of the residue matrix $A_i$ of $A(z)$ near $u_i$, moreover:

$$(R_i)_{12} \neq 0 \text{ if } \theta_i \in \mathbb{N}, \quad (R_i)_{11} = (R_i)_{22} = (R_i)_{21} = 0,$$

$$(R_i)_{21} \neq 0 \text{ if } -\theta_i \in \mathbb{N}, \quad (R_i)_{11} = (R_i)_{22} = (R_i)_{12} = 0.$$
A branch of the logarithm in the function $\Psi_4$ must be chosen.

We now look at what happens when we continue $\Psi_\infty$ analytically along paths in $\mathbb{C}$. A useful notion is:

**Definition 15.** Two paths $\gamma_1$ and $\gamma_2$ are homotopic if there exists a continuous deformation of one path to the other, namely there exists a continuous function:

$$T: [0,1] \times [0,1] \rightarrow \mathbb{C}\{u_1, \ldots, u_{n+2}\},$$

such that $T(0, t) = \gamma_1(t)$ and $T(1, t) = \gamma_2(t)$ for all $t \in [0,1]$, and $T(s, 0) = \gamma_1(0) = \gamma_2(0)$ and $T(s, 1) = \gamma_1(1) = \gamma_2(1)$ for all $s \in [0,1]$.

This is an equivalence relation that permits to identify curves that can be transformed one into the other in a continuous way on $\Sigma_{n+3}$, i.e. the Riemann sphere with $n+3$ boundary components. The following Theorem ensures that if we extend analytically our solution along two homotopic paths with the same end points, then we obtain the same extension:

**Theorem 16.** Suppose $\Psi(u)$ be a solution of (2.8) defined in an open set $U \in \mathbb{C}\{u_1, \ldots, u_{n+2}\}$. Consider $a, b \in U$ and two homotopic paths $\gamma_1$ and $\gamma_2$ with the same endpoints, then $\Psi$ can be analytically continued along each path. Let $\gamma[\Psi]$ denote the analytic continuation of $\Psi$ along the path $\gamma$, then $\gamma_1[\Psi](b) = \gamma_2[\Psi](b)$ and this is again a solution of (2.8).

Thanks to this Theorem, in order to fully characterize the analytic continuation of the fundamental solution $\Psi_\infty$, we fix a basis in the fundamental group:

$$\pi_1 (\mathbb{C}\{u_1, \ldots, u_{n+2}\}, \infty),$$

and we study the analytic continuation of $\Psi_\infty$ along elements of the basis. We perform branch cuts on the Riemann sphere along the $n + 2$ segments.
and fix a basis of generators for the fundamental group, see Figure 2.1. The segments have all the same direction and they are ordered according to the order of the points $u_1, \ldots, u_{n+2}$. A generator can be thought like a path $\gamma_i$ starting and ending at $\infty$, that goes around the singularity $u_i$ in the clockwise direction, leaving the other singular points lying outside.

The product of these $n+2$ loops is equivalent to the loop that encircle the pole at infinity but taken in the counter-clockwise direction:

$$\gamma_1 \cdots \gamma_{n+2} = \gamma_{\infty}^{-1}. \quad (2.12)$$

Fix a fundamental solution $\Psi$ of (2.8). By Theorem 16 and the relation between fundamental solutions, the analytic continuation $\gamma_i[\Psi]$ gives rise to a unique $2 \times 2$ invertible matrix $M_i$, called monodromy matrix, such that:

$$\gamma_i[\Psi] = \Psi M_i. \quad (2.13)$$

It is natural to associate to each analytic continuation of $\Psi$, along a gener-
ator of the fundamental group an element of $\text{GL}_2(\mathbb{C})$:

$$\gamma_i \mapsto M_i,$$  \hspace{1cm} (2.14)

this map is a group anti-homomorphism and, moreover, it is a representation of the fundamental group:

**Definition 17.** The image of $\pi_1 (\mathbb{C}\setminus \{u_1, \ldots, u_{n+2}\}, u_\infty)$ under the map (2.14) is a subgroup of $\text{GL}_2(\mathbb{C})$ that is called *monodromy group*.

From relation (2.12) we get the following cyclic property for the generators of the monodromy group:

$$M_\infty M_{n+2} \ldots M_1 = 1.$$  \hspace{1cm} (2.15)

**Remark 18.** A change of the base point, and the consequent change of basis, leads to a conjugation of the representation by an invertible constant matrix.

**Definition 19.** We call monodromy data the set:

$$\text{MD} := \{M_1, \ldots, M_{n+2}, R_1, \ldots, R_{n+2}\},$$  \hspace{1cm} (2.16)

where matrices $R_i$ are defined in Theorem 14.

If we take the fundamental solution $\Psi_\infty$, defined locally near $\infty$, and another one $\Psi_j$, defined in a neighbourhood of $u_j$, then the former can be analytically continued along the path $\gamma_j$ until a neighbourhood of $u_j$ is reached. Again the two fundamental solutions are related by right multiplication by a constant invertible $2 \times 2$ matrix $C_j$:

$$\Psi_\infty = \Psi_j C_j,$$  \hspace{1cm} (2.17)
called connection matrix. The connection matrix links the local monodromy at \( u_j \) with the global monodromy, in the basis defined by the fundamental matrix \( \Psi_{\infty} \), as follows:

\[
M_j = C_j^{-1} e^{2\pi i j} e^{R_j} C_j.
\]

(2.18)

### 2.3 Isomonodromic deformations

We now deform our system by keeping fixed the monodromy data (2.16). Consider the initial linear system:

\[
\frac{d\Psi^0}{dz} = \sum_{k=1}^{n+2} \frac{A_k^0}{z - u_k^0} \Psi^0,
\]

(2.19)

fix, as above, a basis \( \gamma_1, \ldots, \gamma_{n+2} \) for the fundamental group:

\[
\pi_1 \left( \mathbb{C} \setminus \{u_1, \ldots, u_{n+2}, \infty\} \right),
\]

and a fundamental matrix solution \( \Psi_{\infty} \), near \( \infty \). Isomonodromic deformations are described by the following Theorem, which proof can be found in [Mal91] and [Sib90]:

**Theorem 20.** There exists an open neighbourhood \( U \subset \mathbb{C}^{n+2} \) of the point \( u^0 = (u_1^0, \ldots, u_{n+2}^0) \) such that, for any \( u = (u_1, \ldots, u_{n+2}) \in U \), there exists a unique \((n+2)\)-tuple:

\[
(A_1(u), \ldots, A_{n+2}(u)),
\]
of analytic matrix valued functions such that:

\[ A_j(u^0) = A_j^0, \quad i = 1, \ldots, n + 2, \quad (2.20) \]

and with respect to the same basis of loops \( \gamma_1, \ldots, \gamma_{n+2} \), the monodromy data (2.16) of the Fuchsian system:

\[ \frac{d}{dz} \Psi = \sum_{k=1}^{n+2} \frac{A_k(u)}{z - u_k} \Psi, \quad (2.21) \]

coincides with the given \( M_1, \ldots, M_{n+2} \) and \( R_1, \ldots, R_{n+2} \). Furthermore, the monodromy group \( \langle M_1, \ldots, M_{n+2} \rangle \) is supposed to be irreducible and \( M_i \neq \pm \mathbb{1} \), for \( i = 1, \ldots, n + 2, \infty \). The matrices \( A_j(u) \) are the solutions of the Cauchy problem with the initial data \( A_j^0 \) for the following Schlesinger equations:

\[ \frac{\partial}{\partial u_j} A_i = \frac{[A_i, A_j]}{u_i - u_j}, \quad \frac{\partial}{\partial u_i} A_i = -\sum_{j \neq i} \frac{[A_i, A_j]}{u_i - u_j}. \quad (2.22) \]

The solution \( \Psi_\infty^0 \) of (2.19), in the form (2.10), can be uniquely continued, for \( z \neq u_i \) \( i = 1, \ldots, n + 2 \), to an analytic function \( \Psi_\infty(z, u) \) with \( u \in U \) with \( \Psi_\infty(z, u^0) = \Psi_\infty^0 \). This continuation is the local solution of the Cauchy problem with the initial data \( \Psi_\infty^0 \) for the following system that is compatible to the system (2.21):

\[ \frac{\partial}{\partial u_i} \Psi = -\frac{A_i(u)}{z - u_i} \Psi. \quad (2.23) \]

The functions \( A_i(u) \) and \( \Psi_\infty(z, u) \) can be continued analytically to global meromorphic functions on the universal coverings of:

\[ \mathbb{C}^{n+2} \setminus \{diags\} := \{(u_1, \ldots, u_{n+2}) \in \mathbb{C}^{n+2} \mid u_i \neq u_j \text{ for } i \neq j \}, \quad (2.24) \]
and:

\[ \{(z, u_1, \ldots, u_{n+2}) \in \mathbb{C}^{n+3} \mid u_i \neq u_j \text{ for } i \neq j \text{ and } z \neq u_i, \ i = 1, \ldots, n + 2\} , \]

respectively.

We recall here the main result about the solvability of the inverse monodromy problem in dimension two, the following result was proven by Dekkers in [Dek79] and by Bolibruch in [Bol97]:

**Theorem 21.** Given matrices \( M_1, \ldots, M_{n+2} \in \text{SL}_2(\mathbb{C}) \) satisfying (2.15), with:

\[
M_x = \begin{pmatrix}
    e^{i\pi \theta_x} & 0 \\
    0 & e^{-i\pi \theta_x}
\end{pmatrix}, \quad \theta_x \in \mathbb{C},
\]

and matrices \( R_1, \ldots, R_{n+2} \in \text{SL}_2(\mathbb{C}) \), then in a neighbourhood \( U \) of \( u^0 = (u_1^0, \ldots, u_{n+2}^0) \in \mathbb{C}^{n+2} \setminus \{\text{diags}\} \), there exists \( (u_1, \ldots, u_{n+2}) \in U \) and a Fuchsian system:

\[
\frac{d}{dz} \Psi = \sum_{k=1}^{n+2} \frac{A_k(u)}{z - u_k}\Psi,
\]

with \( M_1, \ldots, M_{n+2} \) and \( R_1, \ldots, R_{n+2} \) as monodromy data and \( u_1, \ldots, u_{n+2} \) as poles.

Now, we show how to reduce the Schlesinger equations to the Garnier system \( G_n \). Since Schlesinger equations are invariant under simultaneous diagonal conjugation of matrices \( A_i \) for \( i = 1, \ldots, n + 2 \), we introduce \( 2n \) co-ordinates on the space of solutions of the Schlesinger equations with respect to the equivalence relation:

\[ A_i \sim D^{-1}A_iD, \]
where $D$ is a $2 \times 2$ diagonal matrix.

Let $a_{ij}(z, u)$ denote the $ij$-element of $A(z, u)$, then $a_{12}(z, u)$ has the form:

$$a_{12}(z, u) = \sum_{i=1}^{n+2} \frac{a_{12}^i}{z - u_i},$$

(2.26)

and its denominator is a polynomial of degree $n$ in the variable $z$. Define $\nu_1, \ldots, \nu_n$ to be the roots of this polynomial, i.e.:

$$a_{12}(\nu_k, u) = 0, \; k = 1, \ldots, n,$$

and $n$ quantities $\rho_k$:

$$\rho_k := \sum_{i=1}^{n+2} \frac{a_{11}^i + \theta_i}{\nu_k - u_i}, \; k = 1, \ldots, n. \tag{2.27}$$

In this way we introduce $2n$ coordinates $(\nu_1, \ldots, \nu_n, \rho_1, \ldots, \rho_n)$ on the space of the solutions of the above Schlesinger equations, as stated in the following Theorem, due to Iwasaki et al. [Iwa91]:

**Theorem 22.** If the $n + 2$ tuple $(A_1(u), \ldots, A_{n+2}(u))$ of $2 \times 2$ matrix is a solution for the Schlesinger system:

$$\frac{\partial}{\partial u_j} A_i = \frac{[A_i, A_j]}{u_i - u_j}, \quad \frac{\partial}{\partial u_i} A_i = -\sum_{j \neq 1} \frac{[A_i, A_j]}{u_i - u_j},$$

(2.28)

then the functions $(\nu_1(u), \ldots, \nu_n(u), \rho_1(u), \ldots, \rho_n(u))$ with $u = (u_1, \ldots, u_{n+2})$, where $\nu_k(u)$ are the roots of (2.26) and the $\rho_k(u)$ are defined in (2.27), determine $A_1, \ldots, A_n$ uniquely up to global diagonal conjugation and the $\rho_k(u)$ satisfy the Garnier system $G_n$.

As a consequence of Theorem 22, we can regard $G_n$ as the system that
governs the isomonodromic deformations of the Fuchsian system (2.8), this is why in the following we will refer directly to $\mathcal{G}_n$.

### 2.4 Braid group $B_n$ and analytic continuation

We now show that the structure of analytic continuation of a solution of $\mathcal{G}_n$ is given by the action of the braid group $B_{n+2}$ (see Definition 8) over the monodromy matrices $(M_1, \ldots, M_{n+2})$, as it was firstly introduced in Dubrovin-Mazzocco in [DM00] for $n = 1$.

Consider a point $u^0 = (u_1^0, \ldots, u_{n+2}^0) \in \mathbb{C}^{n+2}\setminus\{\text{diags}\}$ and a solution $(\nu_1(u), \ldots, \nu_n(u), \rho_1(u), \ldots, \rho_n(u))$ of $\mathcal{G}_n$ in a neighbourhood of $u^0$. We perform $n+2$ cuts on $\mathbb{C}^{n+2}\setminus\{\text{diags}\}$, and we choose a basis of loops $\gamma_1, \ldots, \gamma_{n+2}$ for $\pi_1(\mathbb{C}^{n+2}\setminus\{\text{diags}\}, u^0)$, as in Figure 2.1. In this way, a branch of our solution is fixed and, by Theorem 22, to this branch we can associate $n+2$ monodromy matrices $M_1, \ldots, M_{n+2}$. Suppose now to continue analytically the solution of $\mathcal{G}_n$ along a loop:

$$\beta \in \pi_1(\mathbb{C}^{n+2}\setminus\{\text{diags}\}, u^0),$$

then a new branch is reached with new associated monodromy matrices $M'_1, \ldots, M'_{n+2}$. This action of the fundamental group $\pi_1(\mathbb{C}^{n+2}\setminus\{\text{diags}\}, u^0)$ over the monodromy matrices extends naturally to the action of the pure braid group $P_{n+2}$, see (1.83) and (1.84). Indeed, it is well known that:

$$\pi_1(\mathbb{C}^{n+2}\setminus\{\text{diags}\}, u^0) \simeq P_{n+2}.$$
In order to simplify the following computations, the action of the pure braid group $P_{n+2}$ is extended to the action of the full braid group $B_{n+2}$:

$$\pi_1(\mathbb{C}^{n+2}\backslash\{\text{diags}\}/S_{n+2}, u^0) \simeq B_{n+2},$$

where $S_{n+2}$ is the symmetric group over $n + 2$ elements.

Consider the $i$-th generator $\sigma_i$ of $B_{n+2}$, then $\sigma_i$ acts changing position of the poles $u_i$ and $u_{i+1}$. We show the new basis of loops $\gamma'_k$, for $k = 1, \ldots, n+2$, of $\pi_1(\mathbb{C}^{n+2}\backslash\{\text{diags}\}/S_{n+2}, u^0)$ in Figure 2.2.

Since deformations are supposed to be isomonodromic, the new monodromy matrices $M'_k$, in the new basis of loops, are a reordering of the old ones:

$$M'_i = M_{i+1}, \ M'_{i+1} = M_i, \ M'_k = M_k, \ k \neq i, i + 1.$$  

The new basis of loops $\gamma'_k$ and the old basis of loops $\gamma_k$ for $k = 1, \ldots, n + 2$,
are related as follows:

\[ \gamma_i = \gamma'_i, \quad \gamma_{i+1} = \gamma'_i \gamma_{i+1} \gamma'_i, \quad \gamma_k = \gamma'_k, \quad k \neq i, i + 1. \]

Since the basis of loops must be fixed once for ever, we need to express the new monodromy matrices in the old basis of loops. This leads to the following expression for the \( i \)-th generator of \( B_{n+2} \):

\[ \sigma_i : (M_1, \ldots, M_i, M_{i+1}, \ldots, M_{n+2}) \mapsto (M_1, \ldots, M_{i+1}, M_{i+1} M_i M_{i+1}^{-1}, \ldots, M_{n+2}). \]

(2.29)

Note that the action of \( \sigma_i \) over the monodromy matrices preserves their conjugacy class and the relation \( M_x M_{n+2} \ldots M_1 = 1 \).

Once we fix a branch of a solution of \( G_n \), at the same time we fix the monodromy matrices and the matrices \( R_1, \ldots, R_{n+2} \). The matrices \( R_i \) remain invariant under the action of the braid group. Therefore describing the braid action on the monodromy data (2.16) is equivalent to describe the same action on the character variety or equivalently on:

\[ \hat{\mathcal{M}}_{G_n} := \{(M_1, \ldots, M_{n+2}) \mid M_i \in \text{SL}_2(\mathbb{C}), \quad M_x M_{n+2} \ldots M_1 = 1\} / \sim, \]

(2.30)

where \( \sim \) is equivalence under global diagonal conjugation.

In order to describe all branches of a solution of the Garnier system \( \mathcal{G}_n \), we continue this solution analytically along every loop of \( \pi_1(\mathbb{C}^{n+2} \setminus \{\text{diags}\}, u^0) \). This is done in terms of action of the pure braid group over \( \hat{\mathcal{M}}_{G_n} \):

\[ P_{n+2} \times \hat{\mathcal{M}}_{G_n} \rightarrow \hat{\mathcal{M}}_{G_n}. \]
Define the set:
\[ O_{P_{n+2}}(m) := \{ \beta(m) | \beta \in P_{n+2} \}, \] (2.31)
being the orbit of an element \( m \in \widehat{M}_{G_n} \) under the action of the pure braid group \( P_{n+2} \).

In this thesis we are interested in the classification of algebraic solutions of the system \( G_n \):

**Definition 23.** A \( G_n \)'s solution \((\nu_1(u), \ldots, \nu_n(u), \rho_1(u), \ldots, \rho_n(u))\) where \( u = (u_1, \ldots, u_n) \) is algebraic if every components \( \nu_i \) solves an equation:
\[ P_i(u_1, \ldots, u_n, \nu_i) = 0, \]
and respectively every component \( \rho_i \) solves equation:
\[ \tilde{P}_i(u_1, \ldots, u_n, \rho_i) = 0, \]
where \( P_i \) and \( \tilde{P}_i \) are polynomials in \( C[u_1, \ldots, u_n, \nu_i] \) and \( C[u_1, \ldots, u_n, \rho_i] \) respectively.

Since an algebraic solution has only a finite number of branches, the monodromy associated to this solution under the action of \( P_{n+2} \) necessarily generate a finite orbit. We formalize this fact in the following Theorem due to Cousin [Cou16]:

**Theorem 24.** If a solution of \( G_n \) is algebraic, then the orbit under the \( P_{n+2} \) action over the monodromy matrices associated to this solution, up to conjugation by a diagonal matrix, is finite.

This means that the problem of classification of algebraic solutions of \( G_n \) can be seen as the problem of classification of finite orbit of the \( P_{n+2} \) action over \( \widehat{M}_{G_n} \).
2.5 Symmetry group $G$

In this Section, we study the symmetries acting on the co-adjoint coordinates $p_i, p_{ij}, p_{ijk}$ defined on the big open subset $\mathcal{U} \subset \mathcal{M}_{G_2}$. The definition of a symmetry for $\mathcal{U} \subset \mathcal{M}_{G_2}$ is the following:

**Definition 25.** A symmetry for $\mathcal{U} \subset \mathcal{M}_{G_2}$ is an invertible map $\Phi : \mathcal{M}_{G_2} \to \mathcal{M}_{G_2}$ such that given an element $p \in \mathcal{U} \subset \mathcal{M}_{G_2}$ and its orbit $O_{P_4}(p)$, the following is true:

$$|O_{P_4}(\Phi(p))| = |O_{P_4}(p)|.$$  

(2.32)

By the above Definition, if $p \in \mathcal{U} \subset \mathcal{M}_{G_2}$ generates a finite $P_4$-orbit of length $N$, then $\Phi(p)$ generates a finite $P_4$-orbit of the same length. This leads to define the following equivalence relation between orbits:

**Definition 26.** The elements $p$ and $p' \in \mathcal{U} \subset \mathcal{M}_{G_2}$ are said to generate equivalent orbits if there exists a symmetry $\Phi$ such that $p' = \Phi(p)$.

First, we introduce the group $\tilde{G}$ of symmetries obtained by birational transformations of the Garnier system $G_2$. Subsequently, in order to obtain the group of symmetries $G$, we extend the group $\tilde{G}$ with simpler transformations that arise on the space of monodromy matrices.

2.5.1 Birational transformations of $G_2$

The birational transformations of the Garnier systems (2.1) were firstly introduced by Kimura et al. [Iwa91], and subsequently studied by Tsuda [Tsu03] and Suzuki [Suz05]. A birational transformation for the Garnier
system $\mathcal{G}_2$ is a map of the form:

$$S : (\nu_1, \rho_1, u_1, \nu_2, \rho_2, u_2, \theta_1, \ldots, \theta_4, \theta_\infty) \mapsto (\tilde{\nu}_1, \tilde{\rho}_1, \tilde{u}_1, \tilde{\nu}_2, \tilde{\rho}_2, \tilde{u}_2, \tilde{\theta}_1, \ldots, \tilde{\theta}_4, \tilde{\theta}_\infty),$$

(2.33)

where $S$ acts birationally on $(\nu_1, \rho_1, u_1, \nu_2, \rho_2, u_2)$ and by linear affine transformation on the five parameters $(\theta_1, \ldots, \theta_4, \theta_\infty)$.

Since these transformations are birational, they send algebraic solutions to algebraic solutions, preserving the number of branches. In particular, this implies that the action of these transformations on finite orbits of the action of the pure braid group $P_4$ over $\mathcal{M}_{\mathcal{G}_2}$ are mapped to finite orbits and their length is preserved. If two orbits are related by such transformation, we say that they are equivalent. In order to characterize the group of symmetries acting on $\mathcal{U} \subset \mathcal{M}_{\mathcal{G}_2}$, we define the group:

$$\tilde{\mathcal{G}} = \langle P_{13}, P_{23}, P_{34}, P_{1\infty} \rangle,$$

(2.34)

of birational transformations of the Garnier system $\mathcal{G}_2$, and thanks to the work of Dubrovin-Mazzocco [DM07], we are able to explicitly write the action of the generators of $\tilde{\mathcal{G}}$ over the monodromy matrices $M_1, M_2, M_3, M_4$ and consequently over the co-adjoint coordinates $p_i, p_{ij}, p_{ijk}$.

We list now all known birational transformations acting on the Garnier system $\mathcal{G}_2$ and compute their effect on the co-adjoint coordinates $p_i, p_{ij}, p_{ijk}$ defined on the big open subset $\mathcal{U} \subset \mathcal{M}_{\mathcal{G}_2}$. Firstly we look at the birational transformations $s_i$ for $i = 1, \ldots, 4, \infty$, that act as change of sign on the parameters $\theta_i$ for $i = 1, \ldots, 4, \infty$: 
\( \begin{align*}
{\tilde{v}}_i &= \nu_i, \ i = 1, 2, \\
{\tilde{\rho}}_1 &= \rho_1 - \frac{\theta_1}{\nu_1}, \\
{\tilde{\rho}}_2 &= \rho_2, \\
{\tilde{u}}_i &= u_i, \ i = 1, 2, \\
{\tilde{\theta}}_1 &= -\theta_1, \\
{\tilde{\theta}}_i &= \theta_i, \ i = 2, 3, 4, \infty,
\end{align*} \)

\( s_1 : \) \hspace{1cm} (2.35)

\( \begin{align*}
{\tilde{v}}_i &= \nu_i, \ i = 1, 2, \\
{\tilde{\rho}}_1 &= \rho_1, \\
{\tilde{\rho}}_2 &= \rho_2 - \frac{\theta_2}{\nu_2}, \\
{\tilde{u}}_i &= u_i, \ i = 1, 2, \\
{\tilde{\theta}}_2 &= -\theta_2, \\
{\tilde{\theta}}_i &= \theta_i, \ i = 1, 3, 4, \infty,
\end{align*} \)

\( s_2 : \) \hspace{1cm} (2.36)

\( \begin{align*}
{\tilde{v}}_i &= \nu_i, \ i = 1, 2, \\
{\tilde{\rho}}_i &= \rho_i - \frac{\theta_i}{u_i(\frac{\nu_1}{\nu_1^2} + \frac{\nu_2}{\nu_2}) - 1}, \ i = 1, 2, \\
{\tilde{u}}_i &= u_i, \ i = 1, 2, \\
{\tilde{\theta}}_3 &= -\theta_3, \\
{\tilde{\theta}}_i &= \theta_i, \ i = 1, 2, 4, \infty,
\end{align*} \)

\( s_3 : \) \hspace{1cm} (2.37)
Lemma 27. Transformations \( s_1, \ldots, s_4, s_8 \) act as the identity on \( p \in \mathcal{M}_{G_2} \).

Proof. Since all transformations (2.39) act as the identity on \( \nu_i \) for \( i = 1, 2 \) and on the independent variables \( u_i \) for \( i = 1, 2 \), then the monodromy remains unchanged and each transformation \( s_i \) acts on \( p \in \mathcal{U} \subset \widehat{\mathcal{M}}_{G_2} \) as the identity.

We consider now transformations acting on both dependent and independent variables permuting the positions of the poles. There are four birational generators \( P_{ij} \):
\[
\begin{align*}
\tilde{\nu}_i &= \frac{u_1 - \nu_i}{u_1 - 1}, \quad i = 1, 2, \\
\tilde{\rho}_i &= -(u_1 - 1)\rho_i, \quad i = 1, 2, \\
\tilde{u}_1 &= \frac{u_1}{u_1 - 1}, \\
\tilde{u}_2 &= \frac{u_1 - u_2}{u_1 - 1}, \\
\tilde{\theta}_1 &= \theta_3, \\
\tilde{\theta}_3 &= \theta_1, \\
\tilde{\theta}_i &= \theta_i, \quad i \neq 1, 3.
\end{align*}
\]
\(P_{13}:\) \hfill (2.40)

\[
\begin{align*}
\tilde{\nu}_i &= \frac{u_2 - \nu_i}{u_2 - 1}, \quad i = 1, 2, \\
\tilde{\rho}_i &= -(u_2 - 1)\rho_i, \quad i = 1, 2, \\
\tilde{u}_1 &= \frac{u_2 - u_1}{u_2 - 1}, \\
\tilde{u}_2 &= \frac{u_2}{u_2 - 1}, \\
\tilde{\theta}_2 &= \theta_3, \\
\tilde{\theta}_3 &= \theta_2, \\
\tilde{\theta}_i &= \theta_i, \quad i \neq 2, 3.
\end{align*}
\]
\(P_{23}:\) \hfill (2.41)

\[
\begin{align*}
\tilde{\nu}_i &= 1 - \nu_i, \quad i = 1, 2, \\
\tilde{\rho}_i &= -\rho_i, \quad i = 1, 2, \\
\tilde{u}_i &= 1 - u_i, \quad i = 1, 2, \\
\tilde{\theta}_3 &= \theta_4, \\
\tilde{\theta}_4 &= \theta_3, \\
\tilde{\theta}_i &= \theta_i, \quad i \neq 3, 4.
\end{align*}
\]
\(P_{34}:\) \hfill (2.42)
We restate Theorem 8.1.2 in [Kim90], where a description of the group generated by \( P_{ij} \) is given:

**Theorem 28.** The group \( \tilde{G} \) generated by:

\[
\tilde{G} := \langle P_{13}, P_{23}, P_{34}, P_{1\infty} \rangle,
\]

is a group of symmetries for the Garnier system \( \mathcal{G}_2 \) and it is isomorphic to the symmetric group with 5 elements, i.e. \( S_5 \).

In order to write down the action of the group \( \tilde{G} \) in terms of co-adjoint coordinates (1.3) defined on the big open subset \( \mathcal{U} \subset \mathcal{M}_{\mathcal{G}_2} \), we need to understand how transformations \( P_{ij} \) act on the monodromy matrices. We calculate this action on a 4-tuple \( (M_1, \ldots, M_4) \) of monodromy matrices following Theorem 1.2 in the work of Dubrovin-Mazzocco [DM07] and afterwards the action in terms of co-adjoint coordinates:

**Lemma 29.** If \( (M_1, M_2, M_3, M_4) \in \mathcal{M}_{\mathcal{G}_2} \) is the monodromy associated to a solution \( (\nu_1, \rho_1, u_1, \nu_2, \rho_2, u_2) \) of \( \mathcal{G}_2 \), then transformations \( P_{13}, P_{23}, P_{34} \) act on

\[
P_{1\infty} : \left\{ \begin{align*}
\tilde{\nu}_i &= \frac{1}{\nu_i - u_1}, & i &= 1, 2, \\
\tilde{\rho}_i &= -\rho_i \nu_i^2 - \frac{2\nu_i^2 - 1}{2} \nu_i, & i &= 1, 2, \\
\tilde{u}_1 &= \infty, \\
\tilde{\infty} &= u_1, \\
\tilde{u}_2 &= \frac{1}{u_2 - u_1}, \\
\tilde{\theta}_1 &= \theta_\infty - 1, \\
\tilde{\theta}_\infty &= \theta_1 + 1, \\
\tilde{\theta}_i &= \theta_i, & i &\neq 1, \infty.
\end{align*} \right.
\]
the monodromy matrices as follows:

\[ P_{13} : (M_1, M_2, M_3, M_4) \mapsto (M_1^{-1}M_2^{-1}M_3M_2M_1, M_2, M_2M_1M_2^{-1}, M_4), \]

\[ P_{23} : (M_1, M_2, M_3, M_4) \mapsto ((M_2^{-1}M_3M_2M_1)^{-1}M_1M_2^{-1}M_3M_2M_1, \]
\[ (M_2M_1)^{-1}M_3M_2M_1, M_2, M_4), \]

\[ P_{34} : (M_1, M_2, M_3, M_4) \mapsto (M_xM_3M_2M_1(M_xM_3M_2)^{-1}, M_2, \]
\[ (M_3M_2M_1M_2^{-1})^{-1}M_4(M_3M_2M_1M_2^{-1}), M_3), \]

while transformation \( P_{1x} \) acts on the monodromy matrices as:

\[ P_{1x} : (M_1, M_2, M_3, M_4) \mapsto (-C_1M_xC_1^{-1}, C_1^{-1}M_2C_1, C_1^{-1}M_3C_1, \]
\[ C_1^{-1}M_4C_1), \]

where \( C_1 \) is the connection matrix defined in Section 2.2.

**Proof.** The proof is a consequence of Theorem 1.2 in the work of Dubrovin-Mazzocco [DM07]. \( \square \)

Finally the action of the group \( \tilde{G} \) in terms of co-adjoint coordinates (1.3) defined on the big open subset \( \mathcal{U} \subset \tilde{\mathcal{M}}_{G_2} \):

**Corollary 30.** The group \( \tilde{G} = \langle P_{13}, P_{23}, P_{34}, P_{1x} \rangle \) acts on \( p \in \mathcal{U} \subset \tilde{\mathcal{M}}_{G_2} \) as follows:

\[ P_{13}(p) = \sigma_2\sigma_1^{-1}\sigma_2^{-1}(p), \]  \hspace{1cm} (2.47)

\[ P_{23}(p) = \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}(p), \]  \hspace{1cm} (2.48)

\[ P_{34}(p) = \sigma_3\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1}(p), \]  \hspace{1cm} (2.49)

\[ P_{1x}(p) = (-p_x, p_2, p_3, p_4, -p_1, -p_1p_4 + p_4p_3 + p_2p_4 - p_2p_x, \]
\[ -p_{21}p_4 + p_{421} + p_{321}p_{43} - p_{3p2x}, p_{32}, -p_{321}, p_{42}, p_{43}, \]
\[ -p_1p_4 + p_{41} + p_{321}p_{432} - p_{32}p_{2x}, p_{432}, -p_{21}, \]
\[ -p_1p_3 + p_{31} + p_{21}p_{32} - p_{2p321}, \]
\[ (2.50) \]

where \( \sigma_i \) are defined in (1.86).

**Proof.** Formulae (2.47)-(2.50) can be proven by straightforward computation applying, respectively, to formulae (2.45)-(2.46) the definition of co-adjoint coordinates and the skein relation. \( \square \)

### 2.5.2 Symmetries of the monodromy matrices

In this Section, we introduce a set of transformations on the space of monodromy matrices:

(i) Transformations that change signs to matrices \( M_i \), for \( i = 1, \ldots, 4 \),

the so-called Schlesinger transformations introduced by Jimbo-Miwa in [JM81]:

\[ (M_1, M_2, M_3, M_4, M_x) \mapsto (\epsilon_1 M_1, \epsilon_2 M_2, \epsilon_3 M_3, \epsilon_4 M_4, \epsilon_1\epsilon_2\epsilon_3\epsilon_4(M_4M_3M_2M_1)^{-1}), \]

\[ (2.51) \]

where \( \epsilon_i = \pm 1 \) for \( i = 1, \ldots, 4 \).

(ii) Permutations of the matrices \( M_i \) for \( i = 1, \ldots, 4 \):

\[ (M_1, M_2, M_3, M_4, M_x) \mapsto (M_{\xi(1)}^{-1}, M_{\xi(2)}^{-1}, M_{\xi(3)}^{-1}, M_{\xi(4)}^{-1}, \]
\[ (M_{\xi(4)}^{-1}, M_{\xi(3)}^{-1}, M_{\xi(2)}^{-1}, M_{\xi(1)}^{-1})^{-1}), \]

\[ (2.52) \]

where \( \xi \) is in a subgroup \( H \subset S_4 \), of the symmetric group over 4
elements. In (2.52), we consider the inversion to be able to refer to the work of Lisovyy and Tykhyy [LT14].

Given \( (M_1, \ldots, M_4) \in \tilde{\mathcal{M}}_{G_2} \), we call sign flips the transformations that change sign of the matrices \( M_i \). They are defined as:

\[
\text{sign}_{(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (\epsilon_1 M_1, \epsilon_2 M_2, \epsilon_3 M_3, \epsilon_4 M_4, \\
\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 (M_4 M_3 M_2 M_1)^{-1}) \tag{2.53}
\]

where \( \epsilon_i = \pm 1 \) for \( i = 1, \ldots, 4 \). The following four sign flips generate all the others:

\[
\text{sign}_1 := \text{sign}_{(-1,1,1,1)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (-M_1, M_2, M_3, M_4, -M_\infty), \\
(2.54)
\]

\[
\text{sign}_2 := \text{sign}_{(1,-1,1,1)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_1, -M_2, M_3, M_4, -M_\infty), \\
(2.55)
\]

\[
\text{sign}_3 := \text{sign}_{(1,1,-1,1)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_1, M_2, -M_3, M_4, -M_\infty), \\
(2.56)
\]

\[
\text{sign}_4 := \text{sign}_{(1,1,1,-1)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_1, M_2, M_3, -M_4, -M_\infty). \\
(2.57)
\]

Sign flips are invertible maps \( \Phi \) that lead to equivalent orbits:

**Proposition 31.** Given a sign flip \( \text{sign} \in \langle \text{sign}_1, \ldots, \text{sign}_4 \rangle \) and a braid
\( \sigma \in B_4 \), then there exists \( \text{sign}' \in \langle \text{sign}_1, \ldots, \text{sign}_4 \rangle \) and \( \sigma' \in B_4 \) s.t.

\[
\sigma \text{sign} = \text{sign}' \sigma'.
\]  
(2.58)

**Proof.** Suppose \( m = (M_1, M_2, M_3, M_4) \in \widehat{M}_{G_2} \), we prove that given a generator \( \sigma_i \) for \( i = 1, 2, 3 \) of the full braid group \( B_4 \), then:

\[
\sigma_i \text{sign}_j = \text{sign}_{j'} \sigma_{i'}. 
\]  
(2.59)

for some choice of the indices \( i, i' = 1, 2, 3 \) and \( j, j' = 1, 2, 3, 4 \). We prove it for \( \sigma_1 \) and \( \text{sign}_1 \). Indeed:

\[
\sigma_1 \text{sign}_1(m) = (M_2, -M_2M_1M_2^{-1}, M_3, M_4) = \text{sign}_2 \sigma_1(m). \]  
(2.60)

In a similar way we can prove the following equations:

\[
\begin{align*}
\sigma_1 \text{sign}_2 &= \text{sign}_1 \sigma_1, \\
\sigma_1 \text{sign}_3 &= \text{sign}_2 \sigma_1, \\
\sigma_1 \text{sign}_4 &= \text{sign}_3 \sigma_1, \\
\sigma_2 \text{sign}_1 &= \text{sign}_1 \sigma_2, \\
\sigma_2 \text{sign}_2 &= \text{sign}_3 \sigma_2, \\
\sigma_2 \text{sign}_3 &= \text{sign}_2 \sigma_2, \\
\sigma_2 \text{sign}_4 &= \text{sign}_4 \sigma_2, \\
\sigma_3 \text{sign}_1 &= \text{sign}_1 \sigma_3, \\
\sigma_3 \text{sign}_2 &= \text{sign}_2 \sigma_3, \\
\sigma_3 \text{sign}_3 &= \text{sign}_4 \sigma_3, \\
\sigma_3 \text{sign}_4 &= \text{sign}_3 \sigma_3.
\end{align*}
\]  
(2.61)
This concludes the proof.

The following result gives the action of sign flips in terms of co-adjoint coordinates and can be proved by straightforward computations:

**Corollary 32.** The action of the generators of the group of *sign flips* in terms of co-adjoint coordinates (1.3), on the big open subset $\mathcal{U} \subset \hat{\mathcal{M}}_{G_2}$, is as follows:

\[
\begin{align*}
\text{sign}_1(p) &= (-p_1, p_2, p_3, p_4, -p_{21}, p_{31}, -p_{32}, -p_{41}, -p_{42}, -p_{43}, -p_{431}, -p_{441}), \\
\text{sign}_2(p) &= (p_1, -p_2, p_3, -p_4, -p_{21}, p_{31}, -p_{32}, p_{41}, -p_{42}, p_{43}, -p_{431}, -p_{432}, p_{431}, -p_{441}), \\
\text{sign}_3(p) &= (p_1, p_2, -p_3, -p_4, -p_{21}, -p_{31}, -p_{32}, p_{41}, -p_{42}, p_{43}, -p_{431}, -p_{432}, -p_{431}, p_{441}), \\
\text{sign}_4(p) &= (p_1, p_2, p_3, -p_4, -p_{21}, p_{31}, p_{32}, -p_{41}, -p_{42}, -p_{43}, -p_{431}, -p_{432}, -p_{431}, -p_{441}).
\end{align*}
\]

(2.62) (2.63) (2.64) (2.65)

At this point we introduce permutations on a 4-tuple of monodromy matrices $(M_1, \ldots, M_4) \in \hat{\mathcal{M}}_{G_2}$. These permutations are generated by:

\[
\begin{align*}
(12)(34) : (M_1, M_2, M_3, M_4) &\mapsto (M_2^{-1}, M_1^{-1}, M_4^{-1}, M_3^{-1}, M_2 M_1 M_4 M_3), \\
(1234) : (M_1, M_2, M_3, M_4) &\mapsto (M_4, M_1, M_2, M_3, (M_3 M_2 M_1 M_4)^{-1}).
\end{align*}
\]

(2.66) (2.67)
As in the case of sign flips also (2.66)-(2.67) can be considered as invertible maps $\Phi$ that lead to equivalent orbits (see Definitions 37 and 26):

**Proposition 33.** Given a permutation $\xi \in \langle (12)(34), (1234) \rangle$ and a braid $\sigma \in B_4$, then there exists $\xi' \in \langle (12)(34), (1234) \rangle$ and $\sigma' \in B_4$ such that:

$$\sigma \xi = \xi' \sigma'.$$  \hspace{1cm} (2.68)

**Proof.** Suppose $m = (M_1, M_2, M_3, M_4) \in \widehat{M}_{G_2}$, we prove that given a generator $\sigma_i$ for $i = 1, 2, 3$ of the full braid group $B_4$, then:

$$\xi \sigma_i = \sigma_i' \xi',$$

for some choice of the indices $i, i' = 1, 2, 3$ and $\xi, \xi' \in \langle (12)(34), (1234) \rangle$. We prove in details that the statement is true for the composition of $\langle 1234 \rangle$ and $\sigma_1$:

$$(1234)\sigma_1(m) = (1234)(M_2, M_2M_1M_2^{-1}, M_3, M_4) =$$

$$= (M_4, M_2, M_2M_1M_2^{-1}, M_3),$$

and:

$$\sigma_2(1234)(m) = \sigma_2(M_4, M_1, M_2, M_3) =$$

$$= (M_4, M_2, M_2M_1M_2^{-1}, M_3),$$

then $\sigma_2(1234) = (1234)\sigma_1$. In a similar way we can prove the following equations:

$$\sigma_1(12)(34) = (12)(34)\sigma_1^{-1},$$  \hspace{1cm} (2.69)
\[\sigma_2(12)(34) = (12)(34)(1234)^3\sigma_2\sigma_3, \quad (2.70)\]
\[\sigma_3(12)(34) = (12)(34)\sigma_3^{-1}, \quad (2.71)\]
\[\sigma_1(1234) = (1234)(1234)\sigma_2^{-1}\sigma_1^{-1}, \quad (2.72)\]
\[\sigma_2(1234) = (1234)\sigma_1, \quad (2.73)\]
\[\sigma_3(1234) = (1234)\sigma_2. \quad (2.74)\]

This concludes the proof. \qed

The following result gives the action of the permutations in terms of co-adjoint coordinates and can be proved by straightforward computations:

**Corollary 34.** The action of the generators \((12)(34)\) and \((1234)\) in terms of co-adjoint coordinates, on the big open subset \(U \subset \bar{\mathcal{M}}_{\bar{G}_2}\), is:

\[(12)(34)(p) = (p_2, p_1, p_4, p_3, p_x, p_{21}, p_{42}, p_{41}, p_{32}, p_{31}, p_{43}, p_{421}, p_{431}, p_{432}, p_{321}), \quad (2.75)\]
\[(1234)(p) = (p_4, p_1, p_2, p_3, p_x, p_{41}, p_{42}, p_{21}, p_{43}, p_{31}, p_{32}, p_{421}, p_{321}, p_{432}, p_{431}). \quad (2.76)\]

We resume the results of this Section in the following Theorem:

**Theorem 35.** The group:

\[G := \langle P_{13}, P_{23}, P_{34}, P_{1x}, \text{sign}_1, \ldots, \text{sign}_4, (12)(34), (1234) \rangle. \quad (2.77)\]

is a group of symmetries for \(U \subset \bar{\mathcal{M}}_{\bar{G}_2}\).

**Proof.** For the subgroup of transformations \(\langle P_{13}, P_{23}, P_{34}, P_{1x} \rangle\), the statement follows by construction. For the subgroup of transformations \(\langle \text{sign}_1, \ldots, \text{sign}_4, (12)(34), (1234) \rangle\), the statement follows by Propositions 31 and 33. \qed
Chapter 3

Input set of the matching procedure

We briefly recall the main idea underlying our methodology. In order to classify all finite orbits of the action:

\[ P_4 \times \mathcal{M}_{G_2} \rightarrow \mathcal{M}_{G_2}, \]  

(3.1)

in Theorem 12, we introduced four restrictions \( H_1, \ldots, H_4 \) to subgroups of \( P_4 \) that are isomorphic to the braid group \( P_3 \). These four restrictions are summarized in Table 1.1. In particular each row of Table 1.1 represents a subset \( \tilde{q}, \tilde{q}, q, \tilde{q} \) of co-adjoint coordinates \( p_i, p_{ij}, p_{ijk} \) that must generate a finite orbit under the “restricted action” of \( P_3 \) over the \( SL_2(\mathbb{C}) \) character variety of the Riemann sphere with four boundary components, i.e. \( \mathcal{M}_{PV I} \).

In Section 3.1, we will remind that the “restricted action” describes the structure of the analytic continuation of algebraic solutions of PVI, the Sixth Painlevé equation, and hence, that each algebraic solution of PVI is associated to a finite orbit of the “restricted action”. 
In this Chapter we want to produce a list of all possible $\tilde{q}, \tilde{q}, \theta, \tilde{q}$, as defined in (1.107), such that their orbit, under the action of $H_1, \ldots, H_4$ respectively, is finite. In other words we want to define the “input set” for the matching procedure introduced in Section 1.4. It is then fundamental, for our purpouses, to know the classification of all algebraic solutions of PVI. Many authors approached this problem with different methods, see [AK02, Boa06, Boa05, Dub96, DM00, Hit95, Hit03, Kit05, LT14]. In the following the major ideas are inspired from the work of Dubrovin-Mazzocco [DM00] and its natural generalisation due to Lisovyy and Tykhyy [LT14].

The classification result of Lisovyy and Tykhyy produced a list of $5 + 45$ distinct finite orbits of the action of $P_3$ over the character variety $\mathcal{M}_{PVI}$. Lisovyy and Tykhyy’s classification is folded up to the action of the group $G_{PVI}$ of symmetries acting on $\mathcal{M}_{PVI}$. The action of the group $G_{PVI}$ will be described in Section 3.2 and we will show that also if $G_{PVI}$ is isomorphic to the affine Weyl group of type $F_4$ (an infinite group), the action of $G_{PVI}$ over the co-adjoint coordinates of an element in $\mathcal{M}_{PVI}$ is finite, see Lemmata 38, 44, 45 and 50.

In Section 3.3, we give explicitly the Lisovyy and Tykhyy’s list of $5 + 45$ orbits. In this list there exists 5 infinite sublists of finite orbits (corresponding to families of parametric solutions of PVI) and one finite sublist of 45 finite orbits, see Table 3.4. It is precisely the latter finite sublist that will be crucial to our method in order to succeed. Since the action of $G_{PVI}$ over the 45 finite orbits is finite, in Section 3.3.1 we define an “expansion algorithm” that given the “folded list” of 45 orbits, it produces the “unfolded” list of all “equivalent” finite orbits under the action of $G_{PVI}$. 
3.1 PVI: Braid group and analytic continuation

The Sixth Painlevé equation is the isomonodromic deformations equation for the Fuchsian system:

\[ \frac{d\Psi}{dz} = \left( A_0 + A_1 + \frac{A_u}{z - u} \right) \Psi, \quad z \in \mathbb{C}\setminus\{0, u, 1\}, \]  

(3.2)

where the singular points 0, u, 1, \( \infty \) are simple poles. The matrices \( A_i \) is defined as:

\[ A_\infty := -A_0 - A_1 - A_u = \frac{1}{2} \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \quad \theta_\infty \in \mathbb{C}^*, \]

and the eigenvalues of \( A_i \) are \( \frac{\pm \theta_i}{2} \) with \( \theta_i \in \mathbb{C} \) for \( i = 0, 1, u \). The \( \theta_i \) are the parameters appearing in the PVI equation (2.6). A solution \( \Psi(z) \) of (3.2) is a multivalued analytic function on the Riemann sphere with four punctures: \( \mathbb{C}\setminus\{0, u, 1\} \). Consider loops \( \gamma_0, \gamma_u, \gamma_1 \) and \( \gamma_\infty \) such that they encircle 0, u, 1, \( \infty \), then we denote the associated monodromy matrices as \( N_1, N_2, N_3 \) and \( N_\infty \), with \( N_i \) in \( \text{SL}_2(\mathbb{C}) \) and such that:

\[ \text{Tr } N_i = 2 \cos \pi \theta_i, \quad i = 1, 2, 3, \infty. \]  

(3.3)

The \( \text{SL}_2(\mathbb{C}) \) character variety of the Riemann sphere with four boundary components is identified with the two dimensional quotient space:

\[ \widehat{\mathcal{M}}_{PVI} := \widehat{\mathcal{M}}_{\phi_i} = \{(N_1, N_2, N_3) \mid N_i \in \text{SL}_2(\mathbb{C}), N_\infty N_3 N_2 N_1 = 1\} / \sim, \]  

(3.4)
where $\sim$ is simultaneous conjugation of a triple by diagonal matrix.

Every element in $\hat{\mathcal{M}}_{PV1}$ can be identified with a 7-tuple of complex values:

$$q := (q_1, q_2, q_3, q_\infty, q_{21}, q_{31}, q_{32}) \in \mathbb{C}^7,$$

where:

$$q_i = \text{Tr} N_i, \ i = 1, 2, 3, \infty,$$

$$q_{ij} = \text{Tr} N_iN_j, \ i, j = 1, 2, 3, \ i > j. \quad (3.6)$$

Moreover, the quantities in (3.5) satisfy the Jimbo-Fricke cubic [Jim82, Boa05]:

$$q_{32}q_{31}q_{21} + q_{32}^2 + q_{31}^2 + q_{21}^2 - \omega_1 q_{32} - \omega_2 q_{31} - \omega_3 q_{21} + \omega_4 - 4 = 0, \quad (3.7)$$

where:

$$\omega_1 := q_1 q_\infty + q_3 q_2,$$

$$\omega_2 := q_2 q_\infty + q_3 q_1,$$

$$\omega_3 := q_3 q_\infty + q_2 q_1,$$

$$\omega_4 := q_3^2 + q_2^2 + q_1^2 + q_\infty^2 + q_3 q_2 q_1 q_\infty. \quad (3.8)$$

It was proven by Iwasaki [Iwa03] that, if $q_1, q_2, q_3, q_\infty$ are treated as parameters, then $(q_{21}, q_{31}, q_{32})$ is a system of coordinates on a big open subset $S \subset \hat{\mathcal{M}}_{PV1}$.

By (2.29), explicit formulae for the action of the full braid group $B_3$ on
the monodromy matrices $N_i$ for $i = 1, \ldots, 4$ are:

$$
\sigma_1^{(PVI)} : (N_1, N_2, N_3) \mapsto (N_2, N_2^{-1}N_1N_2, N_3),
$$

$$
\sigma_2^{(PVI)} : (N_1, N_2, N_3) \mapsto (N_1, N_3, N_3^{-1}N_2N_3). \quad (3.9)
$$

Moreover action (3.9) can be restated in terms of co-adjoint coordinates (3.6):

$$
\sigma_1^{(PVI)} : (q_1, q_2, q_3, q_4, q_{21}, q_{31}, q_{32}) \mapsto (q_2, q_1, q_3, q_4, q_{21}, q_{32}, \omega_2 - q_{31} - q_{32}q_{21}),
$$

$$
\sigma_2^{(PVI)} : (q_1, q_2, q_3, q_4, q_{21}, q_{31}, q_{32}) \mapsto (q_1, q_3, q_2, q_4, q_{31}, q_{32}, \omega_3 - q_{21} - q_{31}q_{32}, q_{32}). \quad (3.10)
$$

By (1.83), we can define the generators of the pure braid group $P_3$ as follows:

$$
P_3 = \langle \beta_{21}^{(PVI)}, \beta_{31}^{(PVI)}, \beta_{32}^{(PVI)} \rangle, \quad (3.11)
$$

where:

$$
\beta_{21}^{(PVI)} := (\sigma_1^{(PVI)})^2,
$$

$$
\beta_{31}^{(PVI)} := (\sigma_2^{(PVI)})^{-1}(\sigma_1^{(PVI)})^2\sigma_2^{(PVI)},
$$

$$
\beta_{32}^{(PVI)} := (\sigma_2^{(PVI)})^2. \quad (3.12)
$$

Note that sub-indices $ij$ in the generator $\beta_{ij}^{(PVI)}$ determine the $q_{ij}$ that is actually fixed during the action of the generator.

Before proceeding, we reformulate action (3.12) in a slightly different way. Given $q$ satisfying the cubic relation (3.7), we can define $(q, \omega)$ as follows:

$$
(q, \omega) := (q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4), \quad (3.13)
$$
where $\omega_i$ for $i = 1, \ldots, 4$ are defined in (3.8). Hence, action (3.12) on $q$ is equivalent to the following action on $(q, \omega)$:

$$
\beta_{21}^{(PVI)}(q, \omega) := (q_{21}, \omega_2 - q_{31} - q_{21}q_{32}, \omega_1 - q_{32} - q_{21}(\omega_2 - q_{31} - q_{21}q_{32}), \omega),
$$

$$
\beta_{31}^{(PVI)}(q, \omega) := (\omega_3 - q_{21} - q_{31}(\omega_1 - q_{21}q_{31} - q_{32}), q_{31}, \omega_1 - q_{21}q_{31} - q_{32}, \omega),
$$

$$
\beta_{32}^{(PVI)}(q, \omega) := (\omega_3 - q_{21} - q_{31}q_{32}, \omega_2 - q_{31} - q_{32}(\omega_3 - q_{21} - q_{31}q_{32}), q_{32}, \omega).
$$

(3.14)

This permits us to identify an orbit with a couple $(q, \omega)$.

The following Lemma, that can be found in [DM00], describes, in a geometric manner, the action of the pure braid group (3.11) and a necessary condition for this action in order to be finite:

**Lemma 36.** Suppose $q \in \hat{M}_{PVI}$ generates a finite $P_3$-orbit, then only two possibilities arise:

(i) Or $q$ satisfies:

$$
q_{ij} = 2 \cos \pi r_{ij}, \ r_{ij} \in \mathbb{Q}, \ 0 \leq r_{ij} \leq 1, \ i, j = 1, 2, 3, \ i > j.
$$

(3.15)

(ii) Or there exists a pure braid $\beta_{ij}^{(PVI)}$, for some choice of indices $i, j = 1, 2, 3, \ i > j$, such that $\beta_{ij}^{(PVI)}(q) = q$. Then $q_{ij}$ is a complex parameter satisfying:

$$
q_{ki} = \frac{(2 \omega_i - \omega_j q_{ij})}{(4 - q_{ij}^2)}, \ q_{kj} = \frac{(2 \omega_i - \omega_j q_{ij})}{(4 - q_{ij}^2)}, \ i, j, k = 1, 2, 3, \ k > i > j.
$$

(3.16)

**Proof.** Without loss of generality, we prove the Lemma for $i = 2$ and $j = 1$. The proof in the other cases can be obtained in a similar way.
Consider the generator $\beta_{21}^{\text{(PVI)}}$ defined in (3.14), it fixes the coordinate $q_{21}$ and it acts as a linear transformation on the variables $(q_{31}, q_{32})$. The cubic relation (3.7) is a conic in $(q_{31}, q_{32})$:

$$q_{31}^2 + q_{32}^2 + q_{21}(q_{31}q_{32}) - \omega_1q_{32} - \omega_2q_{31} - (\omega_3q_{21} - \omega_4 + 4) = 0, \quad (3.17)$$

that is invariant under the action of $\beta_{21}^{\text{(PVI)}}$.

If $q_{21} = \pm 2$, then $r_{21} = 0$ or $r_{21} = 1$ and the statement follows. Hereafter suppose $q_{21} \neq \pm 2$:

(i) The linear action of $\beta_{21}^{\text{(PVI)}}$ on $(q_{31}, q_{32})$ describe a rotation $R$ of $(q_{31}, q_{32})$ on the conic (3.17). If $\theta$ is the angle of the rotation $R$, then $q_{21} = 2 \cos \theta$. Moreover if $\theta$ is a rational multiple of $\pi$, then:

$$\exists \ n \in \mathbb{N} \ \text{s.t.} \ R^n = \text{Id}.$$  

The action of $\beta_{21}^{\text{(PVI)}}$ produce a finite orbit on $(q_{31}, q_{32})$ if and only if $q_{21} = 2 \cos \theta$ where $\theta$ is a rational multiple of $\pi$.

(ii) Suppose $q$ is a fixed point of the braid $\beta_{21}^{\text{(PVI)}}$, i.e. $\beta_{21}^{\text{(PVI)}}(q) = q$, then:

$$q_{31} = \frac{(2\omega_2 - \omega_1q_{21})}{(4 - q_{21}^2)},$$
$$q_{32} = \frac{(2\omega_1 - \omega_2q_{21})}{(4 - q_{21}^2)}.$$  

and $q_{21}$ is a complex parameter.

This concludes the proof.  

As a consequence of the previous Lemma, the classification of all finite orbits of the $P_3$-action on $\mathcal{M}_{\text{PVI}}$ reduces to the classification of triples of
rational angles $\pi r_{ij}$ or fixed points. This classification has been achieved by Lisovyy and Tykhyy in [LT14] modulo action of the symmetry group $G_{PVI}$ that we will describe in the next section.

### 3.2 PVI: symmetry group $G_{PVI}$

In this Section, we introduce the group of symmetries $G_{PVI}$ acting on the affine algebraic variety $\mathfrak{M}_{PVI}$. An element of $G_{PVI}$ is defined as follows:

**Definition 37.** A *symmetry* is an invertible map $\Phi : \mathfrak{M}_{PVI} \to \mathfrak{M}_{PVI}$ such that given an element $q$ and its orbit $O_{P_1}(q)$, the following is true:

$$|O_{P_1}(\Phi(q))| = |O_{P_1}(q)|.$$  

(3.18)

As a consequence, if $q$ generates a finite orbit of length $N$, then $\Phi(q)$ generates a finite orbit of the same length. We say that the orbits generated by $q$ and $\Phi(q)$ are equivalent. As for the Garnier system $\mathcal{G}_2$, we firstly study the group of Bäcklund transformations for PVI, the Sixth Painlevé equation, and subsequently we extend this group with more simpler transformations acting on the space of monodromy matrices, in order to obtain the group of symmetries $G_{PVI}$.

#### 3.2.1 Okamoto transformations of PVI

In this Section we study symmetries of $\mathfrak{M}_{PVI}$ that are derived from the so called group of Bäcklund transformations for PVI. Even if this group is isomorphic to the extended affine Weyl group of type $F_4$, see Okamoto [Oka86], that is an infinite group, we show that the action of the extended affine group $F_4$ is finite when we express it in terms of co-adjoint coordinates
Bäcklund transformations are birational maps sending a solution $\nu(u)$ of PVI with a fixed set of parameters $(\theta_1, \theta_2, \theta_3, \theta_\infty)$ to a new solution $\nu'(u')$ of PVI with a new set of parameters $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_\infty)$. Moreover, since these maps are birational, they send algebraic solutions to algebraic solutions, preserving the number of branches. In particular, these transformations act also on the affine algebraic variety $\tilde{\mathcal{M}}_{G_2}$, sending finite orbits to finite orbits and preserving their length. Two orbits related by such transformations are said to be equivalent.

Okamoto [Oka86] showed that the group of birational canonical transformations of the Hamiltonian system associated to the PVI equation, can be identified by an isomorphism with the extended affine Weyl group of type $F_4$. Usually when the context involves the Sixth Painlevé equation, Bäcklund transformations are referred as Okamoto transformations.

Generators for Bäcklund transformations for PVI are listed in Table 3.1. The first five transformations $s_1, s_2, s_3, s_\infty, s_\delta$ generate a group isomorphic to the affine Weyl group of type $D_4$, while transformations $r_1, r_2, r_3$ generate the Klein four-group $K_4$ and permutations $P_{13}, P_{23}$ generate the symmetric group $S_3$. Enlarging the set of generators of affine group $D_4$ by the generators of $K_4$ and $S_3$ the extended affine Weyl group of type $F_4$ is obtained.

We now describe the action of the extended affine Weyl group of type $F_4$ on the quotient space $\tilde{\mathcal{M}}_{PVI}$ endowed with the co-adjoint coordinates $q$ introduced in Section 3.1. If the action of the pure braid group $P_3$ over $(q, \omega)$, defined in (3.13) is finite, then the action of the extended affine Weyl group of type $F_4$ over $(q, \omega)$ is finite too.

Among all transformations in Table 3.1, we need to be particularly careful on how we treat transformation $s_\delta$, since the parameter $\delta$ can be modified.
Table 3.1: Okamoto bi-rational transformations for Painlevé VI, where:

\[ \delta = \frac{\theta_1 + \theta_2 + \theta_3 + \theta_\infty}{2}. \]

by other generators in Table 3.1. Indeed consider a transformation \( \Phi \) in the extended affine group \( F_4 \) then:

\[ \Phi : (\theta_1, \ldots, \theta_\infty) \mapsto (\Phi(\theta_1), \ldots, \Phi(\theta_\infty)) = (\theta'_1, \ldots, \theta'_\infty) \]

then \( s_\delta = s_{\delta'} \) where:

\[ \delta' = \frac{\theta'_1 + \theta'_2 + \theta'_3 + \theta'_\infty}{2}. \]

We want to develop a strategy in order to deal in an easy way with \( s_\delta \).

The following Lemma appears in the works of Terajima [Ter03] and Inaba, Iwasaki and Sato [IIS04], in particular we propose it here as in [IIS04]:

**Lemma 38.** The quantities \((q, \varpi)\), defined in (3.13), are invariants under the action of the transformations \( s_i \) for \( i = 1, \ldots, \infty, \delta \).

As direct consequence of Lemma 38, the affine group \( D_4 \) (in particular transformation \( s_\delta \)) acts trivially on \((q, \varpi)\). Now we study the action of the remaining generators in Table 3.1 on \((q, \varpi)\):
Theorem 39. The generators of Bäcklund transformations listed in Table 3.1 act on (3.13) as follows:

\[
s_i(q, \omega) \mapsto (q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4), \quad i = 1, 2, 3, \infty, \delta,
\]

\[
r_1(q, \omega) \mapsto (-q_{21}, -q_{31}, q_{32}, \omega_1, -\omega_2, -\omega_3, \omega_4),
\]

\[
r_2(q, \omega) \mapsto (-q_{21}, q_{31}, -q_{32}, -\omega_1, \omega_2, -\omega_3, \omega_4),
\]

\[
r_3(q, \omega) \mapsto (q_{21}, -q_{31}, -q_{32}, -\omega_1, -\omega_2, \omega_3, \omega_4),
\]

\[
P_{13}(q, \omega) \mapsto (q_{32}, \omega_2 - q_{31} - q_{21}q_{32}, q_{21}, \omega_3, \omega_2, \omega_1, \omega_4),
\]

\[
P_{23}(q, \omega) \mapsto (\omega_2 - q_{31} - q_{21}q_{32}, q_{21}, q_{32}, \omega_1, \omega_3, \omega_2, \omega_4).
\]

Proof. The proof for transformations \(s_1, s_2, s_3, s_\infty\) and \(s_\delta\) is a direct consequence of Lemma 38. We proceed with the proof for transformations \(r_1, r_2, r_3\) and \(P_{13}, P_{23}\). In particular we prove in details the statement for \(r_1\) and \(P_{13}\), then for the remaining transformations the proof proceeds in a similar way.

Suppose \(\nu(u)\) is a solution to the PVI equation (2.6). Transformation \(r_1\) leaves the independent variable \(u\) unchanged but not the dependent variable:

\[
\nu(u) \mapsto \tilde{\nu} = \frac{u}{\nu(u)}.
\]

By Theorems 2.2 - 2.2' - 2.2'' in [DM00], in the sectors \(\Sigma_0, \Sigma_1, \Sigma_\infty\) of neighbourhoods of the singular points 0, 1 and \(\infty\) respectively, is as follows:

\[
\nu(u) \sim \begin{cases} 
a_0 u^{1-\sigma_0} + \ldots, & \text{for } u \to 0, u \in \Sigma_0, \\
1 - a_1 (1-u)^{1-\sigma_1} + \ldots, & \text{for } u \to 1, u \in \Sigma_1, \\
\alpha_\infty u^{\sigma_\infty} + \ldots, & \text{for } u \to \infty, u \in \Sigma_\infty,
\end{cases}
\] (3.20)
The asymptotic behaviour (3.20) can be used in order to determine the action of $r_1$ over the quantities $q_{ij}$. Indeed the $q_{ij}$ are related to the exponents of the leading terms in the asymptotic behaviour (3.20) by the identities:

\begin{align*}
q_{21} &= 2 \cos \pi \sigma_0, \quad 0 \leq \sigma_0 < 1, \\
q_{32} &= 2 \cos \pi \sigma_1, \quad 0 \leq \sigma_1 < 1, \\
q_{31} &= 2 \cos \pi \sigma_\infty, \quad 0 \leq \sigma_\infty < 1.
\end{align*}

(3.21) (3.22) (3.23)

We compute now the asymptotic expansion for $\tilde{\nu}$:

$$\tilde{\nu}(u) = \frac{u}{\nu(u)} \sim \begin{cases} 
\frac{1}{a_0} u^{\sigma_0} + \ldots, & \text{for } u \to 0, u \in \Sigma_0, \\
\frac{1}{1-a_1(1-u)^{1-\sigma_1}} u + \ldots, & \text{for } u \to 1, u \in \Sigma_1, \\
\frac{1}{a_\infty} u^{1-\sigma_\infty} + \ldots, & \text{for } u \to \infty, u \in \Sigma_\infty.
\end{cases}$$

(3.24)

By uniqueness of the asymptotic behaviour and the fact that the independent variable $u$ is invariant under $r_1$, we can compare the exponents of the leading terms in (3.20) and (3.24), obtaining $\tilde{\sigma}_0 = 1 - \sigma_0$, $\tilde{\sigma}_1 = \sigma_1$ and $\tilde{\sigma}_\infty = 1 - \sigma_\infty$.

Finally, using equations (3.21)-(3.22), we obtain a change of sign in $q_{21}$ and $q_{31}$. Moreover the action of $r_1$ on quantities $\omega_i$ for $i = 1, 2, 3, 4$, defined in (3.8), can be directly calculated by the action of $r_1$ on the $\theta_i$ as listed in Table 3.1 and relations $q_i = 2 \cos(\pi \theta_i)$ for $i = 1, 2, 3, \infty$.

Consider now transformation $P_{13}$. It acts not only on $\nu(u)$ but also on $u$:

$$\nu(u) \to \tilde{\nu}(u) = 1 - \nu(u),$$

$$u \to \tilde{u} = 1 - u.$$
The action of this transformation can no longer be calculated using the asymptotic behaviours of $\nu(u)$ near the singular points, indeed the asymptotics of $\nu(u)$ are defined locally in proper sectors $\Sigma_i$ for $i = 0, 1, \infty$ but the transformation $P_{13}$ is acting globally by the change of the temporal variable $u$. Following the approach of Guzzetti in [Guz08], consider the Fuchsian system associated to $\nu(u)$:

$$\frac{d\Psi}{dz} = \left[ \frac{A_1}{z-u_1} + \frac{A_2}{z-u_2} + \frac{A_3}{z-u_3} \right] \Psi, \quad (3.25)$$

with $u_1 = 0, u_2 = u, u_3 = 1$ and the Fuchsian system associated to $\tilde{\nu}(\tilde{u})$ obtained applying transformation $P_{13}$:

$$\frac{d\tilde{\Psi}}{d\tilde{z}} = \left[ \frac{\tilde{A}_1}{\tilde{z}-\tilde{u}_1} + \frac{\tilde{A}_2}{\tilde{z}-\tilde{u}_2} + \frac{\tilde{A}_3}{\tilde{z}-\tilde{u}_3} \right] \tilde{\Psi}. \quad (3.26)$$

The two systems are related by a diagonal gauge transformation and the exchange of points $u_1 = 0$ and $u_3 = 1$. This exchange generates a new basis of loops $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ in the fundamental group of the Riemann sphere with 4 boundary components. Since monodromy preserving deformations are considered, the basis $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ can be written in terms of the original basis $\gamma_1, \gamma_2, \gamma_3$. As a consequence, setting $(N_1, N_2, N_3)$ and $(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$ to be respectively the monodromy matrices associated to a fundamental solution of (3.25) and (3.26), then they satisfy:

$$\tilde{N}_1 = N_3, \quad (3.27)$$

$$\tilde{N}_2 = N_3 N_2 N^{-1}_3, \quad (3.28)$$

$$\tilde{N}_3 = N_3 N_2 N^{-1}_2 N^{-1}_3, \quad (3.29)$$
and:

\[
\begin{align*}
\tilde{q}_1 &= \text{Tr } \tilde{N}_1 = \text{Tr } N_3 = q_3, \\
\tilde{q}_2 &= \text{Tr } \tilde{N}_2 = \text{Tr } N_3 N_2 N_3^{-1} = q_2, \\
\tilde{q}_3 &= \text{Tr } \tilde{N}_3 = \text{Tr } N_3 N_2 N_1 N_3^{-1} = q_1, \\
\tilde{q}_x &= \text{Tr } \tilde{N}_3 \tilde{N}_2 \tilde{N}_1 = \text{Tr } N_3 N_2 N_1 = q_x, \\
\tilde{q}_{21} &= \text{Tr } \tilde{N}_2 \tilde{N}_1 = \text{Tr } N_3 N_2 = q_{32}, \\
\tilde{q}_{31} &= \text{Tr } \tilde{N}_3 \tilde{N}_1 = \text{Tr } N_3 N_2 N_1 N_2^{-1} = \omega_2 - q_{31} - q_{21} q_{32}, \\
\tilde{q}_{32} &= \text{Tr } \tilde{N}_3 \tilde{N}_2 = \text{Tr } N_3 N_2 N_1 N_3^{-1} = q_{21}.
\end{align*}
\]

and the action of \(P_{13}\) is obtained.

**Corollary 40.** The group:

\[
\langle s_1, s_2, s_3, s_x, s_8, r_1, r_2, r_3, P_{13}, P_{23} \rangle,
\]

is a group of symmetries for \(\widehat{N}_{PV1}\).

**Proof.** The statement follows by construction.

We want now to show that given \((q, \omega)\) such that its orbit is finite under the action of \(P_3\), then the set of all possible \((q, \omega)\) obtained by acting on all points in the orbit with the group of transformations generated by \(\langle r_1, r_2, r_3, P_{13}, P_{23} \rangle\) is finite too. Before we need few Lemmata, the first one is about the finiteness of group \(\langle r_1, r_2, r_3 \rangle\):

**Lemma 41.** Transformations \(r_i\) for \(i = 1, 2, 3\) generate the Klein four group \(K_4\).

**Proof.** We prove that \(\langle r_1, r_2, r_3 \rangle \cong K_4\). If \(a, b, c\) are the generators of the
group $K_4$, then it has presentation:

$$a^2 = b^2 = c^2 = abc = 1$$

(3.31)

Consider $a = r_1, b = r_2, c = r_3$ then (3.31) is satisfied and the isomorphism follows.

By Lemma 41, since $K_4$ is finite, its action on a finite set produces again a finite set. Now we show that $P_{13}$ and $P_{23}$ act as braids:

**Lemma 42.** Transformations $P_{13}$ and $P_{23}$ are such that:

$$P_{13}(q, \omega) = \sigma_1^{(PVI)} \sigma_2^{(PVI)} \sigma_1^{(PVI)}(q, \omega),$$

$$P_{23}(q, \omega) = \sigma_2^{(PVI)} -1(q, \omega).$$

(3.32)

**Proof.** Given the definition (3.10) of the generators $\sigma_i^{(PVI)}$ for $i = 1, 2$ of the full braid group $B_3$ and the action of $P_{13}, P_{23}$ given in (3.19), then identities (3.32) follow.

Finally, given $(q, \omega)$ generating a finite orbit under the action of the braid group, the next Lemma ensures that $(q, \omega)$ has finite orbit also under the action of the group $\langle r_1, r_2, r_3, P_{13}, P_{23} \rangle$. However, before we proceed with the next Lemma, we need some consideration about the classification of Lisovyy and Tykhyy [LT14]:

**Remark 43.** Note that in [LT14], if an orbit is finite under the action of the pure braid group $P_3$, then it is finite under the action of the full braid group $B_3$. Indeed in [LT14] all orbits are classified under the action of the three generators $x, y, z$ (we keep for the moment the same notations as in [LT14]) of the pure braid group $P_3$ (see their definition in (7) of [LT14]).
If we consider the transformations \( p_{12}q \) (defined in Section 2.2 of [LT14]) and their compositions, we obtain the following identities:

\[
x = (12)\beta_x (123)^2, \\
y = (12)\beta_z, \\
z = (123)(12)(123)\beta_x.
\]

We can now state the Lemma:

**Lemma 44.** If \((q, \omega)\) generates a finite orbit under the action of full braid group \(B_3\) and \(N\) is the length of the orbit, then \((q, \omega)\) generates a finite orbit under the action of the group:

\[
G_{PVI}^{(1)} := \langle r_1, r_2, r_3, P_{13}, P_{23} \rangle,
\]

and the orbit has at most \(4N\) elements.

**Proof.** In order to prove the statement, we firstly prove the following relations:

\[
P_{13}r_1 = r_3P_{13}, \\
P_{13}r_2 = r_2P_{13}, \\
P_{13}r_3 = r_1P_{13}, \\
P_{23}r_1 = r_1P_{23}, \\
P_{23}r_2 = r_3P_{23}, \\
P_{23}r_3 = r_2P_{23}.
\]

Thanks to (3.37), we are allowed to split the action of the whole group \(\langle r_1, r_2, r_3, P_{13}, P_{23} \rangle\), into the two separate actions of groups \(\langle P_{13}, P_{23} \rangle\) and
\( \langle r_1, r_2, r_3 \rangle \). Since \(< P_{13}, P_{23} > \subset B_3\), then:

\[
|O_{\langle P_{13}, P_{23} \rangle}(q, \omega)| = N, \quad N \in \mathbb{N}.
\] (3.38)

Moreover, by Lemma 41:

\[
|O_{\langle r_1, r_2, r_3 \rangle}(q, \omega)| = 4,
\] (3.39)

and relations (3.37), if we act on each element in \( O_{\langle P_{13}, P_{23} \rangle}(q, \omega) \) with \( < r_1, r_2, r_3 > \) we obtain:

\[
|O_{\langle r_1, r_2, r_3, P_{13}, P_{23} \rangle}(q, \omega)| \leq 4N.
\] (3.40)

This completes the proof. \( \square \)

By Lemmata 38 and 44 the action of extended affine group \( F_4 \) in terms of \( (q, \omega) \) reduces to the action of group \( \langle r_1, r_2, r_3, P_{13}, P_{23} \rangle \) and if \( |O_{B_3}(q, \omega)| < \infty \) then also the orbit generated acting on \( (q, \omega) \) with \( \langle r_1, r_2, r_3, P_{13}, P_{23} \rangle \) will be finite.

We focus in the next part on how we can calculate the action of affine group \( D_4 \) generated by \( s_i \) for \( i = 1, \ldots, \delta \) over \( (q_1, q_2, q_3, q_4) \). As said at the beginning of the Section, in general transformations of affine group \( D_4 \) don’t act trivially on (3.5) because of the particular nature of \( s_\delta \). Anyhow, suppose we know \( (q, \omega) \), then \( (q_1, q_2, q_3, q_4) \) must be a solution of (3.8) for the given \( \omega \). Moreover, due to invariance of \( (q, \omega) \) under the action of transformations \( s_i \) for \( i = 1, \ldots, \delta \), we expect that solutions \( (q_1, q_2, q_3, q_4) \) could be also related by the lift of some transformations in the affine \( D_4 \) group in Table 3.1, to co-adjoint coordinates \( q \). This observation is formalized in Proposition 10 in the work of Lisovyy and Tykhyy [LT14] where transfor-
mations of the affine group $D_4$ linking solutions $(q_1, q_2, q_3, q_\infty)$ of (3.8) are explicitly given by the authors. Following Lemma recalls Proposition 10 in [LT14]:

**Lemma 45.** Suppose $\omega_1, \omega_2, \omega_3, \omega_4$ are given and consider system (3.8) in the variables $q_1, q_2, q_3, q_\infty$, then this system could have at most 24 solutions and any two solutions are related by transformations on the $\theta_i$ for $i = 1, \ldots, \infty$ of the affine group $D_4$. The 24 transformations are:

\[ id, \]  
\[ (s_1s_2s_3s_\infty s_\delta)^2, \]  
\[ s_6s_1s_2s_\delta s_3 s_\infty, \]  
\[ s_6s_1s_2s_\delta s_2 s_\infty, \]  
\[ (s_1s_2s_3s_\infty s_\delta)^2(s_6s_1s_2s_\delta s_3 s_\infty), \]  
\[ (s_1s_2s_3s_\infty s_\delta)^2(s_6s_1s_2s_\delta s_2 s_\infty), \]  
\[ (s_6s_1s_2s_\delta s_3 s_\infty)(s_6s_1s_2s_\delta s_2 s_\infty), \]  
\[ (s_1s_2s_3s_\infty s_\delta)^2(s_6s_1s_2s_\delta s_3 s_\infty)(s_6s_1s_2s_\delta s_2 s_\infty), \]  
\[ s_\delta \]  
\[ (s_1s_2s_3s_\infty s_\delta)^2 s_\delta, \]  
\[ s_6s_1s_2s_\delta s_3 s_\infty s_\delta, \]  
\[ s_6s_2s_\delta s_2 s_\infty s_\delta, \]  
\[ (s_1s_2s_3s_\infty s_\delta)^2(s_6s_1s_2s_\delta s_3 s_\infty)s_\delta, \]  
\[ (s_1s_2s_3s_\infty s_\delta)^2(s_6s_1s_2s_\delta s_2 s_\infty)s_\delta, \]  
\[ (s_6s_1s_2s_\delta s_3 s_\infty)(s_6s_1s_2s_\delta s_2 s_\infty)s_\delta, \]  
\[ (s_1s_2s_3s_\infty s_\delta)^2(s_6s_1s_2s_\delta s_3 s_\infty)(s_6s_1s_2s_\delta s_2 s_\infty)s_\delta, \]  
\[ s_6s_1. \]
\[(s_1 s_2 s_3 s_x s_6) s_6 s_1, \quad (3.58)\]
\[s_6 s_1 s_2 s_6 s_3 s_x s_6 s_1, \quad (3.59)\]
\[s_6 s_1 s_3 s_6 s_2 s_x s_6 s_1, \quad (3.60)\]
\[(s_1 s_2 s_3 s_x s_6)^2 (s_6 s_1 s_2 s_6 s_3 s_x) s_6 s_1, \quad (3.61)\]
\[(s_1 s_2 s_3 s_x s_6)^2 (s_6 s_1 s_3 s_6 s_2 s_x) s_6 s_1, \quad (3.62)\]
\[s_6 s_1 s_2 s_6 s_3 s_x s_6 s_1, \quad (3.63)\]
\[(s_1 s_2 s_3 s_x s_6)^2 (s_6 s_1 s_2 s_6 s_3 s_x) (s_6 s_1 s_3 s_6 s_2 s_x) s_6 s_1. \quad (3.64)\]

### 3.2.2 Symmetries of monodromy matrices

In the following we remind the reader about other trivial symmetries on the space of monodromy matrices:

(i) Independent sign changes $\epsilon_i = \pm 1$ of the matrices $N_i$ for $i = 1, 2, 3$, due to Schlessinger transformations on the Fuchsian system $(3.2)$ studied by Jimbo and Miwa in [JM81]:

\[
(N_1, N_2, N_3, N_x) \mapsto (\epsilon_1 N_1, \epsilon_2 N_2, \epsilon_3 N_3, \epsilon_1 \epsilon_2 \epsilon_3 (N_3 N_2 N_1)^{-1}).
\]

(ii) Permutations of the matrices $N_i$ for $i = 1, 2, 3$:

\[
(N_1, N_2, N_3, N_x) \mapsto (N_{\xi(1)}^{-1}, N_{\xi(2)}^{-1}, N_{\xi(3)}^{-1}, (N_{\xi(3)} N_{\xi(2)} N_{\xi(1)})^{-1}),
\]

where $\xi$ is any permutation in $S_3$, the symmetric group over $3$ elements.
Given \( n = (N_1, N_2, N_3) \in \widehat{M}_{PVI} \), we call the transformations that change sign of the matrices \( N_i \) *sign flips* and they are defined as:

\[
\text{sign}_{(\epsilon_1, \epsilon_2, \epsilon_3)} : (N_1, N_2, N_3, N_\infty) \mapsto (\epsilon_1 N_1, \epsilon_2 N_2, \epsilon_3 N_3, \epsilon_1 \epsilon_2 \epsilon_3 N_\infty),
\]

where \( \epsilon_k = \pm 1 \) and we included the action on the monodromy matrix \( N_\infty \) as well. The following three sign flips generate all the others:

\[
\begin{align*}
\text{sign}_1 &:= \text{sign}_{(-1,1,1)} : (N_1, N_2, N_3, N_\infty) \mapsto (-N_1, N_2, N_3, -N_\infty), \quad (3.65) \\
\text{sign}_2 &:= \text{sign}_{(1,-1,1)} : (N_1, N_2, N_3, N_\infty) \mapsto (N_1, -N_2, N_3, -N_\infty), \quad (3.66) \\
\text{sign}_3 &:= \text{sign}_{(1,1,-1)} : (N_1, N_2, N_3, N_\infty) \mapsto (N_1, N_2, -N_3, -N_\infty), \quad (3.67)
\end{align*}
\]

and they satisfy:

\[
\begin{align*}
\text{sign}_1^2 &= \text{sign}_2^2 = \text{sign}_3^2 = 1, \\
\text{sign}_1 \text{sign}_2 &= \text{sign}_2 \text{sign}_1, \\
\text{sign}_1 \text{sign}_3 &= \text{sign}_3 \text{sign}_1, \\
\text{sign}_2 \text{sign}_3 &= \text{sign}_3 \text{sign}_2, \quad (3.68)
\end{align*}
\]

as a consequence, the group of sign flips is finite and it is isomorphic to the group \( C_2 \times C_2 \times C_2 \), where \( C_2 \) is the cyclic group of order 2.

We need following Lemma in order to prove that sign flips are symmetries of \( \widehat{M}_{PVI} \):

**Lemma 46.** Given \( \text{sign} \in \langle \text{sign}_1, \text{sign}_2, \text{sign}_3 \rangle \) and a braid \( \sigma \in B_3 \), then there exists \( \text{sign}' \in \langle \text{sign}_1, \text{sign}_2, \text{sign}_3 \rangle \) and \( \sigma' \in B_3 \) such that:

\[
\sigma \text{sign} = \text{sign}' \sigma'.
\]
Proof. Given \( n = (N_1, N_2, N_3) \in \mathcal{M}_{PV/I} \) we prove the result on the generators \( \sigma_i \) for \( i = 1, 2 \) of the full braid group \( B_3 \), i.e. we show that:

\[
\sigma_i \text{sign}_j = \text{sign}_{j'} \sigma_{i'},
\]

for some choice of the indices \( i, i' = 1, 2 \) and \( j, j' = 1, 2, 3 \). Suppose we consider \( \sigma_1 \) and \( \text{sign}_1 \), then:

\[
\sigma_1 \text{sign}_1(n) = \sigma_1(-N_1, N_2, N_3) = (N_2, -N_2N_1N_2^{-1}, N_3) = \\
= \text{sign}_2(N_2, N_2N_1N_2^{-1}, N_3) = \text{sign}_2 \sigma_1(n).
\]

(3.69)

In a similar way we can prove all the following equations:

\[
\sigma_1 \text{sign}_2 = \text{sign}_1 \sigma_1, \\
\sigma_1 \text{sign}_3 = \text{sign}_3 \sigma_1, \\
\sigma_2 \text{sign}_1 = \text{sign}_1 \sigma_2, \\
\sigma_2 \text{sign}_2 = \text{sign}_3 \sigma_2, \\
\sigma_2 \text{sign}_3 = \text{sign}_2 \sigma_2.
\]

(3.70)

This conclude the proof. \( \Box \)

In Table 3.2 we summarize the action of the sign flips in terms of the co-adjoint coordinates \( q \) and the quantities \( \omega_i \), as defined in (3.8).

<table>
<thead>
<tr>
<th>\text{sign}_1</th>
<th>q_1</th>
<th>q_2</th>
<th>q_3</th>
<th>q_x</th>
<th>q_{21}</th>
<th>q_{31}</th>
<th>q_{32}</th>
<th>\omega_1</th>
<th>\omega_2</th>
<th>\omega_3</th>
<th>\omega_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{sign}_2</td>
<td>-q_1</td>
<td>q_2</td>
<td>q_3</td>
<td>-q_x</td>
<td>-q_{21}</td>
<td>-q_{31}</td>
<td>-q_{32}</td>
<td>\omega_1</td>
<td>-\omega_2</td>
<td>-\omega_3</td>
<td>\omega_4</td>
</tr>
<tr>
<td>\text{sign}_3</td>
<td>q_1</td>
<td>-q_2</td>
<td>q_3</td>
<td>-q_x</td>
<td>-q_{21}</td>
<td>q_{31}</td>
<td>-q_{32}</td>
<td>-\omega_1</td>
<td>\omega_2</td>
<td>-\omega_3</td>
<td>\omega_4</td>
</tr>
</tbody>
</table>

Table 3.2: Action of the sign flips in terms of the co-adjoint coordinates \( q \).
Corollary 47. The group \( \langle \text{sign}_1, \text{sign}_2, \text{sign}_3 \rangle \) is a group of symmetries for \( \hat{M}_{PV_1} \).

Proof. The statement is a consequence of Lemma 46 and Table 3.2. \( \square \)

At this point we introduce the permutations on the elements of a triple of monodromy matrices in \( \hat{M}_{PV_1} \). The symmetric group on three elements \( S_3 = \{\text{id}, (12), (13), (23), (123), (132)\} \) is generated by \( (123) \) and \( (12) \), i.e. \( S_3 = \langle (12), (123) \rangle \). We describe the action of the \( S_3 \) on \( n \) defining its two generators \( (123) \) and \( (12) \) as:

\[
(123) : (N_1, N_2, N_3, N_\infty) \mapsto (N_3, N_1, N_2, (N_2N_1N_3)^{-1}) , \quad (3.71)
\]

\[
(12) : (N_1, N_2, N_3, N_\infty) \mapsto (N_2^{-1}, N_1^{-1}, N_3^{-1}, (N_3N_1N_2)^{-1}) . \quad (3.72)
\]

The action on the monodromy matrices of the entire group \( S_3 \) is:

\[
(12) : (N_1, N_2, N_3, N_\infty) \mapsto (N_2^{-1}, N_1^{-1}, N_3^{-1}, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1}) , \quad (3.73)
\]

\[
(13) : (N_1, N_2, N_3, N_\infty) \mapsto (N_3^{-1}, N_2^{-1}, N_1^{-1}, (N_1^{-1}N_2^{-1}N_3^{-1})^{-1}) , \quad (3.74)
\]

\[
(23) : (N_1, N_2, N_3, N_\infty) \mapsto (N_1^{-1}, N_3^{-1}, N_2^{-1}, (N_2^{-1}N_3^{-1}N_1^{-1})^{-1}) , \quad (3.75)
\]

\[
(123) : (N_1, N_2, N_3, N_\infty) \mapsto (N_3, N_1, N_2, (N_2N_1N_3)^{-1}) , \quad (3.76)
\]

\[
(132) : (N_1, N_2, N_3, N_\infty) \mapsto (N_2, N_3, N_1, (N_1N_3N_2)^{-1}) . \quad (3.77)
\]

As in the case of sign flips, we need following Lemma in order to prove that permutations are symmetries of \( \hat{M}_{PV_1} \):
Lemma 48. Given $\xi \in S_3$ and $\sigma \in B_3$, then there exists $\xi' \in S_3$ and $\sigma' \in B_3$ such that:

$$\sigma \xi = \xi' \sigma'.$$

Proof. Given $n = (N_1, N_2, N_3) \in \hat{\mathcal{M}}_{PVI}$ we prove the result on the generators $\sigma_i^{(PVI)}$ for $i = 1, 2, 3$ of the full braid group $B_3$, i.e. we show that:

$$\xi \sigma_i^{(PVI)} = \sigma_{i'}^{(PVI)} \xi', \tag{3.78}$$

for some choice of the indices $i, i' = 1, 2, 3$ and $\xi, \xi' \in \langle (12), (123) \rangle$.

We prove (3.78) for $(12)$ and $\sigma_2^{(PVI)}$:

$$\sigma_2(12)(N_1, N_2, N_3, N_\infty) =$$

$$= \sigma_2(N_2^{-1}, N_1^{-1}, N_3^{-1}, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1}) =$$

$$= (N_2^{-1}, N_3^{-1}, N_3^{-1}N_1^{-1}N_3, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1}).$$

The triple of monodromy matrices is in $GL(2)^3/\text{GL}(2)$ then:

$$(N_2^{-1}, N_3^{-1}, N_3^{-1}N_1^{-1}N_3, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1}) =$$

$$= N_3(N_2^{-1}, N_3^{-1}, N_3^{-1}N_1^{-1}N_3, (N_3^{-1}N_1^{-1}N_2^{-1})^{-1})N_3^{-1} =$$

$$= (N_3N_2^{-1}N_3^{-1}, N_3^{-1}, N_1^{-1}, (N_1^{-1}N_2^{-1}N_3^{-1})^{-1}),$$

Now, if we consider $\langle 23 \rangle = (123)(12)$ and $\sigma_2^{(PVI)}$, then:

$$\langle 23 \rangle(12)\sigma_2(N_1, N_2, N_3, N_\infty) =$$

$$= \langle 23 \rangle(12)(N_1, N_3, N_3N_2N_3^{-1}, N_\infty) =$$

$$= \langle 23 \rangle(N_3^{-1}, N_1^{-1}, N_3N_2^{-1}N_3^{-1}, (N_3N_2^{-1}N_3^{-1})^{-1}N_1^{-1}N_3^{-1}) =$$

$$= (N_3N_2^{-1}N_3^{-1}, N_3^{-1}, N_1^{-1}, (N_1^{-1}N_2^{-1}N_3^{-1})^{-1}).$$
and (3.78) follows.

The following relations can be proven in a similar way:

\[
\begin{align*}
\sigma_2(12) &= (123)(12)\sigma_2, \\
\sigma_1(123) &= (132)\sigma_1^{-1}, \\
\sigma_1(12) &= (12)\sigma_1^{-1}, \\
\sigma_2(123) &= (123)\sigma_1, \\
\sigma_2^{-1}(12) &= (12)(123)\sigma_1^{-1}, \\
\sigma_2^{-1}(123) &= (123)\sigma_1^{-1}, \\
\sigma_1^{-1}(123) &= \sigma_2, \\
\sigma_1^{-1}(12) &= (12)\sigma_1.
\end{align*}
\]

This completes the proof.

The action of permutations \( \langle P_{13}, P_{23} \rangle \) is given, in terms of \( q \) and \( \omega_i \), in Table 3.3.

<table>
<thead>
<tr>
<th></th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( q_3 )</th>
<th>( q_5 )</th>
<th>( q_{21} )</th>
<th>( q_{31} )</th>
<th>( q_{32} )</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (12) )</td>
<td>( q_2 )</td>
<td>( q_1 )</td>
<td>( q_3 )</td>
<td>( q_5 )</td>
<td>( q_{21} )</td>
<td>( q_{31} )</td>
<td>( q_{32} )</td>
<td>( \omega_1 )</td>
<td>( \omega_2 )</td>
<td>( \omega_3 )</td>
<td>( \omega_4 )</td>
</tr>
<tr>
<td>( (123) )</td>
<td>( q_3 )</td>
<td>( q_1 )</td>
<td>( q_2 )</td>
<td>( q_5 )</td>
<td>( q_{21} )</td>
<td>( q_{31} )</td>
<td>( q_{32} )</td>
<td>( \omega_3 )</td>
<td>( \omega_1 )</td>
<td>( \omega_2 )</td>
<td>( \omega_4 )</td>
</tr>
</tbody>
</table>

Table 3.3: Action of the permutations in terms of the co-adjoint coordinates \( q \).

Corollary 49. The group \( \langle (12), (123) \rangle \) is a group of symmetries for \( \hat{\mathcal{M}}_{PVI} \).

**Proof.** The statement is a consequence of Lemma 48 and Table 3.3.

Next Lemma ensures us that sign flips (3.65)-(3.67) and permutations (3.71)-(3.72) generate a group and that this is a finite group:
Lemma 50. The group:

\[ G_{PVI}^{(2)} := \langle \text{sign}_1, \text{sign}_2, \text{sign}_3, (123), (12) \rangle, \]  

(3.79)

is a finite group of 48 elements.

Proof. In order to prove that the group \( G_{PVI}^{(2)} \) has 48 elements, we proceed proving the following relations between the generators:

\[
\begin{align*}
\text{sign}_1(123) &= (123)\text{sign}_3, \\
\text{sign}_2(123) &= (123)\text{sign}_1, \\
\text{sign}_3(123) &= (123)\text{sign}_2, \\
\text{sign}_1(12) &= (12)\text{sign}_2, \\
\text{sign}_2(12) &= (12)\text{sign}_1, \\
\text{sign}_3(12) &= (12)\text{sign}_3.
\end{align*}
\]  

(3.80)

As a consequence, by relations (3.80) and direct computation, the statement follows. \( \square \)

3.3 PVI: Classification of finite orbits

In this Section we resume the classification result, about all algebraic solutions of PVI, achieved by Lisovyy and Tykhyy in [LT14]. In particular, for each family of algebraic solutions, we describe the associated finite orbits of the action of \( P_3 \) over the Riemann sphere with four boundary components.

Each orbit will be presented as a couple \((q, \omega)\), defined in (3.13), and the action of \( P_3 \) is explicitly given in (3.14).

Four families of algebraic solutions of PVI can be distinguished:
• Okamoto solutions.

• three Kitaev-Hitchin solutions.

• Cayley-Picard solutions.

• 45 exceptional solutions.

The list of all finite orbits associate to these families of solutions is:

**Okamoto solutions.** Each orbit (orbits I in [LT14]) associated to this family of solutions consists of one point \((\varphi, \omega)\):

\[
\{(a, b, c, \omega)\},
\]

where \(a, b, c \in \mathbb{C}\) are free parameters and \(\omega_1, \omega_2, \omega_3, \omega_4\) satisfy:

\[
\begin{align*}
\omega_1 &= 2c + ab, \\
\omega_2 &= 2b + ac, \\
\omega_3 &= 2a + bc, \\
\omega_4 &= 4 + 2abc + a^2 + b^2 + c^2.
\end{align*}
\]

**Definition 51.** \(O\) is the set of all the *equivalent* orbits that satisfy (3.81) and (3.82).

**Hitchin-Kitaev solutions.** In this case we distinguish three sub-families of finite orbits:

• K-Type II;

• K-Type III;

• K-Type IV.
Orbits of **K-Type II** (orbits II in [LT14]) consist of two points:

\[
\{(0, 0, a, \omega), (0, 0, b, \omega)\},
\]

(3.83)

where \(a, b \in \mathbb{C}, a \neq b\) is free parameter and \(\omega_1, \omega_2, \omega_3, \omega_4\) satisfy:

\[
\begin{align*}
\omega_1 &= a + b, \\
\omega_2 &= \omega_3 = 0, \\
\omega_4 &= 4 + ab.
\end{align*}
\]

(3.84)

**Definition 52.** \(K_{II}\) is the set of all the *equivalent* orbits that satisfy (3.83) and (3.84).

Orbits of **K-Type III** (orbits III in [LT14]) consist of three points:

\[
\{(0, 0, 1, \omega), (a, 0, 1, \omega), (0, a, 1, \omega)\},
\]

(3.85)

where \(a \in \mathbb{C}, a \neq 0\) is free parameter and \(\omega_1, \omega_2, \omega_3, \omega_4\) satisfy:

\[
\begin{align*}
\omega_1 &= 2, \\
\omega_2 &= \omega_3 = a, \\
\omega_4 &= 5.
\end{align*}
\]

(3.86)

**Definition 53.** \(K_{III}\) is the set of all the *equivalent* orbits that satisfy (3.85) and (3.86).

Finally, orbits of **K-Type IV** (orbits IV in [LT14]) consist of four points:

\[
\{(1, 1, 1, \omega), (a, 1, 1, \omega), (1, a, 1, \omega), (1, 1, a, \omega)\},
\]

(3.87)
where \( a \in \mathbb{C} \), \( a \neq 1 \) is free parameter and \( \omega_1, \omega_2, \omega_3, \omega_4 \) satisfy:

\[
\begin{align*}
\omega_1 &= \omega_2 = \omega_3 = a + 2, \\
\omega_4 &= 3(a + 2).
\end{align*}
\] (3.88)

**Definition 54.** \( K_{IV} \) is the set of all the *equivalent* orbits that satisfy (3.87) and (3.88).

**Cayley-Picard solutions.** The orbits associated to this family of solutions can be generated from the points:

\[
(-2 \cos \pi (a + b), 2 \cos \pi a, 2 \cos \pi b, \omega), \ a, b \in \mathbb{Q},
\] (3.89)

and \( \omega_1, \omega_2, \omega_3, \omega_4 \) satisfy:

\[
\omega_1 = \omega_2 = \omega_3 = \omega_4 = 0.
\] (3.90)

For this particular family the length of orbits varies with the choice of parameters \( a \) and \( b \).

**Definition 55.** \( CP \) is the set of all the *equivalent* orbits that satisfy (3.89) and (3.90).

**Orbits associated to the 45 exceptional solutions.** We summarize the 45 representative of the associated orbits in Table 3.4 (that is exactly Table 5 in [LT14]). The first column identifies the orbit while the second one indicates how many points there are in the orbit. The 4-tuple in the central column gives the values of the parameters (3.8) and in the last column the values \( n_{ij} \) are such that:

\[
q_{ij} = 2 \cos \pi n_{ij}, \ i, j = 1, 2, 3, \ i > j.
\] (3.91)
<table>
<thead>
<tr>
<th>( n )</th>
<th>( \omega_1, \omega_2, \omega_1, 4 - \omega_2 )</th>
<th>( (n_{31}, n_{32}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((0, 1, 1, 0))</td>
<td>((2, 3, 1, 3/3))</td>
</tr>
<tr>
<td>2</td>
<td>((3, 2, 2, 3))</td>
<td>((1, 3, 1, 3/3))</td>
</tr>
<tr>
<td>3</td>
<td>((1, 0, 0, 2))</td>
<td>((1, 2, 1, 3/3))</td>
</tr>
<tr>
<td>4</td>
<td>((\sqrt{2}, 0, 0, 1))</td>
<td>((1, 4, 1, 3/3))</td>
</tr>
<tr>
<td>5</td>
<td>((3, 2 \sqrt{2}, 3, 2, -4))</td>
<td>((1, 2, 1/4, 1/4))</td>
</tr>
<tr>
<td>6</td>
<td>((1 - \sqrt{5}, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, -2 + \sqrt{5}))</td>
<td>((4, 5, 1/3, 3/3))</td>
</tr>
<tr>
<td>7</td>
<td>((1 + \sqrt{5}, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, -2 - \sqrt{5}))</td>
<td>((2, 5, 1/3, 3/3))</td>
</tr>
<tr>
<td>8</td>
<td>((1, 1, 1, 0))</td>
<td>((1/2, 1/2, 1/2))</td>
</tr>
<tr>
<td>9</td>
<td>((2, 0, 0, 0))</td>
<td>((0, 1/2, 3/2))</td>
</tr>
<tr>
<td>10</td>
<td>((1, \sqrt{2}, \sqrt{2}, 0))</td>
<td>((0, 1/2, 1/2))</td>
</tr>
<tr>
<td>11</td>
<td>((\frac{1 + \sqrt{5}}{2}, 1, 1, -\frac{\sqrt{5} + 1}{2}))</td>
<td>((1/3, 1/2, 1/2))</td>
</tr>
<tr>
<td>12</td>
<td>((\frac{1 - \sqrt{5}}{2}, 1, 1, \frac{\sqrt{5} - 1}{2}))</td>
<td>((1/3, 1/2, 1/2))</td>
</tr>
<tr>
<td>13</td>
<td>((2 - \sqrt{5}, 2 - \sqrt{5}, 2 - \sqrt{5}, 2 + \sqrt{5}))</td>
<td>((4, 5/3, 5/3))</td>
</tr>
<tr>
<td>14</td>
<td>((2 + \sqrt{5}, 2 + \sqrt{5}, 2 + \sqrt{5}, 2 - \sqrt{5}))</td>
<td>((2, 5/1, 5/1))</td>
</tr>
<tr>
<td>15</td>
<td>((1, 0, 0, 1))</td>
<td>((1/3, 1/3, 2/3))</td>
</tr>
<tr>
<td>16</td>
<td>((3 - \sqrt{5}, 3 - \sqrt{5}, 3 - \sqrt{5}, 3 + \sqrt{5}))</td>
<td>((3, 5/3, 5/3))</td>
</tr>
<tr>
<td>17</td>
<td>((3 + \sqrt{5}, 3 + \sqrt{5}, 3 + \sqrt{5}, 3 - \sqrt{5}))</td>
<td>((1/5, 1/5, 1/5))</td>
</tr>
<tr>
<td>18</td>
<td>((-\frac{\sqrt{5} - 1}{2}, -\frac{\sqrt{5} - 1}{2}, -\frac{\sqrt{5} - 1}{2}, 0))</td>
<td>((1/2, 1/2, 1/2))</td>
</tr>
<tr>
<td>19</td>
<td>((\frac{\sqrt{5} + 1}{2}, -\frac{\sqrt{5} + 1}{2}, -\frac{\sqrt{5} + 1}{2}, 0))</td>
<td>((1/2, 1/2, 1/2))</td>
</tr>
<tr>
<td>20</td>
<td>((0, 0, 0, 3))</td>
<td>((2, 3/1, 4/1))</td>
</tr>
<tr>
<td>21</td>
<td>((1, 0, 0, 2))</td>
<td>((0, 1/4, 3/4))</td>
</tr>
<tr>
<td>22</td>
<td>((2, \sqrt{5}, \sqrt{5}, -2))</td>
<td>((1, 5/2, 5/2))</td>
</tr>
<tr>
<td>23</td>
<td>((\frac{3 + \sqrt{5}}{2}, \frac{3 + \sqrt{5}}{2}, \frac{3 + \sqrt{5}}{2}, -\sqrt{5}))</td>
<td>((2, 5/2, 5/2))</td>
</tr>
<tr>
<td>24</td>
<td>((\frac{3 - \sqrt{5}}{2}, -\frac{3 - \sqrt{5}}{2}, -\frac{3 - \sqrt{5}}{2}, \sqrt{5}))</td>
<td>((4/5, 4/5, 4/5))</td>
</tr>
<tr>
<td>25</td>
<td>((\frac{\sqrt{5} + 1}{2}, -\frac{\sqrt{5} + 1}{2}, -\frac{\sqrt{5} + 1}{2}, 1/0))</td>
<td>((1/2, 1/2, 1/2))</td>
</tr>
<tr>
<td>26</td>
<td>((\frac{\sqrt{5} - 1}{2}, -\frac{\sqrt{5} - 1}{2}, -\frac{\sqrt{5} - 1}{2}, \sqrt{5} - 1))</td>
<td>((1/2, 3/5, 3/5))</td>
</tr>
<tr>
<td>27</td>
<td>((\frac{\sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}, -\sqrt{5} - 1))</td>
<td>((1/2, 1/5, 1/5))</td>
</tr>
<tr>
<td>28</td>
<td>((\frac{\sqrt{5}}{2}, -\sqrt{5} - 1, -\sqrt{5} - 1, \sqrt{5} - 1))</td>
<td>((3/5, 4/5, 4/5))</td>
</tr>
<tr>
<td>29</td>
<td>((\frac{\sqrt{5}}{2}, -1 + \sqrt{5}, 1 + \sqrt{5}, -\sqrt{5} - 1))</td>
<td>((1/5, 2/5, 2/5))</td>
</tr>
<tr>
<td>30</td>
<td>((0, 0, 0, 2))</td>
<td>((2/3, 2/3, 2/3))</td>
</tr>
<tr>
<td>31</td>
<td>((2, 2, 2, -1))</td>
<td>((0, 1/5, 3/5))</td>
</tr>
<tr>
<td>32</td>
<td>((1 - 2 \cos 2\pi/7, 1 - 2 \cos 2\pi/7, 1 - 2 \cos 2\pi/7, 4 \cos 2\pi/7))</td>
<td>((6/7, 5/7, 5/7))</td>
</tr>
<tr>
<td>33</td>
<td>((1 - 2 \cos 4\pi/7, 1 - 2 \cos 4\pi/7, 1 - 2 \cos 4\pi/7, 4 \cos 4\pi/7))</td>
<td>((2/7, 4/7, 4/7))</td>
</tr>
<tr>
<td>34</td>
<td>((1 - 2 \cos 6\pi/7, 1 - 2 \cos 6\pi/7, 1 - 2 \cos 6\pi/7, 4 \cos 6\pi/7))</td>
<td>((4/7, 6/7, 6/7))</td>
</tr>
<tr>
<td>35</td>
<td>((\frac{3 - \sqrt{5}}{2}, 0, 0, 1 + \sqrt{5}))</td>
<td>((0/1, 3/2, 3/2))</td>
</tr>
<tr>
<td>36</td>
<td>((\frac{3 + \sqrt{5}}{2}, 0, 0, 1 - \sqrt{5}))</td>
<td>((0, 1/3, 2/3))</td>
</tr>
<tr>
<td>37</td>
<td>((\frac{1 + \sqrt{5}}{2}, -\frac{1 + \sqrt{5}}{2}, -\frac{1 + \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}))</td>
<td>((2/3, 3/5, 3/5))</td>
</tr>
<tr>
<td>38</td>
<td>((\frac{1 + \sqrt{5}}{2}, -\frac{1 + \sqrt{5}}{2}, -\frac{1 + \sqrt{5}}{2}, 1/1))</td>
<td>((2/3, 1/5, 1/5))</td>
</tr>
<tr>
<td>39</td>
<td>((4, 1, 1, 1))</td>
<td>((1/5, 1/2, 1/2))</td>
</tr>
<tr>
<td>40</td>
<td>((\frac{3 - \sqrt{5}}{2}, 0, 0, 3 - \sqrt{5}))</td>
<td>((2/3, 2/3, 2/3))</td>
</tr>
<tr>
<td>41</td>
<td>((\frac{3 + \sqrt{5}}{2}, 0, 0, 3 + \sqrt{5}))</td>
<td>((2/3, 2/3, 2/3))</td>
</tr>
<tr>
<td>42</td>
<td>((0, 0, 0, 2))</td>
<td>((0, 1/5, 3/5))</td>
</tr>
<tr>
<td>43</td>
<td>((0, 0, 0, 2))</td>
<td>((0, 1/5, 3/5))</td>
</tr>
<tr>
<td>44</td>
<td>((0, 0, 0, 2))</td>
<td>((0, 1/5, 3/5))</td>
</tr>
<tr>
<td>45</td>
<td>((0, 0, 0, 3))</td>
<td>((1/2, 1/5, 2/5))</td>
</tr>
</tbody>
</table>

Table 3.4: 45 Exceptional solutions.
We define the following set:

**Definition 56.** $E_{45}$ is the set of all the *equivalent* orbits obtained from Table 3.4.

It is important to note that, depending on the family of algebraic solutions of PVI, there is a different number of associated finite orbits. Let us resume this fact:

- The set $O$ of orbits associated to Okamoto solutions is an infinite set.
- The sets $K_{II}, K_{III}, K_{IV}$ of orbits associated to Hitchin-Kitaev solutions are infinite sets.
- The set $CP$ of orbits associated to Cayley-Picard solutions is an infinite set.
- The set $E_{45}$ of orbits associated to the 45 exceptional solutions in Table 3.4 is a finite set.

Since the sets $O, K_{II}, K_{III}, K_{IV}$ and $CP$ are infinite, for the moment we focus on the set $E_{45}$.

### 3.3.1 Expansion algorithm for Table 3.4

In the last part of this Section we explain how, given an element $(q, \omega)$ in Table 3.4, we can generate the set $E_{45}$ of all equivalent orbits under the groups of symmetries studied in the Section 3.2.

The group $G_{PV1}$ of Okamoto transformations of the Sixth Painlevé equation acts as $K_3 \rtimes S_3$ on $\omega = (\omega_1, \ldots, \omega_4)$ [LT14]. Extending this action to $q = (q_{21}, q_{31}, q_{32})$, we obtain Theorem 39. Moreover, we observe that $P_{13}$ and $P_{23}$ are elements of the braid group $B_3$, and since we act only on points
that have finite orbits under the action of the braid group (see Remark 43),
the action of the whole group $F_4$ produces a finite set of values. All these
values will be in the form $(q, \omega)$, in order to extract $q_1, q_2, q_3$ and $q_x$ we use
the fact that we can consider the relations (3.8) as a system of equations in
$q_1, q_2, q_3$ and $q_x$ and that each $q_i$ has the form:

$$q_i = 2 \cos \pi \theta_i, \quad i = 1, 2, 3, \infty.$$  

One particular solution of equations (3.8) is listed in [LT14] in terms of $\theta_1, \theta_2, \theta_3, \theta_x$ for each point in the Table (3.4). We can then compute all other
solutions $q_1, q_2, q_3$ and $q_x$ by using Lemma 41.

Consider $(q, \omega)$ in the Lisovyy and Tykhyy’s sublist summarized in Table
3.4, then the following *expansion Algorithm* generates all equivalent orbits:

**Algorithm 1.**

For every line of Table 3.4, consider $(q, \omega)$ and a solution $(q_1, q_2, q_3, q_x)$
of system (3.8):

1. Apply to $(q_1, q_2, q_3, q_x, q_{21}, q_{31}, q_{32}, \omega)$ all 48 transformations of the
group $G_{PV1}^{(2)}$. Save the results in a set $E_0$.

   For every element $(q'_1, q'_2, q'_3, q'_x, q'_{21}, q'_{31}, q'_{32}, \omega') \in E_0$:

2. Generate the orbit of $(q'_1, q'_2, q'_3, q'_x, q'_{21}, q'_{31}, q'_{32}, \omega')$ under the action of
the group $B_3$. Save the result in a set $E_1$.

   For every element $(q''_1, q''_2, q''_3, q''_x, q''_{21}, q''_{31}, q''_{32}, \omega'') \in E_1$:

3. Apply to $(q''_1, q''_2, q''_3, q''_x, q''_{21}, q''_{31}, q''_{32}, \omega'')$ all the 24 transformations listed
in Lemma 45 and save the result in the set $E_2$. 

For every element $(q''_1, q''_2, q''_3, q''_x, q''_21, q''_31, q''_32, \omega'') \in E^2$:

4. Generate the $P_3$-orbit of $(q''_1, q''_2, q''_3, q''_x, q''_21, q''_31, q''_32, \omega'')$ and save the result in the set $E_{45}$.

Once the Algorithm ends, due to Lemmata 38, 44, 45, 50, the set $E_{45}$ will contain only a finite number of equivalent orbits.

**Remark 57.** Consider $(q_1, q_2, q_3, q_x, q_{21}, q_{31}, q_{32})$, then in Algorithm 1, the order we apply the transformations of groups $G^{(1)}_{PV_I}$ and $G^{(2)}_{PV_I}$, defined respectively in (3.36) and (3.79), is not relevant. Indeed, the following relations hold true:

\[
\begin{align*}
P_{13}\text{sign}_1 &= \text{sign}_3 P_{13}, \\
P_{13}\text{sign}_2 &= \text{sign}_2 P_{13}, \\
P_{13}\text{sign}_3 &= \text{sign}_1 P_{13}, \\
P_{23}\text{sign}_1 &= \text{sign}_1 P_{23}, \\
P_{23}\text{sign}_2 &= \text{sign}_3 P_{23}, \\
P_{23}\text{sign}_3 &= \text{sign}_2 P_{23}, \\
P_{13}(123) &= (123)^2 P_{23}, \\
P_{23}(123) &= \beta_{32}^{(PVI)} P_{13}, \\
P_{13}(12) &= (123)(12) P_{13} P_{23} P_{13}, \\
P_{23}(12) &= (12) P_{23} P_{23} P_{13}, \\
r_1\text{sign}_1 &= \text{sign}_1 r_1, \\
r_1\text{sign}_2 &= \text{sign}_1 \text{sign}_3 r_1, \\
r_1\text{sign}_3 &= \text{sign}_2 \text{sign}_1 r_1, \\
r_2\text{sign}_1 &= \text{sign}_2 \text{sign}_3 r_2, \\
r_2\text{sign}_2 &= \text{sign}_2 r_2,
\end{align*}
\]
\[ r_2 \text{sign}_3 = \text{sign}_2 \text{sign}_1 r_2, \]
\[ r_3 \text{sign}_1 = \text{sign}_2 \text{sign}_3 r_3, \]
\[ r_3 \text{sign}_2 = \text{sign}_1 \text{sign}_3 r_3, \]
\[ r_3 \text{sign}_3 = \text{sign}_3 r_3, \]
\[ r_1(12) = (12)r_2, \]
\[ r_1(123) = (12)r_3, \]
\[ r_2(12) = (12)r_1, \]
\[ r_2(123) = (12)r_1, \]
\[ r_3(12) = (12)r_3, \]
\[ r_3(123) = (12)r_2. \]

In the next Chapter we will describe Algorithms that implement the matching procedure over co-adjoint coordinates \( \hat{q}, \bar{q}, \tilde{q}, \bar{q} \) over \( M_{PV1} \), that eventually leads to the classification of \( p_i, p_{ij}, p_{ijk} \) in the open subset \( U \subset \tilde{\mathcal{M}}_{G_2} \), that possibly will generate a finite orbit under the action of the pure braid group \( P_4 \).
Chapter 4

Matching

In this Chapter, we explain how to implement our methodology. We are classifying finite orbits of action (1.1) in the following way: if $p \in \mathcal{U} \subset \mathcal{M}_{g_2}$, then there exist four restrictions $H_1, \ldots, H_4$, see Theorem 12, such that each restriction acts on $p$ as the pure braid group $P_3$ over $\mathcal{M}_{PV_I}$, i.e., the $\text{SL}_2(\mathbb{C})$ character variety of the Riemann sphere with four boundary components. In particular each subgroup acts respectively on projections $\hat{q}, \tilde{q}, \bar{q}, \tilde{q}$ as summarized in Table 1.1.

If $p$ generates a finite orbit under the action (1.1), then the orbits of $\hat{q}, \tilde{q}, \bar{q}, \tilde{q}$ under the restricted action of the respective $H_i$ must be finite too. In the previous Chapter, we described the list of all such $\hat{q}, \tilde{q}, \bar{q}, \tilde{q}$: the list is infinite (this is an issue in order to develop our method), but it contains a finite sublist, called $E_{45}$ (see Definition 56), that will be crucial in the classification presented in this thesis.

In Section 4.1, we propose a procedure that, given three projection points, produces points $p$ that satisfy the necessary conditions to generate a finite orbit under the action of three of the restrictions $H_1, \ldots, H_4$: we call these points candidate points.
In Sections 4.2 and 4.3, we introduce algorithms that produce the set \( C \) of candidate points \( p \) such that:

(C1) Three over four projections are in the set \( E_{45} \).

(C2) Two over four projections are in the set \( O \) and one of the remaining projections is in the set \( E_{45} \).

(C3) Two over four projections are in the set \( E_{45} \) and one of the remaining two projections is in the set \( O \).

Moreover, all of these algorithms exploit the fact that \( E_{45} \) is finite and therefore, they generate a set \( C \) that will be finite too.

Afterwards, since the set \( C \) will contain only a finite number of elements, we extract from \( C \) a list of points such that they produce finite orbits under the action of the pure braid group \( P_3 \). Finally we present a list of 54 finite orbits up to the action of the group \( G \), i.e. the group of symmetries of \( G_2 \) discussed in Chapter 2. The list of finite orbits is presented in Table 4.2.

### 4.1 Outline of the procedure

In order to better describe the set \( C \), we introduce the following Definition:

**Definition 58.** A point \( p \) such that its four projections \( \tilde{q}, \tilde{q}, \tilde{q}, \tilde{q} \), defined in (1.114), generate finite orbits under the action of \( P_3 \) is said to be a candidate point.

In this Section, we propose a procedure to construct all candidate points \( p \) in the big open subset \( U \subset \mathcal{M}_{G_2} \). Note that, to generate a candidate point \( p \), it is not necessary to know all four projections \( \tilde{q}, \tilde{q}, \tilde{q}, \tilde{q} \). Indeed, looking at Table 1.1, if we know only three projections over four, then only one
Table 4.1: Matching with three points: elements on the same column must be equal.

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
<th>$p_{21}$</th>
<th>$p_{31}$</th>
<th>$p_{32}$</th>
<th>$p_{41}$</th>
<th>$p_{42}$</th>
<th>$p_{43}$</th>
<th>$p_{432}$</th>
<th>$p_{421}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
<td>$\tilde{q}_3$</td>
<td>$\tilde{q}_{21}$</td>
<td>$\tilde{q}_{31}$</td>
<td>$\tilde{q}_{32}$</td>
<td>$\tilde{q}_{41}$</td>
<td>$\tilde{q}_{42}$</td>
<td>$\tilde{q}_{43}$</td>
<td>$\tilde{q}_{432}$</td>
<td>$\tilde{q}_{421}$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
<td>$\tilde{q}_3$</td>
<td>$\tilde{q}_{21}$</td>
<td>$\tilde{q}_{31}$</td>
<td>$\tilde{q}_{32}$</td>
<td>$\tilde{q}_{41}$</td>
<td>$\tilde{q}_{42}$</td>
<td>$\tilde{q}_{43}$</td>
<td>$\tilde{q}_{432}$</td>
<td>$\tilde{q}_{421}$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
<td>$\tilde{q}_3$</td>
<td>$\tilde{q}_{21}$</td>
<td>$\tilde{q}_{31}$</td>
<td>$\tilde{q}_{32}$</td>
<td>$\tilde{q}_{41}$</td>
<td>$\tilde{q}_{42}$</td>
<td>$\tilde{q}_{43}$</td>
<td>$\tilde{q}_{432}$</td>
<td>$\tilde{q}_{421}$</td>
</tr>
</tbody>
</table>

$p_{ijk}$ will be missing, but we can calculate it choosing appropriately one of the four relations $f_1, \ldots, f_4$, defined in (1.53)-(1.56). This leads to state a matching procedure on three points. For example, we can define the following matching procedure for the three points $\tilde{q}, \tilde{q}, \tilde{q}$:

**Procedure 1.**

1. Consider $(\tilde{q}, \tilde{q}, \tilde{q}) \in \mathcal{F} \times \mathcal{F} \times \mathcal{F}$.

2. Check if $\tilde{q}, \tilde{q}, \tilde{q}$ satisfy relations given by the columns of Table 4.1, then go to the next Step, otherwise go to Step 1.

3. Calculate the two roots $p_{321}^{(i)}$, for $i = 1, 2$, of the equation (1.53) in which we express $p_1, p_2, p_3, p_{21}, p_{31}, p_{32}$ in terms of co-adjoint coordinates $\tilde{q}, \tilde{q}, \tilde{q}$:

$$
p_{321}^2 + p_{321}(\tilde{q}_1 \tilde{q}_1 \tilde{q}_2 - \tilde{q}_{21} \tilde{q}_2 - \tilde{q}_1 \tilde{q}_{21} - \tilde{q}_1 \tilde{q}_{21}) +
\tilde{q}_1^2 + \tilde{q}_2^2 - \tilde{q}_1 \tilde{q}_1 \tilde{q}_{21} + \tilde{q}_{21}^2 + \tilde{q}_2^2 - \tilde{q}_1 \tilde{q}_2 \tilde{q}_{21} + \tilde{q}_{21}^2 - \tilde{q}_1 \tilde{q}_2 \tilde{q}_{21} +
\tilde{q}_{21} \tilde{q}_{21} \tilde{q}_{21} + \tilde{q}_{21}^2 - 4 = 0.
$$

(4.1)

For each $i = 1, 2$: 
4. For each root $p_{321}^{(i)}$, determine the value of $p_{x}^{(i)}$ using equation (1.116) written in terms of co-adjoint coordinates $\hat{q}, \tilde{q}, \bar{q}$ and $p_{321}^{(i)}$, using identities (1.108),(1.109),(1.110) and (1.111) as follows:

$$
p_{x}^{(i)} = \frac{1}{2}(\hat{q}_{1}\hat{q}_{1}\hat{q}_{2}\hat{q}_{3} - \hat{q}_{21}\hat{q}_{2}\hat{q}_{3} - \tilde{q}_{1}\tilde{q}_{21}\tilde{q}_{3} + p_{321}^{(i)}\tilde{q}_{3} - \hat{q}_{1}\tilde{q}_{2}\tilde{q}_{3} + \hat{q}_{21}\tilde{q}_{3}1 - q_{21}\tilde{q}_{3}1 + \hat{q}_{2}\tilde{q}_{x} - \tilde{q}_{1}\hat{q}_{i}\bar{q}_{3}2 + \tilde{q}_{21}\hat{q}_{3}2 + \hat{q}_{1}\bar{q}_{x} + \tilde{q}_{1}\hat{q}_{x}).
$$

(4.2)

5. Use identities (1.108),(1.109),(1.110) and (1.111) in order to determine the other components of $p_{x}^{(i)}$.

6. If $p_{x}^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise go to Step 1.

7. Save $p_{x}^{(i)}$ in the set $\mathcal{C}$, and go to Step 1.

The procedure ends when all possible choices of three points $\hat{q}, \tilde{q}, \bar{q} \in \mathcal{M}_{PV\ell}$ are exhausted.

Note that, since $\mathcal{FO}$ is not a finite set, this procedure may never end. However, we will adapt it in different cases in such a way to avoid this problem. For the sake of clarity, let us for the moment suppose that $\mathcal{FO}$ is finite.

In order to obtain the big set $\mathcal{C}$ of all candidate points, other three procedures similar to Procedure 1 must be developed. We summarize these three matching procedures on three points as follows:

(P1.1) Matching procedure with input triple $(\hat{q}, \tilde{q}, \bar{q})$: output set $\mathcal{C}$.

(P1.2) Matching procedure with input triple $(\hat{q}, \tilde{q}, \bar{q})$: output set $\mathcal{C}$.

(P1.3) Matching procedure with input triple $(\hat{q}, \tilde{q}, \bar{q})$: output set $\mathcal{C}$.
(P1.4) Matching procedure with input triple \((\hat{q}, \tilde{q}, \check{q})\): output set \(\tilde{C}\).

In order to construct the set \(C\), the union of all the above four sets \(\tilde{C}, \hat{C}, \check{C}, \bar{C}\) must be taken:

\[
C = \tilde{C} \cup \hat{C} \cup \check{C} \cup \bar{C}.
\]  

(4.3)

As we are going to show in the next Lemma, it is enough to know only one of the sets \(\tilde{C}, \hat{C}, \check{C}, \bar{C}\) to generate the whole set \(C\):

**Lemma 59.** Consider \(m \in \widetilde{\mathcal{M}}_{G_2}\) and the permutation \((1234)\), introduced in Section 2.5:

\[
(1234)(M_1, M_2, M_3, M_4, M_x) = (M_4, M_1, M_2, M_3, (M_3 M_2 M_1 M_4)^{-1}),
\]

that acts on the co-adjoint coordinates of \(m\), in the big open subset \(\mathcal{U} \subset \widetilde{\mathcal{M}}_{G_2}\), as follows:

\[
(1234)(p) = (p_4, p_1, p_2, p_3, p_x, p_{41}, p_{42}, p_{21}, p_{43}, p_{31}, p_{32}, p_{421}, p_{321}, p_{431}, p_{432}).
\]

If \(\tilde{C}, \hat{C}, \check{C}, \bar{C}\) are the sets of candidate points \(p\) obtained running respectively procedures (P1.1),(P1.2),(P1.3),(P1.4), then:

\[
(1234)(\tilde{C}) = \check{C},
\]  

(4.4)

\[
(1234)(\hat{C}) = \bar{C},
\]  

(4.5)

\[
(1234)(\check{C}) = \hat{C},
\]  

(4.6)

\[
(1234)(\bar{C}) = \tilde{C}.
\]  

(4.7)

**Proof.** We proceed proving the statement of the Theorem for (4.4), then in a similar way the statement can be proved for (4.5)-(4.7).

Consider \(\tilde{n}, \hat{n}, \check{n}, \bar{n} \in \widetilde{\mathcal{M}}_{PV1}\) and \(m \in \widetilde{\mathcal{M}}_{G_2}\). Apply to \(\tilde{n}, \hat{n}, \check{n}, \bar{n}\) the match-
ing procedure stated in (ii) in Theorem 12. If we don’t consider the projection \( \tilde{q} \), the matching for procedure (P1.1) is given by the following relations:

\[
\begin{align*}
\tilde{N}_1 &= M_2, \quad \tilde{N}_2 = M_3, \quad \tilde{N}_3 = M_4, \quad \tilde{N}_\infty = (M_4 M_3 M_2)^{-1}, \\
\tilde{N}_1 &= M_1, \quad \tilde{N}_2 = M_3, \quad \tilde{N}_3 = M_4, \quad \tilde{N}_\infty = (M_4 M_3 M_1)^{-1}, \\
\tilde{N}_1 &= M_1, \quad \tilde{N}_2 = M_2, \quad \tilde{N}_3 = M_4, \quad \tilde{N}_\infty = (M_4 M_2 M_1)^{-1}.
\end{align*}
\]

Consider \( m' = (1234)(m) \), then the above matching procedure becomes:

\[
\begin{align*}
\tilde{N}_1' &= M_1, \quad \tilde{N}_2' = M_2, \quad \tilde{N}_3' = M_3, \quad \tilde{N}_\infty' = (M_3 M_2 M_1)^{-1}, \\
\tilde{N}_1' &= M_4, \quad \tilde{N}_2' = M_2, \quad \tilde{N}_3' = M_3, \quad \tilde{N}_\infty' = (M_3 M_2 M_4)^{-1}, \\
\tilde{N}_1' &= M_4, \quad \tilde{N}_2' = M_1, \quad \tilde{N}_3' = M_3, \quad \tilde{N}_\infty' = (M_3 M_1 M_4)^{-1}.
\end{align*}
\]

where \( \tilde{n}', \tilde{n}, \tilde{n}' \in \hat{M}_{PV1} \). After the relabelling:

\[
\tilde{N}_i = \tilde{N}_{i}', \quad \tilde{N}'_i = \tilde{N}_i', \quad \tilde{N}_i = \tilde{N}_\infty', \quad i = 1, 2, 3, \infty,
\]

we obtain the relaxed matching procedure for algorithm (P1.2) that produces the set \( \hat{C} \).

In the following, when proposing a matching on three points, we will generate the set \( \hat{C} \), then we will construct the big set \( C \) of all candidate points, applying Lemma 59.

Now, we need to determine which points in \( C \) lead to a finite orbit of the \( P_4 \)-action. As mentioned above, for the moment we suppose the set \( \mathcal{FO} \) to be finite (this is not true but we will see how to adapt our procedures), consequently the set \( C \) will be finite and we can develop a way to check if \( p \in C \) may or may not generate a finite orbit, based on the following Lemma:
Lemma 60. Assume $C$ finite and let $p \in C$ a candidate point, then its orbit is finite if and only if $\beta(p) \in C$ for every braid $\beta \in P_4$.

Proof. Suppose $\beta(p) \in C$ for every $\beta \in P_4$, then the orbit is finite since $C$ is finite too. Vice versa, suppose $p$ has a finite $P_4$-orbit, then for every $\beta$, $\beta(p)$ must have a finite orbit. Hence, $\beta(p)$ must be an element of $C$. \hfill \square

We briefly give an explanation on how we are going to operatively use this Lemma. Indeed, in the set $C$, to select the finite orbits is equivalent to find the subset $C_0 \subset C$ such that:

$$C_0 = \{p \in C \mid \beta(p) \in C, \ \beta \in P_4\}. \quad (4.9)$$

As mentioned before, the group $P_4$ is an infinite group and, accordingly to this fact, we cannot implement an algorithm able to deal with every pure braid $\beta$ in $P_4$. Nevertheless, $P_4$ is finitely generated:

$$P_4 = \langle \beta_{21}, \beta_{31}, \beta_{32}, \beta_{41}, \beta_{42}, \beta_{43} \rangle, \quad (4.10)$$

where generators $\beta_{ij}$ are defined in (1.87). Now, we explain how we can check which element $p \in C$ generates a finite $P_4$-orbit. Since every braid $\beta \in P_4$ can be thought as an ordered combination of generators $\beta_{ij}$ (and their inverses too), we can introduce the so-called braid word, namely:

$$\beta = \underbrace{\beta_{i_1j_1} \ldots \beta_{i_nj_n}}_n, \quad (4.11)$$

where $n$ indicates the length of the word. Consider $p \in \mathcal{U} \subset \hat{\mathcal{M}}_{\mathcal{G}_2}$ and a
Chapter 4

braid $\beta \in P_4$ such that $\beta(p) \notin \mathcal{C}$. Moreover, consider the following notation:

$$
p^{(0)} = p,
\quad p^{(1)} = \beta_{ij}(p^{(0)}),
\quad \vdots
\quad p^{(n)} = \beta(p) = \beta_{ij}(p^{(n-1)}) = \beta_{ij} \ldots \beta_{ij}(p^{(0)}).
$$

Since we supposed $p^{(n)} \notin \mathcal{C}$, we need to delete $p^{(n)}$ from the set $\mathcal{C}$ and also the element $p_{n-1}$ and so on, till when $p^{(0)} = p$ is deleted from $\mathcal{C}$.

We will then find, in the set $\mathcal{C}_0$, all the points $p$ having finite $P_4$-orbit and we further factorize by the group of symmetries $G$ introduced in Section 2.5.

In the next Sections we are going to adapt these procedures to different cases to account for the fact that $\mathcal{F}_0$ is actually an infinite set.

### 4.2 Matching with three of the PVI 45 exceptional algebraic solutions

In this Section, we give an algorithm that produces the finite set $\mathcal{C}_{E_{45} \times E_{45} \times E_{45}}$ of all candidate points $p$ such that three over four projections $\hat{q}, \tilde{q}, \bar{q}, \tilde{q}$ defined in (1.114), are in the set $E_{45}$, i.e. the set of all equivalent orbits generated from Table 3.4.

We adapt Procedure 1 in such a way it can process the following triples:

$$
(\hat{q}, \tilde{q}, \bar{q}) \in E_{45} \times E_{45} \times E_{45}.
$$

The set $E_{45}$ is generated from Table 3.4, using the expansion Algorithm 1.
and it is finite. After generating $E_{45}$, the following algorithm produces the set of candidate points $\hat{C}_{E_{45} \times E_{45} \times E_{45}}$:

**Algorithm 2.**

1. Consider $(\tilde{q}, \tilde{q}, \tilde{q}) \in E_{45} \times E_{45} \times E_{45}$.

2. Check if $\tilde{q}, \tilde{q}, \tilde{q}$ satisfy relations given by the columns of Table 4.1, then go to the next Step, otherwise go to Step 1.

3. Determine the values $p_{321}^{(i)}$, for $i = 1, 2$, using equation (4.1).
   
   For each $i = 1, 2$:

4. Calculate the values of $p_{32}^{(i)}$ using equation (1.116).

5. Use identities (1.111), (1.108), (1.110) and (1.109) in order to determine the other components of $p^{(i)}$.

6. If $p^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise go to Step 1.

7. Save $p^{(i)}$ in the set $\hat{C}_{E_{45} \times E_{45} \times E_{45}}$, and go to Step 1.

Since $E_{45}$ is a finite set, the set $\hat{C}_{E_{45} \times E_{45} \times E_{45}}$ will be finite too. Finally the big set $C_{E_{45} \times E_{45} \times E_{45}}$ can be generated by Lemma 59 as follows:

$$
C_{E_{45} \times E_{45} \times E_{45}} = \hat{C}_{E_{45} \times E_{45} \times E_{45}} \bigcup_{i=1}^{3} (1234)^{i}(\hat{C}_{E_{45} \times E_{45} \times E_{45}}).
$$

The set $C_{E_{45} \times E_{45} \times E_{45}}$ contains all candidate points $p \in \mathcal{U} \subset \mathcal{M}_{92}$ such that three projections (1.114) are in the set $E_{45}$, whereas the remaining projection could be in any other sets: $O, K_{II}, K_{III}, K_{IV}, CP$ and $E_{45}$, see
Definitions (51) - (56) respectively. There are 3,355,200 candidate points in the set $C_{E_{45} \times E_{45} \times E_{45}}$.

4.3 Matching with Okamoto solutions

By Definition 51, the set $O$ is the set of all orbits related to the infinite family of algebraic solutions of Okamoto type for the PVI equation. The set $O$ is itself an infinite set: it will be crucial to adapt the matching procedure in such a way that the number of required steps is still finite.

We briefly recall which points $p$ are not relevant in our classification: we are going to exclude both cases when the monodromy group is reducible or there exists an index $i = 1, \ldots, 4, \infty$ such that $M_i = \pm \mathbb{1}$. Indeed, if the monodromy group is reducible the associated solution of $G_2$ can be reduced to classical solutions in terms of Lauricella hypergeometric functions, as proved by Mazzocco in the [Maz01a], while if $M_i = \pm \mathbb{1}$ for some index $i$, then, again following [Maz01a], the solution of $G_2$ can be reduced to solution of PVI. We formalize this fact in the following Definition:

**Definition 61.** A point $p$ is not relevant if the associated monodromy group is reducible or there exists an index $i = 1, \ldots, 4, \infty$ such that $M_i = \pm \mathbb{1}$.

The first result of this Section is:

**Theorem 62.** If a point $p \in \widetilde{\mathcal{M}}_{G_2}$ defined in (1.4) is such that any three of its four projections $\hat{q}, \bar{q}, \bar{q}, \hat{q}$, defined in (1.114), are in the set $O$ of all orbits related to the family of Okamoto solutions then the point $p$ is not relevant.

**Theorem 63.** Suppose $m = (M_1, M_2, M_3, M_4) \in \widetilde{\mathcal{M}}_{G_2}$ with co-adjoint coordinates $p$, defined in (1.4), in the big open subset $U \subset \widetilde{\mathcal{M}}_{G_2}$, defined in
(1.38). Then, \( M_x = \pm 1 \) if and only if \( p \) satisfies:

\[
p_4 = \pm p_{321}, \ p_x = \pm 2, \ p_{41} = \pm p_{32}, \ p_{42} = \pm (p_{321}p_2 + p_3p_1 - p_{31} - p_{21}p_{32}),
\]
\[
p_{43} = \pm p_{21}, \ p_{432} = \pm p_1, \ p_{431} = \pm p_2, \ p_{421} = \pm p_3.
\]

Consequently, all points \( p \) satisfying hypothesis of Theorems 62-63 will be irrelevant to our classification (and then excluded from it), as they are dealt with in [Maz01a].

Before proving Theorem 62, we will enunciate some more results allowing us to further restrict our input of Okamoto points into the matching procedure. All proofs are postponed to Section 4.5. To this aim we need the following two definitions:

**Definition 64.** The set \( O_{ID} \) is the set of all the \( q \in O \) such that the associated triple of monodromy matrices \( n \in \hat{\mathcal{M}}_{PVI} \) admits one matrix equals to \( \pm 1 \).

**Definition 65.** The set \( O_{RED} \) is the set of all the \( q \in O \) such that if we consider the associated triple of monodromy matrices \( n \in \hat{\mathcal{M}}_{PVI} \) then the monodromy group \( \langle N_1, N_2, N_3 \rangle \) is reducible, i.e. it admits a common subspace of dimension one.

Definitions 64 and 65 are given in terms of monodromy matrices: in the following, we will work both with triples \( n = (N_1, N_2, N_3) \in \hat{\mathcal{M}}_{PVI} \) and with 4-tuples of matrices \( m = (M_1, M_2, M_3, M_4) \in \hat{\mathcal{M}}_{G_2} \) and the associated co-adjoint coordinates \( q \) and \( p \), introduced in Sections 3.1 and 1.2 respectively.

We are ready to state the results in the following five Lemmata:

**Lemma 66.** If a point \( p \in \hat{\mathcal{M}}_{G_2} \), defined in (1.4), is such that one of its four projections \( \hat{q}, \hat{q}, \hat{q}, \hat{q} \), defined in (1.114), is in the set \( O_{ID} \) and another one projection is in the set \( O_{RED} \) then such point \( p \) is not relevant.
Lemma 67. Let $q$ be the co-adjoint coordinates on $\mathcal{M}_{PV1}$. If $q$ is in the set $O_{\text{RED}}$, then $q$ satisfies:

$$
\begin{align*}
q_{ij} &= \frac{1}{2}(q_{ii}q_{jj} - \epsilon_i \epsilon_j s_is_j), \quad i > j, \quad i, j = 1, 2, 3, \\
q_{ik} &= \frac{1}{4}(q_{i1}q_{23} - \epsilon_1 \epsilon_2 s_1s_2q_3 - \epsilon_1 \epsilon_3 s_1s_3q_2 - \epsilon_2 \epsilon_3 s_2s_3q_1)
\end{align*}
$$

(4.16)

where $s_k = \sqrt{4-q_k^2}$ and $\epsilon_k = \pm 1$ for $k = 1, 2, 3$.

Lemma 68. Suppose $p \in \mathcal{U} \subset \mathcal{M}_{G_2}$, defined in (1.4), and $q$ being co-adjoint coordinates on $\mathcal{M}_{PV1}$ of one over four projections $\hat{q}, \tilde{q}, \bar{q}, \tilde{q}$, defined in (1.114). If $q$ is in the set $O_{\text{ID}}$, then $q$ satisfies:

$$
\begin{align*}
q_{ij} &= \pm q_k, \\
q_{ik} &= \pm 2.
\end{align*}
$$

(4.17)

where $i, j, k = 1, 2, 3$ with $i > j$ and $k \neq i, k \neq j$.

Lemma 69. Suppose $p \in \mathcal{U} \subset \mathcal{M}_{G_2}$, defined in (1.4), is such that any two of its four projections $\hat{q}, \tilde{q}, \bar{q}, \tilde{q}$, defined in (1.114), are in the set $O_{\text{RED}}$ and $q$ being co-adjoint coordinates on $\mathcal{M}_{PV1}$ of one of the remaining projections, then there exists a couple of indices $(i, j), (i', j')$ with one index in $(i, j)$ equal to one index in $(i', j')$ such that:

$$
\begin{align*}
q_{ij}^2 + q_i^2 + q_j^2 - q_{ij}q_iq_j - 4 &= 0, \quad i > j, \quad i, j = 1, 2, 3, \\
q_{i'j'}^2 + q_{i'}^2 + q_{j'}^2 - q_{i'j'}q_{i'}q_{j'} - 4 &= 0, \quad i' > j', \quad i', j' = 1, 2, 3.
\end{align*}
$$

(4.18)

Lemma 70. If a point $p \in \mathcal{U} \subset \mathcal{M}_{G_2}$, defined in (1.4), is such that two of its three projections $\hat{q}, \tilde{q}, \bar{q}$, defined in (1.114), are in the set $O_{\text{ID}}$, then, if $\epsilon = \pm 1$, the following cases hold:
(i) If \( \bar{q}, \bar{q} \in O_{ID} \), then \( \bar{q} \) must satisfy:

\[
\bar{q}_2 = \bar{\epsilon} \bar{\epsilon} \bar{q}_1, \quad \bar{q}_{32} = \bar{\epsilon} \bar{\epsilon} \bar{q}_{31},
\]

(4.19)

and \( p \) is such that:

\[
p_1 = \bar{q}_1, \quad p_2 = \bar{\epsilon} \bar{q}_{31}, \quad p_3 = \bar{\epsilon} \bar{\epsilon} \bar{q}_1, \quad p_4 = \bar{\epsilon} \bar{q}_3, \quad p_{21} = \bar{\epsilon} \bar{q}_3, \quad p_{31} = \bar{\epsilon} \bar{q}_{21}, \quad p_{32} = \bar{\epsilon} \bar{q}_3,
\]

\[
p_{41} = \bar{q}_{31}, \quad p_{42} = \bar{\epsilon} \bar{q}_1, \quad p_{43} = \bar{\epsilon} \bar{\epsilon} \bar{q}_{31}, \quad p_{432} = \bar{\epsilon} 2, \quad p_{431} = \bar{\epsilon} \bar{q}_x, \quad p_{421} = \bar{\epsilon} 2.
\]

(4.20)

(ii) If \( \hat{q}, \bar{q} \in O_{ID} \), then \( \hat{q} \) must satisfy:

\[
\hat{q}_2 = \hat{\epsilon} \hat{\epsilon} \hat{q}_1, \quad \hat{q}_{32} = \hat{\epsilon} \hat{\epsilon} \hat{q}_{31},
\]

(4.21)

and \( p \) is such that:

\[
p_1 = \hat{q}_1, \quad p_2 = \hat{\epsilon} \hat{\epsilon} \hat{q}_1, \quad p_3 = \hat{\epsilon} \hat{q}_{31}, \quad p_4 = \hat{\epsilon} \hat{q}_3, \quad p_{21} = \hat{\epsilon} \hat{q}_3, \quad p_{31} = \hat{\epsilon} \hat{q}_3, \quad p_{32} = \hat{\epsilon} \hat{q}_3,
\]

\[
p_{41} = \hat{q}_{31}, \quad p_{42} = \hat{\epsilon} \hat{q}_{31}, \quad p_{43} = \hat{\epsilon} \hat{q}_1, \quad p_{432} = \hat{\epsilon} 2, \quad p_{431} = \hat{\epsilon} 2, \quad p_{421} = \hat{\epsilon} \bar{q}_x.
\]

(4.22)

(iii) If \( \bar{q}, \bar{q} \in O_{ID} \), then \( \hat{q} \) must satisfy:

\[
\hat{q}_2 = \hat{\epsilon} \hat{\epsilon} \hat{q}_1, \quad \hat{q}_{32} = \hat{\epsilon} \hat{\epsilon} \hat{q}_{31},
\]

(4.23)

and \( p \) is such that:

\[
p_1 = \hat{\epsilon} \hat{q}_{31}, \quad p_2 = \hat{q}_1, \quad p_3 = \hat{\epsilon} \hat{\epsilon} \hat{q}_1, \quad p_4 = \hat{\epsilon} \hat{q}_3, \quad p_{21} = \hat{\epsilon} \hat{q}_3, \quad p_{31} = \hat{\epsilon} \hat{q}_3, \quad p_{32} = \hat{\epsilon} \hat{q}_3,
\]

\[
p_{41} = \hat{\epsilon} \hat{q}_1, \quad p_{42} = \hat{q}_{31}, \quad p_{43} = \hat{\epsilon} \hat{\epsilon} \hat{q}_{31}, \quad p_{432} = \hat{q}_x, \quad p_{431} = \hat{\epsilon} 2, \quad p_{421} = \hat{\epsilon} 2.
\]

(4.24)
Lemmata 67,68,70 lead to the development of additional matching algorithms in order to complete our classification for the cases when these points are included.

Suppose that two over three projections are in the set $O_{\text{RED}}$ and one projection is in the set $E_{45}$. By Lemma 69 the projection in $E_{45}$ must satisfy conditions (4.18), for an appropriate choice of indices $(i, j)$ and $(i', j')$. It turned out that actually there are no elements in $E_{45}$ satisfying (4.18). As a consequence of this fact, there are no points $p \in U \subset \widehat{\mathcal{M}}_{G_2}$ with two projections in the set $O_{\text{RED}}$ and at least one of the remaining two projections in the set $E_{45}$, so the case in which one projection is in $E_{45}$ and two projections are in $O$ is classified by the set $C_{E_{45} \times O_{\text{ID}} \times O_{\text{ID}}}$ of all candidate points $p$ such that one over the four projections $\hat{q}, \tilde{q}, \bar{q}, \tilde{q}$ is in the set $E_{45}$ and two of the remaining projections are in the set $O_{\text{ID}}$.

To construct this set we proceed as follows: firstly we construct the set $\tilde{C}_{E_{45} \times O_{\text{ID}} \times O_{\text{ID}}}$, where one over the three projections $\hat{q}, \tilde{q}, \bar{q}$ is in the set $E_{45}$ and two of the remaining projections are in the set $O_{\text{ID}}$, then, applying Lemma 59 we generate the whole set $C_{E_{45} \times O_{\text{ID}} \times O_{\text{ID}}}$.

The set $\tilde{C}_{E_{45} \times O_{\text{ID}} \times O_{\text{ID}}}$ is the union of the following three sets of candidate points $p$:

(A3.1) $\tilde{C}_{E_{45} \times O_{\text{ID}} \times O_{\text{ID}}}$: candidate points $p$ with $\hat{q}, q, \bar{q} \in O_{\text{ID}}$, $\tilde{q} \in E_{45}$.

(A3.2) $\tilde{C}_{E_{45} \times O_{\text{ID}} \times O_{\text{ID}}}$: candidate points $p$ with $\hat{q}, \tilde{q} \in O_{\text{ID}}$, $q \in E_{45}$.

(A3.3) $\tilde{C}_{E_{45} \times O_{\text{ID}} \times O_{\text{ID}}}$: candidate points $p$ with $q, \bar{q} \in O_{\text{ID}}$, $\tilde{q} \in E_{45}$.

We state the algorithm that generates the subset (A3.1), then in a similar way algorithms for subsets (A3.2) and (A3.3) can be derived:

Algorithm 3.
1. Consider $\bar{q} \in E_{45}$.

2. Check if $\bar{q}$ satisfies:

$$
\begin{align*}
\bar{q}_2 &= \bar{e} \bar{q}_1, \\
\bar{q}_{32} &= \bar{e} \bar{q}_{31},
\end{align*}
$$

then go to the next Step, otherwise go to Step 1.

3. Determine the components of $p$ involved in identities (4.20).

4. Determine the values $p_{321}^{(i)}$, for $i = 1, 2$, using equation (4.1).

For each $i = 1, 2$:

5. Calculate the values of $p_{1234}^{(i)}$ using equation (1.116).

6. Use identities given by the columns of Table 4.1 in order to determine the other components of $p^{(i)}$.

7. If $p^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise Step 1.

8. Save $p^{(i)}$ in the set $\tilde{C}_{E_{45} \times O_{ID} \times O_{ID}}$, and go to Step 1.

When Algorithm 3 and the algorithms for subsets (A3.2) and (A3.3) end, the following set is obtained:

$$
\tilde{C}_{E_{45} \times O_{ID} \times O_{ID}} = \tilde{C}_{E_{45} \times O_{ID} \times O_{ID}} \cup \tilde{C}_{E_{45} \times O_{ID} \times O_{ID}} \cup \tilde{C}_{E_{45} \times O_{ID} \times O_{ID}},
$$

then, by Lemma 59, we generate the set $C_{E_{45} \times O_{ID} \times O_{ID}}$ of all candidate points $p$ such that one over the four projections $\hat{q}, \tilde{q}, \bar{q}, \tilde{q}$ is in the set $E_{45}$ and two of the remaining projections are in the set $O_{ID}$:

$$
C_{E_{45} \times O_{ID} \times O_{ID}} = \tilde{C}_{E_{45} \times O_{ID} \times O_{ID}} \bigcup_{i=1}^{3} (1234)^i (\tilde{C}_{E_{45} \times O_{ID} \times O_{ID}}),
$$
where permutation $(1234)$ is defined in (2.67). There are $6,337$ candidate points in the set $C_{E_{45} \times O_{45} \times O_{45}}$.

The following algorithm generates the set $C_{E_{45} \times E_{45} \times O_{45}}$ of all candidate points $p$ such that one over the four projections $\hat{q}, \tilde{q}, \tilde{q}, \tilde{q}$ is in the set $O_{45}$ and two of the remaining projections are in the set $E_{45}$.

In order to obtain $C_{E_{45} \times E_{45} \times O_{45}}$, we construct the set $\tilde{C}_{E_{45} \times E_{45} \times O_{45}}$, where one over three projections $\hat{q}, \tilde{q}, \tilde{q}$ is in $O_{45}$ and the remaining two are in $E_{45}$, afterwards, by Lemma 59, we can construct the big set $C_{E_{45} \times E_{45} \times O_{45}}$.

Note that the set $\tilde{C}_{E_{45} \times E_{45} \times O_{45}}$ is the union of three subsets of candidate points:

(A4.1) $\tilde{C}_{E_{45} \times E_{45} \times O_{45}}$: candidate points $p$ with $\hat{q}, \tilde{q} \in E_{45}$, $\tilde{q} \in O_{45}$.

(A4.2) $\tilde{C}_{E_{45} \times E_{45} \times O_{45}}$: candidate points $p$ with $\hat{q}, \tilde{q} \in E_{45}$, $\hat{q} \in O_{45}$.

(A4.3) $\tilde{C}_{E_{45} \times E_{45} \times O_{45}}$: candidate points $p$ with $\tilde{q}, \tilde{q} \in E_{45}$, $\hat{q} \in O_{45}$.

We describe in detail the algorithm that generates the subset (A4.1), then in a similar way algorithms for subsets (A4.2) and (A4.3) can be derived:

**Algorithm 4.**

1. Consider $\hat{q}, \tilde{q} \in E_{45} \times E_{45}$.

2. Check if $\hat{q}, \tilde{q}$ satisfy relations given by the columns of the first two rows of Table 4.1 then go to the next Step, otherwise go to Step 1.

3. Calculate $p_{31}$ and $p_{431}$ using Table 4.1 and conditions (4.16):

$$p_{31} = q_2 = \frac{1}{2}(q_1 q_2 - \epsilon_1 \epsilon_2 s_1 s_2) = \frac{1}{2}(p_1 p_2 - \epsilon_1 \epsilon_2 s_1 s_2), \quad (4.27)$$

$$p_{431} = q_4 = \frac{1}{4}(p_1 p_2 p_4 - \epsilon_1 \epsilon_2 s_1 s_2 p_4 - \epsilon_1 \epsilon_4 s_1 s_4 p_2 - \epsilon_2 \epsilon_4 s_2 s_4 p_1) =$$
\[ = \frac{1}{4} (q_1 q_2 q_3 - \epsilon_1 \epsilon_2 s_1 s_2 q_3 - \epsilon_1 \epsilon_3 s_1 s_3 q_2 - \epsilon_2 \epsilon_3 s_2 s_3 q_1). \]  

(4.28)

4. Determine the values \( p^{(i)}_{321} \), for \( i = 1, 2 \), using equation (4.1).

For each \( i = 1, 2 \):

5. Calculate the values of \( p^{(i)}_{32} \) using equation (1.116).

6. Use identities given by the columns of Table 4.1 in order to determine the other components of \( p^{(i)} \).

7. If \( p^{(i)} \) satisfies equations (1.58)-(1.67) then go to the next Step, otherwise Step 1.

8. Save \( p^{(i)} \) in the set \( \tilde{C}_{E_{45} \times E_{45} \times O_{RED}} \), and go to Step 1.

When Algorithm 4 and the algorithms for subsets (A4.2) and (A4.3) end, the following set is obtained:

\[ \tilde{C}_{E_{45} \times E_{45} \times O_{RED}} = \tilde{C}_{E_{45} \times E_{45} \times O_{RED}} \cup \tilde{C}_{E_{45} \times E_{45} \times O_{RED}} \cup \tilde{C}_{E_{45} \times E_{45} \times O_{RED}}. \]

then, by Lemma 59, we generate the set \( \tilde{C}_{E_{45} \times E_{45} \times O_{RED}} \) of all candidate points \( p \) with one over four projections in the set \( O_{RED} \) and two over three of the remaining projections are in the set \( E_{45} \):

\[ \tilde{C}_{E_{45} \times E_{45} \times O_{RED}} = \tilde{C}_{E_{45} \times E_{45} \times O_{RED}} \bigcup_{i=1}^{3} (1234)^i (\tilde{C}_{E_{45} \times E_{45} \times O_{RED}}), \]

(4.29)

where permutation \( (1234) \) is defined in (2.67). There are 342, 368 candidate points in the set \( \tilde{C}_{E_{45} \times E_{45} \times O_{RED}} \).

Last algorithm generates the set \( \tilde{C}_{E_{45} \times E_{45} \times O_{ID}} \) of all candidate points \( p \) such that one projection is in the set \( O_{ID} \) and two of the remaining three pro-
jections are in the set $E_{45}$. Considerations similar to the previous case apply. Indeed in order to obtain $C_{E_{45} \times E_{45} \times O_{ID}}$, we construct the set $\widetilde{C}_{E_{45} \times E_{45} \times O_{ID}}$ where one over three projections $\hat{q}, \bar{q}, \bar{q}$ is in $O_{ID}$ and the remaining two are in $E_{45}$. Thereafter, by Lemma 59, we construct the whole set $C_{E_{45} \times E_{45} \times O_{ID}}$. The set $\widetilde{C}_{E_{45} \times E_{45} \times O_{ID}}$ is the union of three subsets of candidate points $p$:

\begin{align*}
(A5.1) & \quad \widetilde{C}_{E_{45} \times E_{45} \times O_{ID}}: \text{candidate points } p \text{ with } \hat{q}, \bar{q} \in E_{45}, \bar{q} \in O_{ID}. \\
(A5.2) & \quad \widetilde{C}_{E_{45} \times E_{45} \times O_{ID}}: \text{candidate points } p \text{ with } \hat{q}, \bar{q} \in E_{45}, \bar{q} \in O_{ID}. \\
(A5.3) & \quad \widetilde{C}_{E_{45} \times E_{45} \times O_{ID}}: \text{candidate points } p \text{ with } \bar{q}, \bar{q} \in E_{45}, \hat{q} \in O_{ID}.
\end{align*}

We describe in detail algorithm that generates subset (A5.1) and in a similar way algorithms for subsets (A5.2) and (A5.3) can be derived:

**Algorithm 5.**

1. Consider $\hat{q}, \bar{q} \in E_{45} \times E_{45}$.

2. Check if $\hat{q}, \bar{q}$ satisfy relations relations given by the columns of the first two rows of Table 4.1 then go to the next Step, otherwise go to Step 1.

3. Calculate $p_{31}$ and $p_{431}$ using Table 4.1 and conditions (4.16):

\begin{align*}
    p_{31} &= q_{21} = \pm q_3 = \pm p_4, \quad (4.30) \\
    p_{431} &= q_{xc} = \pm 2. \quad (4.31)
\end{align*}

4. Determine the values $p^{(i)}_{321}$, for $i = 1, 2$, using equation (4.1).

For each $i = 1, 2$: 


5. Calculate the values of $p^{(i)}_p$ using equation (1.116).

6. Use identities given by the columns of Table 4.1 in order to determine the other components of $p^{(i)}$.

7. If $p^{(i)}$ satisfies equations (1.58)-(1.67) then go to the next Step, otherwise go to Step 1.

8. Save $p^{(i)}$ in the set $\tilde{C}_{E_{45} \times E_{45} \times O_{ID}}$, and go to Step 1.

When Algorithm 5 and algorithms for subsets (A5.2) and (A5.3) end, we obtain:

$$\tilde{C}_{E_{45} \times E_{45} \times O_{ID}} = \bigcup_{i=1}^{3} (1234)^i (\tilde{C}_{E_{45} \times E_{45} \times O_{ID}})$$

then, by Lemma 59, we generate the set $C_{E_{45} \times E_{45} \times O_{ID}}$ of all candidate points $p$ with one over four projections in the set $O_{ID}$ and two over three of the remaining projections are in the set $E_{45}$:

$$C_{E_{45} \times E_{45} \times O_{ID}} = \tilde{C}_{E_{45} \times E_{45} \times O_{ID}} \cup \tilde{C}_{E_{45} \times E_{45} \times O_{ID}} \cup \tilde{C}_{E_{45} \times E_{45} \times O_{ID}}$$

where permutation (1234) is defined in (2.67). There are 245,760 candidate points in the set $C_{E_{45} \times E_{45} \times O_{ID}}$.

### 4.4 List of finite orbits

Consider $p \in \mathcal{U} \subset \mathcal{M}_{G_2}$ and its four projections $\tilde{q}, \tilde{q}, \tilde{q}, \tilde{q}$. We recall that, in this thesis, we construct candidate points $p \in \mathcal{U} \subset \mathcal{M}_{G_2}$ such that:

(C1) Three projections are in the set $E_{45}$. 
(C2) Two projections are in the set O and one is in the set $E_{45}$.

(C3) Two projections are in the set $E_{45}$ and one is in the set O.

By Theorem 62, the case (C4) is not relevant in our classification, since Mazzocco dealt with it in [Maz01a]. Moreover, we exclude the following cases:

- By Theorem 63, we exclude candidate points such that $p$ satisfies conditions (4.15).
- By Lemma 66, we exclude candidate points such that they have one projection in $O_{ID}$, see Definition (64), and one projection in $O_{RED}$, see Definition (65).

We obtained Algorithms such that they generate the following candidate points:

(C1) Algorithm 2 produces the set of candidate points $C_{E_{45} \times E_{45} \times E_{45}}$.

(C2) Algorithm 3 produces the set of candidate points $C_{E_{45} \times O_{ID} \times O_{ID}}$.

(C3.1) Algorithm 4 produces the set of candidate points $C_{E_{45} \times E_{45} \times O_{RED}}$.

(C3.2) Algorithm 5 produces the set of candidate points $C_{E_{45} \times E_{45} \times O_{ID}}$.

The finite set $C$ of all candidate points $p$ classified in this thesis is:

$$C = C_{E_{45} \times E_{45} \times E_{45}} \cup C_{E_{45} \times O_{ID} \times O_{RED}} \cup C_{E_{45} \times E_{45} \times O_{ID}} \cup C_{E_{45} \times O_{ID} \times O_{ID}}, \quad (4.33)$$

and it contains 3,460,685 candidate points.

Among these points, we need to delete all points in the big open subset $U$, defined in (1.38), that satisfy relations (4.15) in Theorem 63, since they are not relevant.
Algorithm 6.

1. Consider \( p \in C \).

2. If \( p \) satisfies relations (4.15) go to next Step, otherwise save \( p \) in \( C' \).

3. If \( p \) satisfies at least one of the following conditions:

   (i) \( (p_{21}^2 - 4)g(p_{21}, p_3, p_{321})g(p_2, p_1, p_{21})g(p_{21}, p_4, p_{421}) \neq 0 \).

   (ii) \( (p_{31}^2 - 4)g(p_{31}, p_4, p_{431})g(p_{3}, p_1, p_{31})g(p_{31}, p_2, p_{321}) \neq 0 \).

   (iii) \( (p_{32}^2 - 4)g(p_{32}, p_4, p_{432})g(p_{3}, p_2, p_{32})g(p_{32}, p_1, p_{321}) \neq 0 \).

   then the point \( p \) is in the open set \( U \), defined in (1.38) and it is not relevant, otherwise save \( p \) in \( C' \).

This step permits us to eliminate 173,545 and the resulting set \( C' \) has 3,287,140 elements.

Remark 71. During the execution of the previous algorithm we discard also the following point \( p \):

\[
p = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2),
\]

since the monodromy associated is reducible, as proved in [DM00].

Next, as consequence of Lemma 60, we apply the following Algorithm in order to eliminate all points that don’t produce finite orbits:

Algorithm 7.

1. Consider \( p \in C' \).
2. Apply to it all the generators (1.87) of $P_4$:
\[
\beta_{21}(p) = p^{(1)}, \ldots, \beta_{32}(p) = p^{(6)}.
\] (4.34)

3. If there exists an $i = 1, \ldots, 6$ such that $p^{(i)} \notin C'$ then delete $p$ from the set $C'$ and go to Step 1, otherwise save $p$ in $C_0$ and go to Step 1.

This algorithm ends when in the set $C'$ there are no more elements to delete, and it produces a set $C_0$ with 1,095,712 elements that generate finite orbits under the $P_4$-action. Finally, we can factorize the set $C_0$ modulo the action of the pure braid group $P_4$:
\[
C_1 := C_0/P_4.
\]

as follows:

**Algorithm 8.**

For every $p \in C_0$:

1. Save $p \in C_1$.

2. Since $p$ has a finite $P_4$-orbit by construction. Calculate $|O_{P_4}(p)|$.

3. Delete $|O_{P_4}(p)|$ from $C_0$.

Since the set $C_0$ is finite, the algorithm ends. This algorithm produces the set $C_1$, that contains 17,946 finite orbits of the $P_4$-action.

At this point, our aim is to factorize the set $C_1$ by the action of the group of symmetries $G$, introduced in Section 2.5, where $G$ is an infinite and non commutative group. This obviously poses a problem. However, thanks to
the fact that $G$ acts as a finite group on $(p_1, p_2, p_3, p_4, p_8)$ and preserves
the length of a $P_4$-orbit, we are able to set up an algorithm to achieve the
factorization we are looking for.

First of all, we factorize by the action of the finite subgroup:

$$\langle \text{sign}_1, \ldots, \text{sign}_4, (12)(34), (1234) \rangle \subset G, \quad (4.35)$$

to obtain the set $C'_2$. The set $C'_2$ is finite and it contains 122 points. We
do this factorization first as it reduces dramatically from 4,275 to 122 the
number of orbits to be processed afterwards. Next, we subdivide the set
$C'_2$ into subsets that contain orbits of the same length and have the same
$(p_1, p_2, p_3, p_4, p_8)$ modulo change of signs or permutations. Indeed, thanks
to the fact that the action of $G$ preserves the length of an orbit and that
the $(p_1, p_2, p_3, p_4, p_8)$ remain invariant during this action, only points within
the same subset can be related by a transformation in $G$.

Then, in each subset, for all the elements in the subset, we apply a
transformation in the subgroup (4.35) extended with the generator $P_{1\infty}$, in
such a way that every element $p$ in the subset will have the same ordered
$(p_1, p_2, p_3, p_4, p_8)$. We do this step by hand, actually explicitly calculating
the needed transformation.

In each of the subsets, where every element has the same ordered $(p_1, p_2, p_3,$
$p_4, p_8)$, we look for symmetries in $G$ relating the elements. In particular, we
relate elements in the same subset with transformations in the subgroup:

$$\langle P_{13}, P_{23}, P_{34} \rangle \subset G. \quad (4.36)$$

Since elements in the same subset are orbits with the same length and same
$(p_1, p_2, p_3, p_4, p_8)$, the action of the group of transformations (4.36) reduces
to the action of the pure braid group $P_4$, that in this case is finite by construction. In the following, we state the factorization algorithm. Firstly, we factorize with respect to the finite group (4.35):

**Algorithm 9.**

1. Consider $p \in C_1$.

2. Remove from $C_1$ the set $O_{P_4}(p)$ and save $p$ in the set $C'_1$.

3. Apply to $p$ all transformations in $\langle \text{sign}_1, \ldots, \text{sign}_4 \rangle$ and save the result in the set $A_0$.

   For every $p' \in A_0$:

   4. Apply to $p'$ all transformations in $\langle (12)(34), (1234) \rangle$ and save the result in the set $A_1$.

   For every $p'' \in A_1$:

   5. If $p''$ is in $C_1$, then $O_{P_4}(p)$ and $O_{P_4}(p'')$ are equivalent. Remove $O_{P_4}(p'')$ from $C_1$. If $p''$ is not in $C_1$, apply again the current Step to the next $p''$ in $A_1$.

6. If all possible choices of $p''$ in $A_1$ are exhausted go to Step 1.

This algorithm ends when all choices of points $p$ in the finite set $C_1$ are exhausted. The set $C'_2$, created in Step 2, will contain 122 points.

Now, we are going to further factorize the set $C'_2$, as anticipated above, firstly subdividing $C'_2$ in subset which elements are orbits with same length and with the same $(p_1, p_2, p_3, p_4, p_5)$ modulo change of signs or permutations.
Algorithm 10.

1. Consider \( p \in C'_2 \), with \(|\mathcal{O}_{P_4}(p)| = N, N \in \mathbb{N} \).

2. Save \( p \) in a set \( A_N \).

3. Remove \( p \) from \( C'_2 \).

   For every \( p' \in C'_2 \):

4. If \( p' \) is such that:
   
   - \(|\mathcal{O}_{P_4}(p')| = N \).
   
   - \((p_1, p_2, p_3, p_4, p_\infty)\) and \((p'_1, p'_2, p'_3, p'_4, p'_\infty)\) differ by change of signs or permutations.

   Save \( p' \) in \( A_N \) and remove \( p' \) from \( C'_2 \), otherwise apply again this Step to another \( p' \in C'_2 \).

Since the set \( C'_2 \) is finite, this algorithm ends when there are no more elements in \( C'_2 \). This algorithm generates a finite list of 54 subsets \( A_N \), where \( N \) is such that for every \( p \in A_N \) we have \(|\mathcal{O}_{P_4}(p)| = N \).

Next, in each subset \( A_N \), we apply transformations generated by the subgroup (4.35) extended with the generator \( P_{1\infty} \), in such a way that every element in the same subset will have the same ordered \((p_1, p_2, p_3, p_4, p_\infty)\). Afterwards, we quotient each subset with the action of the subgroup of transformations \(< P_{13}, P_{23}, P_{34} >\), that inside each subset acts as the pure braid group \( P_4 \).
Algorithm 11.

For every subset $A_N$:

1. Consider $p \in A_N$ and save it in the set $C_2$.

2. Remove $p$ from $A_N$.

3. Act with the subgroup:

$$\langle \text{sign}_1, \ldots, \text{sign}_4, (12)(34), (1234), P_{1x} \rangle \subseteq G,$$

...to each element in the set $A_N$, producing a new set $A'_N$ in such a way that every element $p'$ in $A'_N$ will have:

$$(p'_1, p'_2, p'_3, p'_4, p'_x) = (p_1, p_2, p_3, p_4, p_x).$$

For every $p' \in A'_N$:

4. Generate the orbit of $p'$ under the action of the subgroup $\langle P_{13}, P_{23}, P_{34} \rangle$.

If $p$ is in this orbit, then $O_{P_4}(p)$ and $O_{P_4}(p')$ are equivalent. Apply again this Step to another $p' \in A'_N$, otherwise save $p'$ in $C_2$ and apply again this Step to another $p' \in A'_N$.

5. When all choices of $p' \in A'_N$ are exhausted, go to Step 1.

Since the number of subsets $A_N$ is 54, and each subset has a finite number of elements, this algorithm ends when there are no more subsets $A_N$ to process. Finally, Algorithm 11 generates a set $C_2$, that contains 54 elements and hence the classification of all finite orbits with points $p$
Matching 127

satisfying conditions (C1), (C2), (C3.1), (C3.2). We summarize the content of the set $C_2$, in Table 4.2.

**Remark 72.** During the factorization algorithm, we apply the generators of $G$ with a specific order. As a consequence, we are factorizing only with respect to a subgroup of the group of symmetries $G$. However, the set $C_2$ contains the factorization we were looking for. Indeed, we recall that: under the action of the group $P_4$ the parameters $p_i$, for $i = 1, \ldots, 4, \infty$, remain constant (see the definition of the generators of $P_4$ given in (1.87)), moreover, the group $G$ acts finitely on the parameters $p_i$, for $i = 1, \ldots, 4, \infty$ and $G$ preserves the length of a finite $P_4$-orbit. We checked that every two orbits in the set $C_2$, satisfy:

- If they have same length and parameters $(p_1, p_2, p_3, p_4, p_{\infty})$ and $(p'_1, p'_2, p'_3, p'_4, p'_{\infty})$ respectively, then there does not exist a transformation $\Phi \in G$ such that $\phi(p_i) = p'_i$.

- If two orbits have same parameters $p_i$, for $i = 1, \ldots, 4, \infty$, then the two orbits have different lengths.
Table 4.2: The 54 finite orbits.

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### 4.5 Proofs of Theorems 62-63 and Lemmata 66-70

In this Section, we give proofs of Theorems 62-63 and Lemmata 66-69. Firstly, we proceed with the proof of Theorems 62-63:

**Proof of Theorem 62.** In order to prove the statement, we distinguish three cases:

(i) Firstly, we prove that given \( p \) with three projections over four in the set \( O_{ID} \), then \( p \) is *not relevant*. In particular, it is enough to consider \( m \in \tilde{\mathcal{M}}_{G_2} \) and the following three projections defined in (1.113):

\[
\tilde{n} = (M_1, M_2, M_3) \in O_{ID},
\]

\[
\hat{n} = (M_2, M_3, M_4) \in O_{ID},
\]
\( \tilde{n} = (M_1, M_2, M_4) \in O_{\text{ID}}, \)

and all other cases follows since they differ from this case only by a permutation of the matrices \( M_i \), see Lemma 59. If any of \( M_i = \pm 1 \), then we conclude. If not, we are left with the following case:

\[
\tilde{N}_\infty = M_3M_2M_1 = \tilde{\epsilon} \mathbb{1}, \quad (4.37)
\]

\[
\tilde{N}_\infty = M_4M_3M_2 = \tilde{\epsilon} \mathbb{1}, \quad (4.38)
\]

\[
\tilde{N}_\infty = M_4M_2M_1 = \epsilon \mathbb{1}, \quad (4.39)
\]

where \( \epsilon = \pm 1 \). Then by equations (4.37) and (4.38):

\[
M_1M_3M_2 = \tilde{\epsilon} \tilde{\epsilon} M_4M_3M_2 \Leftrightarrow M_4 = \tilde{\epsilon} \tilde{\epsilon} M_1, \quad (4.40)
\]

and by equations (4.37) and (4.39):

\[
M_3M_2M_1 = \tilde{\epsilon} \tilde{\epsilon} M_4M_2M_1 \Leftrightarrow M_3 = \tilde{\epsilon} \tilde{\epsilon} M_4, \quad (4.41)
\]

then \( M_3 = \tilde{\epsilon} \tilde{\epsilon} M_1 \). As a consequence, equation (4.39) becomes:

\[
M_4M_2M_1 = \tilde{\epsilon} \mathbb{1} \Leftrightarrow \tilde{\epsilon} \tilde{\epsilon} M_1M_2M_1 = \tilde{\epsilon} \mathbb{1} \Leftrightarrow M_2 = \tilde{\epsilon} \tilde{\epsilon} \tilde{\epsilon} M_1^{-2}, \quad (4.42)
\]

and finally:

\[
m = (M_1, \tilde{\epsilon} \tilde{\epsilon} \tilde{\epsilon} M_1^{-2}, \tilde{\epsilon} \tilde{\epsilon} M_1, \tilde{\epsilon} \tilde{\epsilon} M_1), \quad (4.43)
\]

which is reducible. Therefore \( p \) is not relevant.

(ii) Suppose \( p \) is such that three projections over four are in the set \( O_{\text{RED}} \), then \( p \) has associated reducible monodromy group. Given \( m \in \mathcal{M}_{\mathbb{G}_2} \), it
is enough to consider the following three projections defined in (1.113):

\[ \hat{n} = (M_1, M_2, M_3) \in O_{\text{RED}}, \]
\[ \check{n} = (M_2, M_3, M_4) \in O_{\text{RED}}, \]
\[ \tilde{n} = (M_1, M_2, M_4) \in O_{\text{RED}}, \]

and all other cases follows since they differ from this case only by a permutation of the matrices \( M_i \), see Lemma 59. Then:

- \( M_1, M_2, M_3 \) have \( \tilde{v} \) as common eigenvector.
- \( M_2, M_3, M_4 \) have \( \hat{v} \) as common eigenvector.
- \( M_1, M_2, M_4 \) have \( \check{v} \) as common eigenvector.

All the matrices \( M_i \) for \( i = 1, \ldots, 4 \) are \( 2 \times 2 \) matrices, as a consequence each matrix \( M_i \) can have at most two distinct eigenvectors: the matrix \( M_2 \) that appear in all the three projections, has \( \tilde{v}, \hat{v} \) and \( \check{v} \) as eigenvectors then one of the following identities must hold:

\[ \tilde{v} = \hat{v} \text{ or } \check{v} = \tilde{v} \text{ or } \hat{v} = \check{v}. \quad (4.44) \]

We can freely chose any of identities (4.44), so that \( M_1, \ldots, M_4 \) have a common eigenvector, making the monodromy group reducible.

(iii) When there are three projections in \( O \), not all of the same type, we apply Lemma 66. This concludes the proof.

\[ \square \]

**Proof of Theorem 63.** Suppose \( m = (M_1, M_2, M_3, M_4) \in \tilde{\mathcal{M}}_{G_2} \) with:

\[ M_m = (M_4 M_3 M_2 M_1)^{-1} = \pm 1, \quad (4.45) \]
Then, applying the trace operator and the skein relation to (4.45), we obtain relations (4.15). This concludes the first part of the proof.

Suppose $m = (M_1, M_2, M_3, M_4) \in \widehat{\mathcal{M}}_{G_2}$, with co-adjoint coordinates $p$ in the big open subset $\mathcal{U} \subset \widehat{\mathcal{M}}_{G_2}$. By the definition of the big open subset $\mathcal{U}$, and since $p$ satisfies relations (4.15), then it is straightforward computation to check that the matrix $M_x = \pm 1$ in the charts $\mathcal{U}_{jk}^{(i)}$, for $i = 0, 1, 2$, defined in the statement of Theorem 3. This concludes the proof.

Next we give the proofs of Lemmata 66-70:

Proof of Lemma 66. Consider $m \in \widehat{\mathcal{M}}_{G_2}$ and the following two distinct generic projections:

\begin{align*}
(M_i, M_j, M_k) &\in O_{ID}, \; i > j > k, \; i, j, k = 1, \ldots, 4, \quad (4.46) \\
(M_{i'}, M_{j'}, M_{k'}) &\in O_{RED}, \; i' > j' > k', \; i', j', k' = 1, \ldots, 4. \quad (4.47)
\end{align*}

If either $M_i, M_j, M_k$ is equal to $\pm 1$, then we conclude, otherwise suppose:

\[ (M_iM_jM_k)^{-1} = \pm 1. \quad (4.48) \]

Moreover, suppose the monodromy group associated to the triple $(M_{i'}, M_{j'}, M_{k'})$ is reducible, then matrices $M_{i'}, M_{j'}, M_{k'}$ have a common eigenvector $v$. There always exist two indices in (4.46) and in (4.47) that are equal, without loss of generality, suppose $i \neq i'$, $j = j'$ and $k = k'$, then equation (4.48) can be written as:

\[ M_i = \pm (M_{j'}M_{k'})^{-1}. \]
The last equation implies that $M_i$ has $v$ as eigenvector, then the monodromy group $< M_i, M_j, M_k >$ is reducible. This concludes the proof.

Proof of Lemma 67. Suppose $q$ are the co-adjoint coordinates on $\mathcal{M}_{\text{PVI}}$ of the triple $n = (N_1, N_2, N_3)$. Since the monodromy group $< N_1, N_2, N_3 >$ is reducible, we can suppose the three matrices $N_1, N_2, N_3$ to be upper triangular. Then $N_1, N_2, N_3$ have the eigenvalues on the diagonal and since $\text{eigenv}(N_i) = \exp(\epsilon_i \pi \theta_i)$, where $\epsilon_i = \pm 1$, the following formulae hold:

\[
\text{Tr}(N_i N_j) = 2 \cos(\pi(\epsilon_i \theta_i + \epsilon_j \theta_j)), \ i, j = 1, 2, 3, \ i > j,
\]

\[
\text{Tr}(N_3 N_2 N_1) = 2 \cos(\pi(\epsilon_1 \theta_1 + \epsilon_2 \theta_2 + \epsilon_3 \theta_3)).
\]

(4.49)

(4.50)

Applying trigonometric identities and being $q$ the co-adjoint coordinates of $n$ in $\mathcal{M}_{\text{PVI}}$, we get:

\[
q_{ij} = \frac{1}{2}(q_i q_j - \epsilon_i \epsilon_j s_i s_j), \ i, j = 1, 2, 3, \ i > j,
\]

(4.51)

\[
q_{\infty} = \frac{1}{4}(q_1 q_2 q_3 - \epsilon_1 \epsilon_2 s_1 s_2 q_3 - \epsilon_1 \epsilon_3 s_1 s_3 q_2 - \epsilon_2 \epsilon_3 s_2 s_3 q_1),
\]

(4.52)

where $s_l := \sqrt{4 - q_l^2}$ for $l = 1, 2, 3$. This concludes the proof.

Proof of Lemma 68. Suppose $q$ are the co-adjoint coordinates on $\mathcal{M}_{\text{PVI}}$ of the projection of $p$ that is supposed to be in the set $O_{1D}$. Moreover, suppose the triple $n = (N_1, N_2, N_3)$ is associated to $q$. If any of the $N_i$ is equal to $\pm 1$, by the matching procedure, we end up with a point $p$ that is not relevant, therefore, we avoid this case, otherwise, assume $N_{\infty} = N_3 N_2 N_1 = \pm 1$, then:

\[
N_1 = \pm (N_3 N_2)^{-1}, \ N_2 = \pm (N_1 N_3)^{-1}, \ N_3 = \pm (N_2 N_1)^{-1}.
\]

(4.53)

Being $q$ the co-adjoint coordinates of $n$ on $\mathcal{M}_{\text{PVI}}$, by straightforward com-
putation, we get:
\[
\begin{align*}
q_{21} &= \pm q_3, \\
q_{31} &= \pm q_2, \\
q_{32} &= \pm q_1, \\
q_\infty &= \pm 2.
\end{align*}
\] (4.54)

This concludes the proof.

Proof of Lemma 69. We prove the statement if \( p \) has projections \( \hat{q}, \tilde{q}, \bar{q} \) such that two projections are in the set \( O_{\text{RED}} \). There are three distinct cases: we are going to prove in detail the case when \( \hat{q}, \tilde{q} \in O_{\text{RED}} \) then remaining cases can be proven in a similar way.

Given \( m \), the two projections \( \hat{n}, \tilde{n} \in O_{\text{RED}} \) are such that:

\[
\hat{n} = (M_2, M_3, M_4), \quad \tilde{n} = (M_1, M_2, M_4).
\] (4.55)

Since monodromy groups \( < M_2, M_3, M_4 > \) and \( < M_1, M_2, M_4 > \) are reducible, then \( M_2, M_4 \) are diagonal and \( M_1, M_3 \) can be supposed, without loss of generality, upper and lower triangular respectively, and each matrix will have its own eigenvalues on the diagonal. Recall that \( \text{eigenv}(M_k) = \exp(\epsilon_k \pi \theta_k) \) where \( \epsilon_k = \pm 1 \). Therefore, by Lemma 67, the following relations hold:

\[
\begin{align*}
\text{Tr}(M_iM_j) &= 2 \cos(\pi(\epsilon_i \theta_i + \epsilon_j \theta_j)), \quad i > j, \quad i, j = 2, 3, 4, \quad (4.56) \\
\text{Tr}(M_{i'}M_{j'}) &= 2 \cos(\pi(\epsilon_{i'} \theta_{i'} + \epsilon_{j'} \theta_{j'})), \quad i' > j', \quad i', j' = 1, 3, 4. \quad (4.57)
\end{align*}
\]

Consider the remaining projection \( \bar{n} = (M_1, M_3, M_4) \in \mathcal{M}_{PVl} \), with associated co-adjoint coordinates \( \bar{q} \), then since relations (4.56)-(4.57) hold respectively for \( i = 4, j = 1 \) and \( i' = 4, j' = 3 \), using the trigonometric identities
and matching (1.115), we get:

\[
\begin{align*}
\bar{q}_{41} &= \frac{1}{2}(\bar{q}_4\bar{q}_1 - \epsilon_4\epsilon_1\bar{s}_4\bar{s}_1), \\
\bar{q}_{43} &= \frac{1}{2}(\bar{q}_4\bar{q}_3 - \epsilon_4\epsilon_3\bar{s}_4\bar{s}_3),
\end{align*}
\]  

(4.58) (4.59)

where \( \bar{s}_k := \sqrt{4 - q_k^2} \) for \( k = 1, 3, 4 \). Then equations (4.58)-(4.59) can be written as:

\[
\begin{align*}
\bar{q}_{41}^2 + \bar{q}_1^2 + \bar{q}_1^2 - \bar{q}_{41}\bar{q}_4\bar{q}_1 - 4 &= 0, \\
\bar{q}_{43}^2 + \bar{q}_1^2 + \bar{q}_3^2 - \bar{q}_{43}\bar{q}_4\bar{q}_3 - 4 &= 0,
\end{align*}
\]

and this concludes the proof.

\[\square\]

Proof of Lemma 70. We prove the statement for the case (i), then all the other cases can be proved in a similar way. Suppose \( \bar{q}, \tilde{q} \in \mathbb{O}_{\text{ID}} \), then the only relevant case for our classification (see the beginning of the previous Section) is the following case:

\[
(M_4M_3M_2)^{-1} = \varepsilon \mathbb{1}, \quad (M_4M_2M_1)^{-1} = \tilde{\varepsilon} \mathbb{1},
\]

(4.60)

where \( \varepsilon = \pm 1 \). Therefore relations relations (4.19) and (4.20) follow from Lemma 68 and the matching (1.108),(1.109),(1.110). This concludes the proof.

\[\square\]
Chapter 5

Outlook

In this thesis a list of 54 finite orbits of the action (10) of the pure braid group $P_4$ on the $SL_2(C)$ character variety of the Riemann sphere with five boundary components is presented in Table 4.2. The list is folded up to the action of the group of symmetries $G$ introduced in Chapter 2. Due to the identification of each action of the restriction $H_i$ (determined by the rows in Table 1.1) with the finite action of $P_3$ over the $SL_2(C)$ character variety of $\Sigma_4$, we can associate to each restriction an algebraic solution of PVI (see [DM00, Iwa03, Cou16, LT14]). Then in the list of 54 finite orbits each orbit’s member has the following properties:

- no more than one restriction (determined by the rows of Table 1) is associated to algebraic solutions of PVI obtained by the pull-back of the hypergeometric equation, see Doran [Dor01] and Andreev-Kitaev [AK02],
- no more than one restriction corresponds to the so-called Picard solutions of PVI, see the work of Picard [Pic89] and Mazzocco [Maz01b].

Moreover, we do not allow any orbit’s member such that:
• One restriction is associated to algebraic solutions of PVI obtained by the pull-back of the hypergeometric equation and another restriction is associated to the so-called Picard solutions of PVI.

Many open questions remain. If we consider the parametrization result given in Theorem (3), Lemma (4) and Proposition (5), we could reconstruct, up to global conjugation, the monodromy matrices associated to a candidate point, using the matching procedure (given in Section 4.1) only on two points $q$.

This means that we could extend the classification result given in this thesis to finite orbits whose members can have up to two projections, defined in (1.114), of Picard or Hitching-Kitaev type. This computation is theoretically possible but it is extremely technical and would require many technical Lemmata in order to cover all sub-cases that we decided not to include them in this thesis.

Another direction of research is to use our method to classify all finite orbits of the action of the pure braid group $P_n$ on the $SL_2(C)$ character variety of the Riemann sphere with $n + 1$ boundary components for $n > 4$, or in other words all algebraic solutions of the Garnier system $G_{n-2}$. We expect that the matching procedure can be adapted in order to work in this case too. For generic $n > 4$, the number of restrictions to the action of the pure braid group $P_3$ over $M_{PVI}$ will be $\binom{n}{3}$, consequently many more necessary conditions to be satisfied are introduced in order to produce a candidate point.

In our case, for $n = 4$, we rely on a finite extended list $E_{45}$ of 86,768 points $q$ producing only 54 finite orbits. Since the extended list $E_{45}$ remains the same, and the number of necessary conditions increases with $n$, we expect that the resulting classification list will contain less and less finite
orbits.
Bibliography


