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POSITION DEPENDENT RANDOM MAPS IN ONE AND HIGHER DIMENSIONS

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Abstract. A random map is a discrete-time dynamical system in which one of a number of transformations is randomly selected and applied on each iteration of the process. In this paper, we study random maps with position dependent probabilities on the interval and on a bounded domain of \( \mathbb{R}^n \). Sufficient conditions for the existence of an absolutely continuous invariant measure for random map with position dependent probabilities on the interval and on a bounded domain of \( \mathbb{R}^n \) are the main results of this note.

1. Introduction

Let \( \tau_1, \tau_2, ..., \tau_K \) be a collection of transformations from \( X \) to \( X \). Usually, the random map \( T \) is defined by choosing \( \tau_k \) with constant probability \( p_k, p_k > 0, \sum_{k=1}^{K} p_k = 1 \). The ergodic theory of such dynamical systems was studied in [9] and in [8] (See also [7]).

There is a rich literature on random maps with position dependent probabilities with \( \tau_1, \tau_2, ..., \tau_K \) being continuous contracting transformations (see [10]).

In this paper, we deal with piecewise monotone transformations \( \tau_1, \tau_2, ..., \tau_K \) and position dependent probabilities \( p_k(x), k = 1, ..., K, p_k(x) > 0, \sum_{k=1}^{K} p_k(x) = 1 \), i.e., the \( p_k \)'s are functions of position. We point out that studying such dynamical systems was first introduced in [4] where sufficient conditions for the existence of an absolutely continuous invariant measure were given. The conditions in [4] are applicable only when \( \tau_1, \tau_2, ..., \tau_K \) are \( C^2 \) expanding transformations (see [4] for details). In this paper, we prove the existence of an absolutely continuous invariant measure for a random map \( T \) on \( [a, b] \) under milder conditions (see section 4, Conditions (A) and (B)). Moreover, we prove the existence of an absolutely continuous invariant measure for a random map \( T \) on \( S \), where \( S \) is a bounded domain of \( \mathbb{R}^n \) (see section 6, Condition (C)).

The paper is organized in the following way: In section 2, following the ideas of [4], we formulate the definition of a random map \( T \) with position dependent probabilities and introduce its Perron-Frobenius operator. In section 3, we prove properties of the Perron-Frobenius operator of \( T \). In section 4, we prove the existence of an absolutely continuous invariant measure for \( T \) on \( [a, b] \). In section 5, we give an example of a random map \( T \) which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A)

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and (B). In section 6, we prove the existence of an absolutely continuous invariant measure for $T$ on a bounded domain of $\mathbb{R}^n$. In section 7, we give an example of a random map in $\mathbb{R}^n$ that preserves an absolutely continuous invariant measure.

2. PRELIMINARIES

Let $(X, \mathcal{B}, \lambda)$ be a measure space, where $\lambda$ is an underlying measure. Let $\tau_k : X \to X$, $k = 1, ..., K$ be piecewise one-to-one, non-singular transformations on a common partition $\mathcal{P}$ of $X : \mathcal{P} = \{I_1, ..., I_q\}$ and $\tau_{ki} = \tau_k|_{I_i}, i = 1, ..., q$. $k = 1, ..., K$ ($\mathcal{P}$ can be found by considering finer partitions). We define the transition function for the random map $T = \{\tau_1, ..., \tau_K; p_1(x), ..., p_K(x)\}$ as follows:

\[
\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x) \chi_A(\tau_k(x)),
\]

where $A$ is any measurable set and $\{p_k(x)\}_{k=1}^{K}$ is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^{K} p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in X$ and $\chi_A$ denotes the characteristic function of the set $A$. We define $T(x) = \tau_k(x)$ with probability $p_k(x)$ and $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)$ with probability $p_{k_N}(\tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ ... \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)$. The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_\mu$ on measures on $(X, \mathcal{B})$ defined by

\[
\mathbb{P}_\mu(A) = \int \mathbb{P}(x, A) d\mu(x) = \sum_{k=1}^{K} \int p_k(x) \chi_A(\tau_k(x)) d\mu(x)
\]

\[
= \sum_{k=1}^{K} \int \tau_k^{-1}(A) p_k(x) d\mu(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} \sum_{k=1}^{K} \int \tau_{k,i}^{-1}(A) p_k(x) d\mu(x)
\]

We say that measure $\mu$ is $T$-invariant iff $\mathbb{P}_\mu = \mu$, i.e.,

\[
\mu(A) = \sum_{k=1}^{K} \int \tau_k^{-1}(A) p_k(x) d\mu(x), \quad A \in \mathcal{B}.
\]

If $\mu$ has density $f$ with respect to $\lambda$, the $\mathbb{P}_\mu$ has also a density which we denote by $P_{Tf}$. By change of variables, we obtain

\[
\int_A P_{Tf}(x) d\lambda(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int \tau_k^{-1}(A) p_k(x) f(x) d\lambda(x)
\]

\[
= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_A p_k(\tau_k^{-1}x) f(\tau_k^{-1}x) \frac{1}{J_{k,i}(\tau_k^{-1})} d\lambda(x)
\]

where $J_{k,i}$ is the Jacobian of $\tau_{k,i}$ with respect to $\lambda$. Since this holds for any measurable set $A$ we obtain an a.e. equality:

\[
(P_{Tf})(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} p_k(\tau_k^{-1}x) f(\tau_k^{-1}x) \frac{1}{J_{k,i}(\tau_k^{-1})} \chi_{\tau_k(I_i)}(x)
\]

or

\[
(P_{Tf})(x) = \sum_{k=1}^{K} P_{\tau_k} (p_k f)(x)
\]
where \( P_{\tau_k} \) is the Perron-Frobenius operator corresponding to the transformation \( \tau_k \) (see [1] for details). We call \( P_T \) the Perron-Frobenius of the random map \( T \). The main tool in this paper is the Perron-Frobenius of \( T \) which has very useful properties.

3. Properties of the Perron-Frobenius operator of \( T \)

The properties of \( P_T \) resemble the properties of the classical Perron-Frobenius operator of a single transformation.

**Lemma 3.1.** \( P_T \) satisfies the following properties:

(i) \( P_T \) is linear;

(ii) \( P_T \) is non-negative, i.e., \( f \geq 0 \) implies \( P_T f \geq 0 \);

(iii) \( P_T f = f \iff \mu = f \cdot \lambda \) is \( T \)-invariant;

(iv) \( \| P_T f \|_1 \leq \| f \|_1 \), where \( \| \cdot \|_1 \) denotes the \( L^1 \) norm;

(v) \( P_{T \circ R} = P_T \circ P_R \). In particular, \( P_T^N f = P_T^N f \).

**Proof.** The proofs of (i)-(iv) are analogous to that for single transformation. For the proof of (v), let \( T \) and \( R \) be two random maps corresponding to \( \{ \tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K \} \) and \( \{ \zeta_1, \zeta_2, \ldots, \zeta_L; r_1, r_2, \ldots, r_L \} \) respectively. We define \( \{ \tau_k \}_{k=1}^K \) and \( \{ \zeta_l \}_{l=1}^L \) on a common partition \( \mathcal{P} \). We have

\[
P_T(P_T f) = P_T \left( \sum_{k=1}^K P_{\tau_k}(p_k f) \right) = \sum_{l=1}^L \sum_{k=1}^K \sum_{i=1}^q r_i \zeta^{-1}_k |P_{\tau_k}(p_k f)| \zeta^{-1}_k \frac{1}{J_{\zeta_n}(\zeta^{-1}_k)} \chi_{\zeta_n}(I_i)
\]

(3.1)

\[
= \sum_{k=1}^K \sum_{l=1}^L \sum_{j=1}^q \sum_{i=1}^q r_i \zeta^{-1}_k p_k(\tau^{-1}_{k,j} \circ \zeta^{-1}_k) f(\tau^{-1}_{k,j} \circ \zeta^{-1}_k) \chi_{\tau_l(I_j)}(\zeta^{-1}_k) \chi_{\zeta_n}(I_i)
\]

\[
= \sum_{i=1}^L \sum_{k=1}^K P_{\tau_k} (p_k \zeta^{-1}_l f) = P_{T \circ R} f.
\]

\[ \Box \]

4. The existence of absolutely continuous invariant measure on \([a, b]\)

Let \((I, \mathcal{B}, \lambda)\) be a measurable space, where \( \lambda \) is normalized Lebesgue measure on \( I = [a, b] \). Let \( \tau_k : I \to I, k = 1, \ldots, K \) be piecewise one-to-one and differentiable, non-singular transformations on a partition \( \mathcal{P} \) of \( I : \mathcal{P} = \{ I_1, \ldots, I_q \} \) and \( \tau_{ki} = \tau_k |_{I_i}, i = 1, \ldots, q, k = 1, \ldots, K \). Denote by \( V(\cdot) \) the standard one dimensional variation of a function, and by \( BV(I) \) the space of functions of bounded variations on \( I \) equipped with the norm \( \| \cdot \|_{BV} = V(\cdot) + \| \cdot \|_1 \).

Let \( g_k(x) = \frac{p_k(x)}{|f_k|}, k = 1, \ldots, K \). We assume that

**Condition (A):** \( \sum_{k=1}^K g_k(x) < \alpha < 1, x \in I \), and

**Condition (B):** \( g_k \in BV(I), k = 1, \ldots, K \).

Under the above conditions our goal is to prove:

\[ V_I P_T^N f \leq AV_I f + B\| f \|_1 \]
for some $n \geq 1$, where $0 < A < 1$ and $B > 0$. The inequality (4.1) guarantees the existence of a $T$-invariant measure absolutely continuous with respect to Lebesgue measure and the quasi-compactness of operator $P_T$ with all the consequences of this fact, see [1]. We will need a number of lemmas:

**Lemma 4.1.** Let $f \in BV(I)$. Suppose $\tau : I \to J$ is differentiable and $\tau'(x) \neq 0$, $x \in I$. Set $\phi = \tau^{-1}$ and let $g(x) = \frac{d\phi(x)}{d\tau'(x)} \in BV(I)$. Then

$$V_I(f(\phi)g(\phi)) \leq (V_I f + \sup_I f)(V_I g + \sup_I g).$$

**Proof.** First, note that we have dropped all the $k, i$ indices to simplify the notation. Then, the proof follows in the same way as in Lemma 3 of [9]. \qed

**Lemma 4.2.** Let $T$ satisfy conditions (A) and (B). Then for any $f \in BV(I)$,

$$V_TP_T f \leq AV_T f + B\|f\|_1,$$

where

$$A = 3\alpha + \max_{1 \leq i \leq q} \sum_{k=1}^K V_I g_k;$$

and

$$B = 2\beta \alpha + \beta \max_{1 \leq i \leq q} \sum_{k=1}^K V_I g_k,$$

where $\beta = \max_{1 \leq i \leq q} (\lambda(I_k))^{-1}$.

**Proof.** First, we will refine partition $\mathcal{P}$ to satisfy additional condition. Let $\eta > 0$ be such that $\sum_{k=1}^K (g_k(x) + \epsilon_k) < \alpha$ whenever $|\epsilon_k| < \eta$, $k = 1, \ldots, K$. Since $g_k$, $k = 1, \ldots, K$ are of bounded variation we can find a finite partition $K$ such that for any $k = 1, \ldots, K$,

$$|g_k(x) - g_k(y)| < \eta,$$

for $x, y$ in the same element of $K$. Instead of the partition $\mathcal{P}$ we consider a join $\mathcal{P} \cup K$. Without restricting generality of our considerations, we can assume that this is our original partition $\mathcal{P}$. Then, we have

$$\max_{1 \leq i \leq q} \sum_{k=1}^K \sup_{x \in I_k} g_k(x) < \alpha.$$  

We have $V_I(P_T f) = V_I(\sum_{k=1}^K P_{\tau_k}(p_k f))$. We will estimate this variation. Let $\phi_{k,i} = \tau_{k,i}^{-1}$, $k = 1, \ldots, K$, $i = 1, \ldots, q$. We have

$$V_I \left( \sum_{k=1}^K P_{\tau_k}(p_k f) \right) = V_I \left( \sum_{k=1}^K \sum_{i=1}^q f(\phi_{k,i})g_k(\phi_{k,i})\lambda_{\tau_k(I_i)} \right)$$

$$\leq \sum_{k=1}^K \sum_{i=1}^q \|f(a_{i-1})\| g_k(a_{i-1}) + |f(a_i)| g_k(a_i)|$$

$$+ \sum_{k=1}^K \sum_{i=1}^q V_{\tau_k(I_i)}|f(\phi_{k,i})g_k(\phi_{k,i})|. $$

We have

$$\max_{1 \leq i \leq q} \sum_{k=1}^K \sup_{x \in I_k} g_k(x) < \alpha.$$
First, we estimate the first sum on the right hand side of (4.4):

\[ \sum_{k=1}^{K} \sum_{i=1}^{q} \left[ |f(a_{i-1})|g_k(a_{i-1})| + |f(a_i)|g_k(a_i) \right] \]

\[
= \sum_{i=1}^{q} \left[ |f(a_{i-1})| \left( \sum_{k=1}^{K} |g_k(a_{i-1})| \right) + |f(a_i)| \left( \sum_{k=1}^{K} |g_k(a_i)| \right) \right] \\
\leq \alpha \left( \sum_{i=1}^{q} (|f(a_{i-1})| + |f(a_i)|) \right) \\
\leq \alpha \left( \sum_{i=1}^{q} \left( V_i f + (\lambda(I_i))^{-1} \int_{I_i} f d\lambda \right) \right) = \alpha (V_I f + \beta \| f \|_1). \tag{4.5} \]

We now estimate the second sum on the right hand side of (4.4). Using Lemma 4.1 we obtain:

\[ \sum_{k=1}^{K} \sum_{i=1}^{q} V_{\tau_k(I_i)}[f(\phi_{k,i})g_k(\phi_{k,i})] \leq \sum_{k=1}^{K} \sum_{i=1}^{q} \left( V_i f + \sup_{I_i} \right) \left( V_i g_k + \sup_{I_i} g_k \right) \]

\[
\leq \sum_{i=1}^{q} \left( 2 V_i f + \beta \int_{I_i} f d\lambda \right) \left( \max_{1 \leq i \leq q} \sum_{k=1}^{K} \left( V_i g_k + \sup_{I_i} g_k \right) \right) \\
\leq (2 \beta \| f \|_1) \left( \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_i g_k + \alpha \right). \tag{4.6} \]

Thus, using (4.5) and (4.6), we obtain

\[ V_I P_I f \leq \left( 3 \alpha + \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_i g_k \right) V_I f + \left( 2 \beta \alpha + \beta \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_i g_k \right) \| f \|_1. \tag{4.7} \]

\[ \square \]

In the following two lemmas we show that constants \( \alpha \) and \( \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_i g_k \) decrease when we consider higher iterations \( T^n \) instead of \( T \). The constant \( \beta \) obviously increases, but this is not important.

**Lemma 4.3.** Let \( T \) be a random map which satisfies condition (A). Then, for \( x \in I \),

\[ \sum_{w \in \{1,2,\ldots,K\}^N} p_w(x) \frac{|T_w(x)|}{|T^N(x)|} < \alpha^N, \tag{4.8} \]

where \( T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x) \) and \( p_w(x) = p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x) \), define random map \( T^N \).

**Proof.** We have

\[ T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x) \]

with probability

\[ p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x). \]

The maps defining \( T^N \) may be indexed by \( w \in \{1,2,\ldots,K\}^N \). Set

\[ T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x) \]

\[ < \alpha^N. \]
where \( w = (k_1, \ldots, k_N) \), and

\[
p_w(x) = p_{k_N}(\tau_{k_N-1} \circ \cdots \circ \tau_1(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_1(x)) \cdots p_{k_1}(x).
\]

Then,

\[
T'_w(x) = \tau_{k_N}^f(\tau_{k_N-1} \circ \cdots \circ \tau_1(x))\tau_{k_{N-1}}^f(\tau_{k_{N-2}} \circ \cdots \circ \tau_1(x)) \cdots \tau_{k_1}^f(x).
\]

Suppose that \( T \) satisfies condition (A). We will prove (4.8) using induction on \( N \).

For \( N = 1 \), we have

\[
\sum_{w \in \{1, 2, \ldots, K\}} \frac{p_w(x)}{|T_w'(x)|} < \alpha
\]

by condition (A). Assume (4.8) is true for \( N - 1 \). Then,

\[
\sum_{w \in \{1, 2, \ldots, K\}^N} \frac{p_w(x)}{|T_w'(x)|} = \sum_{w \in \{1, 2, \ldots, K\}^{N-1}} \sum_{k=1}^K \frac{p_k(x)p_{w\setminus\{k\}}(\tau_k(x))}{|T_k'(x)| |T_{w\setminus\{k\}}(\tau_k(x))|}
\]

\[
\leq \left( \sum_{k=1}^K \frac{p_k(x)}{|T_k'(x)|} \right) \left( \sum_{w \in \{1, 2, \ldots, K\}^{N-1}} \frac{p_{w\setminus\{k\}}(\tau_k(x))}{|T_{w\setminus\{k\}}(\tau_k(x))|} \right) < \alpha \cdot \alpha^{N-1} = \alpha^N.
\]

\( \square \)

**Lemma 4.4.** Let \( g_w = \frac{p_{w\setminus\{k\}}}{|T_{w\setminus\{k\}}(\tau_k(x))|} \), where \( T_w \) and \( p_w \) are defined in Lemma 4.3, \( w \in \{1, \ldots, K\}^n \). Define

\[
W_1 \equiv \max_{1 \leq i \leq n} \sum_{k=1}^K V_{i}g_k,
\]

and

\[
W_n \equiv \max_{J \in P(n)} \sum_{w \in \{1, \ldots, K\}^n} V_Jg_w,
\]

where \( P(n) \) is the common monotonicity partition for all \( T_w \). Then, for all \( n \geq 1 \)

\[
W_n \leq n\alpha^{n-1}W_1,
\]

where \( \alpha \) is defined in condition (A).

**Proof:** We prove the lemma by induction on \( n \). For \( n = 1 \) the lemma is true by definition of \( W_n \). Assume that the lemma is true for \( n \), i.e.,

\[
W_n \leq n\alpha^{n-1}W_1.
\]
Let $J \in \mathcal{P}^{(n+1)}$ and $x_0 < x_1 < \ldots < x_l$ be a sequence of points in $J$. Then

\begin{equation}
\sum_{w} \sum_{j=0}^{l-1} |g_w(x_{j+1}) - g_w(x_j)| = \sum_{j=0}^{l-1} \sum_{w \in \{1, \ldots, K\}^{n+1}} |g_w(x_{j+1}) - g_w(x_j)| \\
\leq \sum_{j=0}^{l-1} \sum_{w \in \{1, \ldots, K\}^{n}} \sum_{k=1}^{K} |g_w(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_w(\tau_k(x_j))g_k(x_j)| \\
\leq \sum_{j=0}^{l-1} \sum_{w \in \{1, \ldots, K\}^{n}} \sum_{k=1}^{K} |g_w(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_w(\tau_k(x_j))g_k(x_j)| \\
+ \sum_{j=0}^{l-1} \sum_{w \in \{1, \ldots, K\}^{n}} \sum_{k=1}^{K} |g_w(\tau_k(x_{j+1}))g_k(x_j) - g_w(\tau_k(x_j))g_k(x_j)| \\
\leq \alpha^n \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \\
+ \alpha \sum_{j=0}^{l-1} \sum_{w \in \{1, \ldots, K\}^{n}} |g_w(\tau_k(x_{j+1})) - g_w(\tau_k(x_j))| \\
\leq \alpha^n W_1 + \alpha^n W_n \leq \alpha^n W_1 + n\alpha^n W_1 = (n + 1)\alpha^n W_1.
\end{equation}

We used condition (A) and lemma 4.3. □

**Theorem 4.5.** Let $T$ be a random map which satisfies conditions (A) and (B). Then $T$ preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator $P_T$ is quasi-compact on $BV(I)$, see [1].

**Proof.** Let $N$ be such that $A_N = 3\alpha^N + W_N < 1$. Then, by Lemma 4.3,

\begin{equation}
\sum_{w \in \{1, \ldots, K\}^{N}} g_w(x) < \alpha^N, \quad x \in I.
\end{equation}

We refine the partition $\mathcal{P}^{(N)}$ like in the proof of Lemma 4.2, to have

\begin{equation}
\max_{J \in \mathcal{P}^{(N)}} \sum_{w \in \{1, \ldots, K\}^{N}} \sup_{J} g_w < \alpha^N.
\end{equation}

Then, by lemma 4.2, we get

\begin{equation}
\|P_T^N f\|_{BV} \leq A_N\|f\|_{BV} + B_N\|f\|_1,
\end{equation}

where $B_N = \beta_N(2\alpha^N + W_N)$, $\beta_N = \max_{J \in \mathcal{P}^{(N)}} (\lambda(J))^{-1}$. The theorem follows by the standard technique (see [1]). □

**Remark 4.6.** It is enough to assume that condition (A) is satisfied for some iterate $T^m$, $m \geq 1$. 

Remark 4.7. The number of absolutely continuous invariant measures for random maps has been studied in [6]. The proof of [6], which uses graph theoretic methods, goes through analogously in our case; i.e., when $T$ is a random map with position dependent probabilities.

5. Example

We present an example of a random map $T$ which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A) and (B).

Example 5.1. Let $T$ be a random map which is given by \( \{\tau_1, \tau_2; p_1(x), p_2(x)\} \) where

\[
\tau_1(x) = \begin{cases} 
2x & \text{for } 0 \leq x \leq \frac{1}{2}, \\
x & \text{for } \frac{1}{2} < x \leq 1 
\end{cases}
\]

\[
\tau_2(x) = \begin{cases} 
x + \frac{1}{2} & \text{for } 0 \leq x \leq \frac{1}{2}, \\
x - \frac{1}{2} & \text{for } \frac{1}{2} < x \leq 1
\end{cases}
\]

and

\[
p_1(x) = \begin{cases} 
\frac{2}{3} & \text{for } 0 \leq x \leq \frac{1}{2}, \\
\frac{1}{3} & \text{for } \frac{1}{2} < x \leq 1
\end{cases}
\]

\[
p_2(x) = \begin{cases} 
\frac{1}{3} & \text{for } 0 \leq x \leq \frac{1}{2}, \\
\frac{2}{3} & \text{for } \frac{1}{2} < x \leq 1
\end{cases}
\]

Then, $\sum_{k=1}^{2} g_k(x) = \frac{2}{3} < 1$. Therefore, $T$ satisfies conditions (A) and (B). Consequently, by theorem 4.5, $T$ preserves an invariant measure absolutely continuous with respect to Lebesgue measure. Notice that \( \tau_1, \tau_2 \) are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P} : \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. For such maps the Perron-Frobenius operator reduces to a matrix (see [1]). The corresponding matrices are:

\[
P_{\tau_1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix}, \quad P_{\tau_2} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}
\]

Their invariant densities are $f_{\tau_1} = [0, 2]$ and $f_{\tau_2} = [2, 0]$. The Perron-Frobenius operator of the random map $T$ is given by:

\[
P_T = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}
\]

If the invariant density of $T$ is $f = [f_1, f_2]$, normalized by $f_1 + f_2 = 2$ and satisfying equation $fP_T = f$, then $f_1 = \frac{2}{3}$ and $f_2 = \frac{1}{3}$.

6. The existence of absolutely continuous invariant measure in $\mathbb{R}^n$

Let $S$ be a bounded region in $\mathbb{R}^n$ and $\lambda_n$ be Lebesgue measure on $S$. Let $\tau_k : S \to S$, $k = 1, \ldots, K$ be piecewise one-to-one and $C^2$, non-singular transformations on a partition $\mathcal{P}$ of $S : \mathcal{P} = \{S_1, \ldots, S_q\}$ and $\tau_{k;i} = \tau_k |_{S_i}$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Let each $S_i$ be a bounded closed domain having a piecewise $C^2$ boundary of finite $(n - 1)$-dimensional measure. We assume that the faces of $\partial S_i$ meet at angles bounded uniformly away from 0. We will also assume that the probabilities $p_k(x)$
are piecewise $C^1$ functions on the partition $\mathcal{P}$. Let $D\tau_{k,i}^{-1}(x)$ be the derivative matrix of $\tau_{k,i}^{-1}$ at $x$. We assume:

**Condition (C):**

$$\max_{1 \leq k \leq q} \sum_{k=1}^{K} p_k(x) \left\| D\tau_{k,i}^{-1}(\tau_{k,i}(x)) \right\| < \sigma < 1.$$ 

Let $\sup_{x \in \partial \tau(S)} \left\| D\tau_{k,i}^{-1}(x) \right\| := \sigma_{k,i}$ and $\sup_{x \in \tau_i} p_k(x) := \pi_{k,i}$. Using smoothness of $D\tau_{k,i}^{-1}$'s and $p_k$'s we can refine partition $\mathcal{P}$ to satisfy

**Condition (C'):**

$$\sum_{k=1}^{K} \max_{1 \leq j \leq q} \sigma_{k,i} \pi_{k,j} < \sigma < 1.$$ 

Under this condition, our goal is to prove the existence of an a.c.i.m. for the random map $T = \{\tau_1, ..., \tau_K; p_1, ..., p_K\}$. The main tool of this section is the multidimensional notion of variation defined using derivatives in the distributional sense (see [3]):

$$V(f) = \int_{\mathbb{R}^n} \| Df \| = \sup \{ \int_{\mathbb{R}^n} f(\text{div}(g))d\lambda_n : g = (g_1, ..., g_n) \in C_0^1(\mathbb{R}^n, \mathbb{R}^n) \},$$

where $f \in L_1(\mathbb{R}^n)$ has bounded support, $Df$ denotes the gradient of $f$ in the distributional sense, and $C_0^1(\mathbb{R}^n, \mathbb{R}^n)$ is the space of continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ having a compact support. We will use the following property of variation which is derived from [3], Remark 2.14: If $f = 0$ outside a closed domain $A$ whose boundary is Lipschitz continuous, $f|_A$ is continuous, $f|_{\text{int}(A)}$ is $C^1$, then

$$V(f) = \int_{\text{int}(A)} \| Df \| d\lambda_n + \int_{\partial A} |f| d\lambda_{n-1},$$

where $\lambda_{n-1}$ is the $n-1$-dimensional measure on the boundary of $A$. In this section we shall consider the Banach space (see [3], Remark 1.12),

$$BV(S) = \{ f \in L_1(S) : V(f) < +\infty \},$$

with the norm $\|f\|_{BV} = V(f) + \|f\|_1$. We adapt the following two lemmas from [5]. The proofs of Lemma 6.1 and Lemma 6.2 are exactly the same as in [5].

**Lemma 6.1.** Consider $S_i \in \mathcal{P}$. Let $x$ be a point in $\partial S_i$ and $y = \tau_k(x)$ a point in $\partial(\tau_k(S_i))$. Let $J_{k,i}$ be the Jacobian of $\tau_{k,i}|S_i$ at $x$ and $J_{k,i}^0$ be the Jacobian of $\tau_k|S_i$ at $x$. Then

$$\frac{J_{k,i}^0}{J_{k,i}} \leq \sigma_{k,i}.$$ 

\[\Box\]

Let us fix $1 \leq i \leq q$. Let $Z$ denote the set of singular points of $\partial S_i$. Let us construct at any $x \in Z$ the largest cone having a vertex at $x$ and which lies completely in $S_i$. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\beta(S_i) = \min_{x \in Z} \theta(x).$$

Since the faces of $\partial S_i$ meet at angles bounded away from 0, $\beta(S_i) > 0$. Let $\alpha(S_i) = \pi/2 + \beta(S_i)$ and

$$\alpha(S_i) = | \cos(\alpha(S_i)) |.$$
Now we will construct a $C^1$ field of segments $L_y$, $y \in \partial S_i$, every $L_y$ being a central ray of a regular cone contained in $S_i$, with angle subtended at the vertex $y$ greater than or equal to $\beta(S_i)$.

We start at points $y \in Z$, where the minimal angle $\beta(S_i)$ is attained, defining $L_y$ to be central rays of the largest regular cones contained in $S_i$. Then we extend this field of segments to $C^1$ field we want, making $L_y$ short enough to avoid overlapping. Let $\delta(y)$ be the length of $L_y$, $y \in \partial S_i$. By the compactness of $\partial S_i$ we have

$$\delta(S_i) = \inf_{y \in \partial S_i} \delta(y) > 0.$$ 

Now, we shorten $L_y$ of our field, making them all of the length $\delta(S_i)$.

**Lemma 6.2.** For any $S_i$, $i = 1, \ldots, q$, if $f$ is a $C^1$ function on $S_i$, then

$$\int_{\partial S_i} f(y)d\lambda_{n-1}(y) \leq \frac{1}{a(S_i)} \left( \frac{1}{\delta(S_i)} \int_{S_i} f d\lambda_n + V_{\text{Int}}(S_i)(f) \right).$$

\[ \Box \]

Our main technical result is the following:

**Theorem 6.3.** If $T$ is a random map which satisfies Condition (C), then

$$V(P_T f) \leq \sigma(1 + 1/a)V(f) + (M + \frac{\sigma}{a^\alpha})\|f\|_1,$$

where $a = \min\{a(S_i) : i = 1, \ldots, q\} > 0$, $\delta = \min\{\delta(S_i) : i = 1, \ldots, q\} > 0$, $M_{k,i} = \sup_{x \in S_i}(Dp_k(x) - D\tau_{k,i}^{-1}p_k(x))$ and $M = \sum_{k=1}^K \max_{1 \leq i \leq q} M_{k,i}$.

**Proof.** We have $V(P_T f) \leq \sum_{k=1}^K V(P_{T_k}(p_k f))$. We first estimate $V(P_{T_k}(p_k f))$. Let $F_{k,i} = \frac{f(\tau_{k,i}^{-1})p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i})}$, and $R_{k,i} = \tau_{k,i}(S_i)$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Then,

\[(6.1)\]

$$\int_{\mathbb{R}^n} \|DP_{T_k}(p_k f)\|d\lambda_n \leq \sum_{i=1}^q \int_{\mathbb{R}^n} \|D(F_{k,i}\chi_{R_i})\|d\lambda_n$$

$$\leq \sum_{i=1}^q \left( \int_{\mathbb{R}^n} \|D(F_{k,i}\chi_{R_i})\|d\lambda_n + \int_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_i})\|d\lambda_n \right).$$

Now, for the first integral we have,

\[(6.2)\]

$$\int_{\mathbb{R}^n} \|D(F_{k,i}\chi_{R_i})\|d\lambda_n = \int_{R_i} \|D(F_{k,i}\partial p_k)\|d\lambda_n$$

$$\leq \int_{R_i} \|D(f(\tau_{k,i}^{-1}))\| \frac{p_k(\tau_{k,i})}{J_{k,i}(\tau_{k,i})} d\lambda_n + \int_{R_i} \|f(\tau_{k,i}^{-1})D \left( \frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})} \right) \|d\lambda_n$$

$$\leq \int_{R_i} \|D(f(\tau_{k,i}^{-1}))\| \|D\tau_{k,i}^{-1}\| \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} \|p_k(\tau_{k,i}^{-1})\| d\lambda_n + \int_{R_i} \|f(\tau_{k,i}^{-1})\| \frac{M_k}{J_{k,i}(\tau_{k,i})} d\lambda_n$$

$$\leq \sigma_{k,i} \tau_{k,i} \int_{S_i} \|Df\|d\lambda_n + M_k \int_{S_i} \|f\|d\lambda_n.$$
For the second integral we have,
\[
\int_{\mathbb{R}} \|F_{k,i}(D\chi_{R_0})\|d\lambda_n = \int_{\partial R_0} |f(\tau_{-1}^{-1})| \frac{p_k(\tau_{-1}^{-1})}{J_{k,d}(\tau_{-1}^{-1})} d\lambda_{n-1} = \int_{\partial S_i} |f| \frac{P^0_{k,d}}{J_{k,d}} d\lambda_{n-1}.
\]

By Lemma 4.3, \( \frac{P^0_{k,i}}{J_{k,i}} \leq \sigma_{k,i} \). Using Lemma 4.2, we get:
\[
\int_{\mathbb{R}} \|F_{k,i}(D\chi_{R_0})\|d\lambda_n \leq \sigma_{k,i} \bar{\sigma}_{k,i} \int_{\partial S_i} |f|d\lambda_{n-1} 
\leq \frac{\sigma_{k,i} \bar{\sigma}_{k,i}}{a} V_{S_i}(f) + \frac{\sigma_{k,i} \bar{\sigma}_{k,i}}{a\delta} \int_{S_i} |f|d\lambda_n.
\]

Using Condition (C′), summing first over \( i \), we obtain
\[
V(P_{\tau_k}(p_k f)) \leq \left( \max_{1 \leq i \leq g} \sigma_{k,i} \bar{\sigma}_{k,i} \right) (1+1/a) V(f) + \left( \max_{1 \leq i \leq g} \frac{\sigma_{k,i} \bar{\sigma}_{k,i}}{a\delta} \right) ||f||_1,
\]
and then, summing over \( k \) we obtain
\[
V(P_T f) \leq \sigma (1+1/a) V(f) + (M + \frac{\bar{\sigma}}{a\delta}) ||f||_1.
\]

\[\square\]

**Theorem 6.4.** Let \( T \) be a random map which satisfies condition (C). If \( \sigma (1+1/a) < 1 \), then \( T \) preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator \( P_T \) is quasi-compact on \( BV(S) \), see [1].

**Proof.** The proof of the theorem follows by the standard technique (see [1]). \[\square\]

### 7. Example in \( \mathbb{R}^2 \)

In this section, We present an example of a random map which satisfies condition (C) of Theorem 6.3 and thus it preserves an absolutely continuous invariant measure.

**Example 7.1.** Let \( T \) be a random map which is given by \( \{\tau_1, \tau_2; p_1(x), p_2(x)\} \) where \( \tau_1, \tau_2 : I^2 \to I^2 \) defined by:
\[
\tau_1(x_1, x_2) = \begin{cases} 
(3x_1, 2x_2) & \text{for } (x_1, x_2) \in S_1 = \{0 \leq x_1, x_2 \leq \frac{1}{3}\} \\
(3x_1 - 1, 2x_2) & \text{for } (x_1, x_2) \in S_2 = \{ \frac{1}{3} < x_1 \leq \frac{2}{3}, 0 \leq x_2 \leq \frac{1}{3}\} \\
(3x_1 - 2, 2x_2) & \text{for } (x_1, x_2) \in S_3 = \{ \frac{2}{3} < x_1 \leq 1, 0 \leq x_2 \leq \frac{1}{3}\} \\
(3x_1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_4 = \{0 < x_1 \leq \frac{1}{3}, \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\
(3x_1 - 1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_5 = \{ \frac{1}{3} < x_1 \leq \frac{2}{3}, \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\
(3x_1 - 2, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_6 = \{ \frac{2}{3} < x_1 \leq 1, \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\
(3x_1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_7 = \{0 < x_1 \leq \frac{1}{3}, \frac{2}{3} < x_2 \leq 1\} \\
(3x_1 - 1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_8 = \{ \frac{1}{3} < x_1 \leq \frac{2}{3}, \frac{2}{3} < x_2 \leq 1\} \\
(3x_1 - 2, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_9 = \{ \frac{2}{3} < x_1 \leq 1, \frac{2}{3} < x_2 \leq 1\}.
\end{cases}
\]
The derivative matrix of $(\tau_1, \tau_2)$, and the derivative matrix of the Frobenius operator reduces to a matrix and the invariant density is constant on the invariant measure. Notice that $\tau_1, \tau_2$ are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P} = \{S_1, S_2, \ldots, S_9\}$. For such maps the Perron-Frobenius operator reduces to a matrix and the invariant density is constant on the elements of the partition (see [1]). The Perron-Frobenius operator of the random map $T$ is represented by the following matrix

$$
M = \Pi_1 M_1 + \Pi_2 M_2,
$$
where $M_1$, $M_2$ are the matrices of $P_{T_1}$ and $P_{T_2}$ respectively, and $\Pi_1$, $\Pi_2$ are the diagonal matrices of $p_1(x)$ and $p_2(x)$ respectively. Then, $M$ is given by

\begin{equation}
M = p_1 \mathbf{Id}_9 \times \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
+ p_2 \mathbf{Id}_9 \times \begin{pmatrix}
0.215 & 0.215 & 0.215 & 0.215 & 0.216 & 0.216 & 0.216 & 0.216 & 0.216 \\
0.785 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 \\
0.215 & 0.215 & 0.215 & 0.215 & 0.216 & 0.216 & 0.216 & 0.216 & 0.216 \\
0.785 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 \\
0.215 & 0.215 & 0.215 & 0.215 & 0.216 & 0.216 & 0.216 & 0.216 & 0.216 \\
0.785 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 \\
0.215 & 0.215 & 0.215 & 0.215 & 0.216 & 0.216 & 0.216 & 0.216 & 0.216 \\
0.785 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 \\
0.215 & 0.215 & 0.215 & 0.215 & 0.216 & 0.216 & 0.216 & 0.216 & 0.216 \\
0.785 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 & 0.784 \\
\end{pmatrix} \times \begin{pmatrix}
a & a & a & a & a & b & b & b \\
e & e & e & c & c & c & c & c \\
e & e & e & c & c & c & c & c \\
e & e & e & c & c & c & c & c \\
e & e & e & c & c & c & c & c \\
e & e & e & c & c & c & c & c \\
e & e & e & c & c & c & c & c \\
e & e & e & c & c & c & c & c \\
e & e & e & c & c & c & c & c \\
e & e & e & c & c & c & c & c \\
\end{pmatrix},
\end{equation}

where $p_1 = (0.215, 0.216, 0.216, 0.216, 0.216, 0.216, 0.216, 0.216, 0.215)$, $p_2 = (0.785, 0.784, 0.784, 0.784, 0.784, 0.784, 0.784, 0.784, 0.785)$, $\mathbf{Id}_9$ is $9 \times 9$ identity matrix and

\begin{align*}
a &= 0.12306 \\
b &= 0.087222 \\
c &= 0.12311 \\
d &= 0.087111 \\
e &= 0.11111.
\end{align*}

The invariant density of $T$ is

\begin{equation}
f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9), \quad f_i = f_{i|S_i}, \quad i = 1, 2, \ldots, 9,
\end{equation}

normalized by

\begin{equation}
f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 = 9,
\end{equation}

and satisfying equation $fM = f$. Then, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = \frac{9}{629.38}$ and $f_7 = f_8 = f_9 = \frac{0.29739}{3} f_1$.

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