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POSITIVE DEPENDENT RANDOM MAPS IN ONE AND HIGHER DIMENSIONS

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Abstract. A random map is a discrete-time dynamical system in which one of a number of transformations is randomly selected and applied on each iteration of the process. In this paper, we study random maps with position dependent probabilities on the interval and on a bounded domain of \( \mathbb{R}^n \). Sufficient conditions for the existence of an absolutely continuous invariant measure for random map with position dependent probabilities on the interval and on a bounded domain of \( \mathbb{R}^n \) are the main results of this note.

1. Introduction

Let \( \tau_1, \tau_2, ..., \tau_K \) be a collection of transformations from \( X \) to \( X \). Usually, the random map \( T \) is defined by choosing \( \tau_k \) with constant probability \( p_k, \ p_k > 0, \ \sum_{k=1}^{K} p_k = 1 \). The ergodic theory of such dynamical systems was studied in [9] and in [8] (See also [7]).

There is a rich literature on random maps with position dependent probabilities with \( \tau_1, \tau_2, ..., \tau_K \) being continuous contracting transformations (see [10]).

In this paper, we deal with piecewise monotone transformations \( \tau_1, \tau_2, ..., \tau_K \) and position dependent probabilities \( p_k(x), k = 1, ..., K, \ p_k(x) > 0, \ \sum_{k=1}^{K} p_k(x) = 1 \), i.e., the \( p_k \)'s are functions of position. We point out that studying such dynamical systems was first introduced in [4] where sufficient conditions for the existence of an absolutely continuous invariant measure were given. The conditions in [4] are applicable only when \( \tau_1, \tau_2, ..., \tau_K \) are \( C^2 \) expanding transformations (see [4] for details). In this paper, we prove the existence of an absolutely continuous invariant measure for a random map \( T \) on \( [a, b] \) under milder conditions (see section 4, Conditions (A) and (B)). Moreover, we prove the existence of an absolutely continuous invariant measure for a random map \( T \) on \( S \), where \( S \) is a bounded domain of \( \mathbb{R}^n \) (see section 6, Condition (C)).

The paper is organized in the following way: In section 2, following the ideas of [4], we formulate the definition of a random map \( T \) with position dependent probabilities and introduce its Perron-Frobenius operator. In section 3, we prove properties of the Perron-Frobenius operator of \( T \). In section 4, we prove the existence of an absolutely continuous invariant measure for \( T \) on \( [a, b] \). In section 5, we give an example of a random map \( T \) which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A).
and (B). In section 6, we prove the existence of an absolutely continuous invariant measure for \( T \) on a bounded domain of \( \mathbb{R}^n \). In section 7, we give an example of a random map in \( \mathbb{R}^n \) that preserves an absolutely continuous invariant measure.

2. PRELIMINARIES

Let \( (X, \mathcal{B}, \lambda) \) be a measure space, where \( \lambda \) is an underlying measure. Let \( \tau_k : X \to X, k = 1, \ldots, K \) be piecewise one-to-one, non-singular transformations on a common partition \( \mathcal{P} \) of \( X : \mathcal{P} = \{I_1, \ldots, I_q\} \) and \( \tau_{ki} = \tau_k\big|_{I_i}, i = 1, \ldots, q, k = 1, \ldots, K \) (\( \mathcal{P} \) can be found by considering finer partitions). We define the transition function for the random map \( T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots p_K(x)\} \) as follows:

\[
P(x, A) = \sum_{k=1}^{K} p_k(x) \chi_A(\tau_k(x)),
\]

where \( A \) is any measurable set and \( \{p_k(x)\}_{k=1}^{K} \) is a set of position dependent measurable probabilities, i.e., \( \sum_{k=1}^{K} p_k(x) = 1, p_k(x) \geq 0 \), for any \( x \in X \) and \( \chi_A \) denotes the characteristic function of the set \( A \). We define \( T(x) = \tau_k(x) \) with probability \( p_k(x) \) and \( T^N(x) = \tau_k \circ \tau_{kN-1} \circ \cdots \circ \tau_1(x) \) with probability \( p_k(\tau_{kN-1} \circ \cdots \circ \tau_1(x)) \cdot p_{kN}(\tau_{kN-2} \circ \cdots \circ \tau_1(x)) \cdots p_1(x) \). The transition function \( \mathbb{P} \) induces an operator \( \mathbb{P}_\mu \) on measures on \( (X, \mathcal{B}) \) defined by

\[
\mathbb{P}_\mu(A) = \int \mathbb{P}(x, A) d\mu(x) = \sum_{k=1}^{K} \int p_k(x) \chi_A(\tau_k(x)) d\mu(x)
\]

(2.2)

where

\[
\mathbb{P}_\mu(A) = \int \mathbb{P}(x, A) d\mu(x) = \sum_{k=1}^{K} \int p_k(x) \chi_A(\tau_k(x)) d\mu(x)
\]

(2.2)

We say that measure \( \mu \) is \( T \)-invariant if \( \mathbb{P}_\mu = \mu \), i.e.,

\[
\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x), \quad A \in \mathcal{B}.
\]

(2.3)

If \( \mu \) has density \( f \) with respect to \( \lambda \), the \( \mathbb{P}_\mu \) has also a density which we denote by \( P_T f \). By change of variables, we obtain

\[
\int_A P_T f(x) d\lambda(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{ki}^{-1}(A)} p_k(x) f(x) d\lambda(x)
\]

(2.4)

\[
= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_A p_k(\tau_{ki}^{-1}x) f(\tau_{ki}^{-1}x) \frac{1}{J_{k,i}(\tau_{ki})} d\lambda(x)
\]

where \( J_{k,i} \) is the Jacobian of \( \tau_{ki} \) with respect to \( \lambda \). Since this holds for any measurable set \( A \) we obtain an a.e. equality:

\[
(P_T f)(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} p_k(\tau_{ki}^{-1}x) f(\tau_{ki}^{-1}x) \frac{1}{J_{k,i}(\tau_{ki})} \chi_{\tau_k(I_i)}(x)
\]

(2.5)

or

\[
(P_T f)(x) = \sum_{k=1}^{K} P_{\tau_k}(p_k f)(x)
\]

(2.6)
where $P_{\tau_k}$ is the Perron-Frobenius operator corresponding to the transformation $\tau_k$ (see [1] for details). We call $P_T$ the Perron-Frobenius of the random map $T$. The main tool in this paper is the Perron-Frobenius of $T$ which has very useful properties.

3. Properties of the Perron-Frobenius operator of $T$

The properties of $P_T$ resemble the properties of the classical Perron-Frobenius operator of a single transformation.

**Lemma 3.1.** $P_T$ satisfies the following properties:

(i) $P_T$ is linear;
(ii) $P_T$ is non-negative; i.e., $f \geq 0 \implies P_Tf \geq 0$;
(iii) $P_Tf = f \iff mu = f \cdot \lambda$ is $T$-invariant;
(iv) $\|P_Tf\|_1 \leq \|f\|_1$, where $\|\cdot\|_1$ denotes the $L^1$ norm;
(v) $P_{T^N} = P_T \circ P_{T^{N-1}}$. In particular, $P_{T^N}f = P_{T^{N}}f$.

**Proof.** The proofs of (i)-(iv) are analogous to that for single transformation. For the proof of (v), let $T$ and $R$ be two random maps corresponding to $\{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_L; \tau_1, \tau_2, \ldots, \tau_L\}$ respectively. We define $\{\tau_k\}_{k=1}^K$ and $\{\zeta_l\}_{l=1}^L$ on a common partition $\mathcal{P}$. We have

$$P_{R}(P_{T}f) = P_{R} \left( \sum_{k=1}^{K} P_{\tau_k}(p_kf) \right) = \sum_{l=1}^{L} \sum_{k=1}^{K} P_{\zeta_l}(r_l P_{\tau_k}(p_kf))$$

$$= \sum_{l=1}^{L} \sum_{k=1}^{K} \sum_{l=1}^{q} r_l(\zeta_{l,k}^{-1}) P_{\tau_k}(p_kf)(\zeta_{l,k}^{-1}) \frac{1}{J_{\zeta_l,k,i}(\zeta_{l,k}^{-1})} \chi_{\zeta_l,i}(I_l)$$

$$= \sum_{l=1}^{L} \sum_{k=1}^{K} \sum_{l=1}^{q} \sum_{j=1}^{p} r_l(\zeta_{l,k,j}^{-1}) p_k(\tau_{k,j}^{-1} \circ \zeta_{l,k,j}^{-1}) f(\tau_{k,j}^{-1} \circ \zeta_{l,k,j}^{-1})$$

$$\times \frac{1}{J_{\tau_{k,j},l}(\tau_{k,j}^{-1} \circ \zeta_{l,k,j}^{-1})} J_{\zeta_l,k,i}(\zeta_{l,k}^{-1}) \chi_{\tau_{k,j},l}(I_j)$$

$$= \sum_{l=1}^{L} \sum_{k=1}^{K} \sum_{l=1}^{q} P_{\tau_k \circ \zeta_l}(p_k(\zeta_l r_l f)) = P_{T \circ R}f.$$

4. The existence of absolutely continuous invariant measure on $[a, b]$

Let $(I, \mathcal{B}, \lambda)$ be a measure space, where $\lambda$ is normalized Lebesgue measure on $I = [a, b]$. Let $\tau_k : I \to I$, $k = 1, \ldots, K$ be piecewise one-to-one and differentiable, non-singular transformations on a partition $\mathcal{P}$ of $I : \mathcal{P} = \{I_1, \ldots, I_q\}$ and $\tau_{k,i} = \tau_k |_{I_i}$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Denote by $V(\cdot)$ the standard one dimensional variation of a function, and by $BV(I)$ the space of functions of bounded variations on $I$ equipped with the norm $\|\cdot\|_{BV} = V(\cdot) + \|\cdot\|_1$.

Let $g_k(x) = \frac{p_k(x)}{P_{\tau_k}(p_k)}$, $k = 1, \ldots, K$. We assume that

**Condition (A):** $\sum_{k=1}^{K} g_k(x) < \alpha < 1$, $x \in I$, and

**Condition (B):** $g_k \in BV(I)$, $k = 1, \ldots, K$.

Under the above conditions our goal is to prove:

$$V_T P^N f \leq AV_T f + B\|f\|_1$$
for some \( n \geq 1 \), where \( 0 < A < 1 \) and \( B > 0 \). The inequality (4.1) guarantees the existence of a \( T \)-invariant measure absolutely continuous with respect to Lebesgue measure and the quasi-compactness of operator \( P_{T} \) with all the consequences of this fact, see [1]. We will need a number of lemmas:

**Lemma 4.1.** Let \( f \in BV(I) \). Suppose \( \tau : I \to J \) is differentiable and \( \tau'(x) \neq 0 \), \( x \in I \). Set \( \phi = \tau^{-1} \) and let \( g(x) = \frac{p(x)}{p'(\tau(x))} \in BV(I) \). Then

\[
V_{J}(f(\phi)g(\phi)) \leq (V_{I}f + \sup_{I}f)(V_{I}g + \sup_{I}g).
\]

**Proof.** First, note that we have dropped all the \( k, i \) indices to simplify the notation. Then, the proof follows in the same way as in Lemma 3 of [9]. \( \square \)

**Lemma 4.2.** Let \( T \) satisfy conditions (A) and (B). Then for any \( f \in BV(I) \),

\[
(4.2) \quad V_{I}P_{T}f \leq AV_{I}f + B\|f\|_{1},
\]

where

\[
A = 3\alpha + \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I}g_{k};
\]

and

\[
B = 2\beta\alpha + \beta \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I}g_{k},
\]

where \( \beta = \max_{1 \leq i \leq q}(\lambda(I_{k}))^{-1} \).

**Proof.** First, we will refine partition \( P \) to satisfy additional condition. Let \( \eta > 0 \) be such that \( \sum_{k=1}^{K}(g_{k}(x) + \varepsilon_{k}) < \alpha \) whenever \( |x| < \eta, k = 1, \ldots, K \). Since \( g_{k}, k = 1, \ldots, K \) are of bounded variation we can find a finite partition \( K \) such that for any \( k = 1, \ldots, K \)

\[
|g_{k}(x) - g_{k}(y)| < \eta,
\]

for \( x, y \) in the same element of \( K \). Instead of the partition \( P \) we consider a join \( P \vee K \). Without restricting generality of our considerations, we can assume that this is our original partition \( P \). Then, we have

\[
(4.3) \quad \max_{1 \leq i \leq q} \sum_{k=1}^{K} \sup_{x \in \chi_{k}} g_{k}(x) < \alpha.
\]

We have \( V_{I}(P_{T}f) = V_{I}(\sum_{k=1}^{K} \tau_{k}g_{k}(p_{k}f)) \). We will estimate this variation. Let \( \phi_{k,i} = \tau^{-1}_{k,i}, k = 1, \ldots, K, i = 1, \ldots, q \). We have

\[
V_{I} \left( \sum_{k=1}^{K} P_{\tau_{k}}(p_{k}f) \right) = V_{I} \left( \sum_{k=1}^{K} \sum_{i=1}^{q} f(\phi_{k,i})g_{k}(\phi_{k,i})\chi_{\tau_{k}(I_{k})} \right)
\]

\[
\leq \sum_{k=1}^{K} \sum_{i=1}^{q} |f(a_{i-1})||g_{k}(a_{i-1})| + |f(a_{i})||g_{k}(a_{i})| + \sum_{k=1}^{K} \sum_{i=1}^{q} V_{\tau_{k}(I_{k})}[f(\phi_{k,i})g_{k}(\phi_{k,i})].
\]
First, we estimate the first sum on the right hand side of (4.4):

\[
\sum_{k=1}^{K} \sum_{i=1}^{q} |f(a_{i-1})g_k(a_{i-1})| + |f(a_i)g_k(a_i)|
\]

(4.5)

\[
= \sum_{i=1}^{q} \left[ |f(a_{i-1})| \left( \sum_{k=1}^{K} |g_k(a_{i-1})| \right) + |f(a_i)| \left( \sum_{k=1}^{K} |g_k(a_i)| \right) \right]
\]

\[
\leq \alpha \left( \sum_{i=1}^{q} (|f(a_{i-1})| + |f(a_i)|) \right)
\]

(4.6)

\[
\leq \alpha \left( \sum_{i=1}^{q} \left( V_{I_i}f + (\lambda(I_i))^{-1} \int_{I_i} f d\lambda \right) \right) = \alpha (V_I f + \beta \|f\|_1).
\]

We now estimate the second sum on the right hand side of (4.4). Using Lemma 4.1 we obtain:

\[
\sum_{k=1}^{K} \sum_{i=1}^{q} V_{T_k(I_i)}[f(\phi_{k,i})g_k(\phi_{k,i})] \leq \sum_{k=1}^{K} \sum_{i=1}^{q} \left( V_{I_i}f + \sup_{I_i} f \right) \left( V_{I_i}g_k + \sup_{I_i} g_k \right)
\]

(4.7)

\[
\leq (2V_I f + \beta \|f\|_1) \left( \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i}g_k + \alpha \right).
\]

Thus, using (4.5) and (4.6), we obtain

\[
(4.7) \quad V_I P_I f \leq \left( 3\alpha + \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i}g_k \right) V_I f + \left( 2\beta\alpha + \beta \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i}g_k \right) \|f\|_1.
\]

\[\square\]

In the following two lemmas we show that constants \(\alpha\) and \(\max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i}g_k\) decrease when we consider higher iterations \(T^n\) instead of \(T\). The constant \(\beta\) obviously increases, but this is not important.

**Lemma 4.3.** Let \(T\) be a random map which satisfies condition (A). Then, for \(x \in I\),

\[
(4.8) \quad \sum_{w \in \{1, 2, \ldots, K\}^N} p_w(x) T_w(x) |T_w(x)| < \alpha^N,
\]

where \(T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)\) and \(p_w(x) = p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)\), define random map \(T^N\).

**Proof.** We have

\[
T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)
\]

with probability

\[
p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x).
\]

The maps defining \(T^N\) may be indexed by \(w \in \{1, 2, \ldots, K\}^N\). Set

\[
T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)
\]

...
where \( w = (k_1, \ldots, k_N) \), and
\[
p_w(x) = p_{k_N}(\tau_{k_N-1} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x).
\]

Then,
\[
T'_w(x) = \tau'_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x))\tau'_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots \tau'_{k_1}(x).
\]

Suppose that \( T \) satisfies condition (A). We will prove (4.8) using induction on \( N \). For \( N = 1 \), we have
\[
\sum_{w \in \{1, 2, \ldots, K\}} \frac{p_w(x)}{|T'_w(x)|} < \alpha
\]
by condition (A). Assume (4.8) is true for \( N - 1 \). Then,
\[
\sum_{w \in \{1, 2, \ldots, K\}^N} \frac{p_w(x)}{|T'_w(x)|} = \sum_{w \in \{1, 2, \ldots, K\}^{N-1}} \sum_{k=1}^{K} \frac{p_k(x)p_{\tau'(k)(x)}}{|T'_w(x)| |T'_{k}(\tau_k(x))|}
\leq \left( \sum_{k=1}^{K} \frac{p_k(x)}{|\tau'(k)(x)|} \right) \left( \sum_{w \in \{1, 2, \ldots, K\}^{N-1}} \frac{p_{\tau'(w)(x)}}{|T'_{\tau'(w)(x)}|} \right) < \alpha \cdot \alpha^{N-1} = \alpha^N.
\]

\[\square\]

**Lemma 4.4.** Let \( g_w = \frac{p_{\tau(w)}}{|T'_w|} \), where \( T_w \) and \( p_w \) are defined in Lemma 4.3, \( w \in \{1, \ldots, K\}^n \). Define
\[
W_1 \equiv \max_{1 \leq i \leq n} \sum_{k=1}^{K} V_{i} \cdot g_k,
\]
and
\[
W_n \equiv \max_{J \in \mathcal{P}^{(n)}} \sum_{w \in \{1, \ldots, K\}^n} V_{J} \cdot g_w,
\]
where \( \mathcal{P}^{(n)} \) is the common monotonicity partition for all \( T_w \). Then, for all \( n \geq 1 \)
\[
W_n \leq n \alpha^{n-1} W_1, \tag{4.11}
\]
where \( \alpha \) is defined in condition (A).

**Proof.** We prove the lemma by induction on \( n \). For \( n = 1 \) the lemma is true by definition of \( W_n \). Assume that the lemma is true for \( n \), i.e.,
\[
W_n \leq n \alpha^{n-1} W_1. \tag{4.12}
\]
Let $J \in \mathcal{P}^{(n+1)}$ and $x_0 < x_1 < \ldots < x_l$ be a sequence of points in $J$. Then

\begin{equation}
\sum_{w} \sum_{j=0}^{l-1} |g_w(x_{j+1}) - g_w(x_j)| = \sum_{j=0}^{l-1} \sum_{w \in \{1, \ldots, K\}^{n+1}} |g_w(x_{j+1}) - g_w(x_j)|
\end{equation}

\begin{align*}
&\leq \sum_{j=0}^{l-1} \sum_{\pi \in \{1, \ldots, K\}^n} \sum_{k=1}^{K} |g_{\pi}(r_k(x_{j+1}))g_k(x_{j+1}) - g_{\pi}(r_k(x_j))g_k(x_j)| \\
&\leq \sum_{j=0}^{l-1} \sum_{\pi \in \{1, \ldots, K\}^n} \sum_{k=1}^{K} |g_{\pi}(r_k(x_{j+1}))g_k(x_{j+1}) - g_{\pi}(r_k(x_j))g_k(x_j)| \\
&\quad + \sum_{j=0}^{l-1} \sum_{\pi \in \{1, \ldots, K\}^n} \sum_{k=1}^{K} |g_{\pi}(r_k(x_{j+1}))g_k(x_{j}) - g_{\pi}(r_k(x_j))g_k(x_{j})| \\
&\leq \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \sum_{\pi \in \{1, \ldots, K\}^n} g_{\pi}(r_k(x_{j+1})) \\
&\quad + \sum_{j=0}^{l-1} \sum_{k=1}^{K} g_k(x_j) \sum_{\pi \in \{1, \ldots, K\}^n} |g_{\pi}(r_k(x_{j+1})) - g_{\pi}(r_k(x_j))| \\
&\leq \alpha^n \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \\
&\quad + \alpha \sum_{j=0}^{l-1} \sum_{\pi \in \{1, \ldots, K\}^n} |g_{\pi}(r_k(x_{j+1})) - g_{\pi}(r_k(x_j))| \\
&\leq \alpha^n W_1 + \alpha W_n \leq \alpha^n W_1 + n \alpha^n W_1 = (n+1)\alpha^n W_1.
\end{align*}

We used condition (A) and lemma 4.3. \hfill \Box

**Theorem 4.5.** Let $T$ be a random map which satisfies conditions (A) and (B). Then $T$ preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator $P_T$ is quasi-compact on $BV(I)$, see [1].

**Proof.** Let $N$ be such that $A_N = 3\alpha^N + W_N < 1$. Then, by Lemma 4.3,

$$\sum_{w \in \{1, \ldots, K\}^N} g_w(x) < \alpha^N, \quad x \in I.$$  

We refine the partition $\mathcal{P}^{(N)}$ like in the proof of Lemma 4.2, to have

$$\max_{J \in \mathcal{P}^{(N)}} \sum_{w \in \{1, \ldots, K\}^N} \sup_J g_w < \alpha^N.$$  

Then, by lemma 4.2, we get

\begin{equation}
\|P_T^n f\|_{BV} \leq A_N \|f\|_{BV} + B_N \|f\|_1,
\end{equation}

where $B_N = \beta_N(2\alpha^N + W_N)$, $\beta_N = \max_{J \in \mathcal{P}^{(N)}} (\lambda(J))^{-1}$. The theorem follows by the standard technique (see [1]). \hfill \Box

**Remark 4.6.** It is enough to assume that condition (A) is satisfied for some iterate $T^m, m \geq 1$. 
Remark 4.7. The number of absolutely continuous invariant measures for random maps has been studied in [6]. The proof of [6], which uses graph theoretic methods, goes through analogously in our case; i.e., when $T$ is a random map with position dependent probabilities.

5. Example

We present an example of a random map $T$ which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A) and (B).

Example 5.1. Let $T$ be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$ where

\begin{align*}
\tau_1(x) &= \begin{cases}
2x & \text{for } 0 \leq x \leq \frac{1}{2} \\
1 & \text{for } \frac{1}{2} < x \leq 1
\end{cases}, \\
\tau_2(x) &= \begin{cases}
x + \frac{1}{2} & \text{for } 0 \leq x \leq \frac{1}{2} \\
2x - 1 & \text{for } \frac{1}{2} < x \leq 1
\end{cases},
\end{align*}

and

\begin{align*}
p_1(x) &= \begin{cases}
\frac{2}{3} & \text{for } 0 \leq x \leq \frac{1}{2} \\
\frac{1}{3} & \text{for } \frac{1}{2} < x \leq 1
\end{cases}, \\
p_2(x) &= \begin{cases}
\frac{1}{3} & \text{for } 0 \leq x \leq \frac{1}{2} \\
\frac{2}{3} & \text{for } \frac{1}{2} < x \leq 1
\end{cases}.
\end{align*}

Then, $\sum_{k=1}^{2} g_k(x) = \frac{2}{3} < 1$. Therefore, $T$ satisfies conditions (A) and (B). Consequently, by theorem 4.5, $T$ preserves an invariant measure absolutely continuous with respect to Lebesgue measure. Notice that $\tau_1, \tau_2$ are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P} : \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. For such maps the Perron-Frobenius operator reduces to a matrix (see [1]). The corresponding matrices are:

\begin{align*}
P_{\tau_1} &= \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
0 & 1
\end{pmatrix}, & P_{\tau_2} &= \begin{pmatrix}
0 & 1 \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\end{align*}

Their invariant densities are $f_{\tau_1} = [0, 2]$ and $f_{\tau_2} = [2, 0]$. The Perron-Frobenius operator of the random map $T$ is given by:

\begin{equation}
P_T = \begin{pmatrix}
\frac{2}{3} & 0 & 1 \\
0 & 1 & 0 \\
\frac{1}{3} & 1 & \frac{1}{3}
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{3} & 1 & \frac{1}{3}
\end{pmatrix}.
\end{equation}

If the invariant density of $T$ is $f = [f_1, f_2]$, normalized by $f_1 + f_2 = 2$ and satisfying equation $f P_T = f$, then $f_1 = \frac{2}{3}$ and $f_2 = \frac{1}{3}$.

6. The Existence of Absolutely Continuous Invariant Measure in $\mathbb{R}^n$

Let $S$ be a bounded region in $\mathbb{R}^n$ and $\lambda_n$ be Lebesgue measure on $S$. Let $\tau_k : S \to S, k = 1, \ldots, K$ be piecewise one-to-one and $C^2$, non-singular transformations on a partition $\mathcal{P}$ of $S : \mathcal{P} = \{S_1, \ldots, S_q\}$ and $\tau_{k,i} = \tau_k|_{S_i}, i = 1, \ldots, q, k = 1, \ldots, K$. Let each $S_i$ be a bounded closed domain having a piecewise $C^2$ boundary of finite $(n - 1)$-dimensional measure. We assume that the faces of $\partial S_i$ meet at angles bounded uniformly away from 0. We will also assume that the probabilities $p_k(x)$
are piecewise $C^1$ functions on the partition $P$. Let $D\tau_{k,i}^{-1}(x)$ be the derivative matrix of $\tau_{k,i}^{-1}$ at $x$. We assume:

**Condition (C):**

$$\max_{1 \leq i \leq q} \sum_{k=1}^{K} p_k(x)\|D\tau_{k,i}^{-1}(\tau_{k,i}(x))\| < \sigma < 1.$$  

Let $\sup_{x \in \tau_{k,i}(S_k)} \|D\tau_{k,i}^{-1}(x)\| := \sigma_{k,i}$ and $\sup_{x \in S_k} p_k(x) := \tau_{k,i}$. Using smoothness of $D\tau_{k,i}^{-1}$'s and $p_k$'s we can refine partition $P$ to satisfy

**Condition (C'):**

$$\sum_{1 \leq i \leq q} \max_{1 \leq j \leq q} \sigma_{k,i} \tau_{k,i} < \sigma < 1.$$  

Under this condition, our goal is to prove the existence of an a.c.i.m. for the random map $T = \{\tau_1, ..., \tau_K; p_1, ..., p_K\}$. The main tool of this section is the multi-dimensional notion of variation defined using derivatives in the distributional sense (see [3]):

$$V(f) = \int_{\mathbb{R}^n} \|Df\| = \sup\{ \int_{\mathbb{R}^n} f \text{div}(g) d\lambda_n : g = (g_1, ..., g_n) \in C^1_c(\mathbb{R}^n, \mathbb{R}^n) \},$$

where $f \in L_1(\mathbb{R}^n)$ has bounded support, $Df$ denotes the gradient of $f$ in the distributional sense, and $C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ is the space of continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ having a compact support. We will use the following property of variation which is derived from [3], Remark 2.14: If $f = 0$ outside a closed domain $A$ whose boundary is Lipschitz continuous, $f|_A$ is continuous, $f|_{\text{int}(A)}$ is $C^1$, then

$$V(f) = \int_{\text{int}(A)} \|Df\| d\lambda_n + \int_{\partial A} |f| d\lambda_{n-1},$$

where $\lambda_{n-1}$ is the $n-1$-dimensional measure on the boundary of $A$. In this section we shall consider the Banach space (see [3], Remark 1.12),

$$BV(S) = \{ f \in L_1(S) : V(f) < +\infty \},$$

with the norm $\|f\|_{BV} = V(f) + \|f\|_1$. We adapt the following two lemmas from [5]. The proofs of Lemma 6.1 and Lemma 6.2 are exactly the same as in [5].

**Lemma 6.1.** Consider $S_k \in P$. Let $x$ be a point in $\partial S_k$ and $y = \tau_k(x)$ a point in $\partial(\tau_k(S_k))$. Let $J_{k,i}$ be the Jacobian of $\tau_k|_{S_k}$ at $x$ and $J_{0,k,i}$ be the Jacobian of $\tau_k|_{\mathbb{R}^n}$ at $x$. Then

$$\frac{J_{0,k,i}}{J_{k,i}} \leq \sigma_{k,i}.$$

Let us fix $1 \leq i \leq q$. Let $Z$ denote the set of singular points of $\partial S_k$. Let us construct at any $x \in Z$ the largest cone having a vertex at $x$ and which lies completely in $S_k$. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\beta(S_k) = \min_{x \in Z} \theta(x).$$

Since the faces of $\partial S_k$ meet at angles bounded away from 0, $\beta(S_k) > 0$. Let $\alpha(S_k) = \pi/2 + \beta(S_k)$ and

$$\alpha(S_k) = |\cos(\alpha(S_k))|. $$
Now we will construct a $C^1$ field of segments $L_y$, $y \in \partial S_i$, every $L_y$ being a central ray of a regular cone contained in $S_i$, with angle subtended at the vertex $y$ greater than or equal to $\beta(S_i)$.

We start at points $y \in Z$, where the minimal angle $\beta(S_i)$ is attained, defining $L_y$ to be central rays of the largest regular cones contained in $S_i$. Then we extend this field of segments to $C^1$ field we want, making $L_y$ short enough to avoid overlapping.

Let $\delta(y)$ be the length of $L_y$, $y \in \partial S_i$. By the compactness of $\partial S_i$ we have

$$\delta(S_i) = \inf_{y \in \partial S_i} \delta(y) > 0.$$ 

Now, we shorten $L_y$ of our field, making them all of the length $\delta(S_i)$.

**Lemma 6.2.** For any $S_i$, $i = 1, \ldots, q$, if $f$ is a $C^1$ function on $S_i$, then

$$\int_{\partial S_i} f(y)d\lambda_{n-1}(y) \leq \frac{1}{a(S_i)} \left( \frac{1}{\delta(S_i)} \int_{S_i} f d\lambda_n + V_{\text{inf}}(S_i)(f) \right).$$

Our main technical result is the following:

**Theorem 6.3.** If $T$ is a random map which satisfies Condition (C), then

$$V(P_T f) \leq \sigma(1 + 1/a)V(f) + (M + \frac{\sigma}{a})\|f\|_1,$$

where $a = \min\{a(S_i) : i = 1, \ldots, q\} > 0$, $\delta = \min\{\delta S_i : i = 1, \ldots, q\} > 0$, $M_{k, d} = \sup_{x \in S_i}(Dp_k(x) - \frac{DJ_{k, i}}{J_{k, i}}p_k(x))$ and $M = \sum_{k=1}^{K} \max_{1 \leq i \leq q} M_{k, d}$.

**Proof.** We have $V(P_T f) \leq \sum_{k=1}^{K} V(P_{\tau_k}(p_k f))$. We first estimate $V(P_{\tau_k}(p_k f))$. Let $F_{k, i} = p_k^{(i)}(\tau_k^{(i)})^{\frac{1}{J_{k, i}(\tau_k^{(i)})}}$, and $R_{k, i} = \tau_k^{(i)}(S_i), i = 1, \ldots, q, k = 1, \ldots, K$. Then,

$$\int_{\mathbb{R}^n} \|DP_{\tau_k}(p_k f)\|d\lambda_n \leq \sum_{i=1}^{q} \int_{\mathbb{R}^n} \|D(F_{k, i}\chi_{R_i})\|d\lambda_n$$

$$\leq \sum_{i=1}^{q} \left( \int_{\mathbb{R}^n} \|D(F_{k, i}\chi_{R_i})\|d\lambda_n + \int_{\mathbb{R}^n} \|F_{k, i}(D\chi_{R_i})\|d\lambda_n \right).$$

Now, for the first integral we have,

$$\int_{\mathbb{R}^n} \|D(F_{k, i})\chi_{R_i}\|d\lambda_n = \int_{R_i} \|D(F_{k, i})\|d\lambda_n$$

$$\leq \int_{R_i} \|D(f(\tau_k^{(i)})^{\frac{1}{J_{k, i}(\tau_k^{(i)})}})\|d\lambda_n + \int_{R_i} \|f(\tau_k^{(i)})D\left( \frac{p_k(\tau_k^{(i)})}{J_{k, i}(\tau_k^{(i)})} \right)\|d\lambda_n$$

$$\leq \int_{R_i} \|D(f(\tau_k^{(i)})^{\frac{1}{J_{k, i}(\tau_k^{(i)})}})\|\|D\tau_k^{(i)}\|^{\frac{1}{J_{k, i}(\tau_k^{(i)})}}d\lambda_n + \int_{R_i} \|f(\tau_k^{(i)})\|D\left( \frac{M_k}{J_{k, i}(\tau_k^{(i)})} \right)\|d\lambda_n$$

$$\leq \sigma_{k, i}\chi_{k, i} \int_{S_i} \|D f\|d\lambda_n + M_k \int_{S_i} \|f\|d\lambda_n.$$
For the second integral we have,

\begin{equation}
\int_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_0})\|d\lambda_n = \int_{\partial R_0} |f(\tau_{k,i}^{-1})| \frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda_{n-1} = \int_{\partial S_i} |f| p_k f_0 f_{k,i} d\lambda_{n-1}.
\end{equation}

By Lemma 4.3, \( \frac{f_0}{J_{k,i}} \leq \sigma_{k,i} \). Using Lemma 4.2, we get:

\begin{equation}
\int_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_0})\|d\lambda_n \leq \sigma_{k,i} \int_{\partial S_i} |f| d\lambda_{n-1} \leq \frac{\sigma_{k,i} \sigma_{k,i}}{a} V_{S_i}(f) + \frac{\sigma_{k,i} \pi_{k,i}}{a \delta} \int_{S_i} |f| d\lambda_n.
\end{equation}

Using Condition (C'), summing first over \( i \), we obtain

\[ V(P_{\tau_i}(p_kf)) \leq (\max_{1 \leq i \leq q} \sigma_{k,i} \pi_{k,i})(1+1/a)V(f) + (\max_{1 \leq i \leq q} M_{k,i} + \max_{1 \leq i \leq q} \sigma_{k,i} \pi_{k,i})\|f\|_1, \]

and then, summing over \( k \) we obtain

\[ V(P_T f) \leq \sigma(1+1/a)V(f) + (M + \frac{\sigma}{a \delta})\|f\|_1. \]

Theorem 6.4. Let \( T \) be a random map which satisfies condition (C). If \( \sigma(1+1/a) < 1 \), then \( T \) preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator \( P_T \) is quasi-compact on \( BV(S) \), see [1].

Proof. The proof of the theorem follows by the standard technique (see [1]). \( \square \)

7. Example in \( \mathbb{R}^2 \)

In this section, We present an example of a random map which satisfies condition (C) of theorem 6.3 and thus it preserves an absolutely continuous invariant measure.

Example 7.1. Let \( T \) be a random map which is given by \( \{\tau_1, \tau_2; p_1(x), p_2(x)\} \) where \( \tau_1, \tau_2 : I^2 \rightarrow I^2 \) defined by:

\begin{equation}
\tau_1(x_1, x_2) = \begin{cases} 
(3x_1, 2x_2) & \text{for } (x_1, x_2) \in S_1 = \{0 \leq x_1, x_2 \leq \frac{1}{3}\} \\
(3x_1 - 1, 2x_2) & \text{for } (x_1, x_2) \in S_2 = \{\frac{1}{3} < x_1 \leq \frac{2}{3}, 0 \leq x_2 \leq \frac{1}{3}\} \\
(3x_1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_3 = \{\frac{1}{3} < x_1 \leq 1; 0 \leq x_2 \leq \frac{2}{3}\} \\
(3x_1 - 1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_4 = \{0 < x_1 \leq \frac{1}{3}; \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\
(3x_1 - 2, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_5 = \{\frac{1}{3} < x_1 \leq 1; \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\
(3x_1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_6 = \{0 < x_1 \leq \frac{1}{3}; \frac{2}{3} < x_2 \leq 1\} \\
(3x_1 - 1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_7 = \{\frac{1}{3} < x_1 \leq 1; \frac{2}{3} < x_2 \leq 1\} \\
(3x_1 - 2, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_8 = \{\frac{1}{3} < x_1 \leq 1; \frac{2}{3} < x_2 \leq 1\}
\end{cases}
\end{equation}
The derivative matrix of \((7.2)\) on the same Markov partition \((7.6)\) map for this partition invarient measure. Notice that therefore, by theorem 6.4, the random map \(T\) is represented by the following matrix

\[
\begin{pmatrix}
0.215 & 0.785 \\
0.216 & 0.784 \\
0.216 & 0.784 \\
0.216 & 0.784 \\
0.216 & 0.784 \\
0.216 & 0.784 \\
0.215 & 0.785 \\
\end{pmatrix}
\]

The derivative matrix of \((\tau_1, d)^{-1}\), is

\[
\begin{pmatrix}
\frac{1}{3} & 0 \\
0 & \frac{1}{3} \\
\end{pmatrix}
\]

and the derivative matrix of \((\tau_2, d)^{-1}\), is

\[
\begin{pmatrix}
\frac{1}{3} & 0 \\
0 & \frac{1}{3} \\
\end{pmatrix}
\]

Therefore, the Euclidean matrix norm, \(\|D(\tau_1, d)^{-1}\|\) is \(\frac{\sqrt{3}}{3}\), or \(\frac{\sqrt{13}}{6}\) and the Euclidean matrix norm, \(\|D(\tau_2, d)^{-1}\|\) is \(\frac{\sqrt{3}}{3}\). Then

\[
\max_{1 \leq i \leq q} \sum_{k=1}^{K} p_k(x) \|D_{k, i}^{-1}(\tau_k, d(x))\| \leq 0.216 \frac{\sqrt{13}}{6} + 0.785 \frac{\sqrt{2}}{3}
\]

For this partition \(\mathcal{P}\), we have \(a = 1\), which implies

\[
\sigma(1 + 1/a) = 2(0.216 \frac{\sqrt{13}}{6} + 0.785 \frac{\sqrt{2}}{3}) \approx 0.9998 < 1.
\]

Therefore, by theorem 6.4, the random map \(T\) admits an absolutely continuous invariant measure. Notice that \(\tau_1, \tau_2\) are piecewise linear Markov maps defined on the same Markov partition \(\mathcal{P} = \{S_1, S_2, \ldots, S_9\}\). For such maps the Perron-Frobenius operator reduces to a matrix and the invariant density is constant on the elements of the partition (see [1]). The Perron-Frobenius operator of the random map \(T\) is represented by the following matrix

\[
M = \Pi_1 M_1 + \Pi_2 M_2,
\]

\[
(7.6)
\]
where $M_1$, $M_2$ are the matrices of $P_{\tau_1}$ and $P_{\tau_2}$ respectively, and $\Pi_1$, $\Pi_2$ are the diagonal matrices of $p_1(x)$ and $p_2(x)$ respectively. Then, $M$ is given by

\begin{equation}
(7.7) \quad M = p_1 \text{Id}_9 \times \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\end{equation}

\begin{equation}
+ p_2 \text{Id}_9 \times \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
a & a & a & a & b & b & b \\
0 & 0 & 0 & 0 & c & c & c \\
0 & 0 & 0 & 0 & c & c & c \\
0 & 0 & 0 & 0 & c & c & c \\
0 & 0 & 0 & 0 & c & c & c \\
0 & 0 & 0 & 0 & c & c & c \\
0 & 0 & 0 & 0 & c & c & c \\
0 & 0 & 0 & 0 & c & c & c \\
0 & 0 & 0 & 0 & c & c & c \\
\end{pmatrix}
\end{equation}

where $p_1 = (0.215, 0.216, 0.216, 0.216, 0.216, 0.216, 0.216, 0.216, 0.215)$, 
$p_2 = (0.785, 0.784, 0.784, 0.784, 0.784, 0.784, 0.784, 0.784, 0.785)$, $\text{Id}_9$ is $9 \times 9$ identity matrix and

\begin{align*}
a &= 0.12306 \\
b &= 0.087222 \\
c &= 0.12311 \\
d &= 0.087111 \\
e &= 0.111111.
\end{align*}

The invariant density of $T$ is

\begin{equation}
(7.8) \quad f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9), \quad f_i = f_{\lfloor S_i \rfloor}, \quad i = 1, 2, \ldots, 9,
\end{equation}

normalized by

\begin{equation}
(7.9) \quad f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 = 9,
\end{equation}

and satisfying equation $fM = f$. Then, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = \frac{9}{0,29739}$ and $f_7 = f_8 = f_9 = \frac{0,29739}{3}f_1$.

**References**


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