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POSITION DEPENDENT RANDOM MAPS IN ONE AND HIGHER DIMENSIONS

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Abstract. A random map is a discrete-time dynamical system in which one of a number of transformations is randomly selected and applied on each iteration of the process. In this paper, we study random maps with position dependent probabilities on the interval and on a bounded domain of $\mathbb{R}^n$. Sufficient conditions for the existence of an absolutely continuous invariant measure for random map with position dependent probabilities on the interval and on a bounded domain of $\mathbb{R}^n$ are the main results of this note.

1. Introduction

Let $\tau_1, \tau_2, \ldots, \tau_K$ be a collection of transformations from $X$ to $X$. Usually, the random map $T$ is defined by choosing $\tau_k$ with constant probability $p_k$, $p_k > 0$, $\sum_{k=1}^{K} p_k = 1$. The ergodic theory of such dynamical systems was studied in [9] and in [8] (See also [7]).

There is a rich literature on random maps with position dependent probabilities with $\tau_1, \tau_2, \ldots, \tau_K$ being continuous contracting transformations (see [10]).

In this paper, we deal with piecewise monotone transformations $\tau_1, \tau_2, \ldots, \tau_K$ and position dependent probabilities $p_k(x), k = 1, \ldots, K$, $p_k(x) > 0$, $\sum_{k=1}^{K} p_k(x) = 1$, i.e., the $p_k$'s are functions of position. We point out that studying such dynamical systems was first introduced in [4] where sufficient conditions for the existence of an absolutely continuous invariant measure were given. The conditions in [4] are applicable only when $\tau_1, \tau_2, \ldots, \tau_K$ are $C^2$ expanding transformations (see [4] for details). In this paper, we prove the existence of an absolutely continuous invariant measure for a random map $T$ on $[a, b]$ under milder conditions (see section 4, Conditions (A) and (B)). Moreover, we prove the existence of an absolutely continuous invariant measure for a random map $T$ on $S$, where $S$ is a bounded domain of $\mathbb{R}^n$ (see section 6, Condition (C)).

The paper is organized in the following way: In section 2, following the ideas of [4], we formulate the definition of a random map $T$ with position dependent probabilities and introduce its Perron-Frobenius operator. In section 3, we prove properties of the Perron-Frobenius operator of $T$. In section 4, we prove the existence of an absolutely continuous invariant measure for $T$ on $[a, b]$. In section 5, we give an example of a random map $T$ which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A).
and (B). In section 6, we prove the existence of an absolutely continuous invariant measure for $T$ on a bounded domain of $\mathbb{R}^n$. In section 7, we give an example of a random map in $\mathbb{R}^n$ that preserves an absolutely continuous invariant measure.

2. PRELIMINARIES

Let $(X, \mathcal{B}, \lambda)$ be a measure space, where $\lambda$ is an underlying measure. Let $\tau_k : X \to X$, $k = 1, \ldots, K$ be piecewise one-to-one, non-singular transformations on a common partition $\mathcal{P}$ of $X : \mathcal{P} = \{I_1, \ldots, I_q\}$ and $\tau_{ki} = \tau_k|_{I_i}$, $i = 1, \ldots, q$, $k = 1, \ldots, K$ ($\mathcal{P}$ can be found by considering finer partitions). We define the transition function for the random map $T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots, p_K(x)\}$ as follows:

\begin{equation}
(2.1) \quad \mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x) \chi_A(\tau_k(x)),
\end{equation}

where $A$ is any measurable set and $\{p_k(x)\}_{k=1}^{K}$ is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^{K} p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in X$ and $\chi_A$ denotes the characteristic function of the set $A$. We define $T(x) = \tau_k(x)$ with probability $p_k(x)$ and $T^N(x) = \tau_k \circ \tau_{kN-1} \circ \cdots \circ \tau_{k1}(x)$ with probability $p_k(\tau_{kN-1} \circ \cdots \circ \tau_{k1}(x)) \cdot p_{kN-1}(\tau_{kN-2} \circ \cdots \circ \tau_{k1}(x)) \cdots p_{k1}(x)$. The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_\mu$ on measures on $(X, \mathcal{B})$ defined by

\begin{equation}
(2.2) \quad \mathbb{P}_\mu(A) = \int \mathbb{P}(x, A) d\mu(x) = \sum_{k=1}^{K} \int p_k(x) \chi_A(\tau_k(x)) d\mu(x)
\end{equation}

We say that measure $\mu$ is $T$-invariant iff $\mathbb{P}_\mu = \mu$, i.e.,

\begin{equation}
(2.3) \quad \mu(A) = \sum_{k=1}^{K} \int \tau_k^{-1}(A) p_k(x) d\mu(x), \quad A \in \mathcal{B}.
\end{equation}

If $\mu$ has density $f$ with respect to $\lambda$, the $\mathbb{P}_\mu$ has also a density which we denote by $P_Tf$. By change of variables, we obtain

\begin{equation}
(2.4) \quad \int_A P_Tf(x) d\lambda(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{ki}^{-1}(A)} p_k(x) f(x) d\lambda(x)
\end{equation}

where $J_{k,i}$ is the Jacobian of $\tau_{ki}$ with respect to $\lambda$. Since this holds for any measurable set $A$ we obtain an a.e. equality:

\begin{equation}
(2.5) \quad (P_Tf)(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} p_k(\tau_{ki}^{-1} x) f(\tau_{ki}^{-1} x) \frac{1}{J_{k,i}(\tau_{ki}^{-1})} \chi_{\tau_k(I_i)}(x)
\end{equation}

or

\begin{equation}
(2.6) \quad (P_Tf)(x) = \sum_{k=1}^{K} P_{\tau_k} (p_k f)(x)
\end{equation}
where $P_{\tau_k}$ is the Perron-Frobenius operator corresponding to the transformation $\tau_k$ (see [1] for details). We call $P_T$ the Perron-Frobenius of the random map $T$. The main tool in this paper is the Perron-Frobenius of $T$ which has very useful properties.

3. Properties of the Perron-Frobenius operator of $T$

The properties of $P_T$ resemble the properties of the classical Perron-Frobenius operator of a single transformation.

Lemma 3.1. $P_T$ satisfies the following properties:
(i) $P_T$ is linear;
(ii) $P_T$ is non-negative; i.e., $f \geq 0 \implies P_T f \geq 0$;
(iii) $P_T f = f \iff mu = f \cdot \lambda$ is $T$-invariant;
(iv) $\|P_T f\|_1 \leq \|f\|_1$, where $\|\cdot\|_1$ denotes the $L^1$ norm;
(v) $P_{T \circ R} = P_T \circ P_R$. In particular, $P^N_T f = P^{N-1}_T f$.

Proof. The proofs of (i)-(iv) are analogous to that for single transformation. For the proof of (v), let $T$ and $R$ be two random maps corresponding to $\{\tau_1, \tau_2, ..., \tau_K; p_1, p_2, ..., p_K\}$ and $\{\zeta_1, \zeta_2, ..., \zeta_L; r_1, r_2, ..., r_L\}$ respectively. We define $\{\tau_k\}^K_{k=1}$ and $\{\zeta_l\}^L_{l=1}$ on a common partition $\mathcal{P}$. We have

$$P_R(P_T f) = P_R \left( \sum_{k=1}^K P_{\tau_k}(p_k f) \right) = \sum_{l=1}^L \sum_{i=1}^L r_l(\zeta_{l,i}) [P_{\tau_k}(p_k f)](\zeta_{l,i}^{-1}) \frac{1}{J_{\zeta_{l,i}^{-1}}(\zeta_{l,i})} \chi_{\zeta_{l,i}}(I_i)$$

$$= \sum_{l=1}^L \sum_{i=1}^L \sum_{j=1}^J \sum_{k=1}^K r_l(\zeta_{l,i}^{-1}) p_k(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1}) f(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1})$$

$$\times \frac{1}{J_{\tau_{k,j}}(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1})} \chi_{\tau_{k,j}}(I_j)(\zeta_{l,i}^{-1}) \chi_{\zeta_{l,i}}(I_i)$$

$$= \sum_{l=1}^L \sum_{i=1}^L P_{\tau_k \circ \zeta_l}(p_k(\zeta_l) r_l f) = P_{T \circ R} f.$$

\[
\square
\]

4. The existence of absolutely continuous invariant measure on $[a, b]$

Let $(I, \mathcal{B}, \lambda)$ be a measure space, where $\lambda$ is normalized Lebesgue measure on $I = [a, b]$. Let $\tau_k : I \to I$, $k = 1, ..., K$ be piecewise one-to-one and differentiable, non-singular transformations on a partition $\mathcal{P}$ of $I : \mathcal{P} = \{I_1, ..., I_q\}$ and $\tau_{k,i} = \tau_k |_{I_i}$, $i = 1, ..., q$, $k = 1, ..., K$. Denote by $V(\cdot)$ the standard one dimensional variation of a function, and by $BV(I)$ the space of functions of bounded variations on $I$ equipped with the norm $\|\cdot\|_{BV} = V(\cdot) + \|\cdot\|_{1}$.

Let $g_k(x) = \frac{p_k(x)}{\|p_k\|_1}$, $k = 1, ..., K$. We assume that

Condition (A): $\sum_{k=1}^K g_k(x) < \alpha < 1$, $x \in I$.

Condition (B): $g_k \in BV(I)$, $k = 1, ..., K$.

Under the above conditions our goal is to prove:

$$V_1 P^T_{\alpha} f \leq AV_1 f + B \|f\|_1$$

(4.1)
for some \( n \geq 1 \), where \( 0 < A < 1 \) and \( B > 0 \). The inequality (4.1) guarantees the existence of a \( T \)-invariant measure absolutely continuous with respect to Lebesgue measure and the quasi-compactness of operator \( P_T \) with all the consequences of this fact, see [1]. We will need a number of lemmas:

**Lemma 4.1.** Let \( f \in BV(I) \). Suppose \( \tau : I \to J \) is differentiable and \( \tau'(x) \neq 0 \), \( x \in I \). Set \( \phi = \tau^{-1} \) and let \( g(x) = \frac{p(x)}{|\tau'(x)|} \in BV(I) \). Then

\[
V_I(f(\phi)g(\phi)) \leq (V_I f + \sup_I f)(V_I g + \sup_I g).
\]

*Proof.* First, note that we have dropped all the \( k, i \) indices to simplify the notation. Then, the proof follows in the same way as in Lemma 3 of [9]. \( \square \)

**Lemma 4.2.** Let \( T \) satisfy conditions (A) and (B). Then for any \( f \in BV(I) \),

\[
V_I P_T f \leq AV_I f + B\|f\|_1,
\]

where

\[
A = 3\alpha + \max_{1 \leq i \leq q} \sum_{k=1}^K V_I g_k;
\]

and

\[
B = 2\beta \alpha + \beta \max_{1 \leq i \leq q} \sum_{k=1}^K V_I g_k,
\]

where \( \beta = \max_{1 \leq i \leq q} (\lambda(I_k))^{-1} \).

*Proof.* First, we will refine partition \( \mathcal{P} \) to satisfy additional condition. Let \( \eta > 0 \) be such that \( \sum_{k=1}^K (g_k(x) + \varepsilon_k) < \alpha \) whenever \( |\varepsilon_k| < \eta \), \( k = 1, \ldots, K \). Since \( g_k \), \( k = 1, \ldots, K \) are of bounded variation we can find a finite partition \( \mathcal{K} \) such that for any \( k = 1, \ldots, K \)

\[
|g_k(x) - g_k(y)| < \eta,
\]

for \( x, y \) in the same element of \( \mathcal{K} \). Instead of the partition \( \mathcal{P} \) we consider a join \( \mathcal{P} \lor \mathcal{K} \). Without restricting generality of our considerations, we can assume that this is our original partition \( \mathcal{P} \). Then, we have

\[
\max_{1 \leq i \leq q} \sum_{k=1}^K \sup_{x \in I_k} g_k(x) < \alpha.
\]

We have \( V_I (P_T f) = V_I (\sum_{k=1}^K P_{\mathcal{K}}(p_k f)) \). We will estimate this variation. Let \( \phi_{k,i} = \tau^{-1}_{k,i} \), \( k = 1, \ldots, K, \ i = 1, \ldots, q \). We have

\[
V_I \left( \sum_{k=1}^K P_{\mathcal{K}}(p_k f) \right) = V_I \left( \sum_{k=1}^K \sum_{i=1}^q f(\phi_{k,i})g_k(\phi_{k,i})\lambda_{\mathcal{K}}(I_i) \right)
\]

\[
\leq \sum_{k=1}^K \sum_{i=1}^q |f(a_{i-1})||g_k(a_{i-1})| + |f(a_i)||g_k(a_i)|
\]

\[
+ \sum_{k=1}^K \sum_{i=1}^q V_{\mathcal{K}}(I_i)[f(\phi_{k,i})g_k(\phi_{k,i})].
\]
First, we estimate the first sum on the right hand side of (4.4):

\[
\sum_{k=1}^{K} \sum_{i=1}^{q} \left[ |f(a_{i-1})| g_k(a_{i-1}) + |f(a_i)| g_k(a_i) \right]
= \sum_{i=1}^{q} \left[ |f(a_{i-1})| \left( \sum_{k=1}^{K} |g_k(a_{i-1})| \right) + |f(a_i)| \left( \sum_{k=1}^{K} |g_k(a_i)| \right) \right]
\leq \alpha \left( \sum_{i=1}^{q} (|f(a_{i-1})| + |f(a_i)|) \right)
\leq \alpha \left( \sum_{i=1}^{q} \left( V_{I_i} f + (\lambda(I_i))^{-1} \int_{I_i} f d\lambda \right) \right) = \alpha \left( V_{I} f + \beta \| f \|_1 \right).
\]

Thus, using (4.5) and (4.6), we obtain:

\[
\sum_{k=1}^{K} \sum_{i=1}^{q} V_{\tau_k(I_i)} [f(\phi_{k\delta}) g_k(\phi_{k\delta})] \leq \sum_{k=1}^{K} \sum_{i=1}^{q} \left( V_{I_i} f + \sup_{I_i} f \right) \left( V_{I_i} g_k + \sup_{I_i} g_k \right)
\leq \sum_{i=1}^{q} \left( 2V_{I_i} f + \beta \int_{I_i} f d\lambda \right) \left( \max_{1 \leq i \leq q} \sum_{k=1}^{K} \left( V_{I_i} g_k + \sup_{I_i} g_k \right) \right)
\leq (2V_{I} f + \beta \| f \|_1) \left( \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i} g_k + \alpha \right).
\]

Thus, using (4.5) and (4.6), we obtain

\[
V_{I} V_{I} f \leq \left( 3\alpha + \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i} g_k \right) V_{I} f + \left( 2\beta \alpha + \beta \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i} g_k \right) \| f \|_1.
\]

\[
\sum_{w \in \{1, 2, \ldots, K\}^N} \left( \frac{p_w(x)}{T_w(x)} \right) < \alpha^N,
\]

where \(T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)\) and \(p_w(x) = p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)\), define random map \(T^N\).

\[
T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)
\]

with probability

\[
p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x).
\]

The maps defining \(T^N\) may be indexed by \(w \in \{1, 2, \ldots, K\}^N\). Set

\[
T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)
\]
where \( w = (k_1, ..., k_N) \), and

\[
p_w(x) = p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x).
\]

Then,

\[
T'_w(x) = \tau'_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x))\tau'_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots \tau'_{k_1}(x).
\]

Suppose that \( T \) satisfies condition (A). We will prove (4.8) using induction on \( N \).

For \( N = 1 \), we have

\[
\sum_{w \in \{1,2,\ldots,K\}^N} \frac{p_w(x)}{|T'_w(x)|} < \alpha
\]

by condition (A). Assume (4.8) is true for \( N - 1 \). Then,

\[
\sum_{w \in \{1,2,\ldots,K\}^N} \frac{p_w(x)}{|T'_w(x)|} = \sum_{w \in \{1,2,\ldots,N-1\}^N} \left( \sum_{k=1}^{K} \frac{p_k(x)p_w(\tau_k(x))}{|\tau'_k(x)|} \right) \leq \sum_{w \in \{1,2,\ldots,K\}^{N-1}} \left( \sum_{k=1}^{K} \frac{p_k(x)p_w(\tau_k(x))}{|\tau'_k(x)|} \right) < \alpha \cdot \alpha^{N-1} = \alpha^N.
\]

\[\Box\]

**Lemma 4.4.** Let \( g_w = \frac{p_w}{|T'_w|} \), where \( T_w \) and \( p_w \) are defined in Lemma 4.3, \( w \in \{1, ..., K\}^n \). Define

\[
W_1 \equiv \max_{1 \leq i \leq n} \sum_{k=1}^{K} V_i \tau_k g_k,
\]

and

\[
W_n \equiv \max_{J \in \mathcal{P}^{(n)}} \sum_{w \in \{1, ..., K\}^n} V_{J} g_w,
\]

where \( \mathcal{P}^{(n)} \) is the common monotonicity partition for all \( T_w \). Then, for all \( n \geq 1 \)

\[
W_n \leq n\alpha^{n-1}W_1,
\]

where \( \alpha \) is defined in condition (A).

**Proof.** We prove the lemma by induction on \( n \). For \( n = 1 \) the lemma is true by definition of \( W_n \). Assume that the lemma is true for \( n \), i.e.,

\[
W_n \leq n\alpha^{n-1}W_1.
\]
Let $J \in \mathcal{P}^{(n+1)}$ and $x_0 < x_1 < ... < x_I$ be a sequence of points in $J$. Then

\begin{equation}
\sum_{w} \sum_{j=0}^{I-1} |g_w(x_{j+1}) - g_w(x_j)| = \sum_{j=0}^{I-1} \sum_{w \in \{1,...,K\}^n} |g_w(x_{j+1}) - g_w(x_j)|
\end{equation}

\begin{align*}
&\leq \sum_{j=0}^{I-1} \sum_{w \in \{1,...,K\}^n} \sum_{k=1}^{K} \left| g_w(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_w(\tau_k(x_j))g_k(x_j) \right| \\
&\leq \sum_{j=0}^{I-1} \sum_{w \in \{1,...,K\}^n} \sum_{k=1}^{K} \left| g_w(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_w(\tau_k(x_{j}))g_k(x_{j+1}) \right| \\
&\quad + \sum_{j=0}^{I-1} \sum_{w \in \{1,...,K\}^n} \sum_{k=1}^{K} \left| g_w(\tau_k(x_{j+1}))g_k(x_{j}) - g_w(\tau_k(x_{j}))g_k(x_{j}) \right|
\end{align*}

\begin{align*}
&\leq \sum_{j=0}^{I-1} \sum_{k=1}^{K} \left| g_k(x_{j+1}) - g_k(x_{j}) \right| \sum_{w \in \{1,...,K\}^n} g_w(\tau_k(x_{j+1})) \\
&\quad + \sum_{j=0}^{I-1} \sum_{k=1}^{K} g_k(x_{j}) \sum_{w \in \{1,...,K\}^n} \left| g_w(\tau_k(x_{j+1})) - g_w(\tau_k(x_{j})) \right| \\
&\leq \alpha^n \sum_{j=0}^{I-1} \sum_{k=1}^{K} \left| g_k(x_{j+1}) - g_k(x_{j}) \right| \\
&\quad + \alpha \sum_{j=0}^{I-1} \sum_{w \in \{1,...,K\}^n} \left| g_w(\tau_k(x_{j+1})) - g_w(\tau_k(x_{j})) \right|
\end{align*}

\begin{equation}
\leq \alpha^n W_1 + \alpha W_n \leq \alpha^n W_1 + n\alpha^n W_1 = (n+1)\alpha^n W_1.
\end{equation}

We used condition (A) and lemma 4.3. \hfill \Box

**Theorem 4.5.** Let $T$ be a random map which satisfies conditions (A) and (B). Then $T$ preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator $P_T$ is quasi-compact on $BV(I)$, see [1].

**Proof.** Let $N$ be such that $A_N = 3\alpha^N + W_N < 1$. Then, by Lemma 4.3,

\begin{equation}
\sum_{w \in \{1,...,K\}^N} g_w(x) < \alpha^N, \quad x \in I.
\end{equation}

We refine the partition $\mathcal{P}^{(N)}$ like in the proof of Lemma 4.2, to have

\begin{equation}
\max_{J \in \mathcal{P}^{(N)}} \sum_{w \in \{1,...,K\}^N} \sup_J g_w < \alpha^N.
\end{equation}

Then, by lemma 4.2, we get

\begin{equation}
\|P_T^N f\|_{BV} \leq A_N \|f\|_{BV} + B_N \|f\|_1,
\end{equation}

where $B_N = \beta_N(2\alpha^N + W_N)$, $\beta_N = \max_{J \in \mathcal{P}^{(N)}} (\lambda(J))^{-1}$. The theorem follows by the standard technique (see [1]). \hfill \Box

**Remark 4.6.** It is enough to assume that condition (A) is satisfied for some iterate $T^m, m \geq 1$. 

Remark 4.7. The number of absolutely continuous invariant measures for random maps has been studied in [6]. The proof of [6], which uses graph theoretic methods, goes through analogously in our case; i.e., when \( T \) is a random map with position dependent probabilities.

5. Example

We present an example of a random map \( T \) which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A) and (B).

**Example 5.1.** Let \( T \) be a random map which is given by \( \{ \tau_1, \tau_2; p_1(x), p_2(x) \} \) where

\[
\tau_1(x) = \begin{cases} 
2x & \text{for } 0 \leq x \leq \frac{1}{2}, \\
x & \text{for } \frac{1}{2} < x \leq 1 
\end{cases}
\]

(5.1)

\[
\tau_2(x) = \begin{cases} 
x + \frac{1}{2} & \text{for } 0 \leq x \leq \frac{1}{2}, \\
2x - 1 & \text{for } \frac{1}{2} < x \leq 1 
\end{cases}
\]

(5.2)

and

\[
p_1(x) = \begin{cases} 
\frac{2}{3} & \text{for } 0 \leq x \leq \frac{1}{2}, \\
\frac{1}{3} & \text{for } \frac{1}{2} < x \leq 1 
\end{cases}
\]

(5.3)

\[
p_2(x) = \begin{cases} 
\frac{1}{3} & \text{for } 0 \leq x \leq \frac{1}{2}, \\
\frac{2}{3} & \text{for } \frac{1}{2} < x \leq 1 
\end{cases}
\]

(5.4)

Then, \( \sum_{k=1}^{2} g_k(x) = \frac{2}{3} < 1 \). Therefore, \( T \) satisfies conditions (A) and (B). Consequently, by theorem 4.5, \( T \) preserves an invariant measure absolutely continuous with respect to Lebesgue measure. Notice that \( \tau_1, \tau_2 \) are piecewise linear Markov maps defined on the same Markov partition \( \mathcal{P} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\} \). For such maps the Perron-Frobenius operator reduces to a matrix (see [1]). The corresponding matrices are:

\[
P_{\tau_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad P_{\tau_2} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.
\]

Their invariant densities are \( f_{\tau_1} = [0, 2] \) and \( f_{\tau_2} = [2, 0] \). The Perron-Frobenius operator of the random map \( T \) is given by:

\[
P_T = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.
\]

(5.6)

If the invariant density of \( T \) is \( f = [f_1, f_2] \), normalized by \( f_1 + f_2 = 2 \) and satisfying equation \( f P_T = f \), then \( f_1 = \frac{4}{3} \) and \( f_2 = \frac{2}{3} \).

6. The existence of absolutely continuous invariant measure in \( \mathbb{R}^n \)

Let \( S \) be a bounded region in \( \mathbb{R}^n \) and \( \lambda_n \) be Lebesgue measure on \( S \). Let \( \tau_k : S \to S, k = 1, \ldots, K \) be piecewise one-to-one and \( C^2 \), non-singular transformations on a partition \( \mathcal{P} \) of \( S : \mathcal{P} = \{S_1, \ldots, S_q\} \) and \( \tau_{k,i} = \tau_k|_{S_i}, i = 1, \ldots, q, k = 1, \ldots, K \).

Let each \( S_i \) be a bounded closed domain having a piecewise \( C^2 \) boundary of finite \( (n-1) \)-dimensional measure. We assume that the faces of \( \partial S_i \) meet at angles bounded uniformly away from 0. We will also assume that the probabilities \( p_k(x) \)
are piecewise $C^1$ functions on the partition $\mathcal{P}$. Let $D_{\tau_{k,i}}^{-1}(x)$ be the derivative matrix of $\tau_{k,i}^{-1}$ at $x$. We assume:

**Condition (C):**

$$\max_{1 \leq i \leq q} \sum_{k=1}^K p_k(x) \| D_{\tau_{k,i}}^{-1}(\tau_{k,i}(x)) \| < \sigma < 1.$$ 

Let $\sup_{x \in \tau_i(S_i)} \| D_{\tau_{k,i}}^{-1}(x) \| := \sigma_{k,i}$ and $\sup_{x \in S_i} p_k(x) := \pi_{k,i}$. Using smoothness of $D_{\tau_{k,i}}^{-1}$'s and $p_k$'s we can refine partition $\mathcal{P}$ to satisfy

**Condition (C'):**

$$\sum_{k=1}^K \max_{1 \leq i \leq q} \sigma_{k,i} \pi_{k,i} < \sigma < 1.$$ 

Under this condition, our goal is to prove the existence of an a.c.i.m. for the random map $T = \{\tau_1, ..., \tau_K; p_1, ..., p_K\}$. The main tool of this section is the multi-dimensional notion of variation defined using derivatives in the distributional sense (see [3]):

$$V(f) = \int_{\mathbb{R}^n} \| Df \| = \sup \{ \int_{\mathbb{R}^n} f \text{div}(g) d\lambda_n : g = (g_1, ..., g_n) \in C^1_0(\mathbb{R}^n, \mathbb{R}^n) \},$$

where $f \in L_1(\mathbb{R}^n)$ has bounded support, $Df$ denotes the gradient of $f$ in the distributional sense, and $C^1_0(\mathbb{R}^n, \mathbb{R}^n)$ is the space of continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ having a compact support. We will use the following property of variation which is derived from [3], Remark 2.14: If $f = 0$ outside a closed domain $A$ whose boundary is Lipschitz continuous, $\int_A f$ is continuous, $\int_{\text{int}(A)} f$ is $C^1$, then

$$V(f) = \int_{\text{int}(A)} \| Df \| d\lambda_n + \int_{\partial A} |f| d\lambda_{n-1},$$

where $\lambda_{n-1}$ is the $n-1$-dimensional measure on the boundary of $A$. In this section we shall consider the Banach space (see [3], Remark 1.12),

$$BV(S) = \{ f \in L_1(S) : V(f) < +\infty \},$$

with the norm $\| f \|_{BV} = V(f) + \| f \|_1$. We adapt the following two lemmas from [5]. The proofs of Lemma 6.1 and Lemma 6.2 are exactly the same as in [5].

**Lemma 6.1.** Consider $S_i \in \mathcal{P}$. Let $x$ be a point in $\partial S_i$ and $y = \tau_k(x)$ a point in $\partial(\tau_k(S_i))$. Let $J_{k,i}$ be the Jacobian of $\tau_k|_{S_i}$ at $x$ and $J_{k,i}^0$ be the Jacobian of $\tau_k$ at $x$. Then

$$\frac{J_{k,i}^0}{J_{k,i}} \leq \sigma_{k,i}. \quad \square$$

Let us fix $1 \leq i \leq q$. Let $Z$ denote the set of singular points of $\partial S_i$. Let us construct at any $x \in Z$ the largest cone having a vertex at $x$ and which lies completely in $S_i$. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\beta(S_i) = \min_{x \in Z} \theta(x).$$

Since the faces of $\partial S_i$ meet at angles bounded away from 0, $\beta(S_i) > 0$. Let $\alpha(S_i) = \pi/2 + \beta(S_i)$ and

$$\alpha(S_i) = |\cos(\alpha(S_i))|. \quad \square$$
Now we will construct a $C^1$ field of segments $L_y$, $y \in \partial S_i$, every $L_y$ being a central ray of a regular cone contained in $S_i$, with angle subtended at the vertex $y$ greater than or equal to $\beta(S_i)$.

We start at points $y \in Z$, where the minimal angle $\beta(S_i)$ is attained, defining $L_y$ to be central rays of the largest regular cones contained in $S_i$. Then we extend this field of segments to $C^1$ field we want, making $L_y$ short enough to avoid overlapping. Let $\delta(y)$ be the length of $L_y$, $y \in \partial S_i$. By the compactness of $\partial S_i$ we have

$$\delta(S_i) = \inf_{y \in \partial S_i} \delta(y) > 0.$$ 

Now, we shorten $L_y$ of our field, making them all of the length $\delta(S_i)$.

**Lemma 6.2.** For any $S_i$, $i = 1, \ldots, q$, if $f$ is a $C^1$ function on $S_i$, then

$$\int_{DS_i} f(y) d\lambda_{n-1}(y) \leq \frac{1}{a(S_i)} \left( \frac{1}{\delta(S_i)} \int_{S_i} f d\lambda_n + V_{\text{intr}}(S_i)(f) \right).$$

□

Our main technical result is the following:

**Theorem 6.3.** If $T$ is a random map which satisfies Condition (C), then

$$V(P_T f) \leq \sigma(1 + 1/a) V(f) + (M + \frac{\sigma}{a^2}) ||f||_1,$$

where $a = \min\{a(S_i) : i = 1, \ldots, q\} > 0$, $\delta = \min\{\delta(S_i) : i = 1, \ldots, q\} > 0$,

$$M_{k,i} = \sup_{x \in S_i} \left( Dp_k(x) - \frac{D_{\partial S_i} p_k(x)}{J_{k,i}(x)} \right)$$

and $M = \sum_{k=1}^K \max_{1 \leq i \leq q} M_{k,i}$.

**Proof.** We have $V(P_T f) \leq \sum_{k=1}^K V(P_{\tau_k}(p_k f))$. We first estimate $V(P_{\tau_k}(p_k f))$. Let $F_{k,i} = \frac{f(\tau_k^{-1}) p_k(\tau_k^{-1})}{J_{k,i}(\tau_k^{-1})}$, and $R_{k,i} = \tau_k^{-1}(S_i)$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Then,

(6.1)

$$\int_{\mathbb{R}^n} ||D P_k(p_k f)|| d\lambda_n \leq \sum_{i=1}^q \int_{\mathbb{R}^n} ||D(F_{k,i} \chi_{R_i})|| d\lambda_n$$

$$\leq \sum_{i=1}^q \left( \int_{\mathbb{R}^n} ||D(F_{k,i} \chi_{R_i})|| d\lambda_n + \int_{\mathbb{R}^n} ||F_{k,i} || d\lambda_n \right).$$

Now, for the first integral we have,

(6.2)

$$\int_{\mathbb{R}^n} ||D(F_{k,i} \chi_{R_i})|| d\lambda_n = \int_{R_i} ||D(F_{k,i} \chi_{R_i})|| d\lambda_n$$

$$\leq \int_{R_i} ||D(f(\tau_k^{-1}))\frac{p_k(\tau_k^{-1})}{J_{k,i}(\tau_k^{-1})}|| d\lambda_n + \int_{R_i} ||f(\tau_k^{-1})D\left(\frac{p_k(\tau_k^{-1})}{J_{k,i}(\tau_k^{-1})}\right)|| d\lambda_n$$

$$\leq \int_{R_i} ||D(f(\tau_k^{-1}))|| ||D\tau_k^{-1}||\frac{p_k(\tau_k^{-1})}{J_{k,i}(\tau_k^{-1})}|| d\lambda_n + \int_{R_i} ||f(\tau_k^{-1})||\frac{M_k}{J_{k,i}(\tau_k^{-1})} d\lambda_n$$

$$\leq \sigma_{k,i}\tau_{k,i} \int_{S_i} ||Df|| d\lambda_n + M_k \int_{S_i} ||f|| d\lambda_n.$$
For the second integral we have, \[ (6.3) \]
\[ \int_{\mathbb{R}^n} \| F_{k,i}(D\chi_{R_i}) \| d\lambda_n = \int_{\partial R_i} |f(\tau_{k,i}^{-1})| \frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda_{n-1} = \int_{\partial S_i} |f| p_k J_{k,i}^0 d\lambda_{n-1}. \]

By Lemma 4.3, \[ \frac{J_{k,i}^0}{J_{k,i}} \leq \sigma_{k,i}. \] Using Lemma 4.2, we get:
\[ (6.4) \]
\[ \int_{\mathbb{R}^n} \| F_{k,i}(D\chi_{R_i}) \| d\lambda_n \leq \sigma_{k,i} \bar{\tau}_{k,i} \int_{\partial S_i} |f| d\lambda_{n-1} \]
\[ \leq \frac{\sigma_{k,i} \bar{\tau}_{k,i}}{a} V_{S_i}(f) + \frac{\sigma_{k,i} \bar{\tau}_{k,i}}{a \delta} \int_{S_i} |f| d\lambda_n. \]

Using Condition (C'), summing first over \( i \), we obtain
\[ V(P_{\tau}(p_kf)) \leq (\max_{1 \leq k \leq q} \sigma_{k,i} \bar{\tau}_{k,i})(1+1/a)V(f) + (\max_{1 \leq k \leq q} M_{k,i} + \frac{\max_{1 \leq i \leq q} \sigma_{k,i} \bar{\tau}_{k,i}}{a \delta}) \| f \|_1, \]
and then, summing over \( k \) we obtain
\[ V(P_T f) \leq \sigma(1 + 1/a)V(f) + (M + \frac{\sigma}{a \delta}) \| f \|_1. \]

\[ \square \]

**Theorem 6.4.** Let \( T \) be a random map which satisfies condition (C). If \( \sigma(1 + 1/a) < 1 \), then \( T \) preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator \( P_T \) is quasi-compact on \( BV(S) \), see [1].

**Proof.** The proof of the theorem follows by the standard technique (see [1]). \[ \square \]

### 7. Example in \( \mathbb{R}^2 \)

In this section, We present an example of a random map which satisfies condition (C) of theorem 6.3 and thus it preserves an absolutely continuous invariant measure.

**Example 7.1.** Let \( T \) be a random map which is given by \( \{ \tau_1, \tau_2; p_1(x), p_2(x) \} \)
where \( \tau_1, \tau_2 : I^2 \to I^2 \) defined by:
\[ (7.1) \]
\[ \tau_1(x_1, x_2) = \begin{cases} 
(3x_1, 2x_2) & \text{for } (x_1, x_2) \in S_1 = \{ 0 \leq x_1, x_2 \leq \frac{1}{3} \} \\
(3x_1 - 1, 2x_2) & \text{for } (x_1, x_2) \in S_2 = \{ \frac{1}{3} < x_1 \leq \frac{2}{3}, 0 \leq x_2 \leq \frac{1}{3} \} \\
(3x_1 - 2, 2x_2) & \text{for } (x_1, x_2) \in S_3 = \{ \frac{2}{3} < x_1 \leq 1; 0 \leq x_2 \leq \frac{1}{3} \} \\
(3x_1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_4 = \{ 0 < x_1 \leq \frac{1}{3}, \frac{1}{3} < x_2 \leq \frac{2}{3} \} \\
(3x_1 - 1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_5 = \{ \frac{1}{3} < x_1 \leq \frac{2}{3}, \frac{1}{3} < x_2 \leq \frac{2}{3} \} \\
(3x_1 - 2, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_6 = \{ \frac{2}{3} < x_1 \leq 1; \frac{1}{3} < x_2 \leq \frac{2}{3} \} \\
(3x_1 - 1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_7 = \{ 0 \leq x_1 \leq \frac{1}{3}, \frac{2}{3} < x_2 \leq 1 \} \\
(3x_1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_8 = \{ \frac{1}{3} < x_1 \leq \frac{2}{3}, \frac{2}{3} < x_2 \leq 1 \} \\
(3x_1 - 1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_9 = \{ \frac{2}{3} < x_1 \leq 1; \frac{2}{3} < x_2 \leq 1 \}
\end{cases} \]
and the derivative matrix of $(\tau_1, \bar{d})^{-1}$, is

\begin{equation}
\begin{pmatrix}
\frac{1}{3} & 0 \\
0 & 0
\end{pmatrix}
\end{equation}

or

\begin{equation}
\begin{pmatrix}
\frac{1}{3} & 0 \\
0 & 0
\end{pmatrix}
\end{equation}

and the derivative matrix of $(\tau_2, \bar{d})^{-1}$, is

\begin{equation}
\begin{pmatrix}
\frac{1}{3} & 0 \\
0 & 0
\end{pmatrix}
\end{equation}

or

\begin{equation}
\begin{pmatrix}
\frac{-1}{3} & 0 \\
0 & 0
\end{pmatrix}
\end{equation}

Therefore, the Euclidean matrix norm, $\|D(\tau_1, \bar{d})^{-1}\|$ is $\sqrt{\frac{2}{3}}$, or $\sqrt{\frac{13}{6}}$ and the Euclidean matrix norm, $\|D(\tau_2, \bar{d})^{-1}\|$ is $\sqrt{\frac{2}{3}}$. Then

\[
\max_{1 \leq i \leq q} \sum_{k=1}^{K} p_k(x)\|D_{k,i}^{-1}(\tau_{k,i}(x))\| \leq 0.216 \sqrt{\frac{13}{6}} + 0.785 \sqrt{\frac{2}{3}}
\]

For this partition $\mathcal{P}$, we have $a = 1$, which implies

\[
\sigma(1 + 1/a) = 2(0.216 \sqrt{\frac{13}{6}} + 0.785 \sqrt{\frac{2}{3}}) \approx 0.9998 < 1.
\]

Therefore, by theorem 6.4, the random map $T$ admits an absolutely continuous invariant measure. Notice that $\tau_1, \tau_2$ are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P} = \{S_1, S_2, \ldots, S_9\}$. For such maps the Perron-Frobenius operator reduces to a matrix and the invariant density is constant on the elements of the partition (see [1]). The Perron-Frobenius operator of the random map $T$ is represented by the following matrix

\begin{equation}
M = \Pi_1 M_1 + \Pi_2 M_2,
\end{equation}
where $M_1$, $M_2$ are the matrices of $P_{\gamma_1}$ and $P_{\gamma_2}$ respectively, and $\Pi_1$, $\Pi_2$ are the diagonal matrices of $p_1(x)$ and $p_2(x)$ respectively. Then, $M$ is given by

$$ (7.7) \quad M = p_1 \textbf{Id}_9 \times \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} + p_2 \textbf{Id}_9 \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} a & a & a & a & a & b & b & b \\ c & c & c & c & c & d & d & d \\ e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e \end{pmatrix}, $$

where $p_1 = (0.215, 0.216, 0.216, 0.216, 0.215, 0.216, 0.216, 0.216, 0.215)$, $p_2 = (0.785, 0.784, 0.784, 0.784, 0.785, 0.784, 0.784, 0.784, 0.785)$, $\textbf{Id}_9$ is $9 \times 9$ identity matrix and

$$ a = 0.12306 \\
b = 0.087222 \\
c = 0.12311 \\
d = 0.087111 \\
e = 0.11111. $$

The invariant density of $T$ is

$$ (7.8) \quad f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9), \quad f_i = f_i|_{S_i}, \quad i = 1, 2, \ldots, 9, $$

normalized by

$$ (7.9) \quad f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 = 9, $$

and satisfying equation $fM = f$. Then, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = \frac{9}{0.24703}$ and $f_7 = f_8 = f_9 = \frac{0.24703}{3} f_1$.

**References**


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