Position-dependent random maps in one and higher dimensions

This item was submitted to Loughborough University’s Institutional Repository by the/an author.


Metadata Record: [https://dspace.lboro.ac.uk/2134/26320](https://dspace.lboro.ac.uk/2134/26320)

Version: Accepted for publication

Publisher: Polskiej Akademii Nauk, Instytut Matematyczny

Rights: This work is made available according to the conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) licence. Full details of this licence are available at: [https://creativecommons.org/licenses/by-nc-nd/4.0/](https://creativecommons.org/licenses/by-nc-nd/4.0/)

Please cite the published version.
POSITION DEPENDENT RANDOM MAPS IN ONE AND HIGHER DIMENSIONS

WAEL BAHSOUN AND PAWEŁ GÓRA

Abstract. A random map is a discrete-time dynamical system in which one of a number of transformations is randomly selected and applied on each iteration of the process. In this paper, we study random maps with position dependent probabilities on the interval and on a bounded domain of \( \mathbb{R}^n \). Sufficient conditions for the existence of an absolutely continuous invariant measure for random map with position dependent probabilities on the interval and on a bounded domain of \( \mathbb{R}^n \) are the main results of this note.

1. Introduction

Let \( \tau_1, \tau_2, \ldots, \tau_K \) be a collection of transformations from \( X \) to \( X \). Usually, the random map \( T \) is defined by choosing \( \tau_k \) with constant probability \( p_k \), \( p_k > 0 \), \( \sum_{k=1}^{K} p_k = 1 \). The ergodic theory of such dynamical systems was studied in [9] and in [8] (See also [7]).

There is a rich literature on random maps with position dependent probabilities with \( \tau_1, \tau_2, \ldots, \tau_K \) being continuous contracting transformations (see [10]).

In this paper, we deal with piecewise monotone transformations \( \tau_1, \tau_2, \ldots, \tau_K \) and position dependent probabilities \( p_k(x) \), \( k = 1, \ldots, K \), \( p_k(x) > 0 \), \( \sum_{k=1}^{K} p_k(x) = 1 \), i.e., the \( p_k \)'s are functions of position. We point out that studying such dynamical systems was first introduced in [4] where sufficient conditions for the existence of an absolutely continuous invariant measure were given. The conditions in [4] are applicable only when \( \tau_1, \tau_2, \ldots, \tau_K \) are \( C^2 \) expanding transformations (see [4] for details). In this paper, we prove the existence of an absolutely continuous invariant measure for a random map \( T \) on \([a, b] \) under milder conditions (see section 4, Conditions (A) and (B)). Moreover, we prove the existence of an absolutely continuous invariant measure for a random map \( T \) on \( S \), where \( S \) is a bounded domain of \( \mathbb{R}^n \) (see section 6, Condition (C)).

The paper is organized in the following way: In section 2, following the ideas of [4], we formulate the definition of a random map \( T \) with position dependent probabilities and introduce its Perron-Frobenius operator. In section 3, we prove properties of the Perron-Frobenius operator of \( T \). In section 4, we prove the existence of an absolutely continuous invariant measure for \( T \) on \([a, b] \). In section 5, we give an example of a random map \( T \) which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A).
and (B). In section 6, we prove the existence of an absolutely continuous invariant measure for $T$ on a bounded domain of $\mathbb{R}^n$. In section 7, we give an example of a random map in $\mathbb{R}^n$ that preserves an absolutely continuous invariant measure.

2. PRELIMINARIES

Let $(X, \mathcal{B}, \lambda)$ be a measure space, where $\lambda$ is an underlying measure. Let $\tau_k : X \rightarrow X$, $k = 1, \ldots, K$ be piecewise one-to-one, non-singular transformations on a common partition $\mathcal{P}$ of $X : \mathcal{P} = \{I_1, \ldots, I_q\}$ and $\tau_{ki} = \tau_k \mid I_i$, $i = 1, \ldots, q$, $k = 1, \ldots, K$ ($\mathcal{P}$ can be found by considering finer partitions). We define the transition function for the random map $T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots, p_K(x)\}$ as follows:

\begin{equation}
\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x) \chi_A(\tau_k(x)),
\end{equation}

where $A$ is any measurable set and \{$(p_k(x))_{k=1}^{K}$\} is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^{K} p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in X$ and $\chi_A$ denotes the characteristic function of the set $A$. We define $T(x) = \tau_k(x)$ with probability $p_k(x)$ and $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$ with probability \( p_{k_N} \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x) \cdots p_{k_1}(x) \). The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_{\mu}$ on measures on $(X, \mathcal{B})$ defined by

\begin{equation}
\mathbb{P}_{\mu}(A) = \int \mathbb{P}(x, A) d\mu(x) = \sum_{k=1}^{K} \int p_k(x) \chi_A(\tau_k(x)) d\mu(x)
\end{equation}

\begin{equation}
= \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{ki}^{-1}(A)} p_k(x) d\mu(x)
\end{equation}

We say that measure $\mu$ is $T$-invariant iff $\mathbb{P}_{\mu} = \mu$, i.e.,

\begin{equation}
\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x), \quad A \in \mathcal{B}.
\end{equation}

If $\mu$ has density $f$ with respect to $\lambda$, the $\mathbb{P}_{\mu}$ has also a density which we denote by $P_T f$. By change of variables, we obtain

\begin{equation}
\int_{A} P_T f(x) d\lambda(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{ki}^{-1}(A)} p_k(x) f(x) d\lambda(x)
\end{equation}

\begin{equation}
= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{A} p_k(\tau_{ki}^{-1}x) f(\tau_{ki}^{-1}x) \frac{1}{J_{k,i}(\tau_{ki})} d\lambda(x)
\end{equation}

where $J_{k,i}$ is the Jacobian of $\tau_{ki}$ with respect to $\lambda$. Since this holds for any measurable set $A$ we obtain an a.e. equality:

\begin{equation}
(P_T f)(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} p_k(\tau_{ki}^{-1}x) f(\tau_{ki}^{-1}x) \frac{1}{J_{k,i}(\tau_{ki})} \chi_{\tau_k(I_i)}(x)
\end{equation}

or

\begin{equation}
(P_T f)(x) = \sum_{k=1}^{K} \mathbb{P}_{\tau_k}(p_k f)(x)
\end{equation}
where \( P, \tau_k \) is the Perron-Frobenius operator corresponding to the transformation \( \tau_k \) (see [1] for details). We call \( P, \tau \) the Perron-Frobenius of the random map \( T \). The main tool in this paper is the Perron-Frobenius of \( T \) which has very useful properties.

3. Properties of the Perron-Frobenius operator of \( T \)

The properties of \( P, \tau \) resemble the properties of the classical Perron-Frobenius operator of a single transformation.

**Lemma 3.1.** \( P, \tau \) satisfies the following properties:

(i) \( P, \tau \) is linear;
(ii) \( P, \tau \) is non-negative; i.e., \( f \geq 0 \rightarrow P, \tau f \geq 0 \);
(iii) \( P, \tau \) is \( T \)-invariant;
(iv) \( \|P, \tau f\|_1 \leq \|f\|_1 \), where \( \|\cdot\|_1 \) denotes the \( L^1 \) norm;
(v) \( P, T \circ R = P, \tau \circ T \). In particular, \( P, T \circ T = P, T \).

**Proof.** The proofs of (i)-(iv) are analogous to that for single transformation. For the proof of (v), let \( T \) and \( R \) be two random maps corresponding to \( \{\tau_1, \tau_2, ..., \tau_K; p_1, p_2, ..., p_K\} \) and \( \{\zeta_1, \zeta_2, ..., \zeta_L; r_1, r_2, ..., r_L\} \) respectively. We define \( \{\tau_k\}_{k=1}^K \) and \( \{\zeta_l\}_{l=1}^L \) on a common partition \( P \). We have

\[
P, T \circ R = P, \tau \circ T = (P, \tau P, T) = P, \tau (P, T f) = (P, T (P, \tau f)) = (P, T (P, \tau f)) \]

4. The existence of absolutely continuous invariant measure on \([a, b]\)

Let \( (I, \mathcal{B}, \mu) \) be a measure space, where \( \lambda \) is normalized Lebesgue measure on \( I = [a, b] \). Let \( \tau_k : I \rightarrow I \), \( k = 1, ..., K \) be piecewise one-to-one and differentiable, non-singular transformations on a partition \( P \) of \( I : \mathcal{P} = \{I_1, ..., I_q\} \) and \( \tau_{k,i} = \tau_k |_{I_i} \), \( i = 1, ..., q \), \( k = 1, ..., K \). Denote by \( V(\cdot) \) the standard one dimensional variation of a function, and by \( BV(I) \) the space of functions of bounded variations on \( I \) equipped with the norm \( \|\cdot\|_1 = V(\cdot) + \|\cdot\|_1 \).

Let \( g_k(x) = \frac{p_k(x)}{\int_{I_k} p_k(x) dx} \), \( k = 1, ..., K \). We assume that

**Condition (A):** \( \sum_{k=1}^K g_k(x) < \alpha < 1 \), \( x \in I \), and

**Condition (B):** \( g_k \in BV(I) \), \( k = 1, ..., K \).

Under the above conditions our goal is to prove:

\[
V, T P, f \leq AV, T f + B \|f\|_1
\]
for some \( n \geq 1 \), where \( 0 < A < 1 \) and \( B > 0 \). The inequality (4.1) guarantees the existence of a \( T \)-invariant measure absolutely continuous with respect to Lebesgue measure and the quasi-compactness of operator \( P_T \) with all the consequences of this fact, see [1]. We will need a number of lemmas:

**Lemma 4.1.** Let \( f \in BV(I) \). Suppose \( \tau : I \to J \) is differentiable and \( \tau'(x) \neq 0 \), \( x \in I \). Set \( \phi = \tau^{-1} \) and let \( g(x) = \frac{p(x)}{|\tau'(x)|} \in BV(I) \). Then

\[
V_I(f(\phi)g(\phi)) \leq (V_I f + \sup_I f)(V_I g + \sup_I g).
\]

**Proof.** First, note that we have dropped all the \( k, i \) indices to simplify the notation. Then, the proof follows in the same way as in Lemma 3 of [9]. \( \square \)

**Lemma 4.2.** Let \( T \) satisfy conditions (A) and (B). Then for any \( f \in BV(I) \),

\[
V_I P_T f \leq A V_I f + B \|f\|_1,
\]

where

\[
A = 3\alpha + \max_{1 \leq i \leq q} \sum_{k=1}^K V_I g_k;
\]

and

\[
B = 2\beta \alpha + \beta \max_{1 \leq i \leq q} \sum_{k=1}^K V_I g_k,
\]

where \( \beta = \max_{1 \leq i \leq q} (\lambda(I_k))^{-1} \).

**Proof.** First, we will refine partition \( P \) to satisfy additional condition. Let \( \eta > 0 \) be such that \( \sum_{k=1}^K (g_k(x) + \varepsilon_k) < \alpha \) whenever \( |\varepsilon_k| < \eta \), \( k = 1, \ldots, K \). Since \( g_k \), \( k = 1, \ldots, K \) are of bounded variation we can find a finite partition \( K \) such that for any \( k = 1, \ldots, K \)

\[
|g_k(x) - g_k(y)| < \eta.
\]

for \( x, y \) in the same element of \( K \). Instead of the partition \( P \) we consider a join \( P \vee K \). Without restricting generality of our considerations, we can assume that this is our original partition \( P \). Then, we have

\[
\max_{1 \leq i \leq q} \sum_{k=1}^K \sup_{x \in I_k} g_k(x) < \alpha.
\]

We have \( V_I (P_T f) = V_I (\sum_{k=1}^K P_{\eta_k}(p_k f)) \). We will estimate this variation. Let \( \phi_{k,i} = \tau_{k,i}^{-1} \), \( k = 1, \ldots, K, i = 1, \ldots, q \). We have

\[
V_I \left( \sum_{k=1}^K P_{\eta_k}(p_k f) \right) = V_I \left( \sum_{k=1}^K \sum_{i=1}^q f(\phi_{k,i}) g_k(\phi_{k,i}) \lambda\eta_k(I_k) \right)
\]

\[
\leq \sum_{k=1}^K \sum_{i=1}^q [f(a_{i-1})||g_k(a_{i-1})|| + |f(a_i)||g_k(a_i)||]
\]

\[
+ \sum_{k=1}^K \sum_{i=1}^q V_{\eta_k(I_k)}[f(\phi_{k,i}) g_k(\phi_{k,i})].
\]
First, we estimate the first sum on the right hand side of (4.4):

\[
\sum_{i=1}^{q} \left[ \left| f(a_{i-1}) \right| g_k(a_{i-1}) + \left| f(a_i) \right| g_k(a_i) \right]
\]

\[
= \sum_{i=1}^{q} \left[ \left| f(a_{i-1}) \right| \left( \sum_{k=1}^{K} |g_k(a_{i-1})| \right) + \left| f(a_i) \right| \left( \sum_{k=1}^{K} |g_k(a_i)| \right) \right]
\]

\[
\leq \alpha \left( \sum_{i=1}^{q} \left( \left| f(a_{i-1}) \right| + \left| f(a_i) \right| \right) \right)
\]

\[
\leq \alpha \left( \sum_{i=1}^{q} \left( V_I f + (\lambda(I_i))^{-1} \int_{I_i} f d\lambda \right) \right) = \alpha (V_I f + \beta \|f\|_1).
\]

(4.5)

We now estimate the second sum on the right hand side of (4.4). Using Lemma 4.1 we obtain:

\[
\sum_{k=1}^{K} \sum_{i=1}^{q} V_{\tau_k(I_i)} [f(\phi_{k,i})] g_k(\phi_{k,i}) \leq \sum_{k=1}^{K} \sum_{i=1}^{q} \left( V_{I_i} f + \sup_{I_i} f \right) \left( V_{I_i} g_k + \sup_{I_i} g_k \right)
\]

\[
\leq \sum_{i=1}^{q} \left( 2V_{I_i} f + \beta \int_{I_i} f d\lambda \right) \left( \max_{1 \leq i \leq q} \sum_{k=1}^{K} \left( V_{I_i} g_k + \sup_{I_i} g_k \right) \right)
\]

\[
\leq (2V_I f + \beta \|f\|_1) \left( \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i} g_k + \alpha \right).
\]

(4.6)

Thus, using (4.5) and (4.6), we obtain

\[
V_I P_I f \leq \left( 3\alpha + \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i} g_k \right) V_I f + \left( 2\beta\alpha + \beta \max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i} g_k \right) \|f\|_1.
\]

(4.7)

\[\square\]

In the following two lemmas we show that constants \(\alpha\) and \(\max_{1 \leq i \leq q} \sum_{k=1}^{K} V_{I_i} g_k\) decrease when we consider higher iterations \(T^n\) instead of \(T\). The constant \(\beta\) obviously increases, but this is not important.

**Lemma 4.3.** Let \(T\) be a random map which satisfies condition (A). Then, for \(x \in I\),

\[
\sum_{w \in \{1, 2, \ldots, K\}^N} \left| \frac{p_w(x)}{|T_w(x)|} \right| < \alpha^N,
\]

where \(T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_1(x)\) and \(p_w(x) = p_{k_N}(\tau_{k_N} \circ \cdots \circ \tau_1(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-1}} \circ \cdots \circ \tau_1(x)) \cdots p_{k_1}(x)\), define random map \(T^N\).

**Proof.** We have

\[T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_1(x)\]

with probability

\[p_{k_N}(\tau_{k_N} \circ \cdots \circ \tau_1(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-1}} \circ \cdots \circ \tau_1(x)) \cdots p_{k_1}(x)\]

The maps defining \(T^N\) may be indexed by \(w \in \{1, 2, \ldots, K\}^N\). Set

\[T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_1(x)\]
where $w = (k_1, \ldots, k_N)$, and
\[
p_w(x) = p_{k_N}(\tau_{k_N-1} \circ \cdots \circ \tau_1(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_1(x)) \cdots p_{k_1}(x).
\]
Then,
\[
T'_w(x) = \tau_{k_N}'(\tau_{k_N-1} \circ \cdots \circ \tau_1(x))\tau_{k_{N-1}}'(\tau_{k_{N-2}} \circ \cdots \circ \tau_1(x)) \cdots \tau_1'(x).
\]
Suppose that $T$ satisfies condition (A). We will prove (4.8) using induction on $N$. For $N = 1$, we have
\[
\sum_{w \in \{1,2,\ldots,K\}^n} \frac{p_w(x)}{|T'_w(x)|} < \alpha
\]
by condition (A). Assume (4.8) is true for $N - 1$. Then,
\[
\sum_{w \in \{1,2,\ldots,K\}^n} \frac{p_w(x)}{|T'_w(x)|} = \sum_{w \in \{1,2,\ldots,K\}^n} \sum_{k=1}^K \frac{p_k(x) p_{\tau_k(x)}}{|T'_w(x)|\|T'_w(\tau_k(x))\|} \leq \left( \sum_{k=1}^K \frac{p_k(x)}{|\tau_k'(x)|} \right) \left( \sum_{w \in \{1,2,\ldots,K\}^n} \frac{p_{\tau_k(x)}}{|T'_w(\tau_k(x))|} \right) < \alpha \cdot \alpha^{N-1} = \alpha^N.
\]
\[
\square
\]

Lemma 4.4. Let $g_w = \frac{p_w}{|T'_w|}$, where $T_w$ and $p_w$ are defined in Lemma 4.3, $w \in \{1, \ldots, K\}^n$. Define
\[
W_1 \equiv \max_{1 \leq i \leq n} \sum_{k=1}^K V_i g_k,
\]
and
\[
W_n \equiv \max_{J \in \mathcal{P}^n} \sum_{w \in \{1,\ldots,K\}^n} V_J g_w,
\]
where $\mathcal{P}^n$ is the common monotonicity partition for all $T_w$. Then, for all $n \geq 1$
\[
W_n \leq n\alpha^{n-1}W_1,
\]
where $\alpha$ is defined in condition (A).

Proof. We prove the lemma by induction on $n$. For $n = 1$ the lemma is true by definition of $W_n$. Assume that the lemma is true for $n$, i.e.,
\[
W_n \leq n\alpha^{n-1}W_1.
\]
Let $J \in \mathcal{P}^{(n+1)}$ and $x_0 < x_1 < \ldots < x_l$ be a sequence of points in $J$. Then

\begin{equation}
\sum_{w} \sum_{j=0}^{l-1} |g_w(x_{j+1}) - g_w(x_j)| = \sum_{j=0}^{l-1} \sum_{w \in \{1, \ldots, K\}^{n+1}} |g_w(x_{j+1}) - g_w(x_j)| \leq \sum_{j=0}^{l-1} \sum_{\pi \in \{1, \ldots, K\}^n} \sum_{k=1}^{K} |g_{\pi}(\tau_k(x_{j+1})) g_k(x_{j+1})| - g_{\pi}(\tau_k(x_j)) g_k(x_j)|
\end{equation}

\begin{equation}
\leq \sum_{j=0}^{l-1} \sum_{\pi \in \{1, \ldots, K\}^n} \sum_{k=1}^{K} |g_{\pi}(\tau_k(x_{j+1})) g_k(x_{j+1})| - g_{\pi}(\tau_k(x_j)) g_k(x_j)| + \sum_{j=0}^{l-1} \sum_{\pi \in \{1, \ldots, K\}^n} \sum_{k=1}^{K} |g_{\pi}(\tau_k(x_{j+1})) g_k(x_j) - g_{\pi}(\tau_k(x_j)) g_k(x_j)|
\end{equation}

\begin{equation}
\leq \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \sum_{\pi \in \{1, \ldots, K\}^n} g_{\pi}(\tau_k(x_{j+1}))
\end{equation}

\begin{equation}
+ \sum_{j=0}^{l-1} \sum_{k=1}^{K} g_k(x_j) \sum_{\pi \in \{1, \ldots, K\}^n} |g_{\pi}(\tau_k(x_{j+1})) - g_{\pi}(\tau_k(x_j))| \leq \alpha^n \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)|
\end{equation}

\begin{equation}
+ \alpha \sum_{j=0}^{l-1} \sum_{\pi \in \{1, \ldots, K\}^n} |g_{\pi}(\tau_k(x_{j+1})) - g_{\pi}(\tau_k(x_j))| \leq \alpha^n W_1 + \alpha^n W_n \leq \alpha^n W_1 + na^n W_1 = (n + 1) \alpha^n W_1.
\end{equation}

We used condition (A) and lemma 4.3.

\begin{thm}
Let $T$ be a random map which satisfies conditions (A) and (B). Then $T$ preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator $P_T$ is quasi-compact on $BV(I)$, see [1].
\end{thm}

\begin{proof}
Let $N$ be such that $A_N = 3\alpha^N + W_N < 1$. Then, by Lemma 4.3,

\begin{equation}
\sum_{w \in \{1, \ldots, K\}^N} g_w(x) < \alpha^N, \quad x \in I.
\end{equation}

We refine the partition $\mathcal{P}^{(N)}$ like in the proof of Lemma 4.2, to have

\begin{equation}
\max_{J \in \mathcal{P}^{(N)}} \sum_{w \in \{1, \ldots, K\}^N} \sup_J g_w < \alpha^N.
\end{equation}

Then, by lemma 4.2, we get

\begin{equation}
\|P_T f\|_{BV} \leq A_N \|f\|_{BV} + B_N \|f\|_{1},
\end{equation}

where $B_N = \beta_N (2\alpha^N + W_N)$, $\beta_N = \max_{J \in \mathcal{P}^{(N)}} (\lambda(J))^{-1}$. The theorem follows by the standard technique (see [1]).

\begin{rem}
It is enough to assume that condition (A) is satisfied for some iterate $T^m, m \geq 1$.
\end{rem}
Remark 4.7. The number of absolutely continuous invariant measures for random maps has been studied in [6]. The proof of [6], which uses graph theoretic methods, goes through analogously in our case; i.e., when $T$ is a random map with position dependent probabilities.

5. Example

We present an example of a random map $T$ which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A) and (B).

Example 5.1. Let $T$ be a random map which is given by \{\(\tau_1, \tau_2; p_1(x), p_2(x)\)\} where

\[
\tau_1(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq \frac{1}{2}, \\ x & \text{for } \frac{1}{2} < x \leq 1, \end{cases} \tag{5.1}
\]

\[
\tau_2(x) = \begin{cases} x + \frac{1}{2} & \text{for } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{for } \frac{1}{2} < x \leq 1, \end{cases} \tag{5.2}
\]

and

\[
p_1(x) = \begin{cases} \frac{2}{3} & \text{for } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{3} & \text{for } \frac{1}{2} < x \leq 1, \end{cases} \tag{5.3}
\]

\[
p_2(x) = \begin{cases} \frac{1}{3} & \text{for } 0 \leq x \leq \frac{1}{2}, \\ \frac{2}{3} & \text{for } \frac{1}{2} < x \leq 1. \end{cases} \tag{5.4}
\]

Then, \(\sum_{k=1}^{2} g_k(x) = \frac{2}{3} < 1\). Therefore, $T$ satisfies conditions (A) and (B). Consequently, by theorem 4.5, $T$ preserves an invariant measure absolutely continuous with respect to Lebesgue measure. Notice that \(\tau_1, \tau_2\) are piecewise linear Markov maps defined on the same Markov partition \(\mathcal{P} : \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}\). For such maps the Perron-Frobenius operator reduces to a matrix (see [1]). The corresponding matrices are:

\[
P_{\tau_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 \end{pmatrix}, \quad P_{\tau_2} = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{5.5}
\]

Their invariant densities are \(f_{\tau_1} = [0, 2]\) and \(f_{\tau_2} = [2, 0]\). The Perron-Frobenius operator of the random map $T$ is given by:

\[
P_T = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{5.6}
\]

If the invariant density of $T$ is \(f = [f_1, f_2]\), normalized by \(f_1 + f_2 = 2\) and satisfying equation \(fP_T = f\), then \(f_1 = \frac{2}{3}\) and \(f_2 = \frac{1}{3}\).

6. The existence of absolutely continuous invariant measure in \(\mathbb{R}^n\)

Let $S$ be a bounded region in $\mathbb{R}^n$ and $\lambda_n$ be Lebesgue measure on $S$. Let \(\tau_k : S \to S, k = 1, \ldots, K\) be piecewise one-to-one and $C^2$, non-singular transformations on a partition $\mathcal{P}$ of $S : \mathcal{P} = \{S_1, \ldots, S_q\}$ and $\tau_{kj} = \tau_k | S_j$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Let each $S_i$ be a bounded closed domain having a piecewise $C^2$ boundary of finite $(n - 1)$-dimensional measure. We assume that the faces of $\partial S_i$ meet at angles bounded uniformly away from 0. We will also assume that the probabilities $p_k(x)$
are piecewise $C^1$ functions on the partition $\mathcal{P}$. Let $D\tau_{k,i}^{-1}(x)$ be the derivative matrix of $\tau_{k,i}^{-1}$ at $x$. We assume:

**Condition (C):**

$$\max_{1 \leq i \leq q} \sum_{k=1}^{K} p_k(x) \|D\tau_{k,i}^{-1}(\tau_{k,i}(x))\| < \sigma < 1.$$ 

Let $\sup_{x \in \tau_{k,i}(S_i)} \|D\tau_{k,i}^{-1}(x)\| := \sigma_{k,i}$ and $\sup_{x \in S_i} p_k(x) := \pi_{k,i}$. Using smoothness of $D\tau_{k,i}^{-1}$ and $p_k$'s we can refine partition $\mathcal{P}$ to satisfy

**Condition (C'):**

$$\sum_{k=1}^{K} \max_{1 \leq i \leq q} \sigma_{k,i} \pi_{k,i} < \sigma < 1.$$ 

Under this condition, our goal is to prove the existence of an a.c.i.m. for the random map $T = \{\tau_1, ..., \tau_K; p_1, ..., p_K\}$. The main tool of this section is the multidimensional notion of variation defined using derivatives in the distributional sense (see [3]):

$$V(f) = \int_{\mathbb{R}^n} \|Df\| = \sup \{\int_{\mathbb{R}^n} f \text{div}(g) d\lambda_n : g = (g_1, ..., g_n) \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)\},$$

where $f \in L_1(\mathbb{R}^n)$ has bounded support, $Df$ denotes the gradient of $f$ in the distributional sense, and $C_0^1(\mathbb{R}^n, \mathbb{R}^n)$ is the space of continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ having a compact support. We will use the following property of variation which is derived from [3], Remark 2.14: If $f = 0$ outside a closed domain $A$ whose boundary is Lipschitz continuous, $\int_A f$ is continuous, $\int_{\text{int}(A)} f$ is $C^1$, then

$$V(f) = \int_{\text{int}(A)} \|Df\| d\lambda_n + \int_{\partial A} |f| d\lambda_{n-1},$$

where $\lambda_{n-1}$ is the $(n-1)$-dimensional measure on the boundary of $A$. In this section we shall consider the Banach space (see [3], Remark 1.12),

$$BV(S) = \{f \in L_1(S) : V(f) < +\infty\},$$

with the norm $\|f\|_{BV} = V(f) + \|f\|_1$. We adapt the following two lemmas from [5]. The proofs of Lemma 6.1 and Lemma 6.2 are exactly the same as in [5].

**Lemma 6.1.** Consider $S_i \in \mathcal{P}$. Let $x$ be a point in $\partial S_i$ and $y = \tau_k(x)$ a point in $\partial(\tau_k(S_i))$. Let $J_{k,i}$ be the Jacobian of $\tau_k|_{S_i}$ at $x$ and $J_{k,i}^0$ be the Jacobian of $\tau_k|_{S_i}$ at $x$. Then

$$\frac{J_{k,i}^0}{J_{k,i}} \leq \sigma_{k,i}. \quad \square$$

Let us fix $1 \leq i \leq q$. Let $Z$ denote the set of singular points of $\partial S_i$. Let us construct at any $x \in Z$ the largest cone having a vertex at $x$ and which lies completely in $S_i$. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\beta(S_i) = \min_{x \in Z} \theta(x).$$

Since the faces of $\partial S_i$ meet at angles bounded away from 0, $\beta(S_i) > 0$. Let $\alpha(S_i) = \pi/2 + \beta(S_i)$ and

$$\alpha(S_i) = |\cos(\alpha(S_i))|. $$
Now we will construct a $C^1$ field of segments $L_y$, $y \in \partial S_i$, every $L_y$ being a central ray of a regular cone contained in $S_i$, with angle subtended at the vertex $y$ greater than or equal to $\beta(S_i)$.

We start at points $y \in Z$, where the minimal angle $\beta(S_i)$ is attained, defining $L_y$ to be central rays of the largest regular cones contained in $S_i$. Then we extend this field of segments to $C^1$ field we want, making $L_y$ short enough to avoid overlapping. Let $\delta(y)$ be the length of $L_y$, $y \in \partial S_i$. By the compactness of $\partial S_i$ we have

$$\delta(S_i) = \inf_{y \in \partial S_i} \delta(y) > 0.$$ 

Now, we shorten $L_y$ of our field, making them all of the length $\delta(S_i)$.

**Lemma 6.2.** For any $S_i$, $i = 1, \ldots, q$, if $f$ is a $C^1$ function on $S_i$, then

$$\int_{D S_i} f(y) d\lambda_{n-1}(y) \leq \frac{1}{a(S_i)} \left( \frac{1}{\delta(S_i)} \int_{S_i} f d\lambda_n + V_{\inf(S_i)}(f) \right).$$

\[\square\]

Our main technical result is the following:

**Theorem 6.3.** If $T$ is a random map which satisfies Condition (C), then

$$V(P_T f) \leq \sigma(1 + 1/a) V(f) + (M + \frac{\sigma}{a^0}) \|f\|_1,$$

where $a = \min \{a(S_i) : i = 1, \ldots, q\} > 0$, $\delta = \min \{\delta S_i : i = 1, \ldots, q\} > 0$, $M_{k,d} = \sup_{x \in S_i} (D p_k(x) - \frac{D J_{k,d}}{J_{k,d}} p_k(x))$ and $M = \sum_{k=1}^{K} \max_{i \leq q} M_{k,i}$.

**Proof.** We have $V(P_T f) \leq \sum_{k=1}^{K} V(P_{T_k}(p_k f))$. We first estimate $V(P_{T_k}(p_k f))$. Let $F_k, \tau_k = \tau_k(S_i)$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Then,

\begin{equation}
\int_{\mathbb{R}^n} \|D P_{T_k}(p_k f)\| d\lambda_n \leq \sum_{i=1}^{q} \int_{\mathbb{R}^n} \|D (F_k \chi_{R_i})\| d\lambda_n \leq \sum_{i=1}^{q} \left( \int_{\mathbb{R}^n} \|D (F_k \chi_{R_i})\chi_{R_i}\| d\lambda_n + \int_{\mathbb{R}^n} \|F_k \chi_{R_i} D \chi_{R_i}\| d\lambda_n \right).
\end{equation}

Now, for the first integral we have,

\begin{equation}
\int_{\mathbb{R}^n} \|D (F_k \chi_{R_i})\chi_{R_i}\| d\lambda_n = \int_{R_i} \|D (F_k \chi_{R_i})\| d\lambda_n \leq \int_{R_i} \|D f(\tau_k^{-1})\| p_k(\tau_k^{-1}) \| D \chi_{R_i}\| d\lambda_n + \int_{R_i} \|f(\tau_k^{-1})\| D \left( \frac{p_k(\tau_k^{-1})}{J_{k,d}(\tau_k^{-1})} \right) \| D \chi_{R_i}\| d\lambda_n \leq \sigma k \int_{S_i} \|D f\| d\lambda_n + M_k \int_{S_i} \|f\| d\lambda_n.
\end{equation}
For the second integral we have,

\begin{equation}
\int_{\mathbb{R}^n} \| F_{k,i}(D_X R_{i}) \| d\lambda_n = \int_{\partial R_i} |f(\tau_{k,i}^{-1} x)| \frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda_{n-1} = \int_{\partial S_i} |f| p_k \frac{J_{k,i}^0}{J_{k,i}} d\lambda_{n-1}.
\end{equation}

By Lemma 4.3, \( \frac{J_{k,i}^0}{J_{k,i}} \leq \sigma_{k,i} \). Using Lemma 4.2, we get:

\begin{equation}
\int_{\mathbb{R}^n} \| F_{k,i}(D_X R_{i}) \| d\lambda_n \leq \sigma_{k,i} \tau_{k,i} \int_{\partial S_i} |f| d\lambda_{n-1} \leq \frac{\sigma_{k,i} \tau_{k,i}}{a} V_{S_i}(f) + \frac{\sigma_{k,i} \tau_{k,i}}{a \delta} \int_{S_i} |f| d\lambda_n.
\end{equation}

Using Condition (C'), summing first over \( i \), we obtain

\[ V(\Phi_{\tau_{k,i}}(p_k f)) \leq (\max_{1 \leq i \leq q} \sigma_{k,i} \tau_{k,i}) (1 + 1/a) V(f) + \left( \max_{1 \leq i \leq q} M_{k,i} + \frac{\max_{1 \leq i \leq q} \sigma_{k,i} \tau_{k,i}}{a \delta} \right) \| f \|_1, \]

and then, summing over \( k \) we obtain

\[ V(\Phi f) \leq \sigma (1 + 1/a) V(f) + (M + \frac{\sigma}{a \delta}) \| f \|_1. \]

\[ \square \]

**Theorem 6.4.** Let \( T \) be a random map which satisfies condition (C). If \( \sigma (1 + 1/a) < 1 \), then \( T \) preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator \( P_T \) is quasi-compact on \( BV(S) \), see [1].

**Proof.** The proof of the theorem follows by the standard technique (see [1]). \( \square \)

7. Example in \( \mathbb{R}^2 \)

In this section, We present an example of a random map which satisfies condition (C) of theorem 6.3 and thus it preserves an absolutely continuous invariant measure.

**Example 7.1.** Let \( T \) be a random map which is given by \( \{\tau_1, \tau_2; p_1(x), p_2(x)\} \) where \( \tau_1, \tau_2 : I^2 \to I^2 \) defined by:

\begin{equation}
\tau_1(x_1, x_2) = \left\{ \begin{array}{ll}
(3x_1, 2x_2) & \text{for } (x_1, x_2) \in S_1 = \{0 \leq x_1, x_2 \leq \frac{1}{3}\} \\
(3x_1 - 1, 2x_2) & \text{for } (x_1, x_2) \in S_2 = \{\frac{1}{3} < x_1 \leq \frac{1}{2}; 0 \leq x_2 \leq \frac{1}{3}\} \\
(3x_1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_3 = \{\frac{1}{3} < x_1 \leq 1; 0 \leq x_2 \leq \frac{1}{3}\} \\
(3x_1 - 1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_4 = \{0 < x_1 \leq \frac{1}{3}; \frac{1}{2} < x_2 \leq \frac{1}{3}\} \\
(3x_1 - 2, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_5 = \{\frac{1}{3} < x_1 \leq 1; \frac{1}{2} < x_2 \leq \frac{1}{3}\} \\
(3x_1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_6 = \{0 < x_1 \leq \frac{1}{3}; \frac{1}{3} < x_2 \leq 1\} \\
(3x_1 - 1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_7 = \{\frac{1}{3} < x_1 \leq 1; \frac{1}{3} < x_2 \leq \frac{1}{2}\} \\
(3x_1 - 2, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_8 = \{\frac{1}{3} < x_1 \leq 1; 0 < x_2 \leq 1\}
\end{array} \right\}
\end{equation}
Therefore, the Euclidean matrix norm, 
\[ \| D(\tau_{1,\delta})^{-1} \| \]  
and the derivative matrix of \( (\tau_{2,\delta})^{-1} \), is
\[ \| D(\tau_{2,\delta})^{-1} \| \]  
Therefore, the Euclidean matrix norm, \( \| D(\tau_{1,\delta})^{-1} \| \) is \( \frac{\sqrt{2}}{3} \), or \( \frac{\sqrt{3}}{6} \) and the Euclidean matrix norm, \( \| D(\tau_{2,\delta})^{-1} \| \) is \( \frac{\sqrt{2}}{3} \). Then
\[ \max_{1 \leq i < q} \sum_{k=1}^{K} p_k(x)\| D_{k,i}^{-1}(\tau_{k,\delta}(x)) \| \leq 0.216\frac{\sqrt{3}}{6} + 0.785\frac{\sqrt{2}}{3} \]
For this partition \( \mathcal{P} \), we have \( a = 1 \), which implies
\[ \sigma(1 + 1/a) = 2(0.216\frac{\sqrt{3}}{6} + 0.785\frac{\sqrt{2}}{3}) \approx 0.9998 < 1. \]
Therefore, by theorem 6.4, the random map \( T \) admits an absolutely continuous invariant measure. Notice that \( \tau_1, \tau_2 \) are piecewise linear Markov maps defined on the same Markov partition \( \mathcal{P} = \{ S_1, S_2, \ldots, S_9 \} \). For such maps the Perron-Frobenius operator reduces to a matrix and the invariant density is constant on the elements of the partition (see [1]). The Perron-Frobenius operator of the random map \( T \) is represented by the following matrix
\[ M = \Pi_1 M_1 + \Pi_2 M_2, \]
where $M_1$, $M_2$ are the matrices of $P_{\tau_1}$ and $P_{\tau_2}$ respectively, and $\Pi_1$, $\Pi_2$ are the diagonal matrices of $p_1(x)$ and $p_2(x)$ respectively. Then, $M$ is given by

\begin{equation}
M = p_1\mathbf{1}_9 \times \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
+ p_2\mathbf{1}_9 \times \begin{pmatrix}
0.215 & 0.215 & 0.216 & 0.216 & 0.216 & 0.216 & 0.216 & 0.216 & 0.215
\end{pmatrix}
\end{equation}

where $p_1 = (0.215, 0.216, 0.216, 0.216, 0.215, 0.216, 0.216, 0.216, 0.215)$, $p_2 = (0.785, 0.784, 0.784, 0.785, 0.785, 0.784, 0.784, 0.784, 0.785)$, $\mathbf{1}_9$ is $9 \times 9$ identity matrix and

\begin{align*}
0.12306 & = a \\
0.087222 & = b \\
0.12311 & = c \\
0.087111 & = d \\
0.111111 & = e.
\end{align*}

The invariant density of $T$ is

\begin{equation}
f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9), \quad f_i = f_{|S_i|}, \quad i = 1, 2, \ldots, 9,
\end{equation}

normalized by

\begin{equation}
f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 = 9,
\end{equation}

and satisfying equation $fM = f$. Then, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = \frac{9}{(\alpha_{2973})}$, and $f_7 = f_8 = f_9 = \frac{0.02730}{3} f_1$.

References


DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF VICTORIA, PO BOX 3045
STN CSC, VICTORIA, B.C., V8W 3P4, CANADA
E-mail address: wab@math.uvic.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, CONCORDIA UNIVERSITY, 7141 SHERBROOKE
STREET WEST, MONTREAL, QUEBEC H4B 1R6, CANADA
E-mail address: pgoravaz2.concordia.ca